Method of Discrete Orthogonal Basis Restoration

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METHOD OF DISCRETE ORTHOGONAL BASIS RESTORATION


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Related U.S. Application Data

Continuation-in-part of Ser. No. 337,592, Nov. 10, 1994, abandoned.

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Field of Search  382/279, 254; 382/276

References Cited

U.S. PATENT DOCUMENTS

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ABSTRACT

A method is described for utilizing discrete orthogonal basis to restore signal system, such as radio or sound waves and/or image system such as photographs or medical images that become distorted while being acquired, transmitted and/or received. The signal or image systems are of the linear type and may be represented by the equation \([B][o]=[i]\) wherein \([o]\) is an original signal or image, \([i]\) is a degraded signal or image and \([B]\) is a system transfer function matrix. The method involves estimating a signal-to-noise ratio for a restored signal or image. Next, is the selecting of a set of orthogonal basis set functions to provide a stable inverse solution based upon the estimated signal-to-noise ratio. This is followed by removing time and/or spatially varying distortions in the restored system and obtaining an appropriate inverse solution vector.

17 Claims, 7 Drawing Sheets
CONSTRUCT AN ORTHONORMAL BASIS SET 
$[p]_m$ WITH $m=1,...,M$ MEMBERS OF LENGTH $l$.

PROJECT THE ADJOINT OF THE SYSTEM MATRIX 
ONTO EACH $[p]_m$ TO FORM THE VECTOR SET $[b]_m$
$[B][p]_m = [b]_m$.

PERFORM GRAM-SCHMIDT ORTHOGONALIZATION 
ON THE VECTOR SET $[b]_m$ TO YIELD $[c]_m$, 
WHICH IS ORTHONORMAL.

REGROUP THE GRAM-SCHMIDT COEFFICIENTS INTO 
A SET OF CONSTANTS $\tau$ SO THAT
$c_{mk} = \sum_{i=1}^{m} \tau_{mi} b_{ik}$
IS SATISFIED.

USING THE SAME SET OF CONSTANTS $\tau$, THE 
VECTOR SET $[d]_m$ IS CALCULATED BY
$d_{mk} = \sum_{i=1}^{m} \tau_{mi} p_{ik}$

THE INVERSE SOLUTION ESTIMATE $O_k$ IS THEN 
CALCULATED BY
$O_k = \sum_{m=1}^{M} \sum_{k=1}^{IU} c_{mk} d_{mk} i_k$ 
WHERE $i_k$ IS THE FORWARD SOLUTION VECTOR.

FIG. 8
METHOD OF DISCRETE ORTHOGONAL BASIS RESTORATION

This is a continuation-in-part of U.S. patent application Ser. No. 08/337,592, filed on Nov. 10, 1994, entitled "Method of Discrete Orthogonal Basis Restoration", now abandoned.

TECHNICAL FIELD

The present invention relates generally to the field of signal and image restoration and, more particularly to a method of restoring a signal and/or image degraded by time and/or spatially varying transfer functions.

BACKGROUND OF THE INVENTION

It is commonplace for signals, such as radio or sound waves and images such as photographs or medical images to become distorted while being acquired, transmitted and/or received. This phenomenon occurs in various types of radar, sonar, optic, imaging and electronic systems.

As an example, blurring of a photographic image may result from camera and/or object motion at the time of acquisition or may even be produced by the nature of the photographic equipment (e.g. "fish-eye" lens). In medical imaging, the type of equipment used and the way the images are acquired can have remarkable effects on the level of distortion or blurring present in the final images that are interpreted by physicians.

It should further be appreciated that the characteristics of the distorting process may change with time during the acquisition of a signal or may vary with location over different areas of an image. The time and/or spatially varying distortions in a signal and/or image must be removed to restore a signal and/or image to its undistorted form and enhance clarity.

In practical application there are imperfections in the signal or image acquisition process that make it impossible for any method to perfectly recover the original signal or image. Special mathematical techniques may, however, be utilized to closely estimate what the signal or image was before it was degraded. The time and/or spatially-varying nature of some systems makes it particularly difficult to perform a fully accurate restoration. Still, when properly applied such techniques may be utilized to substantially improve the quality of a signal or image so that it more closely approximates the true or undistorted original signal or image.

In order to further understand this process it must be appreciated that signals or images degraded by a linear system may be cast in the operator notation [B] [o]=[i]; where [o] is the original signal or image, [i] is the degraded signal or image, and [B] is the system transfer function matrix. Signal or image restoration is the determination of an approximation [o'] to the original signal [o], given a priori knowledge of the transfer function matrix [B] and the forward solution [i].

The most straightforward means of determining the inverse solution is by application of the transfer function matrix inverse to the forward solution, that is, [B]^-1 [i]=[o']. However, determining [o'] by this approach frequently represents an ill-posed problem as the inverse of the transfer function may not exist (singular matrix) or [B]^-1 may be near-singular. In either case, the inverse solution cannot be determined. Further, even if [B] is invertible, [B]^-1 will frequently be ill-conditioned, meaning that small perturba-

tions in [i] will lead to large perturbations in [o'] when the inverse solution is computed. This leads to unacceptable results. This is because all practical systems have inherent uncertainty in the measurement of [i], as well as added noise, and accordingly, adequate estimation of the inverse solution [o'] is not possible through application of an ill-conditioned transfer function inverse.

To date, many methods have been developed to solve inverse problems arising in imaging processing, optics, geophysics, astronomy, spectroscopy, and other engineering and scientific disciplines. The existence of multiple solutions is primarily due to the fact that no single prior art method provides the best estimate of inverse solution in all practical applications. In fact, most prior art methods have only very specific, limited applications within specialized technical fields.

As [B] [o]=[i] constitutes a linear system, solution by linear methods is an intuitively attractive approach. However, while the solution may be attempted by linear transform methods, such as Fourier transforms, the ill-conditioned nature is not circumvented by these techniques. Furthermore transform techniques are not directly applicable when the transfer function is shift-variant.

Application of transform methods to shift-variant systems have been limited to those cases where the signal or image can be sectioned into regions over which the system may be considered to be stationary. The inverse solutions for these regions are computed by transform techniques, and then spliced back together to form the overall solution. Similar sectioning into assumed stationary regions with inversion by the maximum a posteriori method has also been proposed. This sectioning and reassembly approach ("mosaicing") is, however, highly dependent on the validity of the stationary assumption, the method of reassembly, and on sampling of the forward solution and these considerations all adversely affect restoration results.

Various non-transform methods of linear inverse solution have also been developed. These include Weiner filtering, constrained Weiner filtering, maximum entropy, and pseudo-inversion techniques. These methods are usually applicable to the shift-variant case and they address the ill-conditioned nature of the problem. One drawback of such methods is, however, that they tend not to perform well in the presence of low signal-to-noise ratio (SNR) or on systems with moderate to severely degrading transfer functions. Thus they fail when they are most needed. These linear methods also do not provide super-resolution capability, and the linear iterative methods (e.g. pseudo-inversion, van Clutter's method, maximum entropy) do not have well defined termination points and can have very high memory and computational demands if a large number of iterations are performed. Thus, hardware requirements and processing times are disadvantageously increased.

The shortcomings of existing linear techniques has spawned great interest in non-linear approaches. The non-linear approaches are based on various regularization techniques that incorporate a priori knowledge of various parameters to yield an inverse solution that stabilizes and constrains the inverse solution. The parameter variables (hyperparameters) may include constraints on the form of the solution (such as non-negativity) goodness of fit parameters, statistical parameters, and assumptions of the character of added noise. Non-linear methods are usually applicable to shift-variant systems and may have super-resolution properties. The performance of these approaches is highly dependent on proper choice of the hyperparameters.
needed for the particular method. Furthermore, these approaches are usually iterative with poorly defined criteria
termination. For systems with well defined hyperparameters and termination criteria, and when the computational
burden is not an obstacle, these are usually the preferred method of inverse solution. However, when the a priori
knowledge of the system is inadequate, or when optimal termination of the iterative process is problematic, a linear
method of solution is likely to produce better restoration results.

Another linear method of interest provides deconvolution for stationary systems based on the properties of the system
adjoint operator. Referenced to as deconvolution by the method of orthogonal polynomials (Strizhke IEEE Trans Med
Imaging vol. 9, 1990, pp. 11–23), the crux of this method is the
inner-product property of adjoint operator on vectors. The
method requires a discrete orthogonal basis set. The original
author, however, failed to define the origins of instability or
the criteria for ensuring a stable solution. Accordingly, the
scope of practical applications of this approach is very
limited.

From the above it should be appreciated that a need exists
for a more versatile and effective method of signal and
image restoration suited for a wide range of applications in
various fields.

SUMMARY OF THE INVENTION

Accordingly, it is a primary object of the present invention
to provide an efficient and dependable method for signal
and/or image restoration adapted for a number of specific
applications across a broad number of technical fields.

Another object of the invention is to provide an improved
method for quickly restoring a signal and/or image system
degraded by time and/or spatially varying transfer functions.
Such a system reduces processing time without comprising
the quality of the final or restored image.

Yet another object of the invention is to provide a discrete
orthogonal basis method for quickly restoring a signal or
image to an undistorted form. Advantageously, the method
utilizes a mathematical processing technique requiring rela-
tively small computer memory capacity such as found in
a personal computer, so as to allow ready application by
individuals in many, differing fields utilizing readily available
computer hardware. Further, the method also provides
uncompromising speed of operation and very effective
results.

A still further object of this invention is to provide a
discrete orthogonal basis restoration method particularly
suited to reconstruct and restore nuclear medicine SPECT
images.

Additional objects, advantages and other novel features of
the invention will be set forth in part in the description that
follows and in part will become apparent to those skilled in
the art upon examination of the following or may be learned
with the practice of the invention. The objects and advan-
tages of the invention may be realized and obtained by
means of the instrumentalities and combinations particularly
pointed out in the appended claims.

To achieve the foregoing and other objects, and in accor-
dance with the purposes of the present invention as
described herein, an improved method is provided wherein
discrete orthogonal basis is utilized to restore a signal and/or
image system that is degraded by time and/or spatially
varying transfer functions. Advantageously, the present
method represents a relatively simple inverse solution that
quickly and efficiently restores the system to an undistorted
form. Accordingly, a clearer and more focused signal or
image system results.

The method includes the step of estimating a SNR for a
restored system. Additionally, there is the step of selecting
of a set of orthogonal basis set functions p_m to provide a stable
inverse solution based upon the estimated forward solution
SNR.

More specifically, the estimated SNR of the restored
system is provided by applying a given forward solution
SNR and selected set of orthogonal basis set functions p_m
to a realistic simulation model of the system. The set of
orthogonal basis set functions p_m may be any orthogonal
basis that spans the forward and inverse solution vector
spaces. Such basis set functions include but are not limited
to a group consisting of Hartley, Walsh, Haar, Legendre,
Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre
functions.

