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The Measure Aspect of Quantum Uncertainty, of Entanglement, and the Associated Entropies

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Abstract: Indeterminacy associated with the probing of a quantum state is commonly expressed through spectral distances (metric) featured in the outcomes of repeated experiments. Here, we express it as an effective amount (measure) of distinct outcomes instead. The resulting *μ-uncertainties* are described by the effective number theory whose central result, the existence of a minimal amount, leads to a well-defined notion of intrinsic irremovable uncertainty. We derive μ-uncertainty formulas for arbitrary set of commuting operators, including the cases with continuous spectra. The associated entropy-like characteristics, the μ-entropies, convey how many degrees of freedom are effectively involved in a given measurement process. In order to construct quantum μ-entropies, we are led to quantum effective numbers designed to count independent, mutually orthogonal states effectively comprising a density matrix. This concept is basis-independent and leads to a measure-based characterization of entanglement.

Keywords: quantum uncertainty; effective number; quantum effective number; effective measure; entropy; quantum entanglement; localization

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1. The Outline

In a stark contrast to its classical counterpart, br quantum mechanics builds the element of uncertainty into its notion of state. Indeed, while a definite entity per se, $|\psi\rangle$ becomes chancy upon probing. How do we usefully characterize $|\psi\rangle$ in terms of its uncertainties?

Questions of this type are as old as quantum mechanics and its Copenhagen interpretation [1]. To address the issue in that vein, consider a prototypical quantum measurement experiment on a system with states in N-dimensional Hilbert space. In particular, the system is repeatedly prepared in state $|\psi\rangle$ and the observable associated with a single non-degenerate Hermitian operator $\hat{O}$ is measured, producing a sequence

$$|\psi\rangle \xrightarrow{\text{measure } \hat{O}} \{\,(\,|i_\ell\rangle, O_{i_\ell}\,) \;\mid\; \ell = 1, 2, \dots\} \tag{1}$$

Here $\{\,(\,|i\rangle, O_i\,) \mid i = 1, 2, \dots, N\}$ is the eigensystem of $\hat{O}$, and $(\,|i_\ell\rangle, O_{i_\ell}\,)$ the outcome of the ℓ-th trial, namely the state into which $|\psi\rangle$ collapsed and the measured value.

The uncertainty of $|\psi\rangle$ with respect to its probing by $\hat{O}$ refers to the indeterminacy associated with the stochastic nature of the sequence $\{(\,|i_\ell\rangle, O_{i_\ell}\,)\}$. This feature is commonly characterized by some form of statistical spread in the encountered eigenvalues, where "spread" has the meaning of separation/distance on the spectrum. We will refer to such characteristics of $|\psi\rangle$ as *metric uncertainties* or ρ-uncertainties. (In addition to these "spectral" ρ-uncertainties, one can also use the metric on the Hilbert space to define "state" ρ-uncertainties. If the inner-product metric is used, the latter formulas are simple and elegant due to the fact that all pairs of distinct orthonormal states are equidistant.) Standard deviation is a popular quantifier of this type since it enters the Heisenberg relations [1,2].

In contrast, our aim here is to characterize quantum uncertainty by its amount/measure. More precisely, we seek characteristics conveying how many distinct $(\,|i\rangle, O_i\,)$ effectively

appear in $\{(\,|\,i_\ell\,\rangle, O_{i_\ell}\,)\}$. The larger such effective number, the larger the uncertainty. We will call such characteristics *measure uncertainties* or μ -uncertainties.

The sequence (1) encodes probabilities p_i of encountering $(\,|\,i\,\rangle, O_i\,)$ in a trial. These then determine the value of a given μ -uncertainty. According to quantum mechanics, the experimental analysis will yield $p_i = |\langle\, i\,|\,\psi\,\rangle|^2$. With this being independent of the eigenvalues, μ -uncertainties are functions of the measurement basis only. On the other hand, ρ -uncertainties depend on the entire eigensystem $\{(\,|\,i\,\rangle, O_i\,)\}$ since any metric quantifier is a function of $\{O_i\}$. Denoting the two types of quantifiers as $\mathcal{N}$ and Δ, respectively, we have

$$\mathcal{N} = \mathcal{N}[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,] \qquad , \qquad \Delta = \Delta[\,|\,\psi\,\rangle, \{(\,|\,i\,\rangle, O_i\,)\}\,] = \Delta[\,|\,\psi\,\rangle, \hat{O}\,] \tag{2}$$

where $\{\,(\,|\,i\,\rangle, O_i\,)\,\}$ fully represents $\hat{O}$. In other words, ρ -uncertainties characterize $|\,\psi\,\rangle$ in relation to measurement operators while μ -uncertainties in relation to measurement bases.

To achieve its intended meaning, a properly constructed μ -uncertainty has to be realized by a function that admits the interpretation as a number of states from $\{\,|\,i\,\rangle\}$ contained in $|\,\psi\,\rangle$. The theory of such objects has been developed in Ref. [3]. In fact, the identity-counting functions $\mathcal{N} = \mathcal{N}[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,]$ constructed there directly correspond to μ -uncertainties.

Theoretical structure leading to these quantifiers formalizes the notion of effective number $\mathcal{N}$ assigned to a collection of N objects endowed with probability weights $P = (p_1, p_2, \ldots, p_N)$. The associated analysis simplifies when $\mathcal{N}$ is equivalently treated as a function of counting weights $C = (c_1, c_2, \ldots, c_N)$ where $c_i = N p_i$. The concept is then represented by the set $\mathfrak{N}$ of all effective number functions (ENFs) $\mathcal{N} = \mathcal{N}[C]$, each realizing one consistent effective counting scheme. The effective number theory of Ref. [3] defines $\mathfrak{N}$ axiomatically and then finds it explicitly (Theorem A1). Thus, all ENFs, and hence all valid μ-uncertainties, are known. A short overview of the ideas underlying the effective number theory and of its main results is given in Appendix A.

A consequential structural feature of $\mathfrak{N}$ (Theorem A2) implies the existence of the minimal ENF which is central to our present purposes. More precisely, the function

$$\mathcal{N}_\star[C] \;=\; \sum_{i=1}^{N} \mathfrak{n}_\star(c_i) \qquad\qquad \mathfrak{n}_\star(c) \;=\; \min\{c, 1\} \tag{3}$$

belongs to $\mathfrak{N}$ and $\mathcal{N}_\star[C] \leq \mathcal{N}[C]$ for all C and all $\mathcal{N}$ from $\mathfrak{N}$. (It is easy to see that if the function with this property exists, it has to be unique. Note also that the function $\mathfrak{n}_\star$ is universal in that it does not depend on N.) In other words, there exists a sharp notion of "minimal amount" for collections of objects with probability weights, realized by $\mathcal{N}_\star$. Hence, there is a sharp notion of minimal quantum μ -uncertainty. In explicit terms,

[U_0] *Let* $C = (c_1, c_2, \ldots, c_N)$, $c_i = N\,|\langle\, i\,|\,\psi\,\rangle|^2$, *be the counting weights associated with quantum state* $|\,\psi\,\rangle$ *and the Hilbert space basis* $\{\,|\,i\,\rangle\} \equiv \{\,|\,i\,\rangle \;|\; i = 1, 2, \ldots, N\,\}$. *The* μ*-uncertainty of* $|\,\psi\,\rangle$ *with respect to* $\{\,|\,i\,\rangle\}$ *is at least* $\mathcal{N}_\star[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,] = \mathcal{N}_\star[C]$ *states.*

Given [U_0], we will refer to $\mathcal{N}_\star[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,]$ as the *intrinsic* μ-uncertainty of $|\,\psi\,\rangle$ with respect to $\{\,|\,i\,\rangle\}$. Indeed, this "uncertainty amount" is inherent to the state since it cannot be lowered or removed by the optional change of a quantifier. Its existence reflects the innate nature of uncertainty in quantum mechanics.

