Persistent Currents and Quantum Critical Phenomena in Mesoscopic Physics

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Persistent Currents and Quantum Critical Phenomena in Mesoscopic Physics

ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Kentucky

By

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Lexington, Kentucky

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Lexington, Kentucky

2009

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Persistent Currents and Quantum Critical Phenomena in Mesoscopic Physics

In this thesis, we study persistent currents and quantum critical phenomena in the systems of mesoscopic physics. As an introduction in Chapter 1 we familiarize the reader with the area of mesoscopic physics. We explain how mesoscopic systems are different from quantum systems of single atoms and molecules and bulk systems with an Avogadro number of elements. We also describe some important mesoscopic phenomena.

One of the mathematical tools that we extensively use in our studies is Random Matrix Theory. This theory is not a part of standard physics courses and for educational purposes we provide the basics of Random Matrix Theory in Chapter 2.

In Chapter 3 we study the persistent current of noninteracting electrons in quantum billiards. We consider simply connected chaotic Robnik-Berry quantum billiard and its annular analog. The electrons move in the presence of a point-like magnetic flux at the center of the billiard. For the simply connected billiard, we find a large diamagnetic contribution to the persistent current at small flux, which is independent of the flux and is proportional to the number of electrons (or equivalently the density since we keep the area fixed). The size of this diamagnetic contribution is much larger than the previously studied mesoscopic fluctuations in the persistent current in the simply connected billiard. This behavior of persistent current can ultimately be traced to the response of the angular-momentum $l = 0$ levels (neglected in semiclassical expansions) on the unit disk to a point-like flux at its center. We observe the same behavior for the annular billiard when the inner radius is much smaller than the outer one. We also find that the usual fluctuating persistent current and Anderson-like localization due to boundary scattering are seen when the annulus tends to a one-dimensional ring. We explore the conditions for the observability of this phenomenon.

In Chapter 4 we study quantum critical phenomena in a system of two coupled quantum dots connected by a hopping bridge. Both the dots and connecting region are assumed to be in universal Random Matrix crossover regimes between Gaussian orthogonal and unitary ensembles (defined in Chapter 2). We exploit a diagrammatic approach appropriate for energy separations much larger than the level spacing, to obtain the ensemble-averaged one- and two-particle Greens functions. We find that two main components of the two-particle Green’s function (diffuson and Cooperon) can be described by separate scaling functions. We then use this information to investigate a model interacting system in which one dot has an attractive s-wave reduced Bardeen-Cooper-Schrieffer interaction, while the other is noninteracting but subject to an orbital magnetic field. We find that the critical temperature $T_C$ of the mean-field transition into the superconducting state in the first dot
is non-monotonic in the flux through the second dot in a certain regime of interdot coupling. Likewise, the fluctuation magnetization above the critical temperature is also non-monotonic in this regime, can be either diamagnetic or paramagnetic, and can be deduced from the Cooperon scaling function.

We end this thesis with conclusion in Chapter 5.

KEYWORDS: quantum dot, crossover, quantum criticality, persistent current, quantum billiard
Persistent Currents and Quantum Critical Phenomena in Mesoscopic Physics

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The Graduate School
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DISSERTATION

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Dedicated to my parents.
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CHAPTER 1: INTRODUCTION

Mesoscopic Physics is a fascinating branch of modern physics. It aims to explain the properties of matter on a nanoscopic scale. The prefix ‘meso’ originates from Greek word ‘mesos’, which means middle. Mesoscopic physics is thus a crossover between the microscopic world of atoms and molecules, and the macroscopic world we live in.

In micro physics the primary focus is on the individual properties of the elementary building blocks of matter. These building blocks exhibit properties completely different from those of classical objects and quantum mechanics becomes an indispensable tool that helps to explain phenomena at the atomic level.

On the other hand, all macro objects are composed of large number of atoms and molecules. Inability to follow the dynamics of every particle led to the discovery of new methods of statistical mechanics that allow us to obtain useful information about the system under consideration.

Mesoscopic systems are usually composed of hundreds or thousands of elements (electrons, atoms, etc.). At this point the concepts applied to macroscopic objects, such as average density, temperature, entropy become important. However, the size of the system does not allow us to apply the powerful methods of statistical mechanics in full, where all results are obtained in the thermodynamic limit. The system is also not large enough to wash out the quantum mechanical effects absent in the bulk material. Thus, mesoscopic systems fit the middle between a single element and a large statistical collection of elements.

Mesoscopic physics is interesting from both the academic and the technological point of view. Incorporating properties of individual particles and large statistical ensembles, mesoscopic systems reveal a number of interesting physical phenomena, never observed in the bulk. Some conceptually important mesoscopic phenomena are weak localization, the Aharonov–Bohm effect, universal conductance fluctuations, and persistent current.

It is due to recent advances in technology that controllable experiments on mesoscopic systems became possible. As was mentioned, mesoscopic systems are of the nanoscopic scale. There is a belief that progress in this area is going to help decrease the size of electronic devices from the micro to the nano level. Some of them (quantum dots) in the future may become circuit elements for quantum computers.

1.1 Fabrication of Mesoscopic Materials. Quantum Dots

Before we proceed to the description of mesoscopic phenomena, we would like to familiarize the reader with the fabrication and typical parameters of mesoscopic systems.

Although some effects can be observed in the three-dimensional mesoscopic structures, the majority of experiments are conducted on low-dimensional systems. These are the
systems where the motion of the microscopic degrees of freedom (electrons, photons, etc.) is restricted to two-dimensional regions, one-dimensional channels (quantum wires), or zero-dimensional wells (quantum dots).

Figure 1.1: Sequence of layers in GaAs/AlGaAs semiconductor heterostructure. Adapted from [3].

The generic semiconductor structure used to obtain a two-dimensional electron gas is shown in Fig. 1.1. The two-dimensional electron gas is formed on the interface of the "sandwich" semiconductor structure between GaAs and AlGaAs. Controlled variation of the doping concentration of impurities confines the electrons in a narrow region. For example, the alloy Al\textsubscript{c}Ga\textsubscript{1−c}As consists of a periodic array of arsenic atoms with a fraction \( c \) of aluminum and 1 − \( c \) of gallium. GaAs and AlAs are special cases when coefficient \( c \) is equal to one and zero respectively.

Though the lattice potential in Al\textsubscript{c}Ga\textsubscript{1−c}As is not strictly periodic, it produces a band structure that interpolates between that of GaAs and AlAs. Consequently, the energy gap between conduction and valence bands varies with \( c \). Controlling the \( c \) parameter one can create a potential that confines electron motion in two dimensions. These graded semiconductor structures are prepared in a high vacuum using the Molecular Beam Epitaxy method (MBE).

The formation of the potential well can also be explained in the following way (see Fig. 1.2). The Fermi energy \( E_F \) in AlGaAs is higher than in the GaAs. Because of this, electrons move from the \( n-AlGaAs \) to \( i-GaAs \) leaving in the \( n-AlGaAs \) positively charged donors. The electrostatic potential thus created causes the electronic bands to bend. After equilibrium is established, electrons accumulate near the GaAs − AlGaAs interface in a thin sheet (about 10 nm thick) where the Fermi energy is inside the conduction band.

The primary focus of this work will be on quantum dots. A quantum dot is a small conduction device of the order of 0.1 − 1\( \mu \)m, confining up to several thousands of electrons.
Figure 1.2: Line-up of conduction and valence band in AlGaAs/GaAs heterojunction, (a) before and (b) after charge transfer. Adapted from [2].

Figure 1.3: Conduction band of $n$-type GaAs/$Al_{0.3}Ga_{0.7}As$ heterostructure. Adapted from [1].
Quantum dots are made by gating the two-dimensional electron gas. Electrons are confined to a small region by applying a negative electrostatic potential to metal gates that are created at the top of semiconductor structure by electron-beam lithography (see Fig. 1.4). The dot is coupled to the rest of two dimensional electron sea by two experimentally controlled point contacts, conventionally named the "source" and the "drain".

![Quantum dot structure](image)

Figure 1.4: Quantum dot structure. Adapted from [93].

The non-zero bias between the source and the drain creates a current $I$ through the device. By changing the potential on some of the metallic gates one can control the shape of the dot and the dot’s coupling to the outer electronic sea. This allows us to study a continuous range of experimentally interesting situations.

The electrons of the 2DEG are 50 – 100 nm below the surface. The effective electron mass $m^*$ in $GaAs$ is low compared to bare electron mass $m_e$: $m^* = 0.067m_e$. The density of the electron gas in the plane of motion is $4 \times 10^{11} cm^{-2}$. It corresponds to a Fermi wavelength of $\lambda_F = (2\pi/n_s)^{1/2} \approx 40 nm$, and the Fermi energy is $E_F \sim 14meV$. For comparison, the Fermi wavelength in 3D aluminum is of the order of 0.35nm and the Fermi energy is $\sim 11.63eV$.

The mobility of the carriers is defined as the ratio of electron drift velocity to applied electric field. In $GaAs/AlGaAs$ it is in the range of $10^4 – 10^6 cm^2/Vs$. The mean free path of the electron $l$ is $0.1 – 10\mu m$.

All mesoscopic effects are low temperature effects. At low temperatures the coherence length (the length scale below which quantum coherence effects can be observed) of the electron motion $L_\phi$ becomes larger than the size of the system $L$, which is one of the definitions of mesoscopic system. As an example, to resolve the spectrum of a closed dot one needs to work at as low temperature as $100mK$.

Geometrically, a quantum dot can be lateral or vertical. In a vertical quantum dot the
current goes perpendicular to the plane of the dot. This type of dot is most suitable for spectroscopic studies of a few electrons ($N \leq 20$). These dots have the shape of the regular disk with a harmonic confining potential.

To study many-electron systems lateral dots are used. In these dots the current goes in the plane of the dot. The shape of the dot is an experimentally controllable parameter. Coupling to the leads is also controlled.

In the regime of strong coupling to the leads (open dot) electrons flow freely through the dot when a potential difference is applied. The opposite regime is the weak coupling regime (closed dot) when the high negative potential on the metallic gates creates an effective potential barrier for the electrons, and the only way to go through the dot is by tunneling. Closed dots have a discrete energy spectrum, which is the reason why they sometimes called "artificial atoms".

Quantum dots with some number of impurities or imperfections inside are called disordered dots. An electron travelling through the dot collides with impurities, changing its direction of motion and/or velocity.

Recent advanced experimental techniques allow scientists to produce relatively clean quantum systems with a very low amount of disorder. The electronic motion is then mainly ballistic (ballistic dot).

![Figure 1.5: Schematic representation of (a) diffusive dot and (b) ballistic dot.](image)

For a better understanding of the classification of quantum dots we are going to discuss various length, energy and time scales. A quantum dot is characterized by its size $L$. The motion of an electron is described using Fermi wavelength $\lambda_F$ and mean free path $l$. In the diffusive regime $l$ is much smaller than the size of the system $L$ ($L \ll l$). For weak disorder $\lambda_F \ll l$ and the motion of the electron is treated semiclassically. When $l \gg L$, the motion is ballistic.

There are two more important length scales: the phase coherence length and the localization length. At low temperatures, transport through the mesoscopic structure happens coherently. The coherence length $L_\phi$ is the distance on which electron motion is phase coherent. The main sources of decoherence are the electron-phonon interactions, electron-
electron interactions, and also interactions of electrons with magnetic impurities. All these interactions do not conserve the energy of an electron and lead to the loss of the electron’s phase ‘memory’. In experiments the size of the system $L$ should be smaller than coherence length ($L < L_\phi$) in order to observe quantum effects.

The localization length $\xi$ is associated with disorder. It was discovered that at some disorder strength the wave function of the electron becomes localized around impurities. When that happens, the conducting material turns into an insulator. This is a metal-insulator phase transition. This type of localization is called Anderson localization or strong localization and can occur in any dimension.

To study coherent transport in nanostructures both the localization length $\xi$ and the coherence length $L_\phi$ should be larger than the system size $L$.

The energy scales in quantum dots are the mean level spacing $\Delta$, the diffusive Thouless energy $E_T$ and the ballistic Thouless energy $E_c$. Mean level spacing energy $\Delta$ is the distance between two neighboring energy levels, averaged over the spectrum. The associated time scale $\tau_H \sim h/\Delta$ is called the Heisenberg time. On this time scale particle dynamics is defined by properties associated with the resolution of individual levels.

The Thouless energy is another important energy scale. It is widely used to distinguish among different physical regimes and is connected to the time the particle needs to travel across the dot. In case of diffusive motion the Thouless energy is $E_T = h/\tau_D = hD/L^2$, where $D$ is a diffusive constant and $\tau_D$ is diffusion time. The ballistic Thouless energy is similarly defined as: $E_c = h/\tau_c = hv_F/L$, where $\tau_c$ is the time particle spends to crossing the dot ballistically.

1.2 Some Classic Mesoscopic Effects

The idea of coherent transport plays a central role in mesoscopic physics. Consider first a normal three dimensional metal. Assuming perfect crystal structure the motion of electron is described by extended Bloch waves. In practice, though, the conductor always has some kind of imperfections (impurities, dislocations, defects, etc.) that introduce some disorder potential. Classical transport in metals is explained by the Drude-Sommerfeld theory. According to it, electrons form a degenerate Fermi gas. Electrons in their motion, scatter from impurities, changing momentum and/or energy. Momentum relaxation occurs with a rate $1/\tau$, where $\tau$ is the mean time between successive collisions with impurities. The mean free path $l = v_F\tau$ is usually much longer than the Fermi wavelength $\lambda_F = 2\pi\hbar/(mv_F)$. Under these conditions the motion of the electron can be considered quasi-classically. In electric field $\mathbf{E}$ the electron, on average, acquires momentum $\mathbf{p} = -e\mathbf{E}\tau$. If current density is $\mathbf{j} = -n_0e\mathbf{v}$, with $n_0$ being electron density, then conductivity $\sigma_D$ defined as proportionality coefficient between current and applied electric field ($\mathbf{j} = \sigma_D \mathbf{E}$) reads:
\[ \sigma_D = \frac{n_0 e^2 \tau}{m} \]  

(1.1)

Conductivity is a material parameter. It is worth noticing that \( \sigma_D \) is a local quantity in a sense that current density at point \( r \) is determined by the value of \( E \) at the same point.

When the system size \( L \) is smaller than coherence length \( L_{\phi} \), interference effects become important. Local conductivity loses its meaning and a more suitable parameter to describe transport is the conductance \( \mathcal{G} = I/V \). Anyway, only the conductance can be effectively measured in nanoscale structures. Conductivity and conductance are related to each other by formula \( \mathcal{G} = \sigma L^{d-2} \), where \( d \) is a space dimensionality of conductor.

The quasi-classical description of electron motion says that electrons move along classical trajectories. There is a quantum mechanical phase associated with each trajectory, which is responsible for quantum interference effects. These interference effects introduce corrections to classical transport properties.

In more detail, the quantum transport probability amplitude for a particle to propagate from an initial point \( r_I \) to a final point \( r_F \) in time \( t \) can be formally expressed by the quantum mechanical propagator which is a sum over all possible paths connecting \( r_I \) and \( r_F \) (see Fig. 1.6.a):

\[ G(r_F, r_I; t) = \sum_i A_i e^{i\phi_i}, \]  

(1.2)

where \( A_i \) and \( \phi_i \) are the absolute value and phase of the complex amplitude.

![Figure 1.6](image)

Figure 1.6: Figure (a) shows two possible Feynman paths connecting initial \( r_I \) and final \( r_F \) points of propagation. Figure (b) is an example of self-crossing path that leads to weak localization.

To find the transfer probability density one takes the square of absolute value of the quantum amplitude:

\[ P(r_F, r_I; t) = |G(r_F, r_I; t)|^2 = \sum_i A_i^2 + 2 \sum_{i \neq j} A_i A_j \cos(\phi_i - \phi_j). \]  

(1.3)
For a particular sample the second term in Eq. (1.3) is responsible for quantum interference. Since the phases $\varphi_i$ that belong to different paths are statistically independent ($|\varphi_i - \varphi_j| \gg 2\pi$), after taking an ensemble average this term disappears; one obtains:

$$P_{cl}(r_F, r_I; t) = \sum_i A_i^2. \quad (1.4)$$

$P_{cl}$ is a classical object for it does not contain any quantum-mechanical phases. It resembles the classical probability and is a solution of the diffusion equation. It is called the diffuson mode, or just the diffuson.

Experiment shows that in mesoscopic systems the conductance deviates from its classical value. This happens because there is a special class of trajectories that survive ensemble averaging in Eq. (1.3). These trajectories contain loops as a result of self-intersection (see Fig. 1.6.b).

For one of these loops consider the quantum-mechanical amplitude for the electron to start at point $r$ and return to the same point $r$. If the system possesses time reversal symmetry (TRS), this means that action is invariant with respect to this symmetry. This also means that the path along the loop and its time reversed counterpart in the opposite direction have the same phase. As a result, the probability to return to the same point in such systems is twice as large compared to a system without TRS. The enhanced return probability tends to localize the electron and reduces conductance. This phenomenon is called weak localization as opposed to strong (Anderson) localization discussed earlier.

We can use this heuristic picture of coherent transport to explain several classic mesoscopic phenomena. Consider magneto-conductance measurements performed on a SiGaAs wire (Fig. 1.7).

![Figure 1.7: Results of magneto-conductance measurements in quantum dot. Adapted from [90].](image)

As one can see, the conductance reveals irregular oscillations as a function of magnetic field. The oscillations are random, but reproducible. Reheating and cooling the sample
will give a different pattern of oscillations. Similar behavior can be observed in quantum
dots. The common phenomenon for the two systems is that at fixed low temperature,
the magneto-conductance fluctuates with magnetic field $B$. On average, the conductance
increases as $B$ increases approaching some limiting value. The size of the conductance
fluctuations is universal; it depends neither on the nature nor the geometry of the system.

Qualitatively these effects can be explained as follows. The set of classical trajectories
that contribute to the quantum mechanical propagator depends on the disorder pattern.
Heating the sample rearranges impurities moving them to new locations. The new disorder
pattern generates a different set of trajectories, that leads to a different dependence of the
conductance on magnetic field.

The fact of conductance fluctuations itself is directly related to the sensitivity of quantum
mechanical phases on $B$:

$$\varphi_B = \frac{e}{\hbar c} \oint A \, dl = \frac{e}{\hbar c} \int B \, dS. \quad (1.5)$$

A change of magnetic field by an amount $(\hbar c/e)L^{-2}$ changes configuration space of the
system which induces reproducible fluctuations of the conductance.

The negative magneto-resistance is explained by the fact that at zero magnetic field
in the system with TRS symmetry, weak localization leads to a decrease in the conduc-
tance. Switching on the magnetic field destroys TRS. The time reversed trajectories in the
loops are no longer equivalent and acquire a phase difference. Due to the independence
of quantum phases, the interference effects caused by these paths do not survive ensemble
averaging, which leads to antilocalization and an increase in the conductance.

Aharonov-Bohm oscillations are another manifestation of quantum interference at the
nanoscale level. Imagine a conducting ring threaded by a magnetic flux $\phi$ (see Fig. 1.8)
The magnetic field is applied perpendicular to the plane of the ring. The area where $\mathbf{B} \neq 0$ is completely enclosed by ring; there is no magnetic field inside the conducting material.

For an electron of charge $-e$ in a nonzero magnetic field $\mathbf{B}$ the canonical momentum $\mathbf{p}$ is replaced by $\mathbf{p} + e\mathbf{A}/c$, where $\mathbf{A}$ is a vector potential with $\mathbf{B} = \nabla \times \mathbf{A}$. Thus, the electron acquires an additional phase factor associated with the magnetic vector potential: 
\[
\psi_k(\mathbf{r}) = \exp(ikr + ie \int_1^r \mathbf{A} \cdot d\mathbf{r}/\hbar c).
\]

If the area of the loop is $S$, enclosed magnetic flux is $\phi = BS$. The phase of electron travelling along upper and lower arm of the ring is:
\[
\delta_1 = \delta_1^{(0)} + \frac{e}{\hbar c} \int_1 \mathbf{A} \cdot d\mathbf{l}, \quad (1.6)
\]
\[
\delta_2 = \delta_2^{(0)} + \frac{e}{\hbar c} \int_2 \mathbf{A} \cdot d\mathbf{l}.
\]

The phase difference $\Delta \delta(\phi) = \delta_1 - \delta_2$ between the two trajectories is modulated by the magnetic flux:
\[
\Delta \delta(\phi) = \delta_1^{(0)} - \delta_2^{(0)} - \frac{e}{\hbar c} \int \mathbf{A} \cdot d\mathbf{l} = \Delta \delta^{(0)} - \frac{e}{\hbar c} \int \mathbf{B} d\mathbf{S} = \Delta \delta^{(0)} - 2\pi \frac{\phi}{\phi_0}, \quad (1.7)
\]
where $\phi_0 = hc/e$ is a magnetic flux quantum.

The conductance of the ring is proportional to the transfer probability density:
\[
\mathcal{G}(\phi) \sim |e^{i\delta_1} + e^{i\delta_2}|^2 \sim \mathcal{G}_0 + \delta \mathcal{G} \cos(\Delta \delta^{(0)} - 2\pi \frac{\phi}{\phi_0}). \quad (1.8)
\]

The conductance of the ring is a periodic function of the flux; these conductance oscillations are called *Aharonov-Bohm oscillations*.

In closed mesoscopic structures (rings, billiards) there is a possibility for existence of persistent (non-decaying) electric current. It happens when the electronic wave function is coherently extended over the whole system. Under these conditions the current exists even at finite temperatures which should be contrasted with superconducting current that appears only at low temperatures and results from a different physical phenomenon, the formation of Cooper pairs and condensation.

The mesoscopic *persistent current* can be explained in the framework of the Aharonov-Bohm effect. Consider the experimental setup for the conducting ring similar to the one discussed above. The difference is that now the ring is isolated (disconnected from the leads). Also assume that the circumference $L$ of the ring is less than the coherence length $L_\phi$. In the absence of a magnetic field, the ground state of the system is not current-carrying. The propagation of the electron in the clockwise and anti-clockwise directions
corresponds to motions with different orbital quantum number $\pm l$. For $\mathbf{B} = 0$ the energy levels with $\pm l$ are degenerate.

In a non-zero magnetic field the electron accumulates a different phase upon moving in the clockwise and anti-clockwise directions. The energy of the particle becomes dependent on $\phi$, and this dependence lifts the degeneracy of the $\pm l$ levels. As a consequence, the ground state becomes current-carrying. The existence of the current is purely due to the geometry of the system and will persist even in the presence of disorder (assuming coherent electron motion and no localization).

Figure 1.9: Persistent current in a one dimensional ring for (a) even and (b) odd number of electrons in the system. Part (c) shows the spectrum of the ring in reduced zone representation [91].

The manifestation of Coulomb interactions in charge transport appears in measurements of the conductance through almost closed dots. The conductance in such dots drops below $e^2/h$ and the charge inside the dot becomes quantized. The conductance oscillates as a function of the gate voltage $V_G$ (Fig. 1.10).

Figure 1.10: (a) shows quantum dot with gate voltage controlling electrostatic energy of electrons inside the dot. Part (b) shows Coulomb blockade peaks [92].
In almost closed dots an incoming electron experiences a tunnel barrier between the lead and the quantum dot. At first glance, the charge transport is due to the resonant transmission through the discrete energy levels in the dot. Experimentally, transmission is possible only for particular values of the gate voltage $V_G$ (Fig. 1.10).

The electrostatic energy of electrons inside the dot is $N^2e^2/2C$, where $C$ is the dot’s capacitance (approximately equal to the diameter of the dot). To add one electron into the dot one needs to supply an energy of the order of $Ne^2/C$. For small dots this is larger than the thermal energy $kT$ and the energy $eV_B$ supplied by the battery ($V_B$ is the bias voltage). Therefore, transmission is blocked by the Coulomb repulsion of electrons that are already inside the dot. This phenomenon is called Coulomb blockade.

Nevertheless, by varying the gate voltage $V_G$ one can change the electrostatic energy of electrons and lower the Coulomb blockade barrier. Conduction oscillates as a function of $V_G$. These oscillations are called Coulomb blockade oscillations.

A simple qualitative explanation of these phenomena is as follows. The classical electrostatic energy of $N$ electrons in the dot is $E(N) = N^2e^2/2C - eNV_G$. For fixed $V_G$ the optimum number of electrons will be at the minimum of $E(N)$ and is equal to $V_GC/e$. Because of continuity of $V_G$ this number is not necessarily an integer. The number of electrons on the dot takes the value of the integer closest to $V_GC/e$. When $V_GC = Ne$, the energy minimum is obtained for the state with charge on the dot $Q = Ne$, and the energy of the states with $Q = (N \pm 1)e$ is higher by $e^2/2C$. The density of states develops a gap of $E_G = e^2/C$ around the Fermi energy, blocking the tunneling of electrons into the dot.

![Figure 1.11](image-url)

Figure 1.11: (a),(c) show the situation when the gate voltage $V_G$ is set to hold exactly $N$ electrons in the dot. There is no current in the system. The dot in a state of Coulomb blockade. On (b),(d) gate voltage is set to the value where there is no difference to hold $N$ or $N+1$ electron. At this value of $V_G$ current is maximal [92].

The tunneling of electrons is possible when both states $Q = Ne$ and $Q = (N+1)e$
have the same electrostatic energy \((N + 1)^2e^2/2C - e(N + 1)V_G = N^2e^2/2C - eNV_G\). This condition is met when \(V_G = (N + 1/2)e^2/C\). This allows the tunneling of electrons into the dot at no extra energy cost and leads to a peak in the conductance.

In Chapter 4 we touch the question of the quantum critical fluctuations of magnetization in a system of two coupled quantum dots. Here we elaborate more about this topic.

Consider a system at zero temperature with the Hamiltonian \(H = H_0 + gH_I\), where \(H_0\) and \(H_I\) parts in general do not commute with each other; \(g\) is a dimensionless coupling constant.

The non-commutativity of \(H_0\) and \(H_I\) means that taken separately these parts define different ground states. Let us ask ourselves how the ground state of the system with total Hamiltonian \(H\) evolves as \(g\) changes.

For small values of coupling constant \(0 \leq g \ll 1\) we can ignore the influence of \(gH_I\) and the behavior of the system is defined by \(H_0\). The ground state of the system in this case is \(|GS\rangle_{H_0}\). In opposite limit of \(g \gg 1\), \(H_I\) is dominant and we get \(|GS\rangle_{H_I}\).

It can be seen that as \(g\) grows, the ground state changes from \(|GS\rangle_{H_0}\) to \(|GS\rangle_{H_I}\). If these ground states are macroscopically different in the sense of one having a nonzero order parameter and the other is not, it is said that a quantum phase transition took place for some critical value \(g_c\). Alternatively, the quantum phase transitions are defined as the points of non-analyticity of the free energy of the system with respect to \(g\).

Strictly speaking, quantum phase transitions happen only at \(T = 0\). Nevertheless, studying these transitions (driven by the Heisenberg uncertainty principle) gives valuable information about classical phase transitions (driven by temperature \(T\)). Fig. 1.12 shows two possible "phase diagrams".

![Figure 1.12](image-url)

**Figure 1.12:** Different regimes of quantum phase transition (panel (a)) and quantum crossover (panel (b)) [49,79].
on the coupling constant \( g \) (the example is the quantum rotor model in \( 2 < D < 3 \). In the large-N approximation the magnetic long-range order exists at all temperatures below the transition line \( T_C = T_C(g) \); see Fig. 1.12a). In the other limit, the phase transition takes place only at \( T = 0 \) (e.g. in 1D quantum Ising model magnetic long-range order exists only at \( T = 0 \); see Fig. 1.12b).

In both cases we can identify three distinct areas. Our primary focus is on the region II named the Quantum Critical Regime (QCR). In this regime the order parameter (for phase transitions) or particular observable of the system experiences quantum fluctuations due to the Heisenberg uncertainty principle: \( \langle (\Delta M)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2 \).

In regions I and III, quantum fluctuations are either vanishingly small or can be handled perturbatively due to the fact that the quantum state governed by particular part \( (H_0 \text{ or } H_I) \) of full Hamiltonian \( H \) is almost an eigenstate of that part \( (H_0 \text{ or } H_I) \).

1.3 Fermi Liquid Theory

Apart from Coulomb blockade effects, so far we have been considering phenomena where the leading role was played by quantum coherence effects. We tacitly assumed that electron interactions had only a minor or no effect. This is a rather strong assumption. Interactions among the electrons and possibly other components of the system are always present. The effects of Coulomb interactions, for example, are responsible for many important manifestations in experiments such as charging effects, dephasing and others.

The answer to the question of why in some situations one can ignore interactions, is given by Fermi liquid theory. This theory provides a one-to-one correspondence between the ground and excited states of interacting and non-interacting systems of fermions. It treats the electrons in good metals as weakly interacting quasiparticles. The lifetime of such quasi-particles close to the Fermi surface is large compared to \( \hbar/kT \) and their decay rate is small compared to their energy.

For a non-interacting Fermi gas the ground state is the filled Fermi sea. All states below the Fermi energy \( E_F \) are occupied. Excited states are created when one of the particles with momentum \( \hbar k \) inside Fermi sphere is put into the state with momentum \( \hbar k' \) above Fermi surface. At finite temperature excited states are produced thermally. These thermal excitations smear the Fermi distribution over a width proportional to the temperature \( T \), which in good metals is small compared to \( E_F \).

The Landau theory of Fermi liquids states that in the presence of repulsive interactions, (a) the filled Fermi sea of quasiparticles is a ground state in interacting case and, (b) the concept of quasiparticles persists in interacting case as long as interactions are weak and short-ranged.

The term 'quasi-particle' here is referred to a 'dressed' electron. The Coulomb interaction of electron pushes away other electrons and attracts positive lattice ions. This effect
creates a positively charged cloud around the electron. As it moves, it drags this cloud along with itself. This cloud modifies the physical properties of an electron. The term 'quasi-particle' is used to stress the difference between the properties of 'bare' (real) electron and 'dressed' electron inside a conductor.

Interactions lead to a renormalization of parameters of the theory, such as the effective mass of the electron, for example; the qualitative behavior of various physical quantities remains unchanged.

