SOLVING LINEAR EQUATIONS: A COMPARISON OF CONCRETE AND VIRTUAL MANIPULATIVES IN MIDDLE SCHOOL MATHEMATICS

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SOLVING LINEAR EQUATIONS:
A COMPARISON OF CONCRETE AND VIRTUAL MANIPULATIVES IN MIDDLE
SCHOOL MATHEMATICS

___________________________________
DISSERTATION
___________________________________

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Education in the
College of Education
at the University of Kentucky

By

Robin Lee Magruder

Lexington, KY

Director: Dr. Margaret Mohr-Schroeder, Associate Professor of Mathematics Education

Lexington, KY

2012

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ABSTRACT OF DISSERTATION

SOLVING LINEAR EQUATIONS:
A COMPARISON OF CONCRETE AND VIRTUAL MANIPULATIVES IN MIDDLE SCHOOL MATHEMATICS

The purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative.

Qualitative research methods such as observation, teacher interviews, and student focus group interviews were employed. Quantitative data analysis techniques were used to analyze pretest and posttest data of middle school students (n=76). ANCOVA, analysis of covariance, uncovered statistically significant differences in favor of the control group. Differences in posttest scores, triangulated with qualitative data, suggested that concrete and virtual manipulatives require more classroom time because of administrative issues and because of time needed to learn how to operate the manipulative in addition to necessary time to learn mathematics content. Teachers must allow students enough time to develop conceptual understanding linking the manipulatives to the mathematics represented. Additionally, a discussion of unique benefits and drawbacks of each manipulative sheds light on the use of manipulatives in middle school mathematics.

KEYWORDS: Solving Equations, Virtual Manipulatives, Concrete Manipulatives, Middle School Mathematics, Mixed Methods Research

Robin L. Magruder

November 19, 2012
SOLVING LINEAR EQUATIONS:
A COMPARISON OF CONCRETE AND VIRTUAL MANIPULATIVES IN MIDDLE SCHOOL MATHEMATICS

By

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November 19, 2012
I would like to dedicate this work to my family. My parents love me very much. Jamie, Tyler, Evan, and Lindsey, thank you for your love and support. Thank you for the love given to me by my deceased grandparents, I miss you all every day. Thank you Jesus for the many blessings bestowed upon me.
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CHAPTER I
INTRODUCTION

Solving linear equations is an important algebraic concept. According to the Common Core State Standards for Mathematics (CCSSM) students must be able to solve equations while understanding the process, justifying, and explaining the steps (CCSSO, 2010). The National Council of Teachers of Mathematics (NCTM) places a heavy emphasis on conceptual understanding of solving equations within their standards as well (NCTM, 2000). Algebra tasks are often difficult for students; the transition from concrete mathematics to abstract concepts is partially responsible for this difficulty (Kilpatrick & Izsak, 2008). Solving equations is a particularly important concept in algebra and one that causes confusion for students (Cai & Moyer, 2008).

Researchers advocate the use of concrete and virtual manipulatives in mathematics education as a means of bridging the transition from concrete to abstract mathematics (Boggan, Harper, & Whitmire, 2010; Caglayan & Olive, 2010; Sherman & Bisanz, 2009). According to Puchner, Taylor, O’Donnell, & Fick (2010), “Manipulatives are concrete tools used to create an external representation of a mathematical idea and include items such as unifix cubes and base 10 blocks” (p. 314). Concrete manipulatives may be purchased or created by teachers and students. Virtual manipulatives are applets, or computer programs typically available on websites that students manipulate to better understand a mathematical concept. Virtual manipulatives are often similar to their physical counterparts (Moyer, 2002; NLVM, 2010; Puchner et al., 2010). Both concrete and virtual manipulatives provide unique benefits and challenges in the mathematics classroom.

Although research exhibits mixed results, overall, concrete and virtual manipulatives are proven methods for learning mathematics (Moyer, 2002). Both concrete and virtual manipulatives address the three primary difficulties students face as they solve equations:

1. symbolic understanding (Borenson & Barber, 2008);
2. the meaning of the equal sign (Caglayan & Olive, 2010); and
3. a reliance on procedural knowledge without conceptual understanding (Sherman & Bisanz, 2009).
The general purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative.

**Statement of the Problem**

Students face many challenges as they study algebra. One important area of study within the subject of algebra is solving linear equations. Within the topic of solving linear equations, students struggle to develop symbolic understanding (Kilpatrick & Izsak, 2008; Poon & Leung, 2010), to form an accurate meaning of the equal sign (Knuth, Stephens, McNeil, & Alibali, 2006), and to balance conceptual and procedural knowledge (Capraro & Joffrion, 2006; Siegler, 2003; Star, 2005). Recommendations put forth by the NCTM and standards presented by **CCSSM** include solving equations as important components. Additionally, both organizations advocate modeling with mathematics; one such model is manipulatives. There is a long history of using concrete manipulatives to model mathematics dating back to prehistoric times (Boggan et al., 2010). In contrast, the use of virtual manipulatives in the mathematics classroom is a recent innovation resulting from advances in technology (NLVM, 2010). Researchers have found common and unique benefits and drawbacks to both types of manipulatives. Prior research presented mixed results as to the effectiveness of using manipulatives in elementary school mathematics. But few studies have compared the effectiveness of concrete and virtual manipulatives in the middle school setting. A comparison of student achievement related to solving linear equations with concrete and virtual manipulatives in the context of middle school mathematics is needed to enhance this timely research topic.

**Purpose of the Study**

The purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative.
Research Questions

This quasi-experimental mixed methods research study investigated and compared student achievement as a result of using concrete and virtual manipulatives in two similar middle school mathematics classes within a rural public school compared to student achievement of a control group within the same school not using manipulatives. More specifically, the following research questions were addressed:

1. What differences, if any, exist in student achievement as a result of using concrete or virtual manipulatives as middle school students use them to solve linear equations compared to a control group using learning methods without manipulatives?

2. What are the unique benefits and drawbacks associated with each type of manipulative?

Significance of the Study

This research project mirrors Lesh and Lovitts’ (2000) research type considered “projects that focus on the development of curricular material,” (p. 55). According to the authors, researchers need to go beyond making statements that certain curriculum works, researchers must question why and how curriculum is effective. The authors also suggested that researchers should analyze curriculum for positive and negative aspects. Although concrete manipulatives have been researched in depth, there are areas that would benefit from further research such as the use of manipulatives in middle and high school. The use of technology is increasing dramatically in school and home settings (Schenker, Kratcoski, Lin, Swan, & van ‘t Hooft, 2007). This increased availability of technology makes virtual manipulatives a timely topic of research, but there is less published research related to virtual manipulatives as compared to concrete manipulatives. At least two researchers (Polly, 2011; Suh & Moyer, 2007) have investigated the use of concrete or virtual manipulatives for solving equations in elementary school settings. However, there is little research on these manipulatives in the middle school setting. This mixed methods research study contributed to instructional design for current middle school teachers.

Theoretical Framework

An important aspect of solving equations involves having both procedural and conceptual understanding of the abstract (Capraro & Joffrion, 2006; Star, 2005). Recent
Math Wars, in which researchers and practitioners pose traditional mathematics against reform mathematics, display the competition between the two elements (Reys, 2001; Schoenfeld, 2004). Star (2005) provided a definition for procedural understanding, focusing on understanding symbols and rules. Star (2005) additionally defined conceptual understanding as making connections and creating networks within information.

Star (2005) emphasized the importance of procedural knowledge and specifically addressed solving equations as an example. According to Star (2005), there are only a few standard procedures necessary for solving equations, “adding or subtracting from both sides, combining like terms, distributing or factoring, and multiplying or dividing both sides” (p. 409). Flexibility, according to Star, is the ability to use nonstandard procedures to solve an equation in the most effective way, and is a sign of deep procedural understanding.

Not all researchers agree with Star (2005); for example, Kilpatrick, Swafford, and Findell (2001) emphasized that procedural skills must be accompanied by conceptual understanding. As students learn mathematics, they need to do more than just compute; they need to understand the meaning and purpose of computations (Sriraman & Lesh, 2007). Siegler (2003) discussed pitfalls in mathematics learning that develop as students focus on procedures, rather than concepts. “Even students who do well in algebra classes often do so by treating the equations as exercises in symbol manipulation, without any connection to real-world contexts” (Siegler, 2003, p. 222).

Rittle-Johnson and Alibali (1999) conducted research to determine how conceptual and procedural understanding affected each other among fourth and fifth grade mathematics students (n = 89). Rittle-Johnson and Alibali (1999) described increases in procedural understanding as resulting from increased conceptual understanding. In contrast, Rittle-Johnson and Alibali (1999) did not attribute improvements in conceptual understanding to increases in procedural understanding. As an example, Rittle-Johnson and Alibali (1999) described students who could do arithmetic procedures such as multidigit subtraction correctly, but did not understand conceptual ideas of mathematics such as place value. In contrast, they described students who used conceptual understanding of place value to correctly conduct mathematical procedures such as multidigit subtraction without prior procedural knowledge.
Capraro and Joffrion (2006) echoed Rittle-Johnson and Alibali’s (1999) claims in describing results of their quantitative study of seventh and eighth grade mathematics students \((n = 668)\). More successful students were those with a higher level of conceptual knowledge. Students with conceptual knowledge were *flexible* in their problem solving strategies and methods.

Constructivists advocate for active learning, which allows students to build their own conceptual understanding (Ernest, 1996). “It is clear that learning is not about accumulating random information, memorizing it, and then repeating it on some exam; learning is about understanding and applying concepts, constructing meaning, and thinking about ideas” (Gordon, 2009, p. 743). Application of concepts and construction of meaning evidence conceptual understanding on the part of students.

According to the Cognitive Science Society, cognitive scientists aspire to understand the nature of the human mind (“Cognitive Science,” 2011). Many cognitive scientists operate under the constructivist paradigm, which advocates student strategy selection. Cognitive scientists expect students to develop most efficient strategies and increase understanding through this selection process (Hatano, 1996; Siegler, 2003). Thinking processes advocated by cognitive scientists further evidence conceptual understanding on the part of students.

Researchers and educators who subscribe to both the constructivist paradigm and cognitive science theories advocate the use of manipulatives in mathematics education. The constructivist paradigm supports the use of manipulatives because learning is active and students functioning at concrete developmental levels benefit from the concrete aspect of manipulatives (Uttal, Scudder, & Deloache, 1997). Cognitive science theories support the use of manipulatives as they contribute to the creation and application of prior knowledge (Hiebert & Carpenter, 1992). Procedural and conceptual understanding are both valuable as students learn mathematics. Conceptual understanding is an important component of constructivism and cognitive science; students must make connections and develop understandings, not just memorize a set of facts or procedures. As Star (2005) suggested, there are a limited number of procedures necessary for solving equations. For this reason, conceptual understanding may be more important as students solve linear equations. If students understand concepts such as the meaning of the equal sign, inverse
operations, and the role of constants and coefficients, they may be able to use this conceptual understanding to accurately implement correct procedures while solving equations.

**Definitions of Terms**

The following definitions are provided for terms having special applications to this study. These terms and definitions will be extensively reviewed in Chapter 2 and discussed in Chapter 3.

*Cognitive Science* – The interdisciplinary study of how the mind works and how students learn (Thompson, 1996).

*Conceptual Understanding* – Mathematical teaching and mental constructs that focus on concepts, problem solving, and making connections (Star, 2005).

*Concrete Manipulatives* – Physical items, such as chips, blocks, or geoboards that students physically manipulate to better represent a mathematical concept (Moyer, 2002).

*Concrete Group* – A treatment group of students learning to solve linear equations using algebra tiles.

*Control Group* – Students learning to solve linear equations using learning methods that do not include manipulatives. Concepts such as the meaning of the equal sign, inverse operations, and roles of constants, variables, and coefficients were emphasized.

*Constructivism* – The building of knowledge from previous knowledge structures (Sriraman & Lesh, 2007).

*Experimental Groups* – Two treatment groups learning to solve linear equations with the assistance of algebra tiles and virtual manipulatives created by the National Library of Virtual Manipulatives (NLVM) respectively.

*Linear Equations with One Variable* According to Malloy, Molix-Bailey, Price, and Willard (2008), “A linear equation is an equation in which variables appear as separate terms and neither variable contains an exponent other than one” (p. 355). For the purposes of this study, linear equations containing only one variable and no equations with denominators were considered. For example, 

\[ x + 5 = 8; \ 2x = 16; \ 2x + 5 = 12; \ 2x + 4 = 3x - 6. \]

*Manipulatives* – Physical or virtual objects used by students to represent components of mathematical concepts (Moyer, 2002).
Procedural Understanding – Mathematical teaching and mental constructs that focus on algorithms, rules and procedures (Star, 2007).

Virtual Group – A treatment group of students learning to solve linear equations using virtual manipulatives created by NLVM.

Virtual Manipulatives – Applets, or computer programs typically available on websites that students manipulate to better understand a mathematical concept; virtual manipulatives are often similar to their physical counterparts (Moyer-Packenham, 2010).

Assumptions

1. The three classes participating in the study were provided with similar amounts of learning time, and similar content.
2. The three classes participating in the study were similar in academic achievement prior to the study.

Delimitations

1. It is not possible to test all middle school mathematics students. Thus, the study was limited to the number of students and teachers available to the researcher.
2. The students were confined to predetermined classes.
3. Study was limited to one unit of study (10 instructional days).

Organization of the Study

This research study is organized into five chapters, a bibliography, and appendices as follows:

Chapter I: Introduction
Chapter II: Review of Literature
Chapter III: Methodology
Chapter IV: Data Analysis and Results
Chapter V: Discussions, Conclusions, and Implications

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CHAPTER II

REVIEW OF LITERATURE

Chapter II contains a review of literature on learning theories, algebraic thinking, solving equations, symbolic understanding, meaning of the equal sign, conceptual, and procedural knowledge. It also provides a review of literature related to manipulatives, including concrete and virtual manipulatives. This review of literature provides the foundation for the study.

The researcher conducted an extensive literature review of both virtual and concrete manipulatives. Additionally, the researcher explored theoretical perspectives of mathematics education, algebra, and particularly solving equations which will provide a framework or lens for looking at manipulatives (Figure 2.1). The body of research on concrete manipulatives is much deeper than virtual manipulatives because of the long history of concrete manipulatives as compared to the recent integration of technology in the classroom. The primary sources of information within this review of literature are articles from peer-reviewed journals published within the last ten years. These articles include both qualitative and quantitative data. Additionally, the researcher cited educational handbooks edited by well-respected authors that are leading experts in the field of mathematics education. Occasionally, especially within the review of virtual manipulatives, the researcher included articles written for and by practitioners published in journals by the National Council of Teachers of Mathematics (NCTM). The researcher additionally discussed a mathematics methods textbook and a few books written primarily for a lay audience. Although these books are not peer reviewed, they provided insight into teaching with manipulatives, or the state of algebra education in the United States. A few Internet web sites have been integrated into the review of literature. Finally, the researcher included information from a doctoral dissertation discussing virtual manipulatives.

Most cited authors maintained a postpositivist paradigm, convinced of the accuracy of the results of their research and claims they make based on quantitative research methods. Other researchers represent a constructivist paradigm, theorizing that reality and meaning are derived by individuals in social contexts. These authors primarily relied on qualitative research methods. Two authors, Spielhagen (2011), and Moses and Cobb
(2001) represent the transformative paradigm because they sought to change the way algebra is taught in the United States.

Within the last twenty-five years, qualitative research has been considered an acceptable form of mathematics research and has provided the preponderance of evidence (Simon, 2004). With that in mind, the researcher consulted Patton (1990) who provided a thorough description of high quality and credible qualitative research. The researcher used guidelines from his handbook while selecting qualitative articles. Qualitative researchers must use rigorous techniques to ensure credibility of results and triangulate data. All articles within this review included at least one of Patton’s suggested methods for data triangulation. Schoenfeld (2002) provided standards for mathematics research, stating that the following criteria can be used to evaluate research: “descriptive power, explanatory power, scope, predicative power, rigor and specificity, falsifiability, replicability, generality, trustworthiness, and triangulation” (p. 456). The researcher considered these criteria while selecting each research article within this review of literature.

_Figure 2.1. Solving equations framework._

Within the last twenty-five years, qualitative research has been considered an acceptable form of mathematics research and has provided the preponderance of evidence (Simon, 2004). With that in mind, the researcher consulted Patton (1990) who provided a thorough description of high quality and credible qualitative research. The researcher used guidelines from his handbook while selecting qualitative articles. Qualitative researchers must use rigorous techniques to ensure credibility of results and triangulate data. All articles within this review included at least one of Patton’s suggested methods for data triangulation. Schoenfeld (2002) provided standards for mathematics research, stating that the following criteria can be used to evaluate research: “descriptive power, explanatory power, scope, predicative power, rigor and specificity, falsifiability, replicability, generality, trustworthiness, and triangulation” (p. 456). The researcher considered these criteria while selecting each research article within this review of literature.
Constructivism

A basic tenant of constructivism is active learning. Zoltan Dienes stated, “One of the first things we should do in trying to teach a learner any mathematics is to think of different concrete situations with a common essence. Then . . . children will learn by acting on a situation” (Sriraman & Lesh, 2007, p. 61). This statement emphasized two salient beliefs of many constructivists regarding mathematics learners; students need to learn by doing and they need to understand mathematics in terms of real life (Gordon, 2009).

American psychologist and philosopher William James (1899) suggested optimizing innate instincts toward movement and action by encouraging students to embrace active learning as they are developmentally appropriate. Ernest (1996) described the learning process as active and recursive; activities undertaken by a learner become previous knowledge on which new knowledge is constructed. Ernest stated, “Thus learning is not just a passive absorption of information; rather, it is more interactive” (p. 338). Active learning occurs as a result of having building blocks, puzzles, and counters in early elementary school and having geoboards, play money, and integer chips in middle school classrooms.

Interested in intellectual development, Jean Piaget, one of the most prolific developmental psychologists in the twentieth century, theorized that learning occurred inside the mind, based on external experiences. Implications of Piaget’s theory involve the use of representations, such as manipulatives to assist meaning-making (Wood, Smith, & Grossniklaus, 2001). Manipulatives engage students in active, participatory learning. As students work with manipulatives and think about the relationship between concepts and manipulatives, meaning is created in the minds of learners.

Many constructivists theorize that learners have different needs at different stages of development. According to Piaget, the preoperational stage occurs between ages two and seven. During this developmental state, language, memory, and imaginations develop (Wadsworth, 1996). The concrete operations stage typically includes children approximately seven through eleven years of age; children in this stage benefit from manipulating symbols and concrete objects (Wood et al., 2001). Finally, the formal operations stage as occurs at approximately twelve years of age through adulthood (Wadsworth, 1996). In this formal operations stage, thinkers can successfully use symbols and operate with abstract concepts because their prior experiences with concrete objects
helped them develop schemas. Although students operating in the concrete operations stage benefit the most from the use of manipulatives, the concrete representations assist students in overcoming developmental barriers at all stages of development (Wood et al., 2001).

Jerome Bruner, a social psychologist in the late-twentieth century with deep interest in education, described learning as a process requiring three simultaneous processes, acquisition, transformation, and evaluation. Often the acquisition process requires students to understand something contrary to previous understanding or conception. For example, while solving equations, students must acquire understanding of the equal sign as a symbol of balance. Transformation is the process of analyzing knowledge and making it fit new tasks. For transformation to take place, students need move from the incorrect understanding of the equal sign as a statement of the answer and move toward a correct understanding of the equal sign. The final aspect of learning, according to Bruner (1960) is evaluation. Students must self-reflect on generalizations made during the other two processes. Once students realize the equal sign means equality, it helps them understand that while solving equations, operations must occur on both sides simultaneously. Bruner (1960) summarized the process of learning as “getting facts, manipulating them, and checking one’s ideas” (p. 48).

Seymour Papert (1993), a mathematician, computer programmer, and educator in the late twentieth century argued that concrete thinking can be just as deep as abstract thoughts. Meaning, such as understanding of mathematics concepts, is derived from physical interactions with objects and technology, such as manipulatives (Kafai, 2002). Papert encouraged action, which he believed led to learning (Mason & Johnston-Wilder, 2004). Manipulatives, including concrete and virtual formats, help students create conceptual understanding by developing meaning. Papert described these tools as “objects to think with” (Kafai, 2002, p. 39).

Constructivists emphasize active creation of knowledge, the use of representations, and development of schemas as students learn. Cognitive scientists similarly emphasize the role of active learning and representations. Cognitive scientists additionally acknowledge the important role of prior knowledge in learning.
Cognitive Science

Active learning, prior knowledge, and efficiency are tenants of cognitive science theories. According to Hatano (1996), students learn best when they are learning actively. During the learning process, students must restructure and reorganize information. Hatano (1996) and Hiebert and Carpenter (1992) emphasized the salient role of prior knowledge stating that students learn well when they are able to make a connection to prior information. Mostly working from a postpositivist paradigm, cognitive scientists assume that reality is objective and learners’ reality is discoverable, even if it is different from individual to individual (Siegler, 2003). Under this platform, an assumed benefit of manipulatives is that students are actively creating knowledge; eventually learners take the concrete knowledge and make a transfer to abstract concepts. In this way, memories of using manipulatives become the prior knowledge that students use to make connections and deepen understanding. Hatano (1996) explained, “Knowledge is acquired by construction; it is not acquired by transmission alone” (p. 198). Additionally, conceptual knowledge develops as students break procedures into steps; manipulatives allow students to focus on the steps of the mathematical procedure.

Cognitive scientists emphasize the role of imagery, connections, and representations in mathematics. Thompson (1996) emphasized the role of imagery in constructing knowledge. These concept images could be in the form of visual representations, experiences, or mental pictures. Although William James preceded cognitive scientists, he urged teachers to help students make connections and associations, suggesting that teachers use multiple cues, such as a variety of representations, in this effort (1899). Manipulatives are tools that teachers can use to impose meaning on representations. Symbolic representations within the manipulatives can merge with nonsymbolic representations (constants, variables, and coefficients, for example) and form deep meanings for students (Caglayan & Olive, 2010; Lee & Chin, 2010; McNeil & Uttal, 2009).

Researchers and educators who subscribe to both the constructivist paradigm and cognitive science theories advocate the use of manipulatives in mathematics education. The constructivist paradigm supports the use of manipulatives because learning is active and students functioning at concrete developmental levels benefit from the concrete aspect
of manipulatives (Uttal, Scudder, & Deloache, 1997). Cognitive science theories support the use of manipulatives as they contribute to the creation and application of prior knowledge (Hiebert & Carpenter, 1992).

**Algebraic Thinking**

Algebraic understanding is essential for student success in higher level mathematics courses, yet many students struggle with algebra and algebraic understanding. Algebra is often thought to be the gatekeeper to higher education (Capraro & Joffrion, 2006; Kilpatrick & Iszak, 2008; Moses & Cobb, 2001; Spielhagen, 2011). Many US students struggle as they transition from arithmetic to algebra because elementary mathematics classrooms often do not prepare students for algebraic thinking (Cai & Moyer, 2008; Kilpatrick et al., 2001). Too often, students learn to operate and manipulate algebraic symbols without understanding the meaning behind important concepts such as coefficients, constants, variables, and the equal sign. Additionally, students often fail to understand the meaning and relevance of algebra in their everyday lives (Baek, 2008). Kaput (1999) noted negative aspects of US classrooms in which students learn procedure-oriented algebra. Kaput (1999) described procedure-oriented learning as being disconnected from previous mathematics training and the real-lives of students. When students experience procedure-oriented learning without making connections, algebra is often difficult. “Algebra is difficult for students because the representations are abstract and because the required operations, especially those relating quantities in word-problem situations, conflict with operations students have learned to use through years of modeling with arithmetic” (Kilpatrick & Izsak, 2008, p. 12).

Recommendations presented by NCTM (2000) and **CCSSM** standards suggested by CCSSO (2010) emphasize the importance of algebra and modeling mathematics. One recommendation put forth by NCTM (2000) advocates that students “represent and analyze mathematical situations and structures using algebraic symbols” (p. 222). Writers of NCTM recommendations expect students to successfully solve multi-step equations with variables on both sides by the end of the eighth grade. **CCSSM** standards emphasize competence and conceptual understanding, such as the ability to model mathematics, use appropriate tools strategically, and look for and make use of structure (CCSSO, 2010). With these recommendations and standards in mind, middle school students should
understand the meaning of algebraic symbols, not just perform rote operations with them. Specifically, middle school students should focus on reasoning about expressions and equations, and solve equations successfully.

Solving Equations

According to the CCSSM, middle school students are expected to reason about and solve one-variable equations (CCSSO, 2010). For example, standard 6.EE.7 states, “Students should solve real-world and mathematical problems by writing and solving equations of the form \( x + p = q \) and \( px = q \) for cases in which \( p \), \( q \) and \( x \) are all nonnegative rational numbers” (CCSSO, 2010, p. 43). Standard 8.EE.7 stated, “Students should solve linear equations in one variable” (CCSSO, 2010, p. 54). In addition to specific grade level mathematical standards, CCSSM presents eight standards for mathematical practice, including making sense of problems, modeling with mathematics, and looking for and making use of structure. These standards can be implemented as students learn how to solve equations. Using manipulatives also helps students use these practice standards.

The CCSSM demonstrates that solving equations is an essential component of middle school mathematics and beginning algebra courses. Students often face challenges in mathematics content, especially when trying to make sense of abstract concepts such as solving equations. Specifically, researchers have identified three common challenges that students often face when attempting to solve equations

1. a lack of symbolic understanding of variables and coefficients within an equation (Kilpatrick & Izsak, 2008; Poon & Leung, 2010);
2. a lack of understanding of the meaning of the equal sign (Knuth et al., 2006); and
3. a reliance on procedural knowledge without conceptual understanding (Capraro & Joffrion, 2006; Siegler, 2003; Star, 2005).

Lack of symbolic understanding.

A lack of symbolic understanding on the part of students is problematic. For example, students do not understand nuances such as the differing roles of 2 in the two expressions, \( 2 \) and \( 2x \). In the first example, \( 2 \) is a constant, in the second, \( 2 \) is a coefficient, but often students treat them the same (Poon & Leung, 2010). Poon and Leung (2010) cited students whom simply accept formal rules and techniques of algebra without
understanding concepts as students experiencing a weak curriculum. In their study, grade nine equivalent Hong Kong students \((n = 815)\) from six different schools were given a logic test and an algebra test that included solving equations. Teachers \((n = 44)\) answered questionnaires related to their perception of the difficulty of items on the algebra test. Selected students and teachers were also interviewed by the researchers. By analyzing student errors and comparing them to teacher expectations, the researchers discovered these errors occurred primarily as a result of confusion over the meaning of symbols and operational mistakes. “From the analysis of the data, together with the interviews, we found that students develop their own schema by successfully attempting easier problems and then trying to memorize the so-called strategies without understanding the mathematical theory that underlies them” (Poon & Leung, 2010, p. 54). The large sample size makes Poon and Leung’s data reliable; however, generalizability from one country to another may be questionable. One limitation of Poon and Leung’s study was that authors did not describe learning conditions, only the results of the test. It is not clear if students received conceptual or procedural instruction which led to these results.

