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ALGEBRAIC PROPERTIES OF EDGE IDEALS

Rachelle R. Bouchat
University of Kentucky, rbouchat@ms.uky.edu

ABSTRACT OF DISSERTATION

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A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Rachelle R. Bouchat
Lexington, Kentucky

Director: Dr. Uwe Nagel, Professor of Mathematics
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ALGEBRAIC PROPERTIES OF EDGE IDEALS

Given a simple graph G, the corresponding edge ideal I_G is the ideal generated by the edges of G. In 2007, Hà and Van Tuyl demonstrated an inductive procedure to construct the minimal free resolution of certain classes of edge ideals. We will provide a simplified proof of this inductive method for the class of trees. Furthermore, we will provide a comprehensive description of the finely graded Betti numbers occurring in the minimal free resolution of the edge ideal of a tree. For specific subclasses of trees, we will generate more precise information including explicit formulas for the projective dimensions of the quotient rings of the edge ideals. In the second half of this thesis, we will consider the class of simple bipartite graphs known as Ferrers graphs. In particular, we will study a class of monomial ideals that arise as initial ideals of the defining ideals of the toric rings associated to Ferrers graphs. The toric rings were studied by Corso and Nagel in 2007, and by studying the initial ideals of the defining ideals of the toric rings we are able to show that in certain cases the toric rings of Ferrers graphs are level.

KEYWORDS: edge ideal, Ferrers graph, free resolution, toric ideal, tree

Author's signature: Rachelle R. Bouchat

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By
Rachelle R. Bouchat

Director of Dissertation: Uwe Nagel

Director of Graduate Studies: Qiang Ye

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Dedicated to my best friend and companion, Macaulay.

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1 Introduction

In this thesis we will examine specific classes of planar graphs and monomial ideals that are associated to these graphs. The goal of studying this relationship is to gain information about the algebraic objects by using both graph-theoretical and combinatorial properties of the planar graphs. This area of mathematics, which lies at the intersection of many different areas including commutative algebra, combinatorics and topology, is a source of many interesting open problems. For a thorough introduction to this branch of mathematics, we refer the reader to the book of Miller and Sturmfels (see [13]). In the chapters that follow, we will study properties of the minimal free resolutions of the quotient rings corresponding to the edge ideals of certain classes of simple graphs.

In Chapter 2, we will begin by providing definitions of the objects that will be studied, namely edge ideals, and descriptions of the simple graphs that we wish to study in later chapters. In this chapter we will also introduce some of the techniques that will be used throughout this thesis, including the mapping cone construction of free resolutions from a short exact sequence.

In Chapter 3, we will consider the edge ideals of trees. In particular, we are interested in studying the minimal free resolution associated to this class of edge ideals. When considering trees as a subclass of simple graphs, we notice their relatively simplistic structure. Transferring to the study of the edge ideals of trees, we would expect that the corresponding minimal free resolutions would be relatively simple. However, in [14], Nagel and Reiner show that for the class of edge ideals associated to trees, the Betti numbers corresponding to these edge ideals can be as complicated as desired. For certain classes of ideals associated to simple graphs, Hà and Van Tuyl introduced in [9] an inductive procedure to compute the minimal free resolution of such ideals. In Theorem (3.0.16) we will restrict to the class of trees and show a simplified development of this inductive procedure for this case. We should note that this inductive procedure also illustrates the complexity of computing a minimal free resolution for the edge ideals of trees. However, for the edge ideals of trees we are able to answer the question of when a particular multi-graded Betti number occurs in the minimal free resolution of the corresponding quotient ring. More precisely, we will prove in Theorem (3.0.25) the following comprehensive description of the Betti numbers for the edge ideal of a given tree.

Theorem. *Given a tree T on the vertex set $\{x_0, \ldots, x_n\}$ and a vector $\boldsymbol{a} \in \mathbb{N}^{n+1}$, the following are equivalent.*

(i) $\beta_{i,\boldsymbol{a}}(S/I_T) = 1$

(ii) The subforest of T defined by $\boldsymbol{a}$ is maximal.

In the above theorem, the property of maximality is an algebraic property of the corresponding edge ideal I_T that will be introduced in Definition (3.0.21). The proof of this theorem leads to an implementation of this result in the open-source mathematics' software SAGE [15]. The code for this implementation is written in Python and provided at the end of Chapter 3.

Due to the complexity of the minimal free resolutions of the edge ideals of trees, we consider classes of edge ideals that occur as subclasses of trees. In particular, in Chapter 4, we will consider the edge ideals of paths and a class of graphs that occur as a natural extension of the class of paths. For these special cases we will generate more specific results concerning the minimal free resolutions of the corresponding quotient rings to the edge ideals. Specifically, for the edge ideals of paths we will show the following result in Proposition (4.1.2) concerning the corresponding minimal free resolutions.

Proposition. *Let P_n denote an n-length path. Then*

(i) the length of the minimal free resolution for S/I_{P_n} is $\lceil \frac{2n}{3} \rceil$.

(ii) the Castelnuovo-Mumford regularity of S/I_{P_n} is $\lceil \frac{n}{3} \rceil$.

In particular, this proposition shows that even in the case of the extremely simplistic graphical structure of paths, the minimal free resolutions of the corresponding edge ideals are relatively complicated. Roughly speaking, they have two-thirds of the maximum length of a minimal free resolution of an ideal in the same polynomial ring. Furthermore, we will show in Corollary (4.1.4) the following result concerning the rank of the last module in the minimal free resolution of an edge ideal for a path.

Corollary. *For a path of length n,*

$$\beta_{\lceil \frac{2n}{3} \rceil}(S/I_{P_n}) = \begin{cases} 1 & \text{if } 3 \nmid n \\ \frac{n}{3} + 1 & \text{if } 3 \mid n \end{cases}$$

Both the above corollary and the result concerning the projective dimension and the Castelnuovo-Mumford regularity of a path clearly demonstrate the integral role that divisibility of a path's length by 3 plays in the algebraic structure of the edge ideal as it relates to its minimal free resolution. Since trees can be inductively constructed from paths, we see that divisibility by 3 also plays an important role in the algebraic structure of the edge ideals of trees.

We will also examine the minimal primary decomposition of the edge ideal of an arbitrary simple graph as it relates to the set of all minimal vertex covers of the planar graph. In particular, we will show in Theorem (4.2.6) that there is the following one-to-one correspondence between the set of minimal vertex covers of a simple graph and the set of associated prime ideals of the corresponding edge ideal.

$$\left\{\begin{array}{c}\text{Minimal vertex}\\ \text{covers of a}\\ \text{simple graph } G\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{Associated prime}\\ \text{ideals of } I_G\end{array}\right\}$$

This one-to-one correspondence will allow us to easily determine whether a given prime ideal is indeed an associated prime ideal of the corresponding edge ideal.

The results that we obtain for the edge ideals of trees will be used in Chapter 5 to generate information concerning the minimal free resolution of cycles and more general graphs. In Proposition (5.0.2), we will provide an explicit formula for the length of the minimal free resolution corresponding to the edge ideal of a cycle. In particular, we will see that it is very closely related to the length of the minimal free resolution corresponding to a path. We should note that the results introduced by Hà and Van Tuyl in [9] and proved again in a more simplified manner in Theorem (3.0.16) to inductively construct the minimal free resolution for the corresponding edge ideal of a given tree do not apply to the class of edge ideals of cycles. The inductive construction used to obtain the minimal free resolution for the edge ideal of a given tree was based upon the addition (or removal) of a leaf of the tree.

In Chapter 6, we will consider monomial ideals associated to another class of simple graphs known as Ferrers graphs. In this chapter, we will provide the basic definitions and tools that we will use to study a class of monomial ideals related to the defining ideal of the toric ring that was studied by Corso and Nagel in [3], where they showed that the toric ring is intimately related to the (2×2)-minors of the associated Ferrers tableau. The monomial ideals that we will study in Chapter 7 will occur as initial ideals of the toric ideal generated by the (2×2)-minors of these Ferrers tableaux.

Specifically, in Chapter 7, we will examine two particular term orders and the resulting initial ideals of the defining ideal of the toric ring. One of our goals of considering initial ideals of these toric ideals is to generate information about the original toric ring. In particular, we would like to show that the toric ring of a Ferrers graph is level. The first term order that we will consider is the reverse lexicographic term order. We will see in Example (7.1.6) that the toric generators of the defining ideal of the toric ring constitute a Gröbner basis. However, the resulting initial ideal in the reverse lexicographic term order is not, in general, level. For this reason we will consider another term order which occurs as a modification of the reverse lexicographic term order and was used by Conca, Hoşten, and Thomas in [2] when they considered ideals that occur as $((n-1) \times (n-1))$-minors of matrices of size $n \times n$. As we will see in this case, the initial ideals show much greater promise. In particular, we show in Proposition (7.2.8) that the initial ideals of the toric ideals in this term order are level for specific Ferrers graphs. Consequently, this shows that the original toric rings are also level for these particular Ferrers graphs.

It should be noted that in Chapter 7 the initial ideals studied are generated by square-free quadrics. Relating this back to Chapter 2, we see that these initial ideals correspond to simple graphs with vertex sets corresponding to the variables that divide the minimal generators of the initial ideal. Furthermore, the results that we show concerning the initial ideals of the defining ideals of the toric rings of Ferrers tableaux are also statements about the edge ideals of the simple graphs defined by the minimal generating sets of the initial ideals.

2 Basic Tools and Concepts

The goal of this chapter is to provide definitions of both the graphical structures and the algebraic objects that will be used in the first three chapters and also to introduce some of the basic tools that will be used in the development of the results concerning the algebraic structure of the ideals that will be studied. We begin by providing a brief introduction to the planar graphs that we will be working with.

2.1 Simple Graphs

The following definitions are adaptations of those given in the book of Diestel [5], which provides a thorough introduction to graph theory.

2.1.1 Definition. A *graph* consists of a vertex set $V_G = \{x_0, \ldots, x_n\}$ and a set of edges $E_G \subset V_G \times V_G$. Moreover, if $\{x_i, x_j\} \in E_G$ we will say x_i and x_j are connected by an edge.

2.1.2 Example. Let G denote the following graph.

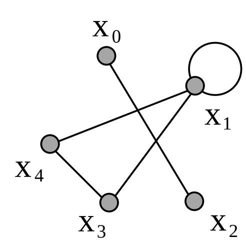


Figure 2.1: A graph on 5 vertices

Then $V_G = \{x_0, x_1, x_2, x_3, x_4\}$ and $E_G = \{\{x_0, x_2\}, \{x_1, x_1\}, \{x_1, x_4\}, \{x_1, x_3\}, \{x_3, x_4\}\}$.

In order to redevelop the inductive procedure for the minimal free resolutions of a particular class of ideals we will need the following definitions concerning the vertices and edges of a graph.

2.1.3 Definition. Let G be a graph.

(i) Two vertices x_i and x_j are called *neighbors* if and only if $\{x_i, x_j\} \in E_G$.

(ii) The *degree of a vertex*, $x_i \in V_G$, is the number of neighbors of the vertex x_i.

(iii) A *subgraph* G' of G is a graph for which both $V_{G'} \subset V_G$ and $E_{G'} \subset E_G$.

(iv) A *path* from vertex x_0 to vertex x_n is a sequence of edges

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$$

that starts at x_0 and ends at x_n. We will require that the x_i are all distinct for $i = 0, \ldots, n$.

(v) A *cycle* is a sequence of edges

$$\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_0\}$$

where the x_i are all distinct for $i = 0, \ldots, n$.

(vi) G is said to be *connected* if any two vertices of G are joined by a path of edges, and a *connected component* is a maximal connected subgraph of G.

(vii) A *loop* of a graph G is an edge of the form $\{x_i, x_i\}$ for some vertex $x_i \in V_G$.

We would like to define an algebraic object associated to a graph, but to provide a well-defined construction we must restrict to a specific class of graphs.

2.1.4 Definition. G is a *simple graph* if it contains no loops, i.e. no vertex is connected to itself via an edge.

2.1.5 Example. If we remove edge $\{x_1, x_1\}$ from the graph in Example (2.1.2), we obtain the following simple graph.

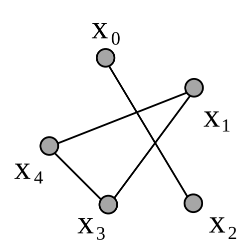


Figure 2.2: A simple graph on 5 vertices

This restriction to the class of simple graphs will enable a one-to-one correspondence between the edges of the graphs and a specific class of ideals known as square-free quadratic monomial ideals.

2.2 Monomial Ideals

In this section, we want to define the algebraic objects that will be related to the simple graphs. We begin by defining the following ideals as in the textbook of Miller and Sturmfels [13].

2.2.1 Definition. Let $S = k[x_0, x_1, \ldots, x_n]$ be the polynomial ring over the field k. Then

(i) a *monomial* in S is a product $\mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ for any vector $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{N}^{n+1}$ of nonnegative integers.

(ii) an ideal $I \subset S$ is called a *monomial ideal* if it can be generated by monomials.

(iii) a monomial $\mathbf{x}^{\mathbf{a}}$ is *square-free* if every coordinate of $\mathbf{a}$ is either 0 or 1.

(iv) an ideal is *square-free* if it can be generated by square-free monomials.

2.2.2 Example. Consider the polynomial ring $S = k[x_0, x_1, x_2, x_3]$. Then

(i) $I = (x_0 x_3^2, x_1^3 x_2)$ is a monomial ideal in S.

(ii) $J = (x_0 x_1, x_2 x_3 - x_0 x_1)$ is also a monomial ideal in S, because J can be generated by monomials as $J = (x_0 x_1, x_2 x_3)$. We can further see that J is actually a square-free monomial ideal.

The study of monomial ideals is an active area of research because of its connections to combinatorics, simplicial topology, and geometry. Specifically, monomial ideals occur as Gröbner degenerations of more general ideals generated by polynomials, and Gröbner basis theory reduces questions regarding systems of polynomial equations down to the combinatorial study of monomial ideals. More generally, monomial ideals form a very important bridge between the areas of commutative algebra and combinatorics. Furthermore, square-free monomial ideals are often referred to as *Stanley-Reisner ideals*, and the connections of these objects to combinatorics arises from their connections to the study of simplicial topology.

In the following section we will examine how to relate simple graphs to square-free monomial ideals.

2.3 Edge Ideals

Given a graph G on the vertex set $\{x_0, \ldots, x_n\}$, we would like to study the algebraic invariants of the ideal whose generators are formed by the edges of the graph. This ideal will reside in the polynomial ring $S := k[x_0, \ldots x_n]$ where k is an arbitrary field and the variables of S correspond to the vertex set of G. However, for this to be a well-defined conversion we must restrict to the class of simple graphs.

