A Rheological Investigation of Asphaltic Materials

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MEMORANDUM

TO: W. B. Drake, Assistant State Highway Engineer
   Chairman, Kentucky Highway Research Committee

RE: "Rheological Investigation of Asphalitic Materials"
   KYHPR-64-20; HPS-HPR-1(25)

The attached report, entitled as above, by W. A. Mossbarger, Jr., Research Engineer, represents a basic preliminary study of visco-elastic principles from the standpoint of their derivation through tensor notations and their application to semi-solid materials which are exemplified largely by bituminous materials. The report is considered to be preliminary inasmuch as it represents a preparatory phase of KYHPR-64-20. The work was begun prior to July 1, 1963, and was continued under HPS-HPR-1(25). The report is being furnished for informational purposes only; and, although comments and suggestions are invited, no specific response is requested. Inasmuch as the work became identified with the Department’s cooperative planning and research program, HPS-HPR-1(25), July 1, 1963, copies of this report are being transmitted to the Bureau of Public Roads in accordance with PPM 50-1.1.

This report has been submitted by Mr. Mossbarger to the Department of Civil Engineering, University of Kentucky, toward the fulfillment of his thesis requirements leading to a Master of Science degree in Civil Engineering. Extensions of this work into the realm of dynamics are in progress. The nature of this study is,
of course, extremely fundamental and highly theoretical and is somewhat limited by the necessity of developing appropriate instrumentation. Nevertheless, basic research in the fundamental mechanics of materials is considered to be very worthwhile from the standpoint of gaining additional insight into the problem of pavement design. The significance of this report is largely in the simplicity of the presentation of mathematical derivations.

Respectfully submitted,

[Signature]
Jas. H. Havens
Director of Research
Secretary, Kentucky Highway Research Committee

Enclosure

cc: Research Committee:
   R. O. Beachamp
   R. L. Campbell
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Research Report

A RHEOLOGICAL INVESTIGATION OF
ASPHALTIC MATERIALS
KYHPR-64-20; HPS-HPR-1(25)

by

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DEPARTMENT OF HIGHWAYS
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in cooperation with the
BUREAU OF PUBLIC ROADS
U. S. Department of Commerce

132 Graham Avenue
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INTRODUCTION

Asphalts, serving as cements or binders in bituminous pavements, behave essentially as purely viscous, mobile, liquids at high temperatures and assume varying degrees of viscoelasticity, plasticity, elasticity, etc., upon cooling. They stiffen to a semi-solid state at cold temperature and do not exhibit a discrete melting or freezing point. Saal (1)* classifies bitumens into three types according to their rheological properties at normal temperatures. They are: 1) those which behave entirely or almost entirely like Newtonian liquids, 2) those which show elastic effects during initial deformation and which exhibit Newtonian flow thereafter, and 3) those which show almost complete resilience within a comparatively slight range of deformation, beyond which they exhibit various degrees of permanent deformation and cease to exhibit proportionality between stress and rate-of-strain. It has been shown (2) that asphalt behavior at cold temperatures is largely independent of time and is therefore similar to the behavior of an elastic solid.

While some of these mechanical properties are desirable, others prove to be detrimental to the structural

*Numbers in parentheses refer to the List of References.
behavior of the bituminous pavement in which the asphalt is used. For example, at low temperatures, most asphalts exhibit the undesirable property of brittleness; at high temperatures, where viscous properties predominate, there is a decrease in the stability of the pavement mixture. In recent years, attempts have been made to improve these and other undesirable properties by using finely dispersed rubber or elastomers as additives to the asphalt cement.

The majority of experimental work conducted on rubberized asphalts has been based upon empirical test methods (3, 4, 5, 6, and 7). The results have indicated that rubber increases flexural resistance, ductility, and resistance to deformation; imparts elastic properties at high temperatures; and decreases brittleness at low temperatures. Although these findings are important, they are relative in the sense that most of them represent degrees of improvement which have been indicated by comparative values. In contrast to this, a more direct approach would be one which provides a knowledge of the mechanical behavior in terms of relationships between the fundamental quantities of stress, strain, and time.

The purpose of this study is to determine these fundamental stress-strain-time relationships for a rubberized asphalt cement when subjected to static stress conditions. The theoretical considerations are viewed
primarily from the standpoint of linear viscoelastic
theory which is based on the general concepts of rheology.
CHAPTER I: THEORY

This section contains brief discussions of the theories which have been developed for the elastic solid, Newtonian liquid and non-Newtonian liquids. These are included in addition to a discussion of linear viscoelastic theory because they contain concepts which are integrally related to viscoelastic theory.

Tensor notation is used throughout because of advantages it offers in the development of these physical theories.

Elastic Solid

The mechanical behavior of a homogenous, linear, elastic or Hooke solid, when written in tensor notation, is expressed by (8 and 9):

\[ \sigma_{ij} = C_{ijpq} \varepsilon_{pq} \quad (1-1) \]

where \( \sigma_{ij} \) and \( \varepsilon_{pq} \) denote the stress and strain tensors respectively, when referred to a Cartesian coordinate system and \( C_{ijpq} \) are constants. If the three mutually perpendicular axes in the coordinate system are denoted by \( x_1, x_2, \) and \( x_3 \), then the stress tensor, \( \sigma_{ij} \), when written in matrix form, appears as:

- 4 -
$$\mathbf{\sigma}_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

where $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{33}$ are normal components and $\sigma_{12}$, $\sigma_{13}$, etc., are shear components. The strain tensor may be written in a similar form. From symmetry of both the stress and strain tensors (i.e., $\sigma_{ij} = \sigma_{ji}$, $\epsilon_{ij} = \epsilon_{ji}$) (8), existence of the strain-energy function, and consideration only of isotropic materials, it can be shown (9) that $C_{ijpq}$ is a fourth-order, isotropic tensor for which the general form is:

$$C_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu (\delta_{ip} \delta_{jq} + \delta_{jp} \delta_{iq}) \quad (1-2)$$

where $\lambda$ and $\mu$ are constants and where

$$\delta_{ij} = \text{"Kronecker delta"}, \text{ defined to be equal to } 1 \text{ if } i = j \text{ and equal to } 0 \text{ if } i \neq j$$

Substitution of this result into Equation (1-1), after simplification, yields:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + \mu (\epsilon_{ij} + \epsilon_{ji}) \quad (1-3)$$

where

$$\epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \quad (1-4)$$
and $\varepsilon_{kk}$ is called the volumetric strain or cubical dilatation. The two constants, $\lambda$ and $\mu$, are known as Lame's constants.

Since both the stress tensor and strain tensor are symmetric, Equation (1-3) further reduces to:

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$  \hspace{1cm} (1-5)$$

This represents the fundamental mathematical relationship between stress and strain for a linear, homogenous, isotropic, elastic solid.

An alternate form of Equation (1-5) may be obtained by setting $i = j$:

$$\sigma_{ii} = \lambda \delta_{ii} \varepsilon_{kk} + 2\mu \varepsilon_{kk}$$  \hspace{1cm} (1-6)$$

Solving for $\varepsilon_{kk}$ from Equation (1-6) and substituting into Equation (1-5) yields:

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu (3\lambda + 2\mu)} \delta_{ij} \sigma_{kk}$$  \hspace{1cm} (1-7)$$

In view of Equation (1-7), it is of interest to consider the following three special stress conditions: 1) pure tension, 2) pure shear, and 3) hydrostatic pressure.

1) Pure tension: If a rod is subjected to an axial stress, $\sigma_{11}$, the stress tensor in matrix form becomes:
\[ \sigma_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

from which it is immediately evident that Equation (1-7) yields:

\[ \varepsilon_{11} = \frac{(\lambda + \mu)}{\mu (3\lambda + 2\mu)} \sigma_{11} \] (1-8)

\[ \varepsilon_{22} = \varepsilon_{33} = -\frac{\lambda}{2\mu (3\lambda + 2\mu)} \sigma_{11} \] (1-9)

\[ \varepsilon_{ij} = 0 \text{ for } i \neq j \]

The theoretical expressions, Equations (1-8) and (1-9), are found to be in complete agreement with experimental results where the ratio \( \sigma_{11}/\varepsilon_{11} \) and the two ratios, \( -\varepsilon_{22}/\varepsilon_{11} \) and \( -\varepsilon_{33}/\varepsilon_{11} \), are found to be constant and are defined to be Young's modulus, \( E \), and Poisson's ratio, \( \nu \), respectively. Thus, we see that:

\[ E = \frac{\mu (3\lambda + 2\mu)}{(\lambda + \mu)} \] (1-10)

and

\[ \nu = \frac{\lambda}{2(\lambda + \mu)} \] (1-11)
2) Pure shear: In pure shear, if we consider
\( \sigma_{12} = \sigma_{21} \) to be non-zero and all other \( \sigma_{ij} = 0 \), the
stress matrix becomes:

\[
\sigma_{ij} = \begin{bmatrix}
0 & \sigma_{12} & 0 \\
\sigma_{12} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and Equation (1-7) merely reduces to the expression:

\[
2\varepsilon_{12} = \sigma_{12}/u
\]  

(1-12)

The ratio of the shear stress to shear strain, \( \sigma_{12}/2\varepsilon_{12} \), * 

*The strain tensor \( \varepsilon_{ij} \), assuming small displacements
and displacement gradients, is defined in terms of dis­placement by the relationship:

\[
\varepsilon_{ij} = 1/2 \left( \frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i} \right)
\]

where

\( u_i = \) displacement

The "classical" definition of shear strain is:

\[
\gamma_{ij} = \frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}
\]

Thus, it is seen that:

\[
\gamma_{ij} = 2\varepsilon_{ij}
\]

which explains why the factor of two is included in
Equation (1-12).
is defined to be the shear modulus or modulus of rigidity, \( G \); thus from Equation (1-12),

\[
G = \mu \tag{1-13}
\]

3) Hydrostatic pressure: For this case, the stress tensor in matrix form is:

\[
\sigma_{ij} = \begin{pmatrix}
-P & 0 & 0 \\
0 & -P & 0 \\
0 & 0 & -P
\end{pmatrix}
\]

Substitution into Equation (1-7) yields:

\[
\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = -\frac{p}{3\lambda + 2\mu} \tag{1-14}
\]

and

\[
\varepsilon_{ij} = 0 \text{ for } i \neq j
\]

Here, the ratio of the pressure, \( p \), to the fractional volume change \((-\varepsilon_{kk})\) is defined to be the bulk modulus, \( K \). Hence,

\[
K = \frac{p}{-\varepsilon_{kk}} = \frac{p}{3p/(3\lambda + 2\mu)}
\]

or

\[
K = \frac{(3\lambda + 2\mu)}{3} \tag{1-15}
\]
By making use of Equations (1-10), (1-11), (1-13), and (1-15) important relationships between the material parameters $E$, $v$, $K$, and $G$ can be established. These are presented in Table I.