The novelty of the above arises largely due to the inclusion of additivity among the defining properties of ENFs [3]. This step is dictated by the intended measure-like nature of effective numbers. In fact, each $\mathcal{N} \in \mathfrak{N}$ extends the counting measure from ordinary sets to those endowed with probability measures. In Section 3, we will construct quantifiers that play this role for the Jordan content (Lebesgue measure of "regular domains" in $\mathbb{R}^D$). Together with the discrete case, this will cover most situations arising in quantum physics. As an elementary example, μ -uncertainties of a Schrödinger particle with respect

to the position basis are effective volumes. In the spinless case, our analysis implies the intrinsic value

$$\mathcal{V}_\star[\psi] \;=\; \int_\Omega \nu_\star(x)\, d^D x \qquad\qquad \nu_\star(x) = \min\{\, V\psi^\star(x)\psi(x)\,, 1\} \tag{4}$$

Here $\psi(x)$ is the wave function of a particle contained in the region $\Omega \subset \mathbb{R}^D$ of finite volume V. The existence of intrinsic μ-uncertainty implies in this case that quantum particle cannot be associated with the effective volume smaller than $\mathcal{V}_\star[\psi]$.

Quantifying the indeterminacy is sometimes approached via entropy. It is thus of theoretical interest to understand the relations between the measure-like and the entropy-like angles on the concept. Here, we start such discussion by conveying μ-uncertainty in an entropy-like manner, which may find uses in the context of field-theoretic and many-body systems. We proceed in analogy with the original Boltzmann approach in classical statistical mechanics [4], where N accessible states of a priori equal probability generate the entropy $\log N$. In our case, N quantum states with arbitrary probabilities effectively represent $\mathcal{N}$ "accessible" ones, leading to $\log \mathcal{N}$ as a Boltzmann-like characteristic we refer to as the μ-entropy. The effective number theory then implies the existence of minimal μ-entropy associated with state $|\,\psi\,\rangle$ and basis $\{\,|\,i\,\rangle\}$, namely

$$\mathcal{S}_\star[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,] \;=\; \log \mathcal{N}_\star[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,] \;=\; \log \mathcal{N}_\star[C] \tag{5}$$

where C has the meaning specified in [U_0]. The motivation for $\mathcal{S}_\star$ is to express μ-uncertainty as the number of degrees of freedom effectively "active" in the measurement (Section 4).

It is natural to ask in this context whether our measure approach can be applied to quantum entanglement. This is indeed the case and the relevant construction is given in Section 5. It is based on a new elementary notion of *quantum effective number* (Definition 3), which is a basis-independent characteristic of a density matrix, expressing the number of states effectively comprising a mixture. In the effective number methodology this exemplifies a context in which it is necessary to take into account that counted objects may "share content", or be generally correlated in a way affecting the total. The resulting measure-based notion of entanglement (*μ-entanglement*) may provide a useful alternative characterization of entangled states. In addition, we use quantum effective numbers to obtain the quantum version of μ-entropy, which is the analog of von Neumann entropy [5].

Before presenting the details of the above outline, we remark that the results of Ref. [3], extended here, may also find fruitful applications in the general area of localization, both in the original Anderson [6,7] and many-body guises [8]. Characterizing states by their intrinsic μ-uncertainty with respect to the position basis invokes a somewhat unusual perspective on this vast topic. This and other applications of μ-uncertainty and quantum effective numbers will be discussed in dedicated forthcoming publications. In Appendix B, we provide tutorial examples of μ-uncertainty in simple situations, both in the discrete and continuum case.

2. μ-Uncertainty

In this and the next section we develop the theory of μ-uncertainty in detail. To that end, we emphasize at the outset that our aim here is not to question the merits of standard metric approach in the analyses of quantum experiments. Rather, our intent is to point out that there exists a complementary, measure outlook on quantum uncertainty that offers new conceptual insights and a different type of practical use. The example of the former is a surprising existence of uniquely-defined intrinsic uncertainty. The latter can be illustrated by the utility of μ-uncertainty in quantum computation. Indeed, the cost of realizing a quantum algorithm is proportional to the effective number of possible collapsed states in its measurement step [3]. Hence, μ-uncertainty can be used in the associated efficiency analysis.

We start by analyzing quantum uncertainty in a general setting. In fact, the discussion of Section 1 needs to be extended in two ways. The first one involves the inclusion of

probing by multiple and possibly degenerate commuting operators. The second one is concerned with the form of μ-uncertainty in situations that require taking the dimension of Hilbert space to infinity, e.g., when removing the regularization cutoffs.

Thus, rather than the prototypical situation of Section 1, consider the experiment involving D commuting operators assembled into a D-tuple $\hat{\mathbf{O}} \equiv (\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_D)$. It is implicitly understood that the eigensubspace decompositions associated with individual operators are distinct so that the redundant setups, such as $(\hat{x}, \hat{x}^2)$, are avoided. Since $\hat{\mathbf{O}}$ does not necessarily represent a complete system, each combination $\mathbf{O}_m = (O_{1,i_1}, O_{2,i_2}, \ldots, O_{D,i_D}) \in \mathbb{R}^D$ of measured individual eigenvalues specifies the subspace $\mathcal{H}_m$ of the underlying N-dimensional Hilbert space $\mathcal{H}$. Collectively, this leads to a decomposition of $\mathcal{H}$ into M orthogonal subspaces

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_M \qquad , \qquad \sum_{m=1}^{M} \dim \mathcal{H}_m = N \tag{6}$$

The set $\mathfrak{N}$ of effective number functions [3] specifies all consistent μ-uncertainties associated with the above experimental setup. Specifically, we have the following definition.

Definition 1. *Let $|\,\psi\,\rangle \in \mathcal{H}$ and let $|\,\chi_m\,\rangle$ be its (non-normalized) projection into subspace $\mathcal{H}_m$ from orthogonal decomposition (6) specified by $\{\mathcal{H}_m\} \equiv \{\,\mathcal{H}_m \mid m = 1, 2, \ldots, M\,\}$. Let further $C = (c_1, c_2, \ldots, c_M)$, $c_m = M\,\langle\,\chi_m\,|\,\chi_m\,\rangle$, be the collection of associated counting weights, and $\mathcal{N} \in \mathfrak{N}$. We refer to $\mathcal{N}[\,|\,\psi\,\rangle, \{\mathcal{H}_m\}\,] \equiv \mathcal{N}[C]$ as the μ-uncertainty of $|\,\psi\,\rangle$ with respect to $\{\mathcal{H}_m\}$ and the effective number function $\mathcal{N}$.*

If $\hat{\mathbf{O}}$ is a complete set of commuting operators, then $M = N$ and the description in terms of a basis ($\{\mathcal{H}_m\} \to \{\,|\,i\,\rangle\}$), utilized in Section 1 becomes convenient. The arguments resulting in [U_0] also lead to the intrinsic μ-uncertainty limits in this general setting. In particular,

[U] *Let C be the M-tuple of counting weights associated with quantum state $|\,\psi\,\rangle$ and the orthogonal decomposition $\{\mathcal{H}_m\}$ of the underlying Hilbert space. The μ-uncertainty of $|\,\psi\,\rangle$ with respect to $\{\mathcal{H}_m\}$ is at least $\mathcal{N}_\star[C]$ states.*

Albeit starting from the experiment specified by probing operators, measure uncertainty only depends on the associated orthogonal decomposition of the Hilbert space. On the other hand, ρ-uncertainties are fully $\hat{\mathbf{O}}$-dependent. To highlight this, consider $\hat{\mathbf{O}}$ involving individual operators of the same physical dimension. Let $\mathbf{O}_m$ be the D-tuple of eigenvalues associated with subspace $\mathcal{H}_m$, and $p_m = \langle\,\chi_m\,|\,\chi_m\,\rangle$ the probability of $|\,\psi\,\rangle$ collapsing into it upon probing. Expressing the ρ-uncertainty as a standard deviation leads to

$$\Delta^2[\,|\,\psi\,\rangle, \hat{\mathbf{O}}\,] = \sum_{m=1}^{M} p_m\, \rho^2(\mathbf{O}_m\,, \langle\hat{\mathbf{O}}\rangle) \qquad , \qquad \langle\hat{\mathbf{O}}\rangle = \sum_{m=1}^{M} p_m\, \mathbf{O}_m \tag{7}$$

where ρ is a metric of choice on $\mathbb{R}^D$. Thus, while $\mathcal{N} = \mathcal{N}[\,|\,\psi\,\rangle, \{\mathcal{H}_m\}\,]$ for any μ-uncertainty, we have $\Delta = \Delta[\,|\,\psi\,\rangle, \{\mathcal{H}_m, \mathbf{O}_m\}\,] = \Delta[\,|\,\psi\,\rangle, \hat{\mathbf{O}}\,]$ in case of ρ-uncertainties.