Next is the step of removing the time and/or spatially
varying distortions in the restored system by obtaining an
inverse solution vector o_k for a one dimensional restoration
wherein:

$$o_k = \sum_{m=1}^{M} c_{mk} \left[ \frac{U_j}{\sum_{l=1}^{M} d_{mk} \alpha_l} \right]$$

wherein:

$$[B]_{mn} = (B)^*_{m} p_m$$

where [B]_mn is the transpose-complex conjugate of the matrix B and p_m is an M member orthogonal basis set and
a are the standard Gram-Schmidt orthogonalization coefficients;

$$c_{mk} = \sum_{l=1}^{M} c_{ml} \alpha_l$$

where:

$$\alpha = \frac{1}{\sum_{l=1}^{M} c_{ml}}$$

Alternatively, the inverse solution vector O_k for a two
dimensional system with separable spatially variant PSF the
inverse solution may be obtained by successive row-column
operations:

$$o_{jk} = \sum_{m=1}^{M} c_{mk} \left[ \frac{U_j U_k}{\sum_{l=1}^{M} d_{mk} \rho_l} \right]$$

wherein:

$$\rho = \frac{1}{\sum_{l=1}^{M} c_{mk} \rho_l}$$

More specifically describing the invention, the estimating
of the signal-to-noise ratio SNR_m is provided by the
formula
wherein $|I_{0_{\text{true}}}|^2$ is the signal power in the original (undegraded) signal or image, $|I_{0_{\text{true}}}|^2$ the inverse solution noise power due to the approximate nature of the inverse solution and $|I_{0_{\text{true}}}|^2$ is the noise power in the inverse solution power due to added noise in the forward solution.

Advantageously, the present method functions to define the origins of instability and behavior in the presence of noise. By applying the method to time-varying systems and using a technique for a priori determination of the SNR inverse solution, it is possible to insure stability and optimal selection of the basis set. Further, the method of inverse solution and SNR estimation may be successfully extended to the restoration of two-dimensional images degraded by spatially variant point spread functions.

The discovery of the origins of instability along with the development of an approach for selection of the optimal basis set to maximize inverse solution SNR, makes the present method a viable linear approach to inverse solution. Advantageously, the method is applicable to both stationary and shift-variant systems, is non-iterative, and is computationally efficient. Thus, the speed of processing and the size of the computer necessary to complete that processing are both reduced. Further, it should be appreciated that the only a priori information required to estimate the SNR of the inverse solution is an estimate of the forward solution signal characteristics and estimate of the inverse solution noise due to a limited basis set. Of course, in some cases the type of instrumentation or acquisition parameters may guide the optimal basis set selection (e.g. Nuclear Medicine SPECT imaging with reconstruction from projections).

As the present method is advantageously applicable to both stationary and shift-variant linear systems in one or more dimensions, potential applications for the present method include medical imaging (e.g. emission tomography, MRI, ultrasound), image processing (lens deblurring, motion artifacts), optics and spectroscopy (light and NMR), geophysics, radar/sonar, and general electronics and electrical engineering problems. Thus, the method is extremely versatile, having application in broad ranging technical fields.

Still other objects of the present invention will become apparent to those skilled in this art from the following description wherein there is described a preferred embodiment of the invention, simply by way of illustration of one of the modes best suited to carry out the invention. As it will be realized, the invention is capable of other different embodiments and its several details are capable of modification in various, obvious aspects all without departing from the invention. Accordingly, the drawings and descriptions will be regarded as illustrative in nature and not as restrictive.

### BRIEF DESCRIPTION OF THE DRAWING

The accompanying drawing incorporated in and forming a part of the specification, illustrates several aspects of the present invention and together with the description serves to explain the principles of the invention. In the drawing:

FIG. 1 graphically shows the original signal (a) is the sum of three unity amplitude sinusoids $(f_1=3\text{ cycles/2}\pi, \phi_1=0.1 \text{ radian}; f_2=7 \text{ cycles/2}\pi, \phi_2=1.0 \text{ radian}; f_3=10 \text{ cycles/2}\pi, \phi_3=0.6 \text{ radian})$. The forward solution (b) results when the time-varying decaying exponential system described by the transfer function $h_0$ acts on the original signal (see equations in Example 1, page 20). Pseudorandom zero-mean noise is added to produce forward solutions with SNR of 20 dB (c) and 10 dB (e). The inverse solutions obtained by the method using a Hartley basis set from 0–10 cycles/2π had SNR of 19.26 dB (d), solid line for 20 dB forward solution, and SNR of 10.33 dB (f), solid line for the 10 dB forward solution. The broken line in (d) and (f) is the original signal.

FIG. 2 is a plot of SNR forward vs. SNR forward (equation (13)) for the system. The solid line represents the predicted relationship between forward and inverse solution SNR for the method using a Hartley basis set over 0–10 cycles/2π. The broken line is for a Hartley basis set extending over 0–14 cycles/2π. The 0–14 cycles/2π basis set provides higher SNR of the inverse solution for high SNR forward due to superior basis representation of the original signal. With lower SNR forward, the broader basis set has greater noise recovery than the more restricted basis set causing inferior SNR forward for the inverse solution.

FIG. 3 is a photograph of an original image of a four quadrant checkerboard with square sizes of four, five, six and seven pixels.

FIG. 4 is a photograph of the forward solution showing severe distortion following degradation by a system with gaussian separable spatially variant point spread function (SSVPSF) that varied radially in width (center FWHM=6 pixels, corner FWHM=3 pixels) and amplitude (center=0.151, corner=0.075), and addition of noise to achieve SNR=20 dB.

FIG. 5 photographically shows the restored image following application of the present method to achieve resolution of all image elements with good contrast recovery.

FIG. 6 photographically demonstrates the added noise in the restored image when the original image is processed in accordance with the present method from a noiseless forward solution.

FIG. 7 is a two dimensional representation of the Gram-Schmidt orthogonalization process in the presence of noise.

FIG. 8 is a flowchart showing the methodology of the present invention.

Reference will now be made in detail to the present preferred embodiment of the invention, an example of which is illustrated in the accompanying drawing.

### DETAILED DESCRIPTION OF THE INVENTION

The method of the present invention for using discrete orthogonal basis to restore a signal and/or image system created by time and/or spatially varying transfer functions will now be described in detail. The method may be applied to restore a signal or image system in a wide variety of technical fields. Stability of the inverse solution may be achieved if the characteristics of the noise in the forward solution may be estimated. For time-varying linear systems having a region of basis function support approximately congruent to the support region of the transfer function, and for which there is sufficient a priori knowledge of the system, the present method provides an efficient and noise tolerant approach to achieve inverse solution.

As previously described, the method of the present invention may be utilized to obtain an inverse solution vector for either one or two dimensional restorations. For purposes of presentation, $m=1.2.3.\ldots\ldots M$ is the index for the vector sets
with M members. The row and column vectors are designated by lowercase letters, and square matrices by uppercase letters. The system matrix transfer function is denoted by B.

Given the linear operation

\[ [B] \vec{a} = [\vec{b}] \]

the purpose of this invention is to recover the length IJ vector \( \vec{b} \), given the forward solution vector \( \vec{a} \) and a prior knowledge of the forward operator \( [B] \). The \( I \times IJ \) matrix \( B \) is constructed using the time (or spatially) varying system transfer function \( b_{mk} \), so that the forward solution \( \vec{i} \) is defined by

\[ \vec{i}_k = \sum_{m=1}^{I} b_{km} \vec{r}_m \]

where \( k = 1,2,3, \ldots ,IJ \).

Recovery of the inverse solution requires two orthogonal function sets related to the adjoint PSF operator. The construction of these function sets (equations (4-6b)) requires a set of \( M \) orthogonal basis set functions \( p_{mk} \) of length IJ. The Hartley basis set, defined by

\[ p_{mk} = \sin \left( \frac{2\pi m - 1}{IJ} k \right) + \cos \left( \frac{2\pi m - 1}{IJ} k \right) \]

is the preferred basis for real-valued systems and will be utilized to illustrate the present method. It should be appreciated, however, that any orthogonal basis that spans the forward and inverse solution vector spaces may be utilized. These include for example, Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

As can be appreciated in viewing FIG. 8, the method begins with the application of the adjoint of the forward operator to each member of the basis set

\[ [B]_m^* [B]_n = \delta_{mn} \]

where \( [B]_m^* \) is the transpose-complex conjugate of the matrix \( B \).

The Gram-Schmidt orthogonalization procedure followed by normalization may be used to construct an orthonormal function set \( c_{mk} \) from \( b_{mk} \). Defining a set of constants \( \tau_{mk} \) (5), where the constants a are the Gram-Schmidt orthogonalization constants, \( c_{mk} \) may be written as a linear combination of \( b_{mk} \) (6a). A second orthogonal function set \( d_{mk} \) can then be constructed as a linear combination of the basis set functions \( p_{mk} \) using the same set of constants \( \tau_{mk} \) (6b).

\[ \tau_{mk} = \frac{c_{mk}}{\sqrt{\sum_{i=1}^{M} c_{mi}^2}} \quad m = 1 \]

\[ \tau_{mk} = \frac{m_i}{\sqrt{\sum_{i=1}^{m} \sum_{k=1}^{I} c_{mk}^2}} \quad m = 1, \ldots ,m - 1 \]

\[ c_{mk} = \sum_{i=1}^{m} \tau_{mi} b_{ik} \]

Recovery of the inverse solution begins with the unit operator for orthonormal functions (7). Reference to (6a) and (4) yields equation (8). The property of adjoint operators on finite-dimension inner product space allows transition to (9) and equation (1) leads to (10). Equation (10) is the operational equation for the inverse solution vector for a one dimensional restoration utilizing the present method.

\[ a_k = \sum_{m} c_{mk} \left[ \sum_{n} \tau_{mn} \left( \sum_{i=1}^{M} D_{ni} \right) \right] = \sum_{m} c_{mk} \left[ \sum_{n} \tau_{mn} \left( \sum_{i=1}^{M} D_{ni} \right) \right] \]

Of course, this inverse solution \( a_k \), like all method of ill-posed problem solution, is an approximation of the true solution. In the noiseless case, the quality of the inverse solution depends primarily on the equality of the representation of the true signal afforded by the chosen finite set of basis functions \( p_{mk} \) where \( m = 1,2,3, \ldots ,M \) is the index for the chosen orthogonal basis set with \( M \) members of length \( k = 1,2,3, \ldots ,IJ \). The recovered vector \( a_k \) may be expressed as a sum of the inverse solutions from the noiseless forward solution and from added noise components (see equation 11).

\[ a_k = \sum_{m} c_{mk} \left[ \sum_{n} D_{ni} \right] = \sum_{m} c_{mk} \left[ \sum_{n} D_{ni} \right] \]

In the Fourier domain, the noise component term in (11) may be expressed by

\[ \sum_{O_{\text{noise}}} = \sum_{m} C_{m} \left[ \frac{1}{N} \sum_{i} D_{0i} \text{noise} \right] \]

which allows an approximation of the inverse solution noise power to be made if the characteristics of the noise are known. The predicted SNR of the inverse solution may be estimated by

\[ \text{SNR}_{\text{pred}} = 10 \log \frac{\text{SNR}_{\text{in}}^2}{\text{SNR}_{\text{in}}^2 + \text{SNR}_{\text{noise}}^2} \]

where \( \text{SNR}_{\text{in}}^2 \) is the signal power in the original (undegraded) signal or image, \( \text{SNR}_{\text{forward}}^2 \) is the inverse solution noise power due to the approximate nature of the inverse solution, and \( \text{SNR}_{\text{noise}}^2 \) is the noise power in the inverse solution due to added noise in the forward solution.

