Method of Discrete Orthogonal Basis Restoration

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METHOD OF DISCRETE ORTHOGONAL BASIS RESTORATION

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References Cited

U.S. PATENT DOCUMENTS
5,249,122 9/1993 Stritzke .................................. 364/413.07

OTHER PUBLICATIONS
Stritzke, P et al.; Funktionsszintigraphie: Eine Einheitliche Methode Zur Quantifizierung Von Stoffwechsel Und Funktio


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ABSTRACT

A method is described for utilizing discrete orthogonal basis to restore signal system, such as radio or sound waves and/or image system such as photographs or medical images that become distorted while being acquired, transmitted and/or received. The signal or image systems are of the linear type and may be represented by the equation [B] [o] = [i] wherein [o] is an original signal or image, [i] is a degraded signal or image and [B] is a system transfer function matrix. The method involves estimating a signal-to-noise ratio for a restored signal or image. Next, is the selecting of a set of orthogonal basis set functions to provide a stable inverse solution based upon the estimated signal-to-noise ratio. This is followed by removing time and/or spatially varying distortions in the restored system and obtaining an appropriate inverse solution vector.

17 Claims, 7 Drawing Sheets
FIG. 1a

FIG. 1b
CONSTRUCT AN ORTHONORMAL BASIS SET $[p]_m$ WITH $m=1, ..., M$ MEMBERS OF LENGTH $1U$

PROJECT THE ADJOINT OF THE SYSTEM MATRIX ONTO EACH $[p]_m$ TO FORM THE VECTOR SET $[b]_m$

$$[B][p]_m=[b]_m$$

PERFORM GRAM–SCHMIDT ORTHOGONALIZATION ON THE VECTOR SET $[b]_m$ TO YIELD $[c]_m$, WHICH IS ORTHONORMAL.

REGROUP THE GRAM–SCHMIDT COEFFICIENTS INTO A SET OF CONSTANTS $\tau$ SO THAT

$$c_{mk} = \sum_{i=1}^{m} \tau_{mi} b_{ik}$$

IS SATISFIED.

USING THE SAME SET OF CONSTANTS $\tau$, THE VECTOR SET $[d]_m$ IS CALCULATED BY

$$d_{mk} = \sum_{i=1}^{m} \tau_{mi} p_{ik}$$

THE INVERSE SOLUTION ESTIMATE $O_k$ IS THEN CALCULATED BY

$$O_k = \sum_{m=1}^{M} \sum_{k=1}^{IU} c_{mk} d_{mk} i_k$$

WHERE $i_k$ IS THE FORWARD SOLUTION VECTOR.
METHOD OF DISCRETE ORTHOGONAL BASIS RESTORATION

This is a continuation-in-part of U.S. patent application Ser. No. 08/337,592, filed on Nov. 10, 1994, entitled "Method of Discrete Orthogonal Basis Restoration", now abandoned.

TECHNICAL FIELD

The present invention relates generally to the field of signal and image restoration and, more particularly to a method of restoring a signal and/or image degraded by time and/or spatially varying transfer functions.

BACKGROUND OF THE INVENTION

It is commonplace for signals, such as radio or sound waves and images such as photographs or medical images to become distorted while being acquired, transmitted and/or received. This phenomenon occurs in various types of radar, sonar, optic imaging and electronic systems.

As an example, blurring of a photographic image may result from camera and/or object motion at the time of acquisition or may even be produced by the nature of the photographic equipment (e.g. "fish-eye" lens). In medical imaging, the type of equipment used and the way the images are acquired can have remarkable effects on the level of distortion or blurring present in the final images that are interpreted by physicians.

It should further be appreciated that the characteristics of the blurring process may change with time during the acquisition of a signal or may vary with location over different areas of an image. These time and/or spatially varying distortions in a signal and/or image must be removed to restore a signal and/or image to its undistorted form and enhance clarity.

In practical application there are imperfections in the signal or image acquisition process that make it impossible for any method to perfectly recover the original signal or image. Special mathematical techniques may, however, be utilized to closely estimate what the signal or image was before it was degraded. The time and/or spatially varying nature of some systems makes it particularly difficult to perform a fully accurate restoration. Still, when properly applied such techniques may be utilized to substantially improve the quality of a signal or image so that it more closely approximates the true or undistorted original signal or image.

In order to further understand this process it must be appreciated that signals or images degraded by a linear system may be cast in the operator notation \([B] [o]=[i]\); where \([o]\) is the original signal or image, \([i]\) is the degraded signal or image, and \([B]\) is the system transfer function matrix. Signal or image restoration is the determination of an approximation \([0']\) to the original signal \([o]\), given a priori knowledge of the transfer function matrix \([B]\) and the forward solution \([i]\).

The most straightforward means of determining the inverse solution is by application of the transfer function matrix inverse to the forward solution, that is, \([B]^{-1}[i]=[o']\). However, determining \([o']\) by this approach frequently represents an ill-posed problem as the inverse of the transfer function may not exist (singular matrix) or \([B]^{-1}\) may be near-singular. In either case, the inverse solution cannot be determined. Further, even if \([B]\) is invertible, \([B]^{-1}\) will frequently be ill-conditioned, meaning that small perturbations in \([i]\) will lead to large perturbations in \([0']\) when the inverse solution is computed. This leads to unacceptable results. This is because all practical systems have inherent uncertainty in the measurement of \([i]\), as well as added noise, and accordingly, adequate estimation of the inverse solution \([0']\) is not possible through application of an ill-conditioned transfer function inverse.

To date, many methods have been developed to solve inverse problems arising in imaging processes, optics, geophysics, astronomy, spectroscopy, and other engineering and scientific disciplines. The existence of multiple solutions is primarily due to the fact that no single prior art method provides the best estimate of inverse solution in all practical applications. In fact, most prior art methods have only very specific, limited applications within specialized technical fields.

As \([B] [o]=[i]\) constitutes a linear system, solution by linear methods is an intuitively attractive approach. However, while the solution may be attempted by linear transform methods, such as Fourier transforms, the ill-conditioned nature of \([B]\) is not circumvented by these techniques. Furthermore transform techniques are not directly applicable when the transfer function is shift-variant.

Application of transform methods to shift-variant systems have been limited to those cases where the signal or image can be sectioned into regions over which the system may be considered to be stationary. The inverse solutions for these regions are computed by transform techniques, and then spliced back together to form the overall solution. Similar sectioning into assumed stationary regions with inversion by the maximum a posteriori method has also been proposed. This sectioning and reassembly approach ("mosaicing") is, however, highly dependant on the validity of the stationary assumption, the method of reassembly, and on sampling of the forward solution and these considerations all adversely effect restoration results.

Various non-transform methods of linear inverse solution have also been developed. These include Weiner filtering, constrained Weiner filtering, maximum entropy, and pseudo-inversion techniques. These methods are usually applicable to the shift-variant case and they address the ill-conditioned nature of the problem. One drawback of such methods is, however, that they tend to not perform well in the presence of low signal-to-noise ratio (SNR) or on systems with severe to moderately degrading transfer functions. Thus they fail when they are most needed. These linear methods also do not provide super-resolution capability, and the linear iterative methods (e.g. pseudo-inversion, van Clutter's method, maximum entropy) do not have well defined termination points and can have very high memory and computational demands if a large number of iterations are performed. Thus, hardware requirements and processing times are disadvantageously increased.

The shortcomings of existing linear techniques has spawned great interest in non-linear approaches. The non-linear approaches are based on various regularization techniques that incorporate a priori knowledge of various parameters to yield an inverse solution that stabilizes and constrains the inverse solution. The parameter variables (hyperparameters) may include constraints on the form of the solution (such as non-negativity) goodness of fit parameters, statistical parameters, and assumptions of the character of the added noise. Non-linear methods are usually applicable to shift-variant systems and may have super-resolution properties. The performance of these approaches is highly dependent on proper choice of the hyperparameters.
needed for the particular method. Furthermore, these approaches are usually iterative with poorly defined criteria for termination. For systems with well-defined hyperparameters and termination criteria, and when the computational burden is not an obstacle, these are usually the preferred method of inverse solution. However, when the a priori knowledge of the system is inadequate, or when optimal termination of the iterative process is problematic, a linear method of solution is likely to produce better restoration results.

Another linear method of interest provides deconvolution for stationary systems based on the properties of the system adjoint operator. Referred to as deconvolution by the method of orthogonal polynomials (Stritzke IEEE Trans Med Imaging vol. 9, 1990, pp. 11–23), the crux of this method is the inner-product property of adjoint operator on vectors. The method requires a discrete orthogonal basis set. The original author, however, failed to define the origins of instability or the criteria for insuring a stable solution. Accordingly, the scope of practical applications of this approach is very limited.

From the above it should be appreciated that a need exists for a more versatile and effective method of signal and image restoration suited for a wide range of applications in various fields.

SUMMARY OF THE INVENTION

Accordingly, it is a primary object of the present invention to provide an efficient and dependable method for signal and/or image restoration adapted for a number of specific applications across a broad number of technical fields.

Another object of the invention is to provide an improved method for quickly restoring a signal and/or image system degraded by time and/or spatially varying transfer functions. Such a system reduces processing time without comprising the quality of the final or restored image.

Yet another object of the invention is to provide a discrete orthogonal basis method for quickly restoring a signal or image to an undistorted form. Advantageously, the method utilizes a mathematical processing technique requiring relatively small computer memory capacity such as found in a personal computer, so as to allow readily available computer hardware. Further, the method also provides uncompromising speed of operation and very effective results.

A still further object of this invention is to provide a discrete orthogonal basis restoration method particularly suited to reconstruct and restore nuclear medicine SPECT images.

Additional objects, advantages and other novel features of the invention will be set forth in part in the description that follows and in part will become apparent to those skilled in the art upon examination of the following or may be learned with the practice of the invention. The objects and advantages of the invention may be realized and obtained by means of the instrumentalities and combinations particularly pointed out in the appended claims.

To achieve the foregoing and other objects, and in accordance with the purposes of the present invention as described herein, an improved method is provided wherein discrete orthogonal basis is utilized to restore a signal and/or image system that is degraded by time and/or spatially varying transfer functions. Advantageously, the present method represents a relatively simple inverse solution that quickly and efficiently restores the system to an undistorted form. Accordingly, a clearer and more focused signal or image system results.

The method includes the step of estimating a SNR for a restored system. Additionally, there is the step of selecting of a set of orthogonal basis set functions \( p_m \) to provide a stable inverse solution based upon the estimated forward solution SNR.

