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Theoretical Study on $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$

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Theoretical study on $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$

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Abstract

The η' meson is associated with the U(1) anomaly. In this paper, a successful effective chiral theory of mesons has been applied to study the anomalous decays of $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$. Contribution of triangle and box diagrams is calculated, which indicates that the box anomaly has a significant contribution to the decay amplitudes. We predict branching ratios of $Br(\eta' \to \pi^+\pi^-\pi^+\pi^-) = \frac{1}{2}Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) = (8.3 \pm 1.2) \times 10^{-5}$, which is in good agreement with BESIII measurement.

1. Introduction

Anomaly is a fantastic phenomenon of quantum field theory. There are Adler-Bell-Jakiew triangle anomaly [1,2] and Chanowitz box anomaly, too [3,4]. On the other hand, the anomalies are described by anomalous Lagrangian: Wess-Zumino-Witten [5,6], Ö. Kaymakcalan et al. [7] and K.C. Chou et al. [8]. It is well known that the η' meson is associated with the U(1) anomaly [9,10]. Besides the anomalous meson processes, there are normal meson processes, for example, $\rho \to \pi\pi$, $\eta' \to \eta\pi\pi$, etc. Chiral Perturbation Theory (ChPT) is used to describe those meson processes with normal parity [11–14].

In the CLEO experiment [15] upper limits are set on the branching ratios for $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$ and in the BESIII experiment [16], the decay modes are observed and branch-

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ing ratios are determined to be $Br(\eta' \to \pi^+\pi^-\pi^+\pi^-) = [8.53 \pm 0.69(stat.) \pm 0.64(syst.)] \times 10^{-5}$ and $Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) = [1.82 \pm 0.35(stat.) \pm 0.18(syst.)] \times 10^{-4}$.

Guo et al. [17] performed a theoretical study of the η' decay into four pions, based on a combination of ChPT and vector meson dominance (VMD). In their work, the decay amplitudes are dominated by the triangle anomaly and predicted results for the branching fractions of $Br(\eta' \to \pi^+\pi^-\pi^+\pi^-) = (1.0 \pm 0.3) \times 10^{-4}$ and $Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) = (2.4 \pm 0.7) \times 10^{-4}$, are in agreement with the experiment.

In Refs. [18,19] an effective chiral theory of pseudoscalar, vector and axial-vector mesons has been proposed. This effective theory has been successfully applied to study different meson processes [18–30]. For example, meson decays with normal parity like $a_1 \to \rho\pi$, $\phi \to K^+K^-$, $\eta' \to \eta\pi^+\pi^-$, $k^\star(892) \to K\pi$, $K_1(1400) \to K^\star(892)\pi$, etc., and anomalous decays $\pi^0 \to \gamma\gamma$, $\eta^{(')} \to \gamma\gamma$, $\eta' \to \rho\gamma$, $\phi \to \eta\gamma$, $\omega \to \pi\pi\pi$, etc., have been evaluated.

In this paper, we use the effective chiral theory [18,19] to study the anomalous decays $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$. We calculate the amplitude of triangle ($\eta' \to \rho\rho$, $\rho \to \pi\pi$) and box ($\eta' \to \rho\pi\pi$, $\rho \to \pi\pi$) diagrams. Based on our results, the box anomaly has a significant contribution to the decay amplitudes. Our theoretical predictions are in good agreement with the experiment [15, 16].

This work is organized as follows: In section 2, we briefly review the effective chiral theory of mesons which has been applied in this paper. In section 3, we calculate $\eta' \to \pi^+\pi^-\pi^+\pi^-$ branching ratio by evaluating triangle and box anomalies, and with the use of isospin relation we connect this branching ratio to $\eta' \to \pi^+\pi^-\pi^0\pi^0$. We summarize our results in section 4.