The Coulomb interaction among electrons is strong and long-ranged. The mechanism of screening of Coulomb potential \( \varphi(r) = q/r \) by conducting electrons makes the resulting interaction potential short-ranged \( \varphi(r) = qe^{-r/r_0}/r \), where \( r_0 \sim \lambda_F \) is a screening length.

In clean metals electrons are fast and screen any charge inhomogeneities from all sides on a short distance of \( \lambda_F \). This is the reason for the success of Fermi Liquid Theory in good three dimensional conductors.

Because the interaction between quasi-particles is weak, it allows us to use perturbation theory for its description. In this sense a Fermi liquid is a weakly perturbed state of a non interacting electron gas.

Fermi liquid theory can break down in low dimensional systems leading to the so-called non Fermi liquid. There is an ongoing debate in mesoscopic scientific community whether the 2D electron gas in mesoscopic systems is a Fermi liquid or not. Experiments show that the behavior of electrons allows us to conclude that electrons constitute a Fermi liquid, or possibly a marginal Fermi liquid that is on the border between a true Fermi liquid and a non Fermi liquid.

### 1.4 Random Matrix Theory

The more technical introduction to Random Matrix Theory (RMT) is given in Chapter 2. Here we just introduce in an informal way the ideas of RMT.

The first physical application of random matrix theory was in nuclear physics. The necessity to explain the behavior of the complex spectrum of compound nuclei led Wigner to the idea of replacing the Hamiltonian of the system with a random matrix. Each element of such a matrix is a random variable with functionally the same probability distribution for all the elements.

It is worth pointing out that the spectrum of a nucleus is completely determined by the corresponding Hamiltonian, that leaves no room for the statistical concepts. Nevertheless, the statistical approach seems to be the only way to deal with the spectral properties of sufficiently complex systems.

Dyson formulated the essence of random matrix theory by saying, "What is here required is a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself. We picture a complex nucleus as a black box in which
a large number of particles are interacting according to unknown laws. The problem then is to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally possible”. In other words, random matrix theory is considered to be a new kind of statistical mechanics. The average over an ensemble of states is replaced by average over an ensemble of Hamiltonians.

Modelling the physical Hamiltonian by a random matrix requires that this matrix be Hermitian, which reduces the number of independent matrix elements.

The other important physical symmetry to account for is Time Reversal symmetry (TRS). All systems fall into three general classes, called Universality classes. To the first class belong systems with Hamiltonians invariant under TRS and spin rotation invariance. The TRS invariant systems with integer spin and broken spin rotation symmetry also reside in this class.

The next class contains systems with broken TR symmetry. In a majority of situations of interest to us this means that the system is in external magnetic field.

In the third class one finds systems with half-integer spin that are invariant under TRS, and not invariant under spin rotations.

Each class is characterized by its Dyson index $\beta$. This index enters all the fundamental formulas of RMT. Depending on the universality class $\beta$ acquires the values 1, 2, and 4 (first, second, and third class respectively).

Let us now address how to use random matrix theory. Working with chaotic, or diffusive systems, or other systems with many degrees of freedom and complex interaction, one is often interested in averaged spectral properties. These averaged quantities allow us to extract useful information about the system such as the density of states, transfer probabilities, and various correlation functions.

RMT has proved to be successful in the description of spectral fluctuation properties of such objects as atomic nuclei, complex atoms and molecules. Unable to explain the locations of resonances in a particular system, it nonetheless correctly explains level correlations on the scale of the mean level spacing $\delta$.

The averaging can be done over a substantial portion of the spectrum. Such an average is always possible if one has experimental data available. Random matrix theory, though, uses a different kind of average. RMT averaging is performed over different realizations of the random Hamiltonian matrix that belongs to a particular symmetry class.

The link between the two types of average is established by the ergodic hypothesis which states that the ensemble average and the spectral average should yield the same result. This hypothesis can be proved in the limit when number of levels $N$ (same as size of random matrix $N \times N$) goes to infinity.

Each symmetry class has a random matrix ensemble associated with it. The ensemble contains matrices with the symmetry of corresponding universality class.
There is one more important aspect of RMT we have not discussed yet. It is the universality of the predictions of random matrix theory. It will be mentioned below that RMT is not limited to nuclear physics and is capable of describing a wide range of physical systems. RMT is parameter-free theory. The only energy scale is mean level spacing $\delta$ that can be eliminated if one measures energy in units of $\delta$. This makes the predictions of random matrix theory universal (independent of system realization).

We can explain this in a different way. The physical reasoning that forces one to consider Hamiltonian matrix elements as stochastic variables can differ from system to system. Thus the probability distributions of the matrix elements are also different. The fundamental property of RMT is that the eigenvalue correlations on the scale of the average level spacing do not depend on the details of probability distribution. All functional forms of probability density give the same unfolded correlation functions. This fact allows to choose computationally the most convenient probability distribution function. The Gaussian distribution is found to be the most suitable one.

Random matrix ensembles with Gaussian probability density for matrix elements are called Gaussian random matrix ensembles. These are Gaussian Orthogonal Ensemble (GOE, $\beta = 1$), Gaussian Unitary Ensemble (GUE, $\beta = 2$), and Gaussian Symplectic Ensemble (GSE, $\beta = 4$).

The analysis of energy spectra of various physical systems show that in some cases energy levels are uncorrelated; it is possible to find two or more energy levels arbitrary close to each other. Alternatively, there are systems frequently found in physics, whose energy levels display a considerable degree of repulsion. RMT is a phenomenological model to simulate such spectral behavior. To compare, the neighbor spacing distribution (the meaning of this function is explained in section 2) for uncorrelated spectrum is described by Poisson distribution $p(s) = \exp(-s)$. The correlated spectrum of GOE systems, for example, is given by formula $p(s) = (\pi s/2) \exp(-\pi s^2/4)$.

The Fig.1.13 shows the Poisson mean level spacing distribution for uncorrelated spectrum, and equivalent mean level spacing distributions for Gaussian ensembles. The Gaussian Symplectic Ensemble ($\beta = 4$) demonstrates the highest degree of level repulsion.

Over the past 50 years RMT has become a paradigm for understanding various phenomena in physics, mathematics, and other branches of modern science. Taking roots in nuclear physics, RMT has been successfully applied to disordered and mesoscopic systems. It also explains the behavior of systems with few degrees of freedom whose classical dynamics is chaotic. Due to the universality of RMT predictions, it is employed in models of interacting fermions, quantum chromodynamics and 2D gravity.
Figure 1.13: The Poisson distribution ($\beta = 0$) exhibits finite probability density for $s = 0$, while for Gaussian RM ensembles this probability goes to zero.

1.5 Large N Approximation

Random Matrix Theory helps to obtain important spectral correlations of complex systems. The results are usually shown as averages of observables over the appropriate Gaussian ensemble. Operationally one computes averages of the one particle Green’s function or the average of a product of several Green’s functions. One can use those to get physically useful information, such as the averaged density of states, or transition probabilities. As part of our work here, we calculate one particle and two particle Green’s functions for the systems of our interest.

The Green’s functions represent transition probability amplitudes for one or several particles to start and finish in particular states. They can be expanded in a series of powers of $H^n$ where $H$ is the Hamiltonian of the system. An example of such an expansion for the one particle Greens function is

$$
\langle \beta | G^{R}(E) | \alpha \rangle = G^{R}_{\alpha\beta}(E) = \left( \frac{1}{E^+ - H} \right)_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{E^+} + \frac{H_{\alpha\beta}}{(E^+)^2} + \frac{H_{\alpha\beta}^2}{(E^+)^3} + \ldots ,
$$

(1.9)

where $E^+ = E + i\eta$ is the energy with small positive part $\eta \to 0^+$.

Often $G^{R}$ is called a retarded propagator. Being a formal solution of the Schrödinger equation, the Green’s function serves as a kernel of linear integral equation that defines the wave function at some later moment of time $t$ if one knows the solution at earlier time $t' (t > t')$. For $t < t'$ the corresponding Green’s function is called advanced propagator. It formally allows to obtain the wave function at early times.

Each term in Green’s function expansion (1.9) has a diagrammatic representation (for
technical details see Appendices B, C). The process of averaging in Eq. (1.9) is performed term by term.

The summation of all terms in expansion like (1.9) does not seem feasible. On the other hand, the results of RMT coincide with experiment when any relevant energy scale in the problem (such as mean level spacing $\delta$, or level correlation distance $\omega = E_i - E_j$) is much smaller than Thouless energy $E_T \approx \delta N \gg \delta, \omega$. The results of Random Matrix Theory become exact in the limit when number of levels $N$ goes to infinity.

It turns out that in the $N \to \infty$ limit when $E, \omega \gg \delta$ only a particular class of diagrams survive averaging. These are the rainbow diagrams for one particle Green’s function (the first term in expansion for self-energy $\Sigma$, see Fig. B.3) and ladder and absolutely crossed diagrams for two particle Green’s function (see Fig. 4.4).

These diagrams give leading contribution in powers of $1/N$. All other diagrams are much smaller when $N \to \infty$.

This important approximation is called the Large-$N$ approximation. We are going to use it extensively in our work.

1.6 Universal Hamiltonian

In this section we define the so-called Universal Hamiltonian. It is called "universal" because in the low energy limit under broad conditions it is capable of describing a wide range of interacting mesoscopic systems. Our derivation follows reference [54].

The universal Hamiltonian consists of a noninteracting part $H_0$, plus the part $H_{int}$ that takes into account electron interactions: $H_U = H_0 + H_{int}$.

In the systems of interest, $H_0 = \sum_{\alpha \sigma} \epsilon_{\alpha} a_{\alpha \sigma}^{\dagger} a_{\alpha \sigma}$ is modelled by random matrix with proper space-time symmetry. The $a_{\alpha \sigma}^{\dagger}$ and $a_{\alpha \sigma}$ are the creation and annihilation operators for state $(\alpha \sigma)$.

In quantum dots or small metallic particles the dynamics of electrons is chaotic due to disorder or boundary conditions. In the limit when $g_1 = E_T / \delta_1 \to \infty$ Random Matrix Theory (RMT) correctly reproduces statistical properties of eigenvalues and eigenfunctions of the classically chaotic systems. The $E_T$ is the Thouless energy, defined by $E_T = \hbar / t_{erg}$, $\delta_1$ is the mean level spacing of the single particle spectrum. The ergodic time $t_{erg}$ is roughly equal to the time that the classical counterpart of the quantum mechanical particle needs to reach the boundaries of the system. For ballistic systems $E_T = \hbar v_F / L$, and for diffusive systems $E_T = \hbar D / L^2$, where $D$ is the diffusion constant and $L$ is characteristic size of the system.

Now the question is how to describe interactions. As was mentioned in previous sections, Fermi liquid theory allows one to treat strongly interacting electrons in metals as weakly interacting quasiparticles. Therefore, we assume the simplest case, where the quasiparticles interact via a short-ranged potential of general form:
\[ H_{\text{int}} = \lambda \delta_1 V \delta(\vec{r}), \]  

(1.10)

\( V \) denotes the volume of the system, and \( \lambda \) is a dimensionless coupling constant.

In second quantization the interacting Hamiltonian is:

\[ H_{\text{int}} = \sum_{\alpha\gamma\mu\nu} M_{\alpha\gamma}^{\mu\nu} a_{\alpha\sigma}^\dagger a_{\gamma\sigma_1}^\dagger a_{\mu\sigma_1} a_{\nu\sigma}. \]  

(1.11)

where the matrix element \( M_{\mu\nu}^{\alpha\gamma} \) has the form:

\[ M_{\mu\nu}^{\alpha\gamma} = \lambda \delta_1 V \int d\vec{r} \phi^*_\alpha(\vec{r}) \phi^*_\gamma(\vec{r}) \phi_\mu(\vec{r}) \phi_\nu(\vec{r}). \]  

(1.12)

When the magnetic field is absent, the system has time reversal symmetry and the eigenfunctions \( \{ \phi_\alpha(\vec{r}) \} \) can be chosen to be real.

In the general case when all the indices \( \alpha, \gamma, \mu, \nu \) are different, the integrand is a highly oscillating function, and the integral in Eq. (1.12) averages to zero. On the other hand, if the indices are pair-wise equal, \( M_{\mu\nu}^{\alpha\gamma} \) is large because integrand is positive-definite. Using the mean value \( \langle \phi^2_\alpha(\vec{r}) \phi^2_\gamma(\vec{r}) \rangle = V^{-2} \), one can find that

\[ M_{\alpha\gamma}^{\alpha\gamma} = M_{\gamma\alpha}^{\alpha\gamma} = \lambda \delta_1. \]  

(1.13)

For further discussion it is convenient to introduce operators for the number of electrons \( \hat{n}_\alpha \) and the spin \( \hat{S}_\alpha \) on the orbital \( \alpha \):

\[ \hat{n}_\alpha = \sum_\sigma a_{\alpha\sigma}^\dagger a_{\alpha\sigma}, \]

\[ \hat{S}_\alpha = \frac{1}{2} \sum_{\sigma\sigma_1} a_{\alpha\sigma}^\dagger a_{\alpha\sigma_1} \sigma_{\sigma_1}, \]  

(1.14)

where \( \sigma_{\sigma_1}^i \) are the Pauli matrices.

Ignoring zero matrix elements the interaction Hamiltonian reduces to the form:

\[ \hat{H}_{\text{int}} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} \]

\[ = \sum_{\alpha\gamma\sigma\sigma_1} [M_{\alpha}^{\alpha\gamma} a_{\alpha\sigma}^\dagger a_{\gamma\sigma_1} a_{\gamma\sigma_1}^\dagger a_{\gamma\sigma} + M_{\gamma}^{\alpha\gamma} a_{\gamma\sigma}^\dagger a_{\gamma\sigma_1} a_{\gamma\sigma_1}^\dagger a_{\gamma\sigma}]. \]  

(1.15)

Using relations (1.13), (1.14) and skipping detailed calculations it can be shown that the Hamiltonian \( H_{\text{int}} \) can be represented with the help of only three operators \( \hat{n}, \hat{S}, \) and \( \hat{T} \):

\[ \hat{H}_{\text{int}} = E_c \hat{n}^2 - J(\hat{S})^2 + \lambda_{\text{BCS}} \hat{T}^\dagger \hat{T}, \]  

(1.16)

where \( \hat{n}, \hat{S}, \) and \( \hat{T} \) are defined as follows:
\[ \hat{\vec{S}} = \sum_{a} \hat{\vec{S}}_{a} = \frac{1}{2} \sum_{\alpha \sigma_{1} \sigma_{2}} a_{\alpha \sigma_{1}}^{\dagger} \vec{\sigma}_{\sigma_{1} \sigma_{2}} a_{\alpha \sigma_{2}} , \]
\[ \hat{n} = \sum_{a} \hat{n}_{a} = \sum_{\alpha \sigma} a_{\alpha \sigma}^{\dagger} a_{\alpha \sigma} , \]
\[ \hat{T} = \sum_{\alpha} a_{\alpha \uparrow}^{\dagger} a_{\alpha \downarrow} . \] (1.17)

This result is not limited to systems with short-ranged interactions and is an essential feature of quantum systems, chaotic in their classical limit.

The constants \( E_{c}, J, \) and \( \lambda_{BCS} \) in Eq. (1.16) are the coupling constants and in the case of preserved time reversal symmetry (zero magnetic field) have values: \( E_{c} = \lambda \delta_{1}/2, \) \( J = 2 \lambda \delta_{1}, \) \( \lambda_{BCS} = \lambda \delta_{1} . \)

The first and second terms in Eq. (1.16) depend on the total number of particles and total spin in the system. Operators \( \hat{n} \) and \( \hat{\vec{S}}^{2} \) commute with each other and with single particle Hamiltonian \( H_{0} \), and are the good quantum numbers.

The last term in Eq. (1.16) depends on the product of Cooper pair creation and annihilation operators. This term leads to the superconducting instability for \( \lambda_{BCS} < 0. \) A non zero magnetic field, creating magnetic flux \( \Phi \approx \Phi_{0} \), breaks time reversal symmetry and destroys the superconducting state. It can be shown that the coupling constant \( \lambda_{BCS} \) is renormalized to zero in this case. Therefore, the superconducting interaction term is absent. In contrast, when the flux of magnetic field is \( \Phi < \Phi_{0} \), the time reversal symmetry is broken only partially and superconducting interactions are present. For weak magnetic fields and temperatures close to \( T_{C} \) the superconducting interaction term describes the physics of the system behavior in the quantum critical regime.

Similarly, if there is a spin-orbit interaction in the system, the second term in Eq. (1.16) is absent (\( J = 0 \)).

The final formula for the universal Hamiltonian reads:
\[ H_{U} = \sum_{\alpha \sigma} \epsilon_{\alpha} a_{\alpha \sigma}^{\dagger} a_{\alpha \sigma} + E_{c} \hat{n}^{2} - J(\hat{\vec{S}})^{2} + \lambda_{BCS} \hat{T}^{\dagger} \hat{T} . \] (1.18)

### 1.7 Overview of the Results

We finish this chapter by giving brief overview of the results of this Thesis. In Chapter 3 we study the behavior of the persistent current for electrons in ballistic billiards subject to a point magnetic flux. We consider a disk billiard and an annulus billiard. The boundaries of the billiards are arbitrary but smooth curves, that cause the ballistic motion of electrons to be chaotic. To leave the classical motion of electrons unchanged, the magnetic field exists as a single flux line in the center of the billiards.
We study the persistent current in a system of noninteracting electrons at low temperatures in the absence of disorder. In this case the net current is the sum of currents contributed by each energy level $I = \sum_k I_k = -\sum_k \partial \epsilon_k / \partial \Phi$. The surprising thing is that large diamagnetic contribution to the persistent current that we found is solely defined by the behavior of energy levels with zero orbital momentum at small magnetic flux. As will be shown later, the persistent current reaches a saturation value as the flux vanishes. That is, it is not proportional to the flux as is the case for the mesoscopic contribution to the persistent current that has been investigated previously. It is also exciting because this contribution to the persistent current was missing in earlier works.

The spectrum of disk billiard is obtained by the method of conformal transformation introduced by Robnik and Berry [42-46]. We generalize this method in a straightforward way to obtain energy levels for the chaotic annulus. Using this approach we obtain ground states for both billiards in the chaotic and integrable regimes. The net persistent current is found as a numerical derivative of the ground state energy for small values of flux. The results of our computations show that for integrable billiards the persistent current is mainly due to $l=0$ levels. The contribution of levels with nonzero orbital momentum $\pm l$ mutually cancel each other because energy levels corresponding to $+l$ and $-l$ have equal and opposite slopes. As a function of reduced magnetic flux $\alpha = \Phi / \Phi_0$ the current has a diamagnetic behavior. We also calculate the dependence of the current on the number of particles for fixed magnetic flux. The result is staircase-like function where each step appears as a new $l=0$ level gets occupied.

For irregular billiards we observe similar behavior, although there are some differences. The chaotic behavior of electrons due to boundary conditions causes each energy eigenstate of chaotic billiard to be a superposition of all the states of regular billiard. Therefore we observe the same diamagnetic persistent current, defined by $l=0$ levels of the regular system. The contribution of $\pm l$ levels is cancelled provided the chaotic dynamics mixes these levels in equal proportions. As a function of the number of particles $N$ the persistent current shows a linear behavior with data points scattered along a straight line of the same slope as for the regular annulus.

The chaotic dynamics of electrons is responsible for smoothing the steps.

Apart from numerical computations, we provide analytical calculations of persistent current for the regular disk billiard in the limit of vanishing magnetic field. We also find the $N$ dependence of current which matches our numerics.

For the regular annulus we consider two limits: the disk billiard and a one-dimensional ring. In the first limit we obtain the energy levels of the annulus expressed as small deviations from the levels of the regular disk. We show that the convergence of the annulus to the disk billiard for fixed small magnetic flux happens only logarithmically. We provide a criterion for this to happen in terms of the $\xi$ and flux $\alpha$, where $\xi = r/R$ is the ratio of the
internal and external annulus radii. Another interesting limit is when $\xi$ is kept fixed and $\alpha \to 0$. For this limit we derive a formula for the persistent current at small $\alpha$. It turns out that the current is a linear function of $\alpha$ and disappears when $\alpha = 0$. As one can see, the limits when $\xi \to 0$ and $\alpha \to 0$ are not interchangeable and lead to physically different results.

In the opposite limit of a one-dimensional ring we derive the expression for energy levels as a function of the width of the ring $\sigma$, the magnetic flux $\alpha$ and the orbital angular momentum $l$. This formula allows us to recover the results for a one-dimensional ring.

In the next Chapter 4 we consider a system of two coupled small metallic grains. Here we address the problem of electron-electron interaction which together with the chaotic character of electron motion in general makes the model difficult to solve. Our model allows us to explore the quantum critical regime where the behavior of the system is defined by an intricate interplay of thermal fluctuations and quantum-mechanical fluctuations (due to Heisenberg uncertainty principle). The basis for our interacting model was set down in reference [49].

We assume that one grain is made of superconducting material and is described by the Gaussian Orthogonal ensemble. The other grain is made of normal metal and is subject to a weak magnetic field that drives it into the crossover between the Gaussian Orthogonal and Unitary ensembles. The connecting bridge between dots belongs to the Gaussian Orthogonal ensemble.

We first consider a mesoscopic noninteracting electron system of two coupled quantum dots. Inside the dots electrons move ballistically hopping between dots. In contrast to previous works on coupled quantum dots we consider the situation when the both dots and connecting region are in the universal crossover regimes between Gaussian Orthogonal and Unitary ensembles. We consider the most general case when each dot and connecting bridge are characterized by their own crossover parameter $X$.

Utilizing a large-N diagrammatic approach we calculate averages of the one and two point Green’s functions for one uncoupled dot in the crossover between GOE and GUE. Based on these results we set up a system of Dyson equations for the one particle Green’s function in the two dot system and solve it in the limit of weak coupling. Next we construct a system of Bethe-Salpeter equations for two-particle Green’s function and solve it. We establish scaling functions that modify the behavior of Green’s functions for partial Time Reversal symmetry breaking. It turns out that the scaling functions depend on the ratio of crossover energy scales in the dots and hopping bridge, and measurement energy $\omega = E_i - E_j$. Namely, the ratio of these energy scales determines if the system belongs to the Gaussian Orthogonal or Unitary ensemble or is in a crossover between them.

We then apply the results obtained for the noninteracting system to study the behavior
of interacting electrons in the two coupled metallic grains. The interaction of electrons in
the low energy limit is described by the Universal Hamiltonian (1.18). We assume that
interactions exist only in the superconducting dot. The structure of our double dot system
allows us to get rid of the charging energy (see Fig. 4.1). As follows from formula (4.6) the
charging energy can be disregarded if the charging energies per particle pair $U_1, U_2$ for the
dots and the interdot Coulomb interaction $U_{12}$ are equal. It can be achieved by making the
dots the same thickness and area, and also by making sure that vertical separation between
the dots is much smaller than their linear size.

There is no spin-orbit coupling in the first dot, and we assume no Stoner interaction as
well. The only terms left are the kinetic term described by Random Matrix Theory and
the superconducting interaction term. We investigate the regime in which the temperature
is just above the mean field critical value $T_c$, so we do not consider phase transition to
superconducting state. This is called the regime of quantum critical fluctuations.

Our main finding is that the critical temperature non-monotonically depends on the
crossover parameter in the normal metal dot. Depending on the strength of coupling be-
tween dots, $T_c$ grows and decreases as a function of the crossover parameter in the second
dot. The interesting fact that we found is that for some interdot coupling magnetic field in
the second dot in fact supports superconducting state in the first dot. That is, the critical
temperature in the first dot grows as the magnetic field in the second dot is increased for
all values of interest.

It is important to remember that there cannot be a real phase transition in the finite sys-
tem like ours. Nevertheless, at sufficiently low temperatures the system reveals substantial
degree of superconducting pairing correlations between electrons. One says that the system
experiences the crossover from normal state to superconducting state as the temperature
is lowered. The experiments show [94–96] the existence of the distinct spectroscopic gap
(much larger than the mean level spacing) for the small superconducting metallic grains
with even number of electrons. By applying external magnetic field this gap can be driven
to zero which is the clear evidence of paramagnetic breakdown of pairing correlations.

Another distinctive result of our model is that orbital magnetization fluctuates around
critical point showing paramagnetic and diamagnetic behavior. It should be compared with
the one dot system where only a diamagnetic behavior of magnetization is observed.

We provide an explanation for these seemingly contradictory effects. On the one hand
the magnetic field in the second dot destroys the superconducting interactions in the first
dot and decreases the critical temperature $T_c$. The coupling between dots also decreases
$T_c$ by allowing interacting electrons to hope to the other dot and, therefore, diluting su-
perconducting interactions. On the other hand, the same magnetic field in the second dot
increases the electron energy in it and causes them to escape to the first dot and increasing
by this superconducting interactions and raising $T_c$. 

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The interplay between these two effects is responsible for the aforementioned fluctuations of magnetization and various dependence of critical temperature on the crossover parameter and interdot coupling.
CHAPTER 2: RANDOM MATRIX THEORY AND CROSSEOVERS

In this chapter we provide some basic facts concerning Random Matrix Theory (RMT). The active development of RMT was began in 1960s by the efforts of Wigner, Dyson and others.

Wigner proposed that the Hamiltonian of a complex system can be modelled by a Hermitian random matrix. His original study was concerned with the excitation spectra of heavy nuclei. The interactions between constituents in many-particle systems like nuclei are very complex. It was believed that the Hamiltonians describing nuclei should behave like a large random matrix.

Almost three decades after Wigner’s original work, Random Matrix Theory was successfully used to describe the behavior of generic quantum systems. Bohigas, Giannoni and Schmit conjectured that the spectral properties of the quantum systems with chaotic dynamics in classical limit should be the same as those for the random matrix.

Today Random Matrix Theory is widely applied in many branches of physics and mathematics.

One uses a random matrix to model the Hamiltonian $H$ of a real physical system. Each element of this matrix is an independent random variable with zero mean value and equal variance. Variances of diagonal and off diagonal elements differ by a numerical factor due to symmetry as will be described below.

Time reversal symmetry (TRS) plays a very important role in physics. The Hamiltonian of particular system can be either invariant or non-invariant with respect to TRS. In quantum mechanics the time reversal operator is a product $T = UK$, where $T$ is time reversal operator, $K$ is a complex-conjugation operator, and $U$ is some unitary operator.

An example of a system without time reversal symmetry is a charged particle in external magnetic field. When TRS is a symmetry of the system, there are two possible situations, when $T^2 = \pm 1$. The $T^2 = +1$ case describes the system of spinless particles in the absence of magnetic field. An example of the system with $T^2 = -1$ is a half-odd-integer particle or an electron with spin-orbit coupling.

Based on the transformation properties under TRS physical systems are divided into Universality classes, labeled by the Dyson index $\beta$:

$\beta = 1$, Systems of spinless particles invariant under TRS. Also systems of particles with integer spin invariant under TR, and with broken spatial rotation symmetry.

$\beta = 2$ Systems with broken TRS.

$\beta = 4$ Systems of particles with half-odd-integer spin with preserved TRS and violating spatial rotation symmetry.
To each symmetry class corresponds an ensemble of matrices with transformation properties compliant with the symmetry of this class.

The ensemble of random matrices with $\beta = 1$ with a Gaussian probability density distribution (the probability distribution will be discussed shortly) constitute the so called Gaussian Orthogonal Ensemble (GOE). The matrices of this ensemble can be chosen real and preserve this property under orthogonal transformation:

$$H_{nm} = H_{mn}, \quad H' = OHO^T,$$

where $O$ is an orthogonal matrix $OO^T = 1$.

The ensemble with $\beta = 2$ is called the Unitary Gaussian Ensemble (GUE). It contains Hermitian matrices that preserve their Hermitian property under unitary transformations:

$$H_{nm} = H_{nm}', \quad H' = UHU^\dagger,$$

where $U$ is a unitary matrix $UU^\dagger = 1$.

Finally, the ensemble with $\beta = 4$ is called the Gaussian Symplectic Ensemble (GSE). Elements of matrices that belong to this ensemble are real quaternions of the form:

$$H_{nm} = (H_0)_{nm} 1 + (H_x)_{nm} \tau_x + (H_y)_{nm} \tau_y + (H_z)_{nm} \tau_z = (H_0)_{nm} 1 + H_{nm} \tau,$$

where $(H_0)_{nm}$ and $H_{nm}$ are real; $1$ is a $2 \times 2$ unit matrix; $\tau$ matrices are related to the Pauli matrices by $\tau = i\sigma$.

The transformation that preserves the form of matrix elements (2.3) are Symplectic transformations defined by

$$H' = SHS^R, \quad S^R = ZS^TZ^{-1} = -ZSZ,$$

where $S$ is a Symplectic matrix $SS^R = 1$, and $Z_{nm} = \delta_{nm}\tau_y$.

The joint probability distribution function $P(H_{11}, H_{12}, \ldots, H_{NN})$ should satisfy several conditions. First, the form of distribution should not depend on the choice of basis. In a system where the energy is the only conserved quantity, any basis of orthonormal functions is equally good. In case there are other conserved quantities, the Hamiltonian has a block-diagonal form and the random matrix is meant to simulate one of these blocks (any one) where all good quantum numbers are fixed.

According to the first requirement:

$$P(H_{11}, H_{12}, \ldots, H_{NN}) = P(H'_{11}, H'_{12}, \ldots, H'_{NN}),$$

where $H' = UHU^\dagger, \quad UU^\dagger = 1$. 

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The second condition is not strictly physical and is used to simplify the situation. According to it the matrix elements should be uncorrelated:

\[ P(H_{11}, H_{12}, \ldots, H_{NN}) = P(H_{11})P(H_{12}) \cdots P(H_{NN}). \]  

(2.6)

To justify the second condition it should be pointed out that applying Renormalization Group methods one can show that in the low energy limit the matrix elements become uncorrelated.

Without loss of generality the function that obeys both requirements has the following form:

\[ P(H_{11}, H_{12}, \ldots, H_{NN}) = Ce^{-A Tr(H^2)}. \]  

(2.7)

The constant is found from normalization condition:

\[ \int P(H_{11}, H_{12}, \ldots, H_{NN}) dH_{11} dH_{12} \ldots dH_{NN} = 1. \]  

(2.8)

Parameter \( A \) can be expressed with help of variance of either diagonal or off-diagonal element:

\[ A = \frac{1}{2\langle H_{nn}^2 \rangle} = \frac{1}{4\langle H_{nm}^2 \rangle}. \]  

(2.9)

To compare predictions of RMT with experiment one usually works with correlated distribution function for eigenvalues, since the energy levels are easily accessible in experiment.