Vlassis (2008) described difficulties students experience with symbolic understanding, emphasizing that students have difficulty with symbolic understanding because of the multiple meanings that mathematical symbols hold. For example, the minus sign can be a unary sign \((-7)\), a binary sign which students cannot further simplify \((2x - 7y)\), or a binary sign that students can further simplify \((7x - 3x)\), or an operation sign \((7 - 3)\). Vlassis (2008) conducted a qualitative study involving eighth grade students exploring their symbolic understanding of the minus sign. Vlassis interviewed 17 students related to the meaning of the minus sign in various algebraic contexts. Students experienced difficulties with equations that created a negative outcome, such as \(-6x = 24\). Secondarily, some students were unable to solve problems such as \(4 - x = 5\) because the negative sign is next to the variable and not the constant. Most students incorrectly transformed the equation to \(x = 5 - 4\). Although Vlassis’ study provided insight into student difficulties that arise from the minus sign, the small sample size makes generalizability difficult.

**Meaning of the equal sign.**

A second common difficulty for students solving equations involves interpreting the equal sign as a do something sign, rather than a symbol of equality (Knuth et al., 2006).
Several authors described a lack of understanding of the equal sign as a pervasive problem associated with algebra (Kieran, 1992; Kilpatrick et al., 2001; Knuth et al., 2006; Rojano, 2002). The equal sign is ubiquitous at all levels of mathematics, but little instructional time is spent describing its meaning (Knuth et al., 2006). Without a proper understanding of equality, difficulties arise as students solve equations.

Student understanding of the equal sign was a topic of research occurring as early as the 1970s (Rojano, 2002). Knuth, Alibali, McNeil, Weinberg, & Stephens (2005) conducted a study of middle school students ($n = 373$) in which they described the meaning of the equal sign within the equation, $3 + 4 = 7$. Student responses were coded as relational, operational, other, or no response. Over fifty percent of sixth and seventh grade students reported operational responses, which were related arriving at an answer. Knuth et al. (2005) revealed that relational student responses increased as students progressed through middle school with over forty percent of eighth graders providing a relational response. Kilpatrick et al. (2001) echoed these findings, stating that many students either conceptualize the equal sign as a separation of the problem and the solution, or as a left to right directional symbol for working out problems. Both of these misconceptions of the equal sign are problematic for solving equations because equations often include variables and constants on both sides of the equation; solving an equation does not occur from left to right.

Within Knuth et al.’s (2006) research, middle school students ($n = 177$) completed a written assessment of algebraic understanding in a quantitative study. Students responded to three questions related to the equal sign and solving equations (Knuth et al., 2006). Student responses were coded as relational, operational, other, or no response. In the first question, students were asked to describe the meaning of the equal sign in the problem, $3 + 4 = 7$. The large majority of the students described the purpose and meaning of the equal sign as operational, which means they expected to announce an answer. Over 50% of sixth and eighth grade students provided an operational definition for the equal sign. Knuth et al. also examined the relationship between how students viewed the equal sign and their mathematical ability, finding that students with higher standardized test scores were statistically significantly more likely to describe the equal sign as a relational symbol.
In the next two questions, students were asked to solve multistep equations with one variable, such as $4m + 10 = 70$. When equations included variables on both sides, students were often unable to understand how to proceed. Student responses were coded as *answer only, no response, guess and test, unwind, algebra*, and *other*. Results indicated that students who defined the equal sign as *relational* were more likely than those who did not to use an *algebraic* strategy to solve the equations. This study was mostly quantitative in nature, but the data was enhanced by student interviews, which provided insight into thinking and meaning-making. “Many elementary and middle school students demonstrate inadequate understanding of the meaning of the equal sign, frequently viewing the symbol as an announcement of a result of an arithmetic operation rather than as a symbol of mathematical equivalence” (Knuth et al., 2006, p. 298). The authors argued that because students did not understand that the equal sign represented a relationship between two quantities, they had difficulty manipulating equations in order to find a solution. A limitation of this study was that researchers did not investigate how students developed their conception of the equal sign. A description of curriculum or classroom activities may have provided insight into ways students developed these conceptions.

**A reliance on procedural knowledge without conceptual understanding.**

Finally, an important aspect of solving equations involves having both procedural and conceptual understanding of the abstract (Capraro & Joffrion, 2006; Star, 2005). Recent *Math Wars*, in which researchers and practitioners pose traditional mathematics against reform mathematics, display the competition between the two elements (Reys, 2001; Schoenfeld, 2004). Star (2005) provided a definition for *procedural understanding*, focusing on understanding symbols and rules. Star (2005) additionally defined *conceptual understanding* as making connections and creating networks within information. Similarly, Rittle-Johnson and Alibali (1999) included action sequences in their definition of *procedural knowledge* and included relationships between knowledge as they defined *conceptual knowledge*. Star (2005) conducted a survey of journals and databases and found an emphasis on conceptual understanding, finding a ratio of 4:1 of articles with conceptual understanding as a topic compared to procedural understanding. Star (2005) emphasized the importance of procedural knowledge and specifically addressed solving equations as an example. According to Star (2005), there are only a few standard procedures necessary for
solving equations, “adding or subtracting from both sides, combining like terms, distributing or factoring, and multiplying or dividing both sides” (p. 409). Flexibility, according to Star, is the ability to use nonstandard procedures to solve an equation in the most effective way, and is a sign of deep procedural understanding. Star disagreed with other researchers who claimed that the only deep understanding is conceptual understanding. Star maintained that the flexible thinking derived from procedural understanding is important to student understanding.

Although Star (2005) maintained that procedural understanding is essential, other researchers emphasized conceptual understanding. For example, Siegler (2003) discussed pitfalls in mathematics learning that develop as students focus on procedures, rather than concepts. Rittle-Johnson and Alibali (1999) randomly selected fifth grade students \((n = 60)\) and assessed understanding of equivalence before and after instruction. Each of three student groups received different treatments as they learned about addition and subtraction equivalence. The control group received no instruction while one treatment group received procedure-oriented instruction and the other group received conceptual-oriented instruction. On post-instruction assessments, students in the conceptual group used more varied strategies than students in the procedural group; however, the students in both groups performed equally-well on the posttest. Rittle-Johnson and Alibali (1999) concluded that gains made by procedural group students did not transfer to improvements in conceptual understanding. “In contrast, gains in conceptual understanding led to fairly consistent improvements in procedural knowledge in this study” (p. 186).

Capraro and Joffrion (2006) echoed Rittle-Johnson and Alibali’s (1999) claims in results of their quantitative study of seventh and eighth grade mathematics students \((n = 668)\) in which students took two forms of an assessment measuring their ability to translate written words into algebraic equations. Student errors were analyzed and randomly chosen students \((n = 5)\) were interviewed. More successful students were those with a higher level of conceptual knowledge. Students with conceptual knowledge were flexible in their problem solving strategies and methods. However, this is not true of students who only have an understanding of procedural skills. Students with procedural knowledge were limited to solving only a few similar problems successfully. “Unfortunately, mere knowledge of procedural skills caused students an inability to apply
methods for solving the problem” (Capraro & Joffrion, 2006, p. 161). A limitation of Capraro and Joffrion’s study was a lack of information about teaching and learning strategies that could have caused these differences.

Although researchers and educators agree that both procedural and conceptual knowledge are important, there is disagreement about the emphasis that should be placed on each type of knowledge within the study of mathematics. Star (2005) stated that flexibility can be derived from deep procedural knowledge. Contrastingly, other researchers emphasized depth of knowledge that results of conceptual understanding. Kilpatrick et al. (2001) emphasized the importance of conceptual understanding for rule-based computations that occur in algebra, noting, however, that U.S. textbooks emphasized rules and procedures to a greater degree than concepts. Kilpatrick et al. (2001) supported conceptual understanding above procedural understanding claiming that an emphasis on rules does not help students create meaning. A lack of understanding of meaning on the part of students, according to Kilpatrick et al. (2001) leads to forgetfulness, a lack of strategy, and inconsistent errors. Finally, Kilpatrick et al. (2001) cautioned that reliance on procedural understanding results in an over-reliance on visual cues, such as manipulatives.

Procedural and conceptual understanding are both valuable as students learn mathematics. Conceptual understanding allows students to make connections and develop understandings, not just memorize a set of facts or procedures. Because as Star (2005) suggested, there are a limited number of procedures necessary for solving equations; conceptual understanding may be more important as students solve equations. If students understand concepts such as the meaning of the equal sign, inverse operations, and the role of constants and coefficients, they may be able to use this conceptual understanding to accurately implement correct procedures while solving equations. Manipulatives assist students in developing conceptual understanding.

**Manipulatives**

Manipulatives can be used to represent abstract concepts, such as algebra, explicitly and concretely; learners understand the abstract by acting in a hands-on manner (Moyer, 2002). According to Puchner et al., “Manipulatives are concrete tools used to create an external representation of a mathematical idea and include items such as unifix cubes and base 10 blocks” (2010, p. 314). The NCTM (2010) endorsed integrating manipulatives in
all levels of mathematics education. Manipulatives, such as Cuisenaire Rods©, balance scales, and algebra tiles can be purchased; other manipulatives can be created by teachers or students. Finally, common objects, such as beans, cereal, and beads can be used as manipulatives in the mathematics classroom. Allowing students to think algebraically by using manipulatives increases conceptual understanding.

This section of the literature review provides a brief history of concrete and virtual manipulatives, and describes specific manipulatives used for solving equations. Additionally, several questions will be addressed. How do concrete and virtual manipulatives increase understanding of the symbolic elements of equations? How do they improve understanding of the equal sign within an equation? To what extent do concrete and virtual manipulatives deepen procedural and conceptual understanding? How effective are concrete manipulatives to their virtual counterparts? Additionally, cautions and teacher considerations for using manipulatives will be examined.

Concrete Manipulatives

According to Boggan et al. (2010), manipulatives have been used to solve mathematical problems throughout world history. For example, ancient Southwest Asians created counting boards, and, the ancient Romans created the abacus, both tools to simplify counting and calculations. More recently, Friedrich Froebel, an educator in Germany during the 1830s, created pattern blocks and geometric blocks for use in Kindergarten (Boggan et al., 2010). Maria Montessori endorsed the use of manipulatives in the early twentieth century inventing several manipulatives to help young students understand mathematics. Now, in the early twenty-first century, the use of manipulatives is still encouraged. NCTM authors advocate the use of representations, such as manipulatives. CCSSM (2011) authors emphasize the use of manipulatives within mathematical practice standards.

Concrete manipulatives for solving equations.

Balance scales are often used to depict solving equations. In their elementary and middle school mathematics methods textbook, Van de Walle and Lovin (2006) suggested that teachers draw balance scales with items on the board in order to have children see that they may take the same thing off both sides in an effort to develop an effective representation of solving equations. Although this is not an example of manipulatives, this
representation closely models manipulatives used for solving equations and acts a bridge between concrete and abstract thinking. For example, Borenson created *Hands-On Equations®*, a research-based set of manipulatives representing equations including a balance scale, pawns representing variables, and dice representing constants. Borenson and Barber (2008) conducted a quantitative study of middle school mathematics students \( n = 243 \) participating in seven lessons using *Hands-On Equations®*. Results of this study indicated a statistically significant difference between pretest and posttest results for participants. However, because Borenson was researching a product which provides his own financial gain, it is wise to be suspect of his claims. Evidence created from a non-biased party would be easier to accept as valid and reliable.

Algebra tiles use distinct items to represent constants and variables, which can be placed on an equality mat to represent solving equations (Figure 2.2). Constants are represented by small yellow squares and variables \((x)\) are represented by yellow rectangles. Negative items are represented by red squares and rectangles. Students solve equations by representing the constants and variables within the equations on equality mats and then removing or adding pieces as necessary to isolate the variables.

![Equation represented with algebra tiles](image)

*Figure 2.2. Equation represented with algebra tiles.*

**Strengthening understanding with concrete manipulatives.**

Researchers investigated the importance of external representations for solving equations. Caglayan and Olive (2010) conducted a qualitative study in which eighth grade students \( n = 24 \) solved equations using cups and tiles to represent variables and constants respectively. Authors collected data using classroom video, and student and teacher interviews for triangulation. Variables and constants were represented with different items
to explicitly differentiate them. Cups represented variables and tiles represented constants, which helped students see that \(2x\) and 2 are distinctly different mathematical concepts. The noticeable difference helped students realize that constants and variables cannot be combined because they are not alike. Caglayan and Olive (2010) concluded that students experienced difficulty linking the physical activities of the manipulatives and the mental operations necessary for solving equations. One limitation of this study was the inability to exhibit subtracting integers with this manipulative model.

A correct understanding of the equal sign is of utmost importance as students learn to solve equations. Students that do not recognize the equal sign as a symbol of relationship between both sides of the equation rely on rules rather than understanding and therefore are prone to errors (Falkner, Levi, & Carpenter, 1999). Using external representations, such as manipulatives, can strengthen this understanding. Sherman and Bisanz (2009) conducted a quantitative study in which second grade students \((n = 48)\) solved problems such as \(5 + 2 = 4 + \_\) in two different ways, either symbolically, such as the example, or with manipulatives representing the problem \((nonsymbolic)\). Manipulatives included cardboard boxes holding wooden cylinders. There was a statistically significant difference in favor of the nonsymbolic representations \((M = 16.92, SD = 4.56)\) over the symbolic representation \((M = 8.33, SD = 4.05)\) in posttest results. One limitation of the study is generality because participants were much younger than typical algebra students. However, the results of the study illustrate the value of nonsymbolic representations, such as manipulatives, to students. Sherman and Bisanz (2009) attributed the differences in student understanding to the physical representation of equality on both sides of the equation using the manipulatives. Students using symbolic representations did not recognize the equal sign as a symbol of equality. Hiebert & Carpenter (1992) suggested that the meaning of the equal sign can be strengthened if students connect the fulcrum of a balance scale with the equal sign of the equation.

**Cautions for using concrete manipulatives.**

Research indicates that students who use manipulatives during mathematics instruction outperform students learning with more traditional methods (Boggan et al., 2010; Moyer, 2002). However, not all researchers agree on the benefits of using manipulatives. Hiebert and Carpenter (1992) provided explanations of potentially negative
results from manipulatives. Hiebert and Carpenter (1992) described the relationship between external and internal representations created by students. According to Hiebert and Carpenter (1992), manipulatives exemplify external representations and how students think about these manipulatives exemplify internal representations. Because of this relationship, the use of manipulatives is encouraged.

Teachers expect students to make connections between concrete representations and abstract concepts as they use manipulatives. However, without necessary background knowledge, some students are unable to make expected connections. In order for endogenous connections to occur, students must reflect on the meaning of the manipulatives.

Hiebert and Carpenter (1992) further contended that manipulatives must closely match the mathematics they represent. When the characteristics of the manipulatives do not match the characteristics of the mathematics, students have difficulty making connections between the two. McNeil and Uttal (2009) agreed, stating that for students to make connections between the actions related to the manipulatives and the symbolic understanding, students must see very explicit connections between the two. Moyer (2002) stated, “The development of the student’s internal representation of ideas, tested on the external representations or manipulatives is at the heart of what it means to learn mathematics” (p. 194).

McNeil and Uttal (2009) reported on a study which mirrors concerns presented by other researchers. Children 30 to 38 months of age were provided a scale model of a room with a hidden object (such as a teddy bear); children were expected to find the hidden object within an actual room based on their experience with the model. The results of this research indicated that not all children made connections between objects and real-life. Based on this evidence, McNeil and Uttal (2009) suggested that without these connections, cognitive load increases as students learn material twice, once with manipulatives, and then again with the abstract concept. McNeil and Uttal (2009) provided interesting information about connections between manipulatives and mathematics. However, a drawback of including this study is the age of study participants. Although the reactions of young children are interesting and informative, the reactions of small children may be different.
than reactions of older students to using manipulatives, so the results are likely not
generalizable to older children.

Borenson and Barber (2008) suggested having students create a written record of
each action undertaken with manipulatives to reduce cognitive load. For solving equations,
this might mean working with the manipulatives while recording steps on paper. A written
record allowed students to see that taking the same thing off both sides with the
manipulatives is equivalent to subtracting from both sides on paper (Borenson & Barber,
2008).

Teaching considerations for concrete manipulatives.

Student success with manipulatives is partially determined by how they are used in
a classroom (Moyer-Packenham, 2010). Boggan et al. (2010) suggested that teachers
provide students the opportunity to play freely with the manipulatives prior to instruction
in order to diminish their appeal as toys. However, Uttal et al. (1997) strongly disagreed
with this idea, stating that manipulatives should not be attractive objects, pointing to
Japanese teachers who do not use novel objects as manipulatives.

Several authors discussed the role of the teacher in helping students use
manipulatives. Cai and Moyer (2008) cautioned teachers to help students transition from
understanding the concrete to understanding the abstract. The transfer of understanding
between the two is not automatic and does not generally occur without the assistance of the
teacher. However, Caglayan and Olive (2010) cautioned that unsuccessful uses of
manipulatives occur if teachers provide too much guidance and do not allow students to
construct meaning in their own way. Cobb, Yackel, and Wood (1992) referred to this
challenge as a teacher’s paradox, stating that the more explicitly a teacher explains
material, less construction is required on the part of the student. Students must create their
own meaning of mathematics, yet, they need assistance from their teachers at the same
time.

Roberts (2007) cautioned teachers to consider student understanding as they select
manipulatives. In an article published in Teaching Mathematics in the Middle School, she
described her experiences searching for the perfect manipulatives for a geometry lesson.
After trying several manipulatives unsuccessfully, she concluded that her prior knowledge
clouded her judgment because it did not match her students’ prior knowledge. Similarly,
Puchner et al. (2010) reminded teachers not to assume that students see connections just because the teacher clearly sees the connection between the manipulative and the symbol. The authors explained,

> Often the teacher so clearly sees how the external representation depicts the idea they are trying to teach, they cannot imagine how the student would not easily form an accurate internal representation from the manipulative. Teachers often falsely assume the manipulative will create the same internal representation for the student. (p. 314)

After conducting qualitative research in which K-8 mathematics teachers ($n = 33$) learned how to integrate manipulatives into the classroom during a summer institute, Puchner et al. (2010) emphasized the importance of letting students create the meaning for the manipulatives. The researchers collected lessons from teachers and identified, isolated, and categorized excerpts which included manipulatives. Teachers in this study identified benefits and concerns with the practice of using manipulatives. A primary observation was a lack of understanding on the part of sixth grade students using manipulatives to represent multiplication. Teachers reported that students used manipulatives in a rote, procedural way without increasing conceptual understanding. Boggan et al. (2010) seemed to agree with this analysis cautioning that when using manipulatives, if students do not develop an understanding of the representation, they simply mimic the actions of the teacher without attaching meaning to the manipulatives or actions.

**Virtual Manipulatives**

Virtual manipulatives, available on the computer, have only existed for approximately a dozen years. Moyer, Bolyard, and Spikell (2002) defined virtual manipulatives as “an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (p. 373). Moyer-Packenham (2010) differentiated virtual manipulatives from static images based on the flexibility of virtual manipulatives, noting that these objects can be moved, stretched, rotated, or changed completely. Many virtual manipulatives were modeled from their concrete manipulative counterparts. Proponents of virtual manipulatives claim that they are superior to or as good as concrete representations, because they can be transformed and manipulated in a similar way to physical representations, yet they are less distracting (Bouck & Flanagan, 2009; Durmus & Karakirik, 2006; Moyer et al., 2002).
Virtual manipulatives for solving equations

A group at Utah State University, sponsored by the National Science Foundation, created the National Library of Virtual Manipulatives (NLVM) providing free access to teachers, parents, and students to many web-based Java applets (2010). Additionally, the NLVM allows teachers to create a virtual classroom using Tracking and Adaptation Tools, in which they can select appropriate manipulatives for their students and customize specific problems for them (Moyer-Packenham, 2010). These tools additionally record class and individual student data to be analyzed by teachers. According to the NLVM, using virtual manipulatives provides students the opportunity to become active participants in learning (2010).

The Algebra Balance Scale virtual manipulative available on the NLVM website, (http://nlvm.usu.edu), uses a balance scale and two different sized blocks, the smaller representing one unit and larger “x-blocks” representing variables. As students place the appropriate items on the scale, it becomes balanced (Figure 2.3). The scale is shown to be out of balance until the problem is represented correctly, which helps students develop a proper understanding of the meaning of the equal sign. Next, students select the operations that they need to implement (add, subtract, multiply, or divide) until the equation is solved correctly. Each step is represented on the balance, and the corresponding new equation is recorded on the screen. The applet generates random problems, or students can create their own equations.

Polly (2011) co-taught a unit to third graders related to solving equations using an applet found on the Illuminations website created by the NCTM. Within the study, Polly noted examples of student reflection and evaluation which occurred as a result of using virtual manipulatives. Polly examined an alternative virtual manipulative for
solving equations and stated that technology allowed students to develop a deeper understanding of mathematics content. A limitation of including Polly’s study is generalizability between third grade students and middle school students.

**Strengthening understanding with virtual manipulatives.**

Several authors ascribed a strengthening of symbolic understanding to the use of virtual manipulatives (Lamberty, 2007; McNeil & Uttal, 2009; Sarama & Clements, 2009). According to Poon and Leung (2010), algebra students lack a clear understanding of the difference between the role of coefficients and whole numbers. Virtual manipulatives help make the distinction between the two clear. For example, if a user attempts to subtract $3x$ from both sides of a scale when the correct operation is to subtract both sides by $3$, the applet does not allow the action. The applet available from the NLVM tells users when proposed actions are not possible. For example in Figure 2.4, when solving the equation $2x - 2 = -2x + 6$, the user attempted to divide by three and the virtual manipulative stated, *Dividing both sides by 3 will not simplify the equation*. Additionally, the applet updates the equation with each step taken by users, strengthening connections between their concrete actions with the manipulatives and the abstract equation they are solving. Within their review of literature related to concrete and virtual manipulatives, Durmas and Karakirik (2006) suggested that the immediate feedback provided by virtual manipulatives allows
students to become familiar with mathematical representations. Moyer-Packenham (2010) noted that feedback provided by some virtual manipulatives highlights important features of the mathematical concept being investigated, thus preventing misconceptions.

![Virtual manipulative feedback](image)

**Figure 2.4.** Virtual manipulative feedback. © 2010 MATTI Assoc. and Utah State University. Used with permission.

Virtual manipulatives help students understand the equal sign because as boxes are placed on the two sides of the equation, the heavier side drops until the equation is balanced. Although Sarama and Clements (2009) were not specifically discussing the Algebra Balance Scale virtual manipulative, the authors referred to a benefit of virtual manipulatives, stating,

> Computer manipulatives can also serve as symbols for mathematical ideas, often better than physical manipulatives can. For example, the computer manipulative can have just the mathematical features that developers wish it to have and just the actions on it that they wish to promote—no additional properties that may be distracting. (p. 148)

The notion of balance as it relates to the equal sign on the Algebra Balance Scale is a good example of this notion.

Actions on virtual manipulatives can be changed, repeated, or undone more simply than with concrete manipulatives (Sarama & Clements, 2009). Virtual manipulatives are more effective than traditional manipulatives because they “may be more manageable, flexible, extensible, and ‘clean’” (Sarama & Clements, 2009, p. 147). Sarama and Clements (2009) emphasized the ability to create connections between mathematics concepts as a key benefit of virtual manipulatives. The authors claimed that after students
finished working with concrete manipulatives, they experience cognitive overload and forget the steps taken to solve the given problem. However, with virtual manipulatives, the steps are often recorded and displayed on the screen. This recording of steps may allow students the opportunity to reflect on their thinking processes, therefore, increasing conceptual understanding.

Martin (2008) stated, “If children’s work with the virtual manipulative is recorded, children can play their actions back, providing greater opportunities for reflection” (p. 270). He described the processing as *representational redescription*. According to Martin, students benefit from working with manipulatives because they allow time for reflection. With time and reflection afforded by representations such as manipulatives, students will learn more advanced mathematical concepts and move away from needing manipulatives. The fact that students can interact with virtual manipulatives by inputting numbers at the same time as manipulating the image strengthens this process. Martin compared the effectiveness of concrete and virtual manipulatives in a Kindergarten mathematics classroom learning about addition. Martin reported that when students used either kind of manipulative, they were more accurate than when working without manipulatives. Martin also noted that regardless of the type of manipulative, both groups of students performed similar actions. In another of Martin’s studies, first grade students \( (n = 31) \) studying division with either concrete or virtual manipulatives all showed improvement between pretest and posttest results (Pretest \( M = 2.51, SD \) not reported, Posttest \( M = 3.27, SD \) not reported). Within this study, Martin noted that students with the lowest pretest scores improved most quickly by using virtual manipulatives. A weakness of Martin’s study was a small number of participants which limits generalizability. Also, because Martin’s study was conducted in elementary classrooms, the results may be different than if they occurred in middle school classes.

**Cautions for using virtual manipulatives.**

Not everyone endorses the use of virtual manipulatives; critics claimed that some virtual manipulatives are not very interactive or motivating (Durmus & Karakirik, 2006). Polly (2011) cautioned teachers to integrate virtual manipulatives only when they make a positive contribution to mathematics lessons. Teachers should not include virtual manipulatives just for the sake of using technology in the classroom. Moyer-Packenham
(2010) reminded teachers that virtual manipulatives should meet the objectives and goals for lessons.

Finally, using virtual manipulatives requires a great deal of planning and organization on the part of the teacher (Burns & Hamm, 2011). Burns and Hamm conducted a comparative study of fourth grade students using concrete and virtual manipulatives to learn fractions. Although the researchers did not find a statistically significant difference between the results of each manipulative, they did report gains in student learning for both groups. Additionally, they made suggestions for optimizing student learning using both types of manipulatives. For example, teachers must make sure that technology is available and software is properly loaded onto computers. The authors suggested that to save class time, teachers may load websites onto the computers prior to class time.

**Teaching considerations for virtual manipulatives.**

In her doctoral dissertation, Lamberty (2007) cited research revealing that some teachers used manipulatives just for fun or for reinforcing previous content; in these cases, manipulatives did not provide deep learning. In an attempt to strengthen student understanding of fractions, Lamberty created an applet modeled after quilts. After Lamberty field tested several iterations of the applet, she concluded that students benefit from the use virtual manipulatives when they are relevant to content and considered a learning tool.

Availability and accessibility are among the noted benefits of virtual manipulatives. As long as schools have adequate computers and reliable Internet access, virtual manipulatives are available in an endless supply for learners (Moyer-Packenham, 2010). Another benefit of virtual manipulatives is accessibility after school and beyond traditional school walls (Durmus & Karakirik, 2006). Finally, there is no clean-up required after using virtual manipulatives (Moyer et al., 2005). One obvious limitation of virtual manipulatives is software constraints. For example, the NLVM applet used for solving equations cannot accept variables larger than nine or fractional variables or constants.