2. Review of the effective chiral theory

It is well known that the current algebra is very successful in the study of hadron physics. Based on current algebra and large N_C expansion of QCD, the Lagrangian of $U(3)_L \times U(3)_R$ chiral field theory of quarks and mesons (0^{-+}, 1^{--} and 1^{++}) has been constructed [18,19]

$$
\begin{aligned}
\mathcal{L} =& \bar{\psi}\{i\gamma\cdot\partial + \gamma\cdot v + eQ\gamma\cdot A + \gamma\cdot a\gamma_5 - mu\}\psi \\
&+ \frac{1}{2}m_0^2(\rho_i^\mu\rho_{\mu i} + \omega^\mu\omega_\mu + a_i^\mu a_{\mu i} + f^\mu f_\mu) \\
&+ \frac{1}{2}m_0^2(K_\mu^{*a}\bar{K}^{*a\mu} + K_1^\mu K_{1\mu}) + m_0^2(\phi^\mu\phi_\mu + f_s^\mu f_{s\mu})
\end{aligned}
\tag{1}
$$

with

$$
\begin{aligned}
u &= exp^{\{i\gamma_5(\pi+K+\eta_8+\eta_0)\}} \\
a_\mu &= \tau_i a_\mu^i + \lambda_a K_{1\mu}^a + (\frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8)f_\mu + (\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8)f_{s\mu} \\
v_\mu &= \tau_i \rho_\mu^i + \lambda_a K_\mu^{*a} + (\frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8)\omega_\mu + (\frac{1}{3} - \frac{1}{\sqrt{3}}\lambda_8)\phi_\mu
\end{aligned}
\tag{2}
$$

where $i = 1, 2, 3$ and $a = 4, 5, 6, 7$. The ψ in Eq. (1) is u, d, s quark fields. The scheme of nonlinear σ-model is used to introduce pseudoscalar mesons into Eq. (1) and parameter m is originated in quark condensation and it leads to the dynamical chiral symmetry breaking. In this Lagrangian (Eq. (1)) meson fields are coupled to the corresponding quark field bilinears. The η_8

and η_0 are octet and singlet, respectively. By assuming the mixing angle $\theta = -20$, η and η' fields are defined as

$$\begin{aligned} \eta &= cos\theta\ \eta_8 - sin\theta\ \eta_0 \\ \eta' &= sin\theta\ \eta_8 + cos\theta\ \eta_0 \end{aligned} \tag{3}$$

Mesons are bound state solutions of QCD and are not independent degrees of freedom. Thus, in Eq. (1) there are no kinetic terms for meson fields. The kinetic terms of the meson fields are generated from quark loops. This theory is an effective theory, therefore, a cut-off is necessary to be introduced [31,32]. In the chiral limit $m_q \to 0$, the cut-off Λ is defined [18]

$$\frac{N_c}{(4\pi)^2}\left\{ln(1+\frac{\Lambda^2}{m^2}) + \frac{1}{1+\frac{\Lambda^2}{m^2}} - 1\right\} = \frac{1}{16}\frac{F^2}{m^2} \tag{4}$$

By normalizing the kinetic terms of pion, η' and ρ fields, physical meson fields are defined [18,19]

$$\frac{2}{f_\pi}\pi \to \pi_{physical}\ , \quad \frac{2\sqrt{2}}{f_\pi}\eta' \to \eta'_{physical}\ , \quad \frac{1}{g}\rho \to \rho_{physical} \tag{5}$$

f_π is the pion decay constant and g is a universal coupling constant which are defined as

$$f_\pi = F^2\big(1 - \frac{2c}{g}\big), \qquad g^2 = \frac{F^2}{6m^2}, \qquad c = \frac{f_\pi^2}{2gm_\rho^2} \tag{6}$$

f_π and g are two inputs and $f_\pi = 186$ MeV and $g = 0.395$ are taken. Thus, the cut-off is determined to be $\Lambda \sim 1.8$ GeV. All the masses of mesons are below the cut-off and the theory is self-consistent. The input values of f_π and g are chosen such that the theory fits the experimental data for different meson processes [20–30].

As shown in Refs. [18,19] the VMD is a natural result of this meson theory instead of an input. According to Sakurai [33], the VMD is revealed from a Lagrangian in which photon and vector mesons are coupled to quarks symmetrically. These symmetries are shown in the Lagrangian (1), in which the photon field is added. At the fourth order in covariant derivative, the $\rho - \gamma$ vertex is derived from the quark vertex of photon and ρ meson [18]

$$\mathcal{L} = -\frac{1}{4}eg(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu \rho_\nu - \partial_\nu \rho_\mu), \tag{7}$$

where A is the photon field. By using Eq. (7) the $\Gamma^{th}(\rho \to e^+e^-) = 6.75$ keV is predicted. The experimental value, as quoted by particle data group (PDG), is $\Gamma^{exp}(\rho \to e^+e^-) = 7.04(6)$ keV [34]. There are similar terms for $\omega - \gamma$ and $\phi - \gamma$ vertex [18,19]. By employing the VMD of $\rho - \gamma$, $\omega - \gamma$, and $\phi - \gamma$, the pion form factor, the form factors of the charged and the neutral kaons are obtained [27,30].