For a given system, an a priori estimate of the inverse solution SNR can be made for various values of SNR_{\text{forward}} using equations (12) and (13). Noise power in the inverse solution due to added noise may be estimated by assigning values to

\[ \sigma_{\text{noise}}^2 \]

(12) based on assumptions of the character and magnitude of added noise. Simulation studies with well modeled noiseless signals allow estimation of intrinsic noise. \( \sigma_{\text{intrinsic}}^2 \) in the
inverse solution. With an estimate of the original signal power, the $\text{SNR}_{\text{pred}}$ for the inverse solution may be computed using equation (13).

The following example is presented for purposes of further illustrating the present invention, but it is not to be considered as limited thereto.

**EXAMPLE 1**

An exponentially decaying transfer function with linearly time-varying amplitude and time constant was chosen for purposes of demonstration. The behavior of the time-varying transfer function $h_{\text{kn}}$ is defined by

$$h_{\text{kn}} = \left(0.5 + 0.5 \frac{n}{N}\right) \exp\left(-\frac{n}{N}\right)$$

The input function (Fig 1(a)) is a summation of three unity amplitude sinusoids of arbitrarily chosen frequency and phase ($f_1$=3 cycles/2π, $f_2$=6 cycles/2π, $f_3$=1.0 radian; $f_4$=10 cycles/2π, $f_5$=0.6 radian). Zero-mean pseudo-random noise was added to the forward solution (Fig 1(b)) to achieve SNRs ($\text{SNR}_{\text{inverse}}=10 \log(\frac{\sigma_{\text{true}}^2}{\sigma_{\text{recovered}}^2})$) of 10.0 and 20.0 dB (Fig. 1 (c) and (e)). The method was performed using the Hartley basis set (equation 3) extending from 0 to 10 cycles/2π. The inverse solutions show good recovery of the original input function, with inverse solution SNRs ($\text{SNR}_{\text{inverse}}=10 \log(\frac{\sigma_{\text{true}}^2}{\sigma_{\text{recovered}}^2})$) of 19.26 dB (Fig. 1 (d)) for the 20 dB forward solution, and 10.33 dB for the 10 dB forward solution (Fig. 1 (f)).

It should also be appreciated that using an assumption of zero-mean white noise, and a predetermined value of $\sigma_{\text{true}}^2$, a plot of $\text{SNR}_{\text{pred}}$ vs. $\text{SNR}_{\text{form}}$ (see Fig. 2) may be constructed for the system described for Hartley basis function bandwidths of 10 and 14 cycles/2π. For the 0–10 cycles/2π Hartley basis set, the $\text{SNR}_{\text{pred}}$ values of 11.39 dB and 17.91 dB (for the 10 dB and 20 dB forward solutions respectively) correspond reasonably well with the experimental $\text{SNR}_{\text{inverse}}$ values of 10.33 dB and 19.26 dB. Fig. 2 illustrates that increasing the recovery bandwidth from 10 to 14 cycles/2π results in improved $\text{SNR}_{\text{pred}}$ when the $\text{SNR}_{\text{form}}$ is high, due to the improvement in representation of the inverse solution afforded by a more complete basis set. However, with lower $\text{SNR}_{\text{form}}$, the effects of increased noise recovery accompanying expansion of the basis set offsets this advantage and results in lowering the $\text{SNR}_{\text{pred}}$ of the inverse solution. This method of $\text{SNR}_{\text{inv}}$ estimation can be performed using training sets of large numbers of simulated signals and noise levels to determine the best selection of basis set for a given application.

The forward solution $J_{pk}$ for two-dimensional separable spatially variant point spread function (SVSPSF) systems may be obtained by successive application of column and row degradation operators to the original image $O_{pk}$. Obtaining the inverse solution for the SVSPSF system by the present method follows the general approach for matrix operators on separable systems. For each row $p=1, 2, \ldots, M$, the blurring matrix across the columns, $B_{\text{l}p}$, is constructed, allowing calculation of the corresponding orthogonal sets $c_{\text{form}}$ and $d_{\text{form}}$ (6). Successive application of the present operational equation (10) to all of the rows yields the $M \times M$ intermediary matrix $J_{pk}$, which has been corrected for the blurring across columns.

$$J_{pk} = \sum_{m=1}^{M} c_{pk}[\frac{1}{10} d_{pk}]$$

$\rho = 1, 2, 3, \ldots, JU$

The present method process is then repeated on each column of the intermediary matrix $J_{pk}$ to remove the blurring across rows. For each column, the appropriate blurring matrix $B_{\text{col}}$ is used to determine the $c_{\text{form}}$ and $d_{\text{form}}$ orthogonal sets needed for the inverse solution. The resultant matrix $O_{pk}$ is the desired two dimensional inverse solution.

Of course, the above equations (14) and (15) are 2-D restoration equations strictly for the "separable, spatially varying point spread function case. $J_{pk}$ is the result of performing 1-D DOBR on all the rows and $O_{pk}$ is the result of performing 1-D DOBR on all the columns of $J_{pk}$.

It should, also, be appreciated that limitation of the basis set bandwidth is required for stability of the inverse solution in the presence of noise. Noise in the restored image may be due to added noise included in the recovery method, or may result from imperfections in the representation of the image by a limited basis set. The inverse solution for a noisy input is the sum of the output of the method applied separately to signal and noise components. For the two-dimensional separable case, the noise power present in the inverse solution is the noise recovered by successive row and column operations.
is likely to provide the best restoration. Large numbers of simulations with training sets of images and noise levels appropriate to the application can be used to determine the optimal basis set for a particular application.

The following example is presented for purposes of further illustrating the present invention, but it is not to be considered as limited thereto.

**EXAMPLE 2**

A 128×128 pixel four quadrant checkerboard pattern with quadrant check sizes of 4.5, 6, and 7 pixels (see photographic FIG. 3) was chosen for illustration of the present method in restoring a SSVPSF system. The light squares were assigned a value of 1.0 and the dark squares 0.3. The multiple high-contrast discontinuities in this image were designed to be particularly challenging for a restoration method using a bandlimited basis set. The spatially varying gaussian transfer function had exponentially-radially varying FWHM of 6 pixels in the image center and FWHM of 3 pixels at the image corners. The PSF located at the center of the image space was assigned an amplitude so that the center PSF was lossless. PSF amplitude declined radially in exponential fashion so that the PSF amplitude in the corners was half of the center PSF amplitude.

Based on the methods described above, the Hartley basis set utilized was restricted to ±18 cycles/2π to assure a stable restoration for an anticipated forward solution SNR of 20 dB. Zero-mean pseudorandom noise was added to the forward solution to achieve a SNR of 20 dB (see photographic FIG. 4). The restored image (see photographic FIG. 5) from this noisy forward solution showed good contrast recovery with resolution of all image details. Due to the presence of method recovered noise the subjective quality is inferior to the noiseless restoration shown in photographic FIG. 6, which is subject only to intrinsic noise, but is a dramatic improvement from the degraded image shown in photographic FIG. 4.

This alternate embodiment described extends directly to two or more dimensions provided that it is cast in “stacked lexicographic” format e.g. A 2×2 image is acted on by a degrading operator to yield a forward solution 2×2 image;
The orthonormal vector set $C_{mk}$ is expressed as a superposition of prior $A_{ik}$ by

$$ C_{mk} = \sum_{i=1}^{m} \sum_{n=1}^{N} C_{i,m} = \sum_{i=1}^{m} \sum_{n=1}^{N} c_{i,m} a_{n} $$ \tag{22} 

A second complex vector set $D_{mk}$ is obtained by the same operation to the orthogonal basis set $P_{ik}$.

$$ D_{mk} = \sum_{i=1}^{m} \sum_{n=1}^{N} d_{i,m} a_{n} $$ \tag{23} 

The vector sets $C_{mk}$ and $D_{mk}$ define the characteristics of the system $[B]$ for frequency domain DOBDR.

In the frequency domain, the completeness relationship [24] for orthonormal vectors may be written as:

$$ O_k = \sum_{m=1}^{M} C_{m,k} \sum_{n=1}^{N} C_{n} a_{n} \Omega_k $$ \tag{24} 

Transferring the complex conjugation to $O_k$, substituting (22) and using the relationship $[F]^{*T}[F]_{mn} = |A|_{mn}$ yields

$$ O_k = \sum_{m=1}^{M} C_{m,k} \sum_{n=1}^{N} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{n=1}^{N} c_{i,m} a_{n} p_{i,n} \Omega_k^{*n} $$ \tag{25} 

which by the property of adjoint operators on inner product spaces [25] is equivalent to

$$ O_k = \sum_{m=1}^{M} C_{m,k} \sum_{n=1}^{N} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{n=1}^{N} c_{i,m} a_{n} p_{i,n} \Omega_k^{*n} $$ \tag{26} 

Reference to (18) and (23) yields the frequency domain DOBDR operational equation

$$ O_k = \sum_{m=1}^{M} C_{m,k} \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{n=1}^{N} c_{i,m} a_{n} D_{mk}^{*} $$ \tag{27} 

$O_k$ is the spectral estimate of the inverse solution. The time domain inverse solution $O_k$ may be obtained by inverse FFT of $O_k$.

The steps where major computational differences exist between the time and frequency domain approaches involve summations that may be limited to regions of time or frequency domain support. In the frequency domain approach the range of summation is restricted to the $M$ discrete frequencies where the composite basis set spectrum in non-zero. The frequency domain DOBDR approach saves $(N-M)$ multiplications and additions for each of the many inner products required by the approach. The reduction in the number of computations is at the expense of substituting complex for real operations. This is not particularly disadvantageous for additions, as current generation microprocessors perform complex and real additions with an equivalent number of clock cycles. Complex multiplications are more time consuming than real multiplications, but for practical DOBDR applications the reduction in computations offsets the increased processor time.

The initial step in the frequency domain approach is the determination of $A_{mk}$. When $[B]$ is time varying, $A_{mk}$ may be estimated by $|DFT|[B]^{*T}[DFT][P]_{mn}$ but this is less efficient than calculating the time domain $[b]_{mn}=[B]^{*T}[p]_{mn}$ and subsequently performing an $N$ point FFT on each vector $[b]_{mn}$. Using this approach, an additional $O(MN\log N)$ complex additions and $O(N/2\log N)$ complex multiplications are required for the frequency domain approach. Determining $C_{mk}$ by Gram-Schmidt orthogonalization requires calculation of $NC=(M^2-M)/2$ coefficients, for which the computational advantage of the frequency domain approach is $2NC(N-N)$ multiplications and additions. Performing the linear combinations of vectors weighted by these coefficients yields computational savings of $NC(N-M)$ multiplications and $(N-M)(M+N-1)$ additions for the frequency domain approach. Normalization of the $C_{mk}$ is $N(M-N)$ multiplications compared to the time-domain approach. The sparsity of the vectors $P_{mk}$, which are non-zero at only two discrete frequencies for $m=1$, and at one frequency (DC) for $m=1$, allow rapid frequency domain computation of the set $D_{mk}$. The computational advantage of the frequency domain approach is $(M+N)(N-2)$ additions and multiplications for the determination of $D_{mk}$.