More specifically, the estimated SNR of the restored system is provided by applying a given forward solution SNR and selected set of orthogonal basis set functions \( p_m \) to a realistic simulation model of the system. The set of orthogonal basis set functions \( p_m \) may be any orthogonal basis that spans the forward and inverse solution vector spaces. Such basis set functions include but are not limited to a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

Next is the step of removing the time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector \( o_k \) for a one dimensional restoration wherein:

\[
o_k = \sum_{m=1}^{M} c_{om} \left[ \frac{U_j}{\sum_{k=1}^{M} d_{omk}} \right]
\]

wherein:

\[
(b)_m = (B)^* (p)_m
\]

where \([B]^*\) is the transpose-complex conjugate of the matrix \( B \) and \( p_m \) is an \( M \) member orthogonal basis set and \( a \) are the standard Gram-Schmidt orthogonalization coefficients.

\[
\sum_{m=1}^{M} \| \sum_{r=1}^{R} c_{m,r} \|_{2} = \sum_{m=1}^{M} \| \sum_{r=1}^{R} c_{m,r} \|_{2}
\]

Alternatively, the inverse solution vector \( O_{pk} \) for a two dimensional system with separable spatially variant PSF the inverse solution may be obtained by successive row-column operations:

\[
o_{pk} = \sum_{n=1}^{N} c_{n,pk} \left[ \sum_{r=1}^{R} d_{n,prk} \right]
\]

wherein:

\[
I_{pk} = \sum_{r=1}^{R} d_{n,prk} \left[ \sum_{r=1}^{R} \sum_{p=1}^{P} d_{n,prk} \right]
\]

More specifically describing the invention, the estimating of the signal-to-noise ratio SNR \( \text{SNR}_{\text{med}} \) is provided by the formula...
wherein $|O_{inv}|^2$ is the signal power in the original (undegraded) signal or image, $|O_{forward}|^2$ the inverse solution noise power due to the approximate nature of the inverse solution and $|O_{noise}|^2$ is the noise power in the inverse solution power due to added noise in the forward solution.

Advantageously, the present method functions to define the origins of instability and behavior in the presence of noise. By applying the method to time-varying systems and using a technique for a priori determination of the SNR inverse solution, it is possible to insure stability and optimal selection of the basis set. Further, the method of inverse solution and SNR estimation may be successfully extended to the restoration of two-dimensional images degraded by spatially variant point spread functions.

The discovery of the origins of instability along with the development of an approach for selection of the optimal basis set to maximize inverse solution SNR, the present method a viable linear approach to inverse solution. Advantageously, the method is applicable to both stationary and shift-variant systems, non-iterative, and is computationally efficient. Thus, the speed of processing and the size of the computer necessary to complete that processing are both reduced. Further, it should be appreciated that the only a priori information required to estimate the SNR of the inverse solution is an estimate of the forward solution noise characteristics and estimate of the inverse solution noise due to a limited basis set. Of course, in some cases the type of instrumentation or acquisition parameters may guide the optimal basis set selection (e.g. Nuclear Medicine SPECT imaging with reconstruction from projections).

As the present method is advantageous applicable to both stationary and shift-variant linear systems in one or more dimensions, potential applications for the present method include medical imaging (e.g. emission tomography, MRI, ultrasound), image processing (lens deblurring, motion artifacts), optics and spectroscopy (light and NMR), geophysics, radar/sonar, and general electronics and electrical engineering problems. Thus, the method is extremely versatile, having application in broad ranging technical fields.

Still other objects of the present invention will become apparent to those skilled in this art from the following description wherein there is shown and described a preferred embodiment of this invention, simply by way of illustration of one of the modes best suited to carry out the invention. As it will be realized, the invention is capable of further embodiment of different embodiments and its several details are capable of modification in various, obvious aspects all without departing from the invention. Accordingly, the drawings and description will be regarded as illustrative in nature and not as restrictive.

BRIEF DESCRIPTION OF THE DRAWING

The accompanying drawing incorporated in and forming a part of the specification, illustrates several aspects of the present invention and together with the description serves to explain the principles of the invention. In the drawings:

FIG. 1 graphically shows the original signal (a) is the sum of three unit amplitude sinusoids ($f_1=3$ cycles/2π, $\phi_1=0.1$ radian; $f_2=7$ cycles/2π, $\phi_2=1.0$ radian; $f_3=10$ cycles/2π, $\phi_3=0.6$ radian). The forward solution (b) results when the time-varying decaying exponential system described by transfer function $h_{3n}$ acts on the original signal (see equations in Example 1, page 20). Pseudorandom zero-mean noise is added to produce forward solutions with SNR of 20 dB (c) and 10 dB (e). The inverse solutions obtained by the method using a Hartley basis set over 0–10 cycles/2π had SNR of 19.26 dB ((d), solid line) for the 20 dB forward solution, and SNR=10.33 dB ((f), solid line) for the 10 dB forward solution. The broken line in (d) and (f) is the original signal.

FIG. 2 is a plot of SNR forward vs. SNR inverse (equation (13)) for the system. The solid line represents the predicted relationship between forward and inverse solution SNR for the method using a Hartley basis set over 0–10 cycles/2π. The broken line is for a Hartley basis set extending over 0–14 cycles/2π. The 0–14 cycles/2π basis set provides higher SNR of the inverse solution for high SNR forward due to superior basis representation of the original signal. With lower SNR forward, the broader basis set has greater noise recovery than the more restricted basis set causing inferior SNR inverse for the inverse solution.

FIG. 3 is a photograph of an original image of a four quadrant checkerboard with square sizes of four, five, six and seven pixels; FIG. 4 is a photograph of the forward solution showing severe distortion following degradation by a system with Gaussian separable spatially variant point spread function (SSVPSF) that varied radially in width (center FWHM=6 pixels, corner FWHM=3 pixels) and amplitude (center=0.151, corner=0.075), and addition of noise to achieve SNR=20 dB;

FIG. 5 photographically shows the restored image following application of the present method to achieve resolution of all image elements with good contrast recovery;

FIG. 6 photographically demonstrates the added noise in the restored image when the original image is processed in accordance with the present method from a noiseless forward solution; and

FIG. 7 is a two dimensional representation of the Gram-Schmidt orthogonalization process in the presence of noise.

FIG. 8 is a flowchart showing the methodology of the present invention.

Reference will now be made in detail to the present preferred embodiment of the invention, an example of which is illustrated in the accompanying drawing.

DETAILED DESCRIPTION OF THE INVENTION

The method of the present invention for using discrete orthogonal basis to restore a signal and/or image system created by time and/or spatially varying transfer functions will now be described in detail. The method may be applied to restore a signal or image system in a wide variety of technical fields. Stability of the inverse solution may be achieved if the characteristics of the noise in the forward solution may be estimated. For time-varying linear systems having a region of basis function support approximately congruent to the support region of the transfer function, and for which there is sufficient a priori knowledge of the system, the present method provides an efficient and noise tolerant approach to achieve inverse solution.

As previously described, the method of the present invention may be utilized to obtain an inverse solution vector for either one or two dimensional restorations. For purposes of presentation, $m=1, 2, 3, \ldots, M$ is the index for the vector sets.
with M members. The row and column vectors are designated by lowercase letters, and square matrices by uppercase letters. The system matrix transfer function is denoted by B.

Given the linear operation

$$[B][o]=[l]$$

(1)

the purpose of this invention is to recover the length IU vector \([o]\), given the forward solution vector \([l]\) and a priori knowledge of the forward operator \([B]\). The IUxIU matrix \(B\) is constructed using the time (or spatially) varying system transfer function \(b_{mk}\) so that the forward solution \(l_k\) is defined by

$$l_k = \sum_{m=1}^{IU} b_{mk} o_m$$

(2)

Recovery of the inverse solution requires two orthogonal function sets related to the adjoint PSF operator. The construction of these function sets (equations (4—6b)) requires a set of M orthogonal basis set functions \(p_{mk}\) of length IU. The Hartley basis set, defined by

$$p_{mk} = \sin \left( \frac{2\pi m - 1}{IU} k \right) + \cos \left( \frac{2\pi m - 1}{IU} k \right)$$

(3)

is the preferred basis for real-valued systems and will be utilized to illustrate the present method. It should be appreciated, however, that any orthogonal basis that spans the forward and inverse solution vector spaces may be utilized. These include for example, Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

As can be appreciated in viewing FIG. 8, the method begins with the application of the adjoint of the forward operator to each member of the basis set

$$[B^T_{inv}[B]]^T[p_{mk}]$$

(4)

where \([B^T_{inv}\) is the transpose-complex conjugate of the matrix \(B\).

The Gram-Schmidt orthogonalization procedure followed by normalization may be used to construct an orthonormal function set \(c_{mk}\) from \(b_{mk}\). Defining a set of constants \(\tau_{mk}\) (5), where the constants \(\tau\) are the Gram-Schmidt orthogonalization constants, \(c_{mk}\) may be written as a linear combination of \(b_{mk}\) (6a). A second orthogonal function set \(d_{mk}\) can then be constructed as a linear combination of the basis set functions \(p_{mk}\) using the same set of constants \(\tau_{mk}\) (6b).

$$\tau_{mk} = \frac{c_{mk}}{\sqrt{\sum \tau_{mk}^2}}$$

(5a)

$$c_{mk} = \frac{1}{\sqrt{\sum \tau_{mk}^2}} \sum_{i=1}^{m-1} \tau_{mi} b_{ik}$$

(5b)

Recovery of the inverse solution begins with the unit operator for orthonormal functions (7). Reference to (6a) and (4) yields equation (O). The property of adjoint operators on finite-dimensional inner product space allows transition to (9) and equation (1) leads to (10). Equation (10) is the operational equation for the inverse solution vector for a one dimensional restoration utilizing the present method.

$$o_k = \sum_{m=1}^{M} c_{mk}$$

(7)

$$o_k = \sum_{m=1}^{M} c_{mk} \left( \sum_{k=1}^{IU} \sum_{n=1}^{M} \tau_{mk} p_{kn} \right)$$

(8)

$$o_k = \sum_{m=1}^{M} c_{mk} \left( \sum_{k=1}^{IU} \sum_{n=1}^{M} \tau_{mk} p_{kn} \right)$$

(9)

$$o_k = \sum_{m=1}^{M} c_{mk} \left[ \frac{1}{IU} \sum_{k=1}^{IU} \sum_{n=1}^{M} \tau_{mk} p_{kn} \right]$$

(10)

Of course, this inverse solution \(o_k\), like all method of ill-posed problem solution, is an approximation of the true solution. In the noiseless case, the quality of the inverse solution depends primarily on the equality of the representation of the true signal afforded by the chosen finite set of basis functions \(p_{mk}\) where \(m=1,2,3\ldots M\) is the index for the chosen orthogonal basis set with M members of length \(k = 1,2,3\ldots IU\). The recovered vector \(o_k\) may be expressed as a sum of the inverse solutions from the noiseless forward solution and from added noise components (see equation 11).

$$o_k = \sum_{m=1}^{M} c_{mk} \left[ \frac{1}{IU} \sum_{k=1}^{IU} \sum_{n=1}^{M} \tau_{mk} p_{kn} \right]$$

(11)

In the Fourier domain, the noise component term in (11) may be expressed by

$$\mathcal{O}(\text{noise}) = \sum_{m=1}^{M} C_{mk} \left[ \frac{1}{NF} \sum_{l=1}^{N} D_{kl} \text{noise} \right]$$

(12)

which allows an approximation of the inverse solution noise power to be made if the characteristics of the intrinsic noise are known. The predicted SNR of the inverse solution may be estimated by

$$\text{SNR}_{pred} = \text{SNR}_{forward}$$

(13)

where \(\text{SNR}_{pred}\) is the signal power in the original (undegraded) signal or image, \(\text{SNR}_{forward}\) is the inverse solution noise power due to the approximate nature of the inverse solution, and \(\text{SNR}_{noise}\) is the noise power in the inverse solution power due to added noise in the forward solution.

For a given system, an a priori estimate of the inverse solution SNR can be made for various values of SNR using equations (12) and (13). Noise power in the inverse solution due to added noise may be estimated by assigning values to

$$I_{\text{noise}}$$

(12) based on assumptions of the character and magnitude of added noise. Simulation studies with well modeled noiseless signals allow estimation of intrinsic noise, \(\text{SNR}_{noise}\), in the
inverse solution. With an estimate of the original signal power, the SNR$_{\text{pred}}$ for the inverse solution may be computed using equation (13).