From this point forward, we will assume that G is a simple graph. This restriction to simple graphs permits the following definition.

2.3.1 Definition. For a graph G with vertex set $\{x_0, \ldots, x_n\}$, the edge ideal of G is the ideal

$$I_G := (x_i x_j \mid \{x_i, x_j\} \in E_G) \subset S := k[x_0, \ldots, x_n].$$

2.3.2 Example. Consider the following graph G.

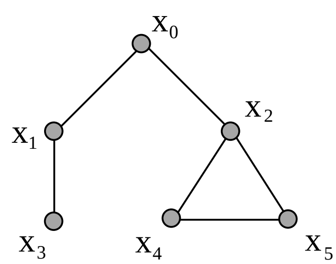


Figure 2.3: A simple graph on 6 vertices

Then the edge ideal corresponding to G is

$$I_G = (x_0x_1, x_0x_2, x_1x_3, x_2x_4, x_2x_5, x_4x_5) \subset k[x_0, x_1, x_2, x_3, x_4, x_5].$$

It should be noted that edge ideals were first introduced by Villarreal in [17] and are a current topic of study in algebraic combinatorics. Connections between the algebraic properties of the edge ideal, I_G, and the combinatorial data associated to the planar graph, G, are an area of active research. The textbook of Miller and Sturmfels (see [13]) provides a thorough introduction to this subject. Furthermore, the following natural one-to-one correspondence illustrates that every square-free quadratic monomial ideal in S arises as an edge ideal of a simple graph on $n+1$ vertices.

$$\left\{ \begin{array}{c} \text{Square-free quadratic monomial} \\ \text{ideals } I \subset S = k[x_0, \ldots, x_n] \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \text{ Simple graphs } G \text{ on } n+1 \text{ vertices } \right\}$$

We would like to study the algebraic invariants of the quotient ring S/I_G. In particular, we would like to study properties of the minimal free resolution of S/I_G. In the next section we provide a brief introduction to minimal free resolutions, but for a more thorough introduction to the theory of minimal free resolutions we refer the reader to the textbook of Eisenbud (see [6]).

2.4 Minimal Free Resolutions

2.4.1 Definition. A *free resolution* of the finitely generated S-module M is an exact sequence, i.e. $\mathrm{im}(\phi_i) = \ker(\phi_{i-1})$, of S-modules

$$\cdots F_i \xrightarrow{\phi_i} F_{i-1} \xrightarrow{\phi_{i-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

where all F_i are finitely generated free S-modules.

It should be noted that in an arbitrary ring R, the free resolution of a finitely generated R-module M does not have to be finite in length. However, if the finitely generated module M is considered in the polynomial ring $S = k[x_0, \ldots, x_n]$, Hilbert proved in [10] that there is a free resolution of M with finite length, but a much more modern proof of this theorem is provided in [6].

2.4.2 Theorem (Hilbert's Syzygy Theorem)**.** *If $S = k[x_0, \ldots, x_n]$, then every finitely generated S-module has a free resolution of length at most $n+1$.*

2.4.3 Example. Consider the graph G below.

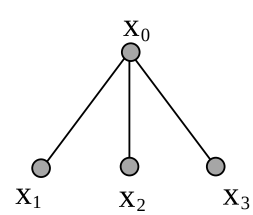


Figure 2.4: The complete bipartite graph $K_{1,3}$

Then $I_G = (x_0x_1, x_0x_2, x_0x_3) \subset S := k[x_0, x_1, x_2, x_3]$, and a free resolution for S/I_G is given by

$$0 \to S \xrightarrow[\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}]{} S^3 \xrightarrow[\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}]{} S^3 \xrightarrow[\begin{bmatrix} x_0x_1 & x_0x_2 & x_0x_3 \end{bmatrix}]{} S$$

$$\longrightarrow S/I_G \longrightarrow 0.$$

It is often useful to record more detailed information in a module's free resolution by considering graded modules and graded resolutions. In preparation for these definitions, we must first let $A = (A, +)$ denote an abelian group with operation $+$.

2.4.4 Definition.

(i) An A-*graded ring* is a ring R with a decomposition

$$R = \bigoplus_{a \in A} [R]_a$$

as a direct sum of graded components such that

$$[R]_a [R]_b \subset [R]_{a+b}$$

for all $a, b \in A$. The elements in $[R]_a$ are called *homogeneous elements of degree* a.

(ii) Let R be an A-graded ring. Then an R-module M is an A-*graded module* if it has a decomposition

$$M = \bigoplus_{a \in A} [M]_a$$

as a direct sum of graded components such that

$$[R]_a [M]_b \subset [M]_{a+b}$$

for all $a, b \in A$

2.4.5 Remark. We will often use the suspension notation, $M(a)$, to denote the A-graded translate of a free R-module M that satisfies

$$[M(a)]_b = [M]_{a+b}$$

for all $a, b \in A$.

2.4.6 Example. Let us consider the polynomial ring $S = k[x_0, \ldots, x_n]$.

(i) Consider the abelian group $(\mathbb{Z}, +)$. Then the *standard grading* (or *coarse grading*) of S is defined by $\deg(\mathbf{x}^{\mathbf{a}}) = a_0 + a_1 + \ldots + a_n$ for each $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$. If we let $S = k[x_0, x_1, x_2, x_3]$, then $\deg(x_0^2 x_2 x_3^3) = 2 + 0 + 1 + 3 = 6$ in the standard grading.

(ii) Consider the abelian group $(\mathbb{Z}^{n+1}, +)$. Then the *fine grading* of S is defined by $\deg(\mathbf{x}^{\mathbf{a}}) = \mathbf{a}$ for each $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$. Thus for $S = k[x_0, x_1, x_2, x_3]$, $\deg(x_0^2 x_2 x_3^3) = (2, 0, 1, 3)$ in the fine grading.

From the previous example, we see that the fine grading carries the information of the standard grading along with a detailed description of the variables contributing to the overall degree of the monomial. In general, we would like to consider monomials in the fine grading, but it often takes considerably more work to keep track of all the degree shifts. For this reason, in the cases where we do not need all of this extra information we will use the standard grading.

We would also like to talk about maps between graded modules; and, in particular, graded free resolutions of graded modules. To do this we must first make the following definition.

2.4.7 Definition. Let M, N be A-graded modules with $a \in A$. Then an *A-graded homomorphism of degree a* is a homomorphism $\phi : M \longrightarrow N$ such that for all homogeneous $m \in M$

$$\deg(\phi(m)) = \deg(m) + a.$$

If $a = 0$, then ϕ is called *degree-preserving*.

2.4.8 Example. Let $S = k[x_0, x_1, x_2]$ considered in the standard grading. Then

$$\Phi_1 : S \xrightarrow{x_0} S$$

is a homomorphism of degree 1. However,

$$\Phi_2 : S(-1) \xrightarrow{x_0} S$$

is a degree-preserving homomorphism.

In the above example, even though the maps perform the same operation to an arbitrary element of S, the definition of Φ_2 is more favorable. This is because Φ_2 encodes the degree transformation that occurs during the mapping in the free module rather than the homomorphism.

Now we may talk of a *graded free resolution* for a finitely generated graded module $M \subset S$. By this, we mean a free resolution of M

$$0 \longrightarrow F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

in which each map, ϕ_i, is degree preserving.

2.4.9 Example. Consider $I = (x_0x_1, x_0x_2, x_0x_3)$ from Example (2.4.3).

(i) The free resolution

$$0 \to S(-4) \xrightarrow[\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}]{} S^3(-3) \xrightarrow[\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}]{} S^3(-2)$$

$$\xrightarrow[\begin{bmatrix} x_0x_1 & x_0x_2 & x_0x_3 \end{bmatrix}]{} S \longrightarrow S/I_G \to 0$$

is a $\mathbb{Z}$-graded (or coarsely graded) free resolution of S/I_G.

(ii) The free resolution

$$0 \to S(-1,-1,-1,-1) \xrightarrow[\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}]{} \begin{matrix} S(-1,-1,-1,0) \\ \bigoplus \\ S(-1,-1,0,-1) \\ \bigoplus \\ S(-1,0,-1,-1) \end{matrix} \xrightarrow[\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}]{}$$

$$\begin{matrix} S(-1,-1,0,0) \\ \bigoplus \\ S(-1,0,-1,0) \\ \bigoplus \\ S(-1,0,0,-1) \end{matrix} \xrightarrow[\begin{bmatrix} x_0x_1 & x_0x_2 & x_0x_3 \end{bmatrix}]{} S \longrightarrow S/I_G \longrightarrow 0$$

is a $\mathbb{Z}^4$-graded (or finely graded) free resolution of S/I_G.

At this point we should note that, in general, free resolutions (and even graded free resolutions) of modules are not unique as illustrated by the following example.

2.4.10 Example. Let $S = k[x_0, x_1]$, and let $M = S/(x_0x_1)$. Then

$$\begin{matrix} 0 \longrightarrow S(-2) \xrightarrow{x_0x_1} S \longrightarrow M \longrightarrow 0 \\ s \longmapsto (x_0x_1)s \end{matrix}$$

and

$$0 \longrightarrow S(-2) \bigoplus S \longrightarrow S^2 \longrightarrow M \longrightarrow 0$$
$$(s,t) \longmapsto ((x_0x_1)s, t)$$

are both $\mathbb{Z}$-graded free resolutions of M.

For uniqueness (up to isomorphism), we must restrict to a graded minimal free resolution, which is a graded free resolution of M with an added restriction on the image of each map.

2.4.11 Definition. Let M be a finitely generated graded S-module.

(i) A *(graded) minimal free resolution* of M is an exact sequence of graded S-modules

$$0 \longrightarrow F_r \xrightarrow{\phi_r} F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

where all F_i are finitely generated free S-modules, all ϕ_i are degree preserving homomorphisms, and $\phi_i(F_i) \subset (x_0, x_1, \ldots, x_n)F_{i-1}$.

(ii) The length of a minimal free resolution of M, r, is called the *projective dimension* of M and is denoted by $\mathrm{pd}(M)$.

(iii) We can write $F_i = \bigoplus_{b \in A} S^{\beta_{i,b}}(-b)$. Then the i^{th}*-Betti number* of M in degree b is denoted by $\beta_{i,b}(M)$ and is the number of copies of $S(-b)$ occurring in the free S-module F_i.

(iv) The *Castelnuovo-Mumford regularity* (or *regularity*) of M

$$\mathrm{reg}(M) = \max\{b - i \mid \beta_{i,b}(M) \neq 0\}.$$

Since each map, ϕ_i, is a map between finitely generated free S-modules, we may represent the maps by their actions on the standard basis elements for each free S-module. In this way, we can represent each map, ϕ_i, by a matrix with the rank of F_i equal to the number of columns of the matrix and the rank of F_{i-1} equal to the number of rows of the matrix. Moreover, if we represent each map in the free resolution by a matrix, we can immediately tell whether or not the free resolution is minimal by looking at the individual entries of the matrices. If none of the matrices' entries are units, then the free resolution is actually a minimal free resolution. Another way of looking at this is to say that a given free resolution is minimal if the matrices representing the maps of the free resolution contain only entries of zeroes or products of the x_i's.

In Example (2.4.10), the first free resolution is actually a minimal free resolution of the module M, whereas the second free resolution of M demonstrates a free resolution of M which is not minimal. In comparing the two resolutions in Example (2.4.10), we can see that the first free resolution of M can be obtained from the second free resolution of M by removing the identity map from the second component of the leftmost map.

The projective dimension is a measure of the length of a minimal free resolution for M, but we can think of the regularity of M as a measure of the "width" of M. Actually, the Castelnuovo-Mumford regularity of M is a measure of how hard it will be to compute a minimal free resolution for M, and it also puts a bound on the largest degree of a matrix entry representing a map in a minimal free resolution for M.

2.4.12 Example. Recall the coarsely graded free resolution of S/I_G where G is the complete bipartite graph $K_{1,3}$ seen in Example (2.4.3).

$$0 \to S(-4) \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} S^3(-3) \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} S^3(-2)$$

$$\xrightarrow{\begin{bmatrix} x_0x_1 & x_0x_2 & x_0x_3 \end{bmatrix}} S \longrightarrow S/I_G \longrightarrow 0$$

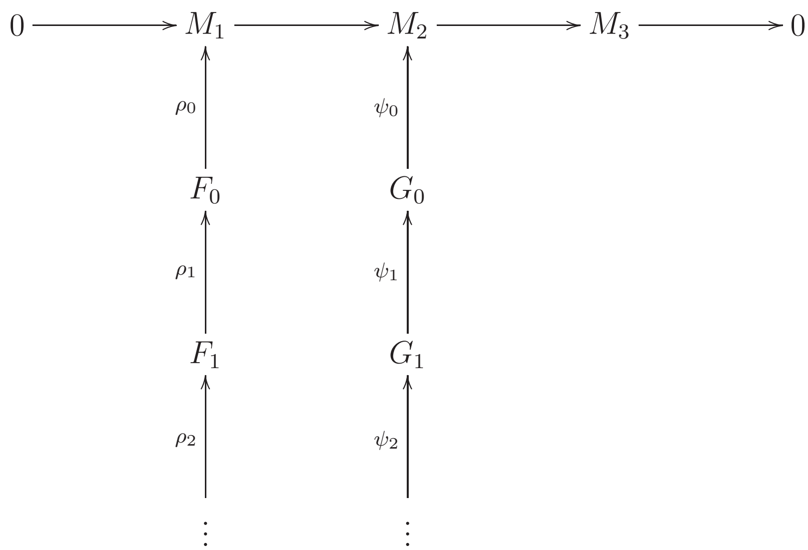
Then we can see that all of the entries of the matrices that represent the maps of the resolutions are either 0 or products of the variables x_i. Hence, this free resolution of S/I_G is actually a minimal free resolution of S/I_G. Also, we see that $\beta_{2,3}(S/I_G) = 3$ but $\beta_{1,3}(S/I_G) = 0$. Furthermore, it is easily seen that $\text{reg}(S/I_G) = 1$.

We want to recover the inductive procedure for obtaining the minimal free resolutions corresponding to edge ideals of simple graphs shown in the paper of Hà and Van Tuyl (see [9]) for a specific class of simple graphs known as trees. This inductive procedure uses the *mapping cone* procedure for short exact sequences as its primary tool.