The expression for the eigenvalue distribution function for all ensembles is expressed by formula:

\[ P(E_1, \ldots, E_N) \sim \prod_{n>m} (E_n - E_m)^\nu \exp(-A \sum_n E_n^2), \]  

(2.10)

where \( \nu \) is universality index with values 1, 2, and 4 for Gaussian Orthogonal, Unitary, and Symplectic Ensembles respectively.

Analysis of the spectra of complex systems shows that levels try to stay apart from each other. This property of energy levels is captured by the previous formula (2.10). The probability to find two levels close to each other goes to zero as \( (E_n - E_m)^\nu \). The greatest level repulsion (not to be confused with spectral rigidity) is observed for Gaussian Symplectic Ensemble.

The density of states of the quantum system is the key to understanding spectral properties and is defined by:

\[ \rho(E) = \sum_n \delta(E - E_n). \]  

(2.11)

One is usually interested in the density of states averaged over the appropriate Gaussian Ensemble, which is given by:

\[ \langle \rho(E) \rangle = \begin{cases} \frac{2N}{\pi W} \sqrt{1 - \left( \frac{E}{W} \right)^2}, & |E| < W \\ 0, & |E| > W \end{cases}. \]  

(2.12)
where $N \times N$ is the size of random matrix, $E$ is the energy, and $2W$ is equal to the bandwidth which is connected to the mean level spacing $\delta$ by relation $W = 2N\delta/\pi$.

Formula (2.12) is called Wigner semicircle law for level density. It is valid for all Gaussian ensembles (GOE, GUE, GSE).

To describe more detailed properties of spectrum there are a number of spectral functions serving different purposes. By far the most useful is nearest neighbor distance distribution function. This function shows the probability for a selected level $E_n$ to find its closest neighbor level ($E_{n-1}$ or $E_{n+1}$) at the distance $S$ irrespective of the positions of all other levels.

The two-level nearest-neighbor distance distribution function is expressed approximately by the formula:

\[
P(S) = \begin{cases} 
\frac{\pi}{2} S \exp\left(-\frac{\pi}{4} S^2\right), & \nu = 1, \quad (GOE) \\
\frac{32}{\pi^2} S^2 \exp\left(-\frac{4}{\pi} S^2\right), & \nu = 2, \quad (GUE) \\
\frac{2^{18}}{3^6 \pi^3} S^4 \exp\left(-\frac{64}{9\pi} S^2\right), & \nu = 4, \quad (GSE) 
\end{cases}
\]  

The calculation of $P(S)$ for a generic system with many levels is too complicated. Nevertheless, formula (2.13) is a very good approximation to the exact result with minor deviations.

Another useful spectral function is $\Sigma^2$ statistics. It measures the rigidity of the spectrum. That is, for a given energy interval it measures the variance of the number of energy levels in this interval. The $\Sigma^2$ statistics is used to estimate universal conductance fluctuations. RMT predicts at low temperatures conductance fluctuations through a mesoscopic system are independent of the nature of the system and the value of the conductance. The value of fluctuation is equal to quantum of the conductance $e^2/h$. This is explained by the fact that fluctuations of conductance happen because of fluctuations of the number of energy levels inside the energy band of width $E_T$ around Fermi energy. For a given mesoscopic system the $\Sigma^2$ statistics helps calculate these fluctuations.

The $\Sigma^2$ statistics can be expressed in a simple way with the help of the 2-point correlation function $R_2(E_1, E_2)$. The general $n$-point correlation function is defined as:

\[
R_n(E_1, E_2, \ldots, E_n) = \frac{N!}{(N-n)!} \int P_N(E_1, E_2, \ldots, E_N) dE_{n+1} \ldots dE_N. \quad (2.14)
\]

The numerical factor $N!/(N-n)!$ accounts all the possibilities to select $n$ levels out of $N$. As one can see, $R_n$ is the probability to find $n$ levels at positions $E_1, \ldots, E_n$ irrespective of the positions of all the other levels.

The 2-point correlation function for GUE and GOE has the form:

\[
R_2(E) = \begin{cases} 
1 - \left(\frac{\sin \pi E}{\pi E}\right)^2, & (GUE) \\
1 - \left(\frac{\sin \pi E}{\pi E}\right)^2 - \left[\frac{\pi}{2} \text{sgn}(E) - \text{Si}(\pi E)\right]\left[\frac{\cos \pi E}{\pi E} - \frac{\sin \pi E}{(\pi E)^2}\right], & (GOE), \quad (2.15)
\end{cases}
\]
where $E = E_2 - E_1$, $\text{Si}(x) = \int_0^x dt \sin t/t$.

Another useful function, related to $R_2$ is two-level cluster function $Y_2(E) = 1 - R_2(E)$.

If the number of levels inside the interval of length $L$ is given by:

$$n(E, L) = \int_{E-L/2}^{E+L/2} \rho(E) dE, \quad \langle n(E, L) \rangle = L,$$

then $\Sigma^2(L)$ statistics is defined as standard deviation:

$$\Sigma^2(L) = \left\langle \left( n(E, L) - \langle n(E, L) \rangle \right)^2 \right\rangle = \langle n(E, L)^2 \rangle - L^2,$$

where we used the fact that $\langle n(E, L) \rangle = L$.

The calculations show that $\Sigma^2$ depends on $L$ as:

$$\Sigma^2(L) = L - 2 \int_0^L (L - E) Y_2(E) dE \approx \frac{2}{\nu \pi^2} \ln(L) + a_\nu + O(L^{-1}),$$

where $a_\nu$ is specific for each Gaussian ensemble.

The situation when the system is described by one of the main RMT ensembles (GOE, GUE or GSE) is rather ideal. There is a growing number of experimental indications that there are systems showing deviations from the standard Gaussian ensembles.

For example, consider a system of spinless particles, originally described by the Gaussian Orthogonal Ensemble, put into weak magnetic field that only slightly breaks the time reversal symmetry.

Therefore, one needs to consider situations when the system belongs neither to the GOE nor to the GUE ensemble. One can say then that the system is in a crossover between the GOE and the GUE.

In our work we consider two systems in crossovers. The first system is in a transition between the Gaussian Orthogonal ensemble and the Gaussian Unitary ensemble. The other system is in a crossover between the Gaussian Orthogonal ensemble and the Gaussian Symplectic ensemble.

As was mentioned, the GOE systems are represented by real symmetric matrices, while GUE systems are described by Hermitian matrices. We model the system in a crossover between the GOE and the GUE by the following matrix:

$$H_{nm} = \frac{S_{nm} + iX A_{nm}}{\sqrt{1 + X^2}},$$

where $S_{nm}$ is a real symmetric matrix and $A_{nm}$ is a real antisymmetric matrix. Both $S_{nm}$ and $A_{nm}$ have the same distribution for matrix elements. One can see that the total Hamiltonian $H$ in (2.19) is a Hermitian matrix.

The parameter $X$ is called a crossover parameter. It is a function of the applied magnetic field that breaks TR symmetry. Depending on the value of $X$ the system is described by
the GOE ensemble \((X = 0)\), GUE ensemble \((X = 1)\), or is in the crossover between two \((0 < X < 1)\). The normalization factor \((1 + X^2)^{-1/2}\) is chosen to keep the mean level spacing \(\delta\) constant for arbitrary \(X\).

As will be discussed below, each crossover parameter \(X\) has a crossover energy scale \(E_X \simeq X^2 E_T\) associated with it. To understand this relation consider a ballistic electron in a mesoscopic ring or a billiard threaded by a magnetic flux. The Aharonov-Bohm phase that electron picks up upon complete single rotation is \(\Delta \phi = 2\pi \Phi / \Phi_0\). For one turn the enclosed flux is proportional to \(\Phi = BL^2\), where \(L\) is the size of the dot and \(B\) is the magnetic field. After \(N\) turns the total flux will be \(\Phi_{\text{total}} = \sqrt{N} \Phi\). The \(\sqrt{N}\) factor appears because the electron with equal probability can go clockwise or counter-clockwise around the dot, thus doing a random walk in the accumulated phase. The minimal phase shift for the electron to notice the presence of the magnetic flux is of the order of \(2\pi\). Then the minimal cumulative flux enclosed by the orbit should be \(\Phi_0 = \sqrt{N} \Phi\). This gives \(N = (\Phi_0 / \Phi)^2\), while the time to make \(N\) turns is \(\tau = LN/v_f\) (for a ballistic/chaotic mesoscopic system). From the Heisenberg uncertainty principle the associated energy scale is:

\[
E_X \approx \frac{\hbar}{\tau} = \frac{E_T}{N} = E_T \left(\frac{\Phi}{\Phi_0}\right)^2,
\]

where \(E_T = \hbar v_f / L\) is the ballistic Thouless energy. For diffusive mesoscopic systems it should be substituted by the diffusive Thouless energy \(E_T \cong \hbar D / L^2\). The crossover parameter \(X\) is proportional to magnetic field. With limiting values zero and one \(X\) can be thought to be proportional \(X \sim \Phi / \Phi_0\). From here we obtain our relation \(E_X \simeq X^2 E_T\).

Knowing the expression for the general matrix element of the system in a crossover \((2.19)\) allows one to calculate averages of pairings of two matrix elements \(\langle H_{nm}H_{st}\rangle\) which are the building blocks in calculations of the averages of the one and two particle Green’s functions.

Now consider the situation when the Hamiltonian of a system of particles with spin-1/2 contains spin-orbit coupling. Without this coupling the system is described by a direct product of two GOE ensembles. When the orbital and spin degrees of freedom are mixed, rotational symmetry is broken, and if the system is TR invariant, it is formally described by GSE ensemble. In the case when we have only a weak spin-orbit coupling, the system is in a crossover. The general matrix element for the system in a transition between the GOE and the GSE reads

\[
H_{nm} = \frac{(H_0)_{nm}1 + iX [(H_x)_{nm} \sigma_x + (H_y)_{nm} \sigma_y + (H_z)_{nm} \sigma_z]}{\sqrt{1 + 3X^2}},
\]

where \((H_0)_{nm}\) and \((H_{x,y,z})_{nm}\) are a real symmetric matrix and a real antisymmetric matrix with the same matrix element probability distribution; \(1\) is a \(2 \times 2\) unit matrix and \(\sigma_{x,y,z}\) are the Pauli matrices. The crossover parameter \(X\) depends on the spin-orbit coupling constant. For \(X = 0, 1\) the system belongs to a pure GOE, GSE ensemble; when \(0 < X < 1\), the
system is in the crossover. As for the previous crossover, the normalization \((1 + 3X^2)^{-1/2}\) keeps the mean level spacing \(\delta\) constant for all values of \(X\).

In our work we are interested in the situation when the crossover parameter is only slightly different from zero \((X \ll 1)\). It will be shown in the following chapters that the true crossover parameter is a combination of \(X\) and relevant energy scales in the system. It really depends on the measurement. As an example, one of the crossover parameters that we will encounter later is \(E_X = 4X^2 N\delta/\pi\). The one and two point Green’s functions depend on the ratio of \(E_X\) and energy difference \(\omega = E_i - E_j\). It will be seen later that when the crossover parameter \(E_X\) is much larger or much smaller than the energy of measurement \(\omega\), the system has properties that characterize it as the one that belongs to one of the pure Gaussian ensembles. We can say that when \(E_X \ll \omega\), the system still belongs to the GOE ensemble. In contrast, when \(E_X \gg \omega\), the system is fully crossed over; that is, it belongs to the opposite ensemble. The energy \(\omega\) can be of the order of mean level spacing. In this case the system is always crossed over. Finally, when \(E_X \sim \omega\), the system truly is in a crossover between the two limiting ensembles.
3.1 Introduction

A resistanceless flow of electrons can occur in mesoscopic systems if the linear size $L$ is less than the phase coherence length $L_\phi$. The simplest example of this is a one-dimensional (1D) metallic ring threaded by a magnetic flux $\Phi$. The thermodynamic relation,

$$I = -\frac{\partial F}{\partial \Phi}$$ (3.1)

defines the persistent current in mks units. At zero temperature, which we will focus on, the free energy $F = E - TS$ becomes by the total ground-state energy $E$.

Persistent currents were first predicted to occur in superconducting rings [4–6]. It was later realized that persistent currents exist in normal metallic rings as well [7–9]. The phenomenon is understood most easily at zero temperature for a ring of noninteracting electrons, where the electronic wave function extends coherently over the whole ring. If the ring is threaded by a solenoidal flux, all physical properties are periodic in applied magnetic flux with a period of the flux quantum $\Phi_0 = h/e$. This is because in the presence of magnetic flux an electron, after complete rotation along the ring, picks up an additional phase $\Delta \delta(\phi) = 2\pi \Phi/\Phi_0$ (see formula (1.7)). A nonzero flux splits the degeneracy between clockwise and anticlockwise moving electrons (for $B = 0$ the electrons with orbital momentums $\pm l$ have the same energy). Upon filling the energy states with electrons, one finds ground states, which have net orbital angular momentum, and net persistent current. Much experimental work has been carried out on ensembles of rings/quantum dots [10, 11] in a flux as well as on single metallic [12–16] or semiconductor quantum dots/rings [17–20]. The subject has a long theoretical history as well [21–31] (for a review see Ref. [32]).

In this chapter we investigate the persistent current of noninteracting electrons in quantum billiards subject to a point flux. Related semiclassical calculations have been carried out in the past for regular (integrable in the absence of flux) (Refs. [33–38]) and chaotic billiards [33, 35, 36, 39]. Numerics have previously been performed on these systems as well [40, 41]. We carry out calculations on the simply connected chaotic Robnik-Berry billiard [42–46] (also known as a Pascal limaçon), obtained by deforming the boundary of the integrable disk, and on an annular analog, which we call the Robnik-Berry annulus (see Fig. 3.1).

The ratio of the inner $r$ to the outer radius $R$ of the annulus ($\xi = r/R$) plays an important role in our analysis, and allows us to go continuously between the simply connected chaotic two-dimensional billiard and a (effectively disordered) quasi-one-dimensional ring. We use
Figure 3.1: Schematic representation of conformal transformation \( w \) of irregular (a) disk and (b) annular billiards into regular counterparts.
this billiard for ease of computation. As will be shown below, for this class of billiards there is a straightforward procedure that allows one to obtain the energy levels and calculate the persistent current. We expect our results to depend on neither the detailed shape of the billiard nor the degree of chaoticity, as will become evident below.

Our main result is that there is a large diamagnetic and flux-independent contribution (for small flux) to the persistent current for $|\Phi| \ll \Phi_0$ in the simply connected billiard, which is proportional to the number of particles and overwhelms the mesoscopic fluctuations, which have been the focus of previous work [21,22,27–29,33–41]. This arises from angular-momentum $l = 0$ states in the integrable disk. These levels respond diamagnetically, with energy increasing linearly with the point flux, for small flux. Due to the periodicity of the spectrum under an integer change of point flux, it follows that there is a similar systematic paramagnetic contribution for the flux tending to an integer from below. In other words, near an integer multiple $n$ of the flux quantum $\Phi_0$, the energy is proportional to $|\Phi - n\Phi_0|$. This behavior is robust under the deformation of the boundary, which makes the dynamics chaotic. As $\xi$ increases from zero, this contribution to the persistent current persists for typical $\Phi/\Phi_0 \simeq 1$ but smoothly decreases in magnitude and becomes negligible for $\xi \to 1$. The precise $\xi$ at which the diamagnetic contribution to the persistent current becomes equal to the typical fluctuating paramagnetic contribution depends on the electron density. For $\xi \neq 0$ and very tiny flux, the diamagnetic contribution to the persistent current varies linearly with $\Phi/\Phi_0$ (see below).

This diamagnetic contribution seems to have been missed in the previous work to the best of our knowledge. The reason is that the semiclassical approximation becomes asymptotically exact as the energy tends to infinity, and in this limit, the spectral density of $l = 0$ states vanishes. Thus, $l = 0$ states are explicitly disregarded [33,35,36,39] in the semiclassical approach since they do not enclose flux. It has been noted in the past that diffraction effects necessitate an inclusion of $l = 0$ states in the sum over periodic orbits on the integrable disk [34] but the connection to persistent currents was not made.

It should be emphasized that since the total persistent current is a sum over the contributions of all levels, the diamagnetic contribution we uncover exists even at very large energies where the levels at the Fermi energy are well approximated by semiclassics.

The robustness of the diamagnetic contribution to the persistent current under deformation can be understood as follows: In the chaotic billiard, each state at a particular energy is roughly a linear combination of states of the regular disk within a Thouless energy ($E_T \simeq h v_F/L$, where $L$ is the linear size of the billiard) of its energy. When the Fermi energy $E_F$ greatly exceeds $E_T$, the contribution of the occupied states does not change much when the boundary is deformed and chaos is introduced.

The above argument also reveals that the diamagnetic contribution we uncover should not depend on the degree of chaoticity of the billiard since it is descended from the response
of the $l = 0$ states on the regular disk/annulus. Different degrees of chaos can only alter the width of the energy window ($\simeq E_T$) over which the states of the regular disk are spread when the disk/annulus is deformed. Again, for $E_F \gg E_T$, the degree of chaos is seen to be unimportant. The effect we describe is completely generic as long as the states are not localized.

The systematic diamagnetic contribution we uncover appears similar to, but is different from, Landau diamagnetism [47] in a finite system, which is a response to a uniform magnetic field. The primary difference is that the orbital magnetization (proportional to the persistent current) in Landau diamagnetism is proportional to the field itself (because the energy goes quadratically with the field strength), whereas the effect we describe is independent of the flux for small flux in the simply connected Robnik-Berry billiard (because the energy goes linearly with the flux (see formula (3.1) and Fig.3.2.a). In the Robnik-Berry annulus with $\xi \to 0$, the energy rises quadratically with the flux for very tiny flux $\Phi \ll \Phi_0 / \log N \xi^{-2}$ but crosses over to the linear behavior characteristic of the simply connected system for larger $\Phi$. Since the flux is point-like and, in the annular case, nonzero only where the electron wave functions vanish, the entire effect is due to Aharonov-Bohm quantum interference. (The annular case is physically cleaner since there are no diffraction effects associated with the point flux, in contrast to the case of the disk [34]). The effect, in the annular case as well, is primarily caused by the $l = 0$ levels deep below $E_F$. Experimental detection is feasible only through the total magnetization and not by conductance fluctuations that are sensitive to the levels within the Thouless shell (lying within $E_T$ of $E_F$). Previous samples have been subjected to a uniform field rather than a point flux [12–15, 17–19] and anyway the ring samples have $\xi$ too large for this effect to be seen. However, we believe that experiments can be designed to observe this effect.

The plan of this chapter is as follows: In Section 3.2 we describe the method we use to calculate the spectrum, and present analytical expressions and numerical results for persistent current in the disk and simply connected Robnik-Berry billiards. In Section 3.3 we generalize the method to the annulus and present our results. Conclusions and implications are presented in Section 3.4.

3.2 Simply connected Robnik-Berry billiard

We begin by briefly describing the procedure to obtain the energy levels $\epsilon_k$ within the billiard, which leads to the persistent current. At zero temperature the net current is the sum of contributions from each occupied level:

$$ I = \sum_k I_k, \quad I_k = -\frac{\partial \epsilon_k}{\partial \Phi}. $$

We consider a quantum billiard without any spatial symmetry. Our quantum billiard is a two dimensional domain surrounded by impenetrable walls. The particle moves ballistically...
inside the billiard. In particular, for computational convenience we work with the Robnik-Berry billiard [43, 44], which is obtained from the unit disk by conformal transformation (see Fig.3.1). The problem of finding energy levels of electron in the original billiard with complicated boundaries is reduced to a problem where the electron moves in the unit disk in a fictitious potential introduced by the following conformal transformation:

$$w(z) = \frac{z + b z^2 + c e^{i\delta} z^3}{\sqrt{1 + 2b^2 + 3c^2}} \quad (3.3)$$

where $w = u + iv$ represents the coordinates in the laboratory coordinate system of irregular billiard, and $z = x + iy$ are the conformally transformed coordinates of the regular unit disk billiard (details are in the Appendix A). The parameters $b$, $c$, and $\delta$ control the shape of the original billiard, and for the values we use, the classical dynamics is mixed, but largely chaotic. It is also straightforward to introduce a point flux that penetrates the center of the unit disk [43–46] (after the conformal transformation; see Appendix A).

As mentioned in the introduction, we use this billiard for ease of computation. The results we find are expected to be completely generic, and also apply to billiards that are not fully chaotic.

Following Robnik and Berry’s method we find 600 energy levels for regular and chaotic billiards for different values of reduced magnetic flux $\alpha = \Phi/\Phi_0$ coming through the billiards. Only the lowest 200 levels are actually used in further calculations since the higher levels become increasingly inaccurate [45, 46]. The persistent current is obtained as a numerical derivative of the ground-state energy for a given number of electrons.

Our numerical results show that for the unit disk billiard the ground-state energy $E_G$ has a non zero slope as $\alpha \to 0$. Thus there is a persistent current in the system for arbitrarily small magnetic flux (see Fig. 3.2a,b).

Qualitatively this behavior can be understood as follows. In the absence of magnetic field energy levels corresponding to orbital quantum numbers $\pm l$, are degenerate. A nonzero $\Phi$ lifts the degeneracy, and for small $\alpha$ the two $\pm l$ levels have slopes that are equal in magnitude and opposite in sign (from Eq. (3.2) the slope of the level is equal to the persistent current carried by the level). Thus, as long as both are occupied, these levels do not contribute to the net persistent current $I$. The only nonzero contribution comes from levels with $l = 0$.

For the unit disk the expression for the persistent current can be derived analytically for small values of magnetic flux. At zero temperature the persistent current due to $k$th level is $I_k = -\partial \epsilon_k / \partial \Phi$ ($\epsilon_k$ is a dimensionless energy, and $I_k$ is persistent current divided by the energy unit $\hbar^2/2mR^2$; see the Appendix A for notations). The solutions of the Schrödinger equation for the unit disk with with zero potential inside are properly normalized Bessel functions $J_{l,n}$ of the first kind

$$\phi_{l,n}(r, \theta) = \frac{J_{|l-\alpha|}(\gamma_n(|l-\alpha|)r)}{\sqrt{\pi} J'_{|l-\alpha|}(\gamma_n(|l-\alpha|))}, \quad (3.4)$$
Figure 3.2: Ground-state energy $E_G$ in units of $10^4 \times \frac{\hbar^2}{2mR^2}$ and persistent current $I$ in units of $\frac{e\hbar}{4\pi mR^2}$ as a function of dimensionless flux for the regular disk (panels a and b), and the simply connected chaotic billiard (panels c,d). The results are for 200 particles.
The energy levels are found from the quantization condition that the wave function turns to zero on the boundaries \( \phi_{l,n}(r = 1, \theta) = 0 \)

\[
J_{|l-\alpha|}(\gamma_n(|l - \alpha|)) = 0, \quad \epsilon_k = \gamma_n^2(|l - \alpha|),
\]

where \( \gamma_n(|l - \alpha|) \) is the \( n \)th root of the Bessel function \( J_{|l-\alpha|} \), and \( \epsilon_k \) is the eigenvalue, corresponding to \( \gamma_n(|l - \alpha|) \).

Then from Eq. (3.2), the persistent current caused by \( k \)th level is:

\[
I_k = - \frac{2e}{\hbar} \gamma_n(|l_k - \alpha|) \frac{\partial \gamma_n(|l_k - \alpha|)}{\partial \alpha}.
\]

To find \( \partial \gamma / \partial \alpha \) we differentiate Eq. (3.5) as

\[
\frac{\partial J_\nu(\gamma)}{\partial \alpha} = \frac{\partial J_\nu(\gamma)}{\partial \nu} \frac{\partial \nu}{\partial \alpha} + \frac{\partial J_\nu(\gamma)}{\partial \gamma} \frac{\partial \gamma}{\partial \alpha} = 0.
\]

(3.7)

For \( l = 0 \) levels, \( \nu = |l - \alpha| = \alpha \). In \( \alpha \to 0 \) limit, for the derivatives of Bessel function, one gets

\[
\frac{\partial J_\nu(\gamma)}{\partial \nu} \bigg|_{\nu=0} = \frac{\pi}{2} N_0(\gamma),
\]

\[
\frac{\partial J_\nu(\gamma)}{\partial \gamma} \bigg|_{\nu=0} = -J_1(\gamma).
\]

(3.8)

Combining Eqs. (3.7) and (3.8) and using relation that, for \( \gamma \gg 1 \) Bessel function \( N_0(\gamma) \approx J_1(\gamma) \) (this approximation works well already for the first root of Eq. (3.5)), we find from (3.7) that \( \frac{\partial \gamma}{\partial \alpha} \big|_{\alpha=0} = \frac{\pi}{2} \), which leads to

\[
I = - \frac{\pi e}{\hbar} \sum_n \gamma_n(0),
\]

(3.9)

where summation is over the levels with orbital quantum number \( l = 0 \).

For large argument values (which is the same as large energies), the quantization condition [Eq. (3.5)] for the unit disk becomes \( \cos(\gamma_n - \pi \alpha/2 - \pi/4) = 0 \), with roots:

\[
\gamma_n = \pi \alpha/2 + \pi/4 + \pi(2n + 1)/2.
\]

(3.10)

With the energy being measured in \( \hbar^2/2mR^2 \) units, the Fermi wave vector is \( k_F = \gamma_{\text{max}} \approx \pi n_{\text{max}} \), where \( n_{\text{max}} \) denotes the largest \( l = 0 \) level. With disk area \( A \) equal to \( \pi \) (\( R = 1 \)), the number of particles in the system is \( N = A \pi k_F^2/(2\pi)^2 = (\pi n_{\text{max}}/2)^2 \). This allows us to find the dependence of the persistent current on number of particles in the system in \( \alpha \to 0 \) limit,

\[
I = - \frac{e}{\hbar} \sum_n (\frac{3\pi}{4} + \pi n) \approx - \frac{e \pi^2}{2\hbar} n_{\text{max}}^2 = - \frac{e}{\hbar} 2N,
\]

(3.11)
where we neglected a subleading term proportional to $n_{\text{max}}$. We remind the reader that the physical persistent current is the expression in formula (3.11) multiplied by energy unit $\hbar^2/2mR^2$.

In Fig. 3.3a the persistent current $I$ is plotted against the number of particles $N$ for magnetic flux $\alpha = 0.01$. The behavior of the current is consistent with Eq. (3.11). That is, for small magnetic flux, it is proportional to $2N$.

For the regular disk (Fig. 3.3a) the persistent current is a set of consecutive steps. Each step appears when the next $l = 0$ level is added to the system. The length of the steps is equal to the number of $l \neq 0$ levels between two adjacent levels with zero orbital quantum number. As one particle is added to the $l \neq 0$ level, it results in persistent current jump. The next level corresponding to $-l$ has opposite slope, and once it is occupied, cancels the contribution of the previous $l$ level to the net persistent current. This explains the ”noise” above each step in Fig. 3.3(a). The noise appears on the level of previous step since the slopes of the current $l = 0$ level and all nearby $l \neq 0$ levels do not differ much. In addition, each step has a small inclination, which is due to the fact that the $l \neq 0$ levels do not cancel each other exactly when $\Phi \neq 0$. For larger magnetic flux the steps become more inclined.

To see that levels with $l \neq 0$ do not contribute to the persistent current at weak magnetic flux, we simply note that the derivative of $\gamma$,

$$\frac{\partial \gamma_n}{\partial \alpha} \bigg|_{\alpha=0} = -\frac{\partial J_\nu(\gamma_n)}{\partial \nu} \frac{\partial \nu}{\partial \alpha} \left/ \frac{\partial J_\nu(\gamma_n)}{\partial \gamma_n} \right|_{\alpha=0}$$

is an odd function of $l$. In the $\alpha \to 0$ limit, the root $\gamma_n(\nu)$ and the derivatives of $J_\nu(\gamma_n)$ in Eq. (3.12) are even functions of $l$, and $\partial \nu/\partial \alpha$ is odd. As a result, the whole expression is an odd function of $l$, which proves the cancellation of $\pm l$ levels in Eq. (3.6).

As was discussed in Sec.3.1, for the chaotic simply connected Robnik-Berry billiard, each eigenstate is a superposition of all $l$ states of regular disk (see Eq. (A.5)), mostly within
a Thouless shell of its energy. Assuming that states with $\pm l$ enter this superposition with equal probability over the ensemble due to the chaotic nature of motion, one can conclude that the ensemble-averaged contribution of these levels to the net current is zero. Thus, as seen in Fig. 3.3(b), the mesoscopic fluctuations due to the $l \neq 0$ levels are overwhelmed by the diamagnetic contribution linear in $N$ for small $\alpha$.

### 3.3 Robnik-Berry annulus

Now we turn our attention to annular billiard. Here there is an additional parameter $\xi$, which is the ratio of the inner radius $r$ to the outer radius $R$ of the regular annulus in the (conformally transformed) $z$ plane. By varying $\xi$ we are able to smoothly go from the simply connected Robnik-Berry billiard to a (effectively) disordered ring in the limit $\xi \to 1$.