In this theory the quark loop is always of order N_C. The meson loops are at higher order in N_c expansion and f_π and g are both of order $\sqrt{N_C}$ [18,19]. The meson physics studied are at the leading order of N_C expansion. In the chiral limit ($m_q \to 0$), the theory is explicitly chiral symmetric. In this limit, f_π and g are two parameters.

Many anomalous processes of mesons have been studied by using this theory. The ABJ anomaly $\pi^0 \to \gamma\gamma$ [1,2] is obtained from both the vertices $\pi\omega\rho$ (WZW anomaly [5,6]) and VMD $\rho - \gamma$ and $\omega - \gamma$ [18]. Other meson anomalies such as; $\eta^{(\prime)} \to \gamma\gamma,\ \eta' \to \omega\gamma,\ \rho \to \eta\gamma,\ \omega \to \eta\gamma,$

etc., have been studied in Refs. [18,19]. These theoretical results are in good agreement with the experimental data.

In this effective chiral theory [18,19] meson resonances are involved. As pointed out in Refs. [11,35,36], the coupling constants of effective chiral Lagrangian for strong interactions are essentially saturated by meson resonance exchange. In this regard, the anomalous decay $\omega \to 3\pi$ is very interesting. In this decay $\omega \to \rho + \pi,\ \rho \to \pi\pi$ and direct $\omega \to 3\pi$ are involved. The amplitude derived in Ref. [18] is the same as the one derived by Ö. Kaymakcalan et al. [4]. The vertex $\omega \to \rho + \pi$ is from the triangle diagram of quarks and the direct vertex $\omega \to 3\pi$ is from the box diagram, which are both in low energies. The vector meson resonance is not involved in the box diagram and as it is shown in Ref. [18], the contribution of the box diagram is small. To be more precise, the contribution of box anomaly to the branching ratio $Br(\omega \to 3\pi)$ is only 5% [18].

At low energies ($E < m_\rho$) the theory (Eq. (1)) goes back to ChPT and the 10 coefficients of ChPT are determined [24,29]. This theory has already been applied to study various aspects of meson physics [18–30] and theoretical predictions are in agreement with the experimental data.

3. Calculation of $\eta' \to \pi^+\pi^-\pi^+\pi^-$ and $\eta' \to \pi^+\pi^-\pi^0\pi^0$

In this paper, the effective chiral theory of mesons [18,19] is applied to study the anomalous decays $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$. Before doing this study it is interesting to mention that the decay width of $\eta' \to \eta\pi\pi$ with normal parity is computed in Ref. [19].

In this process, there are two types of anomalies: 1) triangle anomaly [1,2] of two body decay $\eta' \to \rho\rho$, $\rho \to \pi\pi$; 2) box anomaly [3,4] of three body decay $\eta' \to \rho\pi\pi$, $\rho \to \pi\pi$. The vertex $\eta' \to \rho\rho$ comes from the triangle diagram of quarks and the vertex $\eta' \to \rho\pi\pi$ comes from the box diagram, shown in Fig. 1. The box anomaly proposed by Chanowitz has been applied to study the three body decay of $\eta' \to \pi^+\pi^-\gamma$ in Ref. [37].

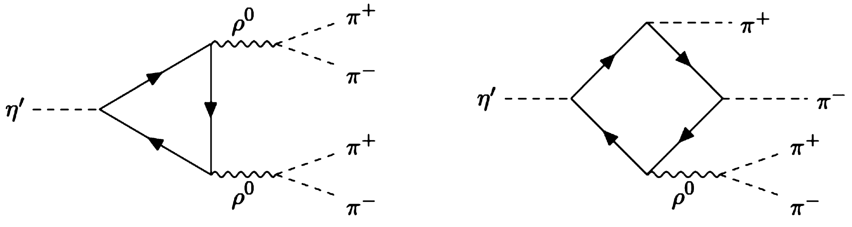

Fig. 1. Feynman diagrams for triangle and box anomalies contributing to $\eta' \to \pi^+\pi^-\pi^+\pi^-$. Feynman diagrams were generated using TikZ-Feynman [38].