**Comparison of Concrete and Virtual Manipulatives**

Durmus and Karakirik (2006) suggested that virtual manipulatives are just as effective as concrete manipulatives and that the two may be interchangeable. Other
researchers made similar claims (Lamberty, 2007; Lee & Chin, 2010; Martin, 2008). One study of particular interest involved two similar classes of third grade students who spent equal time learning how to solve simple equations, one class with concrete, and the other with virtual manipulatives (Suh & Moyer, 2007). In the mixed methods study, the authors collected field notes and interviewed students and teachers. Researchers found unique benefits for both learning experiences; students in both groups showed statistically significant gains on the posttests as compared to pretests. During interviews, students noted the tactile features of concrete manipulatives and reported inventing more original strategies using more mental mathematics than the virtual manipulatives group. Students in the virtual manipulatives group reported a stronger relationship between the virtual manipulatives and the mathematical symbols. Students also stated that the immediate feedback and the step-by-step nature of virtual manipulatives were beneficial.

In another study comparing concrete and virtual manipulatives, Yuan, Lee, and Wang (2010) tested the effectiveness of polyominoes with junior high students ($n = 68$). Each class of students used either concrete or virtual polyominoes during three classroom activities. The purpose of their study was to compare differences in problem solving between the two groups and to measure student attitudes related to virtual manipulatives by conducting pretest and posttests and an attitudes survey. The results of their study indicate no statistically significant difference between the results of the two classes. Researchers noted that both materials were effective, but each resulted in different learning experiences for students. An interesting outcome of their study was a larger amount of time spent on group discussions within the classroom of virtual manipulatives. These students had an easier time creating, decomposing, and recomposing shapes with the virtual manipulatives.

Lamberty (2007) stated that if all other factors are the same, virtual manipulatives can provide more educational gains than concrete manipulatives. Sarama and Clements (2009) emphasized the important aspect of virtual manipulatives was not the physicality, but rather the way students were able to interact with the virtual manipulatives. If students are able to create meaning from the actions, then there is value to manipulatives.

There may be advantages to using virtual manipulatives over concrete manipulatives with symbolic understanding because the equation and the steps are visible on the screen, and are more closely connected to the actions taken (Moyer, 2002). Kaput
(1992) stated that concrete manipulatives create cognitive overload for students because they are keeping up with too many things at once while virtual manipulatives often record steps for students. Yuan et al. (2010) endorsed virtual manipulatives, stating that students can more closely mirror their thought processes with virtual manipulatives than with concrete manipulatives.

Concrete and virtual manipulatives represent the meaning of the equal sign in a similar way, with a physical balance scale; students see when the equation becomes balanced when the same value is on both sides. Finally, concrete and virtual manipulatives both help students develop deeper conceptual understanding as they experience the meaning of operations within equations and make connections between actions and results within an equation (Suh & Moyer, 2007).

While comparing concrete and virtual manipulatives, practical aspects must be considered. Drawbacks of concrete manipulatives include the cost of purchasing them, or the time it takes teachers to create them. Virtual manipulatives are free and available as technology allows within a school and a home. Concrete manipulatives are messy to clean up; virtual manipulative sessions can end at the touch of a button. However, in order to use virtual manipulatives, students must have adequate access to computers.

Conclusions

Researchers described the importance of studying algebra within the mathematics curriculum (Moses & Cobb, 2001). NCTM (2000) and the CCSSM (2011) provide guidelines and standards which include solving equations within middle school curriculum which should include the use of representations and models. While solving equations, students should not just develop procedural understanding, but conceptual understanding must be fostered as related to symbolic understanding and the meaning of the equal sign (Knuth et al., 2006; Poon & Leung, 2010; Siegler, 2003).

Educators have used manipulatives to teach mathematical concepts throughout history, but not all researchers agree on the effectiveness of manipulatives for increasing student understanding. Recently, researchers identified benefits of using manipulatives for teaching solving equations including distinct representations of equation elements (Caglayan & Olive, 2010). Regardless of the mathematical concept, teachers must ensure that students make connections between the manipulatives and the concepts they represent.
Concerns related to manipulatives include the desire on the part of students to play with manipulatives, rather than use them as intended (Boggan et al., 2010). Benefits of virtual manipulatives, as identified by researchers, include flexibility and immediate feedback (Durmas & Karakirik, 2006; Moyer-Packenham, 2010). Concerns related to using virtual manipulatives include availability and accessibility (Moyer et al., 2005). Some researchers suggested that virtual and concrete manipulatives are just as effective and interchangeable (Durmas & Karakirik, 2006). More research is needed to compare the effectiveness of concrete and virtual manipulatives to increase student understanding, especially in middle and high school mathematics. This study will contribute insight into the use of manipulatives in middle school mathematics. Additionally, this study will provide more evidence comparing concrete and virtual manipulatives used to teach solving equations.
CHAPTER III

METHODOLOGY

This chapter describes the methodology used for the quasi-experimental mixed-methods research study. The general purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative.

Quantitative analysis was used to compare pretest and posttest results among the three groups (control, concrete, and virtual). Qualitative analysis was used to understand the unique benefits and challenges students faced as they used each learning method. Qualitative analysis was also used to understand the effectiveness of the treatments as students overcame obstacles to solving simple linear equations. The quantitative and qualitative analysis complement each other; the quantitative data demonstrated differences in performances between the groups while the qualitative data described these differences and provided specific examples that support claims. Further, quantitative data was used to select qualitative focus groups and data analysis included both quantitative and qualitative data.

Brief History of Mixed Methods Research

Mixed methods research has a relatively brief history compared to quantitative and qualitative research. Teddlie and Tashakkori (2009) described mixed methods research as being in its “adolescence phase.” Like many adolescents, mixed methods researchers are in a search for identity, working to define themselves and seeking acceptance from those around them. Teddlie and Tashakkori (2009) traced the roots of quantitative and qualitative research back to Antiquity as they described the informal research methods of Aristotle and other scientists. For example, the scientific assertions made by these scientists were early attempts at quantitative measurements. The passive observations made by such scientists exemplified early qualitative research.

In the early twentieth century, a debate emerged between proponents of positivism and proponents of constructivism (Creswell, 2003; Teddlie & Tashakkori, 2009). Positivists preferred quantitative research methods and sought to verify theories by
implementing scientific research (Creswell, 2003). Constructivists, on the other hand, implemented qualitative methods seeking to understand multiple meanings of experiences derived from social and historic events (Creswell, 2003). Johnson and Onwuegbuzie (2004) describe the differences between quantitative and qualitative researchers as the *incompatibility thesis* which indicates that the two theories should not be mixed. However, with the advent of pragmatic philosophies, mixed methods research “help(ed) bridge the schism” between the two competing paradigms and research methodologies (Johnson & Onwuegbuzie, 2004, p. 15).

Campbell and Fiske are often attributed with implementing the first mixed methods study in 1959 as they used a variety of methods to validate studies related to psychological traits (Creswell, 2003; Johnson, Onwuegbuzie, & Turner, 2007). After conducting this study, Campbell and Fiske encouraged other researchers to integrate multiple data collection techniques they called *multimethod matrix*. However, Teddlie and Tashakkori (2009) looked earlier into research history to the Hawthorne Studies from 1924 to 1932 as an example of mixed methods research. While conducting a study to see if the amount of light influenced worker productivity, researchers collected quantitative data resulting from experiments as well as qualitative data such as interviews and observations (Shuttleworth, 2009; Teddlie & Tashakkori, 2009).

Realizing that all data collection methods have limitations and biases, researchers followed the lead of pioneers and mixed methods research methodologies became more common in the 1960s. Social and behavioral scientists in fields such as education, psychology, nursing, sociology, and library sciences have seen an increase in mixed methods research within the last fifty years (Leech & Onwuegbuzie, 2009). Toward the end of the twentieth century, journals began publishing mixed methods research and in 2003, Teddlie and Tashakkori published the first mixed methods handbook (Johnson et al., 2007; Leech & Onwuegbuzie, 2009).

Creswell (2003) described a change in research practices from quantitative versus qualitative to a continuum with mixed methods falling somewhere in the middle. John Creswell boldly predicted “that the mixed methods paradigm will be the leading research paradigm within the next five years” (Leech & Onwuegbuzie, 2009, p. 266). Because of its strong ties to quantitative and qualitative research, researchers struggle to define mixed
methods research independently (Johnson et al., 2007). For example, the notion of triangulation was described in mixed methods research as early as the 1960s, but not formally defined until 1978 by Denzin. Not only do mixed methods researchers struggle with identity, they also receive criticism from quantitative and qualitative researchers who maintain the incompatibility thesis (Creswell, Shope, Clark, & Green, 2006). However, many mixed methods researchers maintain the fundamental principle of mixed research, which states that researchers should strategically select the best of quantitative and qualitative elements to strengthen data collection (Johnson et al., 2007).

**Mixed Methodology Research Design**

The purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative. Specifically, the following research questions were addressed:

1. What differences exist, if any, in student achievement as a result of using concrete or virtual manipulatives as middle school students use them to solve linear equations compared to a control group using learning methods without manipulatives?

2. What are the unique benefits and drawbacks associated with each type of manipulative?

The researcher chose to conduct a mixed methods research study, working from the Pragmatic Paradigm. Knowledge claims of pragmatists include an orientation to real-world practice and focus on a specific research problem or topic (Creswell, 2003). Pragmatists implement a variety of strategies of inquiry and utilize the strengths of both quantitative and qualitative methodologies. As a researcher in an education setting, quantitative strategies with random experiments were not possible. However, the researcher implemented a quasi-experimental approach as three classes were randomly assigned to either the control group, concrete manipulatives group, or virtual manipulatives group. Within these groups, student responses to and experiences with each treatment were described with qualitative methods.
The value of quantitative data is measurability based on experimental or quasi-experimental research. Educational research often occurs with predetermined groups by schools (Creswell, 2005). Within this research, three similar classes were randomly assigned to the control group, and the two treatment groups. Quantitative methods such as collecting pretest and posttest data and using statistical analysis were used to recognize differences in student performance. Pretest and posttest data was compared to identify growth that occurred as a result of the different learning approaches. Posttest data was also compared to recognize differences between the three groups in conceptual understanding related to the meaning of the equal sign, understand symbols, and solve equations.

The value of qualitative data is the story that it tells. Quantitative data can prove a statistical difference between treatments; qualitative data can paint a picture of the differences. Qualitative data describes student experiences, opinions and explains student learning. Qualitative research is also beneficial for showing how things work, and how processes occur over time. According to Steffe, Thompson, and von Glaserfeld (2000), “A primary purpose for using teaching experiment methodology is for researchers to experience, firsthand, students’ mathematical learning and reasoning” (p. 267). They further described the value of educational research within the classroom, “Looking behind what students say and do in an attempt to understand their mathematical realities is an essential part of a teaching experiment” (p. 270). The researcher wanted to know how students worked with concrete and virtual manipulatives and how students connected their experiences to the abstract concepts they represent. The researcher conducted interviews with the teacher of record prior to and upon completion of the treatment, conducted focus group interviews with students from each of the three groups.

According to Creswell (2003), there are three primary framework elements for mixed methods research, including “philosophical assumptions about what constitutes knowledge claims, general procedures of research called strategies of inquiry, and detailed procedures of data collection, analysis, and writing, called methods” (p. 3). These three elements are all integrated into the framework that follows.

The researcher adopted a Pragmatic Paradigm. Although postpositivism and constructivism have opposing views on what constitutes knowledge claims, a pragmatic
researcher focuses primarily on the problem of the research and seeks the middle ground between the two competing paradigms (Creswell, 2003; Johnson & Onwuegbuzie, 2004).

Strategies of inquiry for this study combined quantitative and qualitative research methods (Figure 3.1). Creswell and Plano-Clark (2007) shared the following definition of mixed methods research,

Mixed methods research is a research design with philosophical assumptions as well as methods of inquiry. As a methodology, it involves philosophical assumptions that guide the direction of the collection and analysis of data and the mixture of qualitative and quantitative approaches in many phases in the research process. As a method, it focuses on collecting, analyzing, and mixing both quantitative and qualitative data in a single study or series of studies. Its central premise is that the use of quantitative or qualitative approaches in combination provides a better understanding of research problems than either approach alone. (p. 5)

This comprehensive definition includes philosophical assumptions which researchers must consider prior to conducting research, as well as data collection procedures and methods. The researcher recognized that both quantitative and qualitative methods have strengths and weaknesses and planned to capitalize on the strengths and minimize the weaknesses of each. First, qualitative data collection methods occurred prior to the interventions as the teacher of record was interviewed. Quantitative data collection occurred prior to and immediately after the intervention. During the intervention, qualitative data collection methods in the form of classroom observations took place. Again after the quantitative data was collected, the researcher collected qualitative data in the form of another teacher interview. The researcher used quantitative data to identify two students in each of the three groups performing at low, middle, and high levels for focus group interviews. Finally, qualitative and quantitative data analysis and results were merged to help the researcher draw conclusions.
Mixed Methods Framework: Embedded Experimental Model

Figure 3.1. Mixed methods design for study (Adapted from Creswell and Clark, 2007).
**Population**

The target population for this study was sixth grade middle grades mathematics students attending a rural public middle school in the Southeast United States ($n = 76$). The research study took place within the first three weeks of the school year. The participants in the study were in three separate, yet similar classes taught by the same teacher of record. The teacher volunteered to participate in the study with the approval of the local school superintendent and school principal. The control group of this experiment used learning methods without manipulatives for solving equations. The control group learned to solve equations by focusing on conceptual topics such as the meaning of the equal sign, inverses, and the roles of constants and variables. The control group also learned procedures and steps for solving equations. The treatment groups learned to solve equations with the same conceptual and procedural emphases, with the additional experience of working with manipulatives. The first treatment group used concrete manipulatives to solve equations and the other treatment group used virtual manipulatives. Each class used approximately the same amount of time and explored the same equations. Due to constraints, the students were not randomly assigned to groups; however, the three classes were randomly assigned to the three learning methods. This placed a limitation on the study. The students in the study represented three of six sixth grade classes in the school.

**Contributions of Pilot Studies**

A pilot study using concrete manipulatives to solve equations was conducted in Spring 2011. Within this pilot study, middle school students ($n = 8$) used researcher-created materials to solve equations using concrete manipulatives for six lessons of approximately fifty minutes each. Students completed pretests and posttests during the pilot study as well. The researcher acted as a teacher-researcher during the pilot study because she was also the teacher of record. While teaching the material, the researcher conducted observations, video recorded all sessions, and collected field notes. The researcher followed-up by interviewing three students in a small focus group setting.

Because of the small sample size, data analysis did not indicate statistically significant differences in pretest and posttest scores. The primary goal in conducting the pilot study was not to collect statistical data, but to practice research techniques and fine-tune the concrete manipulative student materials. The researcher gained understanding
regarding how students used concrete manipulatives and misconceptions that middle school students have related to the meaning of coefficients and constants. The researcher used information gained from this pilot study to revise and modify the student materials, as well as modify interview questions and develop the focus group interview protocol.

A pilot study of virtual manipulatives for solving equations was conducted in Spring 2012. Within this pilot study, middle school students ($n = 22$) used virtual manipulatives and researcher-created materials to solve equations for eight lessons of approximately fifty minutes each, all of which were audio recorded. Prior to conducting the virtual manipulatives field test, IRB approval was obtained from the University of Kentucky. In the virtual manipulatives pilot, the researcher interviewed the teacher prior to the study and afterward. After this pilot study, the researcher conducted a focus group interview with four students. This pilot study was insightful and allowed the researcher to make further adjustments to the teaching materials, interview instruments, and teaching methods.

Data analysis from this pilot study did not indicate statistically significant differences between pretest and posttest scores. However, the researcher collected qualitative data which indicated an increase of understanding related to the meaning of the equal sign as a result of the use of the balance scale in the virtual representation. In this pilot study, students worked with virtual manipulatives mostly at their own pace with little direct instruction from the teacher of record. The researcher realized that students need more guidance from the teacher, so teaching protocols were modified for this research study to reflect this change.

Both pilot studies were informative and beneficial. The researcher practiced different data collection techniques such as video recording, audio recording, and note-taking. This practice allowed the researcher to become more confident in qualitative data collection methods. Also, the researcher modified learning materials by providing more examples to students as a result of feedback provided by students during focus group interviews. The researcher also realized that the teaching protocol should be modified for the virtual group because some students worked well at their own pace, while others struggled to complete tasks in a timely fashion. Pretest and posttest responses allowed the researcher to adjust a few questions in order to make the meaning of the questions more
clear. Finally, comments made during focus group interviews confirmed for the researcher that students benefitted from writing either a numeric or symbol representation of the equations as students solved them.

Although a pilot study with control group materials did not officially occur, the researcher taught solving equations dozen years in various middle school mathematics classrooms. Some of these years included concrete and virtual manipulatives, but all of the years included conceptual and procedural development similar to that in the control group.

**Instrumentation**

**Pre- and Posttests**

The researcher administered pretests prior to interventions and posttests immediately following interventions to all three groups. It took most students approximately forty minutes to complete both the pretest and the posttest. The pretests and posttests were created by the researcher to gauge student ability to solve equations and represent them correctly (see Appendix A). The pretest had a Cronbach’s α of .632, while the posttest had a Cronbach’s α of .766. This indicated the reliability of the assessment was “acceptable” (George & Malloy, 2003). **CCSSM** standards related to solving equations were examined during question development; for example the first twelve questions related to solving simple equations, assessing standards 6.EE.7 and 8.EE.7b (CCSSO, 2010). Of these first twelve equations, the four were additional and subtraction equations such as \( x + 5 = 8 \) and \( x - 2 = 6 \). The next four equations were multiplication equations such as \( 3x = 12 \). There was one multistep equation such as \( 2x + 6 = 12 \) and the final three equations had variables on both sides, such as \( 2x + 3 = 3x + 6 \). Students were asked to provide the solution to the first twelve equations. Students received a score for the number correct out of twelve on this first section of the tests. Two multiple choice questions asked students to identify correct steps in solving an equation and one multiple choice question asked students to identify the correct solution to an equation. The remainder of the test was open-response questions. Multiple choice questions were scored either correct or incorrect.

For open response questions, the researcher also looked at research conducted by authors such as Rittle-Johnson and Alibali (1999) for question design to assess standards such as 6.EE.2b, related to identifying parts of expressions. Three open response questions asked students to create a written representation of their work while solving equations. One
question asked students about the meaning of the equal sign in an arithmetic equation, while another asked students about the meaning of the equal sign within a linear equation. Three questions asked students about the role of constants, coefficients, and variables in equations. One question asked students to describe the process for checking an equation, and the final question asked students about the goal or meaning of solving equations. Pretests and posttests were identical in format; the only difference between them was the actual equations that students solved.

Open response questions related to solving linear equations were scored based on students arriving at a correct solution. These open response questions were also qualitatively evaluated based on criteria such as correctness of representation and explanation provided by students. Open response questions related to the meaning of the equal sign were coded by the researcher as no response, operation response, or balance response. Open response questions related to the roles of constants, coefficients, and variables were coded by the researcher as no response, accurate, or inaccurate.

The pretests and posttests were reviewed by a panel of STEM education doctoral students which provided feedback and suggestions. A university faculty member in instructional design also reviewed the assessments and provided feedback.

Teacher and Student Interviews

The teacher of record was interviewed extensively prior to the research study to gauge previous experience with using manipulatives (virtual and concrete) for teaching mathematics (see Appendix B). Additionally, the teacher explained prior teaching methods and experiences teaching solving equations. The teacher shared anticipated concerns and questions related to the three different treatments. After the interventions, the teacher was again interviewed; sharing perceived benefits and concerns for each treatment (see Appendix C). The teacher addressed concerns and questions that she mentioned in the previous interview.

Six students from each group were interviewed in small focus groups (see Appendix D). In each treatment group, the researcher identified low, medium, and high performing students on posttests and selected one student from each level and conducted two interviews per group, 6 interviews total (two from each group), with a total of 18 students. Students were given the opportunity to discuss benefits and drawbacks of the
specific learning method they used to solve equations. Additionally, they were also asked to solve equations and explain their steps and asked to rank several equations on difficulty and to explain their reasoning.

**Instructional Materials**

The researcher created learning materials for each of the three treatments (see Appendix E for the control group materials, Appendix F for the concrete manipulatives materials, and Appendix G for the virtual manipulatives materials). The researcher considered CCSSM standards and NCTM recommendations while creating student materials. For example, meeting CCSSM standards 6.EE.7 and 8.EE.7b were important goals (CCSSO, 2010). Other standards related to expressions and equations, such as 6.EE.2b, relating to identifying parts of expressions were important as well. CCSSM standards for mathematical practice such as modeling mathematics and using tools strategically were also considered while creating learning materials.

The researcher completed the instructional design process described by Smith and Ragan (2004), which included analysis of learners, learning contexts and learning tasks related to solving equations in middle school. Next, the researcher considered strategies of instructional delivery, organization of materials, and management of materials. Finally, the researcher evaluated materials in pilot studies, reflected and revised materials. The researcher relied upon twelve years of experience teaching middle school mathematics as a basis of material creation. This experience helped the researcher determine what types of equations to include and the order in which to introduce the materials. The researcher additionally consulted several prealgebra textbooks and curriculum for scope and sequence and pacing. The researcher consulted the following teaching materials to investigate uses of solving equations and algebra tiles in the classroom: Balka (1995); Howden (2001); and Witzel and Riccomini (2011).

The researcher included the same equations and the same number of equations for all three groups. The only variance in materials between the three groups was the description of use of manipulatives. Additionally, the students received a set of practice equations to use throughout the study (see Appendix H). All learning materials were reviewed by a panel of three experts in the field of mathematics education who provided
feedback and made suggestions to improve clarity. Additionally, a university faculty member reviewed the materials and provided valuable feedback.

**Collection of Data**

This dissertation research study was conducted in a rural public school in the Southeast United States. As a former teacher in the public school district, the researcher had an established relationship with teachers and administrators in the district who, when contacted, were eager to participate in research. The University of Kentucky IRB approved modifications to the virtual manipulatives pilot study described above to include this research study. Unfortunately, two weeks prior to the beginning of the 2012 school year in which the dissertation research was to take place, the teacher who conducted the pilot study was informed that she was to lose her teaching position due to budgetary restraints. This left the researcher in the position of finding a new teacher willing to participate in the research with very short notice. Fortunately, another teacher volunteered to participate. However, because of the short notice, she was uncomfortable teaching the lessons, but allowed the researcher conduct the research in her sixth grade classroom. At this point, the researcher took on the role of teacher-researcher for the study.

Sixth grade mathematics students \((n = 76)\) in three separate classes participated in the study during the first three weeks of the school year. Each class period lasted 45 minutes and data collection occurred over ten instructional days. Due to constraints, the students were not randomly assigned to groups; however, the three classes were randomly assigned to the three learning methods.

The researcher conducted a face-to-face interview with the teacher of record prior to treatment (see APPENDIX B). Although the teacher of record was not going to teach the students about solving equations, the researcher wanted to gain insight into her prior teaching experiences and knowledge about solving equations. As a middle school mathematics teacher for eighteen years, the teacher of record had insight into general challenges students face as they solve equations. The teacher of record spent the past four years teaching sixth grade students and had keen insight into their overall developmental level. It was revealed during the interview that the teacher considered herself a “traditional” teacher, meaning that she did not often use concrete manipulatives and she had never used virtual manipulatives. The teacher of record was eager to see students learn
with these methods and hoped to implement them into her classroom in the future. Additionally, the interview allowed the teacher to describe preconceived notions related to benefits and challenges attributed to each treatment. During the interview, the teacher of record described concerns related to using concrete manipulatives which resulted from the few times she had implemented them in her classroom. This interview, which lasted approximately forty-five minutes, was audio-recorded and later transcribed by the researcher.

The researcher administered a pretest to all students which included twelve equations for students to solve, as well as multiple choice and open response questions which included such topics as the meaning of the equal sign, the role of constants, and the role of coefficients. The first twelve equations were used for quantitative data analysis and the open response questions were used to evaluate student thinking qualitatively and quantitatively.

On the first day of the research project in the classroom, each student was given a booklet consisting of the same equations and explanations, the only difference being the method for solving the equations. The control group learned to solve equations by focusing on conceptual topics such as the meaning of the equal sign, inverse operations, and the roles of constants and variables. In all groups, the teacher-researcher led class discussions by discussing concepts and providing links to the real lives of students. For example, while discussing the role of the equal sign, the researcher related the equal sign to a fulcrum and to a seesaw. Students shared previous experiences with a seesaw in order to see that the equal sign represents the balance point of an equation. The teacher-researcher encouraged students to actively participate and asked questions to keep students engaged.

The control group also learned procedures and steps for solving equations. The treatment groups learned to solve equations with the same conceptual and procedural emphases, with the additional experience of working with manipulatives. The first treatment group, the concrete group, used algebra tiles. The other treatment group, the virtual group, used virtual manipulatives created by the NLVM. Each group worked through the equations in the booklet with help and instruction provided by the researcher. Students in all classes were asked to show their work and explain their thinking in the space provided for each equation. The booklets were used to evaluate student thinking
qualitatively. As each group progressed through the lessons, the researcher audio recorded most lessons (two lessons were accidently not recorded), and these sessions were later transcribed, coded, and analyzed by the researcher. The researcher decided to audio record the sessions because in a large classroom setting, a video recorder focused on one point in the room does not give the sense of activity in the entire classroom. Rather, the researcher audio recorded the sessions using an iPad which was located at the front of the room, and as the researcher walked around interacting with students, she carried it with her to record these conversations.

Each learning session for the control group and concrete group took place in the teacher of record’s classroom which is approximately twenty feet by thirty feet and includes thirty student desks, a teacher desk, and individual work stations against the walls. The classroom also has a television mounted on the wall and three computers for student use. The researcher used the white board in the front of the room while modeling equations for both the control and the concrete groups.

The researcher used magnetic algebra tiles on the white board to model all equations for the concrete group. Algebra tiles use distinct items to represent constants and variables, which can be placed on an equality mat to represent solving equations. Constants are represented by small yellow squares and variables \((x)\) are represented by yellow rectangles. Negative items are represented by red squares and rectangles. Algebra tiles also contain a large square representing \(x^2\); however, it was not necessary to include it in the students’ sets as the focus was on simple linear equations. Each student had an individual set of algebra tiles to use for solving equations by representing the constants and variables within the equations on equality mats. Equality mats are sheets of paper with two large rectangles on either side with an equal sign separating them. Students placed algebra tiles on both sides of the equality mat to represent both sides of the equation. Students removed or added pieces as necessary to isolate the variables.