The above makes it clear that μ-uncertainties can be viewed as abstract entities which, given a wide variety of possible decompositions $\{\mathcal{H}_m\}$, define a rich collection of characteristics describing $|\,\psi\,\rangle$. They reflect an inherently quantum aspect of the state and have a sharp physical interpretation in terms of quantum experiments. The effective number theory, and [U] in particular, imply that it is meaningful to view $\mathcal{N}_\star[\,|\,\psi\,\rangle, \{\mathcal{H}_m\}\,]$ with varying $\{\mathcal{H}_m\}$ as a complete description of $|\,\psi\,\rangle$ in terms of its μ-uncertainties. It is not known at this time whether a similarly definite structure exists in case of ρ-uncertainties as well.

The above native setup for the theory of μ-uncertainty (finite-dimensional Hilbert space) affords direct applications to many interesting systems, such as those of qbits realizing a quantum computer. However, a transition to infinite case is frequently necessary.

Since $\mathcal{N}$ generically diverges in the process, we will work with the ratio of the effective number to its nominal counterpart, namely the relative μ-uncertainty. More explicitly, consider a regularization procedure involving a sequence of Hilbert spaces $\mathcal{H}^{(k)}$ of growing dimension N_k. At the k-th step of the process, the target state $|\,\psi\,\rangle$ is represented by the vector $|\,\psi^{(k)}\,\rangle$, and the target Hilbert space decomposition $\{\mathcal{H}_m\}$ by the collection $\{\mathcal{H}_m^{(k)}\}$ of M_k subspaces. The relative μ-uncertainty of $|\,\psi\,\rangle$ with respect to $\{\mathcal{H}_m\}$ and $\mathcal{N} \in \mathfrak{N}$ is

$$\mathcal{F}\big[\,|\,\psi\,\rangle,\,\{\mathcal{H}_m\}\,\big] \;\equiv\; \lim_{k\to\infty} \frac{\mathcal{N}\big[\,|\,\psi^{(k)}\,\rangle\,,\{\mathcal{H}_m^{(k)}\}\,\big]}{M_k} \;=\; \lim_{k\to\infty} \frac{\mathcal{N}[\,C_k\,]}{M_k} \tag{8}$$

where C_k is the counting vector associated with $|\,\psi^{(k)}\,\rangle$ and $\{\mathcal{H}_m^{(k)}\}$. Unlike N_k, the number of subspaces M_k does not necessarily grow unbounded in the $k\to\infty$ limit. In fact, the virtue of $\mathcal{F}$ is that it can be used universally: it is applicable to quantum state of arbitrary nature as long as it can be defined via a regularization involving finite-dimensional Hilbert spaces.

3. Continuous Spectra and Effective Uncertainty Volumes

For the purposes of this section, it is convenient to label the subspaces of the Hilbert space decomposition by eigenvalue D-tuples $\mathbf{O}$ of some fixed $\hat{\mathbf{O}}$ generating them as its eigenspaces. Thus, $\mathcal{H}_{\mathbf{O}}$ is the subspace of $\mathcal{H}$ represented by $\mathbf{O} \in \Omega_{\hat{\mathbf{O}}} \subset \mathbb{R}^D$, namely a point in the "spectrum" of $\hat{\mathbf{O}}$. The decomposition itself will be denoted as $\{\mathcal{H}_{\mathbf{O}}\} \equiv \{\mathcal{H}_{\mathbf{O}} \mid \mathbf{O} \in \Omega_{\hat{\mathbf{O}}}\}$.

Upon measurements entailed by the operators in $\hat{\mathbf{O}}$, state $|\,\psi\,\rangle$ undergoes a collapse described by the pair $(\mathcal{H}_{\mathbf{O}}, \mathbf{O})$. While we associated μ-uncertainty with the abundance of distinct $(\mathcal{H}_{\mathbf{O}}, \mathbf{O})$ in repeated experiments, it is also the abundance of $\mathcal{H}_{\mathbf{O}}$ and $\mathbf{O}$ individually because their pairing is one to one. Focusing on $\mathbf{O}$, if the spectra turn continuous upon regularization removal, μ-uncertainty of the target state should thus be expressible in terms of a measure on $\mathbb{R}^D$. In this section, such general expression will be derived.

We use the regularization setup described in connection with the relative μ-uncertainty formula (8), and assume that the spectra of all operators involved in $\hat{\mathbf{O}}^{(k)}$ become continuous in their target $\hat{\mathbf{O}}$. Consider arbitrary $\mathcal{N} \in \mathfrak{N}$ specified by its counting function $\mathfrak{n}$, so that $\mathcal{N}[C] = \sum_i \mathfrak{n}(c_i)$. The corresponding relative μ-uncertainty at k-th regularization step involves the expression

$$\frac{1}{M_k} \sum_{m=1}^{M_k} \mathfrak{n}\big(M_k\, p_{k,m}\big) \;=\; \frac{1}{M_k} \sum_j \sum_{\mathbf{O}_{k,m}\in h^\delta_{\mathbf{o}_j}} \mathfrak{n}\big(M_k\, p_{k,m}\big) \tag{9}$$

where $p_{k,m} = \langle\, \chi_m^{(k)} \mid \chi_m^{(k)}\,\rangle$ with $|\,\chi_m^{(k)}\,\rangle$ the projection of $|\,\psi^{(k)}\,\rangle$ into subspace $\mathcal{H}_{\mathbf{O}_{k,m}}$, i.e., the probability associated with eigenvalue D-tuple $\mathbf{O}_{k,m}$. On the RHS, we introduced a hypercubic grid in $\mathbb{R}^D$ with spacing δ, and grouped individual counts by the elementary hypercube the associated $\mathbf{O}_{k,m}$ falls into ($h^\delta_{\mathbf{o}_j}$ is a hypercube centered at $\mathbf{o}_j$). Note that the j-sum receives non-zero contributions only from hypercubes containing $\mathbf{O}_{k,m}$.

The target relative μ-uncertainty for continuous spectra corresponds to taking $k\to\infty$ followed by $\delta\to 0$ limit of expression (9). Given that each counting function $\mathfrak{n}$ is continuous, and assuming that $\hat{\mathbf{O}}$ is chosen so that the association between $p_{k,m}$ and $\mathbf{O}_{k,m}$ in target $|\,\psi\,\rangle$ becomes expressible via probability density $P = P(\mathbf{o})$ (see below), this limiting procedure is equivalently carried out with

$$\mathcal{F}\big[\,|\,\psi\,\rangle,\,\{\mathcal{H}_{\mathbf{O}}\}\,\big] \;=\; \lim_{\delta\to 0}\lim_{k\to\infty} \sum_j \delta^D\, \frac{M_k^{\mathbf{o}_j,\delta}}{M_k\,\delta^D}\; \mathfrak{n}\left(\frac{M_k\,\delta^D}{M_k^{\mathbf{o}_j,\delta}}\; \frac{\sum\limits_{\mathbf{O}_{k,m}\in h^\delta_{\mathbf{o}_j}} p_{k,m}}{\delta^D} \right) \tag{10}$$

where $M_k^{\mathbf{o}_j,\delta}$ is the number of $\mathbf{O}_{k,m}$ contained in $h^\delta_{\mathbf{o}_j}$. To cast this into a continuous form, we introduce the probability density $P = P(\mathbf{o})$ of encountering $(\mathcal{H}_\mathbf{o}, \mathbf{o})$ in the experiment involving $|\,\psi\,\rangle$ and $\hat{\mathbf{O}}$, as well as the probability density $\pi = \pi(\mathbf{o})$ of $\hat{\mathbf{O}}$-eigenvalue D-tuples

$$P(\mathbf{o}) \;=\; \lim_{\delta\to 0}\lim_{k\to\infty} \frac{\sum\limits_{\mathbf{O}_{k,m}\in h^\delta_\mathbf{o}} p_{k,m}}{\delta^D} \qquad , \qquad \pi(\mathbf{o}) \;=\; \lim_{\delta\to 0}\lim_{k\to\infty} \frac{\sum\limits_{\mathbf{O}_{k,m}\in h^\delta_\mathbf{o}} \frac{1}{M_k}}{\delta^D} \tag{11}$$