Practical situations usually involve applying the DOBDR operational equation for a stationary system $[B]$ to multiple forward solutions. The vector sets $C_{mk}$ and $D_{mk}$ define the DOBDR inverse system for $[B]$, and may be computed, stored, and recalled for each implementation of the operational equation. Storage requirements are $2M^2$ complex numbers for the frequency domain approach and $2(MN)$ real numbers for the time-domain approach, so that a reduction in storage requirements is realized when $M<N/2$. The total operational equation computational advantage for the frequency domain approach is $2MN(M-N)$ multiplications and additions. which is reduced by $O(M\log N)$ additions and $O(N/2\log N)$ multiplications if the FFT of $i_k$ to yield $I_k$ is required. If the spectral estimate of the inverse solution is desired, the approach may be terminated at this point. Obtaining the time-domain solution by inverse FFT of $O_k$ requires an additional $O(M\log N)$ complex additions $O(N/2\log N)$ complex multiplications, and $N$ real multiplications.

An additional advantage of the frequency domain approach is that it is more robust when there are significant perturbations in $[B]$. Errors in transfer function estimation are normally transmitted to the $b_{mk}$ and ultimately have adverse effects on the inverse solution [21]. In the frequency domain approach, the noise components in $[B]$ transmitted to $A_{mk}$ lying outside of the DOBDR bandwidth are not included in the calculations of $C_{mk}$, and therefore are not propagated to the final solution. The inverse solution obtained from the frequency domain approach may be expected to be of higher quality than would be obtained from the time domain approach, especially when large errors in the estimation of $[B]$ are present.

**EXAMPLE 3**

Consider an application with a time duration signal of $N=128$ samples and a DOBDR basis set bandwidth of 0–10 cycles/2π, requiring $M=21$ Hartley basis vectors. For a previously defined stationary system, DOBDR is performed by executing the operational equation (27) with stored values of $C_{mk}$ and $D_{mk}$. The time domain operational equation requires $5376$ real multiplications and $5334$ real additions. The frequency domain operational equation, including the FFT of $i_k$ and IFFT of $O_k$, requires $1778$ complex multiplications, $1736$ complex additions, and $128$ real multiplications, a reduction by a factor of 2.9 in the number of operational equation computations.

Dynamic systems, and initial applications of the frequency domain approach require computation of $C_{mk}$ and
DOBR inverse solution of the forward solution noise component (right hand term in (11)).

Unlike forward solution noise, transfer function perturbations change the characteristics of the vector sets $c_m$ and $d_m$ which define the behavior of the DOBR operational equation. Application of the perturbed transfer function $[B + N]$ to each member of the basis set $p_m$ yields

$$[B + N]p_m = [B]p_m + [N]p_m = [N]p_m$$

As was the case with forward solution noise, it is beneficial to analyze the effects of transfer function noise in the frequency domain. For the time invariant (convolutional) case the calculation of $A_m$ reduces to the point by point multiplication of the Fourier transform of the transfer function and $P_m$. When $[B]$ is a time-varying system, the spectral estimation of $A_m$ is considerably more complex. In the noiseless case, the spectra $A_m$ may be represented by $A_m = |DFT[B]|^2 |DFT[P]_m|$ where $|DFT|$ and $|IDFT|$ are the NP×NP discrete Fourier transform and inverse discrete Fourier transform matrices, and $[P]_m$ is the 1×NP column vector frequency domain representation of the mth basis function $[8]$. In the presence of transfer function noise, $A'_m$ may be estimated by:

$$A'_m = |DFT[B + N]|^2 |DFT[P]_m|$$

$$A'_m = |DFT[B]|^2 |DFT[P]_m| + |DFT[N]|^2 |IDFT|$$

The resultant noise in $A'_m$ is therefore the superposition of $A_m$ and $A_N_m$.

The worst-case scenario will be considered in evaluating the propagation of noise in the frequency domain Gram-Schmidt orthogonalization process. Maximal noise transmission occurs when each $A_N_m$ is orthogonal to $C_{m-1}$ (FIG. 7), and therefore has complete projection onto the vector $C_m$. The Gram-Schmidt coefficients involving the inner products of $A_N_m$ and prior $C_m$ are zero by virtue of orthonormality, greatly simplifying the computational process. Each $C_m$ is then formed from $A_m + A_N_m$ minus the projections of $A_m$ on prior $C_m$.

$$C_1 = A_1 = A_2 + A_1$$

$$C_2 = A_2 + A_N + \frac{<A^*C>}{|C|^2}$$

$$C_m = A_m + A_N - \sum_{j=1}^{m-1} \frac{<A^*C>}{|C|^2}$$

Each $C_m$ is subsequently normalized, which preserves the relative noise contribution to $C_m$ but depending on the amplification properties of the transfer function $[B]$, the normalization may either increase or decrease the magnitude of noise transmitted to $C_m$. For the worst case scenario the total transmitted noise to $C_m$ is

$$\|V_m\| = A_N + \sum_{j=2}^{m} \frac{A_m}{|C|^2}$$

The frequency domain Gram-Schmidt orthogonalization constants

$$\omega_m = \frac{<C^*M_mC>}{|C|^2}$$

can be regrouped and normalized into the set of constants $\phi_m$, so that each $C_m$ is expressed as a linear combination of
The frequency domain operational equation in the presence of transfer function noise is

\[ O_i = \sum_{n=1}^{M} C_{ni} \frac{1}{NP} \sum_{k=1}^{NP} D_{nk} R_{id} \]  

(41)

The vector set \( D'_{nk} \) is constructed to be compatible with the set \( C'_{ni} \) and does not contribute additional noise to the inverse solution. The worst case inverse solution SNR due to transfer function perturbations may be estimated by

\[ \text{SNR}_{\text{worst}} = 10 \log \left( \frac{\text{Signal}_\text{true}^2}{\text{Noise}_\text{true} + \text{Noise}_\text{perturb}} \right) \]  

(42)

In summary, the method of the present invention is equally applicable and extends to both one dimensional and two dimensional signal and image systems. The chief advantage of the present method compared to other SSVPWF, restoration techniques are that preprocessing, matrix inversion, multiple iterations, or assumptions of local PSF variance are not required. As a result, processing times and computer power required for processing are both substantially reduced.

The primary limitation of the present method is that the basis function support of the inverse solution must be nearly congruent to the region of basis support of the transfer function to avoid an unstable inverse solution. Provided that a realistic simulation model exists, however, estimation of the restored image SNR may be made for a given forward solution SNR and chosen basis set. This allows a priori determination of the optimal basis set for a given application and provides an estimate of the anticipated quality of the restored image.

The foregoing description of a preferred embodiment of the invention has been presented for purposes of illustration and description. It is not intended to be exhaustive or to limit the invention to the precise form disclosed. Obvious modifications or variations are possible in light of the above teachings. The embodiment was chosen and described to provide the best illustration of the principles of the invention and its practical application to thereby enable one of ordinary skill in the art to utilize the invention in various embodiments and with various modifications as are suited to the particular use contemplated. All such modifications and variations are within the scope of the invention as determined by the appended claims when interpreted in accordance with the breadth to which they are fairly, legally and equitably entitled.
APPENDIX A

.c harpat.for
.c ..performs discrete orthogonal basis restoration for invariant
.c ..systems
.c .. using Hartley basis set
.c .. calls mat hart inprod

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

.c
.i izz is the maximum number of basis functions
.i ii is the length of the fwd soln vector (ri) and inv. soln
.i vector (h)
.i the basis set is determined and placed in poly
.i the basis set used for the inv soln is ordered transferred
.c to rleg
.c gau is the matrix formulation of the transfer function
.i vector g,
.i and gt is it’s transpose. Note that gau and gt are cast
.c for
.c row*square matrix=row vector fwd soln
.c The gram-schmidt coef are stored in matrix a
.c The tau coef are in tau
.c The vector sets b,c,d with m members correspond to the
.c literature
.c descriptions. f(m) are <d(m),i> and are intermediate
.c step
.c the inverse soln is h
.c npol is the number of basis vectors used
.c the array cnorm holds the normalization factors for each
.c member of the vector set c

implicit integer(i-o)
   parameter(izz=65,ii=128,two*pi=6.283185)
common/c1/delt,rip
common/c2/rp1(ii),rp2(ii)
common/c4/poly(0:izz,ii)
common/c5/iz
common/c6/gau(ii),gt(ii,ii)
common/c6a/g(ii,((2*ii)-1))
common/c7/rphas
real b(izz,ii),c(izz,ii),d(izz,ii)
real a(izz,izz),tau(izz,izz)
real rleg(0:izz,ii)
real f(izz,ii),cnorm(izz)
   real ri(ii),sumc(ii)
   integer ifini
   integer npol
character*13 pre
character*4 suf3,suf2
character*10 filex,fileg,filh,taunn,obax
character*30 taunnx,obasx
character*30 filex,filegx,filein,fileh
character*2 ntrial
character*1 cxans, cyans
pre='c:\fort\data\'
  suf2='.*dec'
suf3='*hdc'
c......compute Hartley Basis Set
   iz=64
        write(6,*),'%%%%%%calling hart%%%%%%
        call hart

c...... normalization of basis
  do 820 m=0,iz
    do 812 kkk=1,iz
       rpi(kkk)=poly(m,kkk)
       rp2(kkk)=poly(m,kkk)
    call inprod
     do 815 kkk=1,iz
        poly(m,kkk)=(poly(m,kkk))/(sqrt(rip))
     815 continue
     820 continue

c......read in the ideal image and the PSP
    write(6,*),'input name for header file xxxxxx.xxx'
    read(5,933)flh
    fileh=pre//flh
    open(25, file=fileh, status='new')
      write(6,*),'input filename of PSP xxxxxx.xxx'
      read(5,933)fileg
      fileg=pre//fileg
    igau=0
    open(20, file=fileg, status='old')
    do 43 i=1,iz
      read(20,*,end=433)gau(i)
      igau=igau+1
    43 continue
    433 close(20)
    933 format(a10)
      write(6,*),'input trial number for this input fn '
      read(5,964)ntrial
    964 format(a22)
      write(6,*),'input filename of fwd soln xxxxxx.xxx'
      read(5,933)filen

c......set the restoration bandwidth
    write(6,*),'high freq c/o'
    read(5,*)ifini
    npol=(2*ifini)+1

c!!!!!!!!!!!!!!!!!!!!!!!!!!!build the selected orthogonal set!!!!!!!!
  c......always include the dc component
     do 24 ii=1,iz
    24 rleg(1,ii)=poly(0,ii)
iptr=0
   do 25 j=1,ifini
   iptr=iptr+2
   jind=(j*2)-1
   do 22 ii=1,iu
      rleg(iptr,ii)=poly(jind,ii)
 22
   rleg(iptr+1,ii)=poly(jind+1,ii)
   continue
25
   continue

   write(6,*)'$$$$$$$ orthonomal setup done $$$$$$

   c...compensate for extra loop trip and truncate to odd #
   igau=igau-1
   figau=figau/2.
   ifig2=igau/2
   if=0.0
   if(fig2.eq.fig)
     igau=igau-1
   endif

   c...call mat to cast the convolution operator in matrix form
   c... note that this is in row vector format

   call mat(igau)

c>>>>>>>calc b(k) using matrix
   approach>>>>>>>>

   write(6,*)'.....calculating b(k)......'