First consider the disk limit $\xi \to 0$. We can derive an analytical expression for $I$ when $\xi$ is small enough that $\xi \gamma_n \ll 1$ and $\gamma_n \gg 1$. For a regular annulus with $R = 1$ ($r = \xi$) the wave function is a linear combination of a Bessel function and a Neumann function. The energy quantization follows from the Dirichlet boundary condition,

$$J_\nu(\gamma_n)N_\nu(\gamma_n \xi) - J_\nu(\gamma_n \xi)N_\nu(\gamma_n) = 0. \quad (3.13)$$

Here $n$ enumerates the roots at fixed index $\nu$. We use the large and small argument expansions for Bessel functions

$$\begin{align*}
J_\nu(z) &\approx \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\pi \nu}{2} - \frac{\pi}{4}\right), \quad z \gg 1 \\
N_\nu(z) &\approx \sqrt{\frac{2}{\pi z}} \sin \left(z - \frac{\pi \nu}{2} - \frac{\pi}{4}\right), \quad z \gg 1 \\
J_\nu(z) &\approx \frac{(z/2)^\nu}{\Gamma(\nu + 1)}, \quad z \ll 1 \\
N_\nu(z) &\approx \cot(\nu \pi) \frac{(z/2)^\nu}{\Gamma(1 + \nu)} - \frac{1}{\sin(\nu \pi)} \frac{(z/2)^{-\nu}}{\Gamma(1 - \nu)}, \quad z \ll 1
\end{align*}$$

(3.14)

to obtain for the $l = 0$ levels:

$$\cot(\gamma_n - \frac{\pi \alpha}{2} - \frac{\pi}{4}) = \frac{1}{\Gamma(1 + \alpha)} \frac{(\gamma_n \xi)^\alpha}{\Gamma(\frac{\alpha}{2})} - \frac{\Gamma(\alpha)}{\pi} \left(\frac{\gamma_n \xi}{2}\right)^{-\alpha}. \quad (3.15)$$

We express the roots for the annulus as a small deviation from the roots for the disk, which we denote as $\gamma_n^{(d)}$; $\gamma_n = \gamma_n^{(d)} + \delta \gamma_n$ with $\gamma_n^{(d)} = \alpha \pi / 2 + \pi / 4 + \pi (2n + 1)/2$. For small $\alpha$ we approximate $\cot(\alpha \pi)$ by $1/(\alpha \pi)$, and for small values of $\delta \gamma_n$, we find

$$\delta \gamma_n = -\frac{\alpha \pi}{2} \left[1 + \coth \left(\alpha \ln \frac{\gamma_n \xi}{2}\right)\right]. \quad (3.16)$$
To derive Eq. (3.16) from Eq. (3.15) we also used the approximation \( \Gamma(\alpha)\Gamma(1 + \alpha) \approx 1/\alpha \).

In Eq. (3.16), for small \( \alpha \), \( \gamma_n \) under logarithm can be safely replaced by its value for the disk \( \gamma_n^{(d)} \). Then roots for the annulus are

\[
\gamma_n = \frac{\pi}{4} + \frac{\pi}{2}(2n + 1) - \frac{\pi}{2}\alpha \coth \left( \alpha \ln \frac{\gamma_n^{(d)}}{2} \right). 
\]  

(3.17)

One can now take various limits of Eq. (3.17). To recover Eq. (3.10) for the disk roots, we keep magnetic flux \( \alpha \) fixed and take the limit \( \xi \to 0 \). As one can see from Eq. (3.17), convergence to the disk limit is slow due to the logarithm, and occurs only for \( \alpha \gg 1/|\log (\gamma_n^{(d)} \xi/2)| \).

Another limit of interest is to keep \( \xi \) fixed and obtain behavior of roots \( \gamma_n \) for small \( \alpha \). For small \( \alpha \ll 1/|\log (\gamma_n^{(d)} \xi/2)| \), the roots \( \gamma_n \) with \( l = 0 \) vanish quadratically with \( \alpha \). Expanding the coth function in Eq. (3.17), we obtain

\[
\gamma_n = \frac{\pi}{4} + \frac{\pi}{2}(2n + 1) - \frac{\pi}{2}\left( 1 + \frac{\alpha^2}{3} \ln^2 \frac{\gamma_n^{(d)}}{2} \right) \ln^{-1} \frac{\gamma_n^{(d)}}{2} \xi. 
\]  

(3.18)

which according to formula (3.6) leads to the persistent current:

\[
I \approx \frac{2\pi e\alpha}{3h} \sum_n \left[ \left( \frac{\pi}{4} + \frac{\pi}{2}(2n + 1) \right) \ln \frac{\gamma_n^{(d)}}{2} - \frac{\pi}{2} \right]. 
\]  

(3.19)

A rough estimate of this sum with the help of the Euler-MacLaurin formula gives

\[
I \approx \frac{\pi^2 e\alpha}{3h} n_{\text{max}}^2 \ln \frac{n_{\text{max}} \xi \pi}{2\sqrt{e}}, 
\]  

(3.20)

where we kept only terms proportional to \( n_{\text{max}}^2 \), and \( e = 2.71828 \ldots \) inside the logarithm denotes Euler’s number and not the electronic charge.

Using the relation \( N = (\pi n_{\text{max}}/2)^2 \) (for small values of \( \xi \), the density of states for the annulus and the disk are practically the same), the persistent current becomes, for small \( \alpha \ll 1/|\log (\gamma_n^{(d)} \xi/2)| \),

\[
I = I^{(d)} \alpha \left[ \frac{N\xi^2}{e} \right], \quad I^{(d)} = -\frac{e}{h} 2N. 
\]  

(3.21)

To approach the limit of a one-dimensional ring, where \( \gamma_n \gg 1 \) and \( \gamma_n \xi \gg 1 \), we return to quantization condition (3.13) and use the following large argument expansion for Bessel functions:

\[
J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left( \cos(z - \frac{\pi \nu}{2} - \frac{\pi}{4}) - \sin(z - \frac{\pi \nu}{2} - \frac{\pi}{4}) \frac{\nu^2 - 1/4}{2z} \right), 
\]

\[
N_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \left( \sin(z - \frac{\pi \nu}{2} - \frac{\pi}{4}) + \cos(z - \frac{\pi \nu}{2} - \frac{\pi}{4}) \frac{\nu^2 - 1/4}{2z} \right). 
\]  

(3.22)

We use formulas (3.22) and quantization condition (3.13) to get an equation for roots:

\[
\sin(\gamma_n \sigma) - \cos(\gamma_n \sigma) \frac{\nu^2 - 1/4}{2\gamma_n \xi} \sigma = 0, 
\]  

(3.23)
where we ignore the term proportional $1/\gamma_n^2$. The quantity $\sigma = 1 - \xi$ is assumed to be much less than unity. For sufficiently small $\sigma$, one can drop the second term in Eq. (3.23) and get $\gamma_n \sigma = \pi n$. To find corrections to this expression, we assume that $\gamma_n \sigma = \pi n + \eta$ with $\eta = \nu^2 - 1/4 - \sigma^2 \ll 1$, and plug it in Eq. (3.23) to obtain the solutions of quantization condition (3.23), which are

$$\gamma_n = \frac{\pi n}{\sigma} + \frac{\nu^2 - 1/4 - \sigma^2}{2\pi n}, \quad \nu = |l - \alpha|. \quad (3.24)$$

The energy spectrum for the annulus in this limit is

$$\epsilon_{n,l} = \gamma_n^2 \approx \left( \frac{\pi n}{\sigma} \right)^2 + (\nu^2 - 1/4). \quad (3.25)$$

The first term in Eq. (3.25) denotes the radial kinetic energy and diverges in $\sigma \rightarrow 0$ limit. This divergence can be absorbed into the chemical potential for the $n = 1$ radial state. The difference between energy levels with radial quantum numbers $n$ is of the order $n(\pi/\sigma)^2$. For $\sigma \rightarrow 0 \Rightarrow \xi \rightarrow 1$, one can assume that all the levels of interest have the radial quantum number $n = 1$ and are labeled only by orbital quantum number $l$. Since our diamagnetic persistent current arises from a large number $\propto \sqrt{N}$ of $l = 0$ levels, it is clear that it vanishes in the limit of a ring.

It is straightforward to show that for a regular annulus the contributions of $\pm l$ levels also cancel each other for small values of $\alpha$. However, levels with $l = 0$ have zero slope when $\alpha \rightarrow 0$. To show this one takes the derivative of quantization condition (3.13):

$$J_{\nu}(\gamma_n \xi)N_{\nu}(\gamma_n) + J_{\nu}(\gamma_n \xi)\dot{N}_{\nu}(\gamma_n) - J_{\nu}(\gamma_n)N_{\nu}(\gamma_n \xi) - J_{\nu}(\gamma_n)\dot{N}_{\nu}(\gamma_n \xi)$$

$$+ \frac{\partial \gamma_n}{\partial \alpha} \left[ \xi \dot{J}_{\nu}(\gamma_n \xi)N_{\nu}(\gamma_n) + J_{\nu}(\gamma_n \xi)\dot{N}_{\nu}(\gamma_n) - J_{\nu}(\gamma_n)N_{\nu}(\gamma_n \xi) - \xi J_{\nu}(\gamma_n)\dot{N}_{\nu}(\gamma_n \xi) \right] = 0, \quad (3.26)$$

where $\dot{A}_\nu(z) = \partial A_\nu(z)/\partial \nu$, and $A'_\nu(z) = \partial A_\nu(z)/\partial z$. When $\alpha \rightarrow 0$, derivatives of Bessel functions become $\dot{J}_{\nu}(z) = \pi N_0(z)/2$, $\dot{N}_{\nu}(z) = -\pi J_0(z)/2$, $J'_\nu(z) = -J_1(z)$, and $N'_\nu(z) = -N_1(z)$. Then all terms outside square brackets in Eq. (3.26) cancel each other. The expression inside brackets in general has a nonzero value, which means $\partial \gamma_n/\partial \alpha = 0$.

In Fig. 3.4 the persistent current in the annular billiard is depicted for different values of the aspect ratio $\xi$. To facilitate the comparison between different values of $\xi$, we keep the area of the annulus the same, thus keeping the average density of states the same. For a regular annulus (Fig. 3.4(a)) for small values of flux, the current is a linear function of $\alpha$. As $\xi$ gets smaller, the diamagnetic contribution to the persistent current increases. This behavior is consistent with Eq. (3.21) that shows linear dependence on $\alpha$ and slow growth as $\xi \rightarrow 0$.

In the regular annulus, for $\xi$ close to unity, the behavior of the persistent current is close, but not identical, to that of a 1D ring. Even for $\xi = 0.9$ there exist several states with...
Figure 3.4: Persistent current $I$ vs reduced magnetic flux $\alpha$ for several values of $\xi$ for $N = 200$ particles. (a) represents current for regular annulus normalized to the same density of states (same area) for different values of $\xi$. The current for the chaotic annulus is depicted in Figure (b).

$l = 0$, which means that our billiard is not purely a 1D ring. The effect of these states on the persistent current is not entirely trivial. For a fixed number of particles in the system, as $\xi$ changes, the number of $l = 0$ levels also changes. As the next $l = 0$ level is added (or expelled), the current experiences a jump. Depending on the occupation of closest $\pm l$ level, for small $\alpha$ the persistent current can be positive or negative. For larger $\alpha$ the current remains diamagnetic.

In the distorted annulus (Fig. 3.4(b)), the persistent current is a linear function of $\alpha$ for small $\alpha$. For larger magnetic flux, one observes nonlinear behavior that can be attributed to level repulsion in the chaotic billiard.

The dependence of the persistent current in the annulus on the number of particles $N$ at fixed $\alpha$ is similar to that in the simply connected billiard. At small $\xi$ the persistent current in the regular annulus is a staircase like function. For the distorted annulus the numerics are scattered around a straight line (see Fig. 3.5).

For larger $\xi$ the magnitude of diamagnetic contribution to the persistent current decreases, and the numerics are dominated by mesoscopic fluctuations. When $\xi \to 1$, the persistent current becomes negligible (see Fig. 3.6(a)) for low occupations. We believe this is a manifestation of Anderson localization due to the boundary scattering. At high energies, when the localization length exceeds the circumference of the annulus, extended states reappear and can carry the persistent current. In Fig. 3.6 we plot the persistent current for two different sets of parameters controlling the shape of annulus. For a large distortion (Fig. 3.6(a)) the current is nonzero only for high energy states beyond $N = 110$.

In Fig. 3.6(b) the parameters $b, c$, and $\delta$ are chosen to make the annulus less distorted, and we see that the threshold for extended states moves to lower energy (about $N = 40$).
Figure 3.5: (Color online) Persistent current $I$ in distorted annulus vs. number of particles $N$ for several values of $\xi$. Magnetic flux $\alpha = 0.17$.

Figure 3.6: Persistent current $I$ vs number of particles $N$ for distorted annulus. Parameters $b$, $c$, and $\delta$ control the shape of billiard.
The positive branch of persistent current in Fig. 3.6(b) can be explained as follows. For the relevant values of \( b, c \) and \( \delta \), the chaotic annulus is almost regular. For small distortions we can loosely speak in terms of states with definite values of orbital momentum (let us say that dominant contribution comes from the state with orbital momentum \( l \)). Then the positive branch signifies the occupation of a \( l > 0 \) state while the addition of another particle into the \( -l \) state brings the total magnetization back to near zero.

### 3.4 Conclusions, caveats, and open questions

We have investigated the behavior of chaotic simply connected and annular billiards penetrated by a point-like flux. The annular billiards are characterized by a dimensionless aspect ratio \( \xi = r/R \), the ratio of the inner \( r \) to the outer radius \( R \). Note that in the annular billiards, the flux exists in a region where the electrons cannot penetrate, and the effects of the flux on the electrons are purely Aharonov-Bohm quantum interference effects. We emphasize this point since in a simply connected billiard, there are diffraction effects associated with a point flux as well [34], and we want to separate those from quantum interference effects.

Our main result is that there is a systematic diamagnetic contribution to the persistent current, which can be traced back to the flux response of the \( l = 0 \) levels of a regular unit disk (or annulus). Even though the number of such \( l = 0 \) levels is submacroscopic \( (\propto \sqrt{N}, \text{where } N \text{ is the number of electrons}) \), the contribution to the persistent current due to these levels is proportional to \( N \) and is independent of the flux for small flux in simply connected billiards. Moreover, it can overwhelm the fluctuating mesoscopic contribution [21–29, 33, 35, 36, 39] from the states in the Thouless shell \( (|E - E_F| \leq E_T) \). This effect is quite distinct from Landau diamagnetism [47]. Near an integer multiple of the flux quantum \( \Phi \approx n\Phi_0 \), the flux dependence of the energy of the simply connected billiard is proportional to \( |\Phi - n\Phi_0| \), which also implies a large paramagnetism as \( \Phi \) approaches \( n\Phi_0 \) from below.

The diamagnetic contribution to the persistent current from \( l = 0 \) states seems to have been missed in the previous work, using the semiclassical sum over periodic orbits [33,35,36,39]. This is understandable since the semiclassical approach becomes exact only as \( E \to \infty \), and in this limit, the \( l = 0 \) states have vanishing spectral density \( \rho_{l=0}(E) \simeq 1/\sqrt{E} \). However, we emphasize that the total persistent current contains the sum over all levels and will indeed behave diamagnetically at small flux (in the simply connected billiard), as we have described. The diamagnetic contribution we uncover is also independent of the degree of chaoticity as long as states are not localized. This is clear from the fact that each exact eigenstate of energy \( E \) of the deformed billiard is roughly a superposition of states of the regular billiard within a Thouless energy \( E_T \) of \( E \). For \( E_F \gg E_T \), the diamagnetic contribution is independent of \( E_T \), and hence on the degree of chaoticity.

For very tiny \( \xi \), the annular Robnik-Berry billiard behaves much like the simply con-
nected one for most values of the dimensionless flux $\alpha = \Phi / \Phi_0 \gg 1 / \log N \xi^{-2}$, with a
diamagnetic contribution to the persistent current that is proportional to the electron
density. However, convergence to the $\xi = 0$ limit is logarithmically slow, and the limits $\alpha \to 0$
and $\xi \to 0$ do not commute. As the aspect ratio $\xi$ increases and the annulus tends to a
one-dimensional ring, the systematic diamagnetic contribution diminishes to zero. For $\xi$
close to one, we also see Anderson localization in the distorted annular billiards, wherein
the persistent currents are negligible below a certain energy (presumably because the localization length for these levels is smaller than the circumference), and become nonzero only beyond a threshold energy.

While we can obtain analytical estimates for the limits $\xi \to 0$ and $\xi \to 1$, it is difficult to
make analytical progress for generic values of $\xi$ (not close to zero or one). However, one can
easily verify from the asymptotic expansions that for generic $\xi$ the diamagnetic contribution
to the persistent current for $\Phi \ll \Phi_0$ goes as

$$I_{dia} \simeq -\frac{\hbar^2}{\pi m r R} \alpha \sqrt{\frac{2N(R-r)}{(R+r)}}$$  \hspace{1cm} (3.27)

where $r$ and $R$ are the inner and outer radii, respectively. This should be compared to the
typical fluctuating persistent current for interacting particles, [23–26] which behaves as

$$I_{fluc} \simeq \frac{E_T}{\Phi_0} \simeq \frac{\hbar^2}{m R \Phi_0} \sqrt{\frac{N}{R(R-r)}}$$  \hspace{1cm} (3.28)

It can be seen that the ratio of the systematic diamagnetic persistent contribution to the
fluctuating contribution is roughly

$$\frac{|I_{dia}|}{|I_{fluc}|} \sim \frac{(R-r)}{r} \frac{\Phi}{\Phi_0}$$  \hspace{1cm} (3.29)

Previous ring samples [12–18, 20] have $(R-r) \ll r$. They are also subject to a uniform
magnetic field rather than a point flux. Despite this, a systematic diamagnetic contribution
at low flux has been detected in recent experiments [16, 20]. However, the experiments are
carried out at finite frequency, and the effects of attractive pair interactions [48, 49] (see
below) or non-equilibrium noise [50] cannot be ruled out.

In order to detect this effect unambiguously, one must work with a material that has no
superconductivity at any temperature to rule out attractive pair interactions. It is also clear
that $\frac{R-r}{r}$ needs to be made as large as possible in order to render this effect easily observable.
Care must be taken that there is no magnetic flux in the region where the electron wave
functions are nonzero in order to maintain the pure Aharonov-Bohm quantum interference
nature of this effect.

Let us now mention some caveats about our work. We have taken only a few ($\approx 200$)
levels into account whereas most experimental samples have a hugely greater number of
levels. However, the physics of the diamagnetic contribution to the persistent current for a particular level concerns only whether that level has \( l = 0 \) or not, and is independent of its relative position in the spectrum. We expect our conclusions to hold for arbitrary densities.

We have considered a point-like flux, which is unachievable in practice. For the annular billiard, all one needs to ensure is that the flux is nonzero only in the central hole of the annulus and is zero in regions where the electron density is nonzero, thus avoiding possible contamination from diffraction effects \[34\]. By gauge invariance, such a situation will be equivalent to the one we study.

We have also ignored the effect of interelectron interactions. For weak repulsive interactions \[51–54\], we expect interactions to modify the effect only slightly because it comes primarily from occupied levels deep within the Fermi sea, which are Pauli blocked from responding to the interactions. However, for strong repulsive interactions \[55–57\], significant corrections to the persistent current \[58\] from electrons in the Thouless shell cannot be ruled out. If the interactions are weak but attractive \[23–26\], the low-energy fluctuations of Cooper pairs become very important \[48,49\], and can produce additional large diamagnetic contributions at low fields.

Similarly, although we have concentrated on the zero-temperature behavior, we expect this effect to persist in quite high temperatures since most of the \( l = 0 \) levels involved lie deep within the Fermi sea.

Finally, it would be interesting to investigate the effects of static disorder within a chaotic billiard, which would induce the system to cross over from a ballistic/chaotic to a disordered (diffusive) system. We hope to address this and other issues in future work.
4.1 Introduction

In this chapter we consider a system of two quantum dots/nanoparticles which are coupled by a hopping bridge. The motion of the electrons inside each dot can be either ballistic or diffusive. In the case of ballistic dots we assume that the dots have irregular shapes leading to classically chaotic motion, so that RMT is applicable.

Here we address the problem of electron-electron interaction. Our model allows us to explore the quantum critical regime where the system behavior is controlled by interplay of thermal and quantum-mechanical fluctuations.

In the model of two coupled quantum dots that we consider, one dot has attractive s-wave reduced Bardeen-Cooper Schrieffer interaction, and the other dot is non-interacting but subject to an orbital magnetic field. We find that the critical temperature $T_C$ is non-monotonic in the flux of the second dot in a certain regime of interdot coupling. We also find that fluctuation magnetization above $T_C$ is non-monotonic in this regime and can be either diamagnetic or paramagnetic.

The results for interacting system are obtained on the basis of the system for non-interacting electrons. We consider two coupled dots system where both dots and connecting region are in universal crossover regime between Gaussian Orthogonal and Unitary ensembles. Utilizing a large-N diagrammatic approach we calculate averages of one and two point Green’s functions. We establish scaling functions that modify the behavior of Green’s functions for partial Time Reversal symmetry breaking. The scaling functions depend on the ratio of crossover energy scales and measurement energy.

As was mentioned above, our system is a subject to a weak magnetic field that can take different values in the dots and the bridge. For weak magnetic flux the spectral properties of the system deviate from those predicted by either the GOE or the GUE [60]. The system is said to be in a crossover [59]. The Hamiltonian for each dot can be decomposed into real symmetric and real antisymmetric matrices:

$$H = \frac{H_S + iXH_A}{\sqrt{1 + X^2}},$$  \hspace{1cm} (4.1)

where $X$ is the crossover parameter [61] which is equal, up to factors of order unity, to $\Phi/\Phi_0$, where $\Phi$ is the magnetic flux through the dot, and $\Phi_0 = h/e$ is the quantum unit of
magnetic flux. Note that the Gaussian orthogonal and unitary ensembles are limiting cases of \( X \to 0 \) and \( X \to 1 \) respectively.

To understand the meaning of the crossover parameter consider the Aharonov-Bohm phase shift picked up by a ballistic electron in a single orbit in the dot:

\[
\Delta \phi = 2\pi \frac{\Phi}{\Phi_0}.
\]

For one turn the flux enclosed by the trajectory is proportional to \( \Phi = BL^2 \), where \( L \) is the size of the dot. After \( N \) turns the total flux is \( \Phi_{\text{total}} = \sqrt{N}\Phi \), where factor \( \sqrt{N} \) originates from the fact that electron has equal probability to make clockwise or counterclockwise orbits, and thus does a random walk in the total flux enclosed. The minimal phase shift for the electron to notice the presence of the magnetic flux is of the order \( 2\pi \), and thus the minimal cumulative flux enclosed by the orbit should be \( \Phi_0 = \sqrt{N}\Phi \). This leads to \( N = (\Phi_0/\Phi)^2 \), while the time to make \( N \) turns is \( \tau = LN/v_f \) (for a ballistic/chaotic dot).

From the Heisenberg uncertainty principle the associated energy scale is:

\[
E_{\text{cross}} \approx \frac{\hbar}{\tau} = \frac{E_T}{N} = E_T \left( \frac{\Phi}{\Phi_0} \right)^2,
\]

(4.3)

where \( E_T \) is the ballistic Thouless energy [62]. For a diffusive dot it should be substituted by the diffusive Thouless energy \( E_T \approx \hbar D/L^2 \). One can see that when \( \Phi \) is equal to \( \Phi_0 \), \( E_X \) is equal to \( E_T \) which means that energy levels are fully crossed over.

In the next sections the reader will encounter many crossover parameters, and thus many crossover energy scales. By a line of argument similar to that leading to Eq. (4.3), it can be shown that to every crossover parameter \( X_i \) there is a corresponding energy scale \( E_{X_i} \approx X_i^2 E_T \).

Our study has a two-fold motivation. The first part comes from works on coupled structures with noninteracting particles in acoustic and electronic systems [65–67], and crossovers [60, 68–71]. We focus on a complete description of the crossover regimes in all three regions (the two dots and the bridge). We define scaling functions for the two particle Green’s function when the time reversal symmetry of the system is partially broken.

While the two particle Green’s function can in general depend separately on \( E_T, E_X \), and the measurement frequency \( \omega \), it turns out that in the universal limit \( \omega, E_X \ll E_T \), it becomes a universal scaling function of the ratio \( E_X/\omega \). The scaling function describes the modification of \( \langle G^R(E + \omega)G^A(E) \rangle \) as one moves away from the “critical” point \( \omega = 0 \). The limits of the scaling function can be understood as follows: If the measurement frequency \( \omega \) is large (small) compared to the crossover energy scale \( E_X \), the \( \langle G^R(E + \omega)G^A(E) \rangle \) takes the form of the GOE (GUE) ensemble correlation function. If \( \omega \sim E_X \), the Green’s function describes the system in crossover regime.

The one particle Green’s function \( \langle G^R(E) \rangle \) is not critical as \( \omega \to 0 \), although it gets modified by the interdot coupling. The two particle Green’s function \( \langle G^R(E + \omega)G^A(E) \rangle \)
always has a diffusion mode [63], that diverges for small $\omega$ in our large-$N$ approximation, which means that our results are valid on scales much larger than mean level spacing. This divergence is not physical and will be cut off by vanishing level correlations for $\omega \ll \delta$ in a more exact calculation [64]. On the other hand, the energy scale $\omega$ should be smaller than Thouless energy of the system for RMT to be applicable. These limitations hold for the crossover energy $E_X$ as well. In what follows we study the regime corresponding to $\delta \ll \omega, E_X \leq E_T$.

The other term that appears in the two particle Green’s function is a Cooperon mode. In general the Cooperon term is gapped if at least one of the crossover parameters is different from zero. In the case when the total Hamiltonian of the system is time reversal invariant, all the crossover parameters are zero and the Cooperon, just like the diffusion, becomes gapless. Finally, when each part of compound system belongs to the GUE (the case when all crossover parameters are much larger than $\omega$) the Cooperon term disappears.

Using parameters analogous to $E_X$ we describe crossover regimes in dots 1 and 2 and the effects of the tunable hopping between them. Varying these parameters allows us to obtain results for various physical realizations, when different parts of the compound system behave as pure GOE, GUE, or belong to the crossover ensemble. In electronic systems it is easy to break time-reversal by turning on an external orbital magnetic flux. In acoustic systems one can break time-reversal by rotating the system or a part thereof. As mentioned before, the system of two dots coupled by hopping has been investigated before using supersymmetry methods [67]. However, the authors considered only the GUE, whereas here we are interested in the full crossover. In fact, the crossover is essential to the second aspect of our work, as will become clear immediately.

The second part of our motivation is the possibility of using the information gained in noninteracting systems to predict the behavior of interacting systems [49, 72–74]. We consider interacting systems controlled by the Universal Hamiltonian [51–54]. For the GOE the Universal Hamiltonian $H_U$ has the form [51–54]

$$H_U = \sum_{\alpha, s} \epsilon_\alpha c_{\alpha, s}^{\dagger} c_{\alpha, s} + \frac{U_0}{2} \hat{N}^2 - JS^2 + \lambda T^\dagger T$$

where $\hat{N}$ is the total particle number, $S$ is the total spin, and $T = \sum c_{\beta, \downarrow}^{\dagger} c_{\beta, \uparrow}$. In addition to the charging energy, $H_U$ has a Stoner exchange energy $J$ and a reduced superconducting coupling $\lambda$. This last term is absent in the GUE, while the exchange term disappears in the GSE. Kinetic energy is described by RMT.

We concentrate on the reduced Bardeen-Cooper-Schrieffer (BCS) coupling $\lambda$ which leads to a mean-field superconducting state when $\lambda < 0$. Previous work [49] sets the context for our investigation. We consider an interacting system which has a single-particle symmetry and a quantum phase transition in the limit $E_T/\delta \to \infty$. An example relevant to us is a superconducting nanoparticle originally in the GOE. It has the reduced BCS interaction
and time-reversal symmetry, and the (mean-field) quantum phase transition is between
the normal and superconducting states and occurs at $\lambda = 0$. Now consider the situation
when the symmetry is softly broken, so that the single-particle dynamics is described by a
crossover RMT ensemble. It can be shown [49] that this step allows us to tune into the many-
body quantum critical regime [77–79] of the interacting system. Thus, the scaling functions
of the noninteracting crossover are transmuted into scaling functions of the interacting
system in the many-body quantum critical regime. In our example, the orbital magnetic
flux breaks the time-reversal symmetry which is crucial to superconductivity. When the
orbital flux increases to a critical value, it destroys the mean-field superconducting state.
Above the critical field, or more generically above the critical temperature, the system is in
the quantum critical regime.

To be more specific, we consider two vertically coupled quantum dots, the first of which
has an attractive reduced BCS coupling, while the second has no BCS coupling. Fig.
4.1 shows the geometry, the reason for which will become clear soon. We apply an orbital
magnetic flux only through (a part of) the second dot, and observe the effect on the coupled
system. Our main results are for the mean-field critical temperature $T_c$ of the system, and
its magnetization in the normal state (above $T_c$) as a function of the flux in the normal
nanoparticle. Such a system could be realized physically without too much difficulty, by,
for example, growing a thin film of normal metal (such as Au) on an insulating substrate,
then a layer of insulator which could serve as the hopping bridge, and finally a thin film of
superconductor(such as Al, which has a mean-field superconducting transition temperature
of around 2.6K). The orbital flux can be applied selectively to the Au layer as shown in Fig.
4.1 by a close pair of oppositely oriented current carrying wires close to the Au quantum
dot, but far from the Al quantum dot.

The reason for this geometry is that we want to disregard interdot charging effects
entirely and concentrate on the BCS coupling. The Hamiltonian for the coupled interacting
system contains charging energies for the two dots and an interdot Coulomb interaction [73],

$$\frac{U_1}{2} N_1^2 + \frac{U_2}{2} N_2^2 + U_{12} N_1 N_2$$

(4.5)
Defining the total number of particles as $N = N_1 + N_2$, and the difference in the number as $n = N_1 - N_2$ the interaction can also be written as

$$\frac{U_1 + U_2 + 2U_{12}}{16}N^2 + \frac{U_1 + U_2 - 2U_{12}}{16}n^2 + \frac{U_1 - U_2}{4}nnN$$

(4.6)

We see that there is an energy cost to transfer an electron from one dot to the other. Our geometry is chosen so as to make $U_1 = U_2 = U_{12}$ as nearly as possible, which can be achieved by making the dots the same thickness and area, and by making sure that their vertical separation is much smaller than their lateral linear size. In this case, since $N$ is constant, we can ignore charging effects entirely. Since our primary goal is to investigate quantum critical effects associated with the BCS pairing interaction, we will assume the above mentioned geometry and ignore charging effects in what follows.

After including the effect of the BCS interaction, we find the surprising result that in certain regimes of interparticle hopping strength, the mean-field transition temperature of the system can increase as the flux through the second quantum dot increases. Indeed, its behavior can be monotonic increasing, monotonic decreasing, or non-monotonic as the flux is increased. We can qualitatively understand these effects by the following considerations. In the absence of orbital flux, hopping between the dots reduces $T_c$ since it “dilutes” the effect of the attractive BCS coupling present only in the first dot. The application of an orbital flux through the second dot has two effects: (i) To raise the energy of Cooper pairs there, thus tending to localize the pairs in the first dot and raise the $T_c$. (ii) To cause time-reversal breaking in the first dot, and reduce $T_c$. The non-monotonicity of $T_c$ arises from the competition between these two effects.