**TABLE I**

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$G$</th>
<th>$v$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$, $v$</td>
<td>$2G(1 + v)$</td>
<td>$G$</td>
<td>$v$</td>
<td>$\frac{2G(1 + v)}{3(1 - 2v)}$</td>
</tr>
<tr>
<td>$K$, $v$</td>
<td>$3K(1 - 2v)$</td>
<td>$\frac{3K(1 - 2v)}{2(1 + v)}$</td>
<td>$v$</td>
<td>$K$</td>
</tr>
<tr>
<td>$E$, $v$</td>
<td>$E$</td>
<td>$\frac{E}{2(1 + v)}$</td>
<td>$v$</td>
<td>$\frac{E}{3(1 - 2v)}$</td>
</tr>
<tr>
<td>$E$, $G$</td>
<td>$E$</td>
<td>$G$</td>
<td>$\frac{E - 2G}{2G}$</td>
<td>$\frac{GE}{3(3G - E)}$</td>
</tr>
<tr>
<td>$E$, $K$</td>
<td>$E$</td>
<td>$\frac{3KE}{9K - E}$</td>
<td>$\frac{3K - E}{6K}$</td>
<td>$K$</td>
</tr>
<tr>
<td>$G$, $K$</td>
<td>$\frac{9KG}{3K + G}$</td>
<td>$G$</td>
<td>$\frac{3K - 2G}{6K + 2G}$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

From the foregoing, it is seen that a knowledge of the stress-strain relationship for an elastic body, when subjected to any stress configuration, requires the de-
Newtonian Liquid

The fundamental stress-strain relationship for a Newtonian or purely viscous liquid (10), which parallels Equation (1-5) for an elastic solid, is:

\[ \sigma_{ij} = -p \delta_{ij} + \lambda \delta_{kk} \delta_{ij} + 2\mu \dot{\varepsilon}_{ij} \]  \hspace{1cm} (1-23)

where the dot denotes differentiation with respect to time. Thus, \( \dot{\varepsilon}_{ij} \) represents rate of strain and \( \delta_{kk} \) represents rate of volume strain. Here, the term, \(-p\delta_{ij}\), is included due to variation of thermodynamic variables. This variation was neglected in the discussion of the elastic solid because it is generally quite small, however it is quite large in the case of liquids and must be considered (see references 9 and 10).

Using the deviatoric and isotropic tensor concept, Equation (1-23) may be reduced to the two equations (11):

\[ \sigma + p = \frac{(3\lambda + 2\mu)}{3} \varepsilon_{ii} \]  \hspace{1cm} (1-24)

and

\[ S_{ij} = 2\mu \dot{\varepsilon}_{ij} \]  \hspace{1cm} (1-25)

Here, the coefficient of bulk viscosity, \( \lambda \), and the coefficient of viscosity, \( \eta \), may be introduced and defined by the relationships:
\[ \xi = \frac{(3\lambda + 2\mu)}{3} \]  

\[ \eta = \mu \]  

Thus Equations (1-24) and (1-25) become:

\[ \sigma + \mathbf{p} = \xi \mathbf{\varepsilon}_{ii} \]  

\[ \mathbf{S}_{ij} = 2\eta \mathbf{\varepsilon}_{ij} \]  

which completely define the stress-strain-time relationship for a Newtonian liquid.

**Non-Newtonian Liquid**

There are a number of liquids for which the simple expression given by Equation (1-29) is not applicable. For example, it has been found that some liquids exhibit a variable coefficient of viscosity which is dependent upon the rate of shear or the magnitude of the shear stress (12, 13, and 8). These are commonly called non-Newtonian liquids and generally show a decrease in the coefficient of viscosity with an increase in the rate of shear.

An example of a material which may be classed as one that exhibits non-Newtonian flow is a Bingham plastic (14 and 15). The mechanical behavior of a Bingham plastic
in simple shear is characterised by the two equations:

\[ 2\dot{e}_{ij} = 0 \quad \text{(for } S_{ij} \leq S_y) \]  
\[ (1-30) \]

and

\[ 2\dot{e}_{ij} = \frac{(S_{ij} - S_y)}{\eta_p} \quad \text{(for } S_{ij} > S_y) \]  
\[ (1-31) \]

where

\[ S_y = \text{yield value of stress} \]
\[ \eta_p = \text{coefficient of plastic viscosity} \]

If a Bingham plastic is stressed below the yield value, \( S_y \), the strain is considered to be elastic. Once the stress exceeds this yield value, the material flows in accordance with the relationship given by Equation (1-31).

Several other relationships have been used to represent the behavior of non-Newtonian materials. Two of these are (12 and 16):

\[ 2\dot{e}_{ij} = A(S_{ij} - S_y)^B \]  
\[ (1-32) \]

and

\[ 2\dot{e}_{ij} = m(S_{ij})^n \]  
\[ (1-33) \]

The quantities \( A, B, m, \) and \( n \) are constants which serve to indicate the complexity of flow. A graphical representation of the flow equations for a Newtonian liquid, Bingham
plastic, and the non-Newtonian materials represented by Equations (1-32) and (1-33) is presented in Figure 1-1.

Figure 1-1. Relationship between Stress and Rate of Strain for Different Liquids.
Viscoelasticity, as the name implies, is concerned with material behavior which may be explained by a combination of linear elastic response and linear viscous response (17). The mathematical formulation of viscoelastic behavior is achieved by establishing a relationship between the stress as a function of time and the strain as a function of time. In the past, seven different methods of specifying this stress-strain-time behavior of viscoelastic materials have been used (18). These are: 1) Maxwell Model, 2) Voigt Model, 3) Operator Equation, 4) Mechanical Impedence Function, 5) Relaxation Curve, 6) Creep Curve, and 7) Dynamic Modulus Curve. It has been shown (18) that all of these methods can be related through transformations — some of which can be rather cumbersome and are of interest only from the theoretical standpoint, and some of which are more practical and provide applicable means of conversion. The methods which are more directly applicable to this experimental study are: 1) Operator Equation, 2) Voigt Model, and 3) Creep Curve. Therefore, in order to avoid unnecessary lengthiness only these three will be discussed.

Operator Equation

The operator equation is probably the most general method of specifying viscoelastic behavior because it
establishes a general relationship between stress and strain regardless of stress conditions. Generally, there are two operator equations (19): one which includes shear effects by relating the stress and strain deviators and another which includes pressure effects by relating pressure and dilatation. These two equations are:

\[ PS_{ij} = 2Qe_{ij} \]  \hspace{1cm} (1-34)

and

\[ P'\sigma_{ii} = 3Q'\varepsilon_{ii} \]  \hspace{1cm} (1-35)

where \( P, P', Q, \) and \( Q' \) are linear, time-derivative, differential operators with constant coefficients of the form:

\[ P = \sum_{n=0}^{\infty} p_n \frac{\partial^n}{\partial t^n} , \quad Q = \sum_{n=0}^{\infty} q_n \frac{\partial^n}{\partial t^n} , \]  \hspace{1cm} (1-36)

where \( p_n \) and \( q_n \) are constants. Of course, it is immediately evident that these two equations are analogous to Equations (1-21) and (1-22) which were written for the elastic solid. If we let

\[ Q/P = G \]  \hspace{1cm} (1-37)

and

\[ Q'/P' = K \]  \hspace{1cm} (1-38)

the equations are identical. Furthermore, since the op-
erators for an elastic solid are of zero order, it is seen that Equations (1-36) yield:

\[ P = P_0, \quad P' = P_0, \quad Q = q_0, \text{ and } Q' = q_0 \]

Again, comparing Equations (1-34) and (1-35) with Equations (1-21) and (1-22) it is seen that

\[ P_0 = 1, \quad p_0 = 1, \quad q_0 = G, \text{ and } q_0' = K \]

for an elastic solid.

This illustrates an important concept which may be utilized to determine the operators for any stress configuration (19, 17, and 20). For example, if the reasonable substitutions,

\[ G \leftrightarrow Q/P \quad (1-39) \]

\[ K \leftrightarrow Q'/P' \quad (1-40) \]

are made in the relationship,

\[ E = \frac{9KG}{3K + G} \quad (1-41) \]

found in Table I, we obtain:

\[ \frac{Q''}{P''} = \frac{9QQ'}{3QP' + QP'} \quad (1-42) \]

which is an expression for the operators, \( Q''/P'' \), for the case of simple tension in terms of the operators for shear
and hydrostatic pressure. This procedure provides a way to utilize solutions which have been developed for elastic theory in the solution of problems which involve viscoelastic materials (19, 17, and 21).

Voigt Model

The use of mechanical models to portray viscoelastic behavior has been related in several instances (18, 17, 20, 22, and 23). This method is particularly advantageous as a means of "pictorially" representing stress-strain-time laws. The two basic elements in any model are the elastic element and viscous element which are depicted as a spring and dashpot respectively (Figure 1-2).

The Voigt element (also referred to as a Kelvin element) consists of an elastic element and a viscous element connected in parallel as shown in Figure 1-3.