Since the sum in the numerator of the latter is $M_k^{\mathbf{o},\delta}/M_k$ we have from (10) that

$$\mathcal{F}\big[\,|\,\psi\,\rangle,\,\{\mathcal{H}_\mathbf{O}\}\,\big] \;=\; \int_{\mathbb{R}^D} d^D\mathbf{o}\;\pi(\mathbf{o})\;\mathfrak{n}\Big(\frac{P(\mathbf{o})}{\pi(\mathbf{o})}\Big) \;=\; \int_{\Omega_{\hat{\mathbf{O}}}} d^D\mathbf{O}\;\pi(\mathbf{O})\;\mathfrak{n}\Big(\frac{P(\mathbf{O})}{\pi(\mathbf{O})}\Big) \tag{12}$$

where the spectral support $\Omega_{\hat{\mathbf{O}}} \subset \mathbb{R}^D$ of $\hat{\mathbf{O}}$ is defined by $\pi(\mathbf{o}) \neq 0$. The integrand vanishes at $\mathbf{o} \notin \Omega_{\hat{\mathbf{O}}}$ since each $\mathfrak{n}$ is bounded, leading to the restriction of the integral to $\Omega_{\hat{\mathbf{O}}}$. Note that we have distinguished the generic variable $\mathbf{o}$ parametrizing entire $\mathbb{R}^D$ from the spectral variable $\mathbf{O}$ labeling the actual continuum of subspaces. Via standard manipulations, one can (formally) write $P(\mathbf{O}) = \langle\,\chi(\mathbf{O})\,|\,\chi(\mathbf{O})\,\rangle$ with $|\,\chi(\mathbf{O})\,\rangle$ the projection of $|\,\psi\,\rangle$ into $\mathcal{H}_\mathbf{O}$.

Several comments regarding the formula (12) are important to make.

(i) Recall that in discrete case we have identified μ-uncertainties with effective number functions $\mathcal{N}\in\mathfrak{N}$. However, in the continuum, where effective number generically loses its direct meaning (diverges), this correspondence becomes facilitated by counting functions $\mathfrak{n}$ of Theorem 1 in Ref. [3]. Thus, in full detail we have $\mathcal{F} = \mathcal{F}[\,|\,\psi\,\rangle,\,\{\mathcal{H}_\mathbf{O}\},\,\mathfrak{n}\,]$ but the last dependence will remain implicit in what follows.

(ii) Since relative μ-uncertainty depends on the Hilbert space decomposition $\{\mathcal{H}_\mathbf{O}\}$ but not on a particular $\hat{\mathbf{O}}$ associated with it, formula (12) should reflect this invariance. To see it, consider relabeling the subspaces $\{\mathcal{H}_\mathbf{O}\}$ as $\{\mathcal{H}_{\mathbf{O}'}\}$, where $\mathbf{O} = f(\mathbf{O}')$ is a one-to-one differentiable map. This defines D-tuple of new operators $\hat{\mathbf{O}}'$, and the associated transformed probability densities P' and π'. The change of variables then confirms

$$\int_{\Omega_{\hat{\mathbf{O}}}} d^D\mathbf{O}\;\pi(\mathbf{O})\;\mathfrak{n}\Big(\frac{P(\mathbf{O})}{\pi(\mathbf{O})}\Big) \;=\; \int_{\Omega_{\hat{\mathbf{O}}'}} d^D\mathbf{O}'\;\pi'(\mathbf{O}')\;\mathfrak{n}\Big(\frac{P'(\mathbf{O}')}{\pi'(\mathbf{O}')}\Big) \tag{13}$$

(iii) How does the additivity, carefully enforced in the regularization process, explicitly translate into Equation (12)? Consider the partition of the spectral support $\Omega \equiv \Omega_{\hat{\mathbf{O}}}$ into subregions Ω_1 and Ω_2, thus specifying both the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the underlying Hilbert space, as well as the operators $\hat{\mathbf{O}}_1$, $\hat{\mathbf{O}}_2$ acting on them, i.e., $\Omega_i \equiv \Omega_{\hat{\mathbf{O}}_i}$. Moreover, spectral probability densities π_i on Ω_i descend from π via

$$\pi_i(\mathbf{O}) = \frac{1}{F_i}\,\pi(\mathbf{O}) \qquad \mathbf{O}\in\Omega_i \qquad , \qquad F_i = \int_{\Omega_i} d^D\mathbf{O}\,\pi(\mathbf{O}) \qquad (F_1 + F_2 = 1) \tag{14}$$

Extending the concatenation notation of Ref [3] to this continuous case, we have equivalently

$$\pi(\mathbf{O}) \;=\; [\,F_1\pi_1 \boxplus F_2\pi_2\,](\mathbf{O}) \;\equiv\; \begin{cases} F_1\,\pi_1(\mathbf{O})\,, & \mathbf{O}\in\Omega_1 \\ F_2\,\pi_2(\mathbf{O})\,, & \mathbf{O}\in\Omega_2 \end{cases} \tag{15}$$

From (12) it then directly follows that

$$\begin{aligned} \mathcal{F}\big[\,|\,\psi\,\rangle,\,\{\mathcal{H}_\mathbf{O}\}\,\big] \;&=\; \mathcal{F}\Big[\,\sqrt{F_1}\,|\,\psi_1\,\rangle \boxplus \sqrt{F_2}\,|\,\psi_2\,\rangle,\,\{\mathcal{H}_{\mathbf{O}_1}\}\cup\{\mathcal{H}_{\mathbf{O}_2}\}\,\Big] \;= \\ &=\; F_1\,\mathcal{F}\Big[\,|\,\psi_1\,\rangle,\,\{\mathcal{H}_{\mathbf{O}_1}\}\,\Big] \;+\; F_2\,\mathcal{F}\Big[\,|\,\psi_2\,\rangle,\,\{\mathcal{H}_{\mathbf{O}_2}\}\,\Big] \quad , \quad \forall\;\; |\,\psi\,\rangle\in\mathcal{H} \end{aligned} \tag{16}$$

where $\boxplus$ was also extended to the elements of mutually orthogonal Hilbert spaces in an obvious manner ($|\psi\rangle = \sqrt{F_1}\,|\psi_1\rangle \boxplus \sqrt{F_2}\,|\psi_2\rangle$), and $\{\mathcal{H}_{\mathbf{O}_i}\}$ is the decomposition of $\mathcal{H}_i$ associated with $\hat{\mathbf{O}}_i$. Relation (16) is precisely the one for composing two fractions of distinct amounts into that of a combined amount ($\mathcal{F} = F_1\,\mathcal{F}_1 + F_2\,\mathcal{F}_2$), and is an equivalent representation of additivity. In terms of probability distributions involved, this reads

$$\begin{aligned}\mathcal{F}\big[\,P,\,\pi\,\big] &= \mathcal{F}\big[\,F_1P_1 \boxplus F_2P_2,\,F_1\pi_1 \boxplus F_2\pi_2\,\big] = \\ &= F_1\,\mathcal{F}\big[\,P_1\,,\pi_1\,\big] + F_2\,\mathcal{F}\big[\,P_2\,,\pi_2\,\big] \quad , \quad \forall\, P \text{ on } \Omega_{\hat{\mathbf{O}}}\end{aligned} \tag{17}$$

(iv) [U] and (12) lead to the notion of minimal μ-uncertainty in the context of continuous spectra. In particular, with the above definitions and notation in place, we have

[U$_c$] *Let $\hat{\mathbf{O}}$ be a D-tuple of Hermitean operators on $\mathcal{H}$ with continuous spectra, and $\{\mathcal{H}_{\mathbf{O}}\}$, $\Omega_{\hat{\mathbf{O}}}$, $\pi = \pi(\mathbf{O})$ the associated spectral characteristics. There exists a minimal relative μ-uncertainty of states from $\mathcal{H}$ with respect to $\{\mathcal{H}_{\mathbf{O}}\}$, assigned by*

$$\mathcal{F}_\star\big[\,|\psi\rangle,\,\{\mathcal{H}_{\mathbf{O}}\}\,\big] \;=\; \int_{\Omega_{\hat{\mathbf{O}}}} d^D\mathbf{O}\;\min\{\pi(\mathbf{O}),P(\mathbf{O})\} \tag{18}$$

where $P = P(\mathbf{O})$ is the probability density of obtaining $\mathbf{O}$ in $\hat{\mathbf{O}}$-measurements of $|\psi\rangle$.