Another quantity of interest above the mean-field $T_c$ is the fluctuation magnetization [83], which corresponds to gapped superconducting pairs forming and responding to the external orbital flux. In contrast to the case of a single quantum dot subjected to an orbital flux, we find that the fluctuation magnetization [83] can be either diamagnetic (the usual case) or paramagnetic. A paramagnetic magnetization results from a free energy which decreases as the flux increases. The origin of this effect is the interplay between the localizing effect of high temperature or the orbital flux in the second dot on the one hand, and the reduced BCS interaction on the other.

The regimes we describe should be distinguished from other superconducting single-particle RMT ensembles discovered in the past decade [84, 85], which apply to a normal mesoscopic system in contact with two superconductors with a phase difference of $\pi$ between their order parameters [84] (so that there is no gap in the mesoscopic system despite Andreev reflection), or to a mesoscopic $d$-wave superconducting system [85]. In our case, the symmetry of the superconducting interaction is $s$-wave. However, the most important difference is that we focus on quantum critical fluctuations, which are inherently many-body, while the RMT classes described previously are single-particle ensembles [84, 85].
Figure 4.2: Dyson equation for the averaged one particle Green’s function.

This chapter is organized as follows. In Section 4.2 we review the basic steps of calculating the one particle and two particle Green’s functions for a single dot. Then in Section 4.3 we present the system of Dyson equations for the one particle Green’s function in the case of two coupled dots and solve it in the limit of weak coupling. In addition, we set up and solve the system of four Bethe-Salpeter equations for the two particle Green’s function. In Section 4.4 we apply our results to the system of superconducting quantum dot weakly coupled to other quantum dot made from a normal metal. We end with our conclusions, some caveats, and future directions in Section 4.5.

4.2 Review of results for a single dot.

Our goal in this section is to calculate the statistics of one and two particle Green’s functions for an uncoupled dot in a GOE→GUE crossover (see Appendix B, and [61] for more details), starting from the series expansion of Green’s function:

\[
\langle \beta | G^R(E) | \alpha \rangle = G^R_{\alpha \beta}(E) = \frac{1}{E^+ - H} \delta_{\alpha \beta} + \frac{H_{\alpha \beta}}{(E^+)^2} + \frac{H^2_{\alpha \beta}}{(E^+)^3} + \ldots
\]

We are interested in averaging this expansion over the appropriate random matrix ensemble. The corresponding Dyson equation for averaged Green’s function is shown on Fig. 4.2. The bold line denotes the averaged propagator \( \langle G^R(E) \rangle \) and regular solid line defines the bare propagator \( \frac{1}{E^+} \) with \( E^+ = E + i\eta \), where \( \eta \) is infinitely small positive number. Here \( \Sigma \) stands for self-energy and is a sum of all topologically different diagrams.

One can solve the Dyson equation by approximating the self-energy by the first leading (sunrise) term and find:

\[
\Sigma = \frac{E}{2} - \frac{i}{2} \sqrt{\left( \frac{2N\delta}{\pi} \right)^2 - E^2},
\]

where \( \delta \) is the mean level spacing. This approximation works only for \( E \gg \delta \). As \( E \) gets comparable with \( \delta \), other terms in expansion for \( \Sigma \) should be taken into account.

Then, the average of the one particle Green’s function is given by:

\[
\langle G^R_{\alpha \beta}(E) \rangle = \frac{\delta_{\alpha \beta}}{\frac{E}{2} + \frac{i}{2} \sqrt{\left( \frac{2N\delta}{\pi} \right)^2 - E^2}}
\]

Next, we repeat the procedure for the averaged two particle Green’s function, which can be represented by the series on Fig. 4.3, where two bold lines on the left hand side denote
\[ \langle G_R(E + \omega)G_A(E) \rangle. \] The leading contribution comes from ladder and maximally crossed diagrams. The sum of these diagrams can be conveniently represented by Bethe-Salpeter equations (a) and (b) on Fig. 4.4. \( \Pi^D \) and \( \Pi^C \) are related to the connected part of two particle Green’s function as shown on Fig. 4.4, (c).

In the limit of \( \omega \) being much smaller than bandwidth \( (\omega \ll N\delta) \), the two particle Green’s function (connected part) is expressed as:

\[ \langle G_{\alpha\gamma}^R(E + \omega)G_{\delta\beta}^A(E) \rangle = \frac{2\pi}{N^2\delta} \delta_{\alpha\beta} \delta_{\gamma\delta} - \frac{i\omega}{1 + iE_X} \]  

The second term is a contribution of maximally crossed diagrams. \( E_X \) is a crossover energy scale, connected to the crossover parameter as \( E_X = 4X^2N\delta/\pi \).

Depending on values of \( E_X \) one can speak of different types of averaging. If \( E_X \ll \omega \), we get average over GOE ensemble, if \( E_X \) is of order \( \omega \), averaging is performed over ensemble being in crossover, and, if \( E_X \gg \omega \), contribution of maximally crossed diagrams can be disregarded, thus going to the limit of the GUE ensemble.

### 4.3 Two coupled dots.

Next we discuss general framework of our calculation and calculate correlation functions for our system of interest, which is two weakly coupled quantum dots (see Appendix C for more technical details). The Hamiltonian for this system can be represented as:

\[ H_{tot} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} + \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix} = \begin{pmatrix} H_1 & V \\ V^\dagger & H_2 \end{pmatrix}. \]
where \( H_1 \) and \( H_2 \) are the Hamiltonians of uncoupled dots 1 and 2. The coupling is realized by a matrix \( V \). The elements of \( H_1, H_2, \) and \( V \) are statistically independent random variables. We assume that both dots and the hopping bridge are in crossover regimes, characterized by parameters \( X_1, X_2, \) and \( \Gamma \) respectively.

In the crossover matrices \( H_i \) and \( V \) are given by:

\[
H_i = \frac{H_i^S + iX_iH_i^A}{\sqrt{1 + X_i^2}}, \quad i = 1, 2; \quad V = \frac{V^R + iV^I}{\sqrt{1 + \Gamma^2}},
\]

(4.12)

where \( H_i^{S,A} \) is a symmetric (antisymmetric) part of \( H_i \), and \( V^R,I \) is real (imaginary) matrix.

In what follows we assume that the bandwidths in dot 1 and dot 2 are the same. That is, \( N_1\delta_1 = N_2\delta_2 \). This should not make any difference in the universal limit \( N \to \infty \).

In addition we introduce the parameter \( \xi \) – the ratio of mean level spacing in two dots:

\[ \xi = \frac{\delta_1}{\delta_2}. \]

For each realization of matrix elements of the Hamiltonian \( H_{tot} \), the Green’s function of this system can be computed as follows:

\[
G = (I \otimes E - H)^{-1} = \begin{pmatrix} E - H_1 & -V \\ -V^\dagger & E - H_2 \end{pmatrix}^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}.
\]

(4.13)

Each element of \( G \) has the meaning of a specific Green’s function. For example, \( G_{11} \) and \( G_{22} \) are the Green’s functions that describe particle propagation in dots 1 and 2 respectively. On the other hand, \( G_{12} \) and \( G_{21} \) are the Green’s functions representing travel from one dot to another.

To find the components of \( G \) we calculate \( (I \otimes E - H)^{-1} \) using the following formula

\[
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} C_1^{-1} & -A_{11}^{-1}A_{12}C_2^{-1} \\ -C_2^{-1}A_{21} & C_2^{-1} \end{pmatrix},
\]

(4.14)

where \( A_{ij} \) are matrices, and \( C_1 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \) and \( C_2 = A_{22} - A_{21}A_{11}^{-1}A_{12} \). For example,

\[
\begin{aligned}
G_{11} &= \left[ (E - H_1) - V(E - H_2)^{-1}V^\dagger \right]^{-1} = G_1 + G_1VG_2V^\dagger G_1 + G_1VG_2V^\dagger G_1VG_2V^\dagger G_1 + \ldots \\
\end{aligned}
\]

(4.15)

where \( G_1 \) and \( G_2 \) are propagators in dot 1 and dot 2 defined by \( G_1 = (E - H_1)^{-1} \) and \( G_2 = (E - H_2)^{-1} \).

To find the ensemble average of \( G_{11} \) one needs to average the whole expansion (4.15) term by term. For coupled dots \( G_{ij} \) interrelated and in large \( N \) approximation can be found from the system of equations on Fig. 4.5.

The bold straight and wavy lines with arrows represent averaged Green’s functions \( \langle G_{\alpha\beta,1}(E) \rangle \) and \( \langle G_{ij,2}(E) \rangle \) respectively, while regular solid lines are bare propagators in dots 1 and 2. The dotted line describes pairing between hopping matrix elements \( V \), and the dashed (wavy) line denotes pairing between matrix elements of \( H_1 \) (\( H_2 \)).
The system on Fig. 4.5 accounts for all possible diagrams without line crossing. Diagrams containing crossed lines of any type are higher order in $1/N$ and can be neglected when $N \to \infty$. If the hopping between dots is zero, this system decouples into two separate Dyson equations for each dot. In the case of weak coupling ($U \ll 1$), where $U$ is a parameter controlling the strength of coupling between dots, this system can be readily solved. As zero approximation, we use results for a single dot.

In this approximation one particle Green’s function for dot 1 and dot 2 are calculated as follows:

$$\langle G^R_{\alpha\beta,1}(E) \rangle = \frac{1}{1 - U \frac{\Sigma_0}{\sqrt{\xi}} E - 2\Sigma_0} \left( \frac{N_1 \delta_1}{\pi} \right) \frac{\delta_{\alpha\beta}}{[\epsilon + i\sqrt{1 - \epsilon^2}]} \left[ \frac{1}{1 + \frac{U \sqrt{\xi}}{2} \left( 1 + i \frac{\epsilon}{\sqrt{1 - \epsilon^2}} \right)} \right]$$

$$\langle G^R_{ij,2}(E) \rangle = \frac{1}{1 - U \frac{\Sigma_0}{\sqrt{\xi}} E - 2\Sigma_0} \left( \frac{N_2 \delta_2}{\pi} \right) \frac{\delta_{ij}}{[\epsilon + i\sqrt{1 - \epsilon^2}]} \left[ \frac{1}{1 + \frac{U \sqrt{\xi}}{2} \left( 1 + i \frac{\epsilon}{\sqrt{1 - \epsilon^2}} \right)} \right],$$

where $\epsilon$ is a dimensionless energy $\epsilon = \pi E/2N\delta$. We used subindex 0 in $\Sigma_0$ and $\langle G^R_0(E) \rangle$ to denote solutions for one uncoupled dot.

In the large $N$ approximation the contribution to the two particle Green’s function comes from ladder diagrams and maximally crossed diagrams. It is convenient to sum them separately. The ladder diagram contribution can be found from the system of equations shown on Fig. 4.6, where $\Pi_D^{ij}$ with proper external lines denote various two particle Green’s functions. As in the case of the one particle Green’s function equations, if the inter-dot coupling is zero, the system reduces to two Bethe-Salpeter equations for uncoupled dots.

The system of four equations on Fig. 4.6 can be broken into two systems of two equations to get:

$$\langle G^R_{\alpha\gamma,1}(E + \omega)G^A_{\beta\delta,1}(E) \rangle_{D1} = \frac{2\pi}{N_1^2 \delta_1} \delta_{\alpha\beta} \delta_{\gamma\delta} g_{D1}$$

$$\langle G^R_{ij,2}(E + \omega)G^A_{kj,2}(E) \rangle_{D2} = \frac{2\pi}{N_2^2 \delta_2} \delta_{ij} \delta_{lk} g_{D2},$$

$$\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dyson_equations}
\caption{The system of Dyson equations for two coupled dots.}
\end{figure}
where $g_D$ are the scaling functions of diffusion terms in dot 1 and dot 2 defined by:

$$
ge_D^1 = \frac{1 + \frac{i}{\sqrt{\xi}} \frac{E_U}{\omega}}{1 + i(\sqrt{\xi} + \frac{1}{\sqrt{\xi}}) \frac{E_U}{\omega}},$$

$$
ge_D^2 = \frac{1 + i\sqrt{\xi} \frac{E_U}{\omega}}{1 + i(\sqrt{\xi} + \frac{1}{\sqrt{\xi}}) \frac{E_U}{\omega}}.$$ (4.17)

Here $E_U = 2UN\delta/\pi$ is the interdot coupling energy scale. These dimensionless functions show how diffusion part is modified due to the coupling to another dot.

Next, the system of equations for the maximally crossed diagrams is shown on Fig. 4.7.

The subsequent solution of this system produces:

$$\langle G_{\alpha\gamma,1}^R(E + \omega)G_{\delta\beta,1}^A(E)\rangle_{C1} = \frac{2\pi}{N_1^2\delta_1} \frac{\delta_{\alpha\delta} \delta_{\gamma\beta} g_{C1}}{-i\omega},$$

$$\langle G_{d,2}^R(E + \omega)G_{k,2}^A(E)\rangle_{C2} = \frac{2\pi}{N_2^2\delta_2} \frac{\delta_{ik} \delta_{lj} g_{C2}}{-i\omega}.$$ (4.18)

where $g_C$ are the scaling functions for Cooperon term defined according to:

$$g_{C1} = \frac{1 + \frac{i}{\sqrt{\xi}} \frac{E_U}{\omega} + \frac{i}{\sqrt{\xi} \omega}}{1 + i \frac{E_X + E_X}{\omega} - \frac{E_X E_U}{\sqrt{\xi} \omega^2} - \frac{\sqrt{\xi} E_X E_U}{\omega^2} + i \left( \sqrt{\xi} + \frac{1}{\sqrt{\xi}} \right) \frac{E_U}{\omega} \left( 1 + i \frac{E_V}{\omega} \right)},$$

$$g_{C2} = \frac{1 + i\sqrt{\xi} \frac{E_U}{\omega} + \frac{i}{\sqrt{\xi} \omega}}{1 + i \frac{E_X + E_X}{\omega} - \frac{E_X E_U}{\sqrt{\xi} \omega^2} - \frac{\sqrt{\xi} E_X E_U}{\omega^2} + i \left( \sqrt{\xi} + \frac{1}{\sqrt{\xi}} \right) \frac{E_U}{\omega} \left( 1 + i \frac{E_V}{\omega} \right)}.$$ (4.19)
Here $E_{X_{1,2}} = 4X_{1,2}^2 N \delta / \pi$, and $E_{\Gamma} = 4\Gamma^2 E_U / (\sqrt{\xi} + \frac{1}{\sqrt{\xi}})$ are the crossover energy scales, describing transition from GOE to GUE ensemble in dot 1 and dot 2, as well as in hopping bridge $V$.

As we determined how the scaling function $g_C$ modifies Cooperon part of two particle Green’s function and depends on the crossover energy scales defined above, we are ready to proceed with write up the connected part of the total two particle Green’s function, which is a sum of diffuson and Cooperon parts:

$$\langle G_{R,1}^{\alpha \gamma,1}(E + \omega) G_{A,1}^{\delta \gamma,1}(E) \rangle = \frac{2\pi}{N_1^2 \delta_1} \frac{\delta_{\alpha \beta} \delta_{\gamma \delta}}{-i \omega} g_{D1} + \frac{2\pi}{N_1^2 \delta_1} \frac{\delta_{\alpha \delta} \delta_{\gamma \beta}}{-i \omega} g_{C1}. \quad (4.20)$$

$$\langle G_{R,2}^{\rho \gamma,2}(E + \omega) G_{A,2}^{\kappa \gamma,2}(E) \rangle = \frac{2\pi}{N_2^2 \delta_2} \frac{\delta_{ij} \delta_{kl}}{-i \omega} g_{D2} + \frac{2\pi}{N_2^2 \delta_2} \frac{\delta_{ik} \delta_{lj}}{-i \omega} g_{C2}, \quad (4.21)$$

where $g_{D1}$, $g_{C1}$, $g_{D2}$, and $g_{C2}$ are defined by formulas (4.17) and (4.19).

In general, the coupling between dots changes the bandwidth of each dot. Corrections to the bandwidth are of the order of $U$ and can be neglected for weak coupling. Calculating approximations to the second order in $U$ one can be ensure that one particle and two particle Green’s functions can be treated perturbatively.

Diagrams on Fig.4.8 show the typical behavior of absolute value and phase of scaling functions $g_D$ and $g_C$ in dot 1. All energy parameters are measured in units of $E_U$.

Next we analyze the temporal behavior of the computed statistical characteristics. The Fourier transform of the two particle Green’s function shows the time evolution of the density matrix of the system. One can observe that the diffuson part of $\langle G^R G^A \rangle$ diverges.

Figure 4.7: The system of Bethe-Salpeter equations for two coupled dots (maximally crossed diagram contribution).
Figure 4.8: Absolute value and phase of diffusion (a,b) and Cooperon (c,d) scaling functions in dot 1. Frequency $\omega$ is measured in units of $E_U$. For these graphs the crossover parameters are: $E_{X_1}/E_U = E_{X_2}/E_U = 1$, $E_\Gamma/E_U = 0.8$, $\xi = 1$. 
for small $\omega$. To get the correct behavior we replace $1/\omega$ with $\omega/(\omega^2 + \eta^2)$, and take $\eta$ to zero in the final result. As for the Cooperon term, it stays regular in the small $\omega$ limit if at least one of the crossover parameters differs from zero.

First of all, we look at the Fourier transform of $\langle GRG^A \rangle$ in the first dot. We have

$$
\langle G^{R}_{\alpha\gamma,1}(t)G^{A}_{\delta\beta,1}(t) \rangle = \delta_{\alpha\beta}\delta_{\gamma\delta} \frac{2}{N_1(1 + \xi)} \left[ \frac{1}{2} + \xi e^{-(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U t} \right]
+ \delta_{\alpha\delta}\delta_{\gamma\beta} \frac{2}{N_1} \left[ 1 + \frac{E_{X_2} + \frac{E_U}{\sqrt{\xi}}}{a_+ - a_-} \left( e^{-ta_-} - e^{-ta_+} \right) \right],
$$

(4.22)

where $a_\pm$ depend on the crossover parameters (see Eq. (D.11) in appendix D).

Then, for the corresponding quantity in the second dot the Fourier transform produces:

$$
\langle G^{R}_{\alpha\gamma,2}(t)G^{A}_{\delta\beta,2}(t) \rangle = \delta_{ij}\delta_{ik} \frac{2\xi}{N_2(1 + \xi)} \left[ \frac{1}{2} + \frac{1}{\xi} e^{-(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U t} \right]
+ \delta_{ik}\delta_{ij} \frac{2}{N_2} \left[ 1 + \frac{E_{X_1} + \sqrt{\xi}E_U}{a_+ - a_-} \left( e^{-ta_-} - e^{-ta_+} \right) \right].
$$

(4.23)

### 4.4 Two coupled metallic quantum dots

In this section we apply the results obtained in the previous sections to an interacting system. We consider two vertically coupled metallic quantum dots, as shown in Fig. 4.1, the first of which is superconducting and the second noninteracting. For simplicity the quantum dots are assumed to have the same level spacing ($\xi = 1$). The calculations presented in this section can be extended to the case $\xi \neq 1$ in a straightforward way. The first (superconducting) quantum dot and the hopping bridge belong to the GOE ensemble. A nonzero orbital magnetic flux penetrating the second (noninteracting) quantum dot drives it into the GOE to GUE crossover described by the crossover energy scale $E_{X_2}$. The other crossover energy scale $E_U$ describes the hopping between the quantum dots. Because of this hopping one can observe a nonzero magnetization in the first particle caused by a magnetic flux through the second particle. Roughly speaking, when the electrons in the first dot travel to the second and return they bring back information about the orbital flux.

We wish to compute the magnetization as a function of orbital flux, as well as the mean-field critical temperature. It should be noted that since the quantum dot is a finite system, there cannot be any true spontaneous symmetry breaking. However, when the mean-field superconducting gap $\Delta_{BCS} \gg \delta$, the mean-field description is a very good one [23, 24, 48]. Recent numerical calculations have investigated the regime $\Delta_{BCS} \simeq \delta$ where quantum fluctuations are strong [86]. We will focus on the quantum critical regime of the system above the mean-field critical temperature/field, so we do not have to worry about symmetry-breaking.
We start with BCS crossover Hamiltonian for the double-dot system including the interactions in the first dot and the hopping between the dots [49]:

\[
H_{BCS X_2} = \sum_{\mu\nu} H^{(1)}_{\mu\nu} c^\dagger_{\mu\nu} c_{\nu\mu} - \lambda T^\dagger T + \sum_{i\sigma} H^{(2)}_{i\sigma} c^\dagger_{i\sigma} c_{i\sigma} + \sum_{\mu\nu} V_{\mu\nu}(c^\dagger_{\mu\nu} c_{\nu\mu} + h.c.)
\]

\[
= \sum_{\mu s} \epsilon_{\mu} c^\dagger_{\mu s} c_{\mu s} - \delta \tilde{\lambda} T^\dagger T,
\] (4.24)

where \( H^{(2)} \) contains the effect of the orbital flux through the second quantum dot. Here \( T, T^\dagger \) are the operators which appear in the Universal Hamiltonian, and are most simply expressed in terms of electron creation/annihilation operators in the original GOE basis of the first dot (which we call \( \mu_0, \nu_0 \)) as

\[
T = \sum_{\mu_0} c_{\mu_0,\downarrow} c_{\mu_0,\uparrow}
\] (4.25)

Now we need to express the operators \( c_{\mu_0,s} \) in terms of the eigenoperators of the combined single-particle Hamiltonian of the system of two coupled dots. The result is

\[
T = \sum_{\mu\nu} M_{\mu\nu} c_{\nu,\uparrow} c_{\mu,\uparrow}, \quad M_{\mu\nu} = \sum_{\mu_0} \psi_{\mu}(\mu_0) \psi_{\nu}(\mu_0),
\] (4.26)

where \( \epsilon_{\mu} \) denotes the eigenvalues of the total system, \( c_{\mu,s} \) operator annihilates electron in the orbital state \( \mu \) with spin \( s \), \( \psi_{\mu}(\mu_0) \) is the eigenvector of the compound system, \( \delta \) is the mean level spacing of a single isolated dot, \( \tilde{\lambda} > 0 \) is the attractive dimensionless BCS coupling valid in region of width \( 2\omega_D \) around the Fermi energy. Note that while the indices \( \mu, \nu \) enumerate the states of the total system, the index \( \mu_0 \) goes only over the states of the first dot, since the superconducting interaction is present only in the first dot.

To study the magnetization of the first quantum dot in the crossover we follow previous work by one of us [49]: We start with the partition function \( Z = Tr(exp - \beta H) \) where \( \beta = 1/T \) is the inverse temperature. We convert the partition function into an imaginary time path integral and use the Hubbard-Stratanovich identity to decompose the interaction, leading to the imaginary time Lagrangian

\[
\mathcal{L} = \frac{|\sigma|^2}{\delta \tilde{\lambda}} - \sum_{\mu,s} \tilde{c}_{\mu,s}(\partial_\tau - \epsilon_{\mu}) c_{\mu,s} + \sigma \tilde{T} + \tilde{\sigma} T
\] (4.27)

where \( \sigma, \tilde{\sigma} \) are the bosonic Hubbard-Stratanovich fields representing the BCS order parameter and \( \tilde{c}, c \) are Grassman fields representing fermions. The fermions are integrated out, and as long as the system does not have a mean-field BCS gap, the resulting action for \( \sigma, \tilde{\sigma} \) can be expanded to second order to obtain

\[
S_{eff} \approx \frac{\delta}{\beta} \sum_n |\sigma(i\omega_n)|^2 \left( \frac{1}{\lambda} - f_n(\beta, E_X, \omega_D) \right)
\] (4.28)
\[ f_n(\beta, E_X, \omega_D) = \delta \sum_{\mu\nu} |M_{\mu\nu}|^2 \frac{1 - N_F(\epsilon_{\mu}) - N_F(\epsilon_{\nu})}{\epsilon_{\mu} + \epsilon_{\nu} - i\omega_n} \]  \hspace{1cm} (4.29)

where \( \omega_n = 2\pi n/\beta \), and the sums are restricted to \( |\epsilon_{\mu}|, |\epsilon_{\nu}| < \hbar \omega_D \). We see that the correlations between different states \( \mu, \nu \) play an important role. Deep in the crossover (for \( E_X \gg \delta \)) we can replace \( |M_{\mu\nu}|^2 \) by its ensemble average [49]. We will also henceforth replace the summations over energy eigenstates by energy integrations with the appropriate cutoffs. In previous work [49] the statistics [72–74] of \( |M_{\mu\nu}|^2 \) was used to obtain analytical results for this expression.

The (interacting part of the) free energy of the system in the quantum critical regime is given by [49]:

\[ \beta F = \sum_n \ln(1 - \tilde{\lambda} f(i\omega_n, \beta, E_{X_2})) \]  \hspace{1cm} (4.30)

where \( f \) is the scaling function given by expression:

\[ f(i\omega_n, \beta, E_{X_2}) = \delta \sum_{\mu\nu} |M_{\mu\nu}|^2 \frac{1 - n_{\mu}(\beta) - n_{\nu}(\beta)}{\epsilon_{\mu} + \epsilon_{\nu} - i\omega_n} \],  \hspace{1cm} (4.31)

\( n_{\nu}(\beta) = (1 + \exp(\beta\epsilon_{\nu}))^{-1} \) is the Fermi-Dirac distribution. We have shifted the energy so that the chemical potential is 0.

Converting this double sum into integral and substituting \( |M_{\mu\nu}|^2 \) by its ensemble average (see Appendices E and F), we get:

\[ f_n = \frac{E_U}{\pi} \int_{-\omega_D}^{\omega_D} d\epsilon_1 d\epsilon_2 \frac{(\epsilon_1 - \epsilon_2)^2 + E_{X_2} E_U + E_{X_2}^2}{((\epsilon_1 - \epsilon_2)^2 - E_{X_2} E_U)^2 + (E_{X_2} + 2E_U)^2(\epsilon_1 - \epsilon_2)^2} \frac{\tanh(\frac{\beta\epsilon_1}{2}) + \tanh(\frac{\beta\epsilon_2}{2})}{\epsilon_1 + \epsilon_2 - i\omega_n} \]  \hspace{1cm} (4.32)

where \( \omega_D \) is the Debye frequency, and \( \beta = 1/k_B T \) is the inverse temperature.

One can decompose the ratio in the first part of integrand into two Lorentzians to get [49]:

\[ f_n = \frac{E_U E_{X_2}^2 + E_U E_{X_2} - E_{E_{1}}^2}{2E_1} \ln \left[ \frac{4(h\omega_D)^2 + \omega_n^2}{C'/\beta^2 + (E_1 + |\omega_n|)^2} \right] + \frac{E_U E_{E_{1}}^2 - E_{X_2}^2 - E_U E_{X_2}}{2E_2} \ln \left[ \frac{4(h\omega_D)^2 + \omega_n^2}{C'/\beta^2 + (E_2 + |\omega_n|)^2} \right].  \hspace{1cm} (4.33)

Here \( C' \approx 3.08 \) and \( E_{1,2} \) depend on crossover energy scales as follows:

\[ E_{1,2}^2 = \frac{1}{2} \left[ (E_{X_2} + 2E_U)^2 - 2E_U E_{X_2} \pm \sqrt{(E_{X_2} + 2E_U)^2(E_{X_2}^2 + 4E_U^2)} \right].  \hspace{1cm} (4.34)\]

The magnetization can then be obtained from the free energy:
\[ M = -\frac{\partial F}{\partial B} = M_{\text{nonint}} + \frac{\bar{\lambda} L^2}{\beta} \frac{\partial E_{X_2}}{\partial \phi} \sum_n \frac{\partial f_n}{\partial E_{X_2}} \left( 1 - \lambda f_n \right), \]  

(4.35)

where \( M_{\text{nonint}} \) is the contribution from noninteracting electrons [29]. We will be interested in the second term, which is the fluctuation magnetization [83].

For illustrative purposes, we use the parameters for Al in all our numerical calculations, with \( \omega_D = 34 \text{meV} \) and \( \bar{\lambda} = 0.193 \). This leads to a mean-field transition temperature \( T_c = 0.218 \text{meV} = 2.6K \) for an isolated Al quantum dot in the absence of magnetic flux.

In all our calculations we evaluate Matsubara sums with a cutoff \( \exp(-|\omega_n|/\omega_D) \). We have verified that changing the cutoff does not qualitatively affect our results, but only produces small numerical changes.

It will be informative to compare the two-dot system with a single dot subject to an orbital magnetic flux [49] (see Fig. 4.9). We draw the reader’s attention to two important features. Firstly, the critical temperature \( T_c \) decreases monotonically with \( E_{X_2} \), resulting from the fact that time-reversal breaking disfavors superconductivity. Secondly, the fluctuation magnetization is always negative, or diamagnetic, resulting from the fact that the free energy monotonically increases as the orbital flux increases.

Now let us turn to our system of two quantum dots coupled by hopping. Before we carry out a detailed analysis, it is illuminating to inspect the behavior of \( E_1, E_2 \) and the coefficients of the two logarithms in Eq. (4.33) (which we call \( A_{1,2} \)) as a function of \( E_{X_2} \). This is shown in Fig. 4.10. \( E_1 \) tends to \( E_{X_2}/2 \) for \( E_{X_2} \ll E_U \), and to \( E_U \) in the opposite limit \( E_{X_2} \gg E_U \). \( E_2 \) tends to \( E_U \) for \( E_{X_2} \ll E_U \), while in the opposite limit \( E_{X_2} \gg E_U \), \( E_2 \rightarrow E_{X_2} \). Both coefficients \( A_{1,2} \) start at \( \frac{1}{2} \) for small \( E_{X_2} \). For \( E_{X_2} \gg E_U \), \( A_1 \rightarrow 1 \), while \( A_2 \rightarrow 0 \).