   c*****use transpose matrix
   iz=(2*ifini)+1
   do 10 k=1,iz
      do 9 ic=1,iu
         sum=0.
         do 8 ir=1,iu
            sum=sum+(rleg(k,ir)*gt(ir,ic))
      8    continue
      b(k,ic)=sum
      continue
   10   continue

   c?????????calc c(k)????????????????????????????????????????????????
   write(6,*)'+++++calculating c(k)+++++'

   do 21 kkk=1,iu
   21   c(1,kkk)=b(1,kkk)
   do 23 m=1,iz
   23   a(m,m)=1.

   c...... calculate Gram-Schmidt coefficients
   do 50 m=2,iz
do 40 j=1,m-1
  do 30 kkk=1,iu
    rp1(kkk)=c(j,kkk)
    rp2(kkk)=rp1(kkk)
    call inprod
    rnorm=rip
  do 29 kkk=1,iu
    rp1(kkk)=c(j,kkk)
    rp2(kkk)=b(m,kkk)
    call inprod
    a(m,j)=-1.*(rip/rnorm)
  40 continue

C....now calculate c(m) by G-S orthogonalization
   do 46 ii=1,iu
     sumc(ii)=0.
   do 49 j=1,m-1
     do 48 kkk=1,iu
       sumc(kkk)=sumc(kkk)+(a(m,j)*c(j,kkk))
     48 continue
     49 continue
   47 continue
   50 continue

C????????????????????????????????????????????????????????????????????????????
C....normalization
   do 720 m=1,iz
     do 712 kkk=1,iu
       rp1(kkk)=c(m,kkk)
       rp2(kkk)=c(m,kkk)
     712 continue
     call inprod
     cnorm(m)=sqrt(rip)
   715 continue
   720 continue
C~~~~~~~~~~~~~~~~~~~~~~~~~~~~~calculate tau~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
   write(6,*)'========calculating tau========'
   do 75 k=1,iz
     tau(k,k)=a(k,k)
   77 do 80 m=2,iz
     80 i=1,m-1
       if(i.ne.m)then
         rsun=0.
     do 83 j=i,m-1
       rsun=rsun+(a(m,j)*tau(j,i))
     83 continue
     continue
   tau(m,i)=rsun
else
  endif
85  continue
90  continue
c normalize taus by dividing by cnorms
  do 94 m=1,iz
do 92 i=1,iz
    tau(m,i)=(tau(m,i)/cnorm(m))
  92  continue
94  continue

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!calculate d(iz,iu)!!!!!!!!!!!!!!!
  write(6,'(*)')!!!!!!!!calculating d(iz,iu)!!!!!!!!'
  do 110 m=1,iz
do 105 i=1,m
do 100 it=1,iu
    d(m,it)=d(m,it)+(tau(m,i)*rleg(i,it))
  100  continue
105  continue
110  continue

filex=pre//'d'/ntrial//'filen(5:6)
  * //fileg(5:6)//suf2
fileh=pre//'ntrial//'filen(5:6)
  * //fileg(5:6)//suf3
filein=pre//'filen(1:10)
filegx=pre//'fileg(1:10)

c:-------------------------------------------------------------------------------------------------
c:-------------------------------------------------------------------------------------------------
c:-------------------------------------------------------------------------------------------------

open(19,file=filein,status='old')
do 32 i=1,iu
  read(19,*,end=519)ri(i)
  continue
519 close(19)

izk=npol
c...clear arrays
  do 190 iy=1,izk
90  f(iy)=0.
do 191 iy=1,iu
191  h(iy)=0.
c:-----------------------------------------------------------------------------compute f*************************************************
  do 120 m=1,izk
    rsum=0.
do 115 i=1,iu
      rsum=rsum+d(m,i)*ri(i))
115  continue
    f(m)=rsum
  write(6,*)m,'c(m) = ',f(m)
120  continue
calculate

\[
do \ 140 \ i=1, \text{iu} \\
\quad \text{rsum}=0. \\
\quad \text{do} \ 130 \ m=1, \text{izk} \\
\quad \quad \text{rsum}=\text{rsum}+(c(m,i)*f(m)) \\
\quad \text{continue} \\
\quad h(i)=\text{rsum} \\
\text{continue} \\
\]

open(18, file=filex, status='new')
\[
do \ 150 \ i=1, \text{iu} \\
\quad \text{write}(18, *) h(i) \\
\quad \text{close}(18) \\
\]

write(6, *) filex

write(25, *) 'iu=', iu
\text{close}(25)

write(6, *) filex

write (27, *) tau(m, i)
\text{close}(27)

\]

write(6, *) ' save to disk d(m) array?' 
\text{read}(5, 916) cyans

write(6, *) ' # rows=' , iz
write(6, *) ' input name of otpt file xxxxxx.xxx ' 
\text{read}(5, 933) obas
\text{obasx}=pre//obas
open(30, file=obasx, status='new')
\[
do \ 501 \ ibas=1, iz
\]
DO 501 KU=1, IU
      WRITE(30, *) ID(IBAS, KU)
      CLOSE(30)
   ENDIF
   WRITE(6, *) ' save to disk C(m) array ? '  
      READ(5, 916) CYANS
      IF(CYANS.EQ.'Y') THEN
      WRITE(6, *) ' input name of opt file xxxxxx.xxx '  
      READ(5, 933) OBAS
      OBASX=PRE//OBAS
      OPEN(30, FILE=OBASX, STATUS='NEW')
      DO 503 IBAS=1, IZ
      DO 503 KU=1, IU
      WRITE(30, *) C(IBAS, KU)
      CLOSE(30)
   ENDIF
      WRITE(6, *) ' save to disk B(m) array ? '  
      READ(5, 916) CYANS
      IF(CYANS.EQ.'Y') THEN
      WRITE(6, *) ' BZ=', IZ
      WRITE(6, *) ' input name of opt file xxxxxx.xxx '  
      READ(5, 933) OBAS
      OBASX=PRE//OBAS
      OPEN(31, FILE=OBASX, STATUS='NEW')
      DO 504 IBAS=1, IZ
      DO 504 KU=1, IU
      WRITE(31, *) B(IBAS, KU)
      CLOSE(31)
   ENDIF
      WRITE(6, *) ' save G-T ? '  
      READ(5, 916) CYANS
      IF(CYANS.EQ.'Y') THEN
      WRITE(6, *) ' input name of opt file xxxxxx.xxx '  
      READ(5, 933) OBAS
      OBASX=PRE//OBAS
      OPEN(30, FILE=OBASX, STATUS='NEW')
      DO 506 IBAS=1, IU
      WRITE(30, *) GT(IBAS, KU), KU=1, IU
      CLOSE(30)
   ENDIF
      STOP
   END
APPENDIX B

c tvpat.for
.c performs discrete orthogonal basis restoration for invariant
c systems
c . using Hartley basis set
c . calls: hart inpod

...................variables:.............
c...........izz is the maximum number of basis functions
c...........iu is the length of the fwd soln vector (rl) and inv. soln

c...........vector (h)
c...........the basis set is determined and placed in poly
c...........the basis set used for the inv soln is ordered transferred
c...........to rreg

c...........tvtfm is the imported matrix transfer function

c...........The gram-schmidt coef are stored in matrix a

c...........The tau coef are in tau

c...........The vector sets b,c,d with m members correspond to the

c...........literature

c...........descriptions, f(m) are <d(m),i> and are an

c...........intermediate step

c...........the inverse soln is h

c...........npol is the number of basis vectors used

c...........the array cnorm holds the normalization factors for each

c...........member of the vector set c

implicit integer(i-o)
parameter(izz=65,iu=128,twopi=6.283185)
common/c1/delt,rip
common/c2/rpl(iu),rp2(iu)
common/c4/poly(0:izz,iu)
common/c5/iz
  common/c6a/g(iu,((2*iu)-1))
common/c11/rphas
real b(izz,iu),c(izz,iu),d(izz,iu)
real a(izz,izz),tau(izz,izz)
real rleg(0:izz,iu)
real f(izz),h(iu),cnorm(izz)
  real ri(iu),sumc(iu)
  real tvtfm(iu,iu)
integer ifini
integer npol
character*13 pre
character*4 suf3,suf2
character*10 filen,filef,flh,taunm,obas
character*30 taumnx,obasx
character*30 filex,filegx,filein,fileh
character*2 ntrial
character*1 cxans,cyans
pre='c:\fort\data\'
suf2='*.dec'
suf3='hdc'
c........compute Hartley Basis Set
iz=64
  write(6,*),'****calling hart*******',
call hart

c.... normalization of basis
c
do 820 m=0,iz
do 812 kkk=1,iu
  rp1(kkk)=poly(m,kkk)
  rp2(kkk)=poly(m,kkk)
call inprod
  do 815 kkk=1,iu
  poly(m,kkk)=(poly(m,kkk))/(sqrt(rip))
continue
820 continue


c....read in the ideal image and the PSF
write(6,*),'input name for header file xxxxxx.xxx'
  read(5,913)filh
fileh=pre//filh
open(25,file=fileh,status='new')

933 format(a10)
write(6,*),'input trial number for this input fn'
  read(5,964)ntrial
964 format(a2)
write(6,*),'input filename of fwd soln xxxxxx.xxx'
  read(5,933)filen

c... set the restoration bandwidth
write(6,*),'high freq c/o'
  read(5,*)ifini
npol=(2*ifini)+1

!! build the selected orthogonal set!!!!!!!
c...always include the dc component
do 24 ii=1,iu
  rleg(ii)=poly(0,ii)
  iptr=0
  do 25 j=1,ifini
    iptr=iptr+2
    jind=(j*2)-1
    do 22 ii=1,iu
      rleg(iptr,ii)=poly(jind,ii)
    continue
  rleg(iptr+1,ii)=poly(jind+1,ii)
22 continue
25 continue

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write(6,'$$$$$$$ orthonomal setup done $$$$$$$

...import the transfer function matrix
write(6,*)'input filename of TV transfer fn matrix
xxxx.xxxx'
read(5,933)fileg
filegx=prefileg
open(20,file=filegx,status='old')
do 43 i=1,iu
read(20,*,end=433)(tvtfm(i,j),j=1,iu)
43 continue
433 close(20)

c>>>>>>>>>>>>>>>>calc b(k) using matrix
approach>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
write(6,*)'.....calculating b(k).......'
c*****use transpose matrix
  iz=(2*infini)+1
  do 10 k=1,iz
  do 9 ic=1,iu
    sum=0.
    do 8 ir=1,iu
      sum=sum+(rleg(k,iz)*tvtfm(ir,ic))
  8 continue
  b(k,ic)=sum
  9 continue
10 continue

c>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
write(6,*)'+++++calculating c(k)+++++
  do 21 kkk=1,iu
  21 c(1,kkk)=b(1,kkk)
  do 23 m=1,iz
  23 a(m,m)=1.