In the theory [18,19], the pion and η' fields have two sources: one from the term u of the Lagrangian (Eq. (1)), and the other from the shift caused by the mixing between the axial-vector and pseudoscalar fields

$$\begin{aligned} a^i_\mu &\to \frac{1}{g}(1-\frac{1}{2\pi^2 g^2})^{-\frac{1}{2}} a^i_\mu - \frac{c}{g}\frac{2}{f_\pi}\partial_\mu \pi^i, \\ f_\mu &\to \frac{1}{g}(1-\frac{1}{2\pi^2 g^2})^{-\frac{1}{2}} f_\mu - \frac{c}{g}\frac{2\sqrt{2}}{f_\pi}\partial_\mu \eta'. \end{aligned} \tag{8}$$

Equations (8) are the result of mixing a_μ and $\partial_\mu \pi^i$, and f_μ and $\partial_\mu \eta'$, which are generated by corresponding quark loop diagrams. The constant $\frac{1}{g}(1-\frac{1}{2\pi^2 g^2})^{-\frac{1}{2}}$ is the normalization constant of the a^i_μ and f_μ fields.

In Ref. [19] the triangle anomaly of the η' is expressed as

$$\begin{aligned}\mathcal{L}_{\eta' vv} =& \frac{N_C}{(4\pi)^2}\frac{4}{g^2}\frac{2\sqrt{2}}{f_\pi}\varepsilon^{\mu\nu\alpha\beta}\eta'\Big\{(\sqrt{\frac{2}{3}}cos\theta+\frac{1}{\sqrt{3}}sin\theta)(\partial_\mu\rho^i_\nu\partial_\alpha\rho^i_\beta+\partial_\mu\omega_\nu\partial_\alpha\omega_\beta)+\\ &(\sqrt{\frac{2}{3}}cos\theta-\frac{2}{\sqrt{3}}sin\theta)\partial_\mu\phi_\nu\partial_\alpha\phi_\beta\Big\},\end{aligned} \tag{9}$$

where the θ is the mixing angle.

The $\eta' \to \gamma\gamma;\ \rho\gamma;\ \omega\gamma$ anomalous decay modes have been studied by using this Lagrangian (9). The theory agrees with experimental data without a new parameter [19].

The triangle anomaly $\eta' \to \rho^0\rho^0, \rho^0 \to \pi^+\pi^-$ is calculated by applying Lagrangian (9) and the amplitude is expressed as Eq. (10)

$$\begin{aligned}T(1) =& \frac{N_C}{(4\pi)^2}\frac{2\sqrt{2}}{f_\pi}\frac{32}{g^2}(\sqrt{\frac{2}{3}}cos\theta+\frac{1}{\sqrt{3}}sin\theta)m_{\eta'}\epsilon_{ijk}k_{1i}k_{2j}k_{3k}\\ &\Big\{\frac{f_{\rho\pi\pi}(q_1^2)}{q_1^2-m_\rho^2+i\sqrt{q_1^2}\Gamma_\rho(q_1^2)}\frac{f_{\rho\pi\pi}(q_2^2)}{q_2^2-m_\rho^2+i\sqrt{q_2^2}\Gamma_\rho(q_2^2)}\\ &-\frac{f_{\rho\pi\pi}(q_3^2)}{q_3^2-m_\rho^2+i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}\frac{f_{\rho\pi\pi}(q_4^2)}{q_4^2-m_\rho^2+i\sqrt{q_4^2}\Gamma_\rho(q_4^2)}\Big\},\end{aligned} \tag{10}$$

where $q_1 = k_1 + k_2;\ q_2 = k_3 + k_4;\ q_3 = k_1 + k_4;\ q_4 = k_2 + k_3$ and $k_i,\ \ i = 1, 2, 3, 4$ are the momentum of the four pions, respectively, and

$$\begin{aligned}f_{\rho\pi\pi} &= \frac{2}{g}\Big\{1+\frac{q^2}{2\pi^2 f_\pi^2}\big[(1-\frac{2c}{g})^2-4\pi^2c^2\big]\Big\},\\ \Gamma(\rho\to\pi\pi) &= \frac{f^2_{\rho\pi\pi}(q^2)}{48\pi}\sqrt{q^2}\big(1-\frac{4m_\pi^2}{q^2}\big)^{\frac{3}{2}},\end{aligned} \tag{11}$$

where q is the momentum of the ρ meson. Taking $q^2 = m_\rho^2$, and using $m_\rho = 775.26$ MeV as quoted by PDG [34], $\Gamma_\rho = 146$ MeV is obtained.