(v) Important special case of formula (12) arises for uniform $\pi(\mathbf{O})$. Among other things, this setting applies to several relevant operators, such as those of position and momentum in quantum mechanics. Thus, let $\Omega_{\hat{\mathbf{O}}}$ occupy a finite volume $V_{\hat{\mathbf{O}}}$ in $\mathbb{R}^D$. A unique feature of uniform $\pi(\mathbf{O}) = 1/V_{\hat{\mathbf{O}}}$ is that the effective fraction of states, quantified by $\mathcal{F}$, also expresses the effective fraction of spectral volume in this case. Indeed, uniformity at the regularized level implies that distinct subspaces represent non-overlapping elementary volumes, and the ratio $\mathcal{N}/N$ becomes the effective volume fraction in the continuum limit. Thus, it is meaningful in this case to define μ-uncertainty (rather than relative μ-uncertainty) and interpret it as the *effective spectral volume*. In particular, from (12) we obtain

$$\mathcal{V}\big[\,|\psi\rangle,\,\{\mathcal{H}_{\mathbf{O}}\}\,\big] \;\equiv\; V_{\hat{\mathbf{O}}}\,\mathcal{F} \;=\; \int_{\Omega_{\hat{\mathbf{O}}}} d^D\mathbf{O}\;\mathfrak{n}\big(V_{\hat{\mathbf{O}}}P(\mathbf{O})\big) \qquad , \qquad V_{\hat{\mathbf{O}}} = \int_{\Omega_{\hat{\mathbf{O}}}} d^D\mathbf{O} \tag{19}$$

Note that the μ-uncertainty of Schrödinger particle with respect to the position basis Equation (4) is a special case of this general relationship.

(vi) The results of this section entail a notable mathematical corollary. Thus, leaving the realm of quantum mechanics for the moment, consider $\Omega \subset \mathbb{R}^D$ with well-defined non-zero Jordan content (ordinary volume), i.e., $0 < \int_\Omega d^Dx = V < \infty$. (Speaking of Jordan content simply means that $\int$ is understood to denote the Riemann integral.) Can we extend the meaning of Jordan content so that, in addition to Ω itself, the volume is assigned to any pair (Ω, P), where $P = P(x)$ is a continuous probability distribution on Ω? The effective number theory [3] provides a positive answer to this question, and Equation (12) the corresponding prescription. Indeed, introducing a Riemann partition of Ω and the associated discrete probability distribution descended from $P(x)$, effective volume fraction associated with counting function $\mathfrak{n}$ can be evaluated. Adopting any sequence of Riemann refinements producing V, one obtains a result that can be read off directly from Equation (12). The conversion from $\mathcal{F}$ to effective volume $\mathcal{V} = V\mathcal{F}$ then leads to the analogue of (19), namely

$$\mathcal{V}[\Omega,P] \;=\; \int_\Omega d^Dx\;\mathfrak{n}\big(VP(x)\big) \;\geq\; \int_\Omega d^Dx\;\min\{VP(x)\,,1\} \;=\; \mathcal{V}_\star[\Omega,P] \tag{20}$$

Here, the first equality specifies all consistent effective volume assignments (labeled by $\mathfrak{n}$). The inequality, valid for all P and all $\mathfrak{n}$, expresses the existence of *minimal effective*

volume quantifier specified by $\mathfrak{n}_\star$ and guaranteed to play this role by Theorem 2 of Ref. [3].

(vii) Finally, consider the case involving both continuous and discrete operators. Thus, let the D-tuple $\hat{\mathbf{O}}$ contain $D_c < D$ operators $\hat{\mathbf{O}}_c$ with continuous spectra upon regularization removal. Expression (12) for relative μ-uncertainty then generalizes into

$$\mathcal{F}\big[\,|\psi\rangle,\,\{\mathcal{H}_{m,\mathbf{O}}\}\,\big] \;=\; \int_{\Omega_{\hat{\mathbf{O}}_c}} d^{D_c}\mathbf{O}\;\sum_{m=1}^{M}\pi_m(\mathbf{O})\;\mathfrak{n}\left(\frac{P_m(\mathbf{O})}{\pi_m(\mathbf{O})}\right) \tag{21}$$

Here, $\mathbf{O}\in\mathbb{R}^{D_c}$ and π_m, P_m are associated with $\mathbf{O}_m\in\mathbb{R}^{D-D_c}$ whose components are discrete target eigenvalues. Note that $\int d^{D_c}\mathbf{O}\;\sum_m \pi_m(\mathbf{O})=1$, and similarly for P_m.

4. μ-Entropy

Following upon our opening discussion in Section 1, we now introduce μ-entropies with the aim of providing a useful alternative way to express μ-uncertainty in systems with many degrees of freedom. Similarly to the familiar cases of Shannon [9] and Rényi [10] entropies, it is convenient to build the primary concept in a discrete setting. The following definition is generic in the sense that it is concerned with the objects of arbitrary nature.

Definition 2. *Let N objects be assigned probabilities (relevance weights) $P=(p_1,\ldots,p_N)$. If $\mathcal{N}\in\mathfrak{N}$ is an effective number function, then $\mathcal{N}[NP]$ defines the μ-uncertainty and*

$$\mathcal{S}[P] \;\equiv\; \log\mathcal{N}[NP] \tag{22}$$

the associated μ-entropy of this collection with respect to $\mathcal{N}$.

As with μ-uncertainties, of prime interest is the minimal μ-entropy, namely

$$\mathcal{S}_\star[P] \;=\; \log\sum_{i=1}^{N}\min\{Np_i,1\} \tag{23}$$

A few points are worth elaborating upon here.

(i) The indeterminacy expressed by $\mathcal{N}[NP]$ can be viewed as the "uncertainty of choice". Indeed, the choice of N equivalent objects is effectively reduced to $\mathcal{N}[NP]$ by virtue of their varied relevance. This motivates a generic interpretation of μ-entropy as the entropy of choice. In the case of quantum measurement, "choice" takes the form of an outcome.

(ii) Unlike the Shannon and Rényi cases, the entropic additivity is not built into μ-entropies. Indeed, the additivity of effective numbers and the entropic additivity have very different roots and motivations. However, similarly to Tsallis entropy [11], this may not preclude its usefulness, even in the context of statistical physics. While the related issues will be studied in a dedicated account, here we point out the corresponding relation for the family of μ-entropies

$$\mathcal{S}_{(\alpha)}[P] \;=\; \log\mathcal{N}_{(\alpha)}[NP] \;=\; \log\sum_{i=1}^{N}\min\{(Np_i)^\alpha,1\} \quad,\quad 0<\alpha\le 1 \tag{24}$$

where $\mathcal{N}_{(\alpha)}\in\mathfrak{N}$ are the canonical ENF representatives introduced in [3]. In particular

$$\mathcal{S}_{(\alpha)}[P\boxtimes Q] \;\ge\; \mathcal{S}_{(\alpha)}[P]+\mathcal{S}_{(\alpha)}[Q] \quad,\quad \forall\,\alpha\,,\,P\,,\,Q \tag{25}$$

as can be shown directly from the corresponding definitions. Here, if $P=(p_1,\ldots,p_N)$ and $Q=(q_1,\ldots,q_M)$, then $P\boxtimes Q$ is the product distribution with probability entries p_iq_j.