c......calculate Gram-Schmidt coefficients
  do 50 m=2,iz
    do 40 j=1,m-1
      do 30 kkk=1,iu
        rp1(kkk)=c(j,kkk)
      30 rp2(kkk)=rp1(kkk)
call inprod
    rnorm=rrip
    do 29 kkk=1,iu
      rp1(kkk)=c(j,kkk)
    29 rp2(kkk)=b(m,kkk)
call inprod

3
a(m,j)=-1.*(rip/rnorm)

continue

c....now calculate c(m) by G-S orthogonalization
    do 46 li=1,4u
        sumc(il)=0.
            do 49 j=1,m-1
                do 48 kkk=1,4u
                    sumc(kkk)=sumc(kkk)+(a(m,j)*c(j,kkk))
                enddo
            enddo
        continue
    enddo
    do 47 kkk=1,4u
        c(m,kkk)=b(m,kkk)+sumc(kkk)
    enddo

continue


c..........................normalization
    do 720 m=1,iz
        do 712 kkk=1,4u
            rpl(kkk)=c(m,kkk)
        enddo
        call inprod
        cnorm(m)=sqrt(rip)
        do 715 kkk=1,4u
            c(m,kkk)=(c(m,kkk))/sqrt(rip)
        enddo
    enddo

continue

write(6,'(a10,1x,5f9.2)'),tau(1:m,1:i)

write(6,*)'='*10,'=encoding tau=*****='
    do 75 k=1,iz
        tau(k,k)=a(k,k)
    enddo
    do 85 m=1,i-1
        do 81 j=1,m-1
            if(j.ne.m) then
                rsum=0.
            enddo
            do 83 j=1,m-1
                rsum=rsum+(a(m,j)*tau(j,i))
            enddo
        continue
        tau(m,i)=rsum
    endif

continue

continue

continue

normalize tau by dividing by cnorms
    do 94 m=1,iz
        do 92 i=1,iz
            tau(m,i)=(tau(m,i)/cnorm(m))
        enddo
    enddo

continue

continue
c~~~~~~~~~~~~~~~~~~~~~~~~~~~~~calculate d(iz,iu)!!!!!!!!!!!!
    write(6,*)'!!!!!!calculating d(iz,iu)!!!!!'
    do 110 m=1,iz
    do 105 i=1,m
    do 100 it=1,iu
        d(m,it)=d(m,it)+(tau(m,i)*rleg(i,it))
    100 continue
    105 continue
    110 continue

filex=pre//d'//trial//filen(5:6)
* //filen(5:6)//suf2
fieln=pre//trial//filen(5:6)
* //filen(5:6)//suf3
filex=pre//filen(1:10)
filex=pre//filen(1:10)

C:.................................................................

C:..................................................................
    open(19,file=filein,status='old')
    do 32 i=1,iu
    read(19,*,end=519)ri(i)
    32 continue
    519 close(19)

izk=npol

C...clear arrays
    do 190 iy=1,izk
    190 f(iy)=0.
    do 191 iy=1,iu
    191 h(iy)=0.

C...........................................................................
    do 120 m=1,izk
    rsum=0.
    do 115 i=1,iu
    rsum=rsum+(d(m,i)*ri(i))
    115 continue
    write(6,*)m,'c(m) = ',f(m)
    120 continue

C...............................................................................

C$$$$$$$$$$$$$$$$$$$$$$calculate h$$$$$$$$$$$$$$$$$$$$
    do 140 i=1,izk
    rsum=rsum+(c(m,i)*f(m))

130 continue
h(i)=rsum
140 continue
c+++++++++++++++++++++c+++++++++++++++++++++c+++++++++++++++++++++c
   open(18,file=filex,status='new')
do 150 i=1,lu
150 write(18,*)h(i)
close(18)

777 continue
write(6,*)filex
c+++++++++++++++++++++c+++++++++++++++++++++c+++++++++++++++++++++c
   write(25,*)'iu= ',iu
close(25)
c+++++++++++++++++++++c+++++++++++++++++++++c+++++++++++++++++++++c
   c:::this section allows various process vectors and matrices to
c:::be saved to disk for later analysis
   write(6,*)' write taufile (y/n)? ',
      read(5,976)cxans
976 format(a1)
   if(cxans.eq.'y')then
      write(6,*)' enter name of taufile ',
      read(5,977)taunn
   format(a10)
   taunnx=pre//taunn
   open(27,file=taunnx,status='new')
   write(6,*)'iz= ',iz
do 833 m=1,iz
do 833 i=1,iz
   write(27,*)tau(m,i)
833 continue
   close(27)
endif
888 write(6,*)' save to disk d(m) array ? ',
      read(5,916)cyans
916 format(a1)
   if(cyans.eq.'y')then
556 write(6,*)' # rows= ',iz
   write(6,*)' input name of otpt file xxxxxx.xxx ',
      read(5,933)obas
   obasx=pre//obas
   open(30,file=obasx,status='new')
do 501 ibas=1,iz
do 501 ku=1,lu
501 write(30,*)d(ibas,ku)
close(30)
endif
   write(6,*)' save to disk c(m) array ? ',
      read(5,916)cyans
if(cyans.eq.'y') then
    write(6,*)' input name of otpt file xxxxxx.xxx '
    read(5,933)obas
    obasx=pre//obas
    open(30,file=obasx,status='new')
    do 503 ibas=1,iz
      do 503 ku=1,iu
        write(30,*)c(ibas,ku)
      close(30)
    endif
    write(6,*)' save to disk b(m) array ? '
    read(5,916)cyans
    if(cyans.eq.'y') then
      write(6,*)'iz=',iz
      write(6,*)' input name of otpt file xxxxxx.xxx '
      read(5,933)obas
      obasx=pre//obas
      open(31,file=obasx,status='new')
      do 504 ibas=1,iz
        do 504 ku=1,iu
          write(31,*)b(ibas,ku)
      close(31)
    endif

C>@>><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
APPENDIX C

c TVFPAT.FOR

c FREQ DOMAIN DOBR THAT TAKES IDFT FOR TIME DOMAIN INV SOLN

c...as most calculations involve complex operations, default
implicit complex(c)

parameter(np=128,izz=128,dcval=11.3137,xval=5.6568,mp2=128)
c...np is the number of time and frequency domain signal points

c...izz is the maximum number of basis functions

c...dcval and xval are used to assign the frequency domain

c...attributes of the Hartley Basis Set

complex cbn(izz,np),ccn(izz,np),csunc(np),cttrue(np)
complex cdn(izz,np),cp(izz,np),ca(izz,izz),ctau(izz,izz)
complex cnorm(izz),cf(izz),ch(np),cfwd(np),cfwdd(np)
real atrue(np2),arec(np2),xo(np)

c...Freq. domain variable names correspond to those used in

c...time domain algorithm

complex cbn(m,k)---->b(m,k), ccn(m,k)<--->c(m,k),
complex cdn(m,k)---->d(m,k)

c the hartley basis is in cp(m,k)

c gram-schmidt coef are in array ca, tau in ctau

c fwd soln-->cfwd. Freq domain inv soln-->ch,

c time domain inv soln-->xo

------------------------------------------------------------------

character*10 tfnm,fnm,tnam,onm
character*13 pre
character*30 tfnmx,fnmx,tnamx,onmx

pre="c:\fort\data\"

900 format(a10)

.........select the DOBR bandwidth
write(6,'(a)') input fmax 'read(5,*)ifmax

.........and compute the number of Hartley Basis vectors to be used
npol=(2*ifmax)+1

...... input the filename for the Fourier transforms of the vector
...... set b(m,k). These are most easily obtained by dumping the
c......vector set b(m,k) computed using time-domain DOBR and doing
......FFT of each member
write(6,'(a)') input TV B(m,k) file (xxxxxx.fcxy) 'read(5,900)tfnm

tfnmx=pre//tfnm

......The FFT of the forward solution is also needed
write(6,'(a)') input FWD soln file (xxxxxx.fcxy) 'read(5,900)tfnm
fnmx=pre//fnm
  open(19, file=fnmx, status='old')
c...cfrw computes running sum of fwd soln pwr
  cfrw=(0.,0.)
do 3 i=1,np
    read(19,*)cfwd(i)
    cfrw=cfrw+(cfwd(i)*conjg(cfwd(i)))
  3 continue
  close(19)
  open(20, file=cfwx, status='old')
do 6 m=1,npol
      do 5 i=1,np
          read(20,*)cnb(m,i)
      5 continue
  6 continue
  close(20)

c...assign Freq. Domain Hartley basis set P(m,k) based on 128 pt
  signal
  c...assign dc val
    cp(1,1)=(dcval,0.)
c...assign spectra for even-indexed basis
    do 10 i=2,npol
      ibin1=(i/2)+1
      ibin2=(np+1)-(i/2)
      cp(i,ibin1)=(xval,-xval)
      cp(i,ibin2)=(xval,xval)
  10 continue
  c...and for odd indexed basis
    do 15 i=3,npol,2
      ibin1=((i-1)/2)+1
      ibin2=(np+1)-((i-1)/2)
      cp(i,ibin1)=(xval,xval)
      cp(i,ibin2)=(xval,-xval)
  15 continue
  c...note that this assignment assumed that the basis ordering of
  haropd
  c...was: 1=dc, 2=1hz, 3=1hz, 4=2hz, 5=2hz....
c????????????????????????????????????????????????????????????????????????
  npxx=ifmax
  write(6,*)'+++++calculating c(k)+++++
  ccn(1,1)=cbn(1,1)
  do 21 kkk=1,npxx
    ccn(1,(np2-kkk+1))=cbn(1,(np2-kkk+1))
  21 ccn(1,(kkk+1))=cbn(1,(kkk+1))
do 23 m=1,npol
  23 ca(m,m)=(1.,0.)
c......calculate ca coef (complex gram-schmidt coef.)
do 50 m=2,npol
do 40 j=1,m-1
    csum2=(0.,0.)
csum2=(ccn(j,1)*(conjg(ccn(j,1))))
do 30 kkk=1,nnxx
    csum2=csum2+(ccn(j,(np2-kkk+1))*(conjg(ccn(j,(np2-kkk+1)))))
    csum2=csum2+(ccn(j,kkk+1)*(conjg(ccn(j,kkk+1))))
30    continue
    cnormx=csum2
c........save the normalization factor for each vector
    c(m)........
cnorm(j)=csqrt(cnormx)
c
    csum2=(0.,0.)
csum2=(ccn(j,1)*conjg(cbn(m,1)))
do 29 kkk=1,nnxx
    csum2=csum2+(ccn(j,kkk+1)*conjg(cbn(m,kkk+1)))
    csum2=csum2+(ccn(j,(np2-kkk+1))*conjg(cbn(m,np2-kkk+1)))
29    continue
    cip=csum2
c(c(m),j)=-1.*(cip/cnormx)
40    continue
    do 46 ii=1,np
46     csumc(ii)=(0.,0.)
do 49 j=1,m-1
    csumc(1)=csumc(1)+(ca(m,j)*ccn(j,1))
do 48 kkk=1,nnxx
    csumc(kkk+1)=csumc(kkk+1)+(ca(m,j)*ccn(j,kkk+1))
    csumc(np2-kkk+1)=csumc(np2-kkk+1)+(ca(m,j)*ccn(j,(np2-kkk+1)))
48     continue
49    continue
    ccn(m,1)=cbn(m,1)+csumc(1)
do 47 kkk=1,nnxx
    ccn(m,kkk+1)=cbn(m,kkk+1)+csumc(kkk+1)
    ccn(m,(np2-kkk+1))=cbn(m,(np2-kkk+1))+csumc(np2-kkk+1)
47    continue
50    continue