Besides the two body decay channel, the decay $\eta' \to \pi^+\pi^-\pi^+\pi^-$ have the channel of three body decay $\eta' \to \rho\pi\pi$. As we mentioned previously, in the study $\omega \to 3\pi$ both the triangle, $\omega \to \rho\pi,\ \rho \to 2\pi$ and the box diagram $\omega \to 3\pi$ are included and the contribution of the box diagram is very small [18]. But, for the $\eta' \to \rho + 2\pi,\ \rho \to 2\pi$, the ρ-resonance boosts the contribution of box diagram [11,35,36]. Thus, the box anomaly [3,4] $\eta' \to \rho^0\pi^+\pi^-,\ \rho^0 \to \pi^+\pi^-$ must be included.

According to Eqs. (1), (8), pions and η' are coupled to pseudoscalar and axial-vector currents and ρ^0 is coupled to vector current. Thus, by permuting the final states in the box diagram $\eta' \to \rho^0\pi^+\pi^-$, we can see that there are 48 box diagrams. The effective Lagrangian of all box diagrams is obtained as

$$\mathcal{L}_{\eta'\rho\pi\pi} = \frac{1}{g}\frac{8\sqrt{2}}{f_\pi^3}\frac{N_C}{3\pi^2}(\frac{1}{2}-\frac{2c}{g}+\frac{c^2}{g^2})\varepsilon^{\mu\nu\alpha\beta}\eta'(\sqrt{\frac{2}{3}}cos\theta+\sqrt{\frac{1}{3}}sin\theta)\partial_\mu\rho^0_\nu\partial_\alpha\pi^+\partial_\beta\pi^-. \tag{12}$$

The amplitude of the box diagrams is calculated by $\rho^0 \to \pi^+\pi^-$

$$\begin{aligned}T(2) =&\frac{1}{g}\frac{8\sqrt{2}}{f_\pi^3}\frac{N_C}{3\pi^2}(\sqrt{\frac{2}{3}}cos\theta+\sqrt{\frac{1}{3}}sin\theta)(\frac{1}{2}-\frac{2c}{g}+\frac{c^2}{g^2})\\ &m_{\eta'}\epsilon_{ijk}k_{1i}k_{2j}k_{3k}\Big\{\frac{f_{\rho\pi\pi}(q_1^2)}{q_1^2-m_\rho^2+i\sqrt{q_1^2}\Gamma_\rho(q_1^2)}+\frac{f_{\rho\pi\pi}(q_2^2)}{q_2^2-m_\rho^2+i\sqrt{q_2^2}\Gamma_\rho(q_2^2)}\\ &-\frac{f_{\rho\pi\pi}(q_3^2)}{q_3^2-m_\rho^2+i\sqrt{q_3^2}\Gamma_\rho(q_3^2)}-\frac{f_{\rho\pi\pi}(q_4^2)}{q_4^2-m_\rho^2+i\sqrt{q_4^2}\Gamma_\rho(q_4^2)}\Big\},\end{aligned} \tag{13}$$

where $q_1 = k_1 + k_2,\ q_2 = k_3 + k_4,\ q_3 = k_1 + k_4,\ q_4 = k_2 + k_3$.

Adding both the triangle and the box diagrams (Eqs. (10), (13)) the total amplitude of the decay $\eta' \to \pi^+\pi^-\pi^+\pi^-$ can be obtained.

The $\eta' \to \pi^+\pi^-\pi^0\pi^0$ is another decay mode and have been measured in [15,16]. The decay mode $\eta' \to \pi^+\pi^-\pi^+\pi^-$ is related to the $\eta' \to \pi^+\pi^-\pi^0\pi^0$ mode by isospin relation. For the triangle diagrams, these decays are from the term

$$\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\rho_\nu^i\partial_\alpha\rho_\beta^i. \tag{14}$$

Ignoring the Lorentz indices we have

$$\begin{aligned}&\rho^i\rho^i = 2\rho^+\rho^- + \rho^0\rho^0,\\ &\rho^+\rho^- \to \pi^+\pi^-2\pi^0,\\ &\rho^0\rho^0 \to 2\pi^+2\pi^-.\end{aligned} \tag{15}$$