The asymptotic regimes \( T, E_{X_2} \ll E_U \) and \( T, E_{X_2} \gg E_U \) can be understood simply. In the first regime, \( E_U \) is the largest energy scale, and far below it the spatial information that there are two distinct quantum dots is lost. The system behaves like a single large dot with a smaller “diluted” superconducting coupling. On the other hand, when \( T, E_{X_2} \gg E_U \), \( A_2 \) is vanishingly small, and the system resembles the isolated first dot with a superconducting coupling \( \bar{\lambda} \) but with a crossover energy \( E_U \). Note that the approach of the energies to the asymptotes is slow, so for a particular value of \( E_U \) it may happen that one cannot realistically approach the asymptotic regime without running into either \( \delta \) at the lower end or \( \omega_D \) at the higher end. Finally, one can envisage situations in which \( E_{X_2} \ll E_U \) but \( T \geq E_U \), for which there are no simple pictures.

The temperature dependence of magnetization per unit volume for different values of crossover parameters \( E_{X_2} \) and \( E_U \) (excluding the part due to noninteracting electrons) is shown in Fig. 4.11.

In the range where magnetization changes significantly, the fluctuation magnetization shows both diamagnetic and paramagnetic behavior. This is in contrast to the case of a single superconducting quantum dot subjected to an orbital flux where the fluctuation
Figure 4.9: Magnetization (per unit volume) in a single dot system as a function of temperature for different values of crossover parameters $E_X$. Panel (d) shows the dependence of the critical temperature on $E_X$.

Figure 4.10: The behavior of Log coefficients in Eq. (4.33) and $E_1,E_2$ as functions of the ratio $E_{X_2}/E_U$.
Figure 4.11: Magnetization (per unit volume) as a function of temperature for different values of crossover parameters $E_{X_2}$ and $E_U$. The fluctuation magnetization is diamagnetic for low $T$ and paramagnetic for high $T$. 
magnetization is always diamagnetic (Fig. 4.9). Close to \( T = 0 \) an increase in temperature makes the fluctuation magnetization more diamagnetic. A further temperature increase changes the fluctuation magnetization from diamagnetic to paramagnetic. For large values of temperature the fluctuation magnetization is paramagnetic and decreasing as \( T \) increases. Another set of diagrams, Fig. 4.12, demonstrates the dependence of the fluctuation magnetization in the first dot on crossover parameter \( E_{X_2} \) in the second dot. Generically, we find that at low \( T \) the fluctuation magnetization is diamagnetic while at high \( T \) it is paramagnetic.

The variation of crossover energy scales \( E_{X_2} \) and \( E_U \) does not change the qualitative behavior of the fluctuation magnetization as a function of \( T \) or \( E_{X_2} \). A paramagnetic magnetization is counterintuitive in superconducting system, because one believes that “an orbital flux is the enemy of superconductivity”, and therefore that the free energy must always increase as the orbital flux increases. This assumption is false for our system. The explanation is fairly simple, as we will see immediately after the results for \( T_c \) have been presented.

The mean-field critical temperature \( T_c \) of transition between normal and superconducting state strongly depends on \( E_{X_2} \) and \( E_U \). As one can see from Fig. 4.13, for very strong hopping \( (E_U \gg T_{c0}) \) between quantum dots \( T_c \) is monotonically decreasing as \( E_{X_2} \) increases. On the other hand, for intermediate hopping \( T_c \) has a maximum as a function of orbital flux, which means that for small values of orbital magnetic flux \( T_c \) increases as the orbital flux increases. Finally, when \( E_U \) is very weak, \( T_c \) monotonically increases as a function of orbital flux through the second quantum dot. This is in contrast to the behavior of a single superconducting quantum dot for which \( T_c \) decreases monotonically as a function of orbital flux.

These counterintuitive phenomena can be understood in terms of the following cartoon picture. One can think of the two dots as two sites, each capable of containing a large number of bosons (the fluctuating pairs). The BCS pairing interaction occurs only on the first site. When there is no magnetic flux, hopping delocalizes the bosons between the two sites, leading to a “dilution” of the BCS attraction and a low critical temperature. The effect of the magnetic flux on the second dot is twofold: (i) Firstly, it gaps the Cooperon of the second dot, which we think of as raising the energy for the bosons to be in the second dot. (ii) Secondly, by virtue of the interdot hopping, a small time-reversal symmetry breaking is produced in the first dot, thereby raising the energy of the bosons there as well. As the flux through the second dot rises, the bosons prefer to be in the first dot since they have lower energy there. The more localized the cooper pairs are in the first dot due to effect (i), the more “undiluted” will be the effect of the BCS attraction \( \lambda \), and the more favored will be the superconducting state. However, effect (ii) produces a time-reversal breaking in the first dot, thus disfavoring the superconducting state. These two competing effects lead to
Figure 4.12: Fluctuation magnetization in the first dot vs crossover parameter $E_{X_2}$ in the second dot for different values of temperature. The fluctuation magnetization is diamagnetic for low $T$ and paramagnetic for high $T$. 
Figure 4.13: Critical temperature as a function of $E_{X_2}$ for several intermediate to strong values (compared to $T_c^0$) of the hopping parameter $E_U$. For larger values of $E_{X_2}$ (not shown on graphs) critical temperature is equal to zero.
Figure 4.14: Behavior of critical temperature $T_C$ as a function of $E_{X_2}$ for small to intermediate values (compared to $T_{c0}$) of $E_U$. 

We can see that:

(a) $E_U = 0.1$ meV

(b) $E_U = 0.3$ meV

(c) $E_U = 0.4$ meV

(d) $E_U = 0.5$ meV

These figures illustrate how $T_C$ changes with $E_{X_2}$ for different values of $E_U$. 

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the varying behaviors of $T_c$ and the fluctuation magnetization versus the orbital flux in the second quantum dot. When the hopping between the quantum dots is weak ($E_U < T_{c0}$), the first effect dominates, and $T_c$ increases with $E_X$. When the hopping is stronger ($E_U \simeq T_{c0}$) the first effect dominates at small orbital flux, and the second at large orbital flux. Finally, at very large hopping ($E_U \gg T_{c0}$), effect (ii) is always dominant.

When considering the magnetization one must take into account the temperature as well, so the picture is more complex. The general feature is that effect (i) which tends to localize the pairs in the first dot also tends to decrease the interacting free energy of the system, which leads to a paramagnetic fluctuation magnetization. Effect (ii), which breaks time-reversal in the first dot, increases the free energy of the system and thus leads to a diamagnetic fluctuation magnetization. Based on our results we infer that at high temperature the coherence of pair hopping is destroyed leading to more localization in the first quantum dot. The consequences of high $T$ are thus similar to that of the effect (i): A lowering of the interacting free energy and a paramagnetic fluctuation magnetization.

We can make this picture a bit more quantitative for the behavior of $T_c$ with respect to $E_X$. Consider once more the scaling function of Eq. (4.33), which we reproduce here for the reader’s convenience

$$f_\alpha(E_X, E_U, T) = \frac{E_U}{2E_1} \frac{E_X^2}{E_2} + \frac{E_U E_X^2 - E_1^2 E_X^2}{E_2^2 - E_1^2} \ln \left[ \frac{4(h\omega_D)^2 + \omega_n^2}{C' + (E_1 + |\omega_n|)^2} \right] + \frac{E_U}{2E_2} \frac{E_2^2 - E_X^2 - E_U E_X^2}{E_2^2 - E_1^2} \ln \left[ \frac{4(h\omega_D)^2 + \omega_n^2}{C' + (E_2 + |\omega_n|)^2} \right]. \quad (4.36)$$

It is straightforward to show that $f_\alpha$ reaches its maximum value for $\omega_n = 0$. The condition for $T_c$ is then

$$\tilde{\lambda}f_0(E_X, E_U, T_c) = 1 \quad (4.37)$$

Let us first set $E_X = 0$. Let us also call the mean-field critical temperature of the isolated first dot in the absence of a magnetic flux $T_{c0}$ (recall that for the parameters pertinent to Al, $T_{c0} = 0.218 \text{meV} \approx 2.6K$). Now there are two possible limits, either $E_U \ll T_{c0}$ or $E_U \gg T_{c0}$. In the first case we obtain

$$T_c(E_U) \simeq T_{c0} \left( 1 - \frac{E_U^2}{\lambda C'T_{c0}^2} + \cdots \right) \quad (4.38)$$

In the second case, $E_U \gg T_{c0}$, we obtain

$$T_c(E_U) \simeq T_{c0} \frac{\omega_D}{E_U} e^{-1/\tilde{\lambda}} \quad (4.39)$$

Note that this can be much smaller than $T_{c0}$ and is an illustration of the “dilution” of the BCS attraction due to the second dot mentioned earlier. Of course, there will be a smooth crossover between the expressions of Eq. (4.38) and Eq. (4.39), so that $T_c$ is always smaller than $T_{c0}$.
Figure 4.15: The behavior of $E_{X_2}^*$ vs $E_U$ for numerical simulation and analytical approximation.

Now under the assumption $E_{X_2}, T_c \ll E_U$ we can solve analytically for $T_c$ to obtain

$$T_c^2(E_{X_2}, E_U) \simeq -\frac{E_{X_2}^2}{4C^2} + \frac{4\omega_D^4}{C^2E_U^2}e^{-4/\lambda e^{-\frac{2E_{X_2}}{E_U}}\left(\frac{1}{\lambda} - \frac{1}{4}\ln \frac{\omega_D}{E_U}\right)}$$  \hspace{1cm} (4.40)

One can further find the maximum of this expression. It turns out that $E_U$ has to be larger than a critical value $E_U^*$ for there to be a maximum.

$$E_U^* = \omega_D e^{\left(\frac{1}{4} - \frac{1}{\lambda}\right)}$$  \hspace{1cm} (4.41)

For our values of the parameters $\omega_D = 34meV$, $\tilde{\lambda} = 0.193$, we find $E_U^* = 0.245meV$. The position of the maximum can now be estimated asymptotically for $E_U > E_U^*$ as

$$E_{X_2}^* \simeq 16e^{-1}E_U^* \left(\frac{E_U^*}{E_U}\right)^3 \ln \frac{E_U}{E_U^*}$$  \hspace{1cm} (4.42)

Fig.4.15 compares the dependence of $E_{X_2}^*$ vs $E_U$ in case of numerical simulation and the one described by Eq. (4.42). For large values of $E_U$ compared to $E_U^*$ the numerically computed curve matches the analytical approximation.

4.5 Discussion

In this chapter we pursued two objectives. We intended to compute noninteracting scaling functions in the GOE$\rightarrow$GUE crossover in a system of two dots coupled by hopping, and to use this information to investigate the properties of an interacting system [49,72–74] in the many-body quantum critical regime [77–79].
We have considered a system of two coupled quantum dots, each of which could have its own time-reversal breaking parameter, coupled by a bridge which could also have time-reversal breaking. For each crossover parameter, there is a corresponding crossover energy scale, which represents the inverse of the time needed for the electron to “notice” the presence of that coupling in the Hamiltonian. We have computed the two particle Green’s functions in the coupled system in a large-$N$ approximation [61], valid when all energies of interest are much greater than the mean level spacing. This allows us to compute the correlations of products of four wave functions belonging to two different energy levels (which have been previously calculated for a single dot for the pure ensembles by Mirlin using supersymmetry methods [87], and for the Orthogonal to Unitary crossover by Adam et al [72]). The two particle Green’s function splits naturally into a diffuson part and a Cooperon part. Each of these parts can be represented as $\frac{1}{\omega}$ times a scaling function, where $\omega$ represents the frequency at which the measurement is being performed. For example, when we use the two particle Green’s function to find the ensemble average of four wave functions belonging to two energies, $\omega$ is the energy difference between the two states. The “scaling” nature of the scaling function is represented by the fact that it depends only on the ratio of $\omega$ to certain crossover energy scales. For the diffuson part the crossover energy $E_U$ is controlled solely by the strength of the hopping between the two dots, while the scaling function for the Cooperon part depends sensitively on the time-reversal breaking in all three parts of the system.

In the second part of the paper, we consider the case when one of the dots has an attractive BCS interaction, implying that it would be superconducting in the mean-field limit at zero temperature if it were isolated, and the other dot has no electron interactions but is penetrated by an orbital magnetic flux. The BCS interaction is one part of the Universal Hamiltonian [51–54], known to be the correct low-energy effective theory [55–57] in the renormalization group [75,76] sense for weak-coupling and deep within the Thouless band $|\varepsilon - \varepsilon_F| \ll E_T$. In order to eliminate complications arising from the charging energy, we consider a particular geometry with the dots being vertically coupled and very close together in the vertical direction, as shown in Fig. 4.1. Our focus is on the quantum critical regime [77–79], achieved by increasing either the temperature or the orbital flux through the second dot. The first dot is coupled by spin-conserving hopping to a second dot on which the electrons are noninteracting. This coupling always reduces the critical temperature, due to the “diluting” effect of the second dot, that is, due to the fact that the electrons can now roam over both dots, while only one of them has a BCS attraction. Thus, the mean-field critical temperature $T_c$ of the coupled system is always less than that of the isolated single superconducting dot $T_{c0}$. This part of the phenomenology is intuitively obvious.

However, when the hopping crossover energy $E_U$ is either weak or of intermediate strength compared to $T_{c0}$, turning on an orbital flux in the second dot can lead to a counter-
intuitive increase in the mean-field critical temperature of the entire system. For very weak hopping, the mean-field $T_c$ monotonically increases with orbital flux through the second dot, reaching its maximum when the second dot is fully time-reversal broken. For intermediate hopping strength, the mean-field $T_c$ initially increases with increasing orbital flux to a maximum. Eventually, as the orbital flux, and therefore the crossover energy corresponding to time-reversal breaking in the second dot increases, the critical temperature once again decreases. For strong hopping $E_U \gg T_{c0}$, $T_c$ monotonically decreases as a function of the orbital flux in the second quantum dot.

We have obtained the detailed dependence of the fluctuation magnetization in the quantum critical regime as a function of the dimensionless parameters $T/E_{X_2}$ and $E_{X_2}/E_U$. Once again, the coupled dot system behaves qualitatively differently from the single dot in having a paramagnetic fluctuation magnetization in broad regimes of $T$, $E_{X_2}$, and $E_U$.

We understand these phenomena qualitatively as the result of two competing effects of the flux through the second dot. The first effect is to raise the energy for Cooper pairs in the second dot, thereby tending to localize the pairs in the first dot, and thus reducing the “diluting” effect of the second dot. This first effect tends to lower the interacting free energy (as a function of orbital flux) and raise the critical temperature. The second effect is that as the electrons hop into the second dot and return they carry information about time-reversal breaking into the first dot, which tends to increase the free energy (as a function of orbital flux) decrease the critical temperature. The first effect dominates for weak hopping and/or high $T$, while the second dominates for strong hopping and/or low $T$. Intermediate regimes are more complex, and display non-monotonic behavior of $T_c$ and the fluctuation magnetization.

It should be emphasized that the quantum critical regime we focus on is qualitatively different from other single-particle random matrix ensembles applicable to a normal mesoscopic system which is gapless despite being in contact with one or more superconducting regions \([84,85]\), either because the two superconductors have a phase difference of $\pi$ in their order parameters \([84]\), or because they are d-wave gapless superconductors \([85]\). The main difference is that we investigate and describe an interacting regime, not a single-particle one. Without the interactions there would be no fluctuation magnetization.

Let us consider some of the limitations of our work. The biggest limitation of the non-interacting part of the work is that we have used the large-$N$ approximation, which means that we cannot trust our results when the energy scales and/or the frequency of the measurement becomes comparable to the mean level spacing. When $\omega \sim \delta$ the wave functions and levels acquire correlations in the crossover which we have neglected. Another limitation is that we have used a particular model for the interdot hopping which is analytically tractable, and is modelled by a Gaussian distribution of hopping amplitudes. This might be a realistic model in vertically coupled quantum dots, or where the bridge has a large
number of channels, but will probably fail if the bridge has only a few channels. These limitations could conceivably be overcome by using supersymmetric methods [63,67].

Coming now to the part of our work which deals with interactions, we have restricted ourselves to the quantum critical regime of the system, that is, when there is no mean-field BCS gap. Of course, a finite system cannot undergo spontaneous symmetry-breaking. However, in mean-field, one still finds a static BCS gap. The paradox is resolved by considering phase fluctuations of the order parameter which restore the broken symmetry [86]. To systematically investigate this issue one needs to analyze the case when the bosonic auxiliary field $\sigma$ in the coupled-dot system acquires a mean-field expectation value and quantize its phase fluctuations.

We have also chosen a geometry in which interdot charging effects can be ignored. However, most experimental systems with superconducting nanoparticles deal with almost spherical particles. For two such nanoparticles coupled by hopping, one cannot ignore charging effects [73,80–82]. We expect these to have a nontrivial effect on the mean-field $T_c$ and fluctuation magnetization of the combined system. We defer this analysis to future work.

In general, one can imagine a wide range of circumstances where changing a crossover parameter in one (noninteracting) dot allows one to softly and tunably break a symmetry in the another (interacting) dot, thereby allowing one access to a quantum critical regime. We hope the present work will be useful in exploring such phenomena.
CHAPTER 5: CONCLUSION

In this thesis we studied two systems of mesoscopic physics. Firstly, we investigated the behavior of the persistent current in quantum billiards subject to a point flux. We also studied a system of two coupled quantum dots which exhibit quantum critical phenomena.

Chapter 3 was dedicated to the study of the persistent current in the chaotic quantum billiards subject to a point magnetic flux. Following Robnik and Berry, we applied a conformal transformation and reduced the problem of finding the spectrum of electrons in an arbitrarily shaped billiard to the situation where the electrons move inside an integrable billiard in the fictitious potential introduced by the conformal transformation.

We found that there is a systematic diamagnetic contribution to the persistent current, that has been missed in previous work. It can be traced back to the flux response of the $l = 0$ levels of a regular unit disk (or annulus). In semiclassical approximation (when the energy tends to infinity) the spectral density of $l = 0$ states vanishes; thus these states were explicitly disregarded as those that do not enclose magnetic flux. We showed that even though the number of such $l = 0$ levels is submacroscopic ($\propto \sqrt{N}$, where $N$ is the number of electrons), the contribution to the persistent current due to these levels is proportional to $N$ and is independent of the flux for small flux in simply connected billiards.

For small values of magnetic flux the diamagnetic contribution overwhelms the fluctuating mesoscopic contribution from the states in the Thouless shell.

The diamagnetic contribution we uncover is also independent of the degree of chaoticity as long as states are not localized. This is clear from the fact that each exact eigenstate of energy $E$ of the deformed billiard is roughly a superposition of states of the regular billiard within a Thouless energy $E_T$ of $E$. For $E_F \gg E_T$, the diamagnetic contribution is independent of $E_T$, and hence on the degree of chaoticity.

Apart from the simply connected disk billiard we also investigated quantum annular billiards. The annular billiard is characterized by a dimensionless aspect ratio $\xi = r/R$, the ratio of the inner ($r$) to the outer radius ($R$). Note that in the annular billiard, the flux exists in a region where the electrons cannot penetrate, and the effects of the flux on the electrons are purely Aharonov-Bohm quantum interference effects.

For very tiny $\xi$, the annular Robnik-Berry billiard behaves much like the simply connected one for most values of the dimensionless flux $\alpha = \Phi/\Phi_0 \gg 1/\log N\xi^{-2}$, with a diamagnetic contribution to the persistent current that is proportional to the electron density. However, convergence to the $\xi = 0$ limit is logarithmically slow, and the limits $\alpha \to 0$ and $\xi \to 0$ do not commute. As the aspect ratio $\xi$ increases and the annulus tends to a one-dimensional ring, the systematic diamagnetic contribution diminishes to zero. For $\xi$ close to one, we also see Anderson localization in the distorted annular billiards, wherein
the persistent currents are negligible below a certain energy (presumably because the localization length for these levels is smaller than the circumference), and become nonzero only beyond a threshold energy.

In Chapter 4 we investigated a system of two coupled dots. We used a model where one of the dots was made from a superconducting material and had an attractive BCS interaction, and the other dot was made from a normal metal without any interactions among electrons. We also assumed that the magnetic field is zero in the domain of the first superconducting dot. In the second dot a weak magnetic field drives it into the GOE to GUE Universal RMT crossover. In order to eliminate complications arising from the charging energy, we consider a particular geometry with the dots being vertically coupled and very close together in the vertical direction, as shown in Fig. 4.1.

In our model we probe the quantum critical regime above mean-field critical temperature and/or field which allows us not to worry about symmetry breaking. Even though there cannot be any true spontaneous symmetry breaking in finite system, the mean-field description is a good one when the bulk superconducting gap $\Delta_{BCS}$ is much larger than the mean level spacing $\delta$ in the system.

For this model we find some unexpected results. The critical temperature $T_C$ in the first dot is a non-monotonic function of the interdot coupling and the strength of magnetic field in the second dot. Furthermore, for some values of interdot coupling $T_C$ is a monotonically growing function of the magnetic field in the second dot. This is surprising since conventionally an increasing magnetic field tends to destroy the superconducting state.

We explain this interesting effect as follows. The coupling between dots reduces the critical temperature due to the "diluting" effect of the second dot, that is, due to the fact that the electrons can now roam over both dots, while only one of them has a BCS attraction. Thus, the mean-field critical temperature $T_C$ of the coupled system is always less than that of the isolated single superconducting dot $T_{C0}$.

The effect of magnetic field in the second dot is twofold. First, it raises the energy for Cooper pairs in the second dot, thereby tending to localize the pairs in the first dot, and thus reducing the "diluting" effect of the second dot. This first effect tends to lower the interacting free energy (as a function of orbital flux) and raise the critical temperature.

Second, when the electrons hop into the second dot and return they carry information about time-reversal breaking into the first dot, which tends to increase the free energy (as a function of orbital flux) decrease the critical temperature. The first effect dominates for weak hopping and/or high $T$, while the second dominates for strong hopping and/or low $T$. Intermediate regimes are more complex, and display non-monotonic behavior of $T_C$ and the fluctuation magnetization.

We also studied the behavior of the magnetization vs temperature in the first dot. It turns out that the conventional expectation of a diamagnetic behavior of magnetization (as
for the one uncoupled dot) is violated. Close to the critical temperature, in the region of quantum critical fluctuations, the magnetization varies non-monotonically with temperature $T$ and crossover parameter $E_{X_2}$ in the second dot showing diamagnetic and paramagnetic behavior. The character and the size of the fluctuation magnetization depends on the interdot coupling.

As a prerequisite for the aforementioned results we calculated the one- and two-point Green’s functions for the system of two coupled dots. We considered the most general case when the both dots and the hopping bridge are in the Universal crossover between Gaussian orthogonal and Gaussian Unitary ensembles. Each part of the system is characterized by its own crossover parameter. We found that in the crossover the one-point Green’s function and the two main parts of the two-point Green’s function get modified by separate scaling functions. These scaling functions depend on the ratios of the crossover energy scales and measurement energy, and describe the transition between the GOE and GUE ensembles as the magnetic field through the system is changed. It is not the values of the crossover parameters themselves that define whether the system belongs to GOE or GUE or is in the crossover between two. The true crossover parameter is the ratio of the crossover energy scales and experimental measurement energy.
Appendix A: NUMERICS FOR ENERGY LEVELS

The idea of this method is as follows [43–46]. In the original $uv$ domain the Schrödinger equation is

$$\frac{1}{2m} (-i\hbar \nabla - qA(u,v))^2 \Psi(u,v) = E\Psi(u,v), \quad q = -e < 0. \quad (A.1)$$

To keep the dynamics of the electron unchanged, it is assumed that the magnetic field exists only at the origin of $uv$ plane inside the billiard. This requires that the vector potential satisfies the condition $\nabla \times A(r) = n \Phi \delta(r)$, where $n$ is a unit vector perpendicular to the plane of the billiard.

The billiard is threaded by a single magnetic-flux tube. The strength of the flux is $\Phi = \alpha \Phi_0$, where $\Phi_0 = \hbar/e$ is the magnetic-flux quantum.

If the vector potential has the form:

$$A(u,v) = \frac{\alpha}{2\pi} \Phi_0 \left( \frac{\partial f}{\partial v}, - \frac{\partial f}{\partial u}, 0 \right), \quad f = \frac{1}{2} \ln |z|^2, \quad (A.2)$$

then with the help of the conformal transformation:

$$w(z) = \frac{z + bz^2 + ce^{i\delta} z^3}{\sqrt{1 + 2b^2 + 3c^2}}, \quad w = u + iv, \quad z = x + iy. \quad (A.3)$$

the Schrödinger equation in polar coordinates of $xy$ plane becomes:

$$\nabla_{r,\theta}^2 \Psi(r,\theta) - \frac{i2\alpha}{r^2} \partial_\theta \Psi(r,\theta) - \frac{\alpha^2}{r^2} \Psi(r,\theta) + \epsilon \left| w'(re^{i\theta}) \right|^2 \Psi(r,\theta) = 0. \quad (A.4)$$

Here the energy $\epsilon$ is measured in units of $\hbar^2/2mR^2$, and the distance is in units of $R$, where $R$ is the radius of the disk in the $xy$ plane. Also, the coefficients $b, c,$ and $\delta$ in Eq. (A.3) are real parameters selected in such a way that the transformation $w(z)$ is nonsingular ($|w'(z)| > 0$) for all values of $z$ inside the disk. The transformation $w(z)$ is a cubic polynomial normalized to preserve the area of the billiard and leave the average density of states invariant. Equation (A.4) should be accompanied by the Dirichlet boundary condition.

To find the energy spectrum, one expands $\Psi(r, \theta)$ in Eq. (A.4) in terms of the eigenstates $\phi_{l,n}(r, \theta)$ of a free electron ($w = 0$) inside the unit disk:

$$\Psi_p(r, \theta) = N_p \sum_{j=1}^{\infty} \frac{c_j^{(p)}}{\gamma_j} \phi_j(r, \theta). \quad (A.5)$$

The index $j = (l, n)$ enumerates levels in ascending order. The normalized function $\phi_{l,n}(r, \theta)$ is

$$\phi_{l,n}(r, \theta) = \frac{J_{|l-\alpha|}(\gamma_n(|l-\alpha|)r)e^{i\theta}}{\sqrt{\pi}J'_{|l-\alpha|}(\gamma_n(|l-\alpha|))}, \quad (A.6)$$
where \( J_{\nu}(r) \) is the Bessel function of the first kind, \( \gamma_{l,n} \) is the \( n \)th root of \( J_{\nu}(r) \), and \( l \) is an orbital quantum number. The coefficients of the expansion in Eq. \((A.5)\) are chosen to make the matrix \( M_{ij} \) (below) Hermitian.

Plugging the expansion \((A.5)\) into Eq. \((A.4)\) after simplification, one gets the matrix equation for the eigenvalue problem:

\[
M_{ij} c_j^{(p)} = \frac{1}{\epsilon_p} c_i^{(p)},
\]

where the matrix \( M \) is

\[
M_{ij} = \left[ \frac{\delta_{ij}}{\gamma_i \gamma_j} + \delta_{l_i,l_j} - 26c e^{-i \delta} I_{ij}^{(2)} + \delta_{l_i,l_j} - 1(4b) I_{ij}^{(1)} + 12b c e^{-i \delta} I_{ij}^{(3)} \right. \\
\left. + \delta_{l_i,l_j} (8b^2 I_{ij}^{(2)} + 18c^2 I_{ij}^{(4)}) + \delta_{l_i,l_j} (4b) I_{ij}^{(1)} + 12b c e^{-i \delta} I_{ij}^{(3)} \\
+ \delta_{l_i,l_j} (26c e^{-i \delta} I_{ij}^{(2)}) \right] / (1 + 2b^2 + 3c^2).
\]

The integrals \( I_{ij}^{(h)} \) have the form

\[
I_{ij}^{(h)} = \frac{\int_0^1 drr^{h+1} J_{\nu_i}(\gamma_i r) J_{\nu_j}(\gamma_j r)}{\gamma_i \gamma_j J_{\nu_i}'(\gamma_i) J_{\nu_j}'(\gamma_j)}.
\]

Along with the simply connected domain (irregular disk), we consider the irregular annulus. Using a similar conformal transformation, we map the annulus with irregular boundaries from the \( uv \) plane onto the regular annulus in the \( xy \) plane with inner radius \( \xi \) and outer radius \( R = 1 \). The area-preserving conformal transformation is

\[
w(z) = \frac{z + bz^2 + ce^{i \delta} z^3}{\sqrt{1 + 2b^2(1 + \xi^2) + 3c^2(1 + \xi^2 + \xi^4)}}.
\]

For this kind of billiard the expansion of \( \Psi(r, \theta) \) is in terms of eigenstates \( \phi_j(r, \theta) \) for the regular annulus

\[
\phi_{l,n}(r, \theta) = \frac{\left[ J_{|l-\alpha|}(\gamma_n(|l-\alpha|)r) - J_{|l-\alpha|}(\gamma_n(|l-\alpha|)\xi) \right.}{N_{|l-\alpha|}(\gamma_n(|l-\alpha|)\xi) N_{|l-\alpha|}(\gamma_n(|l-\alpha|)r)} e^{i \delta} \\
\sqrt{2\pi} \int_{\xi}^1 drr \left[ J_{|l-\alpha|}(\gamma_n(|l-\alpha|)r) - J_{|l-\alpha|}(\gamma_n(|l-\alpha|)\xi) \right. \\
\left. \frac{N_{|l-\alpha|}(\gamma_n(|l-\alpha|)\xi) N_{|l-\alpha|}(\gamma_n(|l-\alpha|)r)}{N_{|l-\alpha|}(\gamma_n(|l-\alpha|)\xi) N_{|l-\alpha|}(\gamma_n(|l-\alpha|)r)} \right]^2.
\]

The counterpart of the matrix \( M \) for the annulus is

\[
M_{ij} = \left[ \frac{\delta_{ij}}{\gamma_i \gamma_j} + \delta_{l_i,l_j} - 26c e^{-i \delta} I_{ij}^{(2)} + \delta_{l_i,l_j} - 1(2b) I_{ij}^{(1)} + 12b c e^{-i \delta} I_{ij}^{(3)} \right. \\
\left. + \delta_{l_i,l_j} (4b^2 I_{ij}^{(2)} + 9c^2 I_{ij}^{(4)}) + \delta_{l_i,l_j} (4b) I_{ij}^{(1)} + 12b c e^{-i \delta} I_{ij}^{(3)} \\
+ \delta_{l_i,l_j} (26c e^{-i \delta} I_{ij}^{(2)}) \right] / (1 + 2b^2(1 + \xi^2) + 3c^2(1 + \xi^2 + \xi^4)).
\]

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where the integrals $I_{ij}^{(h)}$ are defined as

$$I_{ij}^{(h)} = \int_1^1 dr \frac{h+1 \tilde{\phi}_i(r) \tilde{\phi}_j(r)}{\gamma_i \gamma_j}, \quad \tilde{\phi}_i(r) = \sqrt{2\pi} e^{-i\theta} \phi_i(r, \theta).$$  (A.13)
Appendix B: ONE UNCOUPLED DOT

In this Appendix we calculate one particle and two particle Green’s functions for a single dot undergoing the crossover. The strength of magnetic field inside the dot is controlled by crossover parameter \( X \). The Hamiltonian of the system in crossover is:

\[
H = \frac{H_S + iXH_A}{\sqrt{1 + X^2}}, \tag{B.1}
\]

where \( H_{S,A} \) are symmetric and antisymmetric real random matrices with the same variance for matrix elements. Normalization \((1 + X^2)^{-1/2}\) keeps the mean level spacing \( \delta \) fixed as magnetic field changes inside the dot.