The stress-strain relationship for a Voigt element, is obtained by adding the stresses in the two basic elements (22). Thus, using deviatoric components*, the stress

*This is merely a matter of choice since the stress-strain equation may be written for any stress component by substituting the proper material moduli. Of course, in the case of deviatoric components (or shear components), the material moduli for the two basic elements are the shear modulus, G, for the elastic element and the coefficient of viscosity, n, for the viscous element.
Figure 1-2. Basic Elements: (a) Elastic Element; (b) Viscous Element.

Figure 1-3. Viscous Element.
in the elastic element (Equation 1-21) resolves to:

\[ S_{ijE} = 2G\varepsilon_{ij} \]

and the stress in the viscous element (Equation 1-29) is:

\[ S_{ijV} = 2\dot{\varepsilon}_{ij} \]

Therefore, the stress in the Voigt element is:

\[ S_{ij} = 2G\varepsilon_{ij} + 2\dot{\varepsilon}_{ij} \quad (1-43) \]

It is seen from Equation (1-43) that the Voigt element behaves essentially as an elastic solid if it is strained very slowly (\( \dot{\varepsilon}_{ij} \to 0 \)).

It is of particular interest to study the behavior of the Voigt element when it is subjected to a stress which is applied instantaneously to its full value, \( S_0 \), and held constant for a period of time, i.e. a creep test. In this case:

\[ \dot{S}_{ij} = 0 \quad (1-44) \]

Differentiating Equation (1-43) with respect to time, we obtain:

\[ \ddot{S}_{ij} = 2G\ddot{\varepsilon}_{ij} + 2\ddot{\varepsilon}_{ij} \quad (1-45) \]

However, since \( \ddot{S}_{ij} = 0 \), we obtain the homogenous differential equation:
Figures 1-4. (a) Stress vs Time and (b) Strain vs Time for a Voigt Element.

where

\[ k=1,2,...,n \]

Since the elements are connected in series, the stress is the same in each element (22); therefore, this expression represents n-differential equations for the model. Also, the total strain which occurs in the model is the sum of the strain which occurs in each element; therefore, we may write:
Figure 1-5. Generalized Voigt Model.
The combination of Equations (1-49) and (1-50) completely defines the mechanical behavior of the model shown in Figure 1-5. This model is called the generalized Voigt model and represents a viscoelastic solid which exhibits retarded elastic deformation.

The solution of Equation (1-49) for the special conditions of a creep test yields:

\[ 2e_{ij}(k) = \left\{ \frac{S_0}{G_k} \right\} \left[ 1 - \exp\left( \frac{-t}{\tau_k} \right) \right] \]  
(1-51)

where

\[ S_0 = \text{instantaneously applied constant stress} \]
\[ \tau_k = \eta_k/G_k = \text{retardation time of } k\text{th element.} \]

Substituting this result into Equation (1-50), we obtain:

\[ 2e_{ij} = \sum_{k=1}^{n} \left( \frac{S_0}{G_k} \right) \left[ 1 - \exp\left( \frac{-t}{\tau_k} \right) \right] \]  
(1-52)

which represents the strain response of a generalized Voigt model to a constant stress. If the stress, \( S_0 \), is removed after a strain \( \varepsilon_0(k) \) is attained in the \( k \)th element, the strain will decay according to:
\[ e_{ij} = \sum_{k=1}^{n} e_{\alpha(k)} \exp \left(-t/t_k\right) \] (1-53)

The accuracy of representation by a generalized Voigt model depends upon the number of elements the model contains. Consequently, as the number of elements in the generalized model approaches infinity (i.e., \( n \to \infty \)), the ultimate accuracy is attained; and the generalized model response is identical with that of the material response.

As stated earlier, the generalized Voigt model represents a viscoelastic solid. Various other viscoelastic materials may be represented by a generalized model which contains degenerate Voigt elements. For example, a model which contains a Voigt element that has been reduced to a dashpot represents a viscoelastic liquid, whereas one which contains an element that has been reduced to a spring represents a viscoelastic solid that possesses instantaneous elasticity. A material which exhibits instantaneous elasticity, retarded elasticity, and viscous flow may be represented by a generalized Voigt model which contains two degenerate elements as shown in Figure 1-6. The response of this model to a constant stress is obtained by adding the strain in the elastic and viscous elements to the strain in the generalized Voigt model. Thus:
Figure 1-6. Generalized Voigt Model Which Contains Two Degenerate Elements.
Another viscoelastic model of special interest may be obtained by connecting a Bingham-body and generalized Voigt model in series as shown in Figure 1-7. The Bingham-body consists of an elastic element connected in series with a friction element (or St. Venant body) and a viscous element in parallel (23). Therefore, the constant-stress flow equations for the model shown in Figure 1-7 are:

For \( S_0 \leq S_y \),

\[
2e_{ij} = \frac{S_0}{G_0} + \frac{S_0}{\eta_0} t + \sum_{k=1}^{n} \left[ \frac{S_0}{G_k} \left( 1 - \exp \left( \frac{-t}{\tau_k} \right) \right) \right] \tag{1-55}
\]

and

For \( S_0 > S_y \),

\[
2e_{ij} = \frac{S_0}{G_0} + \left[ \frac{(S_0 - S_y)}{\eta_p} \right] t + \sum_{k=1}^{n} \left[ \frac{S_0}{G_k} \left( 1 - \exp \left( \frac{-t}{\tau_k} \right) \right) \right] \tag{1-56}
\]

From Equation (1-56) we see that for the special case where \( S_y = 0 \), we obtain the creep equation for the model shown in Figure 1-6. The results indicated by Equations (1-55) and (1-56) are shown in Figure 1-8, where we see
Figure 1-7. Bingham-Body and Generalized Voigt Model Connected in Series.
that upon application of a load which produces a stress $S_0$, there is an instantaneous elastic strain which is equal to $S_0/G_0$. As time progresses, the strain in the model increases as the sum of the strain which occurs in the Bingham-body and that which occurs in the generalized Voigt model increases. Of course, if $S_0 \leq S_y$, the strain in the Bingham-body is $S_0/G_0$. If after some time, $t$, the load is removed, there will be an instantaneous elastic recovery and a delayed recovery which may be represented by Equation (1-53). Any strain which occurs in the
Bingham-body will appear as permanent strain of magnitude:

\[\frac{(S_0 - S_y)}{\eta_p} t_1\]

If the load is not removed at time, \(t_1\), but instead allowed to act for a sufficient period of time such that

\[t >> t_1, t_2, \ldots, t_n\]  \hspace{1cm} (1-57)

we see that the strain in each element of the generalized Voigt model approaches a constant value of \(\frac{S_0}{G_k}\). Consequently, if the model in Figure 1-7 is subjected to a stress such that \(S_0 > S_y\) for a period of time which satisfies the conditions of Equation (1-57), the rate of strain will approach a steady-state value, as indicated by the dashed line in Figure 1-8, and can be attributed completely to that portion of the model represented by the Bingham-body. This presents a method for determining the value of \(\eta_p\), because, as seen from Equation (1-56), the slope of this straight-line portion of the creep curve is equal to the quantity \(\frac{(S_0 - S_y)}{\eta_p}\).

**Creep Function**

It is often convenient to refer to the flow of a viscoelastic material in terms of a creep function (17), which is defined by:

\[\psi = 2\varepsilon_{ij}/S_0\]  \hspace{1cm} (1-57)
where

$$\psi = \text{creep function}$$

Thus, the creep function is a measure of the strain per unit stress; and, for a generalized Voigt model, is given by:

$$\psi = \sum_{k=1}^{n} \left( \frac{1}{G_k} \right) \left[ 1 - \exp \left( -\frac{t}{\eta_k} \right) \right]$$  \hspace{1cm} (1-58)

Because Equation (1-58) represents that portion of the response which is truly viscoelastic, the creep function for any viscoelastic model is generally defined by this relationship regardless of the number and kind of additional elements the model contains. Therefore, the creep function for each of the models in Figures 1-6 and 1-7 is also represented by Equation (1-58).
CHAPTER II: APPARATUS, MATERIALS, AND PROCEDURE

Apparatus

The apparatus used in this study was a rotating co-axial cylinder viscometer -- designed and constructed by the author. The basic design is similar to that used by other investigators (24, 25, 26, 27, 28, and 29). Essentially, the arrangement consists of an inner cylinder (or bob) and an outer cylinder (or cup) mounted concentrically. The annulus formed by the bob and cup is filled with the test material and sheared by applying a torque to the bob. An alternate design has been used by others (16 and 25), where the cup is rotated and the torque exerted on the bob is measured.

The instrument used in this investigation is shown in Figures 2-1 and 2-2. The bob is attached to a vertical spindle which is supported by means of a thrust bearing. A pulley is attached to the top of the spindle, and provisions are made whereby weights and pulleys subject the bob to torque. The weights are attached in such a manner that a couple is produced, which eliminates bending in the axle. The angle through which the bob rotates is observed directly from a circular scale imprinted on the top of the pulley disk.
Figure 2-1. Photograph of Rotating Co-Axial Cylinder Viscometer.
Figure 2-2. Cross-Section Drawing of Rotating Co-Axial Cylinder Viscometer.
The bob is 3.12 cm. in diameter and is constructed of aluminum. The cup is constructed of a steel cylinder, 5.28 cm. in diameter and rests in a 1/8 in. groove in a brass base plate. The annulus formed by the bob and cup is 4.62 cm. in height. A thin brass plate (approximately 1/8 in. thick) which is attached to the lower end of the bob and the area of the base plate inside the cup were amalgamated with mercury to prevent the test material from sticking to them. Later, it was found that a small excess of mercury poured in the bottom of the cup was more effective and at the same time prevented the test material from flowing under the bob. It was assumed that this virtually eliminated end effects because the material under test was actually floating on the mercury. This provided a completely fluid, floating bearing.