The following example is presented for purposes of further illustrating the present invention, but it is not to be considered as limited thereto.

**EXAMPLE 1**

An exponentially decaying transfer function with linearly time-varying amplitude and time constant was chosen for purposes of demonstration. The behavior of the time-varying transfer function $h_{\text{sn}}$ is defined by

$$h_{\text{sn}}(n) = \left(0.5 + 0.5 \frac{n}{N} \right) e^{-\frac{n}{N}}$$

The input function (Fig 1(a)) is a summation of three unity amplitude sinusoids of arbitrarily chosen frequency and phase ($f_1=3$ cycles/2T, $\phi_1=0$ radians; $f_2=7$ cycles/2T, $\phi_2=1.0$ radians; $f_3=10$ cycles/2T, $\phi_3=0.6$ radians). Zero-mean pseudorandom noise was added to the forward solution (Fig. 1(b)) to achieve SNRs (SNR$_{\text{pred}}=10 \log(\sigma_{\text{forward}}^2/\sigma_{\text{added noise}}^2)$) of 10.0 and 20.0 dB (Fig. 1(c) and (d)). The method was performed using the Hartley basis set (equation 3) extending from 0 to 10 cycles/2T. The inverse solutions show good recovery of the original input function, with inverse solution SNRs (SNR$_{\text{inverse}}=10 \log(\sigma_{\text{forward}}^2/\sigma_{\text{recovered noise}}^2)$) of 19.26 dB (Fig. 1(d)) for the 20 dB forward solution, and 10.33 dB for the 10 dB forward solution (Fig. 1(f)).

It should also be appreciated that using an assumption of zero-mean white noise, and a predetermined value of $\sigma_{\text{noise}}^2$, a plot of SNR$_{\text{pred}}$ vs. SNR$_{\text{forward}}$ (see Fig. 2) may be constructed for the system described for the Hartley basis function bandwidths of 10 and 14 cycles/2T. For the 0–10 cycles/2T Hartley basis set, the SNR$_{\text{pred}}$ values of 11.39 dB and 17.91 dB (for the 10 dB and 20 dB forward solutions respectively) correspond reasonably well with the experimental SNR$_{\text{inverse}}$ values of 10.33 dB and 19.26 dB. FIG. 2 illustrates that increasing the recovery bandwidth from 10 to 14 cycles/2T results in improved SNR$_{\text{pred}}$ when the SNR$_{\text{forward}}$ is high, due to the improvement in representation of the inverse solution afforded by a more complete basis set. However, with lower SNR$_{\text{forward}}$, the effects of increased noise recovery accompanying expansion of the basis set offsets these advantages and results in lowering the SNR$_{\text{pred}}$ of the inverse solution. This method of SNR$_{\text{inverse}}$ estimation can be performed using training sets of large numbers of simulated signals and noise levels to determine the best selection of basis set for a given application.

The forward solution $J_{\text{f}}$ for two-dimensional separable spatially variant point spread function (SSVPSF) systems may be obtained by successive application of column and row degradation operators to the original image $C_{\text{f}}$. Obtaining the inverse solution for the SSVPSF system by the present method follows the general approach for matrix operators on separable systems. For each row $p=1,2,\ldots,\text{JU}$, the blurring matrix across the columns, $[B]_p$, is constructed, allowing calculation of the corresponding orthogonal sets $c_{\text{row}}$ and $d_{\text{row}}$ (6). Successive application of the present operational equation (10) to all of the rows yields the IUxIU intermediary matrix $J_{\text{r}}$, which has been corrected for the blurring across columns.
is likely to provide the best restoration. Large numbers of simulations with training sets of images and noise levels appropriate to the application can be used to determine the optimal basis set for a particular application.

The following example is presented for purposes of further illustrating the present invention, but it is not to be considered as limited thereto.

EXAMPLE 2

A 128×128 pixel four quadrant checkerboard pattern with quadrant check sizes of 4.5, 6, and 7 pixels (see photographic FIG. 3) was chosen for illustration of the present method in restoring a SSVPSF system. The light squares were assigned a value of 1.0 and the dark squares 0.3. The multiple high-contrast discontinuities in this image were designed to be particularly challenging for a restoration method using a bandlimited basis set. The spatially varying gaussian transfer function had exponentially-radially varying FWHM of 6 pixels in the image center and FWHM of 3 pixels at the image corners. The PSF located at the center of the image space was assigned an amplitude so that the center PSF was lossless. PSF amplitude declined radially in exponential fashion so that the PSF amplitude in the corners was half of the center PSF amplitude.

Based on the methods described above, the Hartley basis set utilized was restricted to +/- 18 cycles/2π to assure a stable restoration for an anticipated forward solution SNR of 20 dB. Zero-mean pseudorandom noise was added to the forward solution to achieve a SNR of 20 dB (see photographic FIG. 4). The restored image (see photographic FIG. 5) from this noisy forward solution showed good contrast recovery with resolution of all image elements. Due to the presence of method recovered noise the subjective quality is inferior to the noiseless restoration shown in photographic FIG. 6, which is subject only to intrinsic noise, but is a dramatic improvement from the degraded image shown in photographic FIG. 4.

The limited basis embodiment described extends directly to two or more dimensions provided that it is cast in "stacked lexicographic" format, e.g. A 2×2 image is acted on by a degrading operator to yield a forward solution 2×2 image; in stacked lexicographic notation this may be cast as the one dimensional problem

\[
\begin{bmatrix}
1 & 2 & 3 & 4
\end{bmatrix}
\begin{bmatrix}
A & B & C & D
\end{bmatrix}
\]

This is a common approach in image processing to reduce multidimensional problems to 1-D problems.

As should be appreciated, the above described method for discrete orthogonal basis restoration (DOBR) is a time-domain approach. In an alternative embodiment of the present invention, discrete orthogonal basis restoration is presented as a frequency domain approach for the estimation of the inverse solution vector for linear systems defined by the matrix operation

\[
\begin{bmatrix}
[F] & [o]
\end{bmatrix}
\]

where \([B]\) is an N×N non-singular transfer function matrix and \([o]\) and \([i]\) are length N column and row vectors. In this alternative embodiment, upper case letters denote frequency domain variables while lower case letters denote time-domain variables. For vector sets with two subscripts, such as \(C_{mk}\), the first denotes the set position and the second the discrete time or frequency index.

The normalized Hartley basis set is preferred for this alternative approach of discrete orthogonal basis restoration. The Fourier representation of a Hartley basis vector of frequency \(f_c\) cycles/2\(\pi\) is non zero only at the ±\(f_c\) discrete frequencies. All positive frequency Hartley basis vectors have identical complex amplitude

\[
\frac{N}{2} \left( 1 - \frac{k - 1}{N} \right)
\]

in their non-zero positive frequency bin, and the complex conjugate of this value in the negative frequency bin. Negative frequency Hartley basis vectors are the complex conjugates of the positive frequency spectrum. These relationships facilitate rapid computation and efficient storage of the basis set. The members of the frequency domain basis set, \(F_{mk}\), are ordered such that the DC component is assigned index \(m=1\), even values of the basis set index \(m\) correspond to positive Hartley frequency \(f_c=2\) and odd values of \(m\) correspond to the \(f_c=(m-1)/2\) Hartley frequency. For a selected DOBR bandlimit of \(0-f_{max}\) cycles/2\(\pi\), there are \(M=(2f_{max}+1)\) basis and vectors, so that the composite basis set spectrum is non-zero for \(m=1, \ldots, f_{max}+1\) and \((N-M+1), \ldots, N\). The relative compactness of the frequency domain representation of the basis set is instrumental in the development of an efficient frequency domain approach.

The initial step in the time-domain approach is the application of the adjoint (complex conjugate transpose) of the transfer function \([B]\) to each member of the orthogonal basis set \(F_{mk}\) to yield the vector set \(b_{mk}\). For the time varying case, the M frequency domain row vectors \(A_{mk}\) may be determined by \([F]^T[F]_m=[A]_m\), where the N×N matrix \([F]^T=[DFT][F][IDFT]\) and \([DFT]\) and \([IDFT]\) are the discrete and inverse Fourier transform matrices. Using the properties \([F][F]^T=[T]\) and \([DFT][k][IDFT]=[T]^k\), it follows that \([F][F]^T[F][B][IDFT]\) and noted that the spectrum of the forward solution is given by

\[
\sum_{k=1}^N F_{mk} b_{mk} = t_k
\]

For non-singular \([B]\) the vector set \(A_{mk}\) is linearly independent, but usually not orthogonal. Gram-Schmidt orthogonalization can be performed on the frequency domain vector set \(A_{mk}\) to yield an orthogonal complex set \(C_{mk}\). Regrouping the complex Gram-Schmidt coefficients \(a_{ml}\)

\[
a_{ml} = \sum_{k=1}^N C_{mk} A_{ml}
\]

\(i = 1, \ldots, m - 1\)

\(m = 1, \ldots, M\)

into a set of constants \(c\) defined by

\[
c_{ml} = \sqrt{\sum_{k=1}^N C_{mk} A_{ml}}
\]

\(i = 1, \ldots, m - 1\)

\(m = 1, \ldots, M\)
\[ m = i \]
\[ \omega_m = \frac{1.0}{\sum_{k=1}^{m-1} c_{ma} c_{mb} } \]
\[ i = 1, \ldots, m-1 \]
\[ M \]

allows the orthonormal vector set \( C_{mk} \) to be expressed as a superposition of prior \( A_{mk} \) by
\[ C_{mk} = \sum_{i=1}^{m} \omega_i a_{ik} \]

A second complex vector set \( D_{mk} \) is obtained by applying the same operation to the orthonormal basis set \( P_{ik} \).
\[ D_{mk} = \sum_{i=1}^{m} \omega_i p_{ik} \]

The vector sets \( C_{mk} \) and \( D_{mk} \) define the characteristics of the system \( B \) for frequency domain DOBR.

In the frequency domain, the completeness relationship [24] for orthonormal vectors may be written as:
\[ O_k = \frac{1}{N} \sum_{m=1}^{M} c_{ma} \sum_{n=1}^{N} c_{ma} o_{nk} \]

Transferring the complex conjugation to \( O_k \), substituting (22) and using the relationship \( [F^*F]_m = [A]_m \) yields
\[ O_k = \sum_{m=1}^{M} c_{ma} \left( \frac{1}{N} \sum_{i=1}^{N} \omega_i a_{ik} \right) \sum_{n=1}^{N} p_{nk} \]

which by the property of adjoint operators on inner product spaces [25] is equivalent to
\[ O_k = \sum_{m=1}^{M} c_{ma} \left( \frac{1}{N} \sum_{i=1}^{N} \omega_i a_{ik} \right) \sum_{n=1}^{N} p_{nk} \]

Reference to (18) and (23) yields the frequency domain DOBR operational equation
\[ O_k = \sum_{m=1}^{M} c_{ma} \left( \frac{1}{N} \sum_{i=1}^{N} \omega_i a_{ik} \right) D_{mk}^* \]

\( O_k \) is the spectral estimate of the inverse solution. The time domain inverse solution \( O_k \) may be obtained by inverse FFT of \( O_k \).

