Supplemental Material
for "Skyrmions and Hall Transport"

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Here we drive the general Ward identity, equation (14) of our Letter, in the presence of a uniform external magnetic field B with conserved current J^μ. The momentum density and its conservation equation are modified as in equations (12) and (13). Taking two time derivatives of the retarded Green function $G^{0i,0j}(x^\mu;x'^\mu)=i\theta(x^0-x'^0)\left\langle[T^{0i}(x^\mu),T^{0j}(x'^\mu)]\right\rangle$ gives

$$\begin{aligned}\partial_0\partial_0' G^{0i,0j} &= \partial_n\partial_m' G^{nj,ml} + B\epsilon^j{}_n\partial_0' G^{n,0l} + B\epsilon^l{}_m\partial_0 G^{0j,m} - B^2\epsilon^j{}_n\epsilon^l{}_m G^{n,m}\\ &\quad -\frac{1}{2}(\partial_0-\partial_0')\Big(\delta(x^0-x'^0)C^{0j,0l}\Big)\ ,\end{aligned} \tag{S1}$$

where $G^{a,b}(x^\mu;x'^\mu)=i\theta(x^0-x'^0)\left\langle[V^a(x^\mu),V^b(x'^\mu)]\right\rangle$ are the retarded correlators. The index a can either represent one or two indices, depending on context: e.g., V^a can represent either the momentum density components T^{0i} or a current J^i. Note that the first line in (S1) is a consequence of the conservation equation $\partial_\mu T^{\mu i}=B\epsilon^i_j J^j$, while the second line comes from a contact term with an equal time correlator $C^{a,b}(x^0;\vec{x},\vec{x}')=i\langle[V^a(x^0,\vec{x}),V^b(x^0,\vec{x}')]\rangle$. There are four other terms

$$\frac{1}{2}\delta(x^0-x'^0)\big[\partial_n C^{nj,0l}-\partial_m' C^{0j,ml}+B\left(\epsilon^j{}_n C^{n,0l}-\epsilon^l{}_m C^{0j,m}\right)\big]$$

that vanish in the presence of spacetime translation and rotation symmetries. See Ref. [11] for a discussion of the general case.

Using time translation symmetry, we perform a Fourier transform $\int d(x^0-x'^0)e^{iq_0(x^0-x'^0)}$ on (S1), followed by

$$G^{a,b}(x^0-x'^0;\vec{x},\vec{x}')=\int\frac{dq_0}{2\pi}e^{-iq_0(x^0-x'^0)}\hat{G}^{a,b}(q_0;\vec{x},\vec{x}')$$

using equation (3) and $i[T_B^{0i}(\vec{x}),\mathcal{O}(\vec{x}')]=\partial_i\mathcal{O}(\vec{x})\delta^2(\vec{x}-\vec{x}')$, where T_B is the momentum generator in the presence of the magnetic field given in equation (12). We evaluate the contact term in (S1) as $i\omega C^{0j,0l}=-i\omega\epsilon^{jl}\big[\epsilon^n{}_m\partial_n\left\langle T^{0m}\right\rangle+c-B\rho\big]\delta^2(\vec{x}-\vec{x}')$, where $q_0\equiv\omega$. The first term is incompatible with translation and rotation symmetries, and we discard it. c and $\rho=\left\langle J^0\right\rangle$ are the topological number density and charge density. The Ward identity becomes

$$\begin{aligned}\omega^2\hat{G}^{0j,0l}-i\omega B\epsilon^j{}_n\hat{G}^{n,0l}+i\omega B\epsilon^l{}_m\hat{G}^{0j,m}+B^2\epsilon^j{}_n\epsilon^l{}_m\hat{G}^{n,m}\\ =\partial_n\partial_m'\hat{G}^{nj,ml}-i\omega\epsilon^{jl}(c-B\rho)\delta^2(\vec{x}-\vec{x}')\ .\end{aligned} \tag{S2}$$

Using space translation symmetry, we perform a Fourier transform $\hat{G}^{a,b}(\omega;\vec{x}-\vec{x}')=\int\frac{d^2q}{(2\pi)^2}e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}\widetilde{G}^{a,b}(\omega,\vec{q})$. To extract useful information, we exploit rotational invariance to rewrite the retarded Green's functions in the form

$$\begin{aligned}\widetilde{G}^{a,b}&=\Pi_\delta\delta^{ab}+\Pi_\epsilon\epsilon^{ab}+q^aq^b\Pi_q+(\epsilon^{bo}q^a+\epsilon^{ao}q^b)q_o\Pi_{q\epsilon}\ ,\\ \widetilde{G}^{ij,kl}&=-i\omega\Big[\eta\left(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}-\delta^{ij}\delta^{kl}\right)+\zeta\delta^{ij}\delta^{kl}+\frac{\eta_H}{2}(\epsilon^{ik}\delta^{jl}+\epsilon^{il}\delta^{jk}+\epsilon^{jk}\delta^{il}+\epsilon^{jl}\delta^{ik})\Big]\ .\end{aligned} \tag{S3}$$

Here $\Pi^{TT},\Pi^{TJ},\Pi^{JT},\Pi^{JJ}$ are four different types of form factors, and η,ζ,η_H are the shear, bulk and Hall viscosities, respectively. Plugging them into (S2) and doing some algebra, we arrive at

$$\begin{aligned}&\delta^{jl}\Big[\omega^2\Pi_\delta^{TT}+i\omega B(\Pi_\epsilon^{JT}+\Pi_\epsilon^{TJ})+B^2\Pi_\delta^{JJ}+B^2q^2\Pi_q^{JJ}-i\omega Bq^2(\Pi_{q\epsilon}^{JT}-\Pi_{q\epsilon}^{TJ})\Big]\\ &+\epsilon^{jl}\Big[\omega^2\Pi_\epsilon^{TT}-i\omega B(\Pi_\delta^{JT}+\Pi_\delta^{TJ})+B^2\Pi_\epsilon^{JJ}-\frac{i\omega B}{2}q^2(\Pi_q^{JT}+\Pi_q^{TJ})\Big]\\ &+q^jq^l\Big[\omega^2\Pi_q^{TT}+2i\omega B(\Pi_{q\epsilon}^{JT}-\Pi_{q\epsilon}^{TJ})-B^2\Pi_q^{JJ}\Big]+(\epsilon^{jo}q^l+\epsilon^{lo}q^j)q_o\Big[\omega^2\Pi_{q\epsilon}^{TT}-\frac{i\omega B}{2}(\Pi_q^{JT}-\Pi_q^{TJ})-B^2\Pi_{q\epsilon}^{JJ}\Big]\\ &=\epsilon^{jl}i\omega\big[-c+B\rho-q^2\eta_H\big]-i\omega\delta^{jl}q^2\eta-i\omega q^jq^l\zeta\ .\end{aligned} \tag{S4}$$

We identify $\Pi^{TT}=-i\omega\boldsymbol{\kappa},\Pi^{TJ}=-i\omega\boldsymbol{\alpha},\Pi^{JJ}=-i\omega\boldsymbol{\sigma}$, where $\boldsymbol{\kappa},\boldsymbol{\alpha},\boldsymbol{\sigma}$ are thermal, thermoelectric, and electric conductivity tensors. Matching tensor structures in (S4), we find four independent Ward identities. The resulting equation appears as equation (14).