The equations which describe the flow of a material between co-axial cylinders may be more conviently discussed if the stress and strain are referred to cylindrical, polar coordinates. Thus, if we write the stress and the strain tensors in cylindrical, polar coordinates, we find that in matrix form they appear as (9):

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\
\sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\
\sigma_{zr} & \sigma_{z\theta} & \sigma_{zz}
\end{bmatrix}
\] (2-1)
and

\[ \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{z\theta} & \varepsilon_{zz} \end{bmatrix} \]  

(2-2)

For the case of an incompressible liquid under steady flow conditions (laminar flow) in a co-axial cylinder viscometer, the rate of strain tensor, \( \dot{\varepsilon}_{ij} \), has only two components which differ from zero (30); they are the equal shear components, \( \dot{\varepsilon}_{r\theta} \) and \( \dot{\varepsilon}_{\theta r} \); therefore, the rate-of-strain tensor and its deviatoric component are equal. Thus:

\[ \dot{\varepsilon}_{ij} = \hat{\varepsilon}_{ij} = \begin{bmatrix} 0 & \dot{\varepsilon}_{r\theta} & 0 \\ \dot{\varepsilon}_{r\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  

(2-3)

The corresponding stress tensor and its deviatoric component are:

\[ \sigma_{ij} = \begin{bmatrix} -p & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \]  

(2-4)

and
The pressure, \( p \), in Equation (2-4) may be evaluated through the laws of motion (30) and is immaterial to the determination of the flow response to the shear stress, \( \sigma_{r\theta} \). Therefore, we shall be concerned with the relationship between the deviatoric stress tensor and the deviatoric rate-of-strain tensor. However, before proceeding to a study of this relationship from the standpoint of some of the materials discussed earlier, it is essential that we obtain the following relationships: 1) between the strain, \( \varepsilon_{r\theta} \), and the angle of rotation of the bob, \( \theta \); 2) between the rate of strain, \( \dot{\varepsilon}_{r\theta} \), and the angular velocity, \( \omega \); and 3) between the stress, \( S_{r\theta} \), and the torque, \( T \).

If we denote displacements in a radial and a tangential direction by \( u_r \) and \( u_\theta \) respectively, the strain component, \( \varepsilon_{r\theta} \), in terms of these displacements is (13):

\[
\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \tag{2-6}
\]

where, \( r \), is the radial distance measured from the center of the bob; and \( \theta \) is the angle of rotation. However, since it is assumed that no displacement occurs in the radial direction, \( u_r = 0 \); and Equation (2-6) reduces to:
\[ \varepsilon_{r\theta} = \frac{1}{2} \left( \frac{du_\theta}{dr} - \frac{u_\theta}{r} \right) \]  \hspace{1cm} (2-7)

If we make the substitution:

\[ u_\theta = r \theta \]  \hspace{1cm} (2-8)

into Equation (2-7), we obtain:

\[ \varepsilon_{r\theta} = \left( \frac{1}{2} \right) \left( \frac{d(r \theta)}{dr} - \theta \right) \]  \hspace{1cm} (2-9)

which yields:

\[ \varepsilon_{r\theta} = \left( \frac{1}{2} \right) (rd\theta/dr) \]  \hspace{1cm} (2-10)

Differentiating Equation (2-10) with respect to time, and recalling that

\[ \omega = \frac{d\theta}{dt} = \text{angular velocity}, \]  \hspace{1cm} (2-11)

we obtain:

\[ \dot{\varepsilon}_{r\theta} = \left( \frac{1}{2} \right) r \frac{d\omega}{dr} \]  \hspace{1cm} (2-12)

The shear stress, \( S_{r\theta} \), at a distance, \( r \), from the center of the bob is given by the relationship:

\[ S_{r\theta} = \frac{T}{2\pi r^2 L} \]  \hspace{1cm} (2-13)

where
T = applied torque

and

L = length of the annulus

Thus, Equations (2-10), (2-12), and (2-13) provide the relationships which were desired.

From the earlier discussion regarding Equations (1-29), (1-31), (1-32), (1-33), and (1-52), it can be observed that a general flow equation may be written for any material in the form of the functional relationship:

\[ 2 \varepsilon_{ij} = f(S_{ij}) \]  

(2-14)

Using this functional relationship, we have for the co-axial cylinder:

\[ 2 \varepsilon_{r\theta} = f(S_{r\theta}) \]  

(2-15)

or

\[ r \frac{d\omega}{dr} = f(S_{r\theta}) \]  

(2-16)

Therefore, the angular velocity, \( \omega \), of a layer of material at a distance, \( r \), is given by:

\[ \omega(r) = \int_{a_1}^{r} f \left( \frac{T}{2\pi r^2 L} \right) \frac{dr}{r} \]  

(2-17)

where
\[ a_1 = \text{radius of inner cylinder or bob}. \]

Finally, by integrating between the limits \( a_1 \) and \( a_2 \), where \( a_2 \) equals the radius of the cup, we obtain the following expression for the angular velocity of the bob:

\[
\omega = \int_{a_1}^{a_2} \frac{f \left( \frac{T}{2\pi r^2 L} \right)}{r} \, dr
\]  

(2-18)

In the case of a Newtonian liquid, we see from Equation (1-29) that

\[
2\dot{\theta} = \frac{1}{\eta} (S_{\theta}) = f(S_{\theta})
\]  

(2-19)

Substituting this result into Equation (2-18), we obtain:

\[
\omega = \int_{a_1}^{a_2} \frac{1}{\eta} \left( \frac{T}{2\pi r^2 L} \right) \frac{dr}{r}
\]  

(2-20)

Performing the indicated integration, we obtain:

\[
\omega = \frac{T}{4\pi \eta L} \left( \frac{1}{a_1^2} - \frac{1}{a_2^2} \right)
\]  

(2-21)

which is a linear relationship between the angular velocity of the bob and the applied torque. Consequently, a plot of angular velocity vs. torque will result in a straight line passing through the origin -- the slope being equal to

\[
\frac{C}{\eta}
\]  

(2-22)
where

\[ C = \frac{1}{4\pi L} \left( \frac{1}{a_1^2} - \frac{1}{a_2^2} \right) \]  

(2-23)

If a Bingham plastic is stressed in a co-axial cylinder viscometer, flow does not occur until the shear stress on the layer of material immediately adjacent to the bob exceeds the yield stress value, \( S_y \), of the material. Hence, flow occurs when

\[ S_{a_1} > S_y \]  

(2-24)

where

\[ S_{a_1} = \text{shear stress on layer of material immediately adjacent to the bob} \]

Therefore, since

\[ S_{a_1} = \frac{T}{2\pi a_1^2 L} \]

flow will occur if

\[ T > 2\pi a_1^2 L S_y \]  

(2-25)

Although flow occurs when the applied torque exceeds this value, it is confined to the portion of material immediately adjacent to the bob. As the torque is increased, successive layers of the material flow until finally, when the torque exceeds the value \( 2\pi a_2^2 LS_y \), the entire an-
nulus of material flows. Therefore, the flow characteristics of a Bingham plastic in a co-axial cylinder viscometer must be studied in three stages:

1) When \( T \leq 2\pi a_1^2 L S_y \):

\[ \omega = 0 \]

2) When \( 2\pi a_1^2 L S_y < T \leq 2\pi a_2^2 L S_y \):

From Equation (1-31),

\[ 2\varepsilon_r = \left( \frac{S_r - S_y}{\eta_p} \right) = f(S_r) \]  

Substitution into Equation (2-17) yields:

\[ \omega = \int_{a_1}^{r} \frac{1}{\eta_p} \left( \frac{T}{2\pi r^2 L} - S_y \right) \]  

(2-27)

The upper limit on the integral in Equation (2-27) represents the outermost point in the annulus where the material flows and is therefore obtained from the relationship:

\[ r = \sqrt{\frac{T}{2\pi S_y L}} \]

Hence, integration and substitution of these limits yields:

\[ \omega = \frac{T}{4\pi a_1^2 L \eta_p} - \frac{S_y}{2\eta_p} - \frac{S_y}{2\eta_p} \ln \left( \frac{T}{2\pi a_1^2 L S_y} \right) \]  

(2-28)
nullus of material flows. Therefore, the flow characteristics of a Bingham plastic in a co-axial cylinder viscometer must be studied in three stages:

1) When $T \leq 2\pi a_1^2 L S_y$:

$$\omega = 0$$

2) When $2\pi a_1^2 L S_y < T \leq 2\pi a_2^2 L S_y$:

From Equation (1-31),

$$2\cdot \tau_\theta = \frac{(S \cdot \tau_\theta - S_y)}{\eta_p} = f(S \cdot \tau_\theta) \quad (2-26)$$

Substitution into Equation (2-17) yields:

$$\omega = \int_{a_1}^{r} \frac{1}{\eta_p} \left( \frac{T}{2\pi r^2 L} - S_y \right) \quad (2-27)$$

The upper limit on the integral in Equation (2-27) represents the outermost point in the annulus where the material flows and is therefore obtained from the relationship:

$$r = \sqrt{T/2\pi S_y L}$$

Hence, integration and substitution of these limits yields:

$$\omega = \frac{T}{4\pi a_1^2 \eta_p} - \frac{S_y}{2\eta_p} - \frac{S_y}{2\eta_p} \ln \left( \frac{T}{2\pi a_1^2 L S_y} \right) \quad (2-28)$$
3) When \( T > 2\pi a_2^2 L S_y \):

For this case, the integral in Equation (2-27) is applicable if the upper limit is changed to \( a_2 \). Therefore:

\[
\omega = \int_{a_1}^{a_2} \frac{T}{2\pi r^2 L} - S_y \left. \frac{dr}{r} \right|_{a_1}^{a_2} \quad (2-29)
\]

Integration and substitution of limits yields:

\[
\omega = \frac{C T}{\eta_p} - \frac{S_y}{\eta_p} \ln \left( \frac{a_2}{a_1} \right) \quad (2-30)
\]

A graphical representation of the results indicated by Equations (2-28) and (2-30) is presented in Figure 2-3.