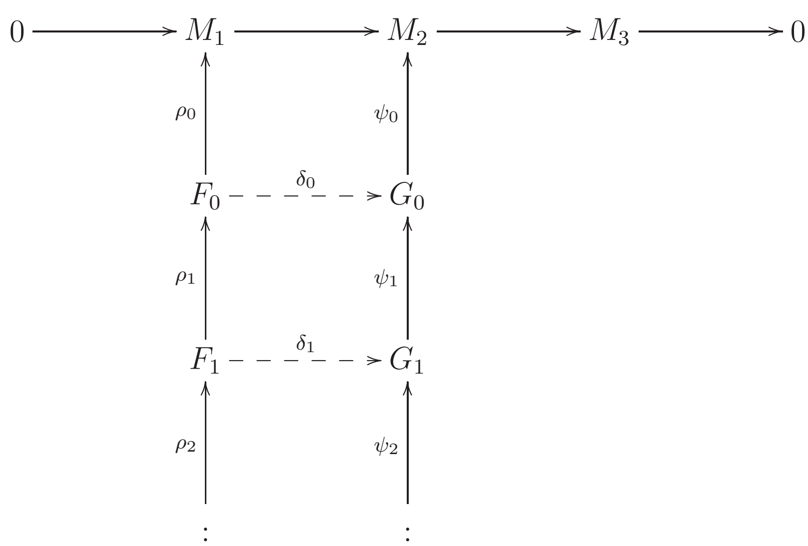
Given a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

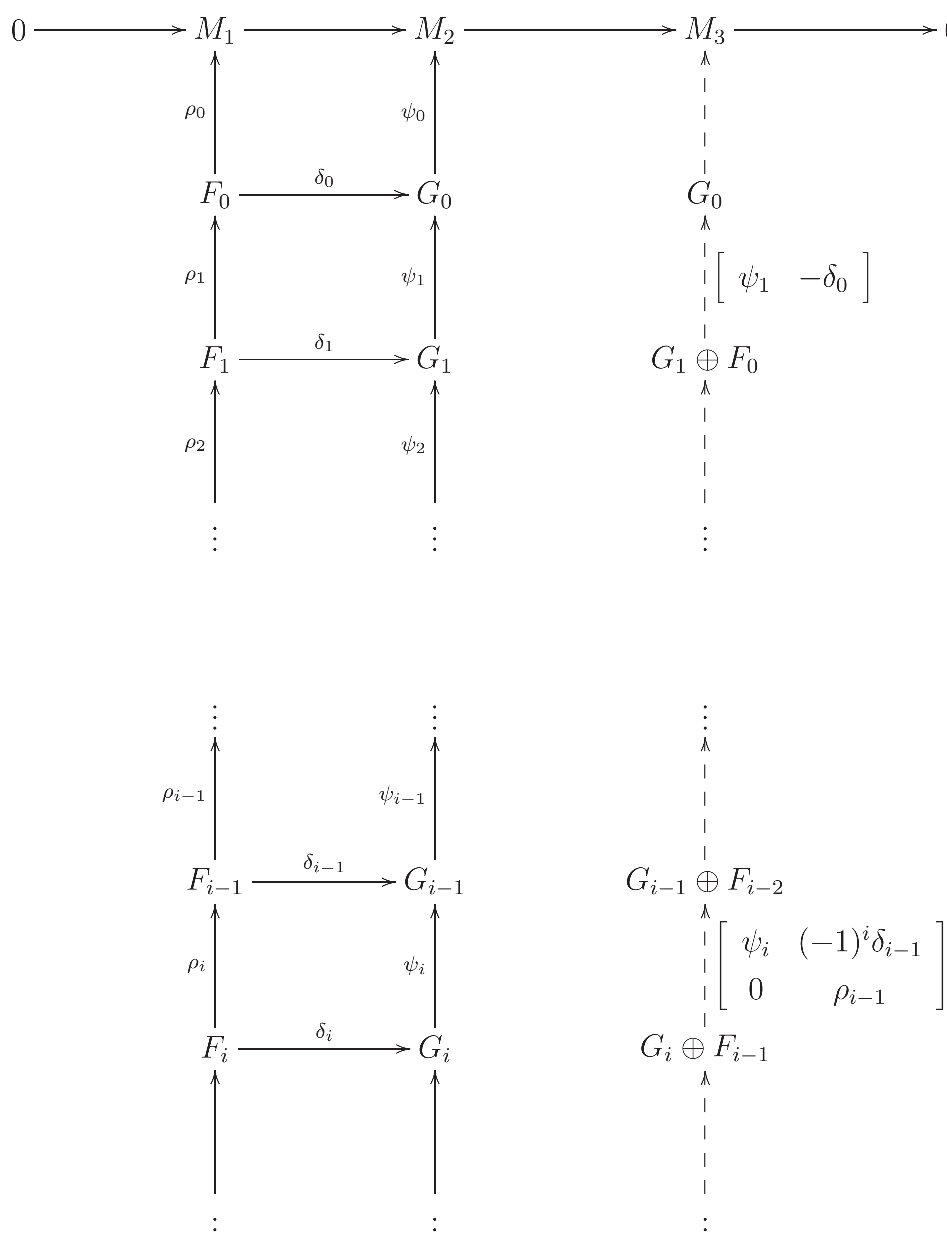
of graded S-modules M_i, the mapping cone construction will enable us to obtain a free resolution for M_3 knowing free resolutions of M_1 and M_2. Knowing the free resolutions of M_1 and M_2 we obtain the following diagram.



where the vertical sequences are free resolutions of M_1 and M_2, respectively. Then there are maps $\delta_i : F_i \to G_i$ such that the squares become commutative and we obtain the following diagram.



Furthermore, the maps ρ_i, δ_i, and ψ_i can be used to construct maps that form a free resolution of the module M_3 as follows.



(2.1)

In general, even if we start with minimal free resolutions for M_1 and M_2, after performing the mapping cone procedure, we do not necessarily generate a minimal free resolution for M_3.

2.4.13 Example. Consider the edge ideal of the cycle of length 5 depicted below.

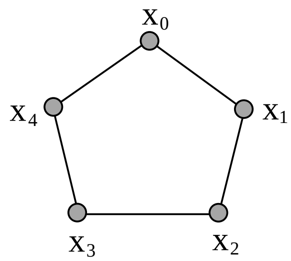


Figure 2.5: The graph of C_5, the 5-cycle

Then

$$I_{C_5} = (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_0x_4) \subset S = k[x_0, \ldots, x_4].$$

Set $J = (x_0x_1, x_1x_2, x_2x_3, x_3x_4)$, and consider the following short exact sequence.

$$0 \longrightarrow S/J : (x_0x_4)(-2) \xrightarrow{x_0x_4} S/J \longrightarrow S/I \longrightarrow 0$$

Then the mapping cone procedure using $\mathbb{Z}$-graded minimal free resolutions of $S/J : (x_0x_4)$ and S/J provides the following free resolution of S/I_{C_5}.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S/J : (x_0x_4)(-2) & \longrightarrow & S/J & \longrightarrow & S/I_{C_5} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & S(-2) & & S & & S & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & S^2(-3) & & S^4(-2) & & S^5(-2) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & S(-4) & & S^3(-3) \oplus S(-4) & & S^5(-3) \oplus S(-4) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & S(-5) & & S(-4) \oplus S(-5) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

However, the above resolution of S/I_{C_5} is not minimal. The minimal free resolution for S/I_{C_5} is given by

$$0 \longrightarrow S(-5) \longrightarrow S^5(-3) \longrightarrow S^5(-2) \longrightarrow S \longrightarrow S/I_{C_5} \longrightarrow 0.$$

In the previous example, we can see that in the mapping cone construction of a free resolution for S/I_{C_5} a copy of $S(-4)$ must be canceled from the bottom-most map to obtain a minimal free resolution for S/I_{C_5}. This copy of $S(-4)$ is often referred to as a *ghost term*.

We are interested in finding cases where there is no cancellation in the mapping cone procedure, i.e. where no ghost terms arise. More precisely, we are interested in cases where the mapping cone procedure applied to minimal free resolutions of M_1 and M_2 and the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

provides a minimal free resolution of M_3.

3 Edge Ideals of Trees

In the case of specific square-free quadratic monomial ideals (or the edge ideals corresponding to certain simple graphs), there is an inductive method to produce the minimal free resolutions of the corresponding quotient rings. Hà and Van Tuyl describe in [9] a method to decompose edge ideals of particular simple graphs to generate information on the algebraic invariants of the corresponding quotient rings S/I_G. The decomposition that is used is based upon the concept of splittable monomial ideals which were originally defined by Eliahou and Kervaire in [7].

3.0.14 Definition. Let I be a monomial ideal in $S = k[x_0, \ldots, x_n]$, and let $\mathcal{G}(I)$ denote the minimal set of monomial generators of I. Then I is *splittable* if I is the sum of two nonzero monomial ideals J and K, i.e. $I = J + K$, such that

(i) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$; and

(ii) there is a *splitting function*

$$\begin{aligned} \mathcal{G}(J \cap K) &\longrightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\longmapsto (\phi(w), \psi(w)) \end{aligned}$$

satisfying

(a) for all $w \in \mathcal{G}(J \cap K)$, $w = \operatorname{lcm}(\phi(w), \psi(w))$; and

(b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\operatorname{lcm}(\phi(S))$ and $\operatorname{lcm}(\psi(S))$ strictly divide $\operatorname{lcm}(S)$.

If J and K satisfy the above conditions, then we say that $I = J + K$ is a *splitting* of I.

This definition, however, is quite cumbersome to use because it says that to determine if an ideal is splittable we must first be able to decompose the ideal into the sum of two ideals with disjoint sets of minimal generators, and then we must satisfy an additional nontrivial restriction on the minimal generators of the ideals involved in the decomposition. However, given an edge of the simple graph G, Hà and Van Tuyl considered in [9] when this edge defines a splitting of the edge ideal I_G, i.e. when

$$\begin{pmatrix} \text{Edge ideal of} \\ \text{the graph } G \end{pmatrix} = \begin{pmatrix} \text{Edge} \end{pmatrix} + \begin{pmatrix} \text{Edge ideal of subgraph} \\ \text{obtained from } G \\ \text{by removing the edge} \end{pmatrix}$$

defines a splitting of I_G. Written more formally, we have the following definition.

3.0.15 Definition. An edge $\{x_i, x_j\}$ of G is a *splitting edge* if

$$I_G = (x_i x_j) + I_{G\setminus\{x_i,x_j\}}$$

defines a splitting of I_G.

In [9], Hà and Van Tuyl remarked that if the simple graph G has a vertex of degree 1, say x_k, then the edge formed by x_k and its neighbor is a splitting edge of I_G. In this case, we can also recover the inductive result concerning the minimal free resolutions of the corresponding quotient rings as proved by Hà and Van Tuyl (see [9]).

3.0.16 Theorem. *Let G be a simple graph with vertex set $V_G = \{x_0, \ldots, x_n\}$ and the added restriction that G has a vertex of degree 1, say x_n. Furthermore, let x_{n-1} be the neighbor of x_n. Then the mapping cone procedure applied to the sequence*

$$0 \to (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(0,\ldots,0,-1,-1) \xrightarrow{x_{n-1}x_n} S/I_{G\setminus\{x_{n-1},x_n\}} \longrightarrow S/I_G \to 0$$

provides a minimal free resolution of S/I_G where

$$I_{G\setminus\{x_{n-1},x_n\}} := (x_i x_j \mid x_i x_j \text{ is a generator of } I_G \text{ and } x_i x_j \neq x_{n-1}x_n)$$

i.e.

$$\beta_{i,\boldsymbol{a}}(S/I_G) = \beta_{i,\boldsymbol{a}}(S/I_{G\setminus\{x_{n-1},x_n\}}) + \beta_{i-1,\boldsymbol{a}}(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)(0,\ldots,0,-1,-1))$$

for all $\boldsymbol{a} \in \mathbb{N}^{n+1}$.

Proof. We first note that since x_n does not divide a minimal generator of $I_{G\setminus\{x_{n-1},x_n\}}$

$$I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}).$$

However, this implies that the exact sequence

$$0 \longrightarrow (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(0,\ldots,0,-1,-1) \longrightarrow S/I_{G\setminus\{x_{n-1},x_n\}} \longrightarrow S/I_G \longrightarrow 0$$

factors as

$$\begin{array}{ccccccc} 0 \longrightarrow & (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(0,\ldots,0,-1,-1) & \xrightarrow{x_{n-1}x_n} & S/I_{G\setminus\{x_{n-1},x_n\}} & \longrightarrow & S/I_G & \longrightarrow 0. \\ & \downarrow{\scriptstyle x_n} & \nearrow{\scriptstyle x_{n-1}} & & & & \\ & (S/I_{G\setminus\{x_{n-1},x_n\}} : x_{n-1})(0,\ldots,-1,0) & & & & & \end{array}$$

Furthermore, let

$$\mathcal{F} : 0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) \longrightarrow 0$$

be a minimal free resolution of $S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1})$, and let

$$\mathcal{G} : 0 \longrightarrow G_t \longrightarrow G_{t-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow S \longrightarrow S/I_{G\setminus\{x_{n-1},x_n\}} \longrightarrow 0$$

be a minimal free resolution of $I_{G\setminus\{x_{n-1},x_n\}}$. Set

$$(\mathbf{-2}) = (0,\ldots,0,-1,-1)$$

and

$$(\mathbf{-1}) = (0,\ldots,0,-1,0).$$

Then we get the following diagram.

$$
\begin{array}{ccccc}
(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n))(\mathbf{-2}) & \xrightarrow{x_n} & (S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}))(\mathbf{-1}) & \longrightarrow & S/I_{G\setminus\{x_{n-1},x_n\}} \\
\uparrow & & \uparrow & & \uparrow \\
S(\mathbf{-2}) & \xrightarrow{x_n} & S(\mathbf{-1}) & \overset{\delta'_1}{\dashrightarrow} & S \\
\uparrow & & \uparrow & & \uparrow \\
F_1(\mathbf{-2}) & \xrightarrow{x_n} & F_1(\mathbf{-1}) & \overset{\delta'_2}{\dashrightarrow} & G_1 \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
F_r(\mathbf{-2}) & \xrightarrow{x_n} & F_r(\mathbf{-1}) & \overset{\delta'_r}{\dashrightarrow} & G_r \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & \vdots \\
 & & & & \uparrow \\
 & & & & G_t \\
 & & & & \uparrow \\
 & & & & 0
\end{array}
$$

From the mapping cone construction illustrated in (2.1), if the mapping cone did not produce a minimal free resolution of S/I_G, then one of the maps in the free resolution would contain a unit. Looking at the factoring of the resolution, we see that this is impossible. If some δ'_i contained a unit, it would be multiplied by x_n upon composition. Hence the induced maps cannot contain a unit and consequently there is no cancelation in the mapping cone, i.e. the mapping cone provides a minimal free resolution of S/I_G. Therefore,

$$\beta_{i,\mathbf{a}}(S/I_G) = \beta_{i,\mathbf{a}}(S/I_{G\setminus\{x_{n-1},x_n\}}) + \beta_{i-1,\mathbf{a}}(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)(0,\ldots,0,-1,-1))$$

for all $\mathbf{a} \in \mathbb{N}^{n+1}$. $\square$

If we take a closer look at $S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ we can see that

$$I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = \mathfrak{a} + (x_0, \ldots, x_s)$$

where $\{x_n, x_0, \ldots, x_s\}$ are the neighbors of x_{n-1} and the generators of $\mathfrak{a}$ are square-free quadrics in $k[x_{s+1}, \ldots, x_{n-2}]$. This shows that $I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ can be realized as a subgraph of G.