The steps where major computational differences exist between the time and frequency domain approaches involve summations that may be limited to regions of time or frequency domain support. In the frequency domain approach the range of summation is restricted to the \( N \) discrete frequencies where the composite basis set spectrum in non-zero. The frequency domain DOBR approach saves \( (N-M) \) multiplications and additions for each of the many inner products required by the approach. The reduction in the number of computations is at the expense of substituting complex for real operations. This is not particularly disadvantageous for additions, as current generation microprocessors perform complex and real additions with an equivalent number of clock cycles. Complex multiplications are more time consuming than real multiplications, but for practical DOBR applications the reduction in computations offsets the increased processor time.

The initial step in the frequency domain approach is the determination of \( A_{mk} \). When \( B \) is time varying, \( A_{mk} \) may be estimated by \( \text{IDFT} \{ [B^*F]_m \} \), but this is less efficient than calculating the time domain \( [b]_m = [B^*F]_m \), and subsequently performing an \( N \) point FFT on each vector \( [b]_m \). Using this approach, an additional \( M(N\log_2 N) \) complex additions and \( M(N/2\log_2 N) \) complex multiplications are required for the frequency domain approach. Determining \( C_{mk} \) by Gram-Schmidt orthogonalization requires calculation of \( NC = (M^2-M)/2 \) coefficients, for which the computational advantage of the frequency domain approach is \( 2NC(N-N) \) multiplications and additions. Performing the linear combinations of vectors weighted by these coefficients yields computational savings of \( NC(N-M) \) multiplications and \( (N-M)(M+NC-1) \) additions for the frequency domain approach. Normalization of the \( C_{mk} \) has savings of \( M(N-M) \) multiplications compared to the time-domain approach. The sparsity of the vectors \( P_{mk} \), which are non-zero at only two discrete frequencies for \( m=1 \), and at one frequency \( (DC) \) for \( m=1 \), allow rapid frequency domain computation of the set \( D_{mk} \). The computational advantage of the frequency domain approach is \( (M+N)(N-2) \) additions and multiplications for the determination of \( D_{mk} \).

Practical situations usually involve applying the DOBR operational equation for a stationary system \( B \) to multiple forward solutions. The vector sets \( C_{mk} \) and \( D_{mk} \) define the DOBR inverse system for \( B \), and may be computed, stored, and recalled for each implementation of the operational equation. Storage requirements are \( 2M^2 \) complex numbers for the frequency domain approach and \( 2(MN) \) real numbers for the time-domain approach, so that a reduction in storage requirements is realized when \( M=1 \). The total operational equation computational advantage for the frequency domain approach is \( 2M(N-M) \) multiplications and additions, which is reduced by \( N\log_2 N \) additions and \( N/2\log_2 N \) multiplications if the FFT of \( i_k \) to yield \( I_k \) is required. If the spectral estimate of the inverse solution is desired, the approach may be terminated at this point. Obtaining the time-domain solution by inverse FFT of \( O_k \) requires an additional \( N\log_2 N \) complex additions \( N/2\log_2 N \) complex multiplications, and \( N \) real multiplications.

An additional advantage of the frequency domain approach is that it is more robust when there are significant perturbations in \( B \). Errors in transfer function estimation are normally transmitted to the \( b_{mk} \) and ultimately have adverse effects on the inverse solution [21]. In the frequency domain approach, the noise components in \( B \) transmitted to \( a_{mk} \) lying outside of the DOBR bandwidth are not included in the calculations of \( C_{mk} \), and therefore are not propagated to the final solution. The inverse solution obtained from the frequency domain approach may be expected to be of higher quality than would be obtained from the time domain approach, especially when large errors in the estimation of \( B \) are present.

**EXAMPLE 3**

Consider an application with a time duration signal of \( N=128 \) samples and a DOBR basis set bandwidth of \( 0-10 \) cycles/2\( \pi \), requiring \( M=21 \) Hartley basis vectors. For a previously defined stationary system, DOBR is performed by executing the operational equation (27) with stored values of \( C_{mk} \) and \( D_{mk} \). The time domain operational equation requires 5376 real multiplications and 5334 real additions. The frequency domain operational equation, including the FFT of \( i_k \), and IFFT of \( O_k \), requires 1778 complex multiplications, 1736 complex additions, and 128 real multiplications, a reduction by a factor of 2.9 in the number of operational equation computations.

Dynamic systems, and initial applications of the frequency domain approach require computation of \( C_{mk} \) and
DOBR inverse solution of the forward solution noise component (right hand term in (11)). Unlike forward solution noise, transfer function perturbations change the characteristics of the vector sets $c_m$ and $d_m$, which define the behavior of the DOBR operational equation. Application of the perturbed transfer function $[B+N]$ to each member of the basis set $p_m$ yields:

$$\left( \begin{array}{c} B+N \end{array} \right) \left[ \begin{array}{c} p_m \end{array} \right] = \left[ \begin{array}{c} d_m \end{array} \right] = \left( \begin{array}{c} D_m \end{array} \right)$$

(31)

As was the case with forward solution noise, it is beneficial to analyze the effects of transfer function noise in the frequency domain. For the time invariant (convolutional) case the calculation of $A_m$ reduces to the point by point multiplication of the Fourier transform of the transfer function and $P_m$. When $[B]$ is a time-varying system, the spectral estimation of $A_m$ is considerably more complex. In the noiseless case, the spectra $A_m$ may be represented by:

$$A_m = |\text{DFT}[B]\text{IDFT}[P_m]|$$

(32)

$$A_m = |\text{DFT}[B]\text{IDFT}(\text{DFT}[N]\text{DFT}[P_m])|$$

(33)

$$A_m = A_m - A_{nm}$$

(34)

The resultant noise in $A_m$ is therefore the superposition of $A_m$ and $A_{nm}$.

The worst-case scenario will be considered in evaluating the propagation of noise in the frequency domain Gram-Schmidt orthogonalization process. Maximal noise transmission occurs when each $A_{nm}$ is orthogonal to $C_{m-1}$ (FIG. 7), and therefore has complete projection onto the vector $C_m$. The Gram-Schmidt coefficients involving the inner products of $A_{nm}$ and prior $C_m$ are zero by virtue of orthogonality, greatly simplifying the computational process. Each $C_m$ is then formed from $A_m + A_{nm}$ minus the projections of $A_m$ on prior $C_m$:

$$C_1 = A_1 = A_{12} + A_1$$

(35)

$$C_2 = A_2 + A_{23} - \frac{\langle A_2 C_1 \rangle}{|C_1|^2}$$

(36)

$$C_m = A_m + A_{nm} - \frac{\langle A_m C_{m-1} \rangle}{|C_{m-1}|^2}$$

(37)

Each $C_m$ is subsequently normalized, which preserves the relative noise contribution to $C_m$ but depending on the amplification properties of the transfer function $[B]$, the normalization may either increase or decrease the magnitude of noise transmitted to $C_m$. For the worst case scenario the total transmitted noise to $C_m$ is

$$\|\mathbf{W}_m\| = A_m + \frac{M}{|C_m|^2}$$

(38)

The frequency domain Gram-Schmidt orthogonalization constants

$$\omega_m = -\frac{\langle C_m A_m \rangle}{|C_m|^2}$$

(39)

can be regrouped and normalized into the set of constants $\phi_m$, so that each $C_m$ is expressed as a linear combination of
A', (39). The \( D_m \) is then calculated using the same set of constants \( \phi_m \) (40).

\[
C_m = \sum_{i=1}^{p} \phi_m A_{ia}
\]

\[
D_m = \sum_{i=1}^{p} \phi_m P_{ia}
\]

The frequency domain operational equation in the presence of transfer function noise is

\[
\Omega_m = \frac{M}{m-1} C_m \frac{1}{NP} \sum_{k=1}^{NP} D_m P_{ia}
\]

The vector set \( D_m \) is constructed to be compatible with the set \( C_m \) and does not contribute additional noise to the inverse solution. The worst case inverse solution SNR due to transfer function perturbations may be estimated by

\[
\text{SNR}_{\text{inv}} = 10 \log \frac{\| \Omega_m \|^2}{\| \Omega_m \|^2 + \| \Psi \|^2}
\]

In summary, the method of the present invention is equally applicable and extends to both one dimensional and two dimensional signal and image systems. The chief advantage of the present method compared to other SSVPSF restoration techniques are that preprocessing, matrix inversion, multiple iterations, or assumptions of local PSF variance are not required. As a result, processing times and computer power required for processing are both substantially reduced.

The primary limitation of the present method is that the basis function support of the inverse solution must be nearly congruent to the region of basis support of the transfer function to avoid an unstable inverse solution. Provided that a realistic simulation model exists, however, estimation of the restored image SNR may be made for a given forward solution SNR and chosen basis set. This allows a priori determination of the optimal basis set for a given application and provides an estimate of the anticipated quality of the restored image.

The foregoing description of a preferred embodiment of the invention has been presented for purposes of illustration and description. It is not intended to be exhaustive or to limit the invention to the precise form disclosed. Obvious modifications or variations are possible in light of the above teachings. The embodiment was chosen and described to provide the best illustration of the principles of the invention and its practical application to thereby enable one of ordinary skill in the art to utilize the invention in various embodiments and with various modifications as are suited to the particular use contemplated. All such modifications and variations are within the scope of the invention as determined by the appended claims when interpreted in accordance with the breadth to which they are fairly, legally and equitably entitled.
APPENDIX A

c harpat.for
..performs discrete orthogonal basis restoration for invariant
..systems
..using Hartley basis set
..calls:mat hart inprod

(.....variables:......

izz is the maximum number of basis functions
iu is the length of the fwd soln vector (ri) and inv. soln
vector (h)

the basis set is determined and placed in poly
to use for the inv soln is ordered transferred
to xleg
gau is the matrix formulation of the transfer function
gt is the transpose. Note that gau and gt are cast
to the row std vector row vector fwd soln
The gram-schmidt coef are stored in matrix a
The tau coef are in tau
The vector sets b, c, d with m members correspond to the

character*30 tau
character*30 obas
character*30 file
character*30 file
character*30 taunmx,obasx
character*30 filex, filegx, filein, fileh
character*2 ntrial
character*1 cxans, cyans

implicit integer(i-o)
parameter(izz=65,iu=128,two=6.283185)
common/c1/delt,rip
common/c2/rp1(iu),rp2(iu)
common/c4/poly(0:izz,iu)
common/c5/iz
common/c6/gau(iu),gt(iu,iu)
common/c6a/g(iu,((2*iu)-1))
common/c11/rphas
real b(izz,iu),c(izz,iu),d(izz,iu)
real a(izz,izz),tau(izz,izz)
real xleg(0:izz,iu)
real f(izz),h(iu),cnorm(izz)
real ri(iu),sumc(iu)
integer ifini
integer npol

character*13 pre
character*4 suf3,suf2
character*10 filex, fileg, filh, taumx, obas
character*30 taunmx, obasx
character*30 filex, filegx, filein, fileh
character*2 ntrial
character*1 cxans, cyans
pre='c:\fort\data\'
  suf2='.dec'
  suf3='hdc'
coperative Hartley Basis Set
  iz=64
    write(6,*)'%% calling hart''''''
    call hart

c... normalization of basis
c
  do 820 m=0,iz
do 812 kkk=1,iu
    rpi(kkk)=poly(m,kkk)
  812
    rp2(kkk)=poly(m,kkk)
call inprod
do 815 kkk=1,iu
  815
    poly(m,kkk)=(poly(m,kkk))/(sqrt(rpi))
do 24 ii=1,iu
  24
    rleg(1,ii)=poly(0,ii)