In order to study the behavior of a linear viscoelastic material represented by a Voigt model, we return to Equation (1-52) and differentiate it with respect to time. Hence:

\[
2 \dot{\varepsilon}_{r0} = S_{r0} \sum_{k=1}^{N} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) = f(S_{r0}) \quad (2-31)
\]

Substitution of this result into Equation (2-18), yields:

\[
\omega = \int_{a_1}^{a_2} \frac{T}{2\pi r^2 L} \left[ \sum_{k=1}^{N} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) \right] \frac{dr}{r} \quad (2-32)
\]

Integrating and substituting limits, we obtain:
Figure 2-3. Relationship between Angular Velocity and Torque for a Bingham Plastic in a Co-Axial Cylinder Viscometer.
\[ \omega = CT \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp\left(\frac{-t}{\tau_k}\right) \]  
\hspace{1cm} \text{(2-33)}

Of course, it is not possible to obtain a \( \omega \text{-vs-} \tau \) plot directly from Equation (2-33) since the angular velocity also varies with the time. However, if a series of angular-velocity-\( \tau \text{-vs-} \)time plots were obtained experimentally for several different torques, it would be found that for each torque the values of \( \omega \) which correspond to equal times will vary linearly with the torque.

From Equation (1-54), we see that for the model shown in Figure 1-6

\[ 2\dot{\theta} = \frac{S_{\theta \theta}}{n_0} + \sum_{k=1}^{n} \frac{S_{\theta \theta}}{\tau_k G_k} \exp\left(\frac{t}{\tau_k}\right) \]  
\hspace{1cm} \text{(2-34)}

Substitution into Equation (2-18) and integration yields:

\[ \omega = \frac{C T}{n_0} + \sum_{k=1}^{n} \frac{C T}{\tau_k G_k} \exp\left(\frac{-t}{\tau_k}\right) \]  
\hspace{1cm} \text{(2-35)}

The equations for the model shown in Figure 1-7 are obtained in a similar manner and are:

1) When \( T \leq 2\pi a_2^2 L S_y \):

\[ \omega = CT \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp\left(\frac{-t}{\tau_k}\right) \]  
\hspace{1cm} \text{(2-36)}

2) When \( 2\pi a_2^2 L S_y < T \leq 2\pi a_2^2 L S_y \):
\[
\omega = \frac{T}{4\pi a_2^2 \ln n_p} - \frac{S_Y}{2\eta_p} - \frac{S_Y}{2\eta_p} \ln \left( \frac{T}{2\pi a_2^2 L S_Y} \right)
\]

\[
+ \frac{C}{\eta_p} \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) \quad (2-37)
\]

3) When \( T > 2\pi a_2^2 L S_Y \):

\[
\omega = \frac{C}{\eta_p} - \frac{S_Y}{\eta_p} \ln \left( \frac{a_2}{a_1} \right)
\]

\[
+ \frac{C}{\eta_p} \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) \quad (2-38)
\]

Equations (2-35) and (2-38) may be written as one equation in terms of steady-state angular velocity and the creep function. For example, in Equation (2-35) we see that the steady-state angular velocity is

\[
\omega_0 = \frac{C}{\eta_0}
\]

and the corresponding term in Equation (2-38) is

\[
\omega_0 = \frac{C}{\eta_0} - \frac{S_Y}{\eta_0} \ln \left( \frac{a_2}{a_1} \right)
\]

Therefore, we may write:

\[
\omega = \omega_0 + \frac{C}{\eta_0} \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) \quad (2-39)
\]

also, from Equation (1-58) we see that
\[
\frac{d\psi}{dt} = \dot{\psi} = \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp \left[ -\frac{t}{\tau_k} \right]
\]  
(2-40)

and, thus, Equation (2-39) becomes:

\[
\omega = \omega_0 + CT \dot{\psi}
\]  
(2-41)

**Materials**

The asphalt used in this study was a standard, paving-grade, 85-100-penetration, asphalt cement which was refined from a mid-continent crude. The base asphalt and two rubber-asphalt blends—one containing 5% rubber solids and one containing 10% rubber solids—were studied. The rubber-asphalt blends were made by mechanically blending a synthetic, oil-extended, rubber latex into the asphalt cement. The latex contained 55% rubber solids and 45% high-flash-point oil; therefore, the rubber-solids content of each blend was figured as a percentage by weight of the asphalt plus the oil contributed by the added latex. The blending was accomplished by heating the asphalt to 280-300°F, adding the latex while the asphalt was being stirred, and stirring the mixture for 15 to 20 minutes afterwards.

Photomicrographs of the control asphalt and the two rubber-asphalt blends are shown in Figure 2-4. The small white circles in Figure 2-4 (a) are air bubbles trapped under the cover glass. In comparing Figures 2-4 (b) and
Figure 2-4. Photomicrographs Showing (a) PAC-5, Control Asphalt, (b) Control Asphalt + 5% Rubber, and (c) Control Asphalt + 10% Rubber.
2-4 (c), it is seen that the dispersion patterns differ somewhat for the two rubber-asphalt blends—the 5% rubber blend exhibiting a finer degree of dispersion. This can be attributed largely to the difference in the percentage of rubber solids since greater percentages of rubber require longer mixing times to obtain fine degrees of dispersion.

**Testing Procedure**

A specimen of test asphalt was heated until it became liquid enough to pour readily (approximately 280°F); the heated specimen was poured into the annulus of the viscometer and allowed to cool in room air for a minimum of 30 minutes. The viscometer, containing the specimen, was then transferred to a constant temperature water bath and maintained at the desired test temperature for a minimum of one hour prior to testing. Also, the viscometer remained in the water bath while each test was being performed.

A test was begun by releasing weights which were attached to the pulley and, at the same instant, recording the time. At subsequent intervals of time, the angle of rotation was recorded by reading the scale which was attached to the pulley. Each test was allowed to continue until a steady-state or near steady-state condition was achieved, i.e., until a linear relationship between angle
of rotation and time was observed. After the steady-state condition was reached, the load was removed, and an attempt was made to record the rebound as a function of time. Of course, due to a slight amount of friction in the apparatus, it was not possible to completely unload the specimen unless a compensating torque, equal to the friction, was applied in the opposite direction at the instant the load was removed. This was attempted; however, little success was achieved because the friction forces could not be determined within the accuracy required for a correction of this type.

Each of the three asphalt samples was tested at temperatures of 39.2, 77, and 104°F and with a minimum of four intensities of torque at each temperature. The magnitude of each torque was chosen so as to give sufficient rotation in a reasonable period of time. Also, the torques selected for each temperature differed by constant increments—thereby, affording a means of checking linearity directly from a plot of the raw data, i.e., angle of rotation vs time.

In order to avoid the possibility of obtaining data affected by previous loading, a new specimen was used for each test.
The experimental data are presented in Figures 3-1, 3-2, and 3-3. These curves show the relationship between the angle of rotation, φ, and time, t, for all the tests performed on the three materials at each temperature. From these creep curves, it is obvious that a combination of steady-state flow and viscoelastic flow was prevalent (compare Figure 1-8) and that the flow pattern could be represented by Equation (2-41). With that concept in mind, the curves were analysed in two steps: 1) analysis of the straight-line or steady-state portion to determine \( w_0 \) and 2) analysis of the curved or viscoelastic portion to determine \( \psi \).

**Steady-State Analysis**

The steady-state, angular velocity, \( w_0 \), was determined from the slope of the straight-line portion of each curve and was plotted as a function of the corresponding torque. The resulting curves are shown in Figure 3-4. By comparing the shape of these curves with the shape of similar curves for the "classical" materials which are known to exhibit steady-state flow properties, it was possible to infer the steady-state flow behavior of the test materials. In Figure (3-4), it is seen that the relationship between the steady-state angular velocity and the torque...
Figure 3-1. Creep Curves Showing the Relationship between Angle of Rotation and Time. Test Temperature, 39.2°F.
Figure 3-2. Creep Curves Showing the Relationship between Angle of Rotation and Time. Test Temperature, 77°F.
Figure 3-3. Creep Curves Showing the Relationship between Angle of Rotation and Time. Test Temperature, 104°F.
Figure 3-4. Relationship between Steady-State, Angular Velocity and Torque. Test Temperatures, 39.2°F, 77°F, and 104°F.
could be satisfactorily approximated by a straight line for all three asphalts at each of the three test temperatures. Therefore, from the earlier discussion (compare Figure 1-1), it may be concluded that the steady-state flow was characteristic of either a Newtonian liquid or a Bingham plastic. In order to distinguish between these two, it was necessary to examine the torque intercepts. For the tests performed at 39.2°F, it was found that extrapolations of each of the straight lines failed to intercept the origin—indicating a yield stress and further characterizing the behavior of a Bingham plastic. Therefore, using the flow relationship for a Bingham plastic (Equation 2-30) as the equation of these lines, the values of \( \eta_p \) and \( S_y \) were calculated. These values are shown in Figure 3-4 for each curve obtained at 39.2°F. The minimum value of the torque necessary to sustain flow was calculated from the relationship:

\[
T = 2\pi a_1^2 L S_y
\]

and is shown as the intercept of the curved, dashed lines in Figure 3-4.

At 77°F, extrapolation of the straight lines yielded a torque intercept for the 10% rubber-asphalt; however, no intercept was obtained for the control and 5% rub-
ber-asphalt. Therefore, the steady-state flow of the 10%-rubber blend, again, was represented by a Bingham plastic, and the other two materials could be represented by a Newtonian liquid (Equation 2-21).

At 104°F, all three materials exhibited flow which was characteristic of a Newtonian liquid.