If we further restrict ourselves to a subclass of simple graphs where each graph in the class has at least one vertex of degree 1, then we will obtain an inductive construction for the minimal free resolution of the corresponding quotient rings. Let us consider the subclass of simple graphs known as trees.

3.0.17 Definition. Let G be a connected simple graph. Then

(i) G is a *tree* if it does not contain a cycle;

(ii) a vertex of degree 1 in a tree is called a *leaf*; and

(iii) a *forest* is a disjoint union of trees.

By definition, every tree has a leaf. Hence, in the case of the edge ideals of trees, Theorem (3.0.16) provides a comprehensive description of the corresponding minimal free resolutions, because the quotient ideal $I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$ can be realized graphically as a subforest of I_G. The following picture illustrates the relationship between I_G, $I_{G\setminus\{x_{n-1},x_n\}}$, and $I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)$.

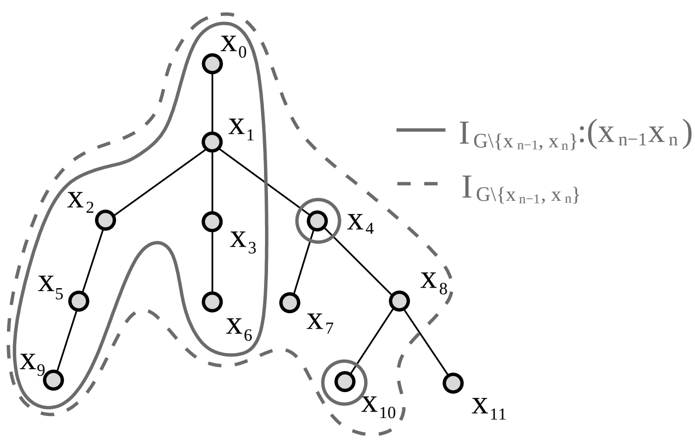


Figure 3.1: A graphical representation of $I_{G\setminus\{x_8,x_{11}\}} : (x_8x_{11})$

Let us now consider the complete bipartite graph $K_{1,n}$ depicted below.

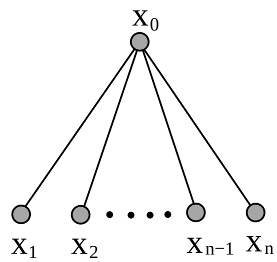


Figure 3.2: The complete bipartite graph $K_{1,n}$

Then $I_{K_{1,n}} = (x_0x_1, x_0x_2, \ldots, x_0x_n) = x_0(x_1, x_2, \ldots, x_n)$, and we can see that $S/I_{K_{1,n}}$ is resolved by the Koszul complex. Therefore, $\mathrm{pd}(S/I_{K_{1,n}}) = n$. Combining this information with Theorem (3.0.16) we get the following rough estimate on the length of a minimal free resolution corresponding to a simple graph.

3.0.18 Corollary. *Let G be a simple graph with vertex set $V_G = \{x_0, \ldots, x_n\}$. Then*

$$\mathrm{pd}(S/I_G) \geq \max\{\deg(x_i) \mid 0 \leq i \leq n\}$$

Proof. Consider the vertex set $V_G = \{x_0, \ldots, x_n\}$ of the graph G, and let x_k be the vertex of highest degree. Now consider the subgraph of G consisting of vertex x_k and its neighbors. Then this subgraph is $K_{1,\deg(x_k)}$. Adding vertices one at a time to $K_{1,\deg(x_k)}$, we can reconstruct the graph G; and hence Theorem (3.0.16) provides

$$\mathrm{pd}(S/I_G) \geq \mathrm{pd}(S/I_{K_{1,\deg(x_k)}}) = \deg(x_k) = \max\{\deg(x_i) \mid 0 \leq i \leq n\}.$$

□

We would also like to take a closer look at the finely graded Betti numbers associated to the edge ideal of a tree. Hochster proved the following result concerning the possible degree shifts for Betti numbers in a minimal free resolution corresponding to a monomial ideal. A proof of this result can be found in the book of Miller and Sturmfels (see [13]).

3.0.19 Proposition (Hochster's Formula (Dual Version))**.** *The nonzero Betti numbers of a monomial ideal $I \subset S$ lie only in square-free degrees, i.e.*

$$\beta_{i,\boldsymbol{a}}(S/I) = 0 \qquad \text{if } a_i \geq 2 \text{ for some } i \in \{1, \ldots, n+1\}$$

where $\boldsymbol{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{N}^{n+1}$.

Using Theorem (3.0.16) and Hochster's formula we can specify further limits on the Betti numbers corresponding to the quotient rings of edge ideals of trees.

3.0.20 Theorem. *Let T_n denote a tree with n edges, then the finely graded Betti numbers*

$$\beta_{i,\boldsymbol{a}}(S/I_{T_n}) \in \{0,1\}$$

for all $\boldsymbol{a} \in \mathbb{N}^{n+1}$.

Proof. Proceed by induction on the number of edges n. If $n = 1$, then $I_{T_1} = (x_0x_1)$. A minimal free resolution for S/I_{T_1} is

$$0 \longrightarrow S(-1,-1) \xrightarrow{x_0x_1} S/I_{T_1} \longrightarrow 0$$

and the claim is true. Assume true for all possible trees with $n-1$ edges. Consider a tree T_n with n edges, then by removing a leaf, say x_n, the remaining subtree is a tree with only $n-1$ edges. Denote it by T_{n-1}. Without loss of generality assume x_{n-1} is the neighbor of x_n. Set $(\textbf{-2}) = (0,\ldots,0,-1,-1)$. Then we have the exact sequence

$$0 \longrightarrow S/(I_{T_{n-1}} : (x_{n-1}x_n))(\textbf{-2}) \longrightarrow S/I_{T_{n-1}} \longrightarrow S/I_{T_n} \longrightarrow 0.$$

Moreover,

$$I_{T_{n-1}} : (x_{n-1}x_n) = (x_0,\ldots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}}$$

where $\{x_0,\ldots,x_s\}$ is the set of neighbors of x_{n-1} and $\{T_{n_0} + \cdots + T_{n_k}\}$ is the set of subtrees of T_n occurring in the graphical representation of the quotient ideal $I_{T_{n-1}} : (x_{n-1}x_n)$ as a subforest of T_n. By the induction hypothesis

$$\beta_{i,\mathbf{a}}(S/I_{T_{n_j}}) \in \{0,1\} \quad \text{for } j = 0,\ldots,k.$$

Since $S/(x_0,\ldots,x_s)$ is resolved by the Koszul complex, $\beta_{i,\mathbf{a}}(S/(x_0,\ldots,x_s)) \in \{0,1\}$. Moreover, since the generators of $(x_0,\ldots,x_s)$ and $I_{T_{n_j}}$ are disjoint for $j = 0,\ldots,k$, the minimal free resolution of $S/((x_0,\ldots,x_s)+I_{T_{n_0}}+\cdots+I_{T_{n_k}})$ is resolved by the tensor product of the minimal free resolutions of $S/(x_0,\ldots,x_s)$ and $S/I_{T_{n_j}}$ for $j = 0,\ldots,k$. Hence,

$$\beta_{i,\mathbf{a}}\left(S/((x_0,\ldots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})\right) \in \{0,1\}.$$

Thus the mapping cone provides

$$\beta_{i,\mathbf{a}}(S/I_{T_n}) = B_{i,\mathbf{a}}(S/I_{T_{n-1}}) + \beta_{i-1,\mathbf{a}}(S/((x_0,\ldots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})(\textbf{-2})).$$

Assume to the contrary that $\beta_{i,\mathbf{a}}(S/I_{T_n}) = 2$, i.e.

$$\beta_{i,\mathbf{a}}(S/I_{T_{n-1}}) = \beta_{i-1,\mathbf{a}}(S/((x_0,\dots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})(\textbf{-2})) = 1.$$

Then Hochster's Formula (3.0.19) implies that $\mathbf{a} = (a_1,\dots,a_{n+1}) \in \{0,1\}^{n+1}$. Thus we obtain the following two cases based upon the value of a_{n+1}.

Case (i): Let $\mathbf{a} = (\dots,1)$. Then $\beta_{i,\mathbf{a}}(S/I_{T_{n-1}}) = 0$, because x_n does not divide any generator of $I_{T_{n-1}}$.

Case (ii): Let $\mathbf{a} = (\dots,0)$. Then $\beta_{i-1,\mathbf{a}}(S/((x_0,\dots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})(\textbf{-2})) = 0$, because the shift of $(\textbf{-2})$ says that any contribution from the minimal free resolution of $\left(S/((x_0,\dots,x_s) + I_{T_{n_0}} + \cdots + I_{T_{n_k}})\right)(\textbf{-2})$ will be in a shift with last two entries $(\dots,1,1)$.

Therefore $\beta_{i,\mathbf{a}}(S/I_{T_n}) \in \{0,1\}$. □

Our goal is to give a comprehensive description of the Betti numbers that occur in the minimal free resolution corresponding to the quotient ring of an edge ideal of a tree. In order to present this description we must first define what it means for a tree to be maximal.

3.0.21 Definition. Let T be a tree. Then T is called *maximal* if

$$\beta_{\mathrm{pd}(S/I_T),\mathbf{d}}(S/I_T) = 1 \text{ where } \mathbf{d} = (1,1,\dots,1),$$

i.e. if the minimal free resolution of S/I_T has the maximal shift.

From the definition we see that the property of maximality is purely an algebraic property dealing with the leftmost Betti number of the minimal free resolution for S/I_T.

3.0.22 Example. Consider P_2, the path of length 2, depicted below.

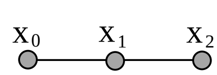


Figure 3.3: The path of length 2

Then a finely graded minimal free resolution for S/I_{P_2} is given by

$$0 \longrightarrow S(-1,-1,-1) \longrightarrow \begin{array}{c} S(-1,-1,0) \\ \bigoplus \\ S(0,-1,-1) \end{array} \longrightarrow S \longrightarrow S/I_{P_2} \longrightarrow 0$$

From this minimal free resolution, we see that P_2 is a maximal graph. Now, let us consider P_3, a path of length 3, depicted below.

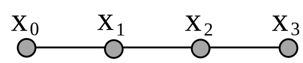


Figure 3.4: The path of length 3

Then a minimal free resolution for S/I_{P_3} is given by

$$0 \longrightarrow \begin{array}{c} S(-1,-1,-1,0) \\ \bigoplus \\ S(0,-1,-1,-1) \end{array} \longrightarrow \begin{array}{c} S(-1,-1,0,0) \\ \bigoplus \\ S(0,-1,-1,0) \\ \bigoplus \\ S(0,0,-1,-1) \end{array} \longrightarrow S \longrightarrow S/I_{P_3} \longrightarrow 0.$$

In this case, we see from the above minimal free resolution that P_3 is not maximal.

When considering the above example, we start to see that the length of a path affects its maximality. In particular, we will see in Chapter 4 that a path is maximal if its length is not divisible by 3. Additionally in Chapter 4, we will describe how to determine maximality by decomposing the planar graph T into smaller subgraphs.

3.0.23 Remark. The above definition of maximality also applies to a forest F, which is a disjoint union of trees. The reasoning is that the corresponding minimal free resolution of a forest is formed by taking the tensor product of minimal free resolutions for each of the disjoint component trees. Therefore, a forest is maximal when all of its component trees are maximal.

Additionally, for a given tree T on the vertex set $\{x_0, \ldots, x_n\}$ we can talk about the *subforest of T defined by a vector $\boldsymbol{a} \in \mathbb{N}^{n+1}$*. This subforest is obtained from T by removing all vertices x_i that have a 0 in the i^{th}-entry of $\mathbf{a}$.

3.0.24 Example. Consider the tree T depicted below.

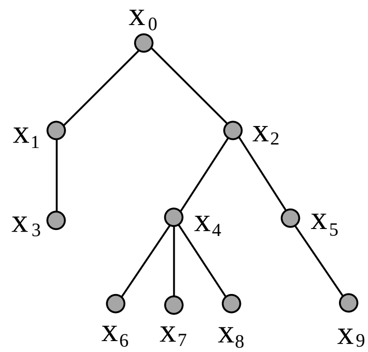


Figure 3.5: A tree on the vertex set $\{x_0, \ldots, x_9\}$.

Then the subforest of T defined by $(0, 1, 1, 1, 0, 1, 0, 1, 0, 1)$ is given by the following graph.

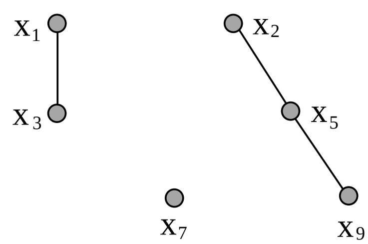


Figure 3.6: The subforest of T defined by $(0, 1, 1, 1, 0, 1, 0, 1, 0, 1)$

Using this idea of a subforest of a tree defined by a vector, we can determine when a particular Betti number will occur in the minimal free resolution corresponding to the quotient ring of the edge ideal of the tree.

3.0.25 Theorem. *Given a tree T on the vertex set $\{x_0, \ldots, x_n\}$ and a vector $\boldsymbol{a} \in \mathbb{N}^{n+1}$, the following are equivalent.*

(i) $\beta_{i,\boldsymbol{a}}(S/I_T) = 1$

(ii) The subforest of T defined by $\boldsymbol{a}$ is maximal.

Proof. Induct on n, the number of edges in the tree. For $n = 1$, the minimal free resolution of $S/I_{T_1} = S/(x_0x_1)$ is given by

$$0 \longrightarrow S(-1, -1) \longrightarrow S \longrightarrow S/I_{T_1} \longrightarrow 0$$

and the claim clearly holds. Assume true for any tree of length $n-1$. Let T_n denote a tree with n edges. Without loss of generality, we will assume that x_n is a leaf of the tree T_n with neighbor x_{n-1}.