We define the retarded one particle Green’s function as follows:

\[
G^{R\alpha\beta}(E) = \left( \frac{1}{E^+ - H} \right)_{\alpha\beta} = \frac{1}{E^+} \left( I + \frac{H}{E^+} + \frac{H^2}{(E^+)^2} + \ldots \right) = \frac{\delta_{\alpha\beta}}{E^+} + \frac{H_{\alpha\beta}}{(E^+)^2} + \frac{H_{\alpha\beta}^2}{(E^+)^3} + \ldots, \tag{B.2}
\]

Here \( H \) is a Hamiltonian, and \( E^+ \) is the energy with infinitely small positive imaginary part \( E^+ = E + i\eta \).

This series has nice graphical representation as shown on Fig. B.1, where the straight solid line represents \( 1/E^+ \) and the dashed line with dot stands for Hamiltonian.

Just as in disordered conductor or quantum field theory the target is not the Green’s function itself, but rather its mean and mean square. We take on random matrix ensemble average of \( G_{\alpha\beta} \). Such averaging assumes knowledge of \( \langle H^n \rangle \), where angular brackets stand for Gaussian ensemble averaging, and \( n = 1, \infty \). For \( n = 1 \) we have \( \langle H \rangle = 0 \), while for \( n = 2 \) the second moment reads:

\[
\langle H_{\alpha\gamma}H_{\delta\beta} \rangle = \frac{\langle H^a_{\alpha\gamma}H^a_{\delta\beta} \rangle - X^2\langle H^s_{\alpha\gamma}H^s_{\delta\beta} \rangle}{1 + X^2} = \frac{N\delta^2}{\pi^2}\delta_{\alpha\beta}\delta_{\gamma\delta} + \left( 1 - X^2 \right) \frac{N\delta^2}{\pi^2}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{B.3}
\]

where we used the following correlators for the Hamiltonian matrix elements:

\[
\langle H^s_{\alpha\gamma}H^s_{\delta\beta} \rangle = \frac{N\delta^2}{\pi^2}[\delta_{\alpha\delta}\delta_{\gamma\beta} \pm \delta_{\alpha\beta}\delta_{\gamma\delta}]. \tag{B.4}
\]

The ‘+’ (‘−’) sign corresponds to the symmetric (antisymmetric) part of the Hamiltonian.

\[
\quad + \quad + \quad + \quad + \ldots
\]

Figure B.1: Series expansion for one particle Green’s function in the system of one uncoupled dot.
All higher moments of $H$ can be computed using Wick’s theorem [88]. Thus, the ensemble averaging of $G^R$ leaves only the terms containing even moments of $H$ (see Fig. B.2).

Then, the expansion on Fig. B.2 can be written in a compact form of Dyson equation (see Fig. 4.2). The bold line denotes the full one particle Green’s function averaged over Gaussian ensemble, and $\Sigma$ is a self-energy, representing the sum of all topologically different diagrams as shown on Fig. B.3. The corresponding algebraic expression for the Dyson equation can be easily written down:

$$G_{\alpha\beta} = \sum_{\nu\mu} G_{\alpha\nu} \Sigma_{\nu\mu} \delta_{\mu\beta} \frac{\delta_{\alpha\beta}}{E^+} + \delta_{\alpha\beta} \frac{E}{E^+},$$

where $G_{\alpha\beta}$ means $\langle G^R_{\alpha\beta}(E) \rangle$. Now, using the fact that $G_{\alpha\beta} = G_{\alpha} \delta_{\alpha\beta}$ and $\Sigma_{\alpha\beta} = \Sigma_{\alpha} \delta_{\alpha\beta}$ (no summation over $\alpha$ implied), one can solve this equation and obtain:

$$G_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{E^+ - \Sigma}.$$  \hspace{1cm} (B.6)

Next we use large $N$ approximation to substitute self-energy by the first term from series expansion on Fig. B.3 (all other terms will be of the order of $O(1/N)$ and can be neglected when $N \to \infty$):

$$\Sigma_{\alpha\beta} = G \sum_{\gamma} \langle H_{\alpha\gamma} H_{\gamma\beta} \rangle \approx \left( \frac{N \delta}{\pi} \right)^2 \frac{\delta_{\alpha\beta}}{E^+ - \Sigma}.$$  \hspace{1cm} (B.7)

Solving Eq. (B.7) for the self-energy we determine:

$$\Sigma = \frac{E}{2} - \frac{i}{2} \sqrt{\left( \frac{2N \delta}{\pi} \right)^2 - E^2}.$$  \hspace{1cm} (B.8)

Consequently, the ensemble average of one particle Green’s function is given by:
\[
\langle G^R_{\alpha\beta}(E) \rangle = \frac{\delta_{\alpha\beta}}{E^2 + \frac{i}{2\sqrt{2}} \left( \frac{2N\delta}{\pi} \right)^2 E^2}; \quad \langle G^A_{\alpha\beta}(E) \rangle = \langle G^R_{\beta\alpha}(E) \rangle^* .
\] (B.9)

Next, to study the two particle Green’s function we notice that the main contributions come from ladder and maximally crossed diagrams (see Fig. 4.3). As for the one particle Green’s function, all other diagrams with crossing lines are of the order \(O(1/N)\). Two bold lines on the left side of equation from Fig. 4.3 stand for the average two particle Green’s function \(\langle G_R(E + \omega)G_A(E) \rangle\).

The analytic expression of the sum of ladder diagrams described by Bethe-Salpeter equation (a) on Fig. 4.4 is:

\[
\Pi^{\alpha\beta,D}_{\delta\gamma} = \frac{N\delta^2}{\pi^2} \delta_{\alpha\delta} \delta_{\beta\gamma} + \left( \frac{N\delta}{\pi} \right)^2 \frac{\Pi^{\alpha\beta,D}_{\delta\gamma}}{F[E,\omega]},
\] (B.10)

where \(\Pi^D\) is a ladder approximation of diffuson part of two particle Green’s function. Here \(F[E,\omega]\) is a product of two inversed averaged one particle Green’s functions and in the limit \(\omega \ll N\delta\) is:

\[
F[E,\omega] = \langle G^R(E + \omega) \rangle^{-1}\langle G^A(E) \rangle^{-1} \approx -\frac{i\omega\delta N}{2\pi} + \left( \frac{N\delta}{\pi} \right)^2 .
\] (B.11)

One can solve this equation taking into account \(\Pi^{\alpha\beta,D}_{\delta\gamma} = \Pi^D \delta_{\alpha\delta} \delta_{\beta\gamma}\):

\[
\Pi^D = \frac{N\delta^2}{\pi^2} \frac{F[E,\omega]}{F[E,\omega] - \left( \frac{N\delta}{\pi} \right)^2} .
\] (B.12)

Multiplying \(\Pi^D\) by \(F^2[E,\omega]\) we arrive at the following expression for the diffuson term:

\[
\langle G^R_{\alpha\gamma}(E + \omega)G^A_{\delta\beta}(E) \rangle_D = \frac{2\pi}{N^2\delta} \delta_{\alpha\beta} \delta_{\gamma\delta} - \frac{i\omega}{1 + \frac{X^2}{N^2\delta}} .
\] (B.13)

Then, we turn our attention to the equation for maximally crossed diagrams (see Fig. 4.4, (b)). \(\Pi^C\) is expressed in terms of \(F[E,\omega]\) as follows:

\[
\Pi^C = \left( 1 - X^2 \right) \frac{N\delta^2}{\pi^2} \frac{F[E,\omega]}{F[E,\omega] - \frac{1 - X^2}{1 + X^2} \left( \frac{N\delta}{\pi} \right)^2} .
\] (B.14)

Assuming \(X\) to be small compared to unity (weak crossover), we evaluate the contribution of maximally crossed diagrams to Green’s function to get:

\[
\langle G^R_{\alpha\gamma}(E + \omega)G^A_{\delta\beta}(E) \rangle_C = -\frac{2\pi}{N^2\delta} \frac{\delta_{\alpha\delta} \delta_{\gamma\beta}}{-i\omega} \frac{1}{1 + \frac{X^2}{\omega}} ,
\] (B.15)

where \(E_X = 4X^2N\delta/\pi\) is a crossover energy scale. Final expression for the connected part of the two particle Green’s function is:
\begin{equation}
\langle G^R_{\alpha\gamma}(E + \omega)G^A_{\delta\beta}(E) \rangle = \frac{2\pi}{N^2\delta} \frac{\delta_{\alpha\beta}\delta_{\gamma\delta}}{-i\omega} + \frac{2\pi}{N^2\delta} \frac{\delta_{\alpha\delta}\delta_{\gamma\beta}}{-i\omega} \frac{1}{1 + i\frac{E_X}{\omega}}.
\end{equation}
(B.16)
Appendix C: TWO COUPLED DOTS

This Appendix contains details of the derivation for statistical properties of the Green’s functions for the two coupled dots connected to each other via hopping bridge $V$. Coupling between dots is weak and characterized by dimensionless parameter $U$. For the system of uncoupled dots the Hilbert space is a direct sum of spaces for dot 1 and dot 2. Hopping $V$ mixes the states from two spaces. The Hamiltonian of the system can be represented as:

$$H_{\text{tot}} = \begin{pmatrix} H_1 & V \\ V^\dagger & H_2 \end{pmatrix}.$$  \hfill (C.1)

For $H_1,2$ and $V$ we have:

$$H_n = \frac{H_n^S + iX_nH_n^A}{\sqrt{1 + X_n^2}}, \quad i = 1,2; \quad V = \frac{V^R + iV^I}{\sqrt{1 + \Gamma^2}}.$$  \hfill (C.2)

Here $S$ ($A$) stands for symmetric (antisymmetric), and $R$ ($I$) means real (imaginary). Below we use Greek indices for dot 1, and Latin indices for dot 2. We also found it convenient to keep bandwidth of both dots the same; that is, $N_1\delta_1 = N_2\delta_2$ with $\xi = \delta_1/\delta_2$.

The following averaged products of matrix elements of $H$ can be obtained:

$$\langle H_{\alpha\gamma}H_{\delta\beta} \rangle = \frac{N_1\delta_1^2}{\pi^2} \delta_{\alpha\beta} \delta_{\gamma\delta} + \left(\frac{1 - X_1^2}{1 + X_1^2}\right) \frac{N_1\delta_1^2}{\pi^2} \delta_{\alpha\delta} \delta_{\gamma\beta}$$

$$\langle H_{ij}H_{lk} \rangle = \frac{N_2\delta_2^2}{\pi^2} \delta_{ij} \delta_{lk} + \left(\frac{1 - X_2^2}{1 + X_2^2}\right) \frac{N_2\delta_2^2}{\pi^2} \delta_{ik} \delta_{lj},$$ \hfill (C.3)

where $X_1$ and $X_2$ are the crossover parameters in dot 1 and 2. Pairings between $V$ matrix elements are:

$$\langle V_{\alpha i}V_{\beta j} \rangle = \langle V_{\alpha i}^\dagger V_{\beta j}^\dagger \rangle = \left(\frac{1 - \Gamma^2}{1 + \Gamma^2}\right) \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} \delta_{\alpha\beta} \delta_{ij}$$

$$\langle V_{\alpha i}V_{\beta j}^\dagger \rangle = \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} \delta_{\alpha\beta} \delta_{ij},$$ \hfill (C.4)

with $\Gamma$ a crossover parameter in hopping bridge. Normalization for $V$ pairing is chosen to coincide with that of $\langle HH \rangle$ when $\xi = 1$.

To determine one particle Green’s function we use the system shown on Fig. 4.5. The straight and wavy bold lines with arrows represent averaged functions $\langle G^R_1(E) \rangle$, $\langle G^R_2(E) \rangle$ in dot 1 and 2, regular lines represent bare propagators, and the rest of the lines describe pairings between $H_{\text{tot}}$ matrix elements (see Fig. C.1).
The corresponding analytical expressions of this system of equations are:

\[ G_1 = \frac{\Sigma_{11} G_1}{E^+} + \frac{\Sigma_{12} G_1}{E^+} + \frac{1}{E^+}, \]
\[ G_2 = \frac{\Sigma_{22} G_2}{E^+} + \frac{\Sigma_{21} G_2}{E^+} + \frac{1}{E^+}, \]

with \( G_1 \) and \( G_2 \) connected to Green’s functions via:

\[ \langle G^R_{\alpha \gamma}, 1 \rangle (E) = G_1 \delta_{\alpha \gamma}, \]
\[ \langle G^R_{il}, 2 \rangle (E) = G_2 \delta_{il}. \]

The self-energies \( \Sigma_{nm} \) are to be determined using standard procedure [63].

We observe, that the system of two linear equations (C) has a solution:

\[ G_1 = \frac{1}{E^+ - \Sigma_{11} - \Sigma_{12}}, \quad G_2 = \frac{1}{E^+ - \Sigma_{22} - \Sigma_{21}}. \]

Here we approximated self-energies by the first term in large \( N \) expansion again. In this approximation evaluation of \( \Sigma_{nm} \) yields:

\[ \Sigma_{11} = \Sigma_{11} \delta_{\alpha \beta} = G_1 \sum_{\gamma} \langle H_{\alpha \gamma} H_{\gamma \beta} \rangle = \left( \frac{N_1 \delta_1}{\pi} \right)^2 \frac{\delta_{\alpha \beta}}{E^+ - \Sigma_{11} - \Sigma_{12}}, \]
\[ \Sigma_{12} = \Sigma_{12} \delta_{\alpha \beta} = G_2 \sum_i \langle V_{\alpha i} V_{\beta i}^\dagger \rangle = \sqrt{N_1 N_2} \frac{\delta_1 \delta_2 U}{\pi^2} \frac{\delta_{\alpha \beta}}{E^+ - \Sigma_{22} - \Sigma_{21}}, \]
\[ \Sigma_{22}^1 = \Sigma_{22}^1 \delta_{ij} = \left( \frac{N_2 \delta_2}{\pi} \right)^2 \frac{\delta_{ij}}{E^+ - \Sigma_{22} - \Sigma_{21}}, \]
\[ \Sigma_{21} = \Sigma_{21} \delta_{ij} = \sqrt{N_1 N_2} \frac{\delta_1 \delta_2 U}{\pi^2} \frac{\delta_{ij}}{E^+ - \Sigma_{11} - \Sigma_{12}}. \]

Thus, to find all \( \Sigma_{nm} \) one needs to solve the following system of equations:

\[
\begin{align*}
\Sigma_{11} (E^+ - \Sigma_{11} - \Sigma_{12}) &= \left( \frac{N_1 \delta_1}{\pi} \right)^2 \\
\Sigma_{12} (E^+ - \Sigma_{22} - \Sigma_{21}) &= \sqrt{N_1 N_2} \frac{\delta_1 \delta_2 U}{\pi^2} \\
\Sigma_{22} (E^+ - \Sigma_{22} - \Sigma_{21}) &= \left( \frac{N_2 \delta_2}{\pi} \right)^2 \\
\Sigma_{21} (E^+ - \Sigma_{11} - \Sigma_{12}) &= \sqrt{N_1 N_2} \frac{\delta_1 \delta_2 U}{\pi^2}.
\end{align*}
\]
Observing that $\Sigma^{21} = U\Sigma^{11}/\sqrt{\xi}$ and $\Sigma^{12} = U\sqrt{\xi}\Sigma^{11}$ we decouple the system given in Eq. (C.5). For example, the pair of first and third equations can be rewritten as:

\begin{align*}
\left(\Sigma^{11}\right)^2 - E\Sigma^{11} + U\sqrt{\xi}\Sigma^{11}\Sigma^{22} &= -\left(\frac{N_1\delta_1}{\pi}\right)^2 \\
\left(\Sigma^{22}\right)^2 - E\Sigma^{22} + \frac{U}{\sqrt{\xi}}\Sigma^{11}\Sigma^{22} &= -\left(\frac{N_2\delta_2}{\pi}\right)^2 .
\end{align*}

(C.6)

For weak coupling the solution can be found by expanding self-energies $\Sigma^{11}$ and $\Sigma^{22}$ in series in $U$. Taking the solution for single dot as zero approximation (below all the solutions for the uncoupled dot will be marked with subscript 0) we get

\begin{align*}
\Sigma^{11} &= \Sigma^{11}_0 + U\Sigma^{11}_1 \\
\Sigma^{22} &= \Sigma^{22}_0 + U\Sigma^{22}_1.
\end{align*}

(C.7)

(C.8)

Note that $N_1\delta_1 = N_2\delta_2$, and $\Sigma^{11}_0 = \Sigma^{22}_0 \equiv \Sigma_0$.

Plugging into the right hand side of Eq. (C.8) in system (C.6) we arrive at:

\begin{align*}
\Sigma^{11} &= \Sigma_0 \left(1 + U\sqrt{\xi}\frac{\Sigma_0}{E^+ - 2\Sigma_0}\right) \\
\Sigma^{22} &= \Sigma_0 \left(1 + \frac{U}{\sqrt{\xi}}\frac{\Sigma_0}{E^+ - 2\Sigma_0}\right) \\
\Sigma^{21} &= \frac{U}{\sqrt{\xi}}\Sigma^{11} \\
\Sigma^{12} &= U\sqrt{\xi}\Sigma^{22}.
\end{align*}

(C.9)

Neglecting the higher powers in $U$ for one particle Green’s functions we finally arrive at the following expressions for the single particle Green’s functions:

\begin{align*}
\langle G^{R}_{\alpha\beta,1}(E) \rangle &= \frac{\langle G^{R}_{\alpha\beta,0}(E) \rangle}{1 - U\sqrt{\xi}\frac{\Sigma_0}{E^+ - 2\Sigma_0}} = \frac{\delta_{\alpha\beta}}{1 + \frac{U}{\sqrt{\xi}}\left(1 + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\right)} \\
\langle G^{R}_{ij,2}(E) \rangle &= \frac{\langle G^{R}_{ij,0}(E) \rangle}{1 - \frac{U}{\sqrt{\xi}}\frac{\Sigma_0}{E^+ - 2\Sigma_0}} = \frac{\delta_{ij}}{1 + \frac{U}{2\sqrt{\xi}}\left(1 + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\right)},
\end{align*}

where $\epsilon = \pi E/2N\delta$.

Now we switch our attention to the calculational procedure for the average of the two particle Green’s functions $\langle G^{R}_{\alpha\gamma,1}(E + \omega)G^{A}_{\beta\delta,1}(E) \rangle$ and $\langle G^{R}_{il,2}(E + \omega)G^{A}_{kj,2}(E) \rangle$. In the limit of large $N_1$ and $N_2$ ladder and maximally crossed diagrams contribute the most. For ladder diagrams we obtain the system of Bethe-Salpeter equations (see Fig. 4.6). For two particle Green’s functions we use notations shown on Fig. C.2.
For the diffusion $\Pi_{nm}^D$ the system of algebraic equations reads:

\[
\begin{align*}
\Pi_{11}^D &= \frac{N_1 \delta_1^2}{\pi^2} + \frac{N_1 \delta_1^2}{\pi^2} \frac{\Pi_{11}^D}{F_1[E, \omega]} + \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} \frac{\Pi_{21}^D}{F_2[E, \omega]} \\
\Pi_{22}^D &= \frac{N_2 \delta_2^2}{\pi^2} + \frac{N_2 \delta_2^2}{\pi^2} \frac{\Pi_{22}^D}{F_2[E, \omega]} + \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} \frac{\Pi_{12}^D}{F_1[E, \omega]} \\
\Pi_{12}^D &= \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} + \frac{N_1 \delta_1^2}{\pi^2} \frac{\Pi_{12}^D}{F_1[E, \omega]} + \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} \frac{\Pi_{22}^D}{F_2[E, \omega]} \\
\Pi_{21}^D &= \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} + \frac{N_2 \delta_2^2}{\pi^2} \frac{\Pi_{21}^D}{F_2[E, \omega]} + \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} \frac{\Pi_{11}^D}{F_1[E, \omega]},
\end{align*}
\]

where $F_1[E, \omega]$ and $F_2[E, \omega]$ are defined as products of inverse averaged one particle Green’s functions in the first and second dots respectively. For small values of $U$ and $\omega$ these functions can be approximated as follows:

\[
\begin{align*}
F_1[E, \omega] &= \langle G_1^R(E + \omega) \rangle^{-1} \langle G_1^A(E) \rangle^{-1} \approx \left( \frac{N_1 \delta_1}{\pi} \right)^2 \left[ 1 + \sqrt{\xi} U - i \bar{\omega} \right] \\
F_2[E, \omega] &= \langle G_2^R(E + \omega) \rangle^{-1} \langle G_2^A(E) \rangle^{-1} \approx \left( \frac{N_2 \delta_2}{\pi} \right)^2 \left[ 1 + \sqrt{\eta} U - i \bar{\omega} \right],
\end{align*}
\]

where $\bar{\omega} = \pi \omega / 2 N \delta$. The system of four equations given by the Eq. (C.10) can be decoupled into the two systems of two equations each. To determine $\Pi_{11}^D$, one solves the system of the first and the last equations of Eq. (C.10) to get:

\[
\begin{align*}
\left( 1 - \frac{N_1 \delta_1}{F_1[E, \omega]} \right)^2 \Pi_{11}^D - \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} \frac{\Pi_{21}^D}{F_2[E, \omega]} &= \frac{N_1 \delta_1^2}{\pi^2} \\
\left( 1 - \frac{N_2 \delta_2}{F_2[E, \omega]} \right)^2 \Pi_{21}^D - \frac{\sqrt{N_1 N_2 \delta_1 \delta_2 U}}{\pi^2} \frac{\Pi_{11}^D}{F_1[E, \omega]} &= \frac{N_1 N_2 \delta_1 \delta_2 U}{\pi^2}.
\end{align*}
\]

Then, solving the resulting system (Eq. (C.12)) and attaching external lines one obtains expression for the two particle Green’s function in dot 1:
\[ \langle G_{\alpha\gamma,1}(E + \omega)G_{\delta,1}^A(E) \rangle_D = \frac{2\pi}{N^2_1\delta_1} \delta_{\alpha\beta}\delta_{\gamma\delta} \frac{1 + i\sqrt{\xi} \frac{E_{\alpha}}{\omega}}{1 + i(\sqrt{\xi} + \frac{1}{\sqrt{\xi}}) \frac{E_\alpha}{\omega}}. \] (C.13)

The corresponding correlator for dot 2 is readily obtained as well:

\[ \langle G_{\alpha,2}(E + \omega)G_{\delta,2}^A(E) \rangle_D = \frac{2\pi}{N^2_2\delta_2} \delta_{ij}\delta_{lk} \frac{1 + i\sqrt{\xi} \frac{E_j}{\omega}}{1 + i(\sqrt{\xi} + \frac{1}{\sqrt{\xi}}) \frac{E_j}{\omega}}. \] (C.14)

For the second part of the Green’s function (which is the sum of maximally crossed diagrams) the system of equations is shown on Fig. 4.7. Transforming this graphical system into the algebraic one, we get:

\[
\begin{align*}
\Pi_{11}^C &= \left(1 - X_1^2\right) \frac{N_1\delta_1^2}{\pi^2} \left(1 + X_1^2\right) \frac{N_1\delta_1^2}{\pi^2} F_1[E, \omega] + \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} F_2[E, \omega] \frac{N_2\Pi_{11}^C}{F_1[E, \omega]} \\
\Pi_{22}^C &= \left(1 - X_2^2\right) \frac{N_2\delta_2^2}{\pi^2} \left(1 + X_2^2\right) \frac{N_2\delta_2^2}{\pi^2} F_1[E, \omega] + \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} F_2[E, \omega] \frac{N_1\Pi_{11}^C}{F_1[E, \omega]} \\
\Pi_{12}^C &= \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} + \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} + \left(1 - X_1^2\right) \frac{N_1\delta_1^2}{\pi^2} N_1\Pi_{11}^C \\
\Pi_{21}^C &= \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} + \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} + \left(1 - X_2^2\right) \frac{N_2\delta_2^2}{\pi^2} F_2[E, \omega]
\end{align*}
\] (C.15)

Once again, the system at hand breaks into systems of two equations each. We proceed by combining the first and the last equations to obtain:

\[
\left[ 1 - \left(\frac{X_1^2}{1 + X_1^2}\right) \frac{N_1\delta_1^2}{\pi^2} F_1[E, \omega] \right]^2 \Pi_{11}^C - \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} \frac{\Pi_{11}^C}{F_2[E, \omega]} \left(1 - X_1^2\right) \frac{N_1\delta_1^2}{\pi^2} F_1[E, \omega] = \left(1 - \Gamma^2\right) \frac{1}{1 + \Gamma^2} \frac{\sqrt{N_1N_2\delta_1\delta_2U}}{\pi^2} F_2[E, \omega]
\] (C.16)

Now we can construct approximations for the expressions, containing crossover parameters. For example, for small values of \(X\) and \(\Gamma\) the solution for \(\Pi_{11}^C\) is expressed as follows:

\[
\Pi_{11}^C = \frac{N_1\delta_1^2}{\pi^2} \left(1 - 2X_1^2\right) \left(\frac{U}{\sqrt{\xi}} - i\tilde{\omega} + 2X_1^2\right) + \left(1 - 4\Gamma^2\right) U^2
\] (C.17)

Next, introducing crossover energy scales:

\[
E_X = 4X^2 \frac{N\delta}{\pi} \quad E_U = 2U \frac{N\delta}{\pi} \quad E_\Gamma = \frac{4\Gamma^2 E_U}{\sqrt{\xi} + \frac{1}{\sqrt{\xi}}}
\] (C.18)
we obtain the solution for $\Pi_{11}^C$ in the following form:

$$
\Pi_{11}^C = \frac{N_1\delta_1^2}{\pi^2} \frac{1}{-i\omega} \left( 1 - \frac{E_{X_1 + X_{2,\omega}}}{\omega^2} + \frac{E_{X_1} E_{X_{2,\omega}}}{\omega^2} + \frac{\sqrt{\xi E_{X_1} E_{U}}}{\omega^2} \right) \left( \sqrt{\frac{1}{\xi}} + \frac{E_{U}}{\omega} \right) \left( 1 + \frac{E_{U}}{\omega} \right).
$$

(C.19)

Then, adding external lines to $\Pi_{11}^C$ for Green’s function we get:

$$
\langle G_{\alpha\gamma,1}(E + \omega) G_{\delta\beta,1}(E) \rangle_C = \frac{2\pi}{N_1^2\delta_1} \frac{\delta_{\alpha\beta} \delta_{\gamma\beta}}{-i\omega} \left( 1 + \frac{\sqrt{\xi E_{U}}}{\omega} + \frac{i E_{X_1}}{\omega} \right) \frac{\frac{1}{\sqrt{\xi}} + \frac{E_{U}}{\omega}}{1 + \frac{E_{U}}{\omega}}.
$$

(C.20)

Similar manipulations for the corresponding correlator of Green’s functions for the second room result in:

$$
\langle G_{\alpha\gamma,2}(E + \omega) G_{\delta\beta,2}(E) \rangle_C = \frac{2\pi}{N_2^2\delta_2} \frac{\delta_{\alpha\beta} \delta_{\gamma\beta}}{-i\omega} \left( 1 + \frac{\sqrt{\xi E_{U}}}{\omega} + \frac{i E_{X_1}}{\omega} \right) \frac{\frac{1}{\sqrt{\xi}} + \frac{E_{U}}{\omega}}{1 + \frac{E_{U}}{\omega}}.
$$

(C.21)

Finally, the connected part of the total two particle Green’s function is obtained as a sum of diffuson and Cooperon parts, yielding:

$$
\langle G_{\alpha\gamma,1}(E + \omega) G_{\delta\beta,1}(E) \rangle = \frac{2\pi}{N_1^2\delta_1} \frac{\delta_{\alpha\beta} \delta_{\gamma\delta}}{-i\omega} \left( 1 + \frac{\sqrt{\xi E_{U}}}{\omega} + \frac{i E_{X_1}}{\omega} \right) \frac{\frac{1}{\sqrt{\xi}} + \frac{E_{U}}{\omega}}{1 + \frac{E_{U}}{\omega}}.
$$

(C.22)

$$
\langle G_{\alpha\gamma,2}(E + \omega) G_{\delta\beta,2}(E) \rangle = \frac{2\pi}{N_2^2\delta_2} \frac{\delta_{\alpha\beta} \delta_{\gamma\beta}}{-i\omega} \left( 1 + \frac{\sqrt{\xi E_{U}}}{\omega} + \frac{i E_{X_1}}{\omega} \right) \frac{\frac{1}{\sqrt{\xi}} + \frac{E_{U}}{\omega}}{1 + \frac{E_{U}}{\omega}}.
$$

(C.23)

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Appendix D: FOURIER TRANSFORM OF TWO PARTICLE GREEN’S FUNCTION

To be able to study temporal behavior of electrons in the RMT system we introduce the Fourier transform of two particle Green’s function. We define it via the following integral:

\[ \langle G_R^{\alpha\gamma}(t)G_A^{\delta\beta}(t) \rangle = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \exp(-i\omega t) \langle G_R^{\alpha\gamma}(E + \omega)G_A^{\delta\beta}(E) \rangle d\omega dE. \]  

(D.1)

To get the correct behavior of the diffusion part for small \( \omega \), we replace \( \frac{1}{\omega} \) by \( \frac{\omega}{\omega^2 + \eta^2} \), where \( \eta \) is infinitesimal positive number. Now we introduce for dot 1:

\[ f_D(\omega) = \frac{2\pi}{N_1^2 \delta_1} \frac{\delta_{\alpha\beta} \delta_{\gamma\delta}}{-i\omega} \left( \frac{1 + i\xi}{\sqrt{\xi} + \frac{\eta}{\sqrt{\xi}}} \right) \left( \frac{E_U}{\omega} \right) \to \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{2\pi}{N_1^2 \delta_1} \frac{i\omega}{\omega^2 + \eta^2} \left( \frac{\omega + i\sqrt{\xi}E_U}{\omega + i(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U} \right) \]

\[ = \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{2\pi}{N_1^2 \delta_1} \frac{i\omega}{(\omega - i\eta)(\omega + i\eta)(\omega + i(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U)}. \]  

(D.2)

The Fourier transform of this diffusion term gives:

\[ f_D(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega t) f_D(\omega) d\omega. \]  

(D.3)

Next steps are the standard steps of integration in complex plane. For \( t > 0 \) one closes contour in lower-half plane. One root is located in upper half plane and two more are located in lower half plane. The integration yields:

\[ f_D(t) = \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{2\pi}{N_1^2 \delta_1} \left[ \frac{(E_U - \eta)e^{-\eta t}}{2(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U - \eta} \right] + \frac{(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})\sqrt{\xi}E_U^2 e^{-(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U t}}{(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})^2 E_U^2 - \eta^2} \].  