**Viscoelastic Analysis**

The curved portion of each creep curve was differentiated graphically (31) to obtain a locus of the angular velocity, \( \omega \), as a function of time. These curves are shown in Figures 3-5, 3-6, and 3-7 and include the combined effects of the steady-state and the viscoelastic flow. In order to separate the two, Equation (2-39) was written in the form:

\[
\frac{\omega - \omega_0}{CT} = \sum_{k=1}^{n} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right)
\]

which indicates further that a plot of the quantity, \( (\omega - \omega_0)/CT \), as a function of time will result in a curve which represents only that portion of the flow which is viscoelastic. Therefore, using the results of the steady-state analysis and the \( \omega \)-vs-\( t \) curves, the average value of \( (\omega - \omega_0)/CT \) was determined as a function of time for
Figure 3-5. Relationship between Angular Velocity and Time. Test Temperature, 39.2°F.
Figure 3-6. Relationship between Angular Velocity and Time. Test Temperature, 77°F.
Figure 3-7. Relationship between Angular Velocity and Time. Test Temperature, 104°F.
each asphalt at each test temperature*. An example of a
typical curve is shown in Figure 3-8 (a) where the average
value of the quantity \( \frac{\omega - \omega_0}{C_T} \) is plotted on the vertical,
logarithmic scale and time is plotted on the horizontal,
arithmetic scale. The values of the parameters, \( T_k \) and
\( G_k \), were determined from these experimental curves using
the following, approximate method (11 and 32). First,
it was assumed that the equation of the experimental
curve could be represented by the first term of the series
in Equation (3-1). Thus, it was assumed that:

\[
\frac{\omega - \omega_0}{CT} = \frac{1}{T_1 G_1} \exp \left( \frac{-t}{T_1} \right) \tag{3-2}
\]

or

\[
\ln \left( \frac{\omega - \omega_0}{CT} \right) = \ln \left( \frac{1}{T_1 G_1} \right) - \frac{t}{T_1} \tag{3-3}
\]

From Equation (3-3), it is seen that a semi-logarithmic
plot of \( \frac{\omega - \omega_0}{C_T} \) vs time should result in a straight
line if this assumption is correct. However, referring
to the example shown in Figure 3-8 (a), it is seen that

*Although the ratio \( \frac{\omega - \omega_0}{C_T} \) is independent of the torque,
as indicated by Equation (3-1), it was found that there
was a slight variation with the torque in the actual data.
Therefore the average value was used. This variation
was attributed to experimental error introduced from the
test instrument and the graphical analysis—since it did
not follow a definite pattern.
Figure 3-8. Typical Curve for (a) First Term Approximation of Viscoelastic Flow and (b) Second and Third Term Approximations of Viscoelastic Flow.
there is considerable deviation from a straight line except within the lower portion of the curve. Therefore, only this portion was approximated by a straight line whose equation is represented by the right side of Equation (3-3). From the slope of this line, the quantity, \(-\frac{1}{\tau_1}\), was calculated; and, from the intercept the quantity, \(\frac{1}{\tau_1}G_1\), was calculated. The differences, \(\Delta_1\), between the ordinates of this straight line and the ordinates of the experimental curve were determined and plotted as a function of time. The resulting curve is shown in Figure 3-8 (b) and labeled "Second Term". It was assumed that this curve could be represented by the second term of the series, Equation (3-1); and, as before, it was found that this was satisfactory for the lower portion of the curve. Therefore, it was approximated by a second straight line having a slope equal to \(-\frac{1}{\tau_2}\) and an intercept equal to \(\frac{1}{\tau_2}G_2\). A second set of differences, \(\Delta_2\), between the ordinates was obtained, and the process repeated until the differences could be approximated by a straight line. In the example shown, the second set of differences yielded points which all fell on a straight line (labeled "Third Term"); and the viscoelastic response could therefore be satisfactorily represented by the sum of three exponential terms. Thus, for the example shown:
\[
\frac{\omega - \omega_0}{CT} = \sum_{k=1}^{3} \frac{1}{\tau_k G_k} \exp\left[\frac{-t}{\tau_k}\right]
\] (3-4)

This analysis was performed on all the test data, and the results are presented in Tables 2, 3, and 4. These tables show the values of the three parameters, \( \tau \), \( n \), and \( G \), for each approximation (indicated by \( k \)). Also, for convenience, a summary of the steady-state results is presented.

Referring to Table 2, it is seen that the viscoelastic response of all three materials at 39.2°F could be satisfactorily represented by three exponential terms. From Table 3, it is seen that only one term was required for the control asphalt at 77°F; two terms were required for the 5% rubber-asphalt; and three terms were required for the 10% rubber-asphalt. No viscoelastic analysis was performed on the data obtained for the control asphalt at 104°F because it appeared to exhibit perfect linearity between the angle of rotation and time throughout the complete test run (see Figure 3-3). However, one exponential term was required for the 5% rubber-asphalt, and two exponential terms were required for the 10% rubber-asphalt at this temperature.

On the bases of the foregoing analysis, the general form of the co-axial viscometer flow equation for each asphalt at each test temperature may be satisfactorily repre-
### TABLE 2
SUMMARY OF RESULTS OBTAINED AT 39.2°F

<table>
<thead>
<tr>
<th>Sample</th>
<th>k</th>
<th>τ (Sec)</th>
<th>G x 10^6 (Dynes/Cm²)</th>
<th>η x 10^8 (Poise)</th>
<th>Steady-State Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Bingham Plastic:</td>
</tr>
<tr>
<td>0% Rubber</td>
<td>1</td>
<td>214</td>
<td>2.96</td>
<td>6.33</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>28.2</td>
<td>6.75</td>
<td>1.91</td>
<td>n_p = 5.46 x 10^8 poise</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.00</td>
<td>60.6</td>
<td>4.85</td>
<td>S_y = 8.23 x 10^3 dyne/cm^2</td>
</tr>
<tr>
<td>5% Rubber</td>
<td>1</td>
<td>213</td>
<td>2.41</td>
<td>5.13</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>26.0</td>
<td>7.48</td>
<td>1.94</td>
<td>n_p = 5.19 x 10^8 poise</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.00</td>
<td>50.4</td>
<td>4.35</td>
<td>S_y = 1.73 x 10^4 dyne/cm^2</td>
</tr>
<tr>
<td>10% Rubber</td>
<td>1</td>
<td>442</td>
<td>1.18</td>
<td>5.24</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>75.6</td>
<td>17.6</td>
<td>13.3</td>
<td>n_p = 5.59 x 10^8 poise</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>16.8</td>
<td>7.36</td>
<td>1.32</td>
<td>S_y = 2.47 x 10^4 dyne/cm^2</td>
</tr>
<tr>
<td>Sample</td>
<td>k</td>
<td>$\tau$ (Sec)</td>
<td>$G \times 10^4$ (Dynes/Cm$^2$)</td>
<td>$\eta \times 10^5$ (Poise)</td>
<td>Steady-State Flow</td>
</tr>
<tr>
<td>---------</td>
<td>----</td>
<td>---------------</td>
<td>--------------------------------</td>
<td>-----------------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>0% Rubber</td>
<td>1</td>
<td>10.6</td>
<td>7.30</td>
<td>6.90</td>
<td>Newtonian Liquid:  [ \eta_0 = 1.15 \times 10^6 \text{ poise} ]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9.75</td>
<td>23.0</td>
<td>23.6</td>
<td></td>
</tr>
<tr>
<td>5% Rubber</td>
<td>1</td>
<td>81.0</td>
<td>12.5</td>
<td>1.55</td>
<td>Newtonian Liquid:  [ \eta_0 = 1.66 \times 10^6 \text{ poise} ]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>81.0</td>
<td>12.5</td>
<td>1.55</td>
<td></td>
</tr>
<tr>
<td>10% Rubber</td>
<td>1</td>
<td>5.05</td>
<td>50.0</td>
<td>99.0</td>
<td>Bingham Plastic: [ \eta_0 = 2.60 \times 10^6 \text{ poise} ]</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>14.5</td>
<td>26.3</td>
<td>18.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>54.9</td>
<td>28.6</td>
<td>5.21</td>
<td>[ S_y = 585 \text{ dynes/cm}^2 ]</td>
</tr>
<tr>
<td>Sample</td>
<td></td>
<td></td>
<td></td>
<td>Steady-State Flow</td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------------------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>k</td>
<td>$\tau$ (Sec)</td>
<td>$G \times 10^4$ (Dynes/Cm$^2$)</td>
<td>$\eta \times 10^5$ (Poise)</td>
<td>Newtonian Liquid: $\eta_0 = 4.35 \times 10^4$ poise</td>
</tr>
<tr>
<td>0% Rubber</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5% Rubber</td>
<td>1</td>
<td>9.34</td>
<td>2.06</td>
<td>1.92</td>
<td>Newtonian Liquid: $\eta_0 = 6.93 \times 10^4$ poise</td>
</tr>
<tr>
<td>10% Rubber</td>
<td>1/2</td>
<td>21.7</td>
<td>1.15</td>
<td>2.50</td>
<td>Newtonian Liquid: $\eta_0 = 1.26 \times 10^5$ poise</td>
</tr>
</tbody>
</table>

TABLE 4
SUMMARY OF RESULTS OBTAINED AT 104°F

| Newtonian Liquid: $\eta_0 = 4.35 \times 10^4$ poise |
| Newtonian Liquid: $\eta_0 = 6.93 \times 10^4$ poise |
| Newtonian Liquid: $\eta_0 = 1.26 \times 10^5$ poise |
### TABLE 4

SUMMARY OF RESULTS OBTAINED AT 104°F

<table>
<thead>
<tr>
<th>Sample</th>
<th>$k$</th>
<th>$\tau$ (Sec)</th>
<th>$G \times 10^4$ (Dynes/Cm$^2$)</th>
<th>$\eta \times 10^5$ (Poise)</th>
<th>Steady-State Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0% Rubber</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Newtonian Liquid: $\eta_0 = 4.35 \times 10^4$ poise</td>
</tr>
<tr>
<td>5% Rubber</td>
<td>1</td>
<td>9.34</td>
<td>2.02</td>
<td>1.92</td>
<td>Newtonian Liquid: $\eta_0 = 6.93 \times 10^4$ poise</td>
</tr>
<tr>
<td>10% Rubber</td>
<td>1</td>
<td>21.7</td>
<td>1.15</td>
<td>2.50</td>
<td>Newtonian Liquid: $\eta_0 = 1.26 \times 10^5$ poise</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.61</td>
<td>3.19</td>
<td>1.47</td>
<td></td>
</tr>
</tbody>
</table>
sented by one of the four following equations:

for \( T > 2\pi a_2^2 LS_y \),

\[
\omega = \frac{CT}{\eta_p} - \frac{Sv}{\eta_p} \ln \left( \frac{a_2}{a_1} \right) + CT \sum_{k=1}^{3} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) \quad (3-6)
\]

\[
\omega = \frac{CT}{\eta_0} + CT \sum_{k=1}^{2} \frac{1}{\tau_k G_k} \exp \left( \frac{-t}{\tau_k} \right) \quad (3-7)
\]

\[
\omega = \frac{CT}{\eta_0} + CT \frac{1}{\tau_1 G_1} \exp \left( \frac{-t}{\tau_1} \right) \quad (3-8)
\]

\[
\omega = \frac{CT}{\eta_1}
\]