c... read in the ideal image and the PSP
  write(6,*)' input name for header file xxxxxxx.xxx '
  read(5,933)flh
  fileh=pre//flh
  open(25, file=fileh, status='new')
    write(6,*)'input filename of PSP xxxxxxx.xxx '
    read(5,933)fileg
  933
    filegx=pre//fileg
    igau=0
  933
    open(20, file=filegx, status='old')
do 43 i=1,iu
  43
    read(20,*1, end=433)gau(i)
  43
    igau=igau+1
    close(20)
    format(a10)
      write(6,*)'input trial number for this input fn '
    read(5,964)ntrial
  964
    format(a2)
      write(6,*)'input filename of fwd soln xxxxxxx.xxx '
    read(5,933)filen
c... set the restoration bandwidth
  write(6,*)' high freq c/o '
  read(5,*)ifini
  npol=(2*ifini)+1

c!!!!!!!!!!!!!!!build the selected orthogonal set!!!!!!!!!!!
c... always include the dc component
do 24 ii=1,iu
  24
    rleg(1,ii)=poly(0,ii)
iptr=0
doi=1,ifini
iptr=iptr+2
jind=(j*2)-1
do 22 ii=1,iu
   rleg(iptr,ii)=poly(jind,ii)
   rleg(iptr+1,ii)=poly(jind+1,ii)
22  continue
write(6,'$$$$$$ orthonomal setup done $$$$$$')
c:........................................................................
c...compensate for extra loop trip and truncate to odd #
   igau=igau-1
   fig2=igau/2.
   if fig2.gt.igau/2.
      igau=igau-1
   endif
   c...call mat to cast the convolution operator in matrix form
c... note that this is in row vector format
call mat(igau)
c>>>>>>>>calc b(k) using matrix
approach>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
   write(6,'')......calculating b(k).....'
c*****use transpose matrix
   iz=(2*ifini)+1
   do 9 ic=1,iz
      do 8 ir=1,iz
         sum=sum+rleg(k,ir)*gt(ir,ic)
      8     continue
      b(k,ic)=sum
   9     continue
10    continue
c>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
c????????????calc c(k)????????????????????????????????????????????????????
   write(6,'')+++++calculating c(k)+++++
   do 21 kkk=1,iz
21   c(1,kkk)=b(1,kkk)
do 23 m=1,iz
23   a(m,m)=1.
c........calculate Gram-Schmidt coefficients
do 50 m=2,iz
do 40 j=1,m-1
    do 30 kkk=1,iu
        rpl(kkk)=c(j,kkk)
    enddo 30
    rpl(kkk)=rpl(kkk)
call inprod
    rnorm=rip
    do 29 kkk=1,iu
        rpl(kkk)=c(j,kkk)
    enddo 29
    rpl(kkk)=b(m,kkk)
call inprod
    a(m,j)=-1.*(rip/rnorm)
40 continue

C........now calculate c(m) by G-S orthogonalization
    do 46 ii=1,iu
        sumc(ii)=0.
    enddo 46
    do 49 j=1,m-1
        do 48 kkk=1,iu
            sumc(kkk)=sumc(kkk)+(a(m,j)*c(j,kkk))
        enddo 48
    enddo 49
    continue
49 continue
    do 47 kkk=1,iu
        c(m,kkk)=b(m,kkk)+sumc(kkk)
    enddo 47
    continue
50 continue

C???????????????????????????????????????????????????????????????????????????
C........normalization
    do 720 m=1,iz
    enddo 720
    do 712 kkk=1,iu
        rpl(kkk)=c(m,kkk)
    enddo 712
    rpl(kkk)=c(m,kkk)
call inprod
    cnorm(m)=sqrt(rip)
    do 715 kkk=1,iu
        c(m,kkk)=(c(m,kkk))/(sqrt(rip))
    enddo 715
    continue
715 continue
    do 720 m=1,iz
        c(m,kkk)=tau(k,k)
        write(6,*)'=='=calculating tau==='
    enddo 75
75 tau(k,k)=a(k,k)
do 90 m=2,iz
    do 85 i=1,m-1
        if(i.ne.m)then
            rsum=0.
        enddo 85
        do 83 j=i,m-1
            rsum=rsum+(a(m,j)*tau(j,i))
        enddo 83
    enddo 90
83 continue
    tau(m,i)=rsum
else
  endif
85 continue
90 continue
C normalize taus by dividing by cnorms
  do 94 m=1,iz
    tau(m,i)=(tau(m,i)/cnorm(m))
92 continue
94 continue
C*******************************************************************************
C!!!!!!!!!!!!!!!!!!!!!!!!!!!calculate d(iz,iu)!!!!!!!!!!!!!!!!!
  write(6,*)'!!!!!!calcuating d(iz,iu)!!!!!'
  do 110 m=1,iz
    do 105 i=1,m
      do 100 it=1,iu
        d(m,it)=d(m,it)+(tau(m,i)*rleg(i,it))
100 continue
105 continue
110 continue
C*******************************************************************************
C:::-----------------------------------------------------------------------------
C/file=xpre//d'//ntrial//filen(5:6)
  * //fileg(5:6)/suf2
  file=xpre//ntrial//filen(5:6)
  * //fileg(5:6)/suf3
  filein=xpre//filen(1:10)
  fileg=xpre//fileg(1:10)
C:::-----------------------------------------------------------------------------
C:::-----------------------------------------------------------------------------
  open(19,filen=filein,status='old')
    do 32 i=1,iu
      read(19,*,end=519)r1(i)
32 continue
519 close(19)
  izk=npol
C...clear arrays
  do 190 iy=1,izk
    f(iy)=0.
  do 191 iy=1,iu
    h(iy)=0.
C########################################################################compute f$########################################################################
  do 120 m=1,izk
    rsun=0.
    do 115 i=1,iu
      rsun=rsun+{d(m,i)*r1(i)}
115 continue
  f(m)=rsun
  write(6,*)m,'c(m) = ',f(m)
120 continue
*calculate h*

do 140 i=1,iu
   rsun=0.
   do 130 m=1,izk
       rsun=rsun+(c(m,i)*f(m))
   130 continue
   h(i)=rsun
140 continue

open(18,file=filex,status='new')
do 150 i=1,iu
150 write(18,*)h(i)
close(18)

write(6,*)filex

write(25,*)'iu= ',iu
close(25)

!!!

this section allows various process vectors and matrices to be saved to disk for later analysis

write(6,*)' write taufile (y/n)? 'read(5,976)cxans

if(cxans.eq.'y')then
   write(6,*)' enter name of taufile 'read(5,977)tau

format(a10)
taumx=pre//taum
open(27,file=taumx,status='new')
write(6,*)'iz= ',iz
do 833 m=1,iz
do 833 i=1,iz
write(27,*)tau(m,i)

continue
close(27)
endif

write(6,*)' save to disk d(m) array ? 'read(5,916)cyans

if(cyans.eq.'y')then
   format(a1)
   write(6,*)' # rows= ',iz
   write(6,*)' input name of otpt file xxxxx.xxx 'read(5,933)obas
   obaxx=pre//obas
   open(30,file=obaxx,status='new')
do 501 ibas=1,iz
do 501 ku=1,iu
501  write(30,*)(ibas,ku)
close(30)
endif
write(6,*),' save to disk c(m) array ? '
read(5,916)cyans
if(cyans.eq.'y')then
write(6,*),' input name of otpt file xxxxxx.xxx '
read(5,933)obas
obasx=pre/obas
open(30, file=obasx,status='new')
do 503 ibas=1,iz
do 503 ku=1,iu
503  write(30,*)(ibas,ku)
close(30)
endif
write(6,*),' save to disk b(m) array ? '
read(5,916)cyans
if(cyans.eq.'y')then
write(6,*),'iz=',iz
write(6,*),' input name of otpt file xxxxxx.xxx '
read(5,933)obas
obasx=pre/obas
open(31, file=obasx,status='new')
do 504 ibas=1,iz
do 504 ku=1,iu
504  write(31,*)(ibas,ku)
close(31)
endif
write(6,*),' save g-T ? '
read(5,916)cyans
if(cyans.eq.'y')then
write(6,*),' input name of otpt file xxxxxx.xxx '
read(5,933)obas
obasx=pre/obas
open(30, file=obasx,status='new')
do 506 ibas=1,iu
506  write(30,*)(gt(ibas,ku),ku=1,iu)
close(30)
endif

C>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
stop
end
APPENDIX B

c tvpat.for
.c performs discrete orthogonal basis restoration for invariant
c.systems
.c using Hartley basis set
c...calls:hart inprod

variables:

c izz is the maximum number of basis functions
c iu is the length of the fwd soln vector (ri) and inv. soln
vector (h)
c c the basis set is determined and placed in poly
c the basis set used for the inv soln is ordered transferred
c to rleg
c tvtfm is the imported matrix transfer function
c The gram-schmidt coef are stored in matrix a
.c The tau coef are in tau
.c The vector sets b,c,d with m members correspond to the
.c literature
.c descriptions, f(m) are <d(m),i> and are an
.c intermediate step
.c the inverse soln is h
.c npol is the number of basis vectors used
.c the array cnorm holds the normalization factors for each
.c member of the vector set c
c
implicit integer(i-o)
parameter(izz=65,iu=128,twopi=6.283185)
common/c1/delt,rip
c
common/c2/rpl(iu),rp2(iu)
c
common/c4/poly(0:izz,iu)
c
common/c5/iz,
c
common/c6a/g(iu,((2*iu)-1))
c
common/c11/rphas
real b(izz,iu),c(izz,iu),d(izz,iu)
c
real a(izz,iu),tau(izz,iu)
c
real rleg(0:izz,iu)
c
real f(izz),h(iu),cnorm(izz)
c
real ri(iu),sumc(iu)
c
real tvtfm(iu,iu)
c
integer ifin1

integer npol
character*13 pre
character*4 suf3,suf2
character*10 filen,fileg,flh,taunn,obas
character*30 taunnx,obasx
character*30 filex,filegx,filein,fileh
character*2 ntrial
character*1 cxans,cyans
pre='c:\fort\data\'
suf2=''.dec'
suf3=.hdc'
c........compute Hartley Basis Set
   iz=64
   write(6,*)'********calling hart********
   call hart

c.... normalization of basis
   do 820 m=0,iz
      do 812 kkk=1,iu
         rp1(kkk)=poly(m,kkk)
      enddo
      rp2(kkk)=poly(m,kkk)
      call inprod
      do 815 kkk=1,iu
         poly(m,kkk)=(poly(m,kkk))/(sqrt(rp1))
      enddo
   enddo
   continue
   820 continue

   c....read in the ideal image and the PSF
   write(6,*)'input name for header file xxxxxx.xxx '
   read(5,993)filh
   fileh=pre//filh
   open(25,file=fileh,status='new')
   933 format(a10)
   write(6,*)'input trial number for this input fn '
   read(5,994)ntrial
   964 format(a2)
   write(6,*)'input filename of fwd soln xxxxxx.xxx '
   read(5,993)filen
   c.... set the restoration bandwidth
   write(6,*)'high freq c/o '
   read(5,*1)ifini
   npol=(2*ifini)+1

   c!!!!!!!!!!!!!!!build the selected orthogonal set!!!!!!!!
   c....always include the dc component
      do 24 ii=1,iu
         rleg(ii,0)=poly(0,ii)
         iptr=0
      enddo
      do 25 j=1,ifini
         iptr=iptr+2
         jind=(j*2)-1
      enddo
      do 22 ii=1,iu
         rleg(iptr+ii)=poly(jind,ii)
      enddo
      rleg(iptr+1,ii)=poly(jind+1,ii)
      22 continue
      25 continue
write(6,*)'$$orthonomal setup done $$'
c                                                       
c...import the transfer function matrix
write(6,*)' input filename of TV transfer fn matrix
xxx.xxx.xxx'
read(5,933)filex
filex=pre//fileg
open(20,file=filex,status='old')
do 43 i=1,iu
read(20,*,end=433)(tvtfm(i,j),j=1,iu)
43 continue
433 close(20)

 c>>>>>>>>>calc b(k) using matrix
 approach>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
 write(6,*)'......calculating b(k)......'
c*****use transpose matrix
  iz=(2*ifini)+1
  do 10 k=1,iz
  do 9 ic=1,iu
  sum=0.
  do 8 ir=1,iu
      sum=sum+(rleg(k,iz)*tvtfm(ir,ic))
 8 continue
  b(k,ic)=sum
 9 continue
10 continue
 c>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
 write(6,*)'++++++calculating c(k)++++++'
do 21 kkk=1,iu
21 c(1,kkk)=b(1,kkk)
do 23 m=1,iz
23 a(m,m)=1.
c......calculate Gram-Schmidt coefficients
do 50 m=2,iz
  do 40 j=1,m-1
  do 30 kkk=1,iu
      rp1(kkk)=c(j,kkk)
 30 rp2(kkk)=rp1(kkk)
call inprod
      rnorm=rip
  do 29 kkk=1,iu
      rp1(kkk)=c(j,kkk)
 29 rp2(kkk)=b(m,kkk)
call inprod
a(m,j)=-1.*(rip/rnorm)
40 continue

c....now calculate c(m) by G-S orthogonalization
46 sumc(ii)=0.