Assume $\beta_{i,\mathbf{a}}(S/I_T) = 1$. Consider the subforest of T defined by $\mathbf{a}$, denote it by $F_{\mathbf{a}}$. Notice that starting from $F_{\mathbf{a}}$ we can add vertices one at a time as leaves to reconstruct the original tree T. Then Theorem (3.0.16) provides that

$$\beta_{i,\mathbf{a}}(S/I_T) = 1 \iff \beta_{\mathrm{pd}(S/I_{F_{\mathbf{a}}}),\mathbf{d_a}}(S/I_{F_{\mathbf{a}}}) = 1 \tag{3.1}$$

where $\mathbf{d_a} = (1, 1, \ldots, 1)$ and has entries corresponding to $\mathbf{a}$. Furthermore, (3.1) implies that $F_{\mathbf{a}}$ is maximal. Hence, $\mathbf{a}$ defines a maximal subforest of T.

Conversely, let us assume that $\mathbf{a}$ defines a maximal subforest F of T_n. Theorem (3.0.16) provides that $\mathbf{a} \neq (\ldots, 0, 1)$.

Case (i): If $\mathbf{a} = (\ldots, 1, 0)$, then $\mathbf{a}$ defines a maximal subforest of the subtree of T corresponding to $T \setminus \{x_{n-1}, x_n\}$. It follows from the induction hypothesis that $\beta_{i,\mathbf{a}}(S/I_{T\setminus\{x_{n-1},x_n\}}) = 1$. Furthermore, the mapping cone procedure illustrated in (2.1) and Theorem (3.0.16) imply that $\beta_{i,\mathbf{a}}(S/I_T) = 1$.

Case (ii): If $\mathbf{a} = (\ldots, 1, 1)$, then from the definition of maximality, we see that a tree T is maximal if and only if the subforest of T defined by $T \setminus \{x_{n-1}, x_n\} : (x_{n-1}x_n)$ is maximal. Hence if $\mathbf{a} = (\ldots, 1, 1)$ defines a maximal subforest of T, then $\mathbf{a}'|_F = [\mathbf{a} - (0, \ldots, 0, 1, 1)]_F$ defines a maximal subforest of F where

$$I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n) = (x_0, \ldots, x_s) + F.$$

Furthermore, the induction hypothesis implies that for some j,

$$\beta_{j,\mathbf{a}'|_F}(S/I_F) = 1.$$

Hence,

$$\beta_{i,\mathbf{a}'}(S/I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)) = 1$$

which implies that

$$\beta_{i,\mathbf{a}}(S/I_{T\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)(0, \ldots, 0, -1, -1)) = 1.$$

The mapping cone procedure (2.1) and Theorem (3.0.16) then imply that

$$\beta_{i+1,\mathbf{a}}(S/I_T) = 1.$$

□

The above proof leads to an algorithm for determining when a given Betti number occurs in the minimal free resolution of the associated quotient ring, and this algorithm can be implemented in Python for use in the open-source math software SAGE (see [15]) via the following code.

```
def MaxTest2(T,depth):
    if depth<0:
        raise ValueError
    for l in T:
        if len(T[l])==1:    #Find the first leaf of the tree
            n=T[l][0]       #Let n be the neighbor of the leaf
            break
    try:
        T.pop(l)    #Remove the leaf
    except NameError:    #Return error if no leaf is found,
                         #i.e. if the graph was not a tree
        raise TypeError , "Graph contains a cycle--Not a tree"
    e=T[n]    #Neighbors of n
    print 'Leaf=',l
    print 'Neighbor=',n
    T.pop(n)
    e.remove(l)   #Remove the leaf from the list of neighbors of n
    for v in e:
        temp1=T[v]
        temp1.remove(n)   #Remove the neighbor from the lists of
                          #its neighbors
        T.pop(v)     #Remove the neighbors' neighbor from the list
        for w in temp1:
            temp2=T[w]
            temp2.remove(v)
            if len(temp2)>0:   #Remove the vertex v from w's list
                T[w]=temp2
            else:
                print 'Path of length 0 (floating vertex)
                       --Corresponding Betti number is 0'
```

```
            return False  #Removing the neighbor, n, left a
                          #floating vertex (path of length 0)
    print 'Tree=', T
    print '======================================================'
    if len(T)==4:   #Forest is either a 3-path, K_{1,3}, or two
                    #disjoint 1-paths
        for x in T:
            if len(T[x])==3:  #K_{1,3}
                print 'Maximal--Corresponding Betti number is 1!!!'
                return True
            if len(T[x])==2:  #3-Path
                print 'Path of length 3--Corresponding Betti number
                       is 0.'
                return False
        print 'Maximal--Corresponding Betti number is 1!!!'
                                                   #Two disjoint 1-paths
        return True
    if len(T)<4:
        print 'Maximal--Corresponding Betti number is 1!!!'
        return True
    return MaxTest2(T,depth-1)
```

It should be noted that the combinations of Theorems (3.0.20) and (3.0.25) provide a comprehensive description of the Betti numbers occurring in a minimal free resolution of the corresponding quotient ring to an edge ideal of a tree. In particular, Theorem (3.0.25) tells us when a particular shift occurs in the minimal free resolution, and Theorem (3.0.20) tells us that if the shift occurs the corresponding Betti number must have multiplicity 1.

4 Specific Classes of Edge Ideals of Trees

In this section we will look at two subclasses of trees. For these two subclasses we will generate more explicit results concerning the associated minimal free resolutions studied in the previous chapter. In particular, we will deduce explicit formulas for the projective dimension of their corresponding quotient rings. We begin by recalling the definition of one of the simplest trees, a path.

4.1 Minimal Free Resolutions of the Edge Ideals of Paths

4.1.1 Definition. A *path* of length n on the vertex set $V_G = \{x_0, \ldots, x_n\}$ is a graph with edge set $E_G = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}\}$.

Graphically, an n-length path is given by

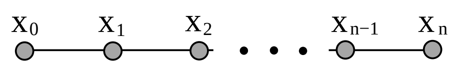


Figure 4.1: The path of length n

and the corresponding edge ideal is given by

$$I_{P_n} = (x_0x_1, x_1x_2, \ldots, x_{n-1}x_n).$$

By restricting to the class of paths, we are able to explicitly write down the projective dimension and regularity of the corresponding quotient ring in terms of the path's length, n.

4.1.2 Proposition. *Let P_n denote an n-length path. Then*

(i) $\mathrm{pd}(S/I_{P_n}) = \left\lceil \frac{2n}{3} \right\rceil$

(ii) $\mathrm{reg}(S/I_{P_n}) = \left\lceil \frac{n}{3} \right\rceil$

Proof. Proceed by induction on length of the path, n. For $n = 1$ the associated edge ideal is $I_{P_1} = (x_0x_1)$. Moreover,

$$0 \longrightarrow S(-2) \xrightarrow{x_0x_1} S \longrightarrow S/I_{P_1} \longrightarrow 0$$

is a minimal free resolution for S/I_{P_1}, and it is clear that $\text{pd}(S/I_{P_1}) = 1 = \left\lceil \frac{2(1)}{3} \right\rceil$ and $\text{reg}(S/I_{P_1}) = 1 = \left\lceil \frac{1}{3} \right\rceil$. Assume the claim holds true for paths of length at most $n-1$. Consider I_{P_n} and the following short exact sequence

$$0 \longrightarrow \left(S/I_{P_{n-1}} : (x_{n-1}x_n)\right)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0.$$

From the mapping cone construction (see (2.1)) and Theorem (3.0.16) we obtain

$$\text{pd}(S/I_{P_n}) = \max\left\{\text{pd}(S/I_{P_{n-1}})\ ,\ \text{pd}(S/I_{P_{n-1}} : (x_{n-1}x_n)) + 1\right\} \tag{4.1}$$

and

$$\text{reg}(S/I_{P_n}) = \max\left\{\text{reg}(S/I_{P_{n-1}})\ ,\ \text{reg}(S/I_{P_{n-1}} : (x_{n-1}x_n)(-2)) - 1\right\}. \tag{4.2}$$

Furthermore, we notice that $I_{P_{n-1}} : (x_{n-1}x_n) = (x_{n-1}) + I_{P_{n-3}}$. Then the induction hypothesis provides the following information

$$\begin{aligned}
\text{pd}(S/I_{P_{n-1}}) &= \left\lceil \frac{2(n-1)}{3} \right\rceil \\
\text{reg}(S/I_{P_{n-1}}) &= \left\lceil \frac{n-1}{3} \right\rceil \\
\text{pd}(S/I_{P_{n-1}} : (x_{n-1}x_n)) &= \left\lceil \frac{2(n-3)}{3} \right\rceil + 1 = \left\lceil \frac{2n}{3} \right\rceil - 1 \\
\text{reg}\left((S/I_{P_{n-1}} : (x_{n-1}x_n)(-2))\right) &= \left\lceil \frac{n-3}{3} \right\rceil + 2.
\end{aligned}$$

Therefore from (4.1) we conclude that

$$\text{pd}\, S/I_{P_n} = \max\left\{\left\lceil \frac{2(n-1)}{3} \right\rceil\ ,\ \left\lceil \frac{2n}{3} \right\rceil\right\} = \left\lceil \frac{2n}{3} \right\rceil$$

and from (4.2) we conclude that

$$\begin{aligned}
\text{reg}(S/I_{P_n}) &= \max\left\{\left\lceil \frac{n-3}{3} \right\rceil + 2 - 1\ ,\ \left\lceil \frac{n-1}{3} \right\rceil\right\} \\
&= \max\left\{\left\lceil \frac{n}{3} \right\rceil\ ,\ \left\lceil \frac{n-1}{3} \right\rceil\right\} \\
&= \left\lceil \frac{n}{3} \right\rceil.
\end{aligned}$$

□

Hilbert's Syzygy Theorem (see Theorem (2.4.2)) says that the longest a minimal free resolution for S/I_{P_n} could be is $n+1$. However, we see that even though a path appears to be rather simple, the projective dimension is already $\lceil \frac{2n}{3} \rceil$. Examining the previous theorem more closely, we see that divisibility of the path's length by 3 has an effect on both the projective dimension and the regularity of S/I_{P_n}. Furthermore, since trees can be constructed inductively from paths by the addition of the appropriate leaves, we see that divisibility by 3 also plays an important role in the algebraic properties of the edge ideal of a tree as it relates to its minimal free resolution.

4.1.3 Example. Consider the path of length 7 depicted below.

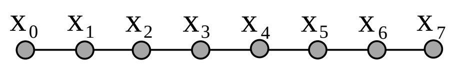


Figure 4.2: The path of length 7

Now let us consider the addition of one leaf to P_7. First let us add the leaf to the third vertex of P_7.

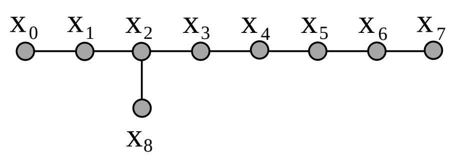


Figure 4.3: The path of length 7 with leaf added to the third vertex

Then the corresponding edge ideal is

$$I = (x_0x_1x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_2x_8)$$

and the minimal free resolution for S/I is given by

$$0 \rightarrow S^3 \rightarrow S^{15} \rightarrow S^{26} \rightarrow S^{21} \rightarrow S^8 \rightarrow S \rightarrow S/I \rightarrow 0$$

Now let us consider the addition of a leaf to the fourth vertex of P_7.

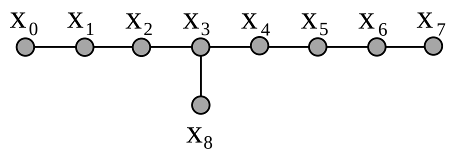


Figure 4.4: The path of length 7 with leaf added to the fourth vertex

Then the corresponding edge ideal is

$$J = (x_0x_1x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_3x_8)$$

and the minimal free resolution for S/J is given by

$$0 \to S \to S^6 \to S^{18} \to S^{27} \to S^{21} \to S^8 \to S \to S/J \to 0$$

We notice that not only are the Betti numbers of S/I and S/J different, but

$$\mathrm{pd}(S/J) = \mathrm{pd}(S/I) + 1.$$

If we take a closer look at the minimal free resolution of S/I_{P_n} we can give more detailed information about the last module in the minimal free resolution courtesy of Theorem (3.0.16). In particular, we have the following result about the overall rank of the left-most free module.

4.1.4 Corollary. *For a path of length n,*

$$\beta_{\lceil \frac{2n}{3} \rceil}(S/I_{P_n}) = \begin{cases} 1 & \text{if } 3 \nmid n \\ \dfrac{n}{3} + 1 & \text{if } 3 \mid n \end{cases}$$

Proof. Induct on the length of the path n. For $n = 1$, $I_{P_1} = (x_0x_1)$ and a minimal free resolution for S/I_{P_1} is given by

$$0 \longrightarrow S(-2) \xrightarrow{x_0x_1} S \longrightarrow S/I_{P_1} \longrightarrow 0.$$

We can see that for P_1, the left-most module of the minimal free resolution for S/I_{P_1} has rank equal to $\beta_1(S/I_{P_1}) = 1$. Assume true for P_{n-1}, and consider the coarsely graded short exact sequence

$$0 \longrightarrow \left(S/I_{P_{n-1}} : (x_{n-1}x_n)\right)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \qquad (4.3)$$

However, $I_{P_{n-1}} : (x_{n-1}x_n) = I_{P_{n-3}} + (x_{n-2})$, and hence (4.3) becomes

$$0 \longrightarrow \big(S/(I_{P_{n-3}} + (x_{n-2}))\big)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{4.4}$$

Since the generators of $I_{P_{n-3}}$ and (x_{n-2}) are disjoint, a minimal free resolution for $S/(I_{P_{n-3}} + (x_{n-2}))$ is formed by the tensor product of minimal free resolutions for $S/I_{P_{n-3}}$ and $S/(x_{n-2})$. Then by considering the mapping cone construction (see (2.1)) and applying Theorem (3.0.16) to (4.4) we obtain

$$\beta_{\lceil \frac{2n}{3} \rceil}(S/I_{P_n}) = B_{\lceil \frac{2n}{3} \rceil}(S/I_{P_{n-1}}) + B_{\lceil \frac{2n}{3} \rceil - 1}(S/I_{P_{n-3}}) + B_{\lceil \frac{2n}{3} \rceil - 2}(S/I_{P_{n-3}}). \tag{4.5}$$

Moreover, the induction hypothesis provides

$$\beta_{\lceil \frac{2n}{3} \rceil}(S/I_{P_{n-1}}) = \begin{cases} 0 & \text{if } 3 \nmid n \\ \beta_{\lceil \frac{2(n-1)}{3} \rceil}(S/I_{P_{n-1}}) & \text{if } 3 \mid n \end{cases}$$

$$= \begin{cases} 0 & \text{if } 3 \nmid n \\ 1 & \text{if } 3 \mid n \end{cases}$$

Furthermore, the induction hypothesis also provides that

$$\beta_{\lceil \frac{2n}{3} \rceil - 1}(S/I_{P_{n-3}}) = \beta_{\lceil \frac{2(n-3)}{3} \rceil + 1}(S/I_{P_{n-3}}) = 0$$

and

$$\beta_{\lceil \frac{2n}{3} \rceil - 2}(S/I_{P_{n-3}}) = \beta_{\lceil \frac{2(n-3)}{3} \rceil}(S/I_{P_{n-3}}) = \begin{cases} 1 & \text{if } 3 \nmid n \\ \dfrac{n-3}{3} + 1 & \text{if } 3 \mid n \end{cases}$$

Then if follows from (4.5) that

$$\beta_{\lceil \frac{2n}{3} \rceil}(S/I_{P_n}) = \begin{cases} 0 + 0 + 1 & \text{if } 3 \nmid n \\ 1 + 0 + \dfrac{n-3}{3} + 1 & \text{if } 3 \mid n \end{cases}$$

$$= \begin{cases} 1 & \text{if } 3 \nmid n \\ \dfrac{n}{3} + 1 & \text{if } 3 \mid n \end{cases}$$

□

Furthermore, if we would like to consider when an n-length path is maximal, we can consider the finely graded Betti numbers. Then Theorem (3.0.16) provides the following result.