(iii) μ-entropy can be used to assess the number of degrees of freedom that become "active" in a given measurement experiment. While this type of role is not foreign to entropies

in general, μ-entropies are based on a proper count of accessible states. Consider a generic situation with K quantum degrees of freedom. Viewed individually, each of them is described by a state in s-dimensional Hilbert space so that the dimension of full state space $\mathcal{H}$ is $N = s^K$. This nominal freedom is generically reduced when analyzing state $|\,\psi\,\rangle$ with respect to a given orthonormal basis $\{\,|\,i\,\rangle\}$ since the probability acquired by $|\,i\,\rangle$ affects its accessibility. To count the effectively accessible states, certain effective number function $\mathcal{N}$ has to be fixed and used for all states and bases. The resulting reduction in states can be viewed as the reduction in the "active" degrees of freedom. We thus define the $\mathcal{N}$-equivalent K_{eq} by

$$N \;=\; s^K \quad\longrightarrow\quad \mathcal{N}[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,] \;=\; s^{K_{eq}[\,|\psi\rangle, \{\,|i\rangle\}\,]} \tag{26}$$

The convenience of μ-entropy is that it directly reflects this relationship. For example, the $\mathcal{N}$-equivalent degree of freedom density is

$$k_{eq}[P] \;=\; \frac{K_{eq}[P]}{K} \;=\; \frac{\mathcal{S}[P]}{\mathcal{S}[P_u]} \qquad , \qquad 0 \le k_{eq} \le 1 \tag{27}$$

where P is the probability distribution associated with $|\,\psi\,\rangle$ and $\{\,|\,i\,\rangle\}$, and P_u the uniform distribution. When the dimension N of the Hilbert space grows unbounded, such as in the process of regularization removal that involves adding the degrees of freedom, it is useful to characterize this growth by the asymptotic power $\mathcal{N}[P_N] \propto N^\alpha$ for $N \to \infty$. Then

$$k_{eq}[P_N] \;=\; \frac{\log \mathcal{N}[P_N]}{\log N} \quad \xrightarrow[\mathcal{N}[P_N] \propto N^\alpha]{N \longrightarrow \infty} \quad \alpha \qquad , \qquad 0 \le \alpha \le 1 \tag{28}$$

The range of α arises due to the fact that $\mathcal{N}$ can grow at most linearly with N.

5. Quantum Effective Numbers, Quantum μ-Entropy and μ-Entanglement

Similarly to naturals, effective numbers were constructed to characterize collections of objects acting as autonomous wholes, i.e., not sharing "parts" with one other. This aspect is generic in situations where counting is normally considered to make sense. Thus, we were justified to use effective numbers to count the states of orthonormal basis, or the subspaces from the orthogonal decomposition of the Hilbert space. Incidentally, these autonomous objects play a crucial role in quantum measurement process, and thus the uncertainty.

When the boundaries between objects become fuzzy and/or their contents can be shared in some manner, counting has to be modified, if at all possible, to accommodate the commonality. In the quantum context, situations of this type arise when inquiring about the state content of a density matrix. Here, we do not mean the abundance of elements from arbitrary fixed basis. (The answer to that question, namely $\sum_{i=1}^{N} \mathfrak{n}(q_i)$ where $q_i = \sum_{j=1}^{J} p_j \,|\langle\, i\,|\,\psi_j\,\rangle|^2$, represents the μ-uncertainty of $\hat{\rho}$ with respect to basis $\{\,|\,i\,\rangle\,\}$, and involves only a direct application of effective counting). Rather, we are interested in a basis-independent characteristic specifying the number of independent states effectively participating in the mixture. Thus, consider a density matrix $\hat{\rho}$, namely

$$\hat{\rho} \;=\; \sum_{j=1}^{J} p_j \,|\,\psi_j\rangle\langle\psi_j\,| \tag{29}$$

where the number J of distinct states $|\,\psi_j\rangle$ from N-dimensional Hilbert space is arbitrary. Recalling that each effective number function $\mathcal{N}$ is uniquely associated with its counting function $\mathfrak{n}$ so that $\mathcal{N}[C] = \sum_{i=1}^{N} \mathfrak{n}(c_i)$ (Theorem 1 of Ref. [3]), we define *quantum effective numbers* associated with $\hat{\rho}$ as follows.

Definition 3. *Let $\hat{\rho}$ be $N \times N$ density matrix and $\mathfrak{n}$ a counting function. Then*

$$\mathcal{Q}[\hat{\rho}, \mathfrak{n}] \equiv \sum_{i=1}^{N} \mathfrak{n}(N\rho_i) = \operatorname{tr} \mathfrak{n}(N\hat{\rho}) \qquad \textit{where} \qquad \hat{\rho}\,|\,i\,\rangle = \rho_i\,|\,i\,\rangle \tag{30}$$

will be referred to as the quantum effective number of $\hat{\rho}$ with respect to $\mathfrak{n}$.

The rationale for the above construct is quite clear. States $|\,\psi_j\rangle$ in definition (29) cannot be directly counted since they are not necessarily orthogonal. However, equivalently expressing $\hat{\rho}$ in terms of its eigenstates gives the latter the role of autonomous components to which effective counting applies. From the mathematical standpoint, the connection between effective numbers and their quantum counterparts is analogous to that of Shannon [9] and von Neumann entropies [5]. To avoid confusion, we emphasize that $\mathcal{Q}$ is not a μ-uncertainty and we do not refer to it such. Rather, it is an useful object that allows us to define quantum μ-entropy and μ-entanglement (see below).

Several comments regarding $\mathcal{Q}$ are important to make.

(i) Quantum effective numbers can be introduced as a well-motivated extension of ordinary effective numbers, as done here, or as an axiomatic construct of its own. Without going into details, we note that the key property of exact additivity, required to be satisfied by $\mathcal{Q}$, concerns combining density matrices defined in mutually orthogonal Hilbert subspaces. Definition 3 manifestly accommodates this feature.

(ii) The notion of minimal effective number applies also to its quantum version. In particular, it follows from Theorem 2 of Ref. [3] that

$$\mathcal{Q}_\star[\hat{\rho}\,] \equiv \mathcal{Q}[\hat{\rho}, \mathfrak{n}_\star] = \sum_{i=1}^{N} \min\{N\rho_i, 1\} \leq \mathcal{Q}[\hat{\rho}, \mathfrak{n}\,] \qquad , \qquad \forall\,\hat{\rho}\,,\ \forall\,\mathfrak{n} \tag{31}$$

Hence, the same reasons that give $\mathcal{N}_\star$ its absolute meaning in case of ordinary effective counting, apply to $\mathcal{Q}_\star$ in the quantum case.

(iii) Quantum effective numbers allow us to express a degree of entanglement between parts of the system as an effective number of states. Thus, given a bipartite system specified by $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, state $|\,\psi\rangle \in \mathcal{H}$, and the associated density matrix $\hat{\rho} = |\,\psi\rangle\langle\psi\,|$, we define

$$\mathcal{Q}^{(e)}[\,|\,\psi\rangle, A, \mathfrak{n}\,] \equiv \mathcal{Q}[\hat{\rho}_A, \mathfrak{n}\,] \qquad , \qquad \hat{\rho}_A = \operatorname{tr}_B \hat{\rho} \tag{32}$$

and refer to $\mathcal{Q}^{(e)}$ as *μ-entanglement* of $|\,\psi\rangle$ with respect to partition specified by A and the counting function $\mathfrak{n}$. Note that $\mathcal{Q}^{(e)}[\,|\,\psi\rangle, A, \mathfrak{n}\,] = \mathcal{Q}^{(e)}[\,|\,\psi\rangle, B, \mathfrak{n}\,]$ by virtue of the Schmidt decomposition argument. The notion of minimal μ-entanglement $\mathcal{Q}^{(e)}_\star[\,|\,\psi\rangle, A\,]$ follows.

(iv) The quantum μ-entropy, namely the μ-entropy associated with a density matrix, is

$$\mathcal{S}[\hat{\rho}, \mathfrak{n}] \equiv \log \mathcal{Q}[\hat{\rho}, \mathfrak{n}] \qquad \text{and} \qquad \mathcal{S}_\star[\hat{\rho}\,] \equiv \log \mathcal{Q}_\star[\hat{\rho}\,] \tag{33}$$

where $\mathcal{S}_\star$ is the minimal entropy quantifier. Similarly to its classical counterpart, the utility of $\mathcal{S}_\star$ is mainly envisioned in many body and field theory applications. The concept of μ-entanglement can be equivalently based on quantum μ-entropy in analogy with the standard quantum information approach to entanglement using von Neumann entropy. In the same way, the general entanglement-related construct of quantum mutual information has a counterpart in the measure-based notion of mutual "state content", which can also be equivalently treated in terms of quantum μ-entropy (33).

6. The Summary

In this work, we proposed and analyzed the approach to quantum uncertainty that characterizes it as an effective total of possible measurement outcomes (μ-uncertainty). Unlike in the case of conventional spectral metric approach (ρ -uncertainty), the mathematical theory governing μ-uncertainties exists. It is the effective number theory of Ref. [3], which implies that there is an amount $\mathcal{N}_\star$ of μ-uncertainty, associated with each quantum state and type of measurement, that cannot be reduced by using a different μ-uncertainty quantifier. Hence, this minimal amount is intrinsic to a quantum situation at hand. Statements [U_0], [U] and [U_c] convey this in various generic contexts of interest.