do 49 j=1,m-1
49 do 48 kkk=1,iu
    sumc(kkk)=sumc(kkk)+(a(m,j)*c(j,kkk))
48 continue
49 continue

do 47 kkk=1,iu
712 c(m,kkk)=b(m,kkk)+sumc(kkk)
715 continue

720 continue

720 continue

c~~~~~~~~~~~~~~calculate taus~~~~~~~~~~~~~~~~~~
720 write(6,*)'=====calculating tau====='
720 do 75 k=1,iz
75 tau(k,k)=a(k,k)
77 do 90 m=2,iz
78 do 87 i=1,m-1
79 if(i.ne.m) then
80 rsum=0.
81 do 83 j=1,m-1
82 rsum=rsum+(a(m,j)*tau(j,i))
83 continue
85 continue
87 endif
88 endif
90 continue

90 continue

90 continue

c normalize taus by dividing by cnorms
94 do 98 m=1,iz
96 do 94 i=1,iz
98 tau(m,i)=(tau(m,i)/cnorm(m))
99 continue
100 continue
c^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
c!^^^^^^^^^^^^^^^^^^^^^^^^calculate d(iz,iu)!!!!!!!!!!!!!
   write(6,*)'!!!!calculating d(iz,iu)!!!!!!!!'
do 110 m=1,iz
do 105 i=i,m
do 100 it=1,iu
   d(m,it)=d(m,it)+(tau(m,i)*rleg(i,it))
100 continue
105 continue
110 continue

c^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
|filex=pre//'d'\ntrial//filen(5:6)
   //fileg(5:6)/\nsuf2
fileh=pre//ntrial//filen(5:6)
   //fileg(5:6)/\nsuf3
filein=pre//filen(1:10)
filegx=pre//fileg(1:10)
c:;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

C:;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
   open(19,file=filein,status='old')
do 32 i=1,iu
   read(19,*,end=519)ri(i)
32 continue
519 close(19)

izk=npol
C...clear arrays
   do 190 iy=1,izk
190 f(iy)=0.
do 191 iy=1,iu
191 h(iy)=0.
c#compute fu# compute fu# compute fu# compute fu# compute fu#
do 120 m=1,izk
   rsum=0.
do 115 i=1,iu
   rsum=rsum+(d(m,i)*ri(i))
115 continue
   f(m)=rsum
   write(6,*,m,'(c(m) = ','f(m)
120 continue

c#compute h#compute h#compute h#compute h#compute h#
do 140 i=1,iu
   rsum=0.
do 130 m=1,izk
   rsum=rsum+(c(m,i)*f(m))

5
130 continue
h(i)=rsum
140 continue
c
open(18,file=filex,status='new')
do 150 i=1,lu
150 write(18,*),h(i)
close(18)

777 continue
write(6,*),filex
write(25,*),iu='
',iu
close(25)
c
be saved to disk for later analysis
write(6,*),' write taufile (y/n)? '
read(5,976)cxans
976 format(a10)
if(cxans.eq.'y') then
write(977)taunn
read(5,977)taunn
977 format(a10)
read(5,977)taunn
open(27,file=taunn,status='new')
write(27,*),iz=' ',iz
do 833 m=1,iz
do 833 i=1,iz
write(27,*),tau(m,i)
833 continue
close(27)
endf
write(6,*),' save to disk d(m) array ? '
read(5,916)cyans
916 format(a10)
if(cyans.eq.'y') then
write(5,933)obas
open(30,file=obas,status='new')
do 501 ibas=1,iz
do 501 ku=1,lu
501 write(30,*),d(ibas,ku)
close(30)
endf
write(6,*),' save to disk c(m) array ? '
read(5,916)cyans
if(cyans.eq.'y')then
write(6,*)' input name of otpt file xxxxxx.xxx '
read(5,933)obas
obasx=pre//obas
open(30,file=obasx,status='new')
do 503 ibas=1,iz
do 503 ku=1,iu
503 write(30,*)c(ibas,ku)
close(30)
endif
write(6,*)' save to disk b(m) array ? '
read(5,916)cyans
if(cyans.eq.'y')then
write(6,*)'iz=',iz
write(6,*)' input name of otpt file xxxxxx.xxx '
read(5,933)obas
obasx=pre//obas
open(31,file=obasx,status='new')
do 504 ibas=1,iz
do 504 ku=1,iu
504 write(31,*)b(ibas,ku)
close(31)
endif
C>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
stop
end
APPENDIX C

c TVFPAT.FOR
c FREQ DOMAIN DOBR THAT TAKES IDFT FOR TIME DOMAIN INV SOLN

c....as most calculations involve complex operations, default
implicit complex(c)

parameter(np=128, izz=128, dcval=11.3137, xval=5.6568, np2=128)
c....np is the number of time and frequency domain signal points

c....zb is the maximum number of basis functions

c....dcval and xval are used to assign the frequency domain

complex cbn(izzard.np), ccn(izzard.np), csumc(izzard.np), ctrue(izzard.np)
complex cdn(izzard.np), cp(izzard.np), ca(izzard.izzard), ctau(izzard.izzard)
complex cnorm(izzard), cf(izzard), ch(izzard.np), cfwd(izzard.np), cfwdn(izzard.np)
real atrue(izzard.np), arec(izzard.np), xo(izzard.np)

c....Freq. domain variable names correspond to those used in
time domain algorithm

c cbn(m,k) ---> b(m,k), ccn(m,k) ---> c(m,k),
cdn(m,k) ---> d(m,k)

c the hartley basis is in cp(m,k)
c gram-schmidt coef are in array ca, tau in ctau
c fwd soln --> cfwd. Freq domain inv soln --> ch,
c time domain inv soln --> xo

........................................

character*10 tfnm, fnm, tnam, onm
character*13 pre
character*30 tfnmx, fnmx, tnamx, onm

pre="c:\fort\data\"
900 format(a10)

c....select the DOBR bandwidth
write(6,'')' input fmax '
read(5,*) ifmax

c....and compute the number of Hartley Basis vectors to be used
npol=2*ifmax+1

c.... input the filename for the Fourier transforms of the vector
c....set b(m,k). These are most easily obtained by dumping the

c....vector set b(m,k) computed using time-domain DOBR and doing

write(6,'')' input TV B(m,k) file (xxxxxx.fcxx) '
read(5,900) tfnm


tfnmx=pre//tfnm

c....The FFT of the forward solution is also needed
write(6,'')' input FWD soln file (xxxxxx.fcxx) '
read(5,900) tfnm

1
c...cfpwr computes running sum of fwd soln pwr
    cfpwr=0.,0.)
do 3 i=1,np
    read(19,*),cfwd(i)
cfpwr=cfpwr+((cfwd(i)*conjg(cfwd(i))))
    continue
    close(19)
do 6 m=1,npol
do 5 i=1,np
    read(20,*),cbn(m,i)
    close(20)

 Cambined Freq. Domain Hartley basis set P(m,k) based on 128 pt signal

c....assign dc val
    cp(1,1)=(dcval,0.)

c....assign spectra for even-indexed basis
    do 10 i=2,npol
        ibin1=(i/2)+1
        ibin2=(np+1)-i/2
        cp(i,ibin1)=(xval,-xval)
        cp(i,ibin2)=(xval,xval)
    continue

c....and for odd indexed basis
    do 15 i=3,npol,2
        ibin1=((i-1)/2)+1
        ibin2=(np+1)-((i-1)/2)
        cp(i,ibin1)=(xval,xval)
        cp(i,ibin2)=(xval,-xval)
    continue

 c....note that this assignment assumed that the basis ordering of hoprd

 c....was: 1=dc, 2+1hz,3=-1hz,4=2hz,5=-2hz....

 c??????????????calculate c(k)????? ?????????????????????????????????????
    npxx=ifmax  
    write(6,*)'begin calculating c(k)++++++'
    ccn(1,1)=cbn(1,1)
    do 21 kkk=1,npxx
        ccn(1,(np2-kkk+1))=cbn(1,(np2-kkk+1))
    21    ccn(1,(kkk+1))=cbn(1,(kkk+1))
    do 23 m=1,npol
    23    ca(m,m)=(1.,0.)