4.1.5 Corollary. *For a path of length n,*

$$\beta_{\lceil \frac{2n}{3} \rceil, \boldsymbol{d}}(S/I_{P_n}) = \begin{cases} 0 & \text{if } 3 \mid n \\ 1 & \text{if } 3 \nmid n \end{cases}$$

where $\boldsymbol{d} = (1, 1, \ldots, 1)$.

Proof. We will proceed by induction on the length of the path, n. For $n = 1$, $I_{P_1} = (x_0 x_1)$, and a finely graded minimal free resolution for S/I_{P_1} is given by

$$0 \longrightarrow S(-1,-1) \xrightarrow{x_0 x_1} S \longrightarrow S/I_{P_1} \longrightarrow 0.$$

In this case, we can clearly see that $\beta_{1,(1,1)}(S/I_{P_1}) = 1$. Assume true for a path of length $n-1$. Set $(\textbf{-2}) = (0, \ldots, 0, -1, -1)$ and consider the exact sequence

$$0 \longrightarrow (S/I_{P_{n-1}} : (x_{n-1}x_n))(\textbf{-2}) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{4.6}$$

However, $I_{P_{n-1}} : (x_{n-1}x_n) = I_{P_{n-3}} + (x_{n-2})$, and hence (4.6) becomes

$$0 \longrightarrow \left(S/I_{P_{n-3}} + (x_{n-2})\right)(\textbf{-2}) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0. \tag{4.7}$$

Considering the left-most module in the minimal free resolution for S/I_{P_n} and the degree shift $(\mathbf{d}) = (1, \ldots, 1)$, the mapping cone construction (see (2.1)) and Theorem (3.0.16) applied to the short exact sequence (4.7) provide the following relationship among Betti numbers.

$$\beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}(S/I_{P_n}) = \beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}(S/I_{P_{n-1}}) + \beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}\left(S/I_{P_{n-3}} + (x_{n-2})\right)(\textbf{-2}) \tag{4.8}$$

Furthermore, since no minimal generator of $I_{P_{n-1}}$ is divisible by x_n

$$\beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}(S/I_{P_{n-1}}) = 0$$

and (4.8) becomes

$$\beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}(S/I_{P_n}) = \beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}\left(S/I_{P_{n-3}} + (x_{n-2})\right)(\textbf{-2}). \tag{4.9}$$

At this point we break into two distinct cases based upon the divisibility of the paths length, n, by 3.

Case (i): If $n \mid 3$, then $(n-3) \mid 3$. Applying the induction hypothesis to (4.9) provides

$$\beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}(S/I_{P_n}) = 0.$$

Case (ii): If $n \nmid 3$, then $(n-3) \nmid 3$. Applying the induction hypothesis to (4.9) provides

$$\beta_{\lceil \frac{2n}{3} \rceil, \mathbf{d}}(S/I_{P_n}) = 1.$$

□

In particular, the previous theorem states that paths are maximal precisely when their length is not divisible by 3. Even though we have not previously considered a path of length 0, i.e. the graph consisting of a single vertex. We will say that such a graph is not maximal by requiring that the polynomial ring S have at least 2 variables. The algorithm presented at the end of Chapter 3 is based upon the decomposition of the original graph into smaller known graphs, in particular into paths and complete bipartite graphs $K_{1,m}$. Then using that $K_{1,m}$ is maximal for all $m \geq 1$ and paths are maximal when $3 \nmid n$, we are able to deduce when an arbitrary tree is maximal. This idea was used in the development of the algorithm shown at the end of Chapter 3. The following example will illustrate the decomposition of a tree to determine its maximality.

4.1.6 Example. We want to determine whether the following tree, T, is maximal.

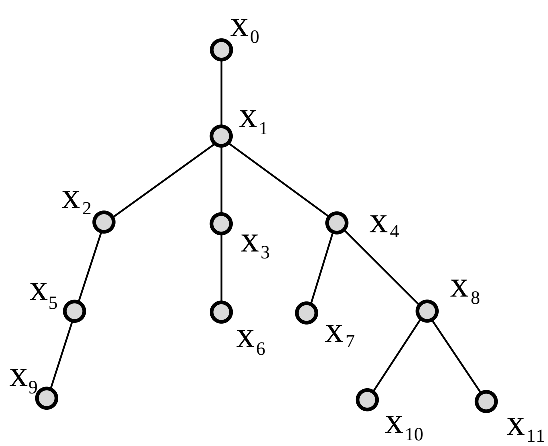


Figure 4.5: A tree on 12 vertices

We first select a leaf of T, say x_9. Then we recall that T is maximal precisely when the subforest defined by $T : (x_5x_9)$ is maximal. Let us call this subforest F_1 and consider its maximality.

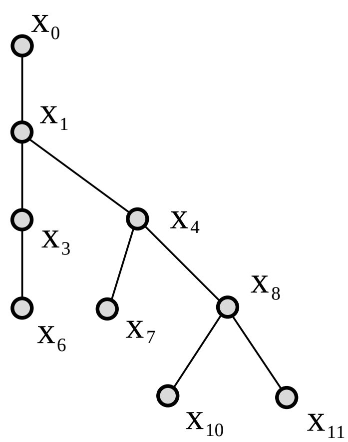


Figure 4.6: The subforest F_1 of T defined by $T : (x_5x_9)$

Now let us select a leaf of F_1, say x_6. We note that for F_1 to be maximal, $F_1 : (x_3x_6)$ must be maximal. Then let us consider the subforest of F_1 defined by $F_1 : (x_3x_6)$, and denote it by F_2.

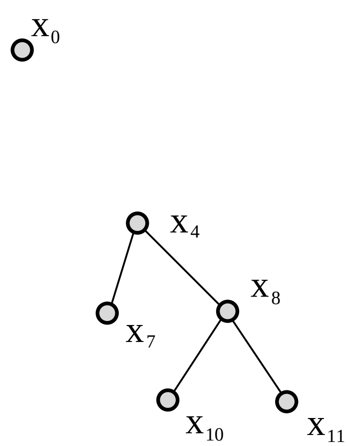


Figure 4.7: The subforest F_2 of F_1 defined by $F_1 : (x_3x_6)$

We recall that for a forest to be maximal, each component tree of the forest must be maximal. However, we notice that the vertex x_0 forms a component tree of F_2. Furthermore, x_0 forms a path of length 0, and hence is not maximal. Therefore, we have determined that the original tree T was not maximal. To verify this, we consider the following Betti diagram for S/I_T obtained from Macaulay 2 (see [8]).

Total :	1	11	38	68	70	42	14	2
0 :	1	–	–	–	–	–	–	–
1 :	–	11	15	6	1	–	–	–
2 :	–	–	23	50	37	11	1	–
3 :	–	–	–	12	32	31	13	2

From this Betti diagram we see that the leftmost module in a minimal free resolution for S/I_T is of rank 2 and has both copies of S in the coarsely graded shift $3+7=10$. However, for T to be maximal, this leftmost module must have a shift of 11. Therefore I_T is not maximal.

The above procedure for determining if a given tree (or forest) is maximal is based upon the idea that a tree, T, is maximal if and only if the subforest defined by $T:(x_{n-1}x_n)$ is maximal where x_n is a leaf of the tree with neighbor x_{n-1}. We also notice that there can be a great advantage in this algorithm by choosing to remove the leaf whose neighbor has the highest degree. However, this is not always the best choice. For instance, in the previous example x_0 would be the leaf whose neighbor has the highest degree, namely 4, but removing x_0 would still result in more than one step to determine whether or not T is maximal. However, if we were to first remove vertex x_{10}, the subforest of T defined by $T:(x_8x_{10})$ is depicted below.

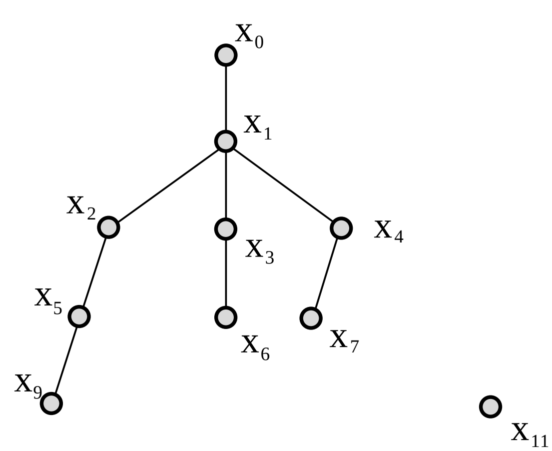


Figure 4.8: The subforest of T defined by $T:(x_8x_{10})$

After this one step, we can already see that the original tree T is not maximal, because x_{11} constitutes a path of length 0, and hence is not maximal.

We would next like to consider when a path is *level*, i.e. when the last module in the minimal free resolution for S/I_{P_n} has only one degree shift.

4.1.7 Proposition. *Let P_n be a path of length n. Then the corresponding edge ideal, I_{P_n}, is level with level shift given by*

$$\begin{cases} n+1 & \text{if } 3 \nmid n \\ n & \text{if } 3 \mid n \end{cases}$$

Proof. We will proceed by induction on the path's length, n. For $n = 1$ and $n = 2$, the claim follows from Corollaries (4.1.5) and (4.1.4). For $n = 3$, $I_{P_3} = (x_0x_1, x_1x_2, x_2x_3)$ is level with level shift 3. Assume the claim holds true for a path of length $n-1$. Then for $n \geq 4$, we have the following exact sequence as in Theorem (3.0.16).

$$0 \longrightarrow S/(I_{P_{n-3}} + (x_{n-2}))(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{P_n} \longrightarrow 0$$

Moreover, Proposition (4.1.2) provides that

$$\begin{aligned} \operatorname{pd}(S/(I_{P_{n-3}} + (x_{n-2}))) &= \left\lceil \frac{2n}{3} \right\rceil - 1 \\ \operatorname{pd}(S/I_{P_{n-1}}) &= \left\lceil \frac{2n-2}{3} \right\rceil. \end{aligned}$$

Then we have the following two cases based upon the divisibility of the path's length by 3.

Case (i): If $3 \nmid n$, then $\operatorname{pd}(S/(I_{P_{n-3}} + (x_{n-2}))) = \operatorname{pd}(S/I_{P_{n-1}})$. Furthermore, by the induction hypothesis $I_{P_{n-3}} + (x_{n-2})$ is level. Hence S/I_{P_n} is also level with level shift $(n-3)+1+1+2 = n+1$.

Case (ii): If $3 \mid n$, then $\operatorname{pd}(S/I_{P_{n-1}}) = \operatorname{pd}(S/(I_{P_{n-3}} + (x_{n-2}))) + 1$. Moreover, the induction hypothesis provides that both $I_{P_{n-3}} + (x_{n-2})$ and $I_{P_{n-1}}$ are level with

$$\begin{aligned} \text{level shift of } S/(I_{P_{n-3}} + (x_{n-2}))(-2) &= (n-3)+1+2 = n \\ \text{level shift of } S/I_{P_{n-1}} &= (n-1)+1 = n. \end{aligned}$$

Therefore, S/I_{P_n} is also level with level shift n.

□

4.2 Minimal Primary Decompositions of the Edge Ideals of Paths

We would next like to study the decomposition of I_{P_n} into an intersection of prime ideals. More formally, this decomposition is referred to as the *minimal primary decomposition* of the ideal I_{P_n} and is defined as follows.

4.2.1 Definition. Let $I \subset S$ be an ideal.

(i) The *radical of* I, denoted $\sqrt{I}$, is the ideal

$$\sqrt{I} = (a \in S \mid a^m \in I \text{ for some } m > 0).$$

(ii) I is a *primary ideal* if $fg \in I$ implies either $f \in I$ or $g^m \in I$ for some $m > 0$.

(iii) A *primary decomposition* of I is an expression of I as a finite intersection of primary ideals, i.e.

$$I = \bigcap_{i=1}^{n} \mathfrak{p}_i.$$

This decomposition is called a *minimal primary decomposition* if we also have the added restrictions that

(a) $\sqrt{\mathfrak{p}_i}$ are all distinct; and

(b) $\bigcap_{j \neq i} \mathfrak{p}_j \not\subset \mathfrak{p}_i$.

In the case of monomial ideals, there is an algorithm to determine a minimal primary decomposition. It is based upon the following relationships.

4.2.2 Lemma. *Let $I, J, K \subset S$ be monomial ideals. Then*

(i) $(I + J) \cap K = (I \cap K) + (J \cap K)$

(ii) $(I \cap J) + K = (I + K) \cap (J + K)$

It should be noted that in general, for arbitrary ideals $I, J, K \subset S$, we have the following relationships

(i) $(I \cap K) + (J \cap K) \subset (I + J) \cap K$

(ii) $(I \cap J) + K \subset (I + K) \cap (J + K)$

The decomposition of monomial ideals presented in Lemma (4.2.2) states that the minimal primary decomposition of a monomial ideal I is actually a decomposition of I into an intersection of prime ideals. For this reason, in the case of monomial ideals, we will use the phrases *minimal primary decomposition* and *minimal prime decomposition* interchangeably.