The conclusion that uncertainty is encoded by quantum formalism at such a basic level via the universal quantifier $\mathcal{N}_\star$ is interesting conceptually. Moreover, its unique explicit form $\mathcal{N}_\star$ is useful from a practical standpoint. In that regard, it is also useful to recall the proposal to characterize state $|\,\psi\,\rangle$ by all $\mathcal{N}_\star[\,|\,\psi\,\rangle, \{\,|\,i\,\rangle\}\,]$, i.e., by the number of basis states from $\{\,|\,i\,\rangle\}$ that $|\,\psi\,\rangle$ effectively resides in, for all bases $\{\,|\,i\,\rangle\}$ [3]. The present discussion casts that into describing $|\,\psi\,\rangle$ by all of its intrinsic μ-uncertainties. This viewpoint gives uncertainty a privileged role in the description of quantum state indeed.

While obvious from our discussion, it may be worth pointing out that Heisenberg relations and statements of minimal μ -uncertainty ([U_0], [U] and [U_c]) offer very different kinds of insight into the nature of quantum uncertainty. Indeed, while Heisenberg relation infers certain minimum which is associated with a pair of incompatible operators and universal with respect to the state involved in simplest cases, the intrinsic μ -uncertainties are the minima associated with each state individually and universal with respect to the operators sharing the same basis. Clearly, more can be said along these lines, both qualitatively and quantitatively, once μ -uncertainties become utilized more fully.

A significant portion of the present work entailed deriving μ-uncertainty expressions in situations where the measurement setup entails an orthogonal decomposition of the Hilbert space labeled by continuous spectral parameters. In particular, Formulas (12) and (21) are the results of the regularization cutoff removals, needed in such cases. The latter represents the most general form of μ-uncertainty, applicable to arbitrary Hilbert space and any of its decompositions specified by a set of commuting Hermitean operators.

It is worth emphasizing that the treatment of uncertainty as a measure became possible by virtue of extending ordinary counting (counting measure) into effective counting (effective counting measure) [3]. Our treatment of continuous spectra here similarly corresponds to extending the notion of Jordan content in $\mathbb{R}^D$ (ordinary volume) to effective Jordan content (effective volume), as expressed by Equation (20). The resulting approach may have uses in applied mathematics, e.g., as a suitable way to define the effective support of a function.

The concept of effective numbers naturally leads to the auxiliary notion of μ-entropy. In the context of quantum states, its motivation mainly relates to convenience in dealing with exponentially growing Hilbert spaces of many-body physics. Working with entropy translates into considering the equivalent number of degrees of freedom and their density, Equations (26) and (27). This approach may be useful in the analysis of thermalization (see, e.g., [8] for a relevant review).

In order to construct quantum μ-entropy and a measure-based approach to quantum entanglement, we have shown how to use effective numbers to analyze the state content of density matrices (Definition 3). A suitable extension is necessary since the states specifying the matrix may not be independent (mutually orthogonal). As is obvious from its intended meaning and the resulting formula (30), this *quantum effective number* is a basis-independent concept. Among other things, it allows us to express quantum entanglement as the effective number of states "generated" in the Hilbert space of one bipartite component due to the influence of the other (32). Substantially more can be said about the ensuing approach to entanglement and to quantum entropy (33), with a dedicated account forthcoming.

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Appendix A. Effective Numbers

In this Appendix we present a short overview of effective number theory [3] which is central to this work. More detailed introductory account is given in Ref. [12].

Consider a collection of N objects whose varied relevance is expressed by means of their probabilities $P = (p_1, p_2, \ldots, p_N)$ or, equivalently, their counting weights $C = (c_1, c_2, \ldots, c_N)$ where $c_i = Np_i$. Effective number theory aims to assign an effective total (count) to all such collections. Any prescription that faithfully accomplishes this is called an effective counting scheme. Associating collections with their weight vectors, specifying a counting scheme becomes equivalent to defining a function $\mathcal{N} = \mathcal{N}[C]$. The domain $\mathcal{W}$ of these effective number functions (ENFs) contains all possible counting vectors of all lengths N.

Given the above setup, the requirements imposed on a valid counting scheme translate into conditions satisfied by ENFs. First, $\mathcal{N}$ has to be a symmetric function of its arguments (condition (S)) which expresses the fact that the effective number cannot change upon reshuffling of objects in the collection. Secondly, $\mathcal{N}$ has to be continuous (C) so that a gradual change of weights does not result in a jump of effective total. Next, there are conditions of boundary type. In particular, when all weights are the same ($c_1 = c_2 = \ldots = c_N = 1$), and thus all objects matter equally, then $\mathcal{N} = N$ as in ordinary counting. On the other hand, when all weight is cumulated in a single object ($c_i = N$ for some i) then $\mathcal{N} = 1$ since the rest of objects do not matter at all. All other assignments must fall between these extremal values, i.e., $1 \leq \mathcal{N} \leq N$. The above requirements are referred to as (B1), (B2) and (B), respectively.

The remaining two conditions, namely the monotonicity (M^-) and additivity (A) shape effective counting schemes in a crucial manner. The former implements an important feature, shared with entropies, that a degree of weight cumulation in the distribution controls the direction of the assignments. In particular, given any two collections of N objects, the one with more cumulated weights cannot be assigned a larger effective number. The ensuing (M^-) monotonicity of ENFs is expressed by

$$\mathcal{N}(\ldots c_i - \epsilon \ldots c_j + \epsilon \ldots) \leq \mathcal{N}(\ldots c_i \ldots c_j \ldots) \tag{A1}$$

for all $c_i \leq c_j$ and $0 \leq \epsilon \leq c_i$. Finally, a key condition that gives the effective counting scheme its measure-like meaning is the additivity formulated as follows. Consider a collection of N objects with weights C and a collection of M other objects with weights B. ENT requires that the operation of merging ($N + M$ objects) results in a collection whose effective total equals the sum of totals assigned to these parts. The associated (A) property of ENFs reads

$$\mathcal{N}\big[C \boxplus B\big] \;=\; \mathcal{N}\big[C\big] \;+\; \mathcal{N}\big[B\big] \quad , \quad \forall\, C, B \in \mathcal{W} \tag{A2}$$

Here the symbol $\boxplus$ represents the operation of concatenation, namely if $C = (c_1, \ldots, c_N)$ and $B = (b_1, \ldots, b_M)$, then $C \boxplus B \equiv (c_1, \ldots, c_N, b_1, \ldots, b_M)$.

With the above in place, the set of all possible effective counting schemes is represented by the $\mathfrak{N}$ of all functions satisfying the conditions (S), (C), (B1), (B2), (B), (M^-) and (A).

Effective number theory then proceeds to explicitly find all elements of $\mathfrak{N}$, namely all ENFs. The following statement specifies the result [3].

Theorem A1. *Function $\mathcal{N}$ on $\mathcal{W}$ belongs to $\mathfrak{N}$ if and only if there exists a real-valued function $\mathfrak{n} = \mathfrak{n}(c)$ on $[0,\infty)$ that is concave, continuous, $\mathfrak{n}(0) = 0$, $\mathfrak{n}(c) = 1$ for $c \geq 1$, and*

$$\mathcal{N}[C] \;=\; \sum_{i=1}^{N} \mathfrak{n}(c_i) \quad , \quad \forall\, C = (c_1, c_2, \ldots, c_N) \in \mathcal{W} \quad , \quad \forall\, N \tag{A3}$$

Such a function $\mathfrak{n}$ associated with $\mathcal{N} \in \mathfrak{N}$ is unique.

Thus, fixing an effective counting scheme amounts to selecting a function $\mathfrak{n} = \mathfrak{n}(c)$ with properties specified by Theorem A1, and assigning the effective number to each collections of objects via prescription (A3).