C...........................normalize
C..........................need the final norm
    crip=(0.,0.)
crip=crip+(ccn(npol,1)*(conjg(ccn(npol,1))))
do 712 kkk=1,nnxx
    c..........................normalize
C..........................need the final norm
    crip=(0.,0.)
crip=crip+(ccn(npol,1)*(conjg(ccn(npol,1))))
do 712 kkk=1,nnxx

crip=crip+(ccn(npol,kkk+1))*(conjg(ccn(npol,kkk+1))))

crip=crip+(ccn(npol,(np2-kkk+1)))*(conjg(ccn(npol,(np2-kkk+1))))
712 continue
cnorm(npol)=csgl(crip)
do 720 m=1,npol
ccn(m,1)=(ccn(m,1))/(cnnor(m))
do 715 kkk=1,npol
ccn(m,kkk+1)=(ccn(m,kkk+1))/(cnnor(m))
ccn(m,(np2-kkk+1))=ccn(m,(np2-kkk+1))/cnnor(m)
715 continue
720 continue

calculate tau
write(6,*)'*********** calculating tau = *****'
do 75 k=1,npol
caua(k,k)=ca(k,k)
do 90 m=2,npol
do 85 i=1,m-1
if(i.ne.m)then
crmsum=(0.,0.)
do 83 j=1,m-1
 crmsum=crmsum-(ca(m,j)*ctau(j,i))
83 continue
ctau(m,1)=crmsum
else
endif
85 continue
90 continue

c normalize tau by dividing by cnorms
94 continue
94 continue

calculate d(iz,np)
write(6,*)'*********** calculating d(iz,np)***********'
do 100 m=1,npol
do 105 i=1,m
c....test for even or odd and assign index
fi=real(i)
itive1=aint(fi/2.)
itive2=aint(fi/2.)
c...if even
if(itive1.eq.itive2)then
 ind1=(i/2)+1
 ind2=(np+1)-(i/2)
 cdm(m,ind1)=cdm(m,ind1)+(ctau(m,i)*cp(i,ind1))
 cdm(m,ind2)=cdm(m,ind2)+(ctau(m,i)*cp(i,ind2))
else
 ind1=((i-1)/2)+1
 cdm(m,ind1)=cdm(m,ind1)+(ctau(m,i)*cp(i,ind1))
if(i.gt.1) then
  ind2=(np+1)-((i-1)/2)
  cdn(m,ind2)=cdn(m,ind2)+(ctau(m,i)*cp(i,ind2))
else
  endif
end if
100  continue
105  continue
110  continue

c:..........................................................
c...this section applies to lab experiments where the true
c...inverse son is known. when unknown, a dummy file can be
c...used and snr values computed may be ignored.

  write(6,*),' input 128 pt spectrum file of true signal '
  read(5,900) tnam
  open(21, file=tnamx, status='old')
  sigp=0.
  do 160 i=1,np
  read(21,*) ctrue(i)
  atrue(i)=real(ctrue(i))*conjg(ctrue(i))
  sigp=sigp+atru(i)
160  continue

c:..........................................................

  do 260 i=1,np
260  cfwdn(i)=cfwd(i)
  do 120 m=1,npol
  crsum=(0.,0.)
  crsum=crsum+(cdn(m,1)*conjg(cfwdn(1))
  do 115 i=1,npxx
    crsum=crsum+(cdn(m,i+1)*conjg(cfwdn(i+1))
    crsum=crsum+(cdn(m,(np2-i-1))*conjg(cfwdn(np2-i+1)))
115  continue
  cf(m)=crsum
120  continue
  do 140 i=1,npxx+1
  crsum=(0.,0.)
  do 130 m=1,npol
  crsum=crsum+(ccn(m,i)*cf(m))
130  continue
  ch(i)=crsum
140  continue
  do 141 i=1,npxx
  crsum=(0.,0.)
  do 131 m=1,npol
  crsum=crsum+(ccn(m,(np2-i+1))*cf(m))
131 continue
ch(np2-i+1)=crsum
141 continue

ccel=0,0.
crsum=crsum+((ch(i)+congg(ch(i))
arec(i)=real(ch(i)+congg(ch(i)))
doi=1,npxx

crsum=crsum+(ch(i)+congg(ch(i)))
arec(i)=real(ch(i)+congg(ch(i)))

crsum=crsum+(ch(np2-i+1)+congg(ch(np2-i+1)))

crsum=crsum+(ch(np2-i+1)+congg(ch(np2-i+1)))

150 continue

write(6,'(a,*)') recovered total pwr= ',crsum

sft=0.
cfret=cttrue(i)-ch(i)

180 snse=sft+real(cfret+congg(cfret))
doi=1,npxx
cfret=cttrue(i+1)-ch(i+1)

180 snse=sft+real(cfret+congg(cfret))
cfret=cttrue(np2-i+1)-ch(np2-i+1)

180 snse=sft+real(cfret+congg(cfret))

180 continue
snrout2=10.*alog10(sign/snse)
write(6,'(a,*)') snr2= ',snrout2

248 do 249 i=npxx+1,np2-npxx
248 ch(i)=(0.,0.)
call fft(ch,128,7,1,0)
do 249 li=1,np

249 xo(li)=real(ch(li))
write(6,'(a,*)') input outname xxxxx.xxx,
read(5,900)onm
onm=pre/onm
open(27,file=onm,status='new')
do 250 li=1,np
write(27,'(a)')xo(li)

250 continue

close(27)
else
endif
continue
stop
end
I claim:

1. A method of using discrete orthogonal basis to restore a signal and/or image system degraded by time and/or spatially varying transfer functions said system being of linear type and represented by the equation \([B] [0]=[i]\) wherein \([0]\) is an original signal or image, \([i]\) is a degraded signal or image and \([B]\) is a system transfer function matrix, comprising:

estimating in estimating means a signal-to-noise ratio for a restored system;

selecting in selecting means a set of orthogonal basis set functions \(\boldsymbol{p}_{\text{m}}\) where \(m=1,2,3 \ldots M\) is the index for the chosen orthogonal basis set with \(M\) members of length \(k=1,2,3 \ldots M\) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and removing in removing means time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector \(\boldsymbol{O}_{k}\) for a one dimensional restoration wherein:

\[
o_{k} = \sum_{m=1}^{M} c_{mk} \left[ \sum_{h=1}^{H} d_{mhl} k \right]
\]

wherein \(d_{mhl}\) is a vector set created by linear combinations of \(p_{m}\) a set of constants formed by linear combinations of \(a\), the standard Gram-Schmidt orthogonalization coefficients, and wherein:

\([B]_{w}^{T} = [c]\)

where \([B]_{w}^{T}\) is the transpose-complex conjugate of the matrix \([B]\) and \([p]_{w}\) is an \(M\) member orthogonal basis set;

\[
\tau_{mk} = \sqrt{\frac{c_{mk}}{\sum_{k=1}^{K} c_{mk}^2}} \quad m = 1, \ldots, M
\]

\[
\tau_{mk} = \frac{1}{m} \left( \sum_{j=1}^{m-1} c_{mjk} m \neq j, m = 1, \ldots, M \right)
\]

\[
c_{mk} = \sum_{l=1}^{M} c_{mlk} k
\]

\[
d_{mhl} = \sum_{l=1}^{M} c_{mlh} k m = 1,2,3 \ldots M
\]

and an inverse solution vector \(\boldsymbol{O}_{pk}\) for a two dimensional restoration wherein:

\[
o_{pk} = \sum_{m=1}^{M} c_{mk} \left[ J_{pk} \right] d_{mhl} k
\]

wherein \(J_{pk}\) is an intermediary matrix of the form:

\[
J_{pk} = \sum_{m=1}^{M} c_{mk} \left[ J_{pk} \right] d_{mhl} k
\]

\(p = 1,2,3, \ldots H\).

2. The method set forth in claim 1, wherein said estimating of the signal-to-noise ratio of the restored system is provided by applying a forward solution signal-to-noise ratio and selected set of orthogonal basis set functions \(\boldsymbol{p}_{\text{m}}\) to a realistic simulation model of said system.

3. The method set forth in claim 1, wherein said set of orthogonal basis set functions \(\boldsymbol{p}_{\text{m}}\) is selected from a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

4. The method set forth in claim 2, wherein said set of orthogonal basis set functions \(\boldsymbol{p}_{\text{m}}\) is selected from a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

5. A method of using discrete orthogonal basis to restore a signal system degraded by a time varying transfer function, comprising:

estimating in estimating means a signal-to-noise ratio for a restored system;

selecting in selecting means a set of orthogonal basis set functions \(\boldsymbol{p}_{\text{m}}\) a set of constants formed by linear combinations of \(a\), the standard Gram-Schmidt orthogonalization coefficients, and wherein:

\([B]_{w}^{T} = [c]\)

where \([B]_{w}^{T}\) is the transpose-complex conjugate of the matrix \([B]\) and \([p]_{w}\) is an \(M\) member orthogonal basis set;

\[
\tau_{mk} = \sqrt{\frac{c_{mk}}{\sum_{k=1}^{K} c_{mk}^2}} \quad m = 1, \ldots, M
\]

\[
\tau_{mk} = \frac{1}{m} \left( \sum_{j=1}^{m-1} c_{mjk} m \neq j, m = 1, \ldots, M \right)
\]

\[
c_{mk} = \sum_{l=1}^{M} c_{mlk} k
\]

\[
d_{mhl} = \sum_{l=1}^{M} c_{mlh} k m = 1,2,3 \ldots M
\]

6. The method set forth in claim 5 wherein said estimating of the signal-to-noise ratio \(\text{SNR}_{\text{prev}}\) is provided by:

\[
\text{SNR}_{\text{prev}} = 10 \log_{10} \frac{\mathbf{I}^{\text{original}}}{\mathbf{I}^{\text{noise}} + \mathbf{I}^{\text{noise}}}
\]

wherein \(\mathbf{I}^{\text{original}}\) is the original signal power, \(\mathbf{I}^{\text{noise}}\) is the noise power due to added noise in the forward solution.
9. A method of using discrete orthogonal basis to restore an image system degraded by time and spatially varying transfer functions, comprising:

- estimating in estimating means a signal-to-noise ratio for a restored system;
- selecting in selecting means a set of orthogonal basis set functions \( p_{mx} \) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
- removing in removing means time and spatially varying distortions in the restored system by obtaining an inverse solution vector \( O_{pk} \) wherein:

\[
O_{pk} = \sum_{m=1}^{M} c_{mpk} \left[ \sum_{k=1}^{K} d_{mpk} f_{pk} \right] \]

\( \kappa = 1, 2, 3, \ldots JU \)

wherein \( d_{mpk} \) is a vector set created by linear combinations of \( p_{mx} \) weighted by \( \tau_{mk} \), a set of constants formed by linear combinations of \( a \), the standard Gram-Schmidt orthogonalization coefficients, and \( J_{pk} \) is an intermediary matrix of the form:

\[
J_{pk} = \sum_{m=1}^{M} c_{mpk} \left[ \sum_{k=1}^{K} d_{mpk} f_{pk} \right] \]

\( \rho = 1, 2, 3, \ldots JU \).