The virtual group learning sessions took place in the school media center, which includes a library in two-thirds of the large room and a computer lab with thirty desktop computers in the remainder of the room. Most students in the virtual group worked individually at computers during this research study, but each day, two students had to share a computer because a few of the computers were not working properly. The
researcher projected the NLVM website on the wall using an LCD projector and also used a white board to demonstrate how to represent equations. The Algebra Balance Scale uses a balance scale and two different sized blocks, the smaller representing one unit and larger “x-blocks” representing variables. As students placed the appropriate items on the scale, it became balanced. The scale is shown to be out of balance until the problem is represented correctly, which helped students develop a proper understanding of the meaning of the equal sign. Next, students selected the operations that they need to implement (add, subtract, multiply, or divide) until the equation was solved correctly. Each step was represented on the balance and the corresponding new equation is recorded on the screen. Each group worked for twelve class periods starting with the pretest on the first day and posttest on the last day. Table 3.1 lists the objectives for each lesson.
Table 3.1
Lesson Objectives

<table>
<thead>
<tr>
<th>Day</th>
<th>Topic</th>
<th>Learning Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>Pretest</td>
<td>Students will complete pretest.</td>
</tr>
<tr>
<td>Two</td>
<td>Balance</td>
<td>Students will demonstrate the balance point of an equation by creating equations which are balanced (For example, $5 \neq 3 + 2$ or $2 + 6 = 2 + 6$).</td>
</tr>
<tr>
<td>Three</td>
<td>Checking an Equation</td>
<td>Students will demonstrate how to check the solution of an equation correctly by substituting a given value for the variable.</td>
</tr>
<tr>
<td>Four and Five</td>
<td>Addition and subtraction equations</td>
<td>Students will represent and solve addition and subtraction equations correctly.</td>
</tr>
<tr>
<td>Six and Seven</td>
<td>Multiplication equations</td>
<td>Students will represent and solve multiplication equations correctly. (Because algebra tiles and virtual manipulatives are unable to represent fractional variables, the researcher chose not to include division equations within the study).</td>
</tr>
<tr>
<td>Eight and Nine</td>
<td>Multistep equations</td>
<td>Students will represent and solve multistep equations correctly.</td>
</tr>
<tr>
<td>Ten and Eleven</td>
<td>Equations with variables on both sides</td>
<td>Students will represent and solve equations with variables on both sides correctly.</td>
</tr>
<tr>
<td>Twelve</td>
<td>Posttest</td>
<td>Students will complete posttest.</td>
</tr>
</tbody>
</table>

While walking around and interacting with students during lessons, the researcher asked students to explain their work and asked specific questions related to understanding solving equations, the meaning of the equal sign, and understanding of symbols. The researcher transcribed and coded audio sessions of all classroom activities at a later date. Upon the conclusion of student learning, on day twelve, each student was given an identical posttest in the same format as the pretest. Students were not allowed to use manipulatives (concrete or virtual) on the posttest, but they were encouraged to create a written representation as needed.

Students received a score for the number correct out of twelve on this first section of the tests. Two multiple choice questions asked students to identify correct steps in
solving an equation and one multiple choice question asked students to identify the correct solution to an equation. The remainder of the test was open-response questions. Multiple choice questions were scored either correct or incorrect.

Three open response questions asked students to create a written representation of their work while solving equations. One question asked students about the meaning of the equal sign in an arithmetic equation, while another asked students about the meaning of the equal sign within a linear equation. Three questions asked students about the role of constants, coefficients, and variables in equations. One question asked students to describe the process for checking an equation, and the final question asked students about the goal or meaning of solving equations. Pretests and posttests were identical in format; the only difference between them was the actual equations that students solved.

Open response questions related to solving equations were scored based on students arriving at a correct solution. These open response questions were also qualitatively evaluated based on criteria such as correctness of representation and explanation provided by students. Open response questions related to the meaning of the equal sign were coded by the researcher as no response, operation response, or balance response. Open response questions related to the roles of constants, coefficients, and variables were coded by the researcher as no response, accurate, or inaccurate. Finally, another category of no response was necessary when students did not answer questions, or stated, “I don’t know”.

The researcher used the quantitative score from posttests to categorize students in each group as low, medium, and high performing. Within each of the groups, the researcher selected two students at each performance level to participate in focus group interviews. The researcher conducted a total of six focus group interviews, two for each group with students in each performance level represented in each interview (n = 18). Each focus group interview lasted approximately thirty-five minutes. The purpose of these interviews was to collect data related to student perception of the treatment methods (see APPENDIX D). What do students see as perceived benefits and challenges related to their treatment method? How did the specific treatment improve understanding of solving equations, the meaning of the equal sign, and understanding of symbols? Did students benefit from recording their work on paper? During focus group interviews, students were asked to use their learning method to model solving equations. The most interactive piece
of each focus group interview was when the researcher provided the students with approximately ten equations. The students were asked to rank the equations based on difficulty. As the students interacted with each other to create the ranking, they naturally talked through the process of solving many of the equations. The interviews were audio recorded and transcribed by the researcher at a later date.

During focus group interviews, the researcher asked each group of students to rank the following equations in order by difficulty and then explain their reasoning:

$$2x = 10; \quad x + 5 = 6; \quad x + 8 = 10; \quad -3x = 9; \quad -2x = 6; \quad x - 5 = 10; \quad 2x - 4 = 6; \quad 2x - 4 = 10;$$

$$2x - 4 = 3x + 6; \quad -2 - x = 4; \quad -3x - 4 = 8.$$ For purposes of this analysis, equations $$-3x = 9, -2x = 6, -2 - x = 4,$$ and $$-3x - 4 = 8$$ were not considered because they involve negative values which proved difficult for sixth grade students to understand and solve correctly.

During the focus group interviews, as students worked together to sort the equations, many of them naturally described how to solve each equation as it was ranked.

Finally, the researcher interviewed the teacher of record to discuss her perceptions and observations of student learning. The interview allowed the teacher to describe perceived benefits and challenges attributed to each treatment. The teacher of record provided unique insight into differences in student achievement resulting from participation in each group, as well as unique benefits and drawbacks of each learning method because she was an observer of all learning activities. Additionally, as students worked to solve equations, she walked around and interacted with students. Like the researcher, she provided assistance and feedback to individual students as needed.

**Analyses of the Data**

**Research Question One**

To answer research question one, *What differences exist, if any, in student achievement as a result of using concrete or virtual manipulatives as middle school students use them to solve linear equations compared to a control group using learning methods without manipulatives?*, the researcher utilized the quantitative data collected from pretest and posttest results, and the qualitative data collected from the teacher and focus group interviews, classroom observations, and image analysis from student work. First, the researcher conducted quantitative statistical analysis comparing pretest and posttest results for each group in an attempt to determine statistically significant
differences in performance between the groups. Data analysis was facilitated by the use of Microsoft Excel and SPSS software. The researcher used Microsoft Excel to organize and display data for each of the classes. The researcher used SPSS software to test to collect descriptive statistics, and additionally to test for skewness and kurtosis, to test for normality, and to conduct statistical analysis.

In addition to quantitative data analysis, the researcher analyzed teacher and student interviews and all classroom observations. The researcher also looked for similarities in student performance; for example, the researcher looked at the reliance of students on written or visual representations as they solved equations. The researcher read and coded all entries in student materials, as well as test responses several times in order to identify themes and discrepant cases. Themes were developed deductively, as the researcher was looking for evidence to answer the first question by reading all transcripts carefully and jotting down ideas. After reading all transcripts again, the researcher made a list of topics; next the topics were clustered by themes. These themes became codes and the researcher read the transcripts again noting evidence of all themes. The researcher developed the list below of themes that informed question one:

- Written/visual representations of equations;
- Student reliance on written representations;
- Use and overuse of rules, procedures, and strategies;
- Correct and incorrect use of manipulatives;
- Student knowledge of constants and variables;
- Student knowledge of inverse operations;
- Student knowledge of equal sign; and
- Student knowledge of purpose of solving equations.

The researcher followed the research cycle suggested by Tashakkori and Teddlie (1998) while looking for themes. The researcher looked at the qualitative data with algebraic concepts in mind, but by looking at the observations and evidence in student books, other themes emerged, such as reliance on written and visual representations. Next, the researcher investigated differences between groups by conducting statistical analysis. She went back to the qualitative analysis and coding to find explanations in differences between group performances.
Research Question Two

To answer research question two, *What are the unique benefits and drawbacks associated with each type of manipulative?*, the researcher analyzed qualitative data collected from interviews, classroom observations, and image analysis from student work. This data analysis was similar to analysis for the first research question in that the researcher began looking at the data with ideas such as time constraints in mind. Further investigation yielded other themes as well. The following themes emerged as a result of coding and data analysis:

- Time considerations (time on-task, time lost);
- Student perseverance and initiative;
- Play/ distraction caused by manipulatives;
- Active and passive learning; and
- Cost and availability of resources.

**Summary of Research Procedures**

The purpose of this embedded quasi-experimental mixed methods research study was to compare student achievement as a result of using concrete and virtual manipulatives as compared to a control group without manipulatives by using solving simple linear equations as the lens. Middle school mathematics students solved equations, focusing on symbolic understanding, including the equal sign, as well as developing conceptual understanding.

Quantitative and qualitative data were collected, analyzed, and triangulated in order to inform the research questions.
CHAPTER IV
DATA ANALYSIS AND RESULTS

Introduction

The purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative. Qualitative methods such as observation, teacher interviews, and student focus group interviews were employed to inform both research questions. Quantitative methods compared results between the groups to inform the first research question. Chapter four provides a presentation of quantitative data in graphic and tabular formats mixed with qualitative results for each of the two research questions.

Research Background

As discussed in chapter three, the researcher conducted a pilot study of the virtual manipulatives material in a rural public school with a certified middle school teacher who was willing to participate in this study. Unfortunately, two weeks before the dissertation research was to take place, the teacher was informed that she no longer had a teaching position because of budgetary constraints in her school district. At this point, the researcher found another certified middle school teacher willing to participate in the research; however, this teacher was uncomfortable teaching the solving equations material using the three different methods with such short notice. The researcher made the decision to continue with the research as the teacher-researcher and provide the instruction to all three groups, thereby providing continuity in content, teacher experience, and familiarity on the part of the instructor.

The researcher is a certified middle school mathematics teacher with over twelve years of teaching experience in public and private school settings with extensive experience with teaching solving equations, as well as teaching with concrete and virtual manipulatives. The researcher had expertise in the subject matter and although she would have preferred not to teach the content, she attempted to avoid any biases toward or against any of the three learning strategies. Although the researcher used concrete and virtual manipulatives frequently to teach solving linear equations, the researcher had more
experience teaching solving equations without manipulatives and with concrete manipulatives because virtual manipulatives are a relatively new strategy. This may be considered a potential bias.

A possible bias in analyzing data from this research was the researcher hoping to see positive differences on behalf of the manipulatives. The researcher attempted to overcome these biases by insisting that all three classes solve the same linear equations and spend approximately the same amount of time on each topic. As a matter of fact, this insistence of similar time frames may have put the manipulative groups at a disadvantage because they needed more time to become familiar with the techniques associated with the manipulatives which resulted in less time solving equations. As unexpected as it was to acquire the role of teacher-researcher, the researcher was able to make the best of the situation by assuring that all students in the study had positive experiences while working within their treatment group.

Data Analysis

Research Question One

What differences exist, if any, in student achievement as a result of using concrete or virtual manipulatives as middle school students use them to solve linear equations compared to a control group using learning methods without manipulatives? The researcher attempted to blend quantitative and qualitative data to answer this question.

The purpose of conducting analysis of covariance, or ANCOVA, was to factor in pretest scores in addition to group membership to determine the outcome variable, posttest scores. The assumption of homogeneity of variance for posttest scores was violated for this set of data. In order to correct this violation and make the analysis robust, the researcher implemented a balanced design in which all three groups had the same number of data by randomly removing students from the concrete group and virtual group. Initially, the researcher removed five students who either had only pretest or only posttest data, reducing the data set to 72 students. Next, the researcher used a random number generator to remove six students from the concrete group and seven students from the virtual group to include 20 students in each group \((n = 60)\). Appendix I includes descriptive statistics of the original data set.
Results of ANCOVA analysis indicated the covariate, pretest scores, was significantly related to posttest scores $F(1, 56) = 5.165, p < .05$. Additionally, both concrete group membership $F(1, 56) = 4.77, p < .05$, and virtual group membership $F(1, 56) = 7.29, p < .05$ were significantly related to group membership (See Table 4.1). The observed power of the pretest score was .608, for the concrete group, .574, and for the virtual group, .756. According to the adjusted $R^2$ value of the model summary, 18.9% of variance was accounted for in the following model:

$$Y = 7.631 + (.25 \cdot \text{pretest}) + (-1.38 \cdot \text{concrete}) + (-1.72 \cdot \text{virtual})$$ (Table 4.2). This model indicates that the pretest scores had a positive effect on posttest scores while concrete group membership and virtual group membership had negative effects on posttest scores. The effect sizes of the covariates were as follows: pretest $\eta^2 = .084$, concrete group $\eta^2 = .078$, and virtual group $\eta^2 = .115$. These results indicated that the effect of virtual group membership was stronger than concrete group membership.

ANCOVA data analysis indicated that posttest scores for the control group were statistically significantly different than posttest scores for students in the concrete and virtual groups. This statistically significant difference was in favor of control group students. The researcher turned to qualitative data analysis to explain these statistically significant differences in performance. While analyzing classroom observations, teacher interviews, and student focus group interviews, the following themes emerged explaining differences in student achievement:

- Student representations of equations;
- Reliance on procedures, strategies, and manipulations;
- Inverse operations;
- Conceptual knowledge of solving equations;
- Meaning of the equal sign;
- Knowledge of constants and coefficients; and
- Knowledge of purpose of solving equations.
Table 4.1
ANCOVA Table Tests of Between Subject Effects

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>F</th>
<th>p</th>
<th>Partial Eta Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corrected Model</td>
<td>3</td>
<td>4.337</td>
<td>.008</td>
<td>.189</td>
</tr>
<tr>
<td>Intercept</td>
<td>1</td>
<td>95.265</td>
<td>&lt;.001</td>
<td>.630</td>
</tr>
<tr>
<td>Virtual</td>
<td>1</td>
<td>7.286</td>
<td>.009</td>
<td>.115</td>
</tr>
<tr>
<td>Concrete</td>
<td>1</td>
<td>4.767</td>
<td>.033</td>
<td>.078</td>
</tr>
<tr>
<td>pretest</td>
<td>1</td>
<td>5.165</td>
<td>.027</td>
<td>.084</td>
</tr>
<tr>
<td>Error</td>
<td>56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>60</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corrected Total</td>
<td>59</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2
ANCOVA Coefficients

<table>
<thead>
<tr>
<th>Variable</th>
<th>B</th>
<th>Std. Error</th>
<th>Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Constant)</td>
<td>7.631</td>
<td>.782</td>
<td></td>
</tr>
<tr>
<td>Pretest</td>
<td>.249</td>
<td>.110</td>
<td>.294</td>
</tr>
<tr>
<td>Virtual</td>
<td>-1.724</td>
<td>.639</td>
<td>-.385</td>
</tr>
<tr>
<td>Concrete</td>
<td>-1.376</td>
<td>.630</td>
<td>-.307</td>
</tr>
</tbody>
</table>

Student representations of equations.

To determine if differences exist in reliance on written representations, the researcher quantitatively analyzed student work on the practice equations for completeness and correctness. The researcher qualitatively analyzed student work looking for examples of correct and incorrect student thinking. Each student had thirteen equations to represent within their booklets. The researcher multiplied that number by the number of participants in each group to determine the possible number of sample equations for each group. In the control group (n = 20), there were 219 practice equations attempted, which represents 84% of the total practice equations. Of the practice equations attempted, 196 (89%) were solved correctly with a correct solution provided. In the concrete group (n = 25), there were 298 practice equations attempted, which represents 92% of the total practice equations. Of the
practice equations attempted, 273 (92%) were solved correctly with a correct solution provided. In the virtual group \((n = 26)\), there were 271 practice equations attempted, which represents 80% of the total practice equations. Of the practice equations attempted, 226 (83%) were solved correctly with a correct solution provided. For the total of all groups, there were 923 practice equations to represent. 788 were attempted (85%). For the total of all groups, 695 equations were solved correctly, which was 88% of the attempted equations. Students in the control group and the concrete group relied on written models and explanations while solving the equations more than students in the virtual group.

As expected, students in the control group represented the equations numerically. In Figure 4.1, the student solved the equation, \(x - 4 = 5\) by adding four to both sides of the equation. The student noted that \(x = 9\), and then checked the equation by substituting the solution into the original equation. The student even noted that the equation was balanced. Many students within the control group took time to check their equations after solving them. The importance of checking equations was emphasized during every learning session within all three groups, yet control group students did so more frequently.

![Figure 4.1. Control group example response in student booklet.](image)

As expected, students in the concrete group represented equations using algebra tiles. Students placed algebra tiles on both sides of the equation mat and then worked to solve the equations. For the equation in Figure 4.2, the student placed one \(x\) and three \(\text{one}\) tiles on the left side of the equation and eight tiles on the right side of the equation. Next, the student “took 1 at a time off until it was uneven.” This statement emphasized the importance of solving equations by doing the same thing on both sides of the equation.
What the student meant by working until it was “uneven” was that there was only one variable left on the left side, there were no common elements between the two sides of the equation.

\[ x + 3 = 8 \]

*Figure 4.2. Concrete group example response in student booklet.*

Students in the virtual group represented equations with virtual manipulatives from the NLVM. As students placed items on the scales, it was unbalanced until the equation was set up properly. In Figure 4.3, the student represented the equation, \( x + 3 = 8 \). The student marked out three from both sides of the equation and noted the solution, \( x = 5 \). This student did not write out a check of the solution of the equation. Students in the virtual group mentioned that this step was not necessary because the virtual manipulative is balanced when the solved correctly.

*Figure 4.3. Virtual group example response in student booklet.*
The first twelve questions on the pretest and posttest asked students to solve equations without any other directions. As expected, on the pretests, no students created a written record for solving equations. On the pretests, students only wrote the solution in the space provided. On the posttest, 14 students in the control group \((n = 20)\) (70\%) wrote steps to at least one equation without being asked to do so. Control group students performed best on the posttest, \((M = 9.1, SD = 1.37)\). On the posttest, 12 students in the concrete group \((n = 25)\) (48\%) drew a model of at least one equation. Concrete group students did not perform as well as control group students, \((M = 7.2, SD = 2.20)\). However, their scores were higher than the posttest scores of the virtual group, \((M = 6.96, SD = 2.55)\). On the posttest, 1 student in the virtual group \((n = 26)\) (4\%) drew a model of at least one equation. It appears that students in the control group relied upon writing steps for solving equations to a greater degree than those in the concrete and virtual groups. Nearly half of the students in the concrete group attempted to sketch algebra tiles, while only one student in the virtual group attempted to sketch the virtual manipulatives on the test. Student performance on posttest as evidenced by group means reflected the level of student reliance on written representation. Students in the control group used more written representations and scored highest while students in the virtual group used the least amount of written representations and scored the poorest.

Students in the control group relied upon writing steps on the posttest to the highest degree, indicating their reliance upon written representations to solve equations because they were not provided any other representations. One student expressed the importance of writing the steps for solving equations,

"It was helpful because it helps you remember the steps to do each one of the problems. I liked it when you left it (work) up on the board and we could look at it and see what you did and how you did it. It helped me figure out my own equations." (Control Group Student, focus group interview, August 22, 2012)

About half of the students in the concrete group relied on written representations to solve equations on the posttest. When asked if drawing representations helped them understand solving equations, one student stated,

"It put an image in my head of what the two sides would look like and it would also help you put the tiles, or drawings and if you have something left over, like \(x + 5\), you take them off and you take them off the other side and it tells you what \(x\) is” (Concrete Group Student, focus group interview, August 22, 2012).
This student expressed the value of making connections between the concrete manipulative, the visual representation, and the mathematics.

A different student in the control group expressed a contrary opinion, “I don’t think drawing helped me because when I drew the x’s, I drew them like squares and I get confused because they look like squares and I get confused and that is how I get them wrong” (Concrete Group Student, focus group interview, August 22, 2012). Another concrete group student indicated that she would prefer the opportunity to solve equations by hand first, and use the manipulatives as a back-up as needed. Still another student indicated she preferred using algebra tiles over solving equations on paper because she would not have to “erase over and over again” (Concrete Group Student, focus group interview, August 22, 2012).

Students in the virtual group had the most difficult time making the transition from the virtual representation to the abstract concept of solving equations. They indicated that solving equations was more difficult on paper than with the virtual manipulatives. A student stated, “Solving equations is much harder on paper because you can see it on the computer. It helps me visualize the equation, but you can’t do it on the paper. I can’t visualize the balloons on paper” (Virtual Group Student, focus group interview, August 22, 2012). Another student stated, “Well, when I solve equations on the computer, I know I am right because the scale is balanced and it says x=, but on the paper, I do not know if I am right or not” (Virtual Group Student, focus group interview, August 22, 2012). These quotes express an important consideration about virtual manipulatives; students relied on the symbolic representations provided by virtual manipulatives. Even though much time in all lessons was spent bridging the virtual manipulatives to the equations they represented, students did not make this connection easily.

When asked if drawing the steps of solving equations was beneficial, one student in the virtual manipulative group stated,

Drawing the scale was easy, but I don’t know how you could solve the equations just by drawing the scales. You are trying to write down the answer, and it is kind of hard to figure out the answer on paper, so you can go to the computer and figure it out, but you can’t really understand why it is. You can be busy adding or subtracting it, but you are not really sure why it is the answer. (Virtual Group Student, focus group interview, August 22, 2012)
This comment indicated that although students spent time working to solve equations, they were not developing conceptual understanding. Many students relied on the elements of the virtual manipulatives to solve the equations without making connections to the mathematics they represented. Instead of making connections between the manipulative and the equations they represent, students were busy manipulating in a trial and error manner, until achieving balance and finding the correct solution.

On three open response questions on the pretests and posttests, students were asked to solve the equations and provide a written representation of their solution. On the posttest, 17 students out of 23 (74%) in the control group solved the equation \(3x + 2 = 3x + 3\) correctly. 19 out of 26 students (73%) in the concrete group solved the equation correctly. 10 out of 27 students (37%) in the virtual group solved the equation correctly. Overall, 46 of 76 students (61%) solved the equation correctly. Posttest results for the second and third equations were less favorable. On the second equation, \(2p + 8 = 12\), only 26% of control group students solved the equation correctly, while 31% of the concrete group solved the equation correctly. Only 15% of the virtual group solved the equation correctly. The most common error was the same among all three groups. Students successfully subtracted eight from both sides of the equation, and rather than stating that \(2p = 4\), students stopped short, stating \(p = 4\). Students had the greatest difficulty on the third open response question related to solving equations, \(7b - 6 = 3b + 2\) because of the negative integer. The CCSSM standards for mathematics expect students to learn how to operate with integers in seventh grade (CCSSO, 2010). These sixth grade students had very limited knowledge of operations with negative integers. For this reason, the researcher did not analyze results of this test question.

These results indicated that students working with manipulatives rely on visual representation to a higher degree than written representations. Written representations were emphasized equally among all three groups, yet the control and concrete groups relied on written representations more than the virtual group. Students in the control group depended on their written representations while they were taking the posttest more than students in the treatment groups. Whereas 70% of control group students attempted at least one equation on the posttest with a written representation, only 48% of concrete group students
attempted at least one equation on the posttest with a written representation. Only 4% of virtual group students attempted at least one equation with a written representation.

Students in the control group described their written efforts as essential to understanding solving equations, while students in the concrete group offered differing opinions on the value of their written work. These results may also indicate that treatment group students had difficulty moving from concrete or virtual representations to abstract one. Students in the virtual group agreed that the written work was much more difficult than solving equations using the manipulative because they were unsure of their results. The fact that treatment group students did not have the manipulatives to use on the test may have adversely affected their posttest scores. In the final interview, the teacher of record, as an observer, indicated that drawing and writing out the steps helped students improve their understanding. She stated, “Writing or drawing the equations just gave students one more opportunity to think about each equation, to make connections between the representation and the algebra” (K. Downs, personal interview, August 21, 2012).

Reliance on rules, strategies, and manipulations.

While teaching students in all three groups to solve equations, the researcher emphasized conceptual understandings such as the meaning of the equal sign, the meanings of constants and coefficients, and the purpose of solving equations. Even with an emphasis on conceptual understanding, students focused on procedural understanding to various degrees within the three groups. Comments made by students and written data collected from student work indicated that students within the control group focused on rules more than students in the treatment groups. At the same time, students in the concrete group developed strategies for solving equations. The use of these rules and strategies were supported by conceptual understanding. Students in the virtual group manipulated equations until arriving at the correct solution. During the final interview, the teacher of record, as an observer, described,

The lessons in the control group strengthened procedural understanding more than anything. The students were very focused on procedures, especially with equations with variables on both sides. While you would give an example, they would watch very closely and imitate those steps because they did not have anything else to fall back on. (K. Downs, personal interview, August 21, 2012)
Within the focus group interviews with control group students, they emphasized steps and rules above all else. During one focus group interview, a student indicated examples worked out on the board by the researcher were essential for her to solve similar practice equations. Another student described solving equations as a series of “steps” that she would have to follow. In a typical student explanation of an equation such as \( x - 3 = 7 \), a concrete group student explained, “Since the opposite of subtract three is add, I added three and negative three plus three cancels out. And I added three to seven and I got ten. And I put ten in and ten minus three equals seven” (Field notes, August 9, 2012). Although this student described her work as steps, this control group student evidenced conceptual understanding of inverse operations and the importance of balance within an equation.

A student in the control group solved the equation \( 7b - 6 = 3b + 2 \) correctly on the posttest (Figure 4.4). Not only did he solve the equation correctly, he wrote out a description of each step that he used in solving this equation. This student had one minor error; in the steps he described, he said that he subtracted \( 3b \) from each side, but in his work, he wrote “\(-b\)” on both sides of the equation. Also, for step five, the student indicated he checked the solution of the equation; however, this check is not written on the book. He may have checked the equation mentally. This student displayed conceptual understanding of inverse operations and balance within an equation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{control_group_explanation.png}
\caption{Control group explanation on posttest.}
\end{figure}

Initially, some students in the control group had difficulty solving simple addition and subtraction equations. For example, while solving the equation \( x + 3 = 8 \), instead of
subtracting three from both sides, six students (26%) subtracted five from both sides and wrote the solution \(x = 5\). This initial misunderstanding shows that students were able to instinctively understand the solution to the equation, yet they were unfamiliar with the necessary procedures to arrive at the solution correctly. In the same equation, \(x + 3 = 8\), three students (14%) in the control group wrote they should subtract three from both sides of the equation, but arrived at the solution \(0 = 5\) because on the left side of the equation, these students subtracted three from three, when they removed the constant, they also incorrectly removed the variable at the same time. As students worked these initial equations, the researcher and teacher of record walked around the room to monitor student progress, encourage successful students, and assist these students to improve their understanding. Eventually, students in the control group became comfortable and capable of solving equations; although they focused on rules, they had conceptual development to back up these procedural understandings.

Students in the concrete group did not describe solving equations as rules and procedures; rather, students in the concrete group developed strategies for solving equations. For example, students in the concrete group developed an understanding of the concept of a “zero pair” while solving many equations involving negative integers. For example, to solve an equation such as \(x - 4 = 5\), a student placed a yellow \(x\) and four negative unit pieces on the left side of the equation while placing five positive unit pieces on the right side of the equation (See Figure 4.5). In order to solve this equation, the student added four positive pieces to both sides of the equation, in so doing, the student created a zero pair on the left side of the equation which could be removed, leaving \(x = 9\) as the solution to the equation.
Concrete group students persisted in their understanding of zero pairs, and a few days later when students were solving multistep equations, students solved the equation $5x + 2 = 7$ by creating zero pairs. Figure 4.6 demonstrates student thinking of creating and removing zero pairs from both sides of the equation $5x + 2 = 7$.