 c....calculate ca coef (complex gram-schmidt coef.)
do 50 m=2,npol
do 40 j=1,m-1  
csum2=0.,0.
   csum2=(ccn(j,1)*conjg(ccn(j,1))))
do 30 kkk=1,npxx
   csum2=csum2+(ccn(j, (np2-kkk+1))*(conjg(ccn(j, (np2-kkk+1)))))
   csum2=csum2+(ccn(j,kkk+1)*(conjg(ccn(j,kkk+1)))))
30 continue
   cnormx=csum2
   c........save the normalization factor for each vector
   c(m)...........
   cnorm(j)=csqrt(cnormx)
c...................
   csum2=0.,0.
   csum2=(ccn(j,1)*conjg(cbn(m,1)))
do 29 kkk=1,npxx
   csum2=csum2+(ccn(j,kkk+1)*conjg(cbn(m,kkk+1)))
csum2=csum2+(ccn(j, (np2-kkk+1))*(conjg(cbn(m, (np2-kkk+1)))))
29 continue
   cip=csum2
   ca(m,j)=-1.*(cip/cnormx)
40 continue
   do 46 ii=1,np
66 csumc(ii)=0.,0.
do 49 j=1,m-1
66 csumc(1)=csumc(1)+(ca(m,j)*ccn(j,1))
do 48 kkk=1,npxx
66 csumc(kkk+1)=csumc(kkk+1)+(ca(m,j)*ccn(j,kkk+1)))
csumc(np2-kkk+1)=csumc(np2-kkk+1)+(ca(m,j)*ccn(j, (np2-kkk+1))))
48 continue
49 continue
   ccn(m,1)=cbn(m,1)+csumc(1)
do 47 kkk=1,npxx
   ccn(m,kkk+1)=cbn(m,kkk+1)+csumc(kkk+1))
cbn(m, (np2-kkk+1))=cbn(m, (np2-kkk+1))+csumc(np2-kkk+1)
47 continue
50 continue
   c?????????????????????????????????????????????????????????????????
c........now normalize
   c........need the final norm
   cip=(0.,0.)
   cip=cip+(ccn(npol,1)*(conjg(ccn(npol,1))))
do 712 kkk=1,npxx
   3
crip=crip+(ccn(npol,kkk+1)*(conjg(ccn(npol,kkk+1))))

crip=crip+(ccn(npol,(np2-kkk+1))*conjg(ccn(npol,(np2-kkk+1))))
712 continue
    cnorm(npol)=sqrt(crip)
    do 720 m=1,npol
        ccn(m,1)=(ccn(m,1))/cnorm(m)
    do 715 kkk=1,npolxx
        ccn(m,kkk+1)=(ccn(m,kkk+1))/cnorm(m)
    ccn(m,(np2-kkk+1))=ccn(m,(np2-kkk+1))/cnorm(m)
715 continue
720 continue
    c**************************calculate taus**************************
    write(6,*)'======calculating tau======'
    do 75 k=1,npol
        ctau(k,k)=ca(k,k)
    do 85 m=1,m-1
        if(i.ne.m) then
            crsum=(0.,0.)
        do 83 j=1,m-1
            crsum=crsum+(ca(m,j)*ctau(j,i))
83        continue
        ctau(m,i)=crsum
    else
        endif
85    continue
90 continue
    c normalize taus by dividing by cnorms
    do 94 m=1,npol
    do 92 i=1,npol
        ctau(m,i)=(ctau(m,i)/cnorm(m))
92    continue
94 continue
    c******************************calculate d(iz,np)******************************
    write(6,*)'!!!!!!calculating d(iz,np)!!!!!!'
    do 110 m=1,npol
    do 105 i=1,m
        fi=real(i)
        itest1=aint(fi/2.)
        itest2=aint(fi/2.)
        c....test for even or odd and assign index
        if(itest1.eq.itest2) then
            ind1=(i/2)+1
            ind2=(np+1)-(i/2)
        cdn(m,ind1)=cdn(m,ind1)+(ctau(m,i)*cp(i,ind1))
        cdn(m,ind2)=cdn(m,ind2)+(ctau(m,i)*cp(i,ind2))
        else
            ind1=((i-1)/2)+1
            cdn(m,ind1)=cdn(m,ind1)+(ctau(m,i)*cp(i,ind1))
if(i.lt.1) then
    ind2=(np+1)-((i-1)/2)
    cdn(m,ind2)=cdn(m,ind2)+(ctau(m,i)*cp(i,ind2))
else
    endif
100 continue
105 continue
110 continue

c:...........................................................
c...this section applies to lab experiments where the true
c...inverse son is known. when unknown, a dummy file can be
c...used and snr values computed may be ignored.
write(6,'*') input 128 pt spectrum file of true signal'
read(5,900)tnam
tnamx=pre//tnam
open(21,file=tnamx,status='old')
siwp=0.
do 160 i=1,np
    read(21,*)ctrew(i)
    atrue(i)=real(ctrue(i)*conjg(ctrue(i)))
    siwp=siwp+atrue(i)
160 continue

c:...........................................................
do 260 i=1,np
cfwdn(i)=cfwd(i)
do 120 m=1,npol
crsun=csun+(cdn(m,1)*conjg(cfwdn(1)))
do 115 i=1,npxx
    csun=csun+(cdn(m,i+1)*conjg(cfwdn(i+1)))
    csun=csun+(cdn(m,(np2-i+1))*conjg(cfwdn(np2-i+1)))
115 continue
cf(m)=csun
120 continue
do 140 i=1,npxx+1
crsun=csun+(ccn(m,i)*cf(m))
do 130 m=1,npol
crsun=csun+(ccn(m,i)*cf(m))
130 continue
ch(i)=csun
140 continue
do 141 i=1,npxx
crsun=csun+(ccn(m,(np2-i+1))*cf(m))
do 131 m=1,npol
crsun=csun+(ccn(m,(np2-i+1))*cf(m))
131  continue  
132  ch(np2-i+1)=crsum  
141  continue  

!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
  crsum=0.0.)  
  crsum=crsum+(ch(i)*conjg(ch(i)))  
  arec(i)=real(ch(i)*conjg(ch(i)))  
  do 150 i=1,npxx  
  crsum=crsum+(ch(i+1)*conjg(ch(i+1)))  
  arec(i+1)=real(ch(i+1)*conjg(ch(i+1)))  
  crsum=crsum+(ch(np2-i+1)*conjg(ch(np2-i+1)))  
  arec(np2-i+1)=real(ch(np2-i+1)*conjg(ch(np2-i+1)))  
150  continue  

  write(6,'(a5,f10.0)') recovered total pwr= ',crsum  

snee=0.  
  cfred=cttrue(i)-ch(i)  
  snee=snee+real(cfred*conjg(cfred))  
  do 180 i=1,npxx  
  cfred=cttrue(i+1)-ch(i+1)  
  snee=snee+real(cfred*conjg(cfred))  
  cfred=cttrue(np2-i+1)-ch(np2-i+1)  
  snee=snee+real(cfred*conjg(cfred))  
180  continue  

  snroute=10.*alog10(sigp/snee)  
  write(6,'(a5,f10.0)') snroute,snroute2  

---

180  continue  

   snroute2=10.*alog10(sigp/snee)  
   write(6,'(a5,f10.0)') snroute,snroute2  

   c...obtain time-domain inverse soln by IDFT  
   write(6,'(a5,f10.0)') perform idft? 1=Y  
   read(5,'i')iiiff  
   if (iiiff.eq.1)then  

---

   do 248 i=npxx+1,np2-npxx  
248  ch(i)=(0.,0.)  
   call fft(ch,128,7,1,0)  
   do 249 li=1,np  
249  xo(li)=real(ch(li))  
   write(6,'(a5,f10.0)') input outname xxxxxx.xxx ,  
   read(5,900)onm  
   onmx=pre//onm  
   open(27,fil=onmx,stat='new')  
   do 250 li=1,np  
   write(27,'*)xo(li)  
250  continue  
   close(27)  
else  
   endif
continue
stop
end
I claim:

1. A method of using discrete orthogonal basis to restore a signal and/or image system degraded by time and/or spatially varying transfer functions said system being of linear type and represented by the equation \([B][o]=[i]\) wherein \([o]\) is an original signal or image, \([i]\) is a degraded signal or image and \([B]\) is a system transfer function matrix, comprising:

   estimating in estimating means a signal-to-noise ratio for a restored system;
   selecting in selecting means a set of orthogonal basis set functions \(p_{mn}\) where \(m=1,2,3\ldots M\) is the index for the chosen orthogonal basis set with \(M\) members of length \(k=1,2,3\ldots\) \(\ldots\) \(M\) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
   removing in removing means time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector \(O_k\) for a one dimensional restoration wherein:

\[
on_k = \sum_{m=1}^{M} c_{mn} \left[ \frac{H}{\sum_{k=1}^{l} d_{mk}} \right]
\]

wherein \(d_{mk}\) is a vector set created by linear combinations of \(p_{mk}\) a set of constants formed by linear combinations of a, the standard Gram-Schmidt orthogonalization coefficients, and wherein:

\[
[b]_{\omega} = [B]^T\ast [p]_{\omega}
\]

where \([B]^T\ast\) is the transpose-complex conjugate of the matrix \([B]\) and \([p]_{\omega}\) is an \(M\) member orthogonal basis set;

\[
\tau_{mk} = \frac{d_{mk}}{\sum_{k=1}^{l} c_{mk}c_{mk}}
\]

\[
\tau_{mk} = \frac{1.0}{\sqrt{\sum_{k=1}^{l} c_{mk}c_{mk}}} m \neq l; i=1, \ldots, m-1;
\]

\[
c_{mk} = \sum_{i=1}^{m} \tau_{mk} d_{ki};
\]

\[
d_{mk} = \sum_{i=1}^{m} \tau_{mk} d_{ki}; m = 1,2,3\ldots M
\]

and an inverse solution vector \(O_{pk}\) for a two dimensional restoration wherein:

\[
on_{pk} = \sum_{m=1}^{M} c_{mpk} \left[ \frac{H}{\sum_{k=1}^{l} d_{mpk}} \right]
\]

wherein \(J_{pk}\) is an intermediary matrix of the form:

\[
J_{pk} = \sum_{m=1}^{M} c_{mpk} \left[ \frac{H}{\sum_{k=1}^{l} d_{mpk}} \right]
\]

\(\rho = 1,2,3\ldots HI\).

2. The method set forth in claim 1, wherein said estimating of the signal-to-noise ratio of the restored system is provided by applying a forward solution signal-to-noise ratio and selected set of orthogonal basis set functions \(p_{mk}\) to a realistic simulation model of said system.

3. The method set forth in claim 1, wherein said set of orthogonal basis set functions \(p_{mk}\) is selected from a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

4. The method set forth in claim 2, wherein said set of orthogonal basis set functions \(p_{mk}\) is selected from a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

5. A method of using discrete orthogonal basis to restore a signal system degraded by a time varying transfer function, comprising:

   estimating in estimating means a signal-to-noise ratio for a restored system;
   selecting in selecting means a set of orthogonal basis set functions \(p_{mk}\) where \(m=1,2,3\ldots M\) is the index for the chosen orthogonal basis set with \(M\) members of length \(k=1,2,3\ldots\) \(\ldots\) \(\ldots\) \(\ldots M\) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
   removing in removing means time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector \(O_{pk}\) wherein:

\[
on_{pk} = \sum_{m=1}^{M} c_{mpk} \left[ \frac{H}{\sum_{k=1}^{l} d_{mpk}} \right]
\]

wherein \(d_{mpk}\) is a vector set created by linear combinations of \(p_{mpk}\) a set of constants formed by linear combinations of a, the standard Gram-Schmidt orthogonalization coefficients, and wherein:

\[
[b]_{\omega} = [B]^T\ast [p]_{\omega}
\]

where \([B]^T\ast\) is the transpose-complex conjugate of the matrix \([B]\) and \([p]_{\omega}\) is an \(M\) member orthogonal basis set;

\[
\tau_{mpk} = \frac{d_{mpk}}{\sum_{k=1}^{l} c_{mpk}c_{mpk}}
\]

\[
\tau_{mpk} = \frac{1.0}{\sqrt{\sum_{k=1}^{l} c_{mpk}c_{mpk}}} m \neq k; i=1, \ldots, m-1;
\]

\[
c_{mpk} = \sum_{i=1}^{m} \tau_{mpk} d_{kip};
\]

\[
d_{mpk} = \sum_{i=1}^{m} \tau_{mpk} d_{kip}; m = 1,2,3\ldots M
\]

6. The method set forth in claim 5 wherein said estimating of the signal-to-noise ratio \(SNR_{peak}\) is provided by:

\[
SNR_{peak} = 10 \log \frac{\|O_{peak}\|^2}{\|O_{noise}\|^2 + \|O_{peak}\|^2}
\]

wherein \(\|O_{peak}\|^2\) is the signal power in the original, undegraded signal/image, \(\|O_{noise}\|^2\) is the inverse solution noise power due to the approximate nature of the inverse solution and \(\|O_{peak}\|^2\) is the noise power in the inverse solution power due to added noise in the forward solution.