10. The method set forth in claim 9, wherein said estimating of the signal-to-noise ratio SNR_{pred} is provided by simulating a noiseless forward solution and determining the intrinsic noise shown as the difference between the restored image and the original image and further considering recovered added noise.

11. The method set forth in claim 10, wherein said estimated signal-to-noise ratio SNR_{pred} is

\[
SNR_{pred} = 10 \log \frac{V_{m_{max}}}{V_{m_{min}}^2 + V_{noise}^2} \]

wherein \( |O_{m_{max}}|^2 \) is the signal power in the original, undegraded signal/image, \( |O_{m_{min}}|^2 \) is the inverse solution noise power due to the approximate nature of the inverse solution and \( |O_{noise}|^2 \) is the noise power in the inverse solution power due to added noise in the forward solution.

12. The method set forth in claim 9, wherein said estimated signal-to-noise ratio SNR_{pred} is

\[
SNR_{pred} = 10 \log \frac{|O_{m_{max}}|^2}{|O_{m_{min}}|^2 + |O_{noise}|^2} \]

wherein \( |O_{m_{max}}|^2 \) is the signal power in the original, undegraded signal/image, \( |O_{m_{min}}|^2 \) is the inverse solution noise power due to the approximate nature of the inverse solution and \( |O_{noise}|^2 \) is the noise power in the inverse solution power due to added noise in the forward solution.

13. A programmable apparatus, comprising:

- means for computing; and
- readable memory defining a process for estimating a signal-to-noise ratio for a restored system; selecting a set of orthogonal basis set functions \( p_{mx} \) where \( m=1, 2, 3, \ldots M \) is the index for the chosen orthogonal basis set with \( M \) members of length \( k=1, 2, 3, \ldots, JU \) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and

\[
O_k = \sum_{m=1}^{M} c_{mk} \left[ \sum_{k=1}^{K} d_{mk} f_k \right] \]

wherein \( d_{mk} \) is a vector set created by linear combinations of \( p_{mx} \) weighted by \( \tau_{mk} \), a set of constants formed by linear combinations of \( a \), the standard Gram-Schmidt orthogonalization coefficients, and

\[
|\beta_m| = |B| \beta_m \]

where \( |B| \) is the transpose-complex conjugate of the matrix \( B \) and \( |\beta_m| \) is an \( M \) member orthogonal basis set.

\[
\tau_{mk} = \frac{\alpha_{mk}}{\sum_{k=1}^{K} \alpha_{mk}} m = i; \quad \sqrt{\frac{1}{\sum_{k=1}^{K} \alpha_{mk}}}
\]

\[
\tau_{mk} = \frac{1.0}{\sum_{k=1}^{K} \alpha_{mk}} m = i; \quad i = 1, \ldots, m - 1;
\]

\[
\alpha_{mk} = \frac{\tau_{mk}}{\sum_{k=1}^{K} \tau_{mk}}
\]

\[
d_{mk} = \frac{\tau_{mk}}{\sum_{k=1}^{K} \tau_{mk}} m = 1, 2, 3, \ldots M
\]

and an inverse solution vector \( O_{pk} \) for a two dimensional restoration wherein:

\[
O_{pk} = \sum_{m=1}^{M} c_{mpk} \left[ \sum_{k=1}^{K} d_{mpk} f_{pk} \right] \]

\( \kappa = 1, 2, 3, \ldots JU \).

14. A programmable apparatus, comprising:

- means for computing; and
- readable memory defining a process for estimating a signal-to-noise ratio for a restored system; selecting a set of orthogonal basis set functions \( p_{mx} \) where \( m=1, 2, 3, \ldots, M \) is the index for the chosen orthogonal basis set with \( M \) members of length \( k=1, 2, 3, \ldots, JU \) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and

\[
O_k = \sum_{m=1}^{M} c_{mk} \left[ \sum_{k=1}^{K} d_{mk} f_k \right] \]

wherein \( d \) is a vector set created by linear combinations of \( p_{mx} \) weighted by \( \tau_{mk} \), a set of constants formed by linear combinations of \( a \), the standard Gram-Schmidt orthogonalization coefficients, and

\[
|\beta_m| = |B| \beta_m \]

where \( |B| \) is the transpose-complex conjugate of the matrix \( B \) and \( |\beta_m| \) is an \( M \) member orthogonal basis set.

\[
\tau_{mk} = \frac{\alpha_{mk}}{\sum_{k=1}^{K} \alpha_{mk}} m = i; \quad \sqrt{\frac{1}{\sum_{k=1}^{K} \alpha_{mk}}}
\]

\[
\tau_{mk} = \frac{1.0}{\sum_{k=1}^{K} \alpha_{mk}} m = i; \quad i = 1, \ldots, m - 1;
\]

\[
\alpha_{mk} = \frac{\tau_{mk}}{\sum_{k=1}^{K} \tau_{mk}}
\]

\[
d_{mk} = \frac{\tau_{mk}}{\sum_{k=1}^{K} \tau_{mk}} m = 1, 2, 3, \ldots M
\]
where \([B]^T\) is the transpose-complex conjugate of the matrix \([B]\) and \([p]^m\) is an \(M\) member orthogonal basis set;

\[
\tau_m = \frac{a_{mn}}{\sqrt{\sum_{k=1}^{N} c_{mn}c_{nk}}} m = i; \quad \frac{1.0}{\sqrt{\sum_{k=1}^{N} c_{mn}c_{nk}}} m \neq i; \quad i = 1, \ldots, m - 1;
\]

\[
c_{mn} = \frac{1}{\sum_{k=1}^{N} a_{mn}a_{nk}}
\]

\[
d_{mn} = \frac{1}{\sum_{k=1}^{N} a_{mn}a_{nk}} m = 1, 2, 3, \ldots M.
\]

15. A programmable apparatus, comprising:
means for comparing; and
readable memory defining a process for estimating means
a signal-to-noise ratio for a restored system;
selecting a set of orthogonal basis set functions \(p^m\) to
provide a stable inverse solution based upon the estimated
signal-to-noise ratio; and
removing time and spatially varying distortions in the
restored system by obtaining an inverse solution vector
\(O^m\) wherein:

\[
O^m = \sum_{m=1}^{M} C_{mn} \frac{1}{N} \sum_{k=1}^{N} D_{nm}^k a_k^m
\]

wherein \(L\) is a fourier transform of the forward solution
and \(O^m\) is a fourier transform of the inverse solution and a
time domain solution may be obtained by

\[
\mathcal{F}^{-1}(O^m) = o_a
\]

wherein

\[
[\beta]^m = [F]^T [P]^m
\]

where

\[
[F] = [DFT][\beta] \text{[IDFT]}
\]

and \([DFT]=\text{discrete Fourier transform and}[\text{IDFT}]=\text{inverse Fourier transform matrices,}

\[
\begin{align*}
\frac{N}{k=1} C_{mn} A_{nk} & - \sum_{k=1}^{N} C_{mn} C_{nk} \\
& = i, \ldots, m - 1 \\
m = 1, \ldots, M
\end{align*}
\]

wherein \(a\) is the complex Gram-Schmidt coefficient and
vector sets \(C\) and \(A\) define the characteristics of the
system \([B]\) for frequency domain discrete orthogonal
basis restoration;

\[
\tau_m = \frac{1.0}{\sqrt{\sum_{k=1}^{N} C_{mn} C_{nk}^m}}
\]

\[
m = i, \ldots, m - 1 \\
m = 1, \ldots, M,
\]

where \(\tau\) is a constant formed by linear combinations of \(a\);

\[
C_{nm} = \frac{1}{\sum_{k=1}^{N} a_{mn}a_{nk}}
\]

and

\[
D_{mn} = \frac{1}{\sum_{k=1}^{N} a_{mn}a_{nk}^m}
\]

16. A method of using discrete orthogonal basis to restore
a signal and/or image system degraded by time and/or
spatially varying transfer functions said system being of
linear type and represented by the equation \([B]\) \([o]=\[i]\)
wherein \([o]\) and \([i]\) are length \(N\) column and row vectors and \([B]\)
is an \(N \times N\) non-singular transfer function matrix, comprising:
estimating in estimating means a signal-to-noise ratio for
a restored system;
selecting in selecting means a set of orthogonal basis set
functions \(p^m\) wherein \(m=1, 2, 3, \ldots M\) is the index for
the chosen orthogonal basis set with \(M\) members of length
\(k=1, 2, 3, \ldots N\) to provide a stable inverse solution based
upon the estimated signal-to-noise ratio; and
removing in removing means time and/or spatially varying
distortions in the restored system by obtaining an
inverse solution vector \(O^m\) for a one dimensional restora-
tion wherein:

\[
O^m = \sum_{m=1}^{M} C_{mn} \frac{1}{N} \sum_{k=1}^{N} D_{nm}^k a_k^m
\]

17. A programmable apparatus, comprising:
means for computing; and
readable memory defining a process for
estimating a signal-to-noise ratio for a restored system;
selecting a set of orthogonal basis set functions \(p^m\)
where \(m=1, 2, 3, \ldots M\) is the index for the chosen orthogonal
basis set with \(M\) members of length \(k=1, 2, 3, \ldots N\) to
provide a stable inverse solution based upon the estimated
signal-to-noise ratio; and
removing time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector $O_k$ for a one dimensional restoration wherein:

$$O_k = \sum_{m=1}^{M} C_{am} \frac{1}{N} \sum_{k=1}^{N} D_{am} s_k$$

wherein $I_k$ is a fourier transform of the forward solution and $O_k$ is a fourier transform of the inverse solution and a time domain solution may be obtained by $\mathcal{F}^{-1}(O_k) = o_k$

wherein

$$[B] = [F]^T [a]$$

where

$$[F] = [\text{DFT}] [\theta] [\text{IDFT}]$$

and $[\text{DFT}]$= discrete fourier transform and $[\text{IDFT}]$= inverse fourier transform matrices.

where $\theta$ is the complex Gram-Schmidt coefficient and vector sets $C$ and $A$ define the characteristics of the system $[B]$ for frequency domain discrete orthogonal basis restoration;

$$\tau_{am} = \frac{1.0}{\sqrt{\sum_{k=1}^{N} C_{am} C_{am}^{*}}}$$

and

$$\tau_{ai} = \frac{1.0}{\sum_{k=1}^{N} C_{am} C_{am}^{*}}$$

where $\tau$ is a constant formed by linear combinations of $a$;

$$C_{am} = \sum_{i=1}^{m-1} \tau_{ai} a_i$$

and

$$D_{am} = \sum_{i=1}^{m} \tau_{ai} a_i$$