(D.4)

As \( \eta \) approaches zero, \( f_D(t) \) becomes:

\[ f_D(t) = \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{2\pi}{N_1^2 \delta_1} \frac{1}{1 + \xi} \left[ \frac{1}{2} + \xi e^{-(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U t} \right]. \]  

(D.5)

The full Fourier transformation includes integration over \( E \) as well. In current approximation, when \( E \) is close to the center of the band, \( \langle G_R^{\alpha\gamma}G_A^{\delta\beta} \rangle \) is independent of \( E \). It will depend on \( E \) if we integrate over the whole bandwidth. The exact dependence of \( \langle G_R^{\alpha\gamma}G_A^{\delta\beta} \rangle \) on \( E \) far from the center of the band is not known. To get correct expression we assume that integration over \( E \) adds to \( \langle G_R^{\alpha\gamma}G_A^{\delta\beta} \rangle \) multiplicative factor \( N_1 \delta_1 \) along with normalization coefficient \( A \). Also, for index pairing \( \alpha = \beta \) and \( \gamma = \delta \), \( G_R^{\alpha\gamma}G_A^{\delta\beta} \) becomes transition probability

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density $P(t)_{\alpha \rightarrow \gamma}$. Using equipartition theorem, for $t \rightarrow \infty$ summation of $P(t)_{\alpha \rightarrow \gamma}$ over $\alpha$ one can get total probability to stay in dot 1. It is equal to $N_1/(N_1 + N_2)$. That is,

$$\sum_{\alpha} \int dE f_D(t) = \frac{N_1}{N_1 + N_2} = \frac{1}{1 + \xi}.$$ \hspace{1cm} (D.6)

Integration over $E$ and summation over $\alpha$ gives the factor of $AN_1^2 \delta_1$. We identify the normalization constant as $A = 1/\pi$. Note, that we did not use Cooperon part $f_C(t)$ to determine normalization constant $A$. The reason for that is chosen index pairing. After the summation over $\alpha$ Cooperon part contribution is of the order $1/N_1$ compared with the diffusion part. After integration over $E$ with proper normalization $f_D(t)$ becomes:

$$f_D(t) = \delta_{\alpha \beta} \delta_\rho \frac{2}{N_1(1 + \xi)} \left[ \frac{1}{2} + \xi e^{-\sqrt{\xi + \frac{1}{\sqrt{\xi}}} E_{UT}} \right].$$ \hspace{1cm} (D.7)

Then we perform the Fourier transform of the Cooperon part:

$$f_C(\omega) = \frac{2\pi}{N_1^2 \delta_1} \frac{\delta_\rho \delta_\beta}{-i\omega - E_{x_2} + E_U/\sqrt{\xi}} \left[ i\omega - E_{x_2} - \frac{E_U}{\sqrt{\xi}} \right] \frac{1 - \frac{E_{x_2}}{\sqrt{\xi}}}{(i\omega)^2 + \frac{E_{x_1} + E_{x_2}^2}{\sqrt{\xi}(i\omega)^2} + \frac{\sqrt{\xi} E_{x_2} E_U}{(i\omega)^2}} + \left( \sqrt{\xi + \frac{1}{\sqrt{\xi}}} \right) \frac{E_U}{i\omega} \left( \frac{E_U}{i\omega} - 1 \right).$$ \hspace{1cm} (D.8)

The $f_C(\omega)$ is a regular function when $\omega$ approaches limiting values, provided at least one of the crossover energy scales $E_{x_1}$, $E_{x_2}$, or $E_U$ differs from zero.

To make $f_C(\omega)$ more suitable for the Fourier transform we manipulate Eq. (D.8) into:

$$f_C(\omega) = -\frac{2\pi}{N_1^2 \delta_1} \frac{\delta_\rho \delta_\beta}{-i\omega - E_{x_2} - \frac{E_U}{\sqrt{\xi}}} \left[ i\omega - E_{x_2} - \frac{E_U}{\sqrt{\xi}} \right] \left[ (i\omega)^2 - \left( (E_{x_1} + \sqrt{\xi} E_U) + \left( E_{x_2} + \frac{E_U}{\sqrt{\xi}} \right) \right) (i\omega) \right. \hspace{1cm} (D.9)

\left. + \left( E_{x_1} E_{x_2} + \frac{E_{x_1} E_U}{\sqrt{\xi}} + E_{x_2} E_U \sqrt{\xi} + (\sqrt{\xi + \frac{1}{\sqrt{\xi}}} E_U E_U) \right) \right]^{-1}.

and observe that the poles of $f_C(\omega)$ are given by

$$i\omega_{\pm} = \frac{(E_{x_1} + \sqrt{\xi} E_U) + (E_{x_2} + \frac{E_U}{\sqrt{\xi}}) \pm \sqrt{\mathcal{D}}}{2}$$ \hspace{1cm} (D.10)

with $\mathcal{D} = ((E_{x_1} + \sqrt{\xi} E_U) - (E_{x_2} + E_U/\sqrt{\xi}))^2 + 4E_U^2(1 - 4\Gamma^2)$. The parameter $\mathcal{D}$ is always positive and $\omega_{\pm}$ are imaginary complex numbers.

It can be proved that $(E_{x_1} + \sqrt{\xi} E_U) + (E_{x_2} + E_U/\sqrt{\xi}) > \sqrt{\mathcal{D}}$ for all values of parameters, which means that the poles are pure imaginary numbers in lower half complex plane:

$$\omega_{\pm} = -i \frac{\left( E_{x_1} + \sqrt{\xi} E_U \right) + \left( E_{x_2} + \frac{E_U}{\sqrt{\xi}} \right) \pm \sqrt{\mathcal{D}}}{2} = -ia_{\pm}, \quad a_+ > a_- > 0.$$ \hspace{1cm} (D.11)
The function $f_C(\omega)$ now reads:

$$f_C(\omega) = -\delta_{\alpha\delta} \delta_{\gamma\beta} \frac{2\pi}{N_1^2} \frac{i\omega - E_{X_2} - E_{U}}{\sqrt{\xi}} (i\omega - a_-)(i\omega - a_+). \quad (D.12)$$

We perform the Fourier transform and use the normalization factor to obtain:

$$f_C(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega t)f_C(\omega)d\omega dE = \delta_{\alpha\delta} \delta_{\gamma\beta} \frac{2}{N_1} \left[ 1 + \frac{E_{X_2} + E_{U}}{\sqrt{\xi}} \left( e^{-ta_-} - e^{-ta_+} \right) \right]. \quad (D.13)$$

Hence, the full expression for the Fourier transform for the two particle Green’s function in the dot are given by:

$$\langle G_{\alpha\gamma}(t)G_{\delta\beta}^A(t) \rangle_{11} = \delta_{\alpha\delta} \delta_{\gamma\beta} \frac{2}{N_1(1 + \xi)} \left[ \frac{1}{2} + \xi e^{-(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U t} \right] + \delta_{\alpha\delta} \delta_{\gamma\beta} \frac{2}{N_1} \left[ 1 + \frac{E_{X_2} + E_{U}}{a_+ - a_-} \left( e^{-ta_-} - e^{-ta_+} \right) \right] \quad (D.14)$$

$$\langle G_{ij}^R(t)G_{lj}^A(t) \rangle_{22} = \delta_{ij} \delta_{kl} \frac{2\xi}{N_2(1 + \xi)} \left[ \frac{1}{2} + \frac{1}{\xi} e^{-(\sqrt{\xi} + \frac{1}{\sqrt{\xi}})E_U t} \right] + \delta_{il} \delta_{kj} \frac{2}{N_2} \left[ 1 + \frac{E_{X_1} + \sqrt{\xi} E_U}{a_+ - a_-} \left( e^{-ta_-} - e^{-ta_+} \right) \right], \quad (D.15)$$

where $a_{\pm}$ is defined through Eq. (D.11).
Appendix E: CORRELATION OF FOUR WAVE FUNCTIONS

In this appendix we obtain correlation of four wave functions \( \langle \psi_n(\alpha)\psi_n^*(\gamma)\psi_m(\delta)\psi_m^*(\beta) \rangle \) for the system of two coupled dots. This has been obtained in a single dot for the pure ensembles by supersymmetry methods by Mirlin [87], and for the GOE→GUE crossover by Adam et al [72]. We consider ensemble average of the following product:

\[
\langle \left[ G^R_{\alpha\gamma}(E) - G^A_{\alpha\gamma}(E) \right] \left[ G^R_{\delta\beta}(E) - G^A_{\delta\beta}(E) \right] \rangle \approx \]

\[
- \langle G^R_{\alpha\gamma}(E)G^A_{\delta\beta}(E) - G^A_{\alpha\gamma}(E)G^R_{\delta\beta}(E) \rangle = -2\delta_{\alpha\beta}\delta_{\gamma\delta} \text{Re}[D_1] + \delta_{\alpha\delta}\delta_{\gamma\beta} \text{Re}[C_1],
\]

(E.1)

where \( D_1 \) and \( C_1 \) are the diffuson and Cooperon expressions from Eq. (4.20). Here we used the fact that ensemble average of \( G^R G^R \) and \( G^A G^A \) are smaller than \( G^R G^A \) and \( G^A G^R \).

On the other hand, we have:

\[
G^R_{\alpha\gamma}(E) - G^A_{\alpha\gamma}(E) = -2\pi i \sum_n \psi_n(\alpha)\psi^*_n(\gamma)\delta(E - E_n),
\]

(E.2)

and

\[
\langle \left[ G^R_{\alpha\gamma}(E + \omega) - G^A_{\alpha\gamma}(E + \omega) \right] \left[ G^R_{\delta\beta}(E) - G^A_{\delta\beta}(E) \right] \rangle \approx \]

\[
- 4\pi^2 \left( \sum_{n,m} \psi_n(\alpha)\psi^*_n(\gamma)\psi_m(\delta)\psi^*_m(\beta) \right) \delta(E - E_n)\delta(E - E_m).
\]

(E.3)

We know that in the crossover components of eigenvalues and eigenvectors are correlated with each other. This correlation is small already on the distances of a few \( \delta \) and can be neglected in the limit \( \omega \gg \delta \), so Eq. (E.3) can be approximated by:

\[
-4\pi^2 \langle \psi_{\tilde{n}}(\alpha)\psi^*_{\tilde{n}}(\gamma)\psi_{\tilde{m}}(\delta)\psi^*_{\tilde{m}}(\beta) \rangle \left( \sum_n \delta(E - E_n) \right) \left( \sum_m \delta(E - E_m) \right).
\]

where \( \tilde{n} \) and \( \tilde{m} \) mark energy levels close to \( E + \omega \) and \( E \) respectively.

The average of the sum is a density of states \( \rho(E) = \langle \sum_n \delta(E - E_n) \rangle = 1/\delta \). Then, we get

\[
\langle \left[ G^R_{\alpha\gamma}(E + \omega) - G^A_{\alpha\gamma}(E + \omega) \right] \left[ G^R_{\delta\beta}(E) - G^A_{\delta\beta}(E) \right] \rangle \approx \]

\[
- \frac{4\pi^2}{\delta^2} \langle \psi_n(\alpha)\psi^*_n(\gamma)\psi_m(\delta)\psi^*_m(\beta) \rangle.
\]

(E.4)

For the two coupled dots we have:
\[ Re[D_1] = \frac{2\pi}{N_1 \delta_1} \frac{\sqrt{\xi} E_U}{\omega^2 + (\sqrt{\xi} + \frac{1}{\sqrt{\xi}})^2 E_U^2} \quad (E.5) \]

In order to calculate \( Re[C_1] \) from Eq. (4.20) we are going to assume that magnetic field is zero in the first dot and in the hopping region \( (E_{X_1} = E_{\Gamma} = 0) \), and the second dot is in GOE to GUE crossover \( (E_{X_2} \sim \omega) \). Then,

\[ Re[C_1] = \frac{2\pi}{N_1^2 \delta_1} \frac{\sqrt{\xi} E_U \omega^2 + (E_U + \sqrt{\xi} E_{X_2}) E_{U} E_{X_2}}{(\omega^2 - \sqrt{\xi} E_U E_{X_2})^2 + (E_{X_2} + (\sqrt{\xi} + \frac{1}{\sqrt{\xi}}) E_U)^2 \omega^2} \quad (E.6) \]

The relation between the mean level spacing \( \delta \) for the system of coupled dots and the mean level spacing in the first uncoupled dot \( \delta_1 \) is as follows. The averaged density of states in coupled system is going to be the sum of densities of each dot: \( \langle \rho \rangle = \langle \rho_1 \rangle + \langle \rho_2 \rangle \), or \( \delta^{-1} = \delta_1^{-1} + \delta_2^{-1} \). Thus, we conclude that \( \delta = \delta_1/(1 + \xi) \).

Finally, we set Eq. (E.1) and Eq. (E.4) equal and obtain correlation of for the wave functions:

\[ \langle \psi_n(\alpha)\psi_n^*(\gamma)\psi_m(\delta)\psi_m^*(\beta) \rangle = \delta_{\alpha\beta}\delta_{\gamma\delta} \frac{\delta_1}{\pi(1 + \xi)^2 N_1^2} \frac{\sqrt{\xi} E_U}{\omega^2 + (\sqrt{\xi} + \frac{1}{\sqrt{\xi}})^2 E_U^2} \\
+ \delta_{\alpha\delta}\delta_{\gamma\beta} \frac{\delta_1 E_U}{\pi(1 + \xi)^2 N_1^2} \frac{\sqrt{\xi} \omega^2 + (\sqrt{\xi} E_{X_2}^2 + E_U E_{X_2})}{(\omega^2 - \sqrt{\xi} E_U E_{X_2})^2 + (E_{X_2} + (\sqrt{\xi} + \frac{1}{\sqrt{\xi}}) E_U)^2 \omega^2} \quad (E.7) \]
To verify the expressions we have obtained for the averaged Green’s functions we use a sum rule.

The pair annihilation (creation) operator $T(T^\dagger)$ in the basis of two uncoupled dots is a sum of two terms belonging to each dot:

$$T = \sum_{\alpha_0} c_{\alpha_0,\downarrow} c_{\alpha_0,\uparrow} + \sum_{i_0} c_{i_0,\downarrow} c_{i_0,\uparrow},$$

$$T^\dagger = \sum_{\alpha_0} c_{\alpha_0}^\dagger c_{\alpha_0,\dagger} + \sum_{i_0} c_{i_0}^\dagger c_{i_0,\dagger}. \quad (F.1)$$

Greek indices go over the states in the first dot, and Latin indices go over the states in the second dot. The subindex 0 denotes the basis of two uncoupled dots.

Our first goal is to calculate the commutator $[T^\dagger, T]$. As operators from different dots anticommute, one gets:

$$[T^\dagger, T] = \sum_{\alpha_0, \beta_0} [c_{\alpha_0,\dagger} c_{\alpha_0,\downarrow}, c_{\beta_0,\downarrow} c_{\beta_0,\dagger}] + \sum_{i_0, j_0} [c_{i_0,\dagger} c_{i_0,\downarrow}, c_{j_0,\downarrow} c_{j_0,\dagger}] = \hat{N}_1 e + \hat{N}_2 e - N_1 - N_2, \quad (F.2)$$

where $\hat{N}_1 e, \hat{N}_2 e$ are the operators of total number of electrons in dot 1 and dot 2, and $N_1, N_2$ are the total number of levels in dot 1 and dot 2.

The expectation value of $[T^\dagger, T]$ in ground state at zero temperature is:

$$\langle [T^\dagger, T] \rangle = \langle \Omega | [T^\dagger, T] | \Omega \rangle = N_e - N. \quad (F.3)$$

$N_e$ and $N$ are the total number of electrons and levels in both dots. This number is conserved when going to another basis.

Now we choose the basis of the system of coupled dots. In this basis $c_{\alpha_0, s} = \sum_m \psi_m(\alpha_0) c_{m, s},$ and $c_{i_0, s} = \sum_m \psi_m(i_0) c_{m, s},$ where $c_{m, s}$ is annihilation operator in new basis. Using this transformation, we rewrite pair destruction operator as follows:

$$T = \sum_{\alpha_0} c_{\alpha_0,\dagger} c_{\alpha_0,\downarrow} + \sum_{i_0} c_{i_0,\dagger} c_{i_0,\downarrow} = \sum_{m_{1,2}} D_{m_{1,2}} c_{m_{1,\dagger}} c_{m_{2,\dagger}}, \quad (F.4)$$

where $D_{m_{1,2}}$ is defined by the following expression:

$$D_{m_{1,2}} = \sum_{m_{1,2}} \left( \sum_{\alpha_0} \psi_{m_1}(\alpha_0) \psi_{m_2}(\alpha_0) + \sum_{i_0} \psi_{m_1}(i_0) \psi_{m_2}(i_0) \right) c_{m_{1,\dagger}} c_{m_{2,\dagger}}$$

$$= \sum_{p_0} \psi_{m_1}(p_0) \psi_{m_2}(p_0). \quad (F.5)$$
The index \( p_0 \) runs over all states in the first and second dots for the basis of uncoupled dots.

In the new basis the \( T, T^\dagger \) operators look like this:

\[
T = \sum_{m_1, m_2} D_{m_1 m_2} c_{m_1, \downarrow} c_{m_2, \uparrow},
\]

\[
T^\dagger = \sum_{m_1, m_2} D_{m_1 m_2}^* c_{m_2, \uparrow}^\dagger c_{m_1, \downarrow}^\dagger.
\] (F.6)

Consequently, in the new basis,

\[
[T^\dagger, T] = \sum_{m_1, m_2, m_3, m_4} D_{m_1 m_2}^* D_{m_3 m_4} [c_{m_2, \uparrow}^\dagger c_{m_1, \downarrow}^\dagger, c_{m_3, \downarrow} c_{m_4, \uparrow}]
\]

\[
= \sum_{m_2, m_4} \left( \sum_{m_1} D_{m_1 m_2}^* D_{m_1 m_4} \right) c_{m_2, \uparrow}^\dagger c_{m_4, \uparrow}^\dagger - \sum_{m_1, m_3} \left( \sum_{m_2} D_{m_1 m_2}^* D_{m_3 m_2} \right) c_{m_3, \downarrow} c_{m_1, \uparrow}^\dagger. \] (F.7)

One can go further and use completeness condition \( \sum_m \psi_m^*(p_0) \psi_m(n_0) = \delta_{p_0 n_0} \) to show that in the new basis the value of commutator is \( \hat{N} - \hat{N} \). Our next goal, however, is to take the disorder average of the vacuum expectation value and to prove the invariance of \( [T^\dagger, T] \).

Taking into account that \( \langle \Omega | c_{m_1, \downarrow}^\dagger c_{m_2, \uparrow} | \Omega \rangle = \delta_{m_1 m_2} \Theta(\mu - E_{m_1}) \) and \( \langle \Omega | c_{m_2, \uparrow}^\dagger c_{m_1, \downarrow}^\dagger | \Omega \rangle = \delta_{m_1 m_2}(1 - \Theta(\mu - E_{m_1})) \), the ground state expectation value for the commutator is:

\[
\langle [T^\dagger, T] \rangle = \langle \Omega | [T^\dagger, T] | \Omega \rangle = \sum_{m_1, m_2} |D_{m_1 m_2}|^2 [2\Theta(\mu - E_{m_1}) - 1], \] (F.8)

where \( \Theta(x) \) is a step function.

Averaging over disorder gives:

\[
\langle [T^\dagger, T] \rangle = 2 \sum_{m_1, m_2} \Theta(\mu - E_{m_1}) \langle |D_{m_1 m_2}|^2 \rangle - \sum_{m_1, m_2} \langle |D_{m_1 m_2}|^2 \rangle \] (F.9)

Converting this into integral, we get:

\[
\langle [T^\dagger, T] \rangle = 2 \int_{-W}^{\mu} \int_{-W}^{W} dE_1 dE_2 \rho(E_1) \rho(E_2) \langle |D(E_1, E_2)|^2 \rangle
\]

\[
- \int_{-W}^{W} \int_{-W}^{W} dE_1 dE_2 \rho(E_1) \rho(E_2) \langle |D(E_1, E_2)|^2 \rangle \] (F.10)

The density of states \( \rho(E) \) is the Winger’s semicircle law:

\[
\rho(E) = \frac{2N}{\pi W_2} \sqrt{W_2^2 - E^2},
\]

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where $2W$ is the bandwidth and $N$ is the number of states in the system.

To proceed we need to find the ensemble average of the following object:

$$\langle |D_{m_{1}m_{2}}|^{2} \rangle = \sum_{p_{0},n_{0}} \langle \psi_{m_{1}}^{*}(p_{0})\psi_{m_{2}}^{*}(p_{0})\psi_{m_{1}}(n_{0})\psi_{m_{2}}(n_{0}) \rangle. \quad (F.11)$$

Using results of appendix E one can obtain expression for the correlation of four wave functions in the form:

$$\langle \psi_{m_{1}}^{*}(p_{0})\psi_{m_{2}}^{*}(p_{0})\psi_{m_{1}}(n_{0})\psi_{m_{2}}(n_{0}) \rangle = \frac{1}{2\pi^{2}\rho(E_{1})\rho(E_{2})}Re\left[ \sum_{p_{0}n_{0}} \langle G_{n_{0}p_{0}}^{R}(E_{2})G_{n_{0}p_{0}}^{A}(E_{1}) \rangle \right] - \sum_{p_{0}n_{0}} \langle G_{n_{0}p_{0}}^{R}(E_{2})G_{n_{0}p_{0}}^{R}(E_{1}) \rangle]. \quad (F.12)$$

Note, that to get the correct answer for the sum rule one should keep $\langle G^{R}G^{R} \rangle$ term as well. Summation in Eq. (F.12) is performed over the states in both dots.

When the dots have equal mean level spacing $\delta_{1} = \delta_{2} = \delta_{0}$, one particle Green’s function can be found exactly from the system (C.6) without approximation in $U$:

$$\langle G_{p_{0}p_{0}'}^{R}(E) \rangle = \frac{\delta_{p_{0}p_{0}'}}{\frac{E}{2} + \frac{1}{2}\sqrt{W^{2} - E^{2}}} = \frac{2i}{W}e^{i\phi}, \quad (F.13)$$

$$\langle G_{p_{0}p_{0}'}^{A}(E) \rangle = \frac{\delta_{p_{0}p_{0}'}}{\frac{E}{2} - \frac{1}{2}\sqrt{W^{2} - E^{2}}} = \frac{2i}{W}e^{-i\phi},$$

where $W = 2N_{0}\delta_{0}\sqrt{1 + U}/\pi$ is the half bandwidth and $\sin \phi = E/W$. Here both indices $p_{0}$ and $p_{0}'$ belong either to the first or to the second dot.

The sum in Eq. (F.12) can be broken into four sums, when the indices $p_{0},n_{0}$ belong either to the first dot, or to the second dot, or one of the indices go over the states in the first dot, and the other one goes over the states in the second dot.

For example, for $\langle G^{R}G^{A} \rangle$ part we have the following expression:

$$\sum_{p_{0}n_{0}} \langle G_{n_{0}p_{0}}^{R}(E_{2})G_{n_{0}p_{0}}^{A}(E_{1}) \rangle =$$

$$N_{0}\left(\frac{2}{W}\right)^{2}(1 + U)\frac{(1 + U)e^{-i\phi_{21}} - \zeta}{[(1 + U)e^{-i\phi_{21}} - 1][(1 + U)e^{-i\phi_{21}} - \zeta] - U^{2}}$$

$$+ N_{0}\left(\frac{2}{W}\right)^{2}(1 + U)\frac{(1 + U)e^{-i\phi_{21}} - 1}{[(1 + U)e^{-i\phi_{21}} - 1][(1 + U)e^{-i\phi_{21}} - \zeta] - U^{2}}$$

$$+ 2N_{0}\left(\frac{2}{W}\right)^{2}(1 + U)\frac{U}{[(1 + U)e^{-i\phi_{21}} - 1][(1 + U)e^{-i\phi_{21}} - \zeta] - U^{2}}. \quad (F.14)$$

Here $\phi_{21} = \phi_{2} - \phi_{1}$, and $\zeta = (1 - X_{2}^{2})/(1 + X_{2}^{2})$. 

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The first term in Eq. (F.14) is the contribution of \( \langle G^R \rangle \langle G^A \rangle \) plus the Cooperon part of two particle Green’s function in the first dot. The second term describes contribution of free term and Cooperon part in the second dot. The last term is a sum of transition parts from dot 1 to dot 2 and vice versa. It appears that these transition terms are equal, which explains coefficient 2 in front of the last term in Eq. (F.14).

Summation of the \( \langle G^R G^R \rangle \) gives similar result:

\[
\sum_{p_0 n_0} \langle G^R_{n_0 p_0} (E_2) G^R_{n_0 p_0} (E_1) \rangle = -N_0 \left( \frac{2}{W} \right)^2 (1 + U) e^{-i \psi_{21}} + \zeta \frac{(1 + U)e^{-i \psi_{21}} + 1}{[(1 + U)e^{-i \psi_{21}} + 1][1 + U]e^{-i \psi_{21}} + \zeta} - \frac{U}{2},
\]

where \( \psi_{21} = \phi_2 + \phi_1 \).

In principle, there should be terms corresponding to diffusons in dot 1 and dot 2. However, these terms after summation over \( p_0, n_0 \) are \( 1/N_0 \) smaller than the others and in the large \( N_0 \) limit can be neglected.

Although one can use Eq. (F.10) to verify the sum rule, it is more convenient to work with derivative of Eq. (F.10) over \( \mu \) at \( \mu = 0 \).

It gives:

\[
\frac{\partial}{\partial \mu} \langle [T^{\dagger}, T] \rangle_{\mu=0} = 2 \rho(0) \int_{-W}^{W} dE_2 \rho(E_2) \langle |D(E_1 = 0, E_2)|^2 \rangle. \tag{F.16}
\]

On the other hand, this expression should be equal to:

\[
\frac{\partial}{\partial \mu} (N_e - N) = \frac{\partial}{\partial \mu} (2 \frac{N}{2} + 2 \int_0^\mu \rho(E) dE - N) = 2 \rho(\mu). \tag{F.17}
\]

Comparison of Eq. (F.16) and (F.17) at \( \mu = 0 \) results in the following condition for the sum rule:

\[
\int_{-W}^{W} dE_2 \rho(E_2) \langle |D(E_1 = 0, E_2)|^2 \rangle = 1. \tag{F.18}
\]

The integral in Eq. (F.18) was computed numerically and matched the unity with high accuracy.
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Fall ’01–Spring ’03  PHY 213, General Physics II, David Harman, Ph.D. Taught 3 labs/week, maintained regular office hours, graded lab reports

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Summer ’07  PHY 232, General University Physics II, Valdis Zeps, Ph.D. Taught in a lecture setting, maintained office hours, graded homework and exams

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PUBLICATIONS


Zelyak, O., Murthy G.  *Diamagnetic persistent currents for electrons in ballistic billiards subject to a point flux.* Phys. Rev. B 78, 125305 (2008); (also arxiv:0806.0826)

Zelyak, O., Murthy G.  *Spin-Orbit Coupling, Random Matrix Crossover, and Quantum Critical Fluctuations in Quantum Dots (proposed title), in preparation*

PROFESSIONAL PRESENTATIONS

Zelyak, O. (2007)  *Interactions, Fluctuation Magnetic and Crossover in the System of Two Coupled Quantum Dots.* Presented to faculty and students of the University of Kentucky, Department of Physics.
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**RESEARCH EXPERIENCE**

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2004–present Symmetry Crossovers in Mesoscopic Systems with Fermi-Liquid Interactions:
— The study of the quantum fluctuations in the system of coupled quantum dots in regimes of softly broken time reversal symmetry and spin-rotational symmetry.
— Exploring zero dimensional quantum phase transitions.
— Correspondence between the energy scale of the quantum fluctuations and the scale of the soft symmetry breaking.

Persistent Currents in Quantum Chaotic Billiards, Non-Interacting Mesoscopic Systems:
— The study of the appearance and behavior of persistent currents in quantum billiards threaded by magnetic flux in the regime of hard chaos.

2003–2004 Study of the Mechanical Properties of Single & Multi-Walled Carbon NanoTubes (S & MWCNT) and Bulk Amorphous Carbon. This work was supported by NASA grant # NCC2-5541:
Techniques used: Molecular dynamics simulations with a Brenner reactive potential (used to simulate bonding in hydrocarbon molecules). The samples were modeled with varying boundary conditions and thermal scenarios
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1993–1998  
Study of phase transitions in models of quantum crystals with an anharmonic interaction potential.  
Techniques used: *Functional analysis, Combinatorics, Cluster expansion, Probability theory.*

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Quality Assurance Department for the software developer MacKiev, a renowned company that specializes in adopting & maintaining Apple technologies in their products.  
Software packages tested:  
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  — *Worldbook* (encyclopedia software)

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Scientific Software Analysis / Tester, details above  
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CNC Machinist  
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Proficient with Mathematica, with both symbolical and numerical calculations.
- Creation of notebooks for finding roots of transcendental functions and for obtaining the energy spectra of Quantum Chaotic Hamiltonians.

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