Students in the virtual group displayed mastery over the software they were using rather than developing conceptual or procedural understanding of solving equations. The following vignette provided insight into student thinking as a male student solved the equation, $-4x + 4 = -2x + 2$ within a matter of seconds. When asked by the researcher to explain his thinking or reasoning, the student had great difficulty.

**Student:** I can add $4x$, but...OK, now let’s see, I can minus two, and I divide by two and $x = 3$.

**Researcher:** How do you know that the solution is correct?

**Student:** I was going to say because when it is balanced, the scale says $x = 1$, or
That is how you know you are right.

**Researcher:** How did you know the first thing to do was to add $4x$?

**Student:** Because there was a negative $x$, so I add *four* $x$ to take away the balloons, and I just get $x$’s and boxes which is much more easier than having balloons.

**Researcher:** Well, that was good thinking, after that step, why did you subtract two from both sides?

**Student:** Well, I subtracted two because, well, I can’t think about that.

At this point, the student was unable to proceed with an explanation of an equation that one minute earlier, he solved rather quickly and without difficulty using the virtual manipulatives. At this point, the researcher asked him to solve another equation, $3x - 5 = 4$, while discussing each step as he proceeded. The student solved the equation by adding five to both sides of the equation and dividing by three. He announced the result, $x = 3$, and stated,

> And you know you are right because the virtual manipulative says $x = 3$. It gives you the solution. There is one $x$ on the left side and three boxes on the right; that is how I know I am right” (Field notes, August 22, 2012).

The researcher chose to further probe the student’s understanding of solving equations by asking him how he would know he had the correct solution without the virtual manipulatives. The student suggested drawing the blocks, stating, “Because if you drew it, one block equals 3. There are only three little blocks and one big $x$ block” (Field notes, August 22, 2012). Then he added, “You could do the problem backwards and see if it is right” (Field notes, August 22, 2012). The researcher sensed that the student was close to the idea of checking the equation, but he was having a difficult time explaining the process. The researcher explained, “Let me show you something, since you said the solution was $x = 3$, go into the equation and substitute *three* for the variable, what is three times *three* minus *five*?” (Field notes, August 22, 2012). The student enthusiastically responded, “Ooh! I did that in my head before, I did that before. I get it” (Field notes, August 22, 2012). This vignette typifies virtual group students who relied on the manipulative to solve the equations at the cost of building conceptual or procedural understanding.

While using virtual manipulatives, students were able to correctly solve equations; however, for at least some of the students, the solutions were not accompanied by a
conceptual or procedural understanding. Rather, random attempts were made at isolating the variable, revealing more of a trial and error strategy. In the final interview, the teacher of record, as an observer, agreed stating, “Some students seemed to just push buttons until something worked, without really thinking about the math that it represented” (K. Downs, personal interview, August 21, 2012).

Students in the three groups displayed reliance on rules, strategies, and manipulations to varying degrees. Control group students frequently described the work at solving equations as rules, but their explanations evidenced conceptual understanding. Concrete group students developed strategies such as “zero pairs” which they were able to use consistently throughout the material. Control and concrete group students developed conceptual understand to support the use of rules and procedures. Finally, some virtual group students manipulated equations represented by virtual manipulatives in a “trial and error” fashion until arriving at a correct solution. Manipulations on the part of virtual group students did not result in much conceptual development.

**Inverse operations.**

Students in all three groups developed understanding of inverse operations within the study of solving addition and subtraction equations. Student booklets described zero pairs; for example, \(-5 + 5 = 0\). Students were formally introduced to the *Addition and Subtraction Property of Equality* within the material. Further, when students learned how to solve multiplication equations, the idea of “undoing” by using the inverse operation was introduced. Throughout the lessons, students or the researcher used the phrases *zero pair*, *inverse*, or *opposite* a total of 65 times in the control group, 37 times in the concrete group, and 25 times in the virtual group. Although the phrases were nearly twice as common with the control group as the two treatment groups, unique elements of the manipulatives allowed students to understand inverse operations. According to the teacher of record in the final interview, as an observer,

At first, students in the control group struggled to solve equations with variables on both sides. They just did not seem to really understand what to do; how to operate within the equation. Students in the concrete group, though, benefitted because they could actually see the same variables (or constants) on both sides of the equation. That helped them know to remove them. The virtual group was able to see similar items on both sides of the equation too. They got a better feel for inverses, a better feel for negatives and positives and how they work together. (K. Downs, personal interview, August 21, 2012)
As the teacher of record, as an observer, noted, students in the control group had several common errors related to inverse operations. Among them were using the inverse operation on the same side of the equation twice as in Figure 4.7. This student attempted to solve the equation \(-2x + 7 = 3x + 2\) by first adding \(2x\) to both sides of the equation and correctly identifying \(7 = 5x + 2\) as an equivalent equation. Next, this student realized that to isolate the variable, she must subtract \(two\). However, she subtracted \(two\) from \(two\) and \(two\) from \(5x\). This error indicated that she recognized the inverse operation, but failed to realize that inverse operations must occur on both sides of the equal sign. Additionally, this error indicated a lack of understanding of the distinct roles of constants and coefficients.

![Figure 4.7](image.png)

*Figure 4.7. Control group error on same side of equation on posttest.*

The lack of recognition of correct inverse operations was another common error of students in the control group, as demonstrated in Figure 4.8. This student attempted to solve the equation \(3 + 2x = 5\). He correctly identified the first step and subtracted \(three\) from both sides of the equation, with \(2x = 2\) as the result. However, this student encountered difficulties because he used subtract \(two\) as the inverse of multiply \(two\), claiming the solution was \(x = 0\). Neither of the students with the last two errors checked the solutions to the equations.
Although the teacher of record, as an observer, described difficulties experienced by control group students; not all control group students shared these difficulties. Figure 4.9 exemplifies work from a student in the control group whom used inverse operations to solve the equation, $3 + 2x = 5$ correctly. This student first subtracted three from both sides of the equation with the result being $2x = 2$, then the student divided both sides of the equation by two, finding the solution $x = 1$. This student additionally checked the equation by substituting one for the variable.

Students in the concrete group benefitted from the physical representations of equations. While solving the equation, $3x - 3 = 6$, in front of the class, a female student described the steps she took stating,

First you put $3x$’s and three reds on the left and six yellows on the right. The reds are negative, six yellow pieces are positive. Add three yellow pieces to each side
because these make zero pairs on the left side. Now the equation is $3x = 9$ and each $x$ equals three. (Field notes, August 16, 2012)

The student quoted above confidently solved the multistep equation with assistance of positive and negative numbers being represented with different color algebra tiles. Another student in the concrete group solved the equation, $2x = 8$ in Figure 4.10. This student realized that in order to solve the equation in which the variable was multiplied by two, he would have to do the inverse of multiplication, which he recognized as division. As discussed in the previous section, students in the concrete group successfully used zero pairs to solve equations. These zero pairs helped students represent inverse operations of addition and subtraction.

$$2x = 8$$

![Figure 4.10. Concrete group inverse operations in student booklet.](image)

A student in the virtual group solved the same equation, $2x = 8$ in Figure 4.11. Students in the virtual group benefitted from the representation on the virtual manipulatives because they had to identify and click the operation they wanted to use. For example, in order to solve this equation, the student had to select divide, and then 2 for the virtual manipulative to divide the equation by two. In the final interview, the teacher of record, as an observer, described a solid understanding of inverse operations among students in the virtual group, stating,

They (virtual group students) really seemed to understand that to solve the equation, they could take away the same thing from both sides because students could actually see the same blocks sitting on both sides of the scale and watch them disappear (K. Downs, personal interview, August 21, 2012).
Although having to name the operation helped students understand inverse operations, students in the virtual group made errors with inverse operations as they solved equations. For example, after successfully representing the equation $3x - 2 = x + 4$ on the virtual manipulatives, a female student subtracted two from both sides of the equation, which made the equation more complex. She stated, “I messed up, I subtracted two from both sides” (Field notes, August 17, 2012). The researcher asked the student for the inverse of -2, and the student stated, “OK, I will add two to both sides and the equation will be the same as it was again” (Field notes, August 17, 2012). This exchange was an example of flexibility ascribed to virtual manipulatives. It was easy for this student to correct her mistake by adding the inverse. Also, this exchange demonstrates how a physical representation allowed the student to recognize her error.

Based on posttest data and qualitative evidence, students in all three groups increased understanding of inverse operations by participating in the study. Students in the control group had difficulties initially as they solved equations; but as students developed conceptual understanding of inverse operations, control group students became more successfully using inverse operations to solve equations. Students in the concrete group were assisted by the visual representation of objects and most were able to proceed with inverse operations correctly because they developed conceptual understanding of zero pairs. Students in the virtual group benefitted from having to select the operations on the screen and seeing the results of the selected operations.
Meaning of the equal sign.

Researchers agreed that conceptual understanding of the equal sign is essential to solving equations (Kieran, 1992; Kilpatrick et al., 2001; Knuth et al., 2006; Rojano, 2002). During the study, the researcher emphasized balance as a meaning of the equal sign to all three groups. The researcher provided an analogy of a see saw for all three groups during the first day of the study. Additionally, the researcher described the fulcrum of a scale as the balancing point, suggesting that the equal sign played the role of the fulcrum in an equation. Within the course of the study, students or the researcher mentioned the phrases equal sign, balance(d), see saw or fulcrum 183 times with the phrases mentioned 59 times in the control group, 59 times in the concrete group, and 65 times in the virtual group.

During the initial interview before the researcher study took place, the teacher of record reflected on her eighteen years as a middle school teacher and expressed concern about student lack of understanding of the equal sign stating,

I do not think they (students) understand the equal sign. That was really made aware to me when I moved to the sixth grade. I know just because you talk to them about greater than, less than, and what equals really means, it is like they see, but they don’t really understand that what is on one side is the same as on the other side, it is just two different representations. My experience the last two years is that students do not understand what the equal sign means. It is just amazing that they do not grasp that. I think it is something that teachers just assume, they have seen it since kindergarten or first grade, but they have never understood that it (the equal sign) means a relationship. (K. Downs, teacher interview, August 1, 2012)

The teacher of record, in her final interview, described perceived benefits of the manipulatives stating,

The (concrete group) students could actually see that it is the same on both sides. The pieces on both sides are physically the same, so they can see that they can take them away. For virtual manipulatives, balance was really emphasized. Students really understood as they saw the scale go up and down as things were placed on and removed from the scale. (K. Downs, teacher interview, August 21, 2012)

The researcher investigated previous studies conducted by Rittle-Johnson and Alibali (1999) related to student understanding of the equal sign. The researcher used similar coding techniques as Rittle-Johnson and Alibali (1999) to score pretest and posttest questions related to the meaning of the equal sign. On pretest and posttests, the researcher asked what the equal sign means in the arithmetic problem $2 + 5 = 7$. The researcher considered responses such as “it tells you the answer”, or “it tells you to add” as operation
or answer responses. The researcher considered responses such as “the same thing on both sides”, or “the balance point” as balance responses. Finally, another category of no response was necessary when students did not answer the question, or stated “I don’t know”. The next question on pretests and posttests asked students the meaning of the equal sign in the equation $2x + 4 = 12$. Results for each question are found in Table 4.3.
Table 4.3
Comparison of Responses Related to the Meaning of the Equal Sign

<table>
<thead>
<tr>
<th>Response</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Control</td>
<td>Concrete</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 + 5 = 7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Operational/ Answer</td>
<td>13 (57%)</td>
<td>3 (12%)</td>
</tr>
<tr>
<td>Balance</td>
<td>1 (4%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>No Response</td>
<td>9 (39%)</td>
<td>23 (88%)</td>
</tr>
<tr>
<td>2x + 4 = 12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Operational/ Answer</td>
<td>6 (26%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Balance</td>
<td>1 (4%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>No Response</td>
<td>16 (70%)</td>
<td>24 (92%)</td>
</tr>
</tbody>
</table>

Posttest data did not indicate much understanding of the equal sign as a balance point. For all groups, there was an increase in balance responses between pretest and posttest responses on the algebraic equation. More students described the equal sign as related to balance in the control and concrete group related to the arithmetic equation.

Despite the negative results on pretests and posttests, during focus group interviews, students in all groups described the equal sign as related to balance. In the control group focus group interview, a student described the equal sign,

I think the equal sign means that whatever is on the left side is going to be the same as whatever is on the right side. It is like a balance beam, like 5x + 4 is going to equal whatever 3x + 2 equals. (Control Group Student, focus group interview, August 22, 2012)

Other students within the control group focus group interview agreed and reiterated these statements.

As students solved equations using concrete manipulatives, they placed items on an equation mat to represent the components of each equation. There was an equal sign between the two sides of the equation mat. The act of physically adding or removing items from both sides of the equation seemed to help students understand the importance of
balance within equations. During a focus group interview of concrete group students, a female student claimed that the equal sign comes at the end of the equation and “you have to figure out what the answer will be” (Concrete Group Student, focus group interview, August 22, 2012). Another student related the equal sign to equivalent fractions, but in the end shared, “It has different numbers on the scale, but the $x$ and the $x$ on the other side equals the same thing” (Concrete Group Student, focus group interview, August 22, 2012). An interesting response by a concrete group student was “That it shows they will both be the same in the end, both of the number things, the tiny equations are equal” (Concrete Group Student, focus group interview, August 22, 2012). This student developed a misconception, but in the end, she realized that both sides of the equation were equivalent.

Students in the virtual group had the advantage of actually using a working scale to solve equations. As elements of an equation were placed on the scale, it became balanced until the equation was represented correctly. As soon as an equation was represented correctly, the scale balanced and the student was allowed to proceed. All actions enacted by the student occurred on both sides of the equation. These visual clues helped students understand the meaning of the equal sign as a balance point within an equation. A virtual group student described the meaning of the equal sign,

It means like both the same, if you put too much, or not enough, it will lower it or higher it. The equal sign also means you have to do the same thing on both sides; it is like the scales of justice. It can have different things on either side, but they are equal. (Virtual Group Student, focus group interview, August 22, 2012)

While providing this explanation, the student was using both hands to gesture a scale rising, falling, and becoming balanced. The visual representation of the scale helped this student understand the balance aspect of the equal sign. While solving the equation $x + 4 = 5$, a student set up the equation as $x$ minus four equals five, because the model did not represent the equation, the scale was not balanced. The student easily realized her error and was able to correctly represent the equation. Other students within the virtual manipulative group made similar errors and were able to recognize their mistakes because of the guidance provided by the virtual manipulative.
Role of constants and coefficients.

Students must understand the distinct roles and meanings of constants and coefficients in order to solve equations successfully (Poon & Leung, 2010). On pretests and posttests, students were asked to describe the meaning of the 2 in the equation $2x + 3 = 11$. The researcher considered responses such as “how many $x$’s there are in the equation” or “multiply the two by the variable” as correct responses. Additionally, the researcher considered responses where students solved the equation correctly or drew a correct representation of the equation as correct. Incorrect responses included such answers as “You add two to the three” or “A two digit number.” Additionally, the researcher tabulated blank responses or responses of “I don’t know.” Table 4.4 displays results for each group on pretests and posttests.

Table 4.4
Comparison of Responses Related to Understanding the Role of Coefficients

<table>
<thead>
<tr>
<th>Response</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Control</td>
<td>Virtual</td>
</tr>
<tr>
<td>Correct</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(26%)</td>
<td>(26%)</td>
</tr>
<tr>
<td>Incorrect</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>(13%)</td>
<td>(23%)</td>
</tr>
<tr>
<td>No Response</td>
<td>14</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>(61%)</td>
<td>(73%)</td>
</tr>
</tbody>
</table>

Based on the data in Table 4.4, students in all three groups increased understanding of the meaning of a coefficient as a result of participating in this study. Students in the control group performed 133% better on the posttest while concrete group students performed 1400% better and virtual group students performed 57% better. It is interesting to note that fewer students in all groups chose to provide no response on posttests than on pretests. This result may indicate more confidence on the part of students in all groups.

During focus group interviews, control group students correctly described the roles of coefficients and constants in the equation $5x + 4 = 3x + 2$. For example, a control group student stated, “It means five times something plus four equals three times something plus two. I think it means there are five $x$’s on the problem and you are adding those five $x$’s plus four and that is it” (Control Group Student, focus group interview, August 22, 2012). Control group students developed conceptual understanding of unique roles of constants and coefficients.
Even though control group students increased correct responses on the posttest related to the role of the coefficient, there were difficulties identified by the researcher. When students solved multistep equations, most equations started with the variable first, then the constant such as $2x + 3 = 5$. However, when students encountered an equation starting with a constant, several students had difficulty proceeding. For example, while solving the equation $3 + 2x = 5$, several students attempted to subtract two from both sides; treating the two as a constant, rather than a coefficient. The confusion arose from the position of the variable in the equation. The researcher reminded students to consider the meaning of the numbers and they were able to realize that they should subtract three from both sides. This was an example of inflexible thinking on the part of control group students who were unable to put aside a rule for the sake of using a mathematic concept properly. This type of error did not occur in treatment groups because of the distinct representations of items with the respective manipulatives.

Another control group student demonstrated a lack of understanding of constants and coefficients on the posttest (Figure 4.12). In this equation, the student subtracted $2x$ from both sides of the equation, but failed to leave the one $x$ in the equation and added the one which should have been a coefficient to the two which was a constant and erroneously solved the equation as $x = 3$.

16) Solve the equation and sketch all steps. $x = \underline{3}$

\[
3x + 2 = 2x + 3
\]

\[
\begin{align*}
\underline{3x} & \quad \underline{2x} \\
1 + x & = +3 \\
3 & =
\end{align*}
\]

*Figure 4.12. Control group coefficient error forgetting the variable on posttest.*

Another student made a similar error (Figure 4.13) by subtracting $4x$ from the left side of the equation while subtracting 4 from the right side of the equation. This student was unable to proceed in solving this equation because they were unable to recognize their
error. On the right side of the box, the researcher helped the student solve the equation correctly by modeling the correct solution.

\[ 4x - 1 = 7 \]

![Figure 4.13. Control group error disappearing variable in student booklet.](image)

A student in the control group made an error on the posttest (Figure 4.14). Rather than reading \(3x\) as a number multiplied by the unknown variable, she saw the three as the tens digit and for an unknown reason, the student thought that the \(x\) was equivalent to \(four\), so the student read the left side of the equation as \(34 + 2\) while reading the right side of the equation as \(24 + 3\). Next, the student found the sum of both sides of the equation and said the variable was equal to \(63\). This error was not unique to control group students as a student in the virtual group made a similar mistake on the posttest. These errors demonstrate a lack of conceptual understanding of the role of constants and coefficients on the part of control group students.

16) Solve the equation and sketch all steps.

\[ \frac{63}{3x + 2 = 2x + 3} \]

![Figure 4.14. Control group error coefficient as place value on posttest.](image)
Students in the concrete group reported that distinct items representing constants and coefficients helped them understand their roles as they solved equations. In the equation, \(5x + 4 = 3x + 2\), a student succinctly described the roles during a focus group interview, “I would say that the five stands for how many \(x\)’s and the four stands for how many ones” (Concrete Group Student, focus group interview, August 22, 2012).

A few errors emerged as the concrete group initially solved equations. As students learned how to solve multiplication equations, several students set up the equation \(2x = 8\) incorrectly. Six students (23%) set up the equation \(2x + 2 = 8\), and three students (12%) set up the equation as \(x + 2 = 8\). The researcher asked students to think about what each piece in the equation represented, so students corrected their mistakes. A second, more common error emerged as students solved multistep equations such as \(5x + 2 = 7\). Students were able to correctly subtract two from both sides of the equation but rather than saying the solution was \(x = 1\), the students were confused because even though students paired one \(x\) with one constant, they saw five on the right side of the equation and mistakenly claimed the solution was \(x = 5\). An example of this error was represented in Figure 4.15. On the posttest, the student removed eight from both sides and \(2x = 4\) was left, but rather than divide both sides of the equation by two, the student claimed the solution was \(x = 4\) (even though the variable in the equation was \(p\)).

![Figure 4.15. Concrete group coefficients example on posttest.](image)

When asked about the role of constants and coefficients in an equation, students in the virtual group referred to the representations they experienced in the virtual manipulative. They recognized that in the equation \(5x + 4 = 3x + 2\), there would be five \(x\)
blocks and *four one* blocks on the left side of the equation. When asked by the researcher the difference between a coefficient and constant, a female student in the virtual group stated, “They are really not that different because they are both boxes, just the $x$ is bigger” (Field notes, August 17, 2012). This student did not demonstrate a conceptual understanding of the different roles of constants and coefficients.

A student in the virtual group made an error similar to a student in the control group by misunderstanding the role of a coefficient as a number in the tens digit. In the equation $x + 7 = 2x + 5$, the student solved the equation correctly and discovered that $x = 2$, but when the researcher questioned the student, she stated, “If I plug in 2 where the $x$’s are, both sides will be the same, 27” (Field notes, August 17, 2012). The researcher further probed the student’s understanding and she stated that *twenty-two* plus *five* equals *twenty-seven*.

Students in all three groups increased conceptual understanding of constants and coefficients as demonstrated by differences in pretests and posttests. This increase of understanding was assisted by representations from the manipulatives and emphasis placed on the elements of an equation by the researcher. The concrete group experienced the most growth (1400% increase) on posttest scores related to the meaning of coefficients which may be the result of distinct representations of variables and constants. Control group participants improved posttest scores 133% over pretest scores on the question related to the meaning of the coefficient. Students in the virtual group experienced a 57% increase in posttest scores over pretest scores on the question related to the meaning of the coefficient. As students set up the equations, they used different sized boxes to represent $x$’s and constants. At least one student failed to see the difference in the meanings and only described the difference in size.

**Purpose of solving equations.**

As the researcher designed the pretest and posttest, she wanted to see if students understood the purpose of solving equations and phrased the question as, “What is the purpose, meaning, or goal of solving equations?” Apparently the question was not phrased well because students did not respond correctly. No student correctly described solving equations on the pretest, and only five students correctly described the goal of solving equations as identifying the value of the variable on the posttest. Three students in the
concrete group and two students in the virtual group provided these correct responses. The incorrect responses of students were interesting and sometimes amusing. One student described the purpose,

That you get it right and accomplish what you couldn't have done before. And knowing you got an A or 100% and knowing you did a great job. That is what I think the purpose is oh yeah and knowing you are good at it. (Virtual Group Student, posttest)

Another student described the goal of solving equations as “The purpose for solving equations is so you can be really smart and get a really good career such as a scientist” (Concrete Group Student, posttest). Several students suggested doing their best and learning as the goal of solving equations. Many others emphasized preparing for the future as a goal of solving equations.

Students in the concrete group knew they successfully solved equations when there was only one variable on one side of the equation. In the final interview, the teacher of record, as an observer, described,

They (concrete group students) knew that they needed only one $x$ on one side of the equation to solve it. Some students still had trouble with multiplication equations when they had to figure out what $x$ was, for example, in $2x = 4$, they would say $x$ is four, rather than $x = 2$. (K. Downs, personal interview, August 21, 2012)

Students in the virtual group knew they successfully solved equations because when they were solved correctly, the virtual manipulative displayed the solution. This feedback helped students understand that in order to solve equations, students must isolate the variable. Again, in the final interview, the teacher of record, as an observer stated,

Students (in the virtual group) understood they had to take steps to get the $x$ by itself. They could see that better than in concrete or traditional (control). They understood that the goal of solving equations was to get the $x$ by itself on one side of the scale. (K. Downs, personal interview, August 21, 2012)

On the posttest, few students were able to describe the goal of solving equations. However, while investigating qualitative data, students were able to describe the goal of solving equations. More importantly, students in all three groups evidenced understanding the goal of solving equations by solving many equations correctly on pretests, posttests, and during observations and student focus group interviews.
**Triangulation of data.**

ANOVA analysis determined that posttest scores were statistically significantly related to pretest scores and group membership in favor of the control group. Students in the virtual group performed more poorly than students in the control or concrete groups. After uncovering quantitative differences related to student performance, the researcher turned to qualitative data analysis to uncover explanations for these statistically significant differences. Qualitative analysis uncovered differences between groups related to:

- Student representations of equations;
- Reliance on procedures, strategies, and manipulations;
- Inverse operations;
- Conceptual knowledge of solving equations;
- Meaning of the equal sign;
- Knowledge of constants and coefficients; and
- Knowledge of purpose of solving equations.

Students in the control group depended on written representations the most within the three groups because they had no other representation to utilize. Depending on the written representation helped the control group develop conceptual understanding which they effectively used as they solved equations on the posttests, resulting in statistically significantly different results over concrete and virtual groups.

Students in the concrete group were able to develop strategies such as using zero pairs by manipulating algebra tiles. These strategies evidenced conceptual understanding on the part of concrete group students. Virtual manipulative students were able to solve equations using virtual manipulatives, but they did not develop connections between the manipulatives and the mathematics they represent.

Students in all three groups increased understanding related to inverse operations during the study. Control group students had no representation to depend on, but developed conceptual understanding of inverse operations. Students in the concrete group were able to physically see items on both sides of the equations which guided their attempts to solve. Students in the virtual group named the operations which were necessary as they solved each equation, which was beneficial.
Students in all three groups improved understanding of the equal sign. The concrete group benefitted from the representation of objects on both sides of the equation. As students solved an equation, they had to act on both sides to solve correctly. Students in the virtual group were able to understand equality because of the scale on which they solved equations.

Students in the control group initially had the most difficulty understanding the roles of constants and coefficients within equations. The concrete group increased understanding of these roles because of the unique representations provided by algebra tiles. The virtual manipulatives provided similar benefits. In order to solve equations, the variable must be isolated on one side of the equation. The representations of the concrete and virtual manipulatives helped students see the solution when the variable was by itself on one side. Students in the virtual group additionally received feedback which indicated that their solution was correct.

While investigating student achievement of the three groups of students, the qualitative data mirrored results of quantitative data analysis. Students in the control group performed statistically significantly better on posttests as a result of conceptual understanding developed during the study. Although they did not experience concrete or virtual manipulatives, the emphasis placed on conceptual development helped them understand solving equations. Although concrete and virtual group students did not perform as well as the control group on posttests, each manipulative provided unique benefits and challenges as students used them to solve linear equations. These unique benefits and drawbacks provided by manipulatives were the focus of research question two.

**Research Question Two**

*What are the unique benefits and drawbacks associated with each type of manipulative?* Results of research question one indicated that students in the control group performed statistically significantly better on posttests than their counterparts in the concrete and virtual groups. This finding contradicts results of many researchers who indicated that manipulatives were effective strategies for teaching mathematics (Durmus & Karakirik, 2006; Lamberty, 2007; Lee & Chin, 2010; Martin, 2008). The researcher investigated qualitative and quantitative data to find evidence of benefits and drawbacks
for each manipulative. While coding and analyzing qualitative data, the following themes emerged related to the benefits and drawbacks of each manipulative:

- Time considerations (time on-task, instructional time lost);
- Student perseverance and initiative;
- Play/distractraction caused by manipulatives;
- Engaged learning; and
- Cost and availability of resources.

Although students in the treatment groups did not perform as well as students in the control group on the posttest, qualitative analysis revealed unique differences in student achievement between the three groups.