Equation (3-6) resolves into the form of Equations (2-36) and (2-37) when \( T < 2\pi a_2^2 LS_y \) and when \( 2\pi a_1^2 LS_y < T < 2\pi a_2^2 LS_y \), respectively. The particular equation which applies to a specific material at a specific test temperature may be found by referring to Table 5. Using these equations and the values of the parameters, \( \tau_k, G_k, \) etc. from Tables 2, 3, and 4, check-values or proof-values of \( \omega \) were calculated for various times and plotted in Figures 3-5, 3-6, and 3-7. These points are shown as the open circles (labeled "Model Points") and in most cases show close agreement with the test data.
TABLE 5
VISCOMETER FLOW EQUATIONS OBTAINED FOR TEST MATERIALS

<table>
<thead>
<tr>
<th>Sample</th>
<th>Flow Equations (Numbers refer to preceeding equations)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>T = 39.2°F</td>
</tr>
<tr>
<td>0% Rubber</td>
<td>3-6</td>
</tr>
<tr>
<td>5% Rubber</td>
<td>3-6</td>
</tr>
<tr>
<td>10% Rubber</td>
<td>3-6</td>
</tr>
</tbody>
</table>

General Creep Equations Determined for Test Materials

The general creep equations, showing the relationship between the deviatoric stress and strain tensors, follow immediately from the preceding, co-axial viscometer flow equations. They are:

for \( S_0 \leq S_y \),

\[
2e_{ij} = \frac{S_0}{G_0} + \sum_{k=1}^{3} \frac{S_0}{G_k} \left[ 1 - \exp \left( \frac{-t}{\tau_k} \right) \right]
\]  \hspace{1cm} (3-10a)

\[
\text{for } S_0 > S_y, \hspace{1cm} 2e_{ij} = \frac{S_0}{G_0} + \frac{(S_0 - S_y)}{\eta_p} t
\]

\[
+ \sum_{k=1}^{3} \frac{S_0}{G_k} \left[ 1 - \exp \left( \frac{-t}{\tau_k} \right) \right]
\]  \hspace{1cm} (3-10b)
\[ 2\epsilon_{ij} = \frac{S_0}{n_0} t + \sum_{k=1}^{2} \frac{S_0}{G_k} \left[ 1 - \exp \left( \frac{-t}{\tau_k} \right) \right] \] (3-11)

\[ 2\epsilon_{ij} = \frac{S_0}{n_0} t + \frac{S_0}{G_1} \left[ 1 - \exp \left( \frac{-t}{\tau_1} \right) \right] \] (3-12)

\[ 2\epsilon_{ij} = \left( \frac{S_0}{n_0} \right) t \] (3-13)

The particular equation which corresponds to a specific material at each of the three test temperatures is shown in Table 6. It is important to note that an elastic term, \(\frac{S_0}{G_0}\), was included in Equations (3-10). Of course, this term does not appear in the corresponding co-axial viscometer flow equations because it is not time dependent.
However, since the steady-state flow of the materials represented by Equations (3-10) was established as being representative of a Bingham plastic, it is necessary that the term, \( S_0/G_0 \), be included in order for it to be wholly valid. The parameter, \( G_0 \), may be evaluated from creep-test data, obtained in a co-axial cylinder viscometer, by the following relationship:

\[
G_0 = \frac{CT}{\theta_0} \quad (3-14)
\]

where

\[
C = \text{instrument constant, defined by Equation (2-23)}
\]

\[
T = \text{applied torque}
\]

\[
\theta_0 = "\text{instantaneous" angle of rotation}
\]

Referring to the curves in Figure 3-1 and the 10% rubber curves in Figure 3-2, it is difficult to discern an instantaneous angle of rotation; obviously this angle is quite small—so small, in fact, that it could not be measured by the visual measuring technique used in this study. As a result, the parameter, \( G_0 \), could not be evaluated quantitatively from these tests. However, qualitatively, it may be said that \( G_0 \) is large in magnitude and approaches an infinite value—this is particularly true for the 10% rubber-asphalt at the 77°F test-temperature.
Model Representation of Test Materials

The models which correspond to the general creep equations are shown in Figure 3-9. The materials represented and test temperatures are indicated under each model.

Operator Equations Determined for Materials

The general form of the operator equation is, of course, represented by Equation (1-34); i.e.,

\[ PS_{ij} = 2Q_{eij} \]

However, we are concerned now with the determination of the two operators \( P \) and \( Q \) for each material at each test temperature. This is accomplished by the simultaneous solution of the differential equations for each of the models in Figure 3-9 (18). For example, the model shown in Figure 3-9(c) is governed by the two differential equations:

\[ S_{ij} = 2\eta \dot{e}_{ij}(0) \]  
\[ S_{ij} = 2G_{1}e_{ij}(1) + 2r_{1}\ddot{e}_{ij}(1) \]  

Also, the total strain in the model is given by:

\[ e_{ij} = e_{ij}(0) + e_{ij}(1) \]

From Equations (3-15) and (3-16) we have:
Figure 3-9. Models Representing the Creep Behavior of the Test Materials.
2\epsilon_{ij}(0) = S_{ij}/\eta_0(\partial/\partial t)

and

2\epsilon_{ij}(1) = S_{ij}/[G_1 + \eta_1(\partial/\partial t)]

By substituting these results into Equation (3-17), the operator equation for the model is obtained. Using this method, the operator equations for the models in Figure 3-9 were found; and P and Q were determined therefrom. These values are as follows:

1) For model (a):

\[ P = \sum_{n=0}^{4} p_n(\partial n/\partial t^n) \] (3-18)

where

\[ p_4 = \eta_p n_1 n_2 n_3 / G_0 \]

\[ p_3 = \eta_p (n_1 n_2 + n_1 n_3 + n_2 n_3) + n_1 n_2 n_3 \]
\[ + (\eta_p / G_0)(G_1 n_2 n_3 + G_2 n_1 n_3 + G_3 n_1 n_2) \]

\[ p_2 = \eta_p [G_1(n_2 + n_3) + G_2(n_1 + n_3) \]
\[ + G_3(n_1 + n_2)] + (G_1 n_2 n_3 \]
\[ + G_2 n_1 n_3 + G_3 n_1 n_2) + (\eta_p / G_0)(G_1 G_2 n_1 + G_1 G_2 n_3) \]

\[ p_1 = \eta_p (G_1 G_2 + G_1 G_3 + G_2 G_3) \]
\[ + (G_1 G_3 n_2 + G_2 G_3 n_1 + G_1 G_2 n_3) \]
\[ + (\eta_p / G_0)(G_1 G_2 G_3) \]
\[ P_0 = G_1 G_2 G_3; \]

\[ Q = \sum_{n=0}^{4} q_n (\partial^n Q/\partial t^n) \quad (3-19) \]

where

\[ q_4 = n p n_1 n_2 n_3 \]

\[ q_3 = n p (G_1 n_2 n_3 + G_2 n_1 n_3 + G_3 n_1 n_2) \]

\[ q_2 = n p (G_1 G_3 n_2 + G_2 G_3 n_1 + G_1 G_2 n_3) \]

\[ q_1 = n p G_1 G_2 G_3 \]

\[ q_0 = 0 \]

and

\[ R = \sum_{n=0}^{3} r_n (\partial^n Q/\partial t^n) \quad (3-20) \]

where

\[ r_3 = n_1 n_2 n_3 \]

\[ r_2 = G_1 n_2 n_3 + G_2 n_1 n_3 + G_3 n_1 n_2 \]

\[ r_1 = G_1 G_3 n_2 + G_2 G_3 n_1 + G_1 G_2 n_3 \]

\[ r_0 = G_1 G_2 G_3 \]
The operator, $R$, is the operator for the yield stress, $S_y$; and the general operator equation for this model appears as:

$$PS_{ij} - R S_y = 2Q_{eij} \quad (3-21)$$

2) For model (b):

$$P = \sum_{n=0}^{2} p_n (\partial^n \partial^n t^n) \quad (3-22)$$

where

$$p_2 = n_0 (n_1 + n_2) + n_1 n_2$$
$$p_1 = G_1(n_0 + n_2) + G_2(n_0 + n_1)$$
$$p_0 = G_1 G_2$$

and

$$Q = \sum_{n=0}^{3} q_n (\partial^n \partial^n t^n) \quad (3-23)$$

where

$$q_3 = n_0 n_1 n_2$$
$$q_2 = (G_1 n_2 + G_2 n_1) n_0$$
$$q_1 = G_1 G_2 n_0$$
q_0 = 0

3) For model (c):

\[ P = \sum_{n=0}^{1} p_n(\sigma^n/\sigma^t^n) \]  \hspace{1cm} (3-24)

where

\[ p_1 = n_0 + n_1 \]
\[ p_0 = G_1 \]

and

\[ Q = \sum_{n=0}^{2} q_n(\sigma^n/\sigma^t^n) \]  \hspace{1cm} (3-25)

where

\[ q_2 = n_0 n_1 \]
\[ q_1 = G_1 n_0 \]
\[ q_0 = 0 \]

4) For model (d):

\[ P = p_0 = 1 \]  \hspace{1cm} (3-26)

and
\[ Q = \sum_{n=0}^{1} q_n \left( \frac{\partial^n}{\partial t^n} \right) \]  \hspace{1cm} (3-27)

where

\[ q_1 = n_0 \]

\[ q_0 = 0 \]

The numerical values for the \( p_n \)'s and \( q_n \)'s may be found by substituting the values for \( G \) and \( n \) from Tables 2, 3, and 4 into the equations above.