7. The method set forth in claim 6, wherein said set of orthogonal basis set functions \(p_{mk}\) is selected from a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.

8. The method set forth in claim 5, wherein said set of orthogonal basis set functions \(p_{mk}\) is selected from a group consisting of Hartley, Walsh, Haar, Legendre, Jacobi, Chebyshev, Gegenbauer, Hermite and Laguerre functions.
9. A method of using discrete orthogonal basis to restore an image system degraded by time and spatially varying transfer functions, comprising:
   estimating in estimating means a signal-to-noise ratio for a restored system;
   selecting in selecting means a set of orthogonal basis set functions \( p_{\text{rad}} \) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
   removing in removing means time and spatially varying distortions in the restored system by obtaining an inverse solution vector \( O_{\text{pk}} \) wherein:

\[
\begin{align*}
O_{\text{pk}} &= \frac{M}{m=1} \sum_{\kappa=1}^{M} \frac{\sum_{p=1}^{U} d_{\text{rad}} / \rho}{d_{\text{rad}}} \\
\kappa &= 1, 2, 3, \ldots, \rho \sum_{m=1}^{M}
\end{align*}
\]

wherein \( d_{\text{rad}} \) is a vector set created by linear combinations of \( p_{\text{rad}} \) weighted by \( \kappa \), a set of constants formed by linear combinations of a, the standard Gram-Schmidt orthogonalization coefficients, and \( J_{\text{rad}} \) is an intermediary matrix of the form:

\[
J_{\text{rad}} = \begin{bmatrix}
\frac{1}{\rho} \sum_{p=1}^{U} d_{\text{rad}} / \rho
\end{bmatrix}
\]

\( \rho = 1, 2, 3, \ldots, \rho \).

10. The method set forth in claim 9, wherein said estimating of the signal-to-noise ratio \( \text{SNR}_{\text{rad}} \) is provided by simulating a noiseless forward solution and determining the intrinsic noise shown as the difference between the restored image and the original image and further considering recovered added noise.

11. The method set forth in claim 10, wherein said estimated signal-to-noise ratio \( \text{SNR}_{\text{rad}} \) is

\[
\text{SNR}_{\text{rad}} = 10 \log \left( \frac{\text{I}_{\text{rad}}}{\text{I}_{\text{rad}} + \text{I}_{\text{noise}}^2} \right)
\]

12. The method set forth in claim 9, wherein said estimated signal-to-noise ratio \( \text{SNR}_{\text{rad}} \) is

\[
\text{SNR}_{\text{rad}} = 10 \log \left( \frac{\text{I}_{\text{rad}}}{\text{I}_{\text{rad}} + \text{I}_{\text{noise}}^2} \right)
\]

13. A programmable apparatus, comprising:
   means for computing; and
   readable memory defining a process for estimating a signal-to-noise ratio for a restored system;
   selecting a set of orthogonal basis set functions \( p_{\text{rad}} \) wherein
   \( m=1, 2, 3, \ldots, \rho \) is the index for the chosen orthogonal basis set with M members of length \( k=1, 2, 3, \ldots, \rho \) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
   removing time varying distortions in the restored system by obtaining an inverse solution vector \( O_{\text{pk}} \) for a one dimensional restoration wherein:

\[
O_{\text{pk}} = \frac{M}{m=1} \sum_{\kappa=1}^{M} \frac{\sum_{p=1}^{U} d_{\text{rad}} / \rho}{d_{\text{rad}}} \]

wherein \( d_{\text{rad}} \) is a vector set created by linear combinations of \( p_{\text{rad}} \) weighted by \( \kappa \), a set of constants formed by linear combinations of a, the standard Gram-Schmidt orthogonalization coefficients, and

\[ \text{SNR}_{\text{rad}} = 10 \log \left( \frac{\text{I}_{\text{rad}}}{\text{I}_{\text{rad}} + \text{I}_{\text{noise}}^2} \right) \]

where \( [B]^{*T} \) is the transpose-complex conjugate of the matrix [B] and \( \text{I}_{\text{rad}} \) is an M member orthogonal basis set;

\[ \tau_{\text{rad}} = \frac{1}{\rho} \sum_{m=1}^{M} \text{I}_{\text{rad}} \]

\[ \kappa = 1, 2, 3, \ldots, \rho \]

and an inverse solution vector \( O_{\text{pk}} \) for a two dimensional restoration wherein:

\[
O_{\text{pk}} = \frac{M}{m=1} \sum_{\kappa=1}^{M} \frac{\sum_{p=1}^{U} d_{\text{rad}} / \rho}{d_{\text{rad}}} \]

\( \kappa = 1, 2, 3, \ldots, \rho \)

14. A programmable apparatus, comprising:
   means for computing; and
   readable memory defining a process for estimating a signal-to-noise ratio for a restored system;
   selecting a set of orthogonal basis set functions \( p_{\text{rad}} \) wherein
   \( m=1, 2, 3, \ldots, \rho \) is the index for the chosen orthogonal basis set with M members of length \( k=1, 2, 3, \ldots, \rho \) to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
   removing time varying distortions in the restored system by obtaining an inverse solution vector \( O_{\text{pk}} \) wherein:

\[
O_{\text{pk}} = \frac{M}{m=1} \sum_{\kappa=1}^{M} \frac{\sum_{p=1}^{U} d_{\text{rad}} / \rho}{d_{\text{rad}}} \]

wherein \( d_{\text{rad}} \) is a vector set created by linear combinations of \( p_{\text{rad}} \) weighted by \( \kappa \), a set of constants formed by linear combinations of a, the standard Gram-Schmidt orthogonalization coefficients, and

\[ [B]^{*T} = \text{SNR}_{\text{rad}} \]
where $[B]^T$ is the transpose-complex conjugate of the matrix $[B]$ and $[p]_{lm}$ is an M member orthogonal basis set.

$$T_m = \frac{C_{mk}}{\sqrt{\sum_{k=1}^{K} C_{mk}^2}} \quad m = i; \quad 5$$

$$T_m = \frac{1.0}{\sqrt{\sum_{k=1}^{K} C_{mk}^2}} \quad m \neq i; \quad i = 1, \ldots, M - 1; \quad 10$$

$$C_{mk} = \frac{\sum_{l=1}^{L} T_{ml} p_{l}}{L} \quad 15$$

$$d_{mk} = \sum_{l=1}^{L} T_{ml} p_{l} \quad m = 1, 2, 3 \ldots M. \quad 15$$

15. A programmable apparatus, comprising:

means for comparing; and

readable memory defining a process for estimating means a signal-to-noise ratio for a restored system;

selecting a set of orthogonal basis set functions $p_{mk}$ to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and

removing time and spatially varying distortions in the restored system by obtaining an inverse solution vector $O_k$ wherein:

$$O_k = \sum_{m=1}^{M} C_{mk} \frac{1}{N} \sum_{l=1}^{N} D_{ml} y^*_l \quad 66$$

wherein $y^*_l$ is a fourier transform of the forward solution and $O_k$ is a fourier transform of the inverse solution and a time domain solution may be obtained by

$$\mathcal{F}^{-1}(O_k) = o_k \quad \text{wherein} \quad [\theta]_{lm} = [F]^T [P]_{lm}$$

and $[F]=[DFT][\theta][IDFT]$ and [$DFT$]=[inverse fourier transform matrices,

$$a. \quad \sum_{k=1}^{K} C_{mk} A_{mk} \quad \sum_{k=1}^{K} C_{mk} C_{mk}^* \quad i = 1, \ldots, M - 1 \quad 25$$

$$m = 1, \ldots, M \quad \text{wherein a is the complex Gram-Schmidt coefficient and vector sets C and A define the characteristics of the system [B] for frequency domain discrete orthogonal basis restoration; \quad 25$$

$$b. \quad T_m = \frac{1.0}{\sqrt{\sum_{k=1}^{K} C_{mk} C_{mk}^*}} \quad m = i \quad 35$$

$$T_m = \frac{1.0}{\sqrt{\sum_{k=1}^{K} C_{mk} C_{mk}^*}} \quad m = i \quad 35$$

$$i = 1, \ldots, M - 1 \quad m = 1, \ldots, M \quad 35$$

where $\tau$ is a constant formed by linear combinations of $a$;

$$c. \quad C_{mk} = \sum_{l=1}^{L} T_{ml} A_{lk} \quad 55$$

and

$$d. \quad D_{mk} = \sum_{l=1}^{L} T_{ml} A_{lk} \quad 55$$

$$m = 1, 2, 3 \ldots M. \quad 55$$

16. A method of using discrete orthogonal basis to restore a signal and/or image system degraded by time and/or spatially varying transfer functions said system being of linear type and represented by the equation $[B] [o] = [i]$ wherein $[o]$ and $[i]$ are length N column and row vectors and $[B]$ is an $N \times N$ non-singular transfer function matrix, comprising:

estimating in estimating means a signal-to-noise ratio for a restored system;

selecting in selecting means a set of orthogonal basis set functions $P_{mk}$ where $m = 1, 2, 3 \ldots M$ is the index for the chosen orthogonal basis set with M members of length $k = 1, 2, 3 \ldots N$ to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and

removing in removing means time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector $O_k$ for a one dimensional restoration wherein:

$$I_{jk} = \sum_{l=1}^{L} C_{jl} \left[ \frac{\sum_{k=1}^{K} T_{ml} p_{l}}{L} \right] \quad \rho = 1, 2, 3 \ldots JU. \quad 45$$

17. A programmable apparatus, comprising:

means for computing; and

readable memory defining a process for estimating a signal-to-noise ratio for a restored system;

selecting a set of orthogonal basis set functions $P_{mk}$ where $m = 1, 2, 3 \ldots M$ is the index for the chosen orthogonal basis set with M members of length $k = 1, 2, 3 \ldots N$ to provide a stable inverse solution based upon the estimated signal-to-noise ratio; and
removing time and/or spatially varying distortions in the restored system by obtaining an inverse solution vector $O_k$ for a one dimensional restoration wherein:

$$O_k = \sum_{m=1}^{M} C_{am} \frac{1}{N} \sum_{k=1}^{N} D_{am} s_k$$

wherein $F_k$ is a fourier transform of the forward solution and $O_k$ is a fourier transform of the inverse solution and a time domain solution may be obtained by

$$\mathcal{F}^{-1}(O_k) = o_k$$

wherein

$$[B] = [F]^* [F]$$

where

$$[F] = [DFT][\theta][IDFT]$$


$$a. \quad C_{am} = \frac{1}{N} \sum_{k=1}^{N} C_{am} s_k$$

$$i = 1, \ldots, m - 1$$

wherein $a$ is the complex Gram-Schmidt coefficient and vector sets $C$ and $A$ define the characteristics of the system $[B]$ for frequency domain discrete orthogonal basis restoration;

$$b. \quad \tau_{am} = \frac{1.0}{\sqrt{\sum_{k=1}^{N} C_{am}^2}}$$

$$m = 1, \ldots, M$$

$$\tau_{am} = \frac{1.0}{\sqrt{\sum_{k=1}^{N} C_{am}^2}} \sum_{j=1}^{m-1} \tau_{aj} a_j$$

$$i = 1, \ldots, m - 1$$

$$m = 1, \ldots, M$$

where $\tau$ is a constant formed by linear combinations of $a$;

$$c. \quad C_{am} = \sum_{i=1}^{m} \tau_{am} a_i$$

and

$$d. \quad D_{am} = \sum_{i=1}^{m} \tau_{am} p_i a_i$$

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