**Time considerations.**

Because the researcher kept the amount of time each group spent on solving equations the same, the two treatment groups may have been at a disadvantage. Students in the treatment groups lost classroom time due to administrative tasks as concrete manipulatives were distributed and collected. Virtual group students lost class time as they logged in and out of computers and as webpages were loaded. More importantly, treatment group students were responsible for learning two sets of information. First, students had to learn how to operate their respective manipulatives and then they had to learn how to solve equations. The time students spent learning how to use the algebra tiles or the computer applet was time that they were unable to learn about solving equations. However, as students worked with manipulatives, they may have developed conceptual understanding of topics represented with the manipulatives which may be prior knowledge later as students work to solve equations.

For example, during the third day of the research study, as concrete group students learned how to solve multiplication equations, several students struggled to set up the equation $2x = 8$. Several students initially set up the equation incorrectly, they set up the equation $2x + 2 = 8$, and a few set up the equation as $x + 2 = 8$. It was necessary for the researcher to ask students to think about what each piece in the equation represented and it took time for students to realize the correct representation of the equation included *two* $x$’s on the left side of the equation and *eight ones* on the right side. This time was spent
developing student understanding of algebra tiles which meant that students lost time developing understanding of solving equations.

Students in the virtual manipulatives group similarly lost class time as they learned how to enter an equation into the software correctly. For example, to solve the equation, 2x + 4 = 6, students needed to understand where to place each numeric value within the virtual manipulative (Figure 4.16). The discussion that took place related to setting up this equation emphasized the distinct roles of constants and coefficients. Students lost time learning about solving equations because of the time it took to learn how to operate the virtual manipulative. A few students entered an incorrect equation and solved it. During class discussions, they could not understand why their solution was “correct” according to the virtual manipulative, but different from their classmates. It was when the researcher noticed that they entered the equation incorrectly that they realized their error.

Students in concrete and virtual manipulatives groups lost time to administrative tasks such as distributing algebra tiles and logging in and out of computers daily for the virtual manipulatives. Of the 450 minutes allocated to each group for learning to solve equations, the concrete group lost approximately 60 minutes (13%) to these administrative tasks. The virtual group lost approximately 70 minutes (15%) to these administrative tasks. More time was lost to teaching students how to use the specific manipulatives as well. It is more difficult to estimate this loss of time because conceptual understanding developed even as students learned how to operate the manipulatives.

![Image of NLVM screen for entering equations.](https://www.matti.org/)

*Figure 4.16. NLVM screen for entering equations. © 2010 MATTI Assoc. and Utah State University. Used with permission.*

**Concrete manipulatives.**

The unique representations provided by the concrete manipulatives, engagement on the part of students, and the opportunity to reflect on mathematics were benefits identified
by the researcher. The necessary time for distributing, collecting, and learning how to use the concrete manipulatives, availability of resources, and the urge to play with concrete manipulatives were observed drawbacks to the use of concrete manipulatives for solving equations.

**Benefits of concrete manipulatives.**

While solving equations with concrete manipulatives, students used algebra tiles to represent items within equations. The representation of variables as rectangles and constants as squares helped students differentiate between their roles. After working with the manipulatives, students were able to distinguish between coefficients and constants and realize that they were not interchangeable. For example, in the equation $5x + 4 = 3x + 2$, students were able to recognize that the *five* tells them how many variables were on the left side of the equation, and the *four* tells them how many constants were on the left side of the equation. This was a beneficial representation because students were able to remove the same items from both sides of the equation, leaving $2x$ on the left side. Next, students removed *two* from both sides leaving *two* on the left side of the equation. A student in the concrete group described the algebra tiles representation as “clues” as to how to solve equations. The unique representations of positive numbers and variables as red squares or rectangles helped students as well. While solving the equation $3x - 4 = 8$, a male student in the concrete group explained,

I put $3x$’s over here and *four* red, then *eight* yellows (on the right side) and you need to add positives to make it *zero*. Then you take it away and what you did on that side, you have to do to the other side and that would make it *twelve*. $3x$’s equals *twelve*, so $x$ equals *four*. I know I am right because *three* times *four* is (the same as) *twelve* minus *four* (which is) is *eight*. (Concrete Group Student, focus group interview, August 22, 2012)

This student benefitted from the distinct representation of coefficients and constants as unique representations for negative and positive numbers. After observing all activities, the teacher of record described a perceived benefit of concrete manipulatives as a representation of common elements on both sides which helped students understand how to proceed in solving equations.

As the researcher observed students using algebra tiles to solve equations, she was impressed by the engagement and interest of students. All students but one female student actively participated in classroom activities throughout the study. Most students were
willing to set up the equations and engage the manipulatives in order to solve them. In the ten days of the study, the researcher only had to correct minor behavior issues two times when students became talkative during the last two days. This active participation helped students increase understanding. During focus group interviews, students expressed positive opinions about working with algebra tiles and claimed that working with them helped them understand solving equations better. At the end of a class, a female student approached the researcher and thanked her for the time spent in the classroom. The student stated, “Thank you for your time, I really understand how to solve equations and I am not really that good at math” (Concrete Group Student, personal correspondence, August 17, 2012).

As students worked with the concrete manipulatives, the distinct representations and active engagement allowed students to reflect on their learning and the meaning of the mathematics the manipulatives portrayed. Students were able to recognize errors in their thinking and correct themselves as they worked. One student, referring to the work with the concrete manipulatives and the time spent drawing the equations stated,

It put an image in my head of what the two sides would look like and it would also help you put the tiles or drawings and if you have something left over, like $x + 5$, you take them off and you take them off the other side and it tells you what $x$ is. (Concrete Group Student, focus group interview, August 22, 2012)

The researcher spent time with individual students as they were working with the manipulatives and noted that students in the concrete group required less personal assistance than students in the control group. The concrete group students were able to represent the equations and then use them to develop strategies for solving the equations. Students in the concrete group were willing to persevere in solving equations by using manipulatives to guide their thinking and actions.

**Drawbacks of concrete manipulatives.**

Before implementing concrete manipulatives in the classroom, a teacher must consider the amount of time necessary and the availability of resources. Additionally, the teacher must recognize the possibility that students will misuse the manipulatives and treat them as toys.

It took approximately six minutes of the class time to distribute and collect the algebra tiles each day of the study which resulted in 13% less learning time. More time
consuming, however, was learning how to use the manipulatives. During the first class period, the researcher took the first fifteen minutes to describe the roles of individual pieces. She noted the different shapes of the variables and constants, as well as different colors of positive and negative items. Throughout the research, students were reminded of the meaning of different colors as they placed negative pieces on the scale when they should have placed positive pieces, for instance. As students encountered more complex equations, the researcher took more time to explain how to set up the equations. It was necessary for students to learn how to set up the equations properly and conceptual understanding developed as students learned how to use algebra tiles. During focus group interviews, concrete group students suggested that they would have benefitted from more practice problems and more time to learn about solving equations.

Another drawback of concrete manipulatives is availability. Concrete manipulatives may be purchased or created by teachers or students as financial resources or time allow. Prior to conducting research, the researcher purchased a class set of algebra tiles, a set of magnetic algebra tiles, and accompanying books. The expense for these items was nearly $100. Recent economic conditions may limit the availability of resources for teachers and students. Due to budget constraints, the teacher of record was unable to purchase these manipulatives for her classroom and would not have had them without participating in this research study. Teachers could create algebra tiles with paper, however, this would be time consuming.

Another drawback of concrete manipulatives is constraints of the manipulatives. Teachers can be flexible in their use of algebra tiles. For example, algebra tiles can be used for solving equations, multiplying binomials, counting, sorting, and much more. However, algebra tiles cannot successfully represent fractional values. This constraint makes using algebra tiles to represent division equations difficult.

A final drawback of using concrete manipulatives was the tendency for students to play with and drop the concrete manipulatives. Although most students were actively engaged throughout the study, isolated students were observed using algebra tiles to create checker board patterns, smiley faces, or castles during lessons. In the focus group interview, a male student discussed the fact that he could not resist the temptation to create objects with his algebra tiles. As a matter of fact, as he was demonstrating how to solve an
equation during the focus group interview, one side of his equation looked like a smiley face. On average, nine algebra tiles were dropped on the floor by students each day. Over time, this could result in a decrease in the quantity of algebra tiles available to students.

Concrete manipulative such as algebra tiles provide unique opportunities to represent elements of linear equations. As a result of using concrete manipulatives, students reflected on their actions and persevered in solving linear equations. These benefits assisted concrete group students in performing statistically significantly better on posttests than on pretests. However, drawbacks of concrete manipulatives included learning time lost as manipulatives were passed out and collected and as students learned how to use the manipulatives. Further, some students used concrete manipulatives as toys or dropped them on the floor. Finally, constraints such as availability of resources and inflexibility of manipulatives must be recognized.

**Virtual manipulatives.**

Unique representations and feedback provided by the virtual manipulatives allowed students to persevere in solving equations and think intuitively as they worked. The necessary time for logging into the website and the urge to play with virtual manipulatives were observed drawbacks of using virtual manipulatives.

**Benefits of virtual manipulatives**

Virtual manipulatives provided feedback which helped students understand the mathematics represented. As students set up equations, the scale balanced automatically if set up correctly. Unlike the other learning strategies; students using the virtual manipulates were confident in their set up of each equation because of this feedback. Also, as students solved equations, when they arrived at the correct solution, the scale was balanced and the solution was displayed on the screen, allowing students to know with certainty that their solution was correct. One student explained the value of this feedback,

> If it (the equation) is not balanced, I know I am not right. If it is balanced, I can see when it is right. Also, the computer helps me see that I need to either add, subtract, multiply, or divide (Virtual Group Student, focus group interview, August 22, 2012).

Another student commented,

> The computer helped because if it is wrong, it is unbalanced and I know it. If it is right, I also know it because the equation is balanced. And after I worked out the
equation, it says $x =$, so I know that I am right or wrong. (Virtual Group Student, focus group interview, August 22, 2012)

Students benefitted from the balance aspect of the virtual manipulatives as well as the feedback provided as students selected operations. If students selected an incorrect operation, the screen displayed a message letting them know that their selected operation was not possible. For example, as a male student solved the equation, $3x + 3 = 6$, the student attempted to divide the equation by two, and the virtual manipulative displayed a message that the equation would not be simplified by the action (Figure 4.17). After reading this statement, the student realized he should have divided by three to simplify the equation. Another student added,

If I didn’t know what to do, and I tried different things, and I clicked the wrong thing, it would tell me. I can also go back and fix it by undoing what I just did if it was not right. (Virtual Group Student, focus group interview, August 22, 2012)

Another student described,

How you could take the times and plus and press the button, and it would tell you it is not the right thing to do, like it would tell you it would be too big for the scale. That helped me out the most, trying to figure out the steps to do. (Virtual Group Student, focus group interview, August 22, 2012)

As students completed each step while solving equations, the resulting equation was displayed in the equation window (Figure 4.18). Students in the pilot study conducted by the researcher relied heavily on the equation window as a clue to understand how to solve the equations. During this research study, as the researcher demonstrated and discussed to solve equations, she emphasized the feedback provided by the equation window. Unlike students in the pilot study, however, students in the virtual group of this research study did not utilize the equation window. During focus group interviews, students indicated that they did not find this feedback to be useful. The age difference of a year and a half between pilot study students who were finishing seventh grade and virtual group students who were beginning sixth grade may partially explain the difference.
Other unique representations by virtual manipulatives helped students understand the mathematics represented. Particularly, the representation of negative numbers and variables with balloons and positive numbers with blocks helped students understand the relationship between positive and negative numbers. The representation helped students develop a conceptual understanding of the relationship between positive and negative numbers. A student described the relationship between positive and negative numbers,

Well, they both mean one, but the balloons mean you are taking away one and the boxes mean you are adding one. They both are adding in a different way. The balloons are adding down to negatives and the boxes are adding up to positives. (Field notes, August 17, 2012)
Although this explanation is naïve, this student developed an understanding of the opposite nature of positive and negative numbers as a result of using the virtual manipulatives. Another student had a more developed understanding of absolute value, stating, “The balloons are negative and you have to add however many positives that is the number of negatives and that cancels them out” (Field notes, August 17, 2012).

Using virtual manipulatives allowed students to think intuitively and persevere in solving difficult equations. While observing student work, the researcher noticed that students in the virtual group were willing to keep trying and of the three groups, they were least likely to give up. In the final interview, the teacher of record, as an observer, described student perseverance,

Students in the virtual group attempted all problems. The students were not afraid to try; they would try difficult problems and work without asking for help. Because they had the virtual manipulatives, they were not intimidated.
(K. Downs, personal interview, August 21, 2012)

The researcher also observed that students in the virtual group were more willing and comfortable with helping each other as they solved equations. Many students leaned over to the computer at the next station and offered suggestions as their classmates encountered difficulties. The positive experiences with video games on the part of students seemed to translate into comfort and ease as students used virtual manipulatives making virtual manipulatives a natural experience.

Because the virtual manipulatives group worked in the school media center, occasionally other students and teachers witnessed the virtual manipulatives. On one occasion, eighth grade students were in the media center and they had positive responses to the virtual manipulatives. These eighth grade students described virtual manipulatives as cool and fun. Another day, the school principal walked in the media center and asked students to describe how to use the virtual manipulatives. The school principal complimented students about how hard they were working.

Finally, another benefit of virtual manipulatives included the ability to use virtual manipulatives at home. During the third day of the study, a male student excitedly discussed showing his parents virtual manipulatives at home. A benefit of virtual manipulatives is that they can be accessed at any time in any location with computer and
Internet access. Students can access virtual manipulatives in after-school programs or at home.

**Drawbacks of virtual manipulatives.**

Although the researcher described feedback provided by virtual manipulatives as a strength; she also acknowledged feedback as a drawback. Because virtual manipulatives provide the solution in the form of $x = _$, students were certain their solutions were correct. Rather than making connections between algebra and the virtual manipulatives, some students simply manipulated equations in a trial and error method until arriving at a correct solution. In the final interview, the teacher of record, as an observer, described, “Some students seemed to just push buttons until something worked, without really thinking about the math that it represented” (K. Downs, personal interview, August 21, 2012). Although these students produced correct solutions while using the virtual manipulative, they had great difficulty on the posttest because they failed to develop conceptual understanding while solving equations. Virtual group students had difficulties transitioning from concrete representations to abstract concepts.

While working with virtual manipulatives, several students were distracted by the balloons representing negative constants and variables. These students were continuously busy placing random balloons on the screen, rather than paying attention to instruction. This distraction occurred on a daily basis with approximately five students (19%). Frequently, when the researcher walked around to monitor student work and interact with students, these students had screens full of balloons and little other work. These young students were more interested in playing with onscreen balloons than learning about how to solve equations.

The first day of research, the virtual group was unable to use the computers because the students needed unique login and passwords provided by the teacher. This took the entire class period. This lost time was made up on another day. Each day, it took about five minutes to get all students logged in to their computers and ready to work. At the end of each day, it took approximately two minutes to log off and shut the computers down. This time added up to at least 70 minutes (15%) of lost learning time.

Students in the virtual manipulatives group also lost learning time as they learned how to enter an equation into the software correctly. For example, to solve the equation,
2x + 4 = 6, students needed to understand where to place each numeric value within the virtual manipulative. The discussion that took place related to setting up this equation emphasized the distinct roles of constants and coefficients. Students lost time learning about solving equations because of the time it took to learn how to operate the virtual manipulative. A few students entered an incorrect equation and solved it. During class discussions, they could not understand why their solution was “correct” according to the virtual manipulative, but different from their classmates. It was when the researcher noticed that they entered the equation incorrectly that they realized their error.

A final drawback of virtual manipulatives is the necessity of resources. The teacher of record described the difficulty involved in gaining access to the computer lab which must be shared among all faculty and students. Many schools face a similar limitation of resources which makes well-intentioned use of virtual manipulatives difficult. Also, students can only work within the constraints of the manipulatives. For example, students cannot enter constants or coefficients which are more than ten and fractions cannot be represented using the virtual manipulative. These constraints limit the number and types of equations students can solve.

Even without statistically significant results, students benefitted from using virtual manipulatives. Perseverance and independence because of feedback provided were recognized benefits of using virtual manipulatives. Feedback, however, was also considered a drawback because some students randomly operated without developing conceptual understanding. Time constraints, constraints of the manipulative, and availability of resources were recognized as drawbacks of using virtual manipulatives.

Unique benefits and drawbacks exist for both concrete and virtual manipulatives. Figures 4.19 and 4.20 display unique benefits and drawbacks of each type of manipulative. Concrete and virtual manipulatives increased student understanding by providing a unique representation of constants, variables, and positive and negative numbers. Virtual manipulatives additionally provided an accurate portrayal of the equal sign as a symbol of balance. Students in each treatment group were willing to persevere and use the tools provided to them in an attempt to understand how to solve equations. Drawbacks of concrete and virtual manipulatives include the time it takes for students to learn how to use them. During the study, students spent time learning how to use the manipulatives properly
which reduced the amount of time they spent learning how to solve equations. Even though students in both groups showed a high level of motivation and active learning, some students were distracted by the manipulatives. These students treated the manipulatives as games and toys instead of using them for their intended use.

*Figure 4.19. Benefits of manipulatives.*
Summary

Students in all three groups received identical example and practice equations throughout the study. The only difference between the materials was the learning method. Students in the control group learned how to solve equations by focusing on conceptual understanding related to the meaning of the equal sign and distinct roles of constants and coefficients. The other groups used concrete and virtual manipulatives to emphasize conceptual understanding. Students in the control group statistically significantly outperformed treatment groups on the posttest. These students benefitted from spending time with written representations of the equations. These students developed conceptual understanding which helped them solve equations successfully on the posttest. Students in the control group benefitted from the emphasis on the meaning of the equal sign and zero pairs.

The control group scored statistically significantly better on posttests than students in the concrete group. Concrete group students shared mixed opinions related to the value of written representations. While solving equations, concrete group students were able to utilize effective strategies that were made obvious with the physical representation of the
manipulatives. Distinct items represented constants, variables, and positive and negative numbers for the concrete group strengthening student understanding. However, some students were distracted by playing with the concrete manipulatives, making patterns with them, and dropping the pieces. These students lost learning time because of distribution and collection of the algebra tiles. More significantly though, students invested time in learning the meaning of the pieces, how to model equations, and manipulate them, which meant time lost on learning how to solve equations.

Students in the control group performed statistically significantly better on posttests than students in the virtual group. These results may have occurred because students did not use the virtual manipulatives on the posttests. Students expressed difficulties in solving equations without the manipulatives. Additionally, these students had less time to learn how to solve equations because they had to learn how to use the software and they lost time logging in and out of computers daily. Some students were distracted by the balloons on the screen. The biggest disadvantage faced by the virtual group was a lack of connection between the virtual manipulatives and the mathematics represented. Students engaged the manipulatives, but did not increase understanding. Some students treated the virtual manipulatives as a game and randomly proceeded until arriving at a solution without thinking about the mathematical processes involved.
CHAPTER V

DISCUSSION, CONCLUSIONS, and RECOMMENDATIONS

Discussion

The purpose of this embedded quasi-experimental mixed methods research was to use solving simple linear equations as the lens for looking at the effectiveness of concrete and virtual manipulatives as compared to a control group using learning methods without manipulatives. Further, the researcher wanted to investigate unique benefits and drawbacks associated with each manipulative. Qualitative methods such as observation, teacher interviews, and student focus group interviews were employed to inform both research questions. Posttest scores indicated statistically significantly different scores in favor of the control group over the virtual group and concrete group. The researcher noted differences in learning outcomes related to representations and conceptual understanding related to the meaning of the equal sign and the role of constants and coefficients. The following list includes topics gleaned from this research study:

- Use of written representations while solving linear equations;
- Development of conceptual understanding;
- Unique representations of distinct objects within equations;
- Manipulatives as toys;
- Unique benefits and drawbacks of using concrete and virtual manipulatives; and
- Comparison of concrete and virtual manipulatives.

According to Piaget (1926), students benefit from multiple representations of topics. Martin (2008) expressed a similar view while describing students using virtual manipulatives to learn about addition. Martin (2008) suggested that manipulatives combined with a written recording of the work allow students time to reflect on the actions taken with the manipulatives. Treatment group students did not depend on written representations while solving equations on the posttest as frequently as control group students. Students in the treatment groups depended on the representations created by the manipulatives rather than written representations while students in the control group heavily relied upon written representations as their only tool for solving equations. The dependence on written representations on the part of control group students displayed their use of rules while developing conceptual understanding of solving equations.
While learning to solve equations, students must develop conceptual understanding. Rittle-Johnson and Alibali (1999) emphasized the value of conceptual understanding over procedural understanding, describing the ability of students to transfer conceptual understanding into improved procedural understanding. Rittle-Johnson and Alibali (1999) also pointed out that the transfer does not work in reverse; rarely do students increase conceptual understanding as a result in increased procedural understanding.

Constructivists such as Piaget (1926), Ernest (1996), and Bruner (1960) emphasized the important role of active learning in building meaning and understanding. Piaget (1926) asserted that students must make connections and not receive facts passively. Students in the control group did not use manipulatives, but they earned statistically significantly different scores on posttests as compared to the treatment groups. Their positive results exemplify development of conceptual understanding without using manipulatives.

In order to increase conceptual understanding, the researcher emphasized the meanings of the equal sign, coefficients, constants, and variables to all groups during this research study. More students in all groups described the purpose of the equal sign as a balance point on posttests than on pretests. On pretests, students in all groups frequently described the equal sign as a symbol of operation, or as a statement of the answer. These findings echo researchers who found elementary and middle school students struggled to accurately describe the meaning of the symbol (Knuth et al., 2005; Knuth et al., 2006). Hiebert and Carpenter (1992) suggested that manipulatives must closely match the mathematics they represent. Students in the virtual group experienced an equal sign that very closely represented its mathematical meaning. As students placed items on the scale, the scale became unbalanced, but as all items were correctly placed on the scale, balance was restored. From this point on, students could only complete actions on both sides of the equation.

CCSSO (2010) standards declared that students should not only perform rote operations with algebraic symbols, rather they should have meaningful experiences which develop student understanding. Students in all groups experienced meaningful experiences with algebraic symbols. Students in the treatment groups additionally experienced physical or virtual representations of constants and variables while using manipulatives. Both the
concrete and virtual manipulatives portrayed constants and variables as distinct items and positive and negative items were represented uniquely as well. Caglayan and Olive (2010) described a similar benefit to students in their qualitative study as students realized that constants and variables could not be combined because they were represented by different objects. Students in the traditional group did not have unique representations for constants and variables, but they developed a conceptual understanding of the different roles as evidenced by posttest scores. More students in all groups correctly described the role of the coefficient on posttests than on pretests. According to the results of this study, the benefits of both manipulatives included these unique representations.

As previously mentioned, Caglayan and Olive (2010) used cups and tiles to represent variables and constants as concrete manipulatives in a study in which eighth grade students learned to solve equations. One conclusion made by Caglayan and Olive (2010) was that students benefitted from the unique representations of the components of the equations. This research supported the claim because in focus group interviews, students in the concrete group and virtual group both discussed the distinct items as helpful in understanding the differences between variables and constants, as well as positive and negative numbers. A concern mentioned by some researchers (Caglayan & Olive, 2010; Hiebert & Carpenter, 1992, and Kieran, 1992) was that students develop an overdependence on manipulatives without making connections to the mathematics represented. Puchner et al. (2010) described concerns about student use of manipulatives in a rote procedural manner without making connections to mathematics. Some students in the virtual group fell into this trap. As they worked with virtual manipulatives, rather than developing conceptual understanding, they merely manipulated until arriving at a correct solution. Several students were able to arrive at the correct solution, yet they were unable to explain why the solution was correct, or how they arrived at the solution. This overdependence on the virtual manipulatives partially explains their relatively low posttest scores as students were not able to use the virtual manipulatives while taking assessments.

Researchers disagree regarding how students should be introduced to manipulatives in the classroom. Boggan et al. (2010) and Johnson (1993) suggested that teachers should provide students with the opportunity to play freely with manipulatives prior to instruction in order to diminish their appeal as toys. However, Uttal et al. (1997) strongly disagreed
with this idea, stating that manipulatives should not be attractive objects, pointing to Japanese teachers who do not use novel objects as manipulatives. McNeal and Jarvin (2007) agreed with Uttal et al. (1997) stating that teachers should use manipulatives that are unlike toys and avoid using items that students may consider as toys. Some students in both treatment groups within this study treated manipulatives as toys. Isolated students in the concrete group were observed building castles, making smiley faces, or creating checker board patterns with algebra tiles rather than actively participating in learning activities. During a focus group interview, a male student created a smiley face as he used algebra tiles to illustrate how to solve an equation. More students in the virtual group were distracted by the balloons which represented negative values. Frequently, students had screens full of balloons which made solving equations more difficult and distracting. Interestingly, all cases of playing with manipulatives involved male students. Some students in the virtual group treated the virtual manipulatives in a familiar way, as if they were a video game, rather than a representation of mathematics.

The amount of time involved and the availability of resources were noted drawbacks of using manipulatives documented in this research study. Students in both treatment groups lost classroom time due to administrative issues of disseminating and collecting manipulatives, or logging in and out of computers. The amount of time lost to administrative tasks was approximately the same for both treatments. Time was also lost on learning how to use the manipulatives.

In schools with limited resources, teachers may not have access to necessary concrete manipulatives, or teachers may have difficulty gaining access to computers for periods of time necessary for students to use virtual manipulatives. Similar concerns were noted in other studies; teachers must plan well while using manipulatives in their classrooms and expect it to take additional time (Burns & Hamm, 2011; Johnson, 1993; Moyer, Bolyard, & Spikell, 2002; and Ross & Kurtz, 1993).

Yuan et al. (2010) compared concrete and virtual manipulatives as junior high students used polyominoes and concluded that while each manipulative was effective, noting no statistically significant differences between the two classes, the learning experiences were different. According to Yuan et al. (2010), students using concrete manipulatives focused on problem solving while students using virtual manipulatives spent
more time working with each other. Within this research study, as students solved equations using concrete manipulatives, students reflected on the meanings of the pieces and connected them to mathematics, but generally stayed to themselves with little interaction among students. However, similarly to Yuan et al. (2010), students in the virtual group were frequently observed helping each other set up equations, or solve them correctly.

Conclusions

Within this research study, students in all three groups learned about solving equations during ten instructional days at the beginning of their sixth grade year. Quantitative data indicated statistically significantly differences in posttest scores in favor of the control group as compared to the virtual group, and in favor of the control group as compared to the concrete group. Although the virtual group or concrete group did not perform as well on the posttests as the control group, learning did occur within the groups. Results of this study are tentative based on a small sample size and limited time frame for the study.

A body of research indicates the effectiveness of concrete and virtual manipulatives (Burns & Hamm, 2011; Moyer-Packenham, Salkind, & Bolyard, 2008). Some of these researchers endorsed the use of manipulatives despite the fact that they did not produce statistically significant results (Burns & Hamm, 2011). The results of this research study indicated that concrete and virtual manipulatives were effective strategies for solving equations; but students developing conceptual understanding without manipulatives performed statistically significantly higher than those with manipulatives. Manipulatives should not be considered a replacement for traditional learning styles; rather, manipulatives should be considered an additional tool for students to use as they learn mathematical concepts.