For the box diagrams, the related term can be written as

$$\varepsilon^{\mu\nu\alpha\beta}\epsilon_{ijk}\partial_\mu\rho_\nu^i\partial_\alpha\pi^j\partial_\beta\pi^k. \tag{16}$$

The isospin structure of Eq. (16) is the same as the $\rho\rho$ (one ρ decays to two pions) of Eq. (14). Taking the properties of identical particles (the mass difference between the charged and neutral pions is considered in our uncertainty estimate) this isospin structure predicts

$$Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) = 2Br(\eta' \to \pi^+\pi^-\pi^+\pi^-). \tag{17}$$

To obtain the branching ratios, we insert the numerical values listed below:

1) Pion decay constant: $\frac{1}{\sqrt{2}}f_\pi = 131.5(1.0)$ MeV is taken, where the uncertainty comes from the difference between the input value of f_π and the PDG experimental value [34].

2) Universal coupling constant: $g = 0.395(8)$ with the uncertainty is assigned to make our prediction of the decay rate $\Gamma^{th}(\rho \to e^-e^+) = 6.75(29)$ keV to be in agreement with the PDG value $\Gamma^{exp}(\rho \to e^-e^+) = 7.04(6)$ keV within 1 σ [34].

3) Weighted average pion mass $M_\pi = (3M_{\pi^+} + M_{\pi^0})/4 = 138.4(1.2)$ MeV. To account for isospin breaking effects due to the phase space corrections, we consider the difference between the charged pion mass and weighted average value as uncertainty.

4) Total width of the η': $\Gamma_{\eta'} = 0.196$ MeV [34]. To obtain branching ratios we normalize partial widths by this value.

In Table 1, we summarized our predictions for $\eta' \to \pi^+\pi^-\pi^{+(0)}\pi^{-(0)}$, together with the results of the other theoretical study [17], and the experimental values [16].

Table 1
Comparison of our predictions with those from the other model, and also the experiment.

	$Br(\eta' \to \pi^+\pi^-\pi^+\pi^-) \times 10^{-5}$	$Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) \times 10^{-5}$
experiment [16]	8.53(0.69)(0.64)	18.2(3.5)(1.8)
GKW [17]	10(3)	24(7)
This work (triangle diagrams)	4.2(0.6)	8.3(1.1)
This work (triangle and box diagrams)	8.3(1.2)	16.6(2.4)

The uncertainty of our prediction comes from the uncertainties of the input parameters of the theory, f_π and g, and the weighted average pion mass, which are combined in quadratic. From Table 1 we can see that the contribution of the triangle diagrams (Eq. (10)) is smaller than the experiment [16] and the agreement becomes excellent when the box diagrams are included. In reference [17] the amplitudes are dominated by the triangle anomaly term.

Of course, there is pentagon diagram for the decay $\eta' \to \pi^+\pi^-\pi^+\pi^-$. It has been shown that the coupling constants of effective chiral Lagrangian for strong interactions are saturated by meson resonance exchange [11,35,36]. There is no ρ-resonance in the pentagon diagram and it is believed that its contribution to this decay mode is small. Thus, calculation of pentagon diagrams is not presented in this work. We are certain that the error made thereby is well below our uncertainty estimate.

4. Summary

An effective chiral theory has been applied to study the anomalous decays $\eta' \to \pi^+\pi^- \times \pi^{+(0)}\pi^{-(0)}$. In this work, we have evaluated the triangle and box diagrams and shown that the box anomaly has a significant contribution to the decay amplitudes. Our theoretical predictions: $Br(\eta' \to \pi^+\pi^-\pi^+\pi^-) = \frac{1}{2}Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) = (8.3 \pm 1.2) \times 10^{-5}$, are in good agreement with the experimental data: $Br(\eta' \to \pi^+\pi^-\pi^+\pi^-) = [8.53 \pm 0.69(stat.) \pm 0.64(syst.)] \times 10^{-5}$ and $Br(\eta' \to \pi^+\pi^-\pi^0\pi^0) = [1.82 \pm 0.35(stat.) \pm 0.18(syst.)] \times 10^{-4}$.

CRediT authorship contribution statement

Ehsan Jafari: Conceptualization, Formal analysis, Software, Writing – review & editing. **Bing An Li:** Conceptualization, Methodology, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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