**Creep Function**

The creep functions representing the flow of each material follow immediately from Equations (3-10), (3-11), and (3-12), the difference being only in the value of \( n \).

Thus:

\[ \psi = \sum_{k=1}^{n} \left( \frac{1}{G_k} \right) \left[ 1 - \exp \left( -\frac{t}{\tau_k} \right) \right] \]

where \( n \) assumes the values of three, two, and one respectively.

**Comparison of Three Materials**

At 39.2°F, the most significant factor to be observed is the resemblance in the behavior of the three materials. This may first be noted by the similarity in the over-all appearance of the creep curves (Figure 3-1).
Also, this is reflected in the curves obtained for the steady-state analysis (Figure 3-4) and the corresponding values of the calculated material parameters. Referring to Table 2, it is seen that the coefficients of plastic viscosity, \( \eta_p \), were practically equal. Although the yield stresses differed appreciably—the highest value being approximately three times the lowest value—they were so small that they could be neglected for all practical purposes and uses of the materials. However, the yield stress did increase with the addition of rubber.

Comparing the material parameters obtained in the viscoelastic analysis at 39.2°F, it is seen that the control asphalt and 5% rubber-asphalt again exhibited almost identical characteristics. However, the 10% rubber-asphalt exhibited longer retardation times and lower values for the elastic moduli, \( G_k \). If these elastic moduli are accepted as being an indication of stiffness or rigidity, it may be concluded that these results indicate that the control asphalt and 5% rubber-asphalt were somewhat stiffer than the 10% rubber-asphalt. This idea of stiffness is further supported by Equations (3-10), where it is seen that the strain which can be attributed to the viscoelastic response approaches a limiting value of
\[ S_0\left(\frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3}\right) \]

as \( t \to \infty \). Therefore, larger values for \( G_k \), produce smaller values for this limiting strain—indicating a more rigid material.

From the results obtained at 77°F (Table 3), it is seen that the steady-state viscosity increased with an increase in rubber content. Although a yield stress value was obtained for the 10% rubber-asphalt at this temperature, it is extremely small—the corresponding value obtained at 39.2°F being approximately 100 times as large. The results of the viscoelastic analysis indicate that the complexity of flow increased with rubber content. Also, there appeared to be a wider difference in the three materials at this temperature than was exhibited at 39.2°F. Although the number of elements differed, the over-all effect indicated at this temperature is that the rubber produced longer retardation times and increased the stiffness of the asphalt over that exhibited by the control asphalt.

The results at 104°F also show an increase in the steady-state viscosity with an increase in the rubber content. The viscoelastic analysis shows that the addition of rubber again increased the complexity of flow. Since the control asphalt did not exhibit viscoelastic flow at this temperature, the comparison must be confined to the 5% rubber-asphalt and 10% rubber-asphalt. Comparing the
two, it appears that the 5% rubber-asphalt was stiffer and exhibited a shorter retardation time for the element representing it than that exhibited by the combined effect of the two elements representing the 10% rubber-asphalt.

**Influence of Temperature**

Although the data were not obtained with the idea of studying temperature effects, its influence on the mechanical behavior of the asphalts is quite evident throughout. Perhaps this is best illustrated in Figure 3-10 which shows a plot of log $\eta$ versus log $T$, where $\eta$ signifies the steady-state viscosity as determined from the steady-state flow. The influence of temperature on the viscoelastic portion of the response is indicated by longer retardation times, higher values for the elastic moduli, $G_k$, and higher values for the viscosity coefficient, $\eta$, with lower values for the temperature.
Figure 3-10. Plot of Log Viscosity vs Log Temperature.
CHAPTER IV: CONCLUSIONS AND RECOMMENDATIONS

The following conclusions may be drawn from the results of this study:

1. The mechanical response of an asphalt to a constant stress may be described using linear viscoelastic theory and may be specified in terms of any one of the following, three, equivalent methods:
   a. A mechanical model.
   b. An operator equation.
   c. A creep function.

2. The addition of rubber to asphalt appeared to have little or no effect on the steady-state flow response at low temperatures. It did tend to decrease the stiffness or rigidity and to increase the retardation time of the viscoelastic response if the amount of rubber added were in the order of 10 percent. However, the viscous resistance of the asphalt at this temperature was so large, relative to the elastic resistance offered by the rubber, that it overshadowed the influence of the rubber. As a result, it is surmised that the elastic characteristics imparted by the rubber are subordinate to the viscous (and also the elastic) characteristics of the asphalt at
3. With an increase in temperature, the effect of the rubber was to increase the steady-state viscosity, increase the stiffness, and increase the retardation time of the viscoelastic response. Also, the complexity of flow (as indicated by the number of Voigt elements) increased with the rubber content. This could be conjectured from the photomicrographs (Figure 2-4), where it appears that the 5% rubber is in a discontinuous phase, i.e. the rubber particles are suspended or "floating" in the asphalt as individual, disconnected strands. However, the rubber particles in the 10% rubber sample appear to be more nearly continuous from the standpoint of connected strands in a plane section. As indicated by these results, this continuity is considered to be a desirable condition which increases the ability of the rubber to perform as elastic, reinforcing elements within the asphalt.

4. A rotating co-axial cylinder viscometer apparatus serves very satisfactorily as an instrument for obtaining creep data on asphaltic materials of the type studied. However, improvements over the apparatus used in this study, such as: 1) a more accurate method for measuring the angle of rotation, 2) elimination of as much friction as possible, and 3) improved construction so as to facilitate cleaning and filling, would enhance
the use of this apparatus.

The following are offered as recommendations for future study:

1. Broaden the temperature range used and attempt to determine the effect of temperature.

2. Study more rubber-asphalt blends, using percentages between those employed in this study, and see if a more definite and clear-cut relationship can be established between the material parameters and the rubber content.

3. Study the dynamic response of asphalts by subjecting specimens to stress varying sinusoidally with time and recording the corresponding strain response at different frequencies. For example, if a material is subjected to a stress such that

\[ S_{ij} = S_0 \sin \omega t \]  \hspace{1cm} (4-1)

the strain response will be:

\[ 2e_{ij} = 2e_0 \sin (\omega t - \phi) \]  \hspace{1cm} (4-2)

where

\[ S = \text{maximum amplitude of stress} \]
\[ e = \text{maximum amplitude of strain} \]
\[ \omega = \text{angular frequency} \]
\[ \phi = \text{phase angle by which the strain lags the stress} \]
Applying this to Equation (1-49), the $k$th Voigt element in a generalized Voigt model must respond to a stress of this type in accordance with

$$2G_k e_{ij(k)} + 2\eta_k \hat{e}_{ij(k)} = S_0 \sin \omega t \quad (4-3)$$

The steady-state solution of this equation is (17):

$$2e_{ij(k)} = S_0 \left( \frac{G_k}{G_k^2 + \eta_k^2 \omega^2} \sin \omega t \right.$$

$$- \frac{\eta_k \omega}{G_k^2 + \eta_k^2 \omega^2} \cos \omega t \left. \right) \quad (4-4)$$

This equation is equivalent to Equation (4-2) if

$$\cos \phi_k = \frac{G_k}{(G_k^2 + \eta_k^2 \omega^2)^{1/2}} \quad (4-5)$$

and

$$\sin \phi_k = \frac{\eta_k \omega}{(G_k^2 + \eta_k^2 \omega^2)^{1/2}} \quad (4-6)$$

Substituting these results back into Equation (4-4), the steady-state response of the $k$th Voigt element appears as:

$$2e_{ij(k)} = \frac{S_0}{(G_k^2 + \eta_k^2 \omega^2)^{1/2}} \sin (\omega t - \phi_k) \quad (4-7)$$

Comparing Equations (4-7) and (4-2), it is seen that

$$\frac{S_0}{2e_0} = (G_k^2 + \eta_k^2 \omega^2)^{1/2} \quad (4-8)$$
The ratio, \( S_0/2e_0 \), is commonly called the dynamic shear modulus and, as seen here, it varies as a function of the frequency. Alternately, the stress may be divided into two components -- one in phase and one 90° out of phase with the strain. The dynamic modulus is equal to the absolute value of the complex shear modulus which is defined by the relationship (11 and 24):

\[
G^* = G' + iG''
\]  

(4-9)

where

\[
G^* = \text{complex shear modulus} \\
G' = \text{ratio of the in-phase component of stress to strain} \\
G'' = \text{ratio of the 90° out-of-phase component of stress to strain}
\]

Thus:

\[
S_0/2e_0 = |G^*| = (G'^2 + G''^2)^{1/2}
\]  

(4-10)

or for the \( k \)th element of a generalized Voigt model it is seen that:

\[
G' = G_k \\
G'' = \eta_k \omega
\]

The energy stored is proportional to \( G' \), whereas the energy lost is proportional to \( G'' \). From Equation (1-50) and the results of Equation (4-7), it is seen that the dynamic response of a generalized Voigt model containing
n-Voigt elements may be represented by:

\[ 2e_{ij} = \sum_{k=1}^{n} \frac{S_o}{\sqrt{G_k^2 + n_k^2w^2}} \sin (\omega t - \phi_k) \] (4-11)

4. Study the response of asphaltic mixes to both static and dynamic stresses. Perhaps this should be restricted to fine-grained mixes such as sand-asphalts because of the necessary requirement that the material be homogenous. The effect of aggregate size, within the range used in sand-asphalt mixtures, and concentration should be of interest. It is suspected that the model in Figure 1-7 would represent the majority of asphalt mixes, with the yield stress, \( S_y \), being influenced primarily by the aggregate.
LIST OF REFERENCES