One possible difficulty experienced by treatment groups was cognitive overload. Borenson and Barber (2008) and Kaput (1992) both described concerns related to the use of manipulatives. Students must engage mathematics and operate manipulatives at the same time, thus experiencing cognitive overload. At times, students in both treatment groups seemed to have difficulty working with the manipulative while making sense of equations at the same time. McNeil and Uttal (2009) suggested that students using
manipulatives must learn material twice, once with the manipulatives, and then again with the abstract concept. This overload may have prohibited students from making connections and developing conceptual understanding. Students in both treatment groups may have benefitted from additional time exploring the manipulatives in order to make connections.

McNeil and Jarvin (2007) described concerns related to cognitive overload resulting from the use of manipulatives stating that students can be so focused on the objects themselves that they entirely miss the mathematical meaning which they represent. McNeil and Jarvin (2007) described three possible difficulties students face while using manipulatives, the first they described as nontransparency. Just because the teacher can understand the link between the manipulative and the mathematics, the teacher should not assume that students should be able to as well. Second, McNeil and Jarvin (2007) described limited cognitive resources which may be over-taxed as students use manipulatives. Finally, McNeil and Jarvin (2007) stated that students see manipulatives as familiar objects and are unable to see them in a different role. McNeil and Jarvin (2007) went so far as to claim that because of these reasons, manipulatives may have little impact on student learning.

Students in treatment groups in this research study experienced these difficulties. Initially, a student in the concrete group guessed that the $x$ tile was equal to five because she was looking at the tiles in a similar manner as base ten blocks. The researcher understood that the $x$ block represented the unknown value which made the equation a true statement. In this case, the size or shape of the $x$ block was irrelevant to the researcher, but to the student, the size of the shape was interpreted as meaningful. Considerable time was invested in learning how to operate with respective manipulatives. The cognitive load of understanding how to operate with manipulatives was all that some students could handle. Finally, some students in the virtual group were very comfortable with using computers. In fact, these students could not see the virtual manipulatives as anything but an opportunity to play a video game. In so doing, they did not increase conceptual understanding of solving equations.

Sixth grade students will be exposed to solving equations again for several years as they complete middle and high school mathematics courses. Experiences and conceptual understanding developed from participating in this research study may be prior learning
they use to better understand solving linear equations later. Ernest (1996) described the value of using manipulatives as previous knowledge on which new knowledge is constructed.

In conclusion, the key to learning mathematics is effectively making connections. These connections may occur in a classroom where students focus increasing conceptual understanding without manipulatives. These connections may occur in classrooms with manipulatives. Students must be able to make connections to previous learning experiences and real-life experiences. Students must have enough time to explore and make the mathematics meaningful to themselves. Although quantitative data revealed that students in the control group outperformed students in the treatment groups, most students in all three groups were actively engaged and interested while solving equations. Hopefully, as these students progress through their mathematics careers, they will remember what they learned about solving linear equations and make connections which build conceptual understanding for years to come.

**Recommendations**

After much reflection, the researcher recommends that if this study were to be repeated, three adjustments may be considered. First, rather than holding time as a constant, it would be better to hold learning time constant. All three groups solved the same equations and had the same amount of time. Treatment group students lost learning time as a result of administrative tasks. Additionally, treatment group students had to learn how to use manipulatives, which reduced time they were able to learn about solving equations. During focus group interviews, many students in both treatment groups suggested that they would benefit from more time practicing solving equations.

Second, the researcher may consider adding another treatment group which combines virtual and concrete manipulatives to learn how to solve linear equations. The teacher of record stated that as she teaches solving equations in the future, she plans to combine all three learning methods.

Finally, the researcher suggests that in order to compare the effectiveness of concrete and virtual manipulatives to a control group without manipulatives, a long-term study may be necessary. Insight into unique aspects may become more apparent as a
researcher investigates the different learning methods over a school year in which they are used on a variety of mathematics topics.

While much is known about manipulatives, many unanswered questions still exist. Although a wealth of research exists as to the value of manipulatives in elementary school, there is much less research as to the effectiveness of manipulatives for middle and high school students (Burns & Hamm, 2011; Freer-Weiss, 2006). A potential area of further study related to manipulatives is determining the effectiveness of manipulatives for students with different ability levels. Bouck and Flanagan (2009) stated that manipulatives create a positive effect for students with learning disabilities. Researchers need to investigate the effect on low ability, average ability, and high ability students to see if significant differences exist. Additionally, research must take place which investigates student attitudes related to the use of manipulatives. Clearly, manipulatives have a valid role in mathematics education, yet many unanswered questions still exist regarding their effective usage in the classroom.
APPENDIX A
Pretest and Posttest

Solving Equations Pre Test  Student Code ________________________

Solve the following equations successfully.

1)  \( x + 4 = 11 \)
   \( x = ____ \)

2)  \( x - 8 = 11 \)
   \( x = ____ \)

3)  \( 2 + x = -6 \)
   \( x = ____ \)

4)  \( x - 6 = 7 \)
   \( x = ____ \)

5)  \( 5x = 15 \)
   \( x = ____ \)

6)  \( 2p = 20 \)
   \( p = ____ \)

7)  \( 10q = 10 \)
   \( q = ____ \)

8)  \( 6s = 60 \)
   \( s = ____ \)

9)  \( 3s + 4 = 16 \)
   \( s = ____ \)

10) \( 4x + 2 = 2x \)
    \( x = ____ \)

11) \( 3x + 5 = 4x - 6 \)
    \( x = ____ \)
12) \(5m - 6 = 3m + 2\)

\[ m = \underline{} \]

Select the best answer for the following problems.

13) Following the rules of solving equations, which is the only way to keep an equation balanced?
   A. Add four to one side of an equation and subtract four from the other
   B. Add positive three to one side and negative three to the other
   C. Divide both sides of the equation by three
   D. Multiply one side of the equation by four and divide the other side of the equation by four

14) For the following equation, which is the correct solution?

\[ 8 - x = 3 \]

   A. \(x = 2\)
   B. \(x = -2\)
   C. \(x = -5\)
   D. \(x = 5\)

15) Which step would not help you solve the equation below?

\[ 4t - 4 = 12 \]

   A. Divide both sides by 4
   B. Add 4 to both sides
   C. Subtract 4 from both sides
Visually represent with numbers, words, or sketches the following equations

16) Visually represent with numbers, words, or sketches the equation and a possible next step for solving the equation (you do not have to solve)

\[3x + 5 = 2x + 6\]

17) Visually represent with numbers, words, or sketches the equation and a possible next step for solving the equation (you do not have to solve)

\[2x + 1 = x + 2\]
18) Visually represent with numbers, words, or sketches the equation and a possible next step for solving the equation (you do not have to solve)

\[-3x - 3 = -6\]

19) Solve the equation and visually represent with numbers, words, or sketches, all steps. \(x = \) ______

\[3x + 2 = 2x + 3\]
**Short Answer**

20) In the problem below, the arrow is pointing at a symbol. What does this symbol mean? Can it mean anything else?

\[ 3 + 4 = 7 \]

21) In the problem below, the arrow is pointing at a symbol. What does this symbol mean? Can it mean anything else?

\[ 3x + 4 = 12 \]

22) In the problem below, what is the purpose or meaning of the ‘3’?

\[ 3x + 4 = 12 \]
23) In the problem below, what is the purpose or meaning of the ‘3’?

\[2x + 3 = 11\]

24) In the problem below, what is the purpose or meaning of the ‘x’?

\[3x + 2 = 17\]

25) In your own words and using complete sentences, describe the process for checking an equation.

26) In your own words, explain the steps for solving the following equation. You visually represent with numbers, words, or sketches to support your explanation.

\[2p + 6 = 12\]
27) In your own words, explain the steps for solving the following equation. You may visually represent with numbers, words, or sketches

\[ 7b - 6 = 3b + 2 \]

28) What is the purpose, meaning, or goal of solving equations?
Solving Equations Post Test

Student Code______________________

Solve the following equations successfully.

1) \( x + 5 = 11 \)
   \( x = \) _____

2) \( x - 8 = 11 \)
   \( x = \) _____

3) \( 4 + x = -6 \)
   \( x = \) _____

4) \( x - 3 = 7 \)
   \( x = \) _____

5) \( 3x = 15 \)
   \( x = \) _____

6) \( 4p = 20 \)
   \( p = \) _____

7) \( 10q = 10 \)
   \( q = \) _____

8) \( 5s = 60 \)
   \( s = \) _____

9) \( 2s + 4 = 16 \)
   \( s = \) _____

10) \( 7x + 2 = 5x \)
    \( x = \) _____

11) \( 3x + 5 = 4x - 6 \)
    \( x = \) _____

12) \( 5m - 6 = 3m + 2 \)
    \( m = \) _____
Solving Equations 2
Select the best answer for the following problems.

13) Following the rules of solving equations, which is the only way to keep an equation balanced?
   E. Add three to one side of an equation and subtract three from the other
   F. Add positive two to one side and negative two to the other
   G. Divide both sides of the equation by three
   H. Multiply one side of the equation by two and divide the other side of the equation by two

14) For the following equation, which is the correct solution?
    \[ 5 - x = 3 \]
   E. \( x = 2 \)
   F. \( x = -2 \)
   G. \( x = 8 \)
   H. \( x = -8 \)

15) Which step would not help you solve the equation below?
    \[ 4t - 4 = 12 \]
   D. Divide both sides by 4
   E. Add 4 to both sides
   F. Subtract 4 from both sides
Sketch the following equations (solve only when necessary)

16) Visually represent with numbers, words, or sketches the equation and a possible next step for solving the equation (you do not have to solve)
\[ 2x + 4 = x + 5 \]

17) Visually represent with numbers, words, or sketches the equation and a possible next step for solving the equation (you do not have to solve)
\[ 2x + 5 = x + 6 \]
18) Visually represent with numbers, words, or sketches the equation and a possible next step for solving the equation (you do not have to solve)

\[-3x + 1 = -5\]

19) Solve the equation and visually represent with numbers, words, or sketches all steps.

\[x = \underline{_____}\]

\[3x + 2 = 2x + 3\]
20) In the problem below, the arrow is pointing at a symbol. What does this symbol mean? Can it mean anything else?

\[ 2 + 5 = 7 \]

21) In the problem below, the arrow is pointing at a symbol. What does this symbol mean? Can it mean anything else?

\[ 2x + 4 = 12 \]

22) In the problem below, what is the purpose or meaning of the ‘3’?

\[ 3x + 4 = 12 \]

23) In the problem below, what is the purpose or meaning of the ‘3’?

\[ 2x + 3 = 11 \]
24) In the problem below, what is the purpose or meaning of the ‘x’?

\[ 3x + 2 = 17 \]

25) In your own words and using complete sentences, describe the process for checking an equation.

26) In your own words, explain the steps for solving the following equation. You may visually represent with numbers, words, or sketches to support your explanation.

\[ 2p + 8 = 12 \]

27) In your own words, explain the steps for solving the following equation. You may visually represent with numbers, words, or sketches to support your explanation.

\[ 7b - 6 = 3b + 2 \]

28) What is the purpose, meaning, or goal of solving equations?
APPENDIX B
Teacher Pre-Interview Protocol

1. Describe your prior experiences using manipulatives of any kind to teach mathematics.
2. Describe your prior experiences using virtual manipulatives to teach mathematics.
3. Describe your prior experiences using concrete manipulatives to teach mathematics.
4. What experiences did you have with manipulatives in your teacher training program?
5. Please tell me your work experience teaching mathematics.
6. What are some barriers you might see with teaching mathematics with concrete manipulatives?
7. What are some barriers you might see with teaching mathematics with virtual mathematics?
8. How have you taught students to solve equations in the past?
9. What difficulties do students have as they solve equations?
10. How do students describe the meaning of the equal sign?
11. What difficulties do students have with symbolic understanding?
12. Is there anything else you would like to add?
APPENDIX C
Teacher Post-Interview Protocol

1. Tell me how you integrate technology in your math class on a typical day.
2. What do you see as the value of technology for teachers and students?
3. How often (and how) do you use manipulatives of any kind in your classroom.
4. Compare the perceived value of virtual manipulatives and concrete manipulatives.
5. What benefits do you think your students obtained by completing this unit using virtual manipulatives?
6. What barriers do you think your students faced by completing this unit using virtual manipulatives?
7. What benefits do you think your students obtained by completing this unit using concrete manipulatives?
8. What barriers do you think your students faced by completing this unit using concrete manipulatives?
9. Would you like to add any other comments?
10. In our original interview, you discussed ______________ as a perceived barrier to using concrete/virtual manipulatives. How has your opinion changed?
11. Next year, if you are responsible for teaching solving equations, what methods will you use?
12. What do you see as the perceived benefits of concrete/virtual manipulatives for helping students understand the meaning of the equality symbol?
13. What do you see as the perceived benefits of concrete/virtual manipulatives for helping students understand symbols while solving equations?
14. How have concrete/virtual manipulatives helped students improve conceptual and procedural understanding?
15. Is there anything else you would like to add?
APPENDIX D
Student Focus Group Interview Protocol
Control Group Protocol

1. How often do you use technology in math class? Describe.
2. How often do you use technology in other classes?
3. Describe how you typically use technology outside of school.
4. What experiences have you had with solving equations before we started this study?
5. How comfortable are you now, with solving equations after your recent experiences learning how to solve them?
6. In the equation, \( 5x + 4 = 3x + 2 \) what is the meaning/purpose of the 5 and the 4?
7. In the equation, \( 5x + 4 = 3x + 2 \), what is the meaning/purpose of the equal sign?
8. Was it helpful to write down the process of solving equations as you were going along?
9. What would have made learning how to solve equations easier for you?
10. As you were solving equations, and the equations became more difficult, were you able to think back to what you learned about the easier equations?

11. I have several equations here and I am going to spread them out on the table. I want you to rank them from easiest to hardest to solve. \{ 2x = 10; x + 5 = 6; x + 8 = 10; -3x = 9; -2x = -6; x – 5 = 10; 2x – 4 = 6; 2x – 4 = 10; 2x – 4 = 3x + 6; -2 – x = 4; -3x – 4 = 8 \}
12. Please explain why you ranked the equations in this way.
13. Talk me through the process of solving some of the equations (If they do not already do this naturally.)
Concrete Group Protocol

1. How often do you use technology in math class?  Describe.
2. How often do you use technology in other classes?
3. Describe how you typically use technology outside of school.
4. What experiences have you had with concrete manipulatives before we started this study?
5. Describe how it felt to use the concrete manipulatives.
6. How comfortable are you now, with solving equations after your experiences with concrete manipulatives?
7. In the equation, $5x +4 = 3x+2$ what is the meaning/purpose of the 5 and the 4?
8. In the equation, $5x + 4 = 3x + 2$, what is the meaning/purpose of the equal sign?
9. When you were solving equations using concrete manipulatives, did you draw the equations and steps on the paper? In what order? Was this a beneficial experience?
10. How is solving an equation on a piece of paper related to how you solve an equation with concrete manipulatives? Can you give me an example?
11. What would have made learning how to solve equations easier for you?
12. I have several equations here and I am going to spread them out on the table. I want you to rank them from easiest to hardest to solve.  $\{2x = 10; x + 5 = 6; x + 8 = 10; -3x = 9; -2x = -6; x – 5 = 10; 2x – 4 = 6; 2x – 4 = 10; 2x – 4 = 3x + 6; -2 – x = 4; -3x – 4 = 8\}$
13. Please explain why you ranked the equations in this way.
14. Talk me through the process of solving some of the equations (If they do not already do this naturally.)
Virtual Group Protocol

1. How often do you use technology in math class? Describe.
2. How often do you use technology in other classes?
3. Describe how you typically use technology outside of school.
4. What experiences have you had with virtual manipulatives before we started this study?
5. Describe how it felt to use the virtual manipulatives.
6. How comfortable are you now, with solving equations after your experiences with virtual manipulatives?
7. In the equation, $5x + 4 = 3x + 2$ what is the meaning/purpose of the $5$ and the $4$?
8. In the equation, $5x + 4 = 3x + 2$, what is the meaning/purpose of the equal sign?
9. When you were solving equations using virtual manipulatives, did you draw the equations and steps on the paper? In what order? Was this a beneficial experience?
10. How is solving an equation on a piece of paper related to how you solve an equation with virtual manipulatives? Can you give me an example?
11. Describe how you used the feedback provided by the virtual manipulatives.
12. What would have made learning how to solve equations easier for you?
13. I have several equations here and I am going to spread them out on the table. I want you to rank them from easiest to hardest to solve. { $2x = 10; x + 5 = 6; x + 8 = 10; -3x = 9; -2x = -6; x - 5 = 10; 2x - 4 = 6; 2x - 4 = 10; 2x - 4 = 3x + 6; -2 - x = 4; -3x - 4 = 8$ }
14. Please explain why you ranked the equations in this way.
15. Talk me through the process of solving some of the equations (If they do not already do this naturally.)
16. We will now go to the computer lab and I would like you to explain the process of solving equations using the virtual manipulatives. (I would like you to use the concrete manipulatives to solve a few equations.)
APPENDIX E
Solving Equations with Traditional Materials
Solving Equations

Solving equations in one variable is a very important skill in algebra. Many equations can be solved using this set of strategies which gives a representation to the abstract concept.
Define the Concept of Balance Related to Equations

Solving equations is like working with a scale with the fulcrum as the equal sign. (The fulcrum is the balancing point). Solving equations and scales both require balance.

**Example 1.A**

For example, the problem below is balanced because there are three on the left side and three on the right side. Both sides of the equation must be equal.

\[ 3 = 3 \]

**Example 1.B**

What happens if two is added to both sides of the equation?

\[ 3 + 2 = 3 + 2 \]

The equation remains balanced because \( 3 + 2 = 5 \) on both sides of the scale.

Two (or any number) can be added to both sides and it will remain balanced.

**Practice 1.1**

Add four to both sides of the equation below. What happens?

\[ 4 = 4 \]

Solving equations requires balance.

In a similar way, the same number can be removed from both sides of a scale. In mathematics, removing is the same as subtraction.

**Example 1.C**

In the equation below, \( 5 - 3 = 5 - 3 \). Both sides of the equation are equal because three is removed from both sides.

\[ 5 - 3 = 5 - 3 \]
**Practice 1.2**

Remove two from both sides of the equation below.

\[ 6 = 6 \]

What happens?

All mathematics operations can be done to both sides of the equation.

**Example 1.D**

Both sides of an equation can be multiplied by two and the equation will remain balanced.

\[ 2 \times 4 = 2 \times 4 \]

\[ 8 = 8 \]

**Practice 1.3**

Multiply each side of the equation below by 3.

\[ 4 = 4 \]

What happens?

**Example 1.E**

Both sides of an equation can be divided by the same number and the equation will remain balanced. Divide each side of this equation by two.

\[ 4 \div 2 = 4 \div 2 \]

\[ 2 = 2 \]

**Practice 1.4**

Divide each side of the equation by three.

What happens?

\[ 9 = 9 \]
Demonstrate Checking an Equation

Both sides of a balance scale must have the same value. If the left side of the equals five, the right side must equal _____.

A variable is a letter, often \( x \), which is used to represent an unknown value. The goal of solving equations is to find the value of the variable.

When a teacher is absent from school, the school hires a substitute teacher, someone who stands in the place of the teacher. The variable is similar; it stands in the place of the unknown value. When checking a solution, let the number “stand in” for the variable and see if both sides of the equation are equal.

**Example 2.A**

When the left side of the equation equals five, the right side of the equation must equal five. In the equation \( 5 = x + 4 \), the solution is \( x = 1 \). Substitute 1 for the variable. The solution is correct if both sides of the equation are equal.

\[
x = 1
\]

Check the equation

\[
5 = x + 4
\]

\[
5 = 1 + 4
\]
Example 2.B

Check to see if \( x = 4 \) is the correct solution for the equation \( 2x = 8 \).

\[
\begin{align*}
\text{Check:} & \\
2x &= 8 \\
(2)(4) &= 8
\end{align*}
\]

Example 2.C

Check to see if \( x = 3 \) is the solution to the equation \( x + 4 = 6 \).

3 is not a solution because \( 3 + 4 \neq 6 \).

Practice 2.1

Check to see if \( x = 5 \) is a solution to the equation \( x + 5 = 10 \).

Practice 2.2

Check to see if \( x = 3 \) is a solution to the equation \( 3x = 6 \).

Practice 2.3

For the following equation, which is the correct solution?

\( x + 4 = 7 \)

A. \( x = 4 \)

B. \( x = 7 \)

C. \( x = 3 \)
Solving Addition and Subtraction Equations

Addition and subtraction are inverse operations, which means they are opposites. Adding three is the opposite of subtracting three. The result of inverse operations is zero (often called a zero pair.) For example,

\[ +5 - 5 = 0 \]
\[ -20 + 20 = 0 \]

**Example 3.A**

Solve the equation, \( x + 3 = 6 \).

The goal of solving equations is to isolate the variable. What can you do to get the variable by itself?

Subtract 3 from both sides of the equation.

\[
\begin{align*}
  x + 3 &= 6 \\
  -3 & \quad \text{(subtract 3)} \\
  x &= 3
\end{align*}
\]

For this equation, the solution is \( x = 3 \).

Remember to check the solution by substituting the result for the variable. \( 3 + 3 = 6 \).

**Example 3.B**

Solve the equation, \( 4 + x = 8 \).

To solve the equation, isolate the variable by subtracting 4 from both sides of the equation.
Practice 3.1

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

$x + 3 = 8$

\[
\begin{align*}
4 + x &= 8 \\
-4 &\quad -4 \\
x &= 4
\end{align*}
\]

Check the equation.

\[
\begin{align*}
4 + x &= 8 \\
4 + 4 &= 8
\end{align*}
\]
Practice 3.2

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x + 1 = 5 \]
Example 3.C

Solve the equation, \( x - 5 = 3 \).

The goal of solving equations is to isolate the variable, what can you do to get the variable by itself?

Add positive five to both sides of the equation. \((-5 + 5 = 0, \text{this is called a zero pair})\).

\[
\begin{align*}
x - 5 &= 3 \\
+5 &+5 \\
\hline
x &= 8
\end{align*}
\]

Example 3.D

Solve the equation, \( x - 2 = -4 \).

Add 2 to both sides of the equation to isolate the variable.

\[
\begin{align*}
x - 2 &= -4 \\
+2 &+2 \\
\hline
x &= -2
\end{align*}
\]

Check the equation.

\[
\begin{align*}
x - 5 &= 3 \\
8 - 5 &= 3
\end{align*}
\]

\[
\begin{align*}
x - 2 &= -4 \\
-2 - 2 &= -4
\end{align*}
\]
These equations in this section are examples of the Addition Property and Subtraction Property of Equality which state that you can add (or subtract) both sides of an equation by the same nonzero number and the statement will remain true.

**Practice 3.3**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 4 = 5 \]

---

**Practice 3.4**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 3 = -4 \]
Solving Multiplication and Division Equations

Just as addition and subtraction are inverse operations, multiplication and division are inverse operations. To solve an equation, you may use the inverse operation to “undo” to isolate the variable. In the equation $2x = 4$, the 2 is the coefficient of $x$ because it is the number by which $x$ is multiplied.

Example 4.A

Solve the equation, $2x = 4$.

What can you do to isolate the variable on the left side of the equation?

Divide both sides of the equation by two.

$$\frac{2x}{2} = \frac{4}{2}$$

The solution to the equation is $x = 2$.

Check the equation.

$$2x = 4$$

$$2 \times 2 = 4$$
Example 4.B

Solve the equation, $3x = 9$.

What can you do to isolate the variable? Divide both sides of the equation by three.

$$\frac{3x}{3} = \frac{9}{3}$$

$x = 3$

Check the solution.

$3x = 9$

$3 \times 3 = 9$

Practice 4.1

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

$2x = 8$
**Practice 4.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ 3x = -9 \]

**Example 4.C**

Solve the equation, \( x \div 2 = 3 \).

This equation tells us that half of a variable equals three, so how would you find out what a whole variable equals?

Multiply both sides of an equation by two.

\[ x = 6 \]

Check the equation.

\[ x \div 2 = 3 \]

\[ 6 \div 2 = 3 \]
Example 4.D

Solve the equation, \( x \div 4 = 2 \).

This equation tells us what that one-fourth of a variable equals two, so how would you find out what a whole variable equals?

Multiply both sides of the equation by 4.

\[
x = 16
\]

Check the solution.

\[
x \div 4 = 2
\]

\[
16 \div 4 = 4
\]

These equations in this section are examples of the **Multiplication Property and Division Property of Equality** which state that you can multiply (or divide) both sides of an equation by the same nonzero number and the statement will remain true.

Practice 4.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[
x \div 2 = 8
\]
**Practice 4.4**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x \div 3 = -2 \]
Solving Multistep Equations

Example 5.A

Solve the equation, \(2x + 4 = 8\).

The goal of solving equations is to isolate the variable, what can you do to get the variable alone?

Subtract four from both sides.

\[
\begin{align*}
2x + 4 &= 8 \\
-4 &
\end{align*}
\]

Note the new equation, \(2x = 4\). How can you isolate the variable?

Divide both sides of the equation by two.

\[
\begin{align*}
\frac{2x}{2} &= \frac{4}{2} \\
x &= 2
\end{align*}
\]

Check the solution.

\[
\begin{align*}
2x + 4 &= 8 \\
2*2 + 4 &= 8
\end{align*}
\]

Example 5.B

Solve the equation, \(3x - 3 = 6\).

There are two options to solving this equation, what are two actions that will isolate the variable?
Strategy One:

Add three to both sides to isolate the variables.

\[
\begin{align*}
\frac{3x-3}{\frac{+3}{3}} &= \frac{6}{3} \\
\frac{3x}{3} &= \frac{9}{3}
\end{align*}
\]

Adding three to both sides created a zero pair on the left side of the equation.

Now, divide both sides of the equation by three.

\[
\begin{align*}
\frac{3x}{3} &= \frac{9}{3} \\
x &= \frac{3}{3}
\end{align*}
\]

Strategy 2:

\[3x - 3 = 6\]

Another strategy to solving this equation is to divide both sides by three.

\[
\begin{align*}
\frac{3x-3}{\frac{3}{3}} &= \frac{6}{3} \\
x - 1 &= \frac{2}{3}
\end{align*}
\]

By dividing both sides of the equation, the resulting equation is \[x - 1 = 2\].
Add one to both sides to solve the equation.

\[
\begin{align*}
x - 1 &= 2 \\
+1 &+1 \\
x &= 3
\end{align*}
\]

\[
x = 3
\]

Check the equation.

\[
\begin{align*}
3x - 3 &= 6 \\
3*3 - 3 &= 6
\end{align*}
\]

**Practice 5.1**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[
4x - 1 = 7
\]
**Practice 5.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-5x + 2 = 7\]

**Example 5.C**

Solve the equation, \(4 - x = 6\).

To solve the equation, subtract four from both sides of the equation.

\[
\begin{align*}
4 - x &= 6 \\
-4 &\quad -4 \\
-x &= 2
\end{align*}
\]

The solution to the equation is \(-x = 2\). However, to solve equations, the goal is to find the value of \(x\), not \(-x\). In this example, the coefficient of \(x\) is \(-1\). Multiply both sides of the equation by \(-1\) to make the variable positive.
Example 5.D

Solve the equation, $4 - 2x = 6$.