4.2.3 Example. Consider the monomial ideal

$$J = (x_0^2 x_1, x_1 x_2, x_0 x_2) \subset S := k[x_0, x_1, x_2].$$

Then using the decompositions given in Lemma (4.2.2) we obtain the following primary decomposition of J.

$$\begin{aligned}
J = (x_0^2 x_1, x_1 x_2, x_0 x_2) &= [(x_0^2) \cap (x_1)] + (x_1 x_2, x_0 x_2) \\
&= [(x_0^2) + (x_1 x_2, x_0 x_2)] \cap [(x_1) + (x_1 x_2, x_0 x_2)] \\
&= (x_0^2, x_1 x_2, x_0 x_2) \cap (x_1, x_0 x_2) \\
&= (x_0^2, x_1 x_2, x_0 x_2) \cap (x_0, x_1) \cap (x_1, x_2) \\
&= [(x_0^2, x_1 x_2) + (x_0 x_2)] \cap (x_0, x_1) \cap (x_1, x_2) \\
&= [(x_0^2, x_1 x_2) + [(x_0) \cap (x_2)]] \cap (x_0, x_1) \cap (x_1, x_2) \\
&= (x_0, x_1 x_2) \cap (x_0^2, x_2) \cap (x_0, x_1) \cap (x_1, x_2) \\
&= [(x_0) + [(x_1) \cap (x_2)]] \cap (x_0^2, x_2) \cap (x_0, x_1) \cap (x_1, x_2) \\
&= (x_0, x_1) \cap (x_0, x_2) \cap (x_0^2, x_2) \cap (x_0, x_1) \cap (x_1, x_2) \\
&= (x_0, x_1) \cap (x_0, x_2) \cap (x_0^2, x_2) \cap (x_1, x_2)
\end{aligned}$$

However, this primary decomposition is not minimal because $(x_0^2, x_2) \subset (x_0, x_2)$. Using this containment we obtain the minimal primary decomposition of J, namely

$$J = (x_0, x_1) \cap (x_0, x_2) \cap (x_1, x_2).$$

As the previous example illustrates, this algorithm can become quite tedious when done by hand even for relatively simple monomial ideals. It should also be noted that, in general, we can have many more prime ideals in the decomposition of a monomial ideal $I \subset S$ than there are minimal generators of I. If we specialize to the monomial ideals that arise as edge ideals of simple graphs (i.e. square-free quadratic monomial ideals), we can realize the prime ideals in the decomposition of I_G as minimal vertex covers of the planar graph G, which in turn allows us to quickly determine and verify the prime ideals in the minimal primary decomposition of I_G.

4.2.4 Definition. Let G be a simple graph on the vertex set $V_G = \{x_0, \ldots, x_n\}$. We will also assume that G possesses no isolated vertex, i.e. for each vertex x_i there is an edge e of G with $x_i \in e$. A *vertex cover* of G is a subset $C \subset V_G$ such that, for each edge $\{x_i, x_j\}$ of G, one has either $x_i \in C$ or $x_j \in C$. Such a vertex cover C is called *minimal* if no subset $C' \subset C$ with $C' \neq C$ is a vertex cover of G.

4.2.5 Example. Consider the path of length 5. Then the following represents a minimal vertex cover of P_5.

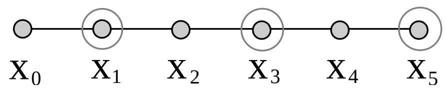


Figure 4.9: A minimal vertex cover of P_5

However, the following is a vertex cover of P_5 that is not minimal.

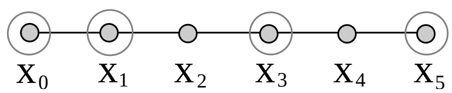


Figure 4.10: A non-minimal vertex cover of P_5

In the next theorem, we examine the relationship between the prime ideals occurring in the minimal prime decomposition of I_G and the minimal vertex covers of the simple graph G.

4.2.6 Theorem.

$$\left\{ \begin{array}{c} \textit{Minimal vertex} \\ \textit{covers of a} \\ \textit{simple graph } G \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Associated prime} \\ \textit{ideals of } I_G \end{array} \right\}$$

Proof. Let V be a minimal vertex cover of a simple graph G with vertex set V_G. We want to show that (V), the ideal whose generators are the vertices contained in V is an associated prime ideal of I_G. Since V is a vertex cover of G, we have that for each edge $\{x_i, x_j\}$ of G either $x_i \in V$ or $x_j \in V$. It follows that $x_i x_j \in (V)$ for each $x_i, x_j \in V_G$. Hence $I_G \subset (V)$.

Conversely, assume that $\mathfrak{p}$ is an associated prime ideal of I_G. Then $I_G \subset \mathfrak{p}$ which implies that $x_i x_j \in \mathfrak{p}$ for each $x_i, x_j \in V_G$. Hence

$$V = \{x_i \mid x_i \text{ is a minimal generator of } \mathfrak{p}\}$$

is a vertex cover for G. To show that V is minimal let us assume the contrary, i.e. that there exists $V' \subset V$ such that V' is a vertex cover of G. Then there would be a prime ideal $\mathfrak{p}' = (V')$ with $I_G \subset \mathfrak{p}' \subset \mathfrak{p}$. However, this would imply that $\mathfrak{p}$ was not minimal, a contradiction. Therefore, V must be a minimal vertex cover. □

If we restrict to the class of paths, the number of minimal vertex covers of P_n (and correspondingly the number of associated prime ideals of I_{P_n}) can be represented by the following recursive formula.

4.2.7 Proposition. *Let $P(n)$ represent the number of minimal vertex covers of P_n. Then*

$$P(n) = P(n-2) + P(n-3)$$

Proof. Proceed by induction of the length of the path, n. The following table illustrates the base case.

n	Minimal Vertex Covers	$P(n)$
0	$\{x_0\}$	1
1	$\{x_0\}$,$\{x_1\}$	2
2	$\{x_1\}, \{x_0, x_2\}$	2
3	$\{x_0, x_2\}, \{x_1, x_2\}, \{x_1, x_3\}$	3

Thus, $P(3) = P(1) + P(0)$. Assume the claim is true for P_{n-1}. We make the following definition.

$$q(n) := |\{\text{Minimal vertex covers of } P_n \text{ that include the vertex } x_n\}|\,.$$

Notice that the above definition for $q(n)$ also means that x_{n-1} is not chosen. Furthermore, we have the following equality.

$$P(n) = q(n) + P(n-2) \tag{4.10}$$

Moreover,

$$\begin{aligned} P(n) &= \begin{pmatrix} \text{Choose } x_{n-1} \\ \text{Can't choose } x_n \end{pmatrix} + \begin{pmatrix} \text{Choose } x_n \\ \text{Can't choose } x_{n-1} \\ \text{Have to choose } x_{n-2} \\ \text{Don't choose } x_{n-3} \end{pmatrix} + \begin{pmatrix} \text{Choose } x_n \\ \text{Can't choose } x_{n-1} \\ \text{Have to choose } x_{n-2} \\ \text{Choose } x_{n-3} \end{pmatrix} \\ &= \quad P(n-2) \quad + \quad q(n-2) \quad + \quad q(n-3) \end{aligned}$$

Using (4.10) we obtain

$$P(n) = P(n-2) + [P(n-2) - P(n-4)] + [P(n-3) - P(n-5)].$$

Finally, applying the induction hypothesis to $P(n-2)$ provides the claim, namely that

$$\begin{aligned} P(n) &= P(n-2) + P(n-4) + P(n-5) - P(n-4) + P(n-3) - P(n-5) \\ &= P(n-2) + P(n-3). \end{aligned}$$

□

Using this recursive formula for the number of prime ideals in the minimal primary decomposition for I_{P_n} we obtain the following explicit formula.

4.2.8 Corollary. *The number of associated prime ideals for I_{P_n} is given by*

$$P(n) = \sum_{i=1}^{3} \frac{(r_i+1)^2}{r_i^n(r_i^3+2)}$$

where $r_1, r_2,$ and r_3 represent the 3 distinct roots of $x^3 + x^2 - 1$.

Proof. The proof follows from Proposition (4.2.7) and uses standard techniques of ordinary differential equations. □

4.3 Minimal Free Resolutions of the Edge Ideals of 3-Spiders

It seems natural to extend from paths to the class of graphs resembling

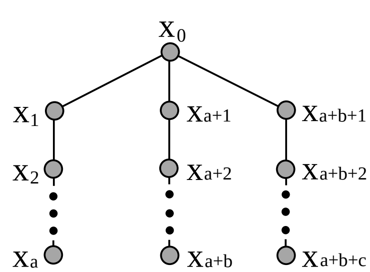


Figure 4.11: A spider with 3 legs

Since this graph resembles a spider with 3 legs, we will call it a 3-spider. This is a natural extension from the class of paths, because if we delete the rightmost leg we would be left with a path of length $a + b$ as illustrated below.

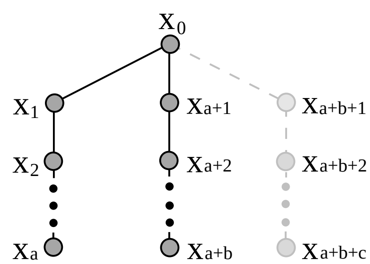


Figure 4.12: A 3-spider as a natural extension of a path

Using the formula for the length of a minimal free resolution corresponding to the quotient ring of a path of length n presented in Proposition (4.1.2) and the mapping cone construction presented in Theorem (3.0.16), we can write an explicit formula for the length of a minimal free resolution corresponding to the quotient ring of a 3-spider.

4.3.1 Proposition. *Let G be the graph of a 3-spider. Then for*

$$
\begin{aligned}
c=1: \quad \mathrm{pd}(S/I_G) &= \left\lceil \frac{2a-1}{3} \right\rceil + \left\lceil \frac{2b-1}{3} \right\rceil + 1 \\
c=2: \quad \mathrm{pd}(S/I_G) &= \begin{cases} \left\lceil \frac{2a-1}{3} \right\rceil + \left\lceil \frac{2b-1}{3} \right\rceil + 1 & \textit{if } a,b \equiv 1 \textit{ mod } 3 \\ \left\lceil \frac{2(a-1)}{3} \right\rceil + \left\lceil \frac{2(b-1)}{3} \right\rceil + 2 & \textit{else} \end{cases} \\
c \geq 3: \quad \mathrm{pd}(S/I_G) &= \left\lceil \frac{2(a+b+c)}{3} \right\rceil + (-1)^r r
\end{aligned}
$$

$$\textit{where } r = \min\{a \textit{ mod } 3, b \textit{ mod } 3, c \textit{ mod } 3\}$$

Proof. Consider the following short exact sequence

$$0 \longrightarrow \left(S/I_{G\setminus\{x_{a+b+c-1},x_{a+b+c}\}} : (x_{a+b+c-1}x_{a+b+c})\right)(-2) \longrightarrow S/I_{G\setminus\{x_{a+b+c-1},x_{a+b+c}\}} \longrightarrow S/I_G \longrightarrow 0.$$

Then the mapping cone construction (see (2.1)) and Theorem (3.0.16) imply that

$$
\begin{aligned}
\mathrm{pd}(S/I_G) = \max\{&\mathrm{pd}(S/I_{G\setminus\{x_{a+b+c-1},x_{a+b+c}\}}), \\
&\mathrm{pd}(S/I_{G\setminus\{x_{a+b+c-1},x_{a+b+c}\}} : (x_{a+b+c-1}x_{a+b+c})) + 1\}.
\end{aligned}
$$

However, we can consider $I_{G\setminus\{x_{a+b+c-1},x_{a+b+c}\}} : (x_{a+b+c-1}x_{a+b+c})$ graphically as follows.

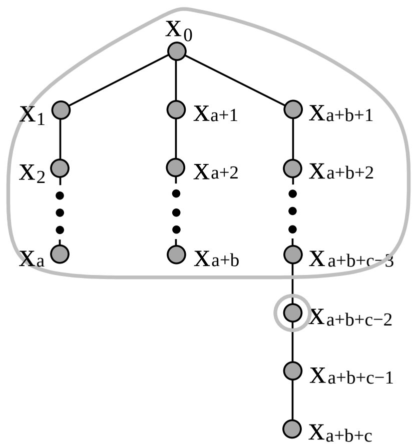


Figure 4.13: The decomposition of a 3-spider

We will proceed by induction on the length of the third leg, c.

$$\begin{aligned}
c = 1 : \operatorname{pd}(S/I_G) = &\ \max\left\{\left\lceil\frac{2(a+b)}{3}\right\rceil, \left\lceil\frac{2(a-2)}{3}\right\rceil + \left\lceil\frac{2(b-2)}{3}\right\rceil + 3\right\} \\
= &\ \max\left\{\left\lceil\frac{2(a+b)}{3}\right\rceil, \left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1\right\} \\
= &\ \left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1 \\
c = 2 : \operatorname{pd}(S/I_G) = &\ \max\left\{\begin{array}{l} \left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1, \\ \qquad\qquad \left\lceil\frac{2(a-1)}{3}\right\rceil + \left\lceil\frac{2(b-1)}{3}\right\rceil + 2 \end{array}\right\} \\
= &\ \begin{cases} \left\lceil\frac{2a-1}{3}\right\rceil + \left\lceil\frac{2b-1}{3}\right\rceil + 1 & \text{if } a, b \equiv 1 \bmod 3 \\ \left\lceil\frac{2(a-1)}{3}\right\rceil + \left\lceil\frac{2(b-1)}{3}\right\rceil + 2 & \text{else} \end{cases}
\end{aligned}$$

$$c=3:\operatorname{pd}(S/I_G)=\begin{cases}\max\left\{\left\lceil\frac{2a-1}{3}\right\rceil+\left\lceil\frac{2b-1}{3}\right\rceil+1,\left\lceil\frac{2(a+b)}{3}\right\rceil+2\right\} & \text{if } a,b\equiv 1 \bmod 3\\ \max\left\{\left\lceil\frac{2(a-1)}{3}\right\rceil+\left\lceil\frac{2(b-1)}{3}\right\rceil+2,\left\lceil\frac{2(a+b)}{3}\right\rceil+2\right\} & \text{else}\end{cases}$$

$$=\left\lceil\frac{2(a+b)}{3}\right\rceil+2$$

$$=\left\lceil\frac{2(a+b+c)}{3}\right\rceil$$

Assume the statement is true for the third length having length $c-1$. Then

$$\begin{aligned}\operatorname{pd}(S/I_G) &= \max\left\{\begin{array}{l}\left\lceil\frac{2(a+b+c-1)}{3}\right\rceil+(-1)^{r'}r',\\ \qquad\left\lceil\frac{2(a+b+c-3)}{3}\right\rceil+(-1)^{r}r+2\end{array}\right\}\\ &= \max\left\{\left\lceil\frac{2(a+b+c-1)}{3}\right\rceil+(-1)^{r'}r',\left\lceil\frac{2(a+b+c)}{3}\right\rceil+(-1)^{r}r\right\}\end{aligned}$$

where $r'=\min\{a \bmod 3, b \bmod 3, (c-1) \bmod 3\}$.