A consequential feature revealed by effective number theory is the existence of a minimal counting scheme. To state the associated full result of Ref. [3], we need to define the function

$$\mathcal{N}_+[C] = \sum_{i=1}^{N} \mathfrak{n}_+(c_i) \quad , \quad \mathfrak{n}_+(c) \;=\; \begin{cases} 0\,, & c = 0 \\ \\ 1\,, & c > 0 \end{cases} \tag{A4}$$

which is not an ENF but represents a useful limiting case. Then

Theorem A2. *Let $\mathcal{N}_\star \in \mathfrak{N}$ and $\mathcal{N}_+ \notin \mathfrak{N}$ be functions defined by (3) and (A4), respectively. Then*

$$(a)\;\; \mathcal{N}_\star[C] \;\leq\; \mathcal{N}[C] \;\leq\; \mathcal{N}_+[C] \quad , \quad \forall\, \mathcal{N} \in \mathfrak{N} \;,\; \forall\, C \in \mathcal{W}$$

$$(b)\;\; \big\{\, \mathcal{N}[C] \;\big|\; \mathcal{N} \in \mathfrak{N} \,\big\} \;=\; [\,\alpha, \beta\,] \quad , \quad \alpha = \mathcal{N}_\star[C] \;,\; \beta = \mathcal{N}_+[C] \;,\; \forall\, C \in \mathcal{W}$$

Apart from the fact that it implies the existence of a lower bound on the effective number that can be assigned to a collection, the essential part (a) is that this bound is facilitated by a function that is an ENF itself ($\mathcal{N}_\star$). This means that the minimal amount is inherent to the concept of effective number which, in turn, leads to to a well-defined notion of intrinsic μ-uncertainty.

Appendix B. Measure Uncertainty in Simple Systems

In this Appendix we illustrate our notion of μ-uncertainty on elementary examples. Starting with the discrete case, consider a generic $N = 2$ system in state $|\,\chi\,\rangle$. What is the intrinsic μ-uncertainty of $|\,\chi\,\rangle$ with respect to the basis $\{\,|\,1\,\rangle, |\,2\,\rangle\}$?

Following the general prescription, we first represent $|\,\chi\,\rangle \;=\; \chi_1\;|\,1\,\rangle + \chi_2\;|\,2\,\rangle$ in this basis. The associated probability and counting vectors can then be expressed as $P = (p, 1-p)$ and $C = (2p, 2-2p)$, respectively, where $p \equiv p_1 = \chi_1^\star \chi_1 \in [0,1]$. From [$U_0$] and the defining Equation (3) we then obtain the intrinsic μ-uncertainty

$$\mathcal{N}_\star[\,|\,\chi\,\rangle, \{\,|\,1\,\rangle, |\,2\,\rangle\}\,] \;=\; \mathcal{N}_\star(p) \;=\; \begin{cases} 1 + 2p & \text{for} \quad 0 \leq p \leq 1/2 \\ 3 - 2p & \text{for} \quad 1/2 < p \leq 1 \end{cases} \quad \text{states} \tag{A5}$$

Notice that $\mathcal{N}_\star(p)$ is symmetric with respect to $p = 1/2$ as one expects, and is by construction continuous. It specifies the intrinsic μ-uncertainty associated with any type of measurement performed on $|\,\chi\,\rangle$ that results in its collapse into $|\,1\,\rangle$ or $|\,2\,\rangle$. When $p = 0$, then $|\,\chi\,\rangle \propto |\,2\,\rangle$ and the μ-uncertainty takes the classical-like value $\mathcal{N}_\star(0) = 1$ state (no collapse/no uncertainty). The maximal μ-uncertainty arises for $p = 1/2$, when basis states are equally represented in $|\,\chi\,\rangle$, and takes the value $\mathcal{N}_\star(1/2) = 2$ states. The other values of intrinsic μ-uncertainty are between these extremes. The significance of *intrinsic* μ-uncertainty in this context is that $\mathcal{N}_\star(p) \;\leq\; \mathcal{N}(p)$ for all p and for all other possible μ-uncertainties $\mathcal{N}$.

We now put the above in contrast to ρ-uncertainties of $|\chi\rangle$. The latter requires specifying the values O_1, O_2 of the measured observable associated with $|1\rangle$ and $|2\rangle$ or, in other words, the operator $\hat{O} = O_1 |1\rangle\langle 1| + O_2 |2\rangle\langle 2|$ to which the ρ-uncertainty $\Delta = \Delta[|\chi\rangle, \hat{O}]$ refers to. This leads to $\Delta = \Delta(p, O_1, O_2)$, namely

$$\Delta^2[|\chi\rangle, \hat{O}] \;=\; \langle\, (\hat{O} - \langle \hat{O} \rangle_{|\chi\rangle})^2 \,\rangle_{|\chi\rangle} \;=\; (O_1 - O_2)^2\, p\,(1-p) \quad \text{(units of } \hat{O})^2 \tag{A6}$$

Here $\langle \ldots \rangle_{|\chi\rangle}$ denotes a mean value in state $|\chi\rangle$.

Elementary results (A5) and (A6) readily illustrate some of the key points concerning the introduced notion of μ-uncertainty. (i) The concept expresses the indeterminacy associated with the statistical pattern involved in quantum-mechanical collapse of $|\chi\rangle$ into $|1\rangle$ or $|2\rangle$. This pattern is common to all measurement experiments represented by non-degenerate operators $\hat{O}$ with eigenvectors $|1\rangle$ and $|2\rangle$. Hence, the expression (A5) is independent of eigenvalues O_1, O_2 and dimensionless. (ii) On the other hand, ρ-uncertainties express how a collapse pattern translates into indeterminacy in one specific quantity $\hat{O}$. As illustrated by (A6), this is necessarily dependent on O_1, O_2 and typically dimensionfull. (iii) Among virtues of intrinsic μ-uncertainty is that it is more universal (in the above sense) and that it is *unique*. Indeed, while there is no fundamental argument against using, e.g., $\Delta/2$ instead of Δ as a quantifier of ρ-uncertainty, effective number theory implies that $\mathcal{N}_\star$ cannot be modified in any way. (iv) The difference in the nature of p-dependence in expressions (A5) and (A6) illustrates that μ-uncertainties indeed provide a very different characterization of quantum indeterminacy than ρ-uncertainties. This of course stems from the fact that the former is based on measure while the latter on metric.

As an elementary example of intrinsic μ-uncertainty in case of continuous spectra, we evaluate it for exponentially decaying wave function on interval $[-R, R]$ and the position basis. Thus, consider the wave function $\psi(x) \propto \exp(-|x|/2\sigma)$ which entails the eigenvalue probability distribution (Such $\psi(x)$ may, for example, be thought of as an approximate ground state of a particle in δ-function potential well in the middle of the interval, with a suitably chosen particle mass and the potential strength.)

$$P(x) = \frac{1}{2\sigma}\, \frac{1}{1 - e^{-R/\sigma}}\, e^{-|x|/\sigma} \tag{A7}$$

The spectral interval itself is clearly uniformly populated and so $\pi(x) = 1/2R$. Inserting this $P(x)$ and $\pi(x)$ into the master formula (12) with $\mathfrak{n} = \mathfrak{n}_\star$, or alternatively using $[\mathrm{U}_c]$ directly (see (18)) one obtains an intrinsic relative μ-uncertainty

$$\mathcal{F}_\star[\psi, \{|x\rangle\}] \;=\; \mathcal{F}_\star(R, \sigma) \;=\; \frac{1}{\mathcal{R}} \left[\log \frac{\mathcal{R}}{1 - e^{-\mathcal{R}}} + 1 + \frac{\mathcal{R}}{1 - e^{-\mathcal{R}}}\, e^{-\mathcal{R}} \right] \quad , \quad \mathcal{R} = R/\sigma \tag{A8}$$

where the terms are ordered in decreasing relevance when in the regime $R \gg \sigma$.

Few basic features of this result may be instructive to point out explicitly. (i) $\mathcal{F}_\star(R, \sigma)$ is a dimensionless characteristic (1-d volume fraction), and one can readily verify in this explicit result that $0 \leq \mathcal{F}_\star \leq 1$. (ii) Note that when $R/\sigma \to 0$ then $\mathcal{F}_\star \to 1$ which is natural since all available positions become equally likely to appear as a result of a measurement. On the other hand, when $R/\sigma \to \infty$, the exponentially decaying probability causes strong suppression of states that can appear, and $\mathcal{F}_\star \to 0$. (iii) One may wish to represent this intrinsic μ-uncertainty as a dimensionful effective 1-d volume, namely $\mathcal{V}_\star = 2R\,\mathcal{F}_\star$. Note that for $R \to \infty$ (fixed σ) the leading contribution to effective volume is $\mathcal{V}_\star \sim 2\sigma \log R/\sigma$. Thus, the linear increase of available eigenvalues and basis states with growing R is only reflected by the logarithmic increase of intrinsic μ-uncertainty due to the exponential suppression of their relevance in the wave function.

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