Solve the equation by subtracting four from both sides of the equation.

\[
\begin{array}{c}
4 - 2x = 6 \\
-4 & -4 \\
-2x & = 2 \\
\end{array}
\]

Solve the equation by dividing both sides of the equation by -2.

\[
\begin{array}{c}
-2x = 2 \\
-2 & -2 \\
\end{array}
\]

\[x = -1\]

Check the solution.

\[4 - 2x = 6\]

\[4 - (2)(-1) = 6\]
Practice 5.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-3x = 9\]

Practice 5.4

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[3 - 2x = -5\]
Solving Equations with Variables on Both Sides

Some multistep equations have variables on both sides of the equation. The goal of solving equations is to isolate the variable on one side of the equation.

Example 6.A

Solve the equation, \(3x - 2 = x - 4\).

\[
\begin{align*}
3x - 2 &= x - 4 \\
-x &= -x \\
2x - 2 &= -4
\end{align*}
\]

Notice that there are identical items on both sides of the equation, so they can be removed. Remove either the \(x\) from both sides, or \(-2\) from both sides, order does not matter.

Now, to eliminate the \(-2\) from both sides, add two, which creates two zero pairs \((-2 + 2 = 0,.)\)

\[
\begin{align*}
2x - 2 &= -4 \\
+2 &+2 \\
2x &= -2
\end{align*}
\]

How can you find the value of \(x\)? Divide both sides of the equation by two.

\[
\frac{2x}{2} = \frac{-2}{2}
\]
Example 6.B

Solve the equation, \(2x - 4 = x + 2\).

Begin solving the equation by removing an \(x\) from both sides of the equation.

\[
\begin{align*}
2x - 4 &= x + 2 \\
-2x + (-4) &= (-x) + (-4) \\
\hline
0 &= 2
\end{align*}
\]

In order to solve the equation, isolate the variable on the left side of the equation by adding four to both sides.

On the left side of the equation, a zero pair is created \((-4 + 4 = 0)\).

\[
\begin{align*}
x - 4 &= 2 \\
+4 &= +4 \\
\hline
x &= 6
\end{align*}
\]
Example 6.C

Solve the equation, \(-3x - 4 = 4x + 3\).

Notice that to solve this equation, there are no common elements on both sides of the equation. Consider using zero pairs to eliminate the variables on the left side of the equation, add \(3x\) to both sides.

\[
\begin{align*}
-3x - 4 &= 4x + 3 \\
+3x &+3x \\
-4 &= 7x + 3
\end{align*}
\]

Notice that the variable is on the right side of the equation, how can you isolate it?

Subtract three from both sides.

\[
\begin{align*}
-4 &= 7x + 3 \\
-3 &= -3 \\
-7 &= 7x
\end{align*}
\]

Divide both sides of the equation by 7.

\[
\begin{align*}
-7 &= 7x \\
7 &\div 7 \\
1 &= x
\end{align*}
\]

Check the solution.

\[
\begin{align*}
-3x - 4 &= 4x + 3 \\
(-3)(-1) - 4 &= 4(-1) + 3 \\
3 - 4 &= -4 + 3 \\
-1 &= -1
\end{align*}
\]
**Practice 6.1**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 4 = 2x - 3 \]
Practice 6.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-2x + 7 = 3x + 2\]
Solving Equations

Solving equations in one variable is a very important skill in algebra. Many equations can be solved using this set of strategies which gives a concrete representation to the abstract concept.

Algebra tiles are used while solving equations and the following pieces represent elements of the equation and the balance mat below is where you will do your work.
Define the Concept of Balance Related to Equations

Solving equations is like working with a scale with the fulcrum as the equal sign. (The fulcrum is the balancing point). Solving equations and scales both require balance.

Example 1.A

For example, the scale below is balanced because there are three squares on the left side and three squares on the right side. An equation is similar. Both sides of the equation must be equal.

Example 1.B

What happens if two is added to both sides of the balance scale?
The scale remains balanced because $3 + 2 = 3 + 2$.

Equations work the same way, two (or any number) can be added to both sides and it will remain balanced.

**Practice 1.1**

Add four to both sides of the scale below. What happens?

The scales should have remained balanced because 4 was added to both sides. Solving equations requires balance.

In a similar way, the same number can be removed from both sides of a scale. In mathematics, removing is the same as subtraction.

**Example 1.C**

On the balance scale below, $5 - 3 = 5 - 3$. Both sides of the equation are equal because three is removed from both sides.
**Practice 1.2**

Remove two from both sides of the balance scale below.

What happens?

All mathematics operations can work with a balance scale if the same operation is done to both sides of the equation.

**Example 1.D**

Both sides of an equation can be multiplied by two and the equation will remain balanced.

\[ 4 \times 2 = 4 \times 2 \]
Practice 1.3

Multiply each side of the equation below by 3.

What happens?

Example 1.E

Both sides of an equation can be divided by the same number and the equation will remain balanced. Divide each side of this equation by two.

$4 \div 2 = 4 \div 2$
**Practice 1.4**

Divide each side of the equation by three.

What happens?
Demonstrate Checking an Equation

Both sides of a balance scale must have the same value. If the left side of the equals five, the right side must equal ______.

A variable is a letter, often $x$, which is used to represent an unknown value. The goal of solving equations is to find the value of the variable.

When a teacher is absent from school, the school hires a substitute teacher, someone who stands in the place of the teacher. The variable is similar; it stands in the place of the unknown value. When checking a solution, let the number “stand in” for the variable and see if both sides of the equation are equal.

**Example 2.A**

When the left side of the equation equals five, the right side of the equation must equal five. In the equation, $5 = x + 4$, the solution is $x = 1$. Substitute 1 for the variable. The solution is correct if both sides of the equation are equal.

$5 = 1 + 4$
Example 2.B

Check to see if $x = 4$ is the correct solution for the equation $2x = 8$.

Since both sides of the equation equal 8, the equation is balanced, so $x = 4$ is the correct solution.

Example 2.C

Check to see if $x = 3$ is the solution to the equation $x + 4 = 6$.

3 is not a solution because $3 + 4 \neq 6$.

Practice 2.1

Check to see if $x = 5$ is a solution to the equation $x + 5 = 10$.

Practice 2.2

Check to see if $x = 3$ is a solution to the equation $3x = 6$. 
Practice 2.3

For the following equation, which is the correct solution?

\[ x + 4 = 7 \]

A. \( x = 4 \)

B. \( x = 7 \)

C. \( x = 3 \)
Solving Addition and Subtraction Equations

Addition and subtraction are **inverse operations**, which means they are opposites. Adding three is the opposite of subtracting three. The result of inverse operations is zero (often called a zero pair.) For example,

\[ +5 - 5 = 0 \]
\[ -20 + 20 = 0 \]

**Example 3.A**

Solve the equation, \( x + 3 = 6 \).

Place \( x \) and three positive integers on the left side of the scale and six positive integers on the right side of the scale.

The goal of solving equations is to isolate the variable. What can you do to get the variable by itself?

Subtract 3 from both sides of the equation.
For this equation, the solution is \( x = 3 \).

Remember to check the solution by substituting the result for the variable.

\[
x + 3 = 6 \\
3 + 3 = 6
\]

**Example 3.B**

Solve the equation, \( 4 + x = 8 \).

Place four positive integers and one variable on the left side of the equation, and eight positive integers on the right side of the equation.

To solve the equation, isolate the variable by subtracting 4 from both sides of the equation.
Practice 3.1

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x + 3 = 8 \]
**Practice 3.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x + 1 = 5 \]

**Example 3.C**

Solve the equation, \( x - 5 = 3 \).
The goal of solving equations is to isolate the variable, what can you do to get the variable by itself? Add positive five to both sides of the equation. \((-5 + 5 = 0, \text{ this is called a zero pair})\). The zero pairs can be removed from the left side of the equation.

Example 3.D

Solve the equation, \(x - 2 = -4\).
To isolate the variable, add 2 to both sides (-2 + 2 = 0, another example of a zero pair.)

The zero pairs can be removed from both sides of the equation.

\[ x = -2 \]

Check the equation.

\[ x - 2 = -4 \]
\[ -2 - 2 = -4 \]

These equations are examples of the **Addition Property and Subtraction Property of Equality** which state that you can add (or subtract) both sides of an equation by the same nonzero number and the statement will remain true.
Practice 3.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 4 = 5 \]

Practice 3.4

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 3 = -4 \]
Solving Multiplication and Division Equations

Just as addition and subtraction are inverse operations, multiplication and division are inverse operations. To solve an equation, you may use the inverse operation to “undo” to isolate the variable. In the equation $2x = 4$, the 2 is the coefficient of $x$ because it is the number by which $x$ is multiplied.

**Example 4.A**

Solve the equation, $2x = 4$.

Place two $x$’s on the left side of the equation and four positive integers on the right side of the equation.

What can you do to isolate the variable on the left side of the equation?

Divide both sides of the equation by two.

Divide the four integers on the right side of the equation equally into 2 groups. There are two integers in each group.
The solution to the equation is $x = 2$.

Check the equation.

$2x = 4$

$2 \times 2 = 4$

**Example 4.B**

Solve the equation, $3x = 9$.

What can you do to isolate the variable? Divide both sides of the equation by three.

Divide the 9 integers equally into three groups. There are three integers in each group.
**Practice 4.1**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

2x = 8
Practice 4.2

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ 3x = -9 \]

Algebra tiles cannot be used to solve division equations because fractional coefficients are not allowed; however, they are solved in a similar fashion.

Example 4.C

Solve the equation, \( x \div 2 = 3 \).

This equation would be represented with half of a variable on the left side of the equation, and three positive integers on the right side of the equation.

This equation tells us that half of a variable equals three, so how would you find out what a whole variable equals?
Multiply both sides of an equation by two.

\[
\begin{align*}
\text{Check the equation.} \\
x \div 2 &= 3 \\
6 \div 2 &= 3
\end{align*}
\]

**Example 4.D**

Solve the equation, \(x \div 4 = 2\).

This equation would be represented with one fourth of a variable on the left side of the equation, and two positive integers on the right side of the equation.

This equation tells us that one-fourth of a variable equals two, so how would you find out what a whole variable equals?

Multiply both sides of the equation by 4.

\[
\begin{align*}
\text{Check the solution.} \\
x \div 4 &= 2 \\
16 \div 4 &= 4
\end{align*}
\]

These equations are examples of the **Multiplication Property and Division Property of Equality** which state that you can multiply (or divide) both sides of an equation by the same nonzero number and the statement will remain true.
Practice 4.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x \div 2 = 8 \]

Practice 4.4

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x \div 3 = -2 \]
Solving Multistep Equations

Example 5.A

Solve the equation, \(2x + 4 = 8\).

Set up the equation properly.

\[
\begin{align*}
\text{Set up the equation properly.} \\
\text{The goal of solving equations is to isolate the variable, what can you do to get the variable alone?} \\
\text{Subtract four from both sides.} \\
\text{Note the new equation, } 2x = 4. \\
\text{How can you isolate the variable?} \\
\text{Divide both sides of the equation by two.}
\end{align*}
\]
Example 5.B

Solve the equation, $3x - 3 = 6$.

Set up the equation properly.

There are two options to solving this equation, what are two actions that will isolate the variable?

**Strategy One:**

Add three to both sides to isolate the variables.
Adding three to both sides created zero pairs which can be removed from the left side of the equation.

Now, divide both sides of the equation by three.

\[
x = 3
\]

Check the equation.

\[
3x - 3 = 6
\]

\[
3 * 3 - 3 = 6
\]
Strategy 2:

$3x - 3 = 6$

Another strategy to solving this equation is to divide both sides by three.

By dividing both sides of the equation, the resulting equation is $x - 1 = 2$.

Add one to both sides to solve the equation.

Check the equation.

$3x - 3 = 6$

$3 \times 3 - 3 = 6$
**Practice 5.1**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ 4x - 1 = 7 \]

**Practice 5.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-5x + 2 = 7 \]
Example 5.C

Solve the equation, \(4 - x = 6\).

Set up the equation properly.

To solve the equation, remove four integers from both sides.

The solution to the equation is \(-x = 2\). However, to solve equations, the goal is to find the value of \(x\), not \(-x\). In this example, the coefficient of \(x\) is -1. Multiply both sides of the equation by -1 to make the variable positive.

Check the equation.

\[
4 - x = 6
\]

\[
4 - (-2) = 6
\]
Example 5.D

Solve the equation, \(4 - 2x = 6\).

Set up the equation properly.

\[-2x = 2\]

Solve the equation by dividing both sides of the equation by -2.

\[x = -1\]

Check the solution.

\[4 - 2x = 6\]

\[4 - (2)(-1) = 6\]
**Practice 5.3**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-3x = 9\]

**Practice 5.4**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[3 - 2x = -5\]
Solving Equations with Variables on Both Sides

Some multistep equations have variables on both sides of the equation. The goal of solving equations is to isolate the variable on one side of the equation.

Example 6.A

Solve the equation, $3x - 2 = x - 4$.

Notice that there are identical items on both sides of the equation, so they can be removed. Remove either the $x$ from both sides, or $-2$ from both sides, order does not matter.

Now, to eliminate the $-2$ from both sides, add two, which creates two zero pairs ($-2 + 2 = 0$) which can be removed.
How can you find the value of $x$?

Divide both sides of the equation by two.

$$x = -1$$

Check the solution.

$$3x - 2 = x - 4$$

$$3(-1) - 2 = (-1) - 4$$

**Example 6.B**

Solve the equation, $2x - 4 = x + 2$.

Begin solving the equation by removing an $x$ from both sides of the equation.

In order to solve the equation, isolate the variable on the left side of the equation by adding four to both sides.
On the left side of the equation, a zero pair is created \((-4 + 4 = 0)\) which can be removed.

\[ x = 6 \]

Check the equation.

\[ 2x - 4 = x + 2 \]

\[ 2(6) - 4 = 6 + 2 \]

**Example 6.C**

Solve the equation, \(-3x - 4 = 4x + 3\).

Notice that to solve this equation, there are no common elements on both sides of the equation. Consider using zero pairs to eliminate the variables on the left side of the equation, add \(3x\) to both sides.
Notice that the variable is on the right side of the equation, how can you isolate it?

Subtract three from both sides.

```
<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
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<th></th>
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</tr>
</thead>
</table>
```

Divide both sides of the equation by 7.

```
<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>
```

Check the solution.

\[-3x - 4 = 4x + 3\]

\[
(-3) (-1) - 4 = 4 (-1) + 3
\]
Practice 6.1

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 4 = 2x - 3 \]

Practice 6.2

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ 3x - 3 = -2x + 2 \]
Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-2x + 7 = 3x + 2\]
APPENDIX G
Solving Equations with Virtual Manipulatives
Solving Equations

Solving equations in one variable is a very important skill in algebra. Many equations can be solved using this set of strategies which gives a concrete representation to the abstract concept.

Virtual manipulatives are used while solving equations and the following pieces represent elements of the equation. All work will occur on balance scale like the one below.

Define the Concept of Balance Related to Equations

Solving equations is like working with a scale with the fulcrum as the equal sign. (The fulcrum is the balancing point). Solving equations and scales both require balance.

Example 1.A

For example, the scale below is balanced because there are three squares on the left side and three squares on the right side. An equation is similar. Both sides of the equation must be equal.
Example 1.B

What happens if two is added to both sides of the balance scale?

The scale remains balanced because $3 + 2 = 3 + 2$.

Equations work the same way, two (or any number) can be added to both sides and it will remain balanced.
**Practice 1.1**

Add four to both sides of the scale below. What happens?

The scales should have remained balanced because 4 was added to both sides. Solving equations requires balance.

In a similar way, the same number can be removed from both sides of a scale. In mathematics, removing is the same as subtraction.

**Example 1.C**

On the balance scale below, 4 – 3 = 4 - 3. Both sides of the equation are equal because three is removed from both sides.
Practice 1.2

Remove two from both sides of the balance scale below.

What happens?

All mathematics operations can work with a balance scale if the same operation is done to both sides of the equation.
Example 1.D

Both sides of an equation can be multiplied by two and the equation will remain balanced.

\[ 3 \times 2 = 3 \times 2 \]

Practice 1.3

Multiply each side of the equation below by 3.

What happens?
Example 1.E

Both sides of an equation can be divided by the same number and the equation will remain balanced, for example, divide both sides of the equation by two.

\[4 \div 2 = 4 \div 2\]

Practice 1.4

Divide each side of the equation by three.

What happens?
Demonstrate Checking an Equation

Both sides of a balance scale must have the same value. If the left side of the equals five, the right side must equal ______.

A variable is a letter, often $x$, which is used to represent an unknown value. The goal of solving equations is to find the value of the variable.

When a teacher is absent from school, the school hires a substitute teacher, someone who stands in the place of the teacher. The variable is similar; it stands in the place of the unknown value. When checking a solution, let the number “stand in” for the variable and see if both sides of the equation are equal.

Example 2.A

When the left side of the equation equals five, the right side of the equation must equal five. In the equation below, the solution is $x = 1$. Substitute 1 for the variable. The solution is correct if both sides of the equation are equal.
**Example 2.B**

Check to see if $x = 4$ is the correct solution for the equation $2x = 8$.

$2x = 8$

Check the equation

$2x = 8$

$(2)(4) = 8$

**Example 2.C**

Check to see if $x = 3$ is the solution to the equation $x + 4 = 6$.

$3$ is not a solution because $3 + 4 \neq 6$.

**Practice 2.1**

Check to see if $x = 5$ is a solution to the equation $x + 5 = 10$.

**Practice 2.2**

Check to see if $x = 3$ is a solution to the equation $3x = 6$. 
Practice 2.3

For the following equation, which is the correct solution?

\[ x + 4 = 7 \]

A. \( x = 3 \)

B. \( x = 7 \)
Solving Addition and Subtraction Equations

Addition and subtraction are inverse operations, which means they are opposites. Adding three is the opposite of subtracting three. The result of inverse operations is zero (often called a zero pair.) For example,

\[ +5 - 5 = 0 \]
\[ -20 + 20 = 0 \]

Example 3.A

Solve the equation, \( x + 3 = 6 \)

Click Create A Problem. Enter the equation, \( 1x + 3 = 6 \) and click Begin.

Place \( x \) and three positive integers on the left side of the scale and six positive integers on the right side of the scale. Click Continue when the equation is set up properly.
The goal of solving equations is to isolate the variable. What can you do to get the variable by itself?

Subtract 3 from both sides of the equation.

Note that the results of all steps are displayed in the equation window. Note that addition, subtraction, multiplication, and division can occur within the Action center.

For this equation, the solution is $x = 3$.

Remember to check the solution by substituting the result for the variable.
Example 3.B

Solve the equation, $4 + x = 8$.

To solve the equation, isolate the variable by subtracting 4 from both sides of the equation.

$4 + x = 8$

$x = 4$

Check the equation.

$4 + x = 8$

$x = 4$

$4 + 4 = 8$

Practice 3.1
Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x + 3 = 8 \]

---

**Practice 3.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x + 1 = 5 \]

---

**Example 3.C**

Solve the equation, \( x - 5 = 3 \).
Enter the equation and place one positive variable and five negative integers (represented by red balloons) on the left side of the scale and three positive integers on the right side of the scale.

The goal of solving equations is to isolate the variable, what can you do to get the variable by itself?

Add positive five to both sides of the equation. \((-5 + 5 = 0, \text{this is called a zero pair}).

Example 3.D
Solve the equation, \( x - 2 = -4 \).

To isolate the variable, add 2 to both sides \((-2 + 2 = 0, \text{ another example of a zero pair.})\)

The equations in this section are examples of the **Addition Property and Subtraction Property of Equality** which state that you can add (or subtract) both sides of an equation by the same nonzero number and the statement will remain true.

*Practice 3.3*
Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 4 = 5 \]
Practice 3.4

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 3 = -4 \]
Solving Multiplication and Division Equations

Just as addition and subtraction are inverse operations, multiplication and division are inverse operations. To solve an equation, you may use the inverse operation to “undo” to isolate the variable. In the equation $2x = 4$, the 2 is the coefficient of $x$ because it is the number by which $x$ is multiplied.

Example 4.A

Solve the equation, $2x = 4$.

Place two $x$’s on the left side of the equation and four positive integers on the right side of the equation.

What can you do to isolate the variable on the left side of the equation?

Divide both sides of the equation by two.
Example 4.B

Solve the equation, $3x = 9$.

What can you do to isolate the variable? Divide both sides of the equation by three.
Practice 4.1

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ 2x = 8 \]
**Practice 4.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[3x = -9\]

Virtual manipulatives cannot be used to solve division equations because fractional coefficients are not allowed; however, they are solved in a similar fashion.

**Example 4.C**

Solve the equation, \(x \div 2 = 3\).

This equation would be represented with half of a variable on the left side of the equation, and three positive integers on the right side of the equation.

This equation tells us that half of a variable equals three, so how would you find out what a whole variable equals?

Multiply both sides of an equation by two.
Example 4.D

Solve the equation, \( x \div 4 = 2 \).

This equation would be represented with one fourth of a variable on the left side of the equation, and two positive integers on the right side of the equation.

This equation tells us that one fourth of a variable equals two, so how would you find out what a whole variable equals?

Multiply both sides of the equation by 4.

These equations are examples of the Multiplication Property and Division Property of Equality which state that you can multiply (or divide) both sides of an equation by the same nonzero number and the statement will remain true.
Practice 4.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x \div 2 = 8 \]

Practice 4.4

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x \div 3 = -2 \]
Example 5.A

Solve the equation, $2x + 4 = 8$.

Enter the equation into the virtual manipulative correctly.

The goal of solving equations is to isolate the variable, what can you do to get the variable alone?

Subtract four from both sides.

Note the new equation in the equation window, $2x = 4$. How can you isolate the variable?
Divide both sides of the equation by two.

Example 5.B

Solve the equation, $3x - 3 = 6$.

Set up the equation properly.
There are two options to solving this equation, what are two actions that will isolate the variable?

**Strategy One:**

Add three to both sides to isolate the variables.

Now, divide both sides of the equation by three.

**Strategy 2:**

Check the equation.

\[3x - 3 = 6\]
\[3 \times 3 - 3 = 6\]
Another strategy to solving this equation is to divide both sides by three.

By dividing both sides of the equation, the resulting equation is $x - 1 = 2$. Add one to both sides to solve the equation.
\[ x = 3 \]

Check the equation.

\[ 3x - 3 = 6 \]

\[ 3 \times 3 - 3 = 6 \]
**Practice 5.1**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[4x - 1 = 7\]

**Practice 5.2**

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-5x + 2 = 7\]

**Example 5.C**
Solve the equation, \( 4 - x = 6 \).

Set up the equation properly.

To solve the equation, remove four integers from both sides.

The solution to the equation is \(-x = 2\). However, to solve equations, the goal is to find the value of \(x\), not \(-x\). In this example, the coefficient of \(x\) is \(-1\). Multiply both sides of the equation by \(-1\) to make the variable positive.
Example 5.D

Solve the equation, \(4 - 2x = 6\).

Set up the equation properly.

Solve the equation by removing four integers from both sides.
-2x = 2. Solve the equation by dividing both sides of the equation by -2.

\[ \frac{-2x}{-2} = \frac{2}{-2} \]

\[ x = -1 \]

Check the solution.

\[ 4 - 2x = 6 \]

\[ 4 - (2)(-1) = 6 \]
Practice 5.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-3x = 9\]

Practice 5.4

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[3 - 2x = -5\]
Some multistep equations have variables on both sides of the equation. The goal of solving equations is to isolate the variable on one side of the equation.

**Example 6.A**

Solve the equation, \(3x - 2 = x - 4\).

Notice that there are identical items on both sides of the equation, so they can be removed.

Remove either the \(x\) from both sides, or \(-2\) from both sides, order does not matter.
Now, to eliminate the -2 from both sides, add two, which creates two zero pairs (-2 + 2 = 0.)
How can you find the value of $x$? Divide both sides of the equation by two.

$x = -1$

Check the solution.

$3x - 2 = x - 4$

$3(-1) - 2 = (-1) - 4$
Example 6.B

Solve the equation, $2x - 4 = x + 2$.

Begin solving the equation by removing an $x$ from both sides of the equation.

In order to solve the equation, isolate the variable on the left side of the equation by adding four to both sides.

On the left side of the equation, a zero pair is created ($-4 + 4 = 0$.)
\[2x - 4 = x + 2\]

Check the equation.

\[2x - 4 = x + 2\]

\[2(6) - 4 = 6 + 2\]

\[x = 6\]
Example 6.C

Solve the equation, \(-3x - 4 = 4x + 3\).

Notice that to solve this equation, there are no common elements on both sides of the equation. Consider using zero pairs to eliminate the variables on the left side of the equation, add \(3x\) to both sides.

Notice that the variable is on the right side of the equation, how can you isolate it?

Subtract three from both sides.
Divide both sides of the equation by 7.

\[ -3x - 4 = 4x + 3 \]

Check the solution.

\[ -3(-1) - 4 = 4(-1) + 3 \]
Practice 6.1

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[ x - 4 = 2x - 3 \]
Practice 6.3

Solve and check the equation below. Sketch the step(s) of your solution in the space provided.

\[-2x + 7 = 3x + 2\]
APPENDIX H

Practice Problems for All Groups
Equations

Solve and check the equations below. Sketch the step(s) of your solution in the space provided.

1. \( x + 4 = 8 \)

2. \( x + 3 = 5 \)

3. \( x - 4 = 3 \)

4. \( x - 6 = -4 \)
5. $4 + x = 2$

6. $-3 + x = 3$

7. $x - 2 = 7$
8. \( x + (-2) = 4 \)
Equations

Solve and check the equations below. Sketch the step(s) of your solution in the space provided.

1. $4x = 8$

2. $3x = 9$

3. $4x = 8$

4. $-2x = -4$
5. \(-3x = 6\)

6. \(x ÷ 2 = 3\)

7. \(x ÷ 2 = 7\)
8. \( x \div (-2) = 4 \)
Solve and check the equations below. Sketch the step(s) of your solution in the space provided.

1. \(-2x + 3 = 1\)

2. \(-3x + 2 = -4\)

3. \(2x + 5 = 9\)

4. \(3x - 4 = 5\)
5. $4 - x = 2$

6. $-3 - x = 3$
7. $3x + 1 = 10$

8. $-2x - 1 = -7$
Solve and check the equations below. Sketch the step(s) of your solution in the space provided.

1. \(-2x - 4 = 2x - 8\)

2. \(2x + 2 = x + 3\)

3. \(x + 7 = 2x + 5\)
4. $3x - 4 = 5$

5. $3x + 1 = 2x + 4$

6. $3x - 5 = -2x + 5$
7. \(-2x + 5 = 2x + 1\)

8. \(-x + 1 = 2x - 5\)
APPENDIX I
ORIGINAL DATA SET DESCRIPTIVE STATISTICS

Table I1
Descriptive Statistics of Original Data Set

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Table I2
Linear Regression on Original Data Set

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a. Dependent Variable: posttest
REFERENCES


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