Case (i): If $c\equiv 0 \bmod 3$, then $(c-1)\equiv 2 \bmod 3$ and $r'=\min\{a \bmod 3, b \bmod 3\}$. However, regardless of the value of r', we see that

$$\begin{aligned}\operatorname{pd}(S/I_G) &= \max\left\{\left\lceil\frac{2(a+b+c-1)}{3}\right\rceil+(-1)^{r'}r',\left\lceil\frac{2(a+b+c)}{3}\right\rceil\right\}\\ &= \left\lceil\frac{2(a+b+c)}{3}\right\rceil.\end{aligned}$$

Case (ii): If $c\equiv 1 \bmod 3$, then $(c-1)\equiv 0 \bmod 3$ and $r'=0$. Furthermore, since $(c-1)\equiv 0 \bmod 3$ implies that $\left\lceil\frac{2(a+b+c-1)}{3}\right\rceil=\left\lceil\frac{2(a+b+c)}{3}\right\rceil-1$, we obtain

$$\begin{aligned}\operatorname{pd}(S/I_G) &= \max\left\{\left\lceil\frac{2(a+b+c-1)}{3}\right\rceil,\left\lceil\frac{2(a+b+c)}{3}\right\rceil+(-1)^{r}r\right\}\\ &= \left\lceil\frac{2(a+b+c)}{3}\right\rceil+(-1)^{r}r.\end{aligned}$$

Case (iii): If $c \equiv 2 \bmod 3$, then $(c-1) \equiv 1 \bmod 3$ and $r' \in \{0,1\}$. However, for either value of r' we see that

$$\begin{aligned}
\operatorname{pd}(S/I_G) &= \max\left\{ \begin{array}{r} \left\lceil \frac{2(a+b+c-1)}{3} \right\rceil + (-1)^{r'} r', \\ \left\lceil \frac{2(a+b+c)}{3} \right\rceil + (-1)^{r} r \end{array} \right\} \\
&= \left\lceil \frac{2(a+b+c)}{3} \right\rceil + (-1)^{r} r.
\end{aligned}$$

□

5 Edge Ideals of Cycles

In the previous chapters, we looked at simple graphs that contained a vertex of degree 1. It is natural to ask when we can extend the previous results to generate information about the edge ideals of simple graphs that do not contain a vertex of degree 1. The simplest of these is a cycle of length n which can be depicted as follows.

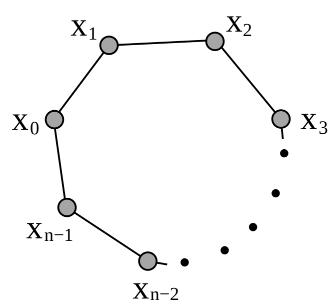


Figure 5.1: The cycle of length n

If we compare an n-cycle, denoted C_n, to a path of length $n-1$ we see the following relationships among both the graphs and the corresponding edge ideals.

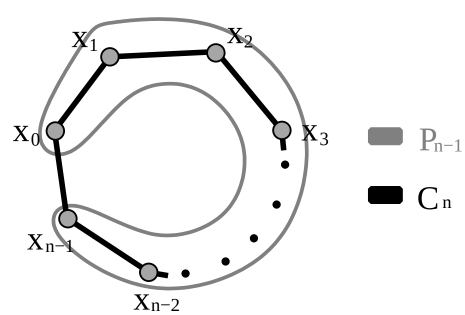


Figure 5.2: The decomposition $I_{C_n} = I_{P_{n-1}} + (x_{n-1}x_0)$

Using this relationship we obtain the following explicit formula for the length of a minimal free resolution corresponding to C_n.

5.0.2 Proposition. *Let C_n denote a cycle of length n. Then*

$$\operatorname{pd}(S/I_{C_n}) = \begin{cases} \left\lceil \dfrac{2n}{3} \right\rceil & \textit{if } 3 \mid (n-1) \\ \left\lceil \dfrac{2(n-1)}{3} \right\rceil & \textit{if } 3 \nmid (n-1) \end{cases}$$

Proof. We want to mimic the procedure used for trees. However, in the case of trees, we removed a leaf of the tree, i.e. a vertex of degree 1. In the case of an n-cycle this is not an option, so we just remove an arbitrary edge, say $\{x_{n-1}, x_0\}$. Upon removing edge $\{x_{n-1}, x_0\}$, we are left with P_{n-1} as shown above in Figure (5.2). We must be careful though, because Theorem (3.0.16) no longer applies. Consider the exact sequence

$$0 \longrightarrow \left(S/I_{P_{n-1}} : (x_{n-1}x_0)\right)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{C_n} \longrightarrow 0. \tag{5.1}$$

Moreover,

$$I_{P_{n-1}} : (x_{n-1}x_0) = (x_1, x_{n-2}) + (x_2x_3, x_3x_4, \ldots, x_{n-4}x_{n-3}) = (x_{n-1}x_0) + I_{P_{n-5}}.$$

Hence, (5.1) becomes

$$0 \longrightarrow \left(S/((x_1, x_{n-2}) + I_{P_{n-5}})\right)(-2) \longrightarrow S/I_{P_{n-1}} \longrightarrow S/I_{C_n} \longrightarrow 0. \tag{5.2}$$

Furthermore, Theorem (4.1.2) provides that

$$\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P_{n-5}})\right) = 2 + \left\lceil \frac{2(n-5)}{3} \right\rceil = \left\lceil \frac{2(n-2)}{3} \right\rceil$$

and

$$\operatorname{pd}\left(S/I_{P_{n-1}}\right) = \left\lceil \frac{2(n-1)}{3} \right\rceil.$$

From the mapping cone construction presented in (2.1), we obtain that

$$\operatorname{pd}(S/I_{C_n}) \leq \max\left\{\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P_{n-5}})\right) + 1, \operatorname{pd}\left(S/I_{P_{n-1}}\right)\right\}.$$

We will proceed by showing that the last module of the free resolution for S/I_{C_n} obtained via the mapping cone construction cannot cancel, i.e. that

$$\operatorname{pd}(S/I_{C_n}) = \max\left\{\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P_{n-5}})\right) + 1, \operatorname{pd}\left(S/I_{P_{n-1}}\right)\right\}.$$

Case (i): If $3 \mid (n-1)$, or $n \equiv 1 \bmod 3$, then P_{n-1} is not maximal. However, in this case, P_{n-5} is maximal. Additionally,

$$\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P_{n-5}})\right) = \operatorname{pd}\left(S/I_{P_{n-1}}\right)$$

and hence there can be no cancellation in the last module of the free resolution for S/I_{C_n} formed from the mapping cone construction. Therefore

$$\operatorname{pd}(S/I_{C_n}) = \left\lceil \frac{2(n-1)}{3} \right\rceil + 1 = \left\lceil \frac{2n}{3} \right\rceil .$$

Case (ii): If $3 \nmid (n-1)$, or $n \equiv 0, 2 \bmod 3$, then P_{n-1} is maximal. Also

$$\operatorname{pd}\left(S/((x_1, x_{n-2}) + I_{P_{n-5}})\right) = \operatorname{pd}\left(S/I_{P_{n-1}}\right) - 1$$

Furthermore, the copy of S with the maximal shift in the last module of the free resolution for S/I_{C_n} obtained via the mapping cone construction cannot cancel, and consequently

$$\operatorname{pd}(S/I_{C_n}) = \left\lceil \frac{2(n-1)}{3} \right\rceil .$$

□

The above proposition says that the length of a minimal free resolution corresponding to C_n agrees with the length of a minimal free resolution of P_{n-1} as long as $3 \nmid (n-1)$. However, in the alternate case, namely when $3 \mid (n-1)$, we see that the length of the minimal free resolution corresponding to C_n agrees with the length of a minimal free resolution for P_n.

In general, we notice that simple graphs are compositions of trees and cycles. As seen in the case of cycles, even though Theorem (3.0.16) does not apply to a general simple graph G, we can still use the short exact sequence

$$0 \longrightarrow \left(S/I_{G\setminus\{x_{n-1},x_n\}} : (x_{n-1}x_n)\right)(-2) \stackrel{x_{n-1}x_n}{\longrightarrow} S/I_{G\setminus\{x_{n-1},x_n\}} \longrightarrow S/I_G \longrightarrow 0$$

where $\{x_{n-1}, x_n\}$ is an arbitrary edge of the graph G. Since we can reconstruct the simple graph G by the addition of edges to subgraphs of G that are paths and cycles, we can generate estimates using the above results on the projective dimension of the more general module S/I_G for an arbitrary simple graph G.

6 Ferrers Graphs and Ferrers Tableaux

In the remaining chapters, we would like to study another class of simple graphs. The graphs that we want to study are related to a class of bipartite graphs known as Ferrers graphs. In preparation for this chapter, we need a few more tools and algebraic properties for a given ideal I considered in a polynomial ring over a field.

6.1 Ferrers Graphs

6.1.1 Definition. A *Ferrers graph* is a bipartite graph, G, on two distinct vertex sets $\mathbf{X} = \{x_1, \ldots, x_n\}$ and $\mathbf{Y} = \{y_1, \ldots, y_m\}$ such that if (x_i, y_j) is an edge of G, then so is (x_p, y_q) for $1 \leq p \leq i$ and $1 \leq q \leq j$. In addition, (x_1, y_m) and (x_n, y_1) are required to be edges of G.

Given a Ferrers graph G, we can associate to G a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$$

where λ_i represents the degree of the vertex x_i for $1 \leq i \leq n$. The conditions for a Ferrers graph imply that

$$\lambda_1 = m \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$$

and thus λ is a partition.

6.1.2 Example. The following is the Ferrers graph with partition $\lambda = (3, 2, 2, 1)$.

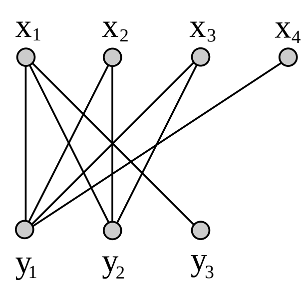


Figure 6.1: The Ferrers graph with partition $\lambda = (3, 2, 2, 1)$

This example illustrates that even with small examples, Ferrers graphs can be relatively complicated to draw. This difficulty can greatly increase with an increase in the cardinality of either of the sets **X** or **Y**. In the case of Ferrers graphs though, we have a simpler way to display the graph's structure while still being able to easily identify the edges and the disjoint vertex sets **X** and **Y**. This method will arise from the partition λ.

We can associate to a given Ferrers graph with partition λ a diagram $\mathbf{T}_\lambda$, called the *Ferrers tableau*, that consists of an array of n rows of cells with λ_i adjacent cells, right justified, in the i^{th}-row.

6.1.3 Example. The Ferrers tableau associated to the Ferrers graph with partition $\lambda = (3, 2, 2, 1)$ seen in the previous example is given by

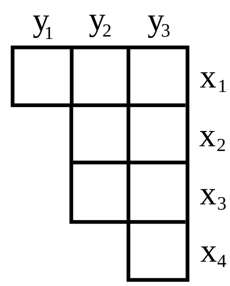


Figure 6.2: The Ferrers tableau with partition $\lambda = (3, 2, 2, 1)$

Then each box in a Ferrers tableau, $\mathbf{T}_\lambda$, represents an edge of the Ferrers graph with partition λ.

As in the case of the graphs of trees and cycles, we are interested in studying algebraic structures related to a given Ferrers graph. In particular, we would link to consider the toric ring of a Ferrers graph as studied by Corso and Nagel in [3].

6.2 Toric Rings Associated to Ferrers Graphs

Given a Ferrers graph G, we can look at the edge ideal corresponding to G. This ideal lies in the polynomial ring $S = k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ where the variables of the polynomial ring S correspond to the vertex set of G. In [3], Corso and Nagel studied the algebraic properties of the *toric ring* $k[G]$ associated to a given Ferrers graph G, where $k[G]$ is the monomial subalgebra generated by the elements $x_i y_j$.

Consider the Ferrers tableau $\mathbf{T} := \mathbf{T}_\lambda$ associated to a Ferrers graph G with partition $\lambda = (\lambda_1, \ldots, \lambda_s, 1, \ldots, 1)$. Now let us consider the subtableau $\mathbf{T}'$ of $\mathbf{T}$

formed by deleting all boxes in the first row beyond the λ_2 one and all boxes in the first column beyond the s one. Then the partition corresponding to $\mathbf{T}'$ is given by $\lambda' = (\lambda_2, \lambda_2, \ldots, \lambda_s)$. By considering the subtableau $\mathbf{T}'$ of $\mathbf{T}$ we have guaranteed that the outer border of the tableau has a minimum thickness of 2. Since the thickness of the outer border of the tableau is at least 2, we can treat the Ferrers tableau like a matrix and consider the (2×2)-minors of the tableau. In particular, we will later see that these 2-minors of the tableau are intimately related to the structure of the toric ring.

6.2.1 Example. Consider the Ferrers tableau $\mathbf{T}_{(5,4,4,2,1,1)}$ shown below.

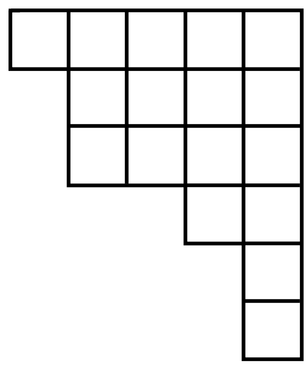


Figure 6.3: The Ferrers tableau with partition $\lambda = (5, 4, 4, 2, 1, 1)$

Then the subtableau $\mathbf{T}'$ of $\mathbf{T}$ has partition $\lambda' = (4, 4, 4, 2)$ and is depicted as

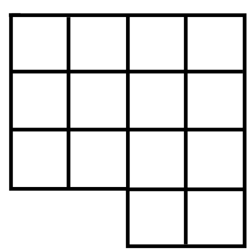


Figure 6.4: The subtableau $\mathbf{T}'$ of $\mathbf{T}_{(5,4,4,2,1,1)}$

In [3], Corso and Nagel studied the structure of the special fiber ring of the edge ideal of a Ferrers graph. In particular, they demonstrated the relationship between the special fiber ring of the edge ideal of a Ferrers graph, the toric ring of a Ferrers graph, and the (2×2)-minors of the associated Ferrers tableau. However, to prepare for their result we will need the following definition.

6.2.2 Definition. Let $I \subset S = k[x_1, \ldots, x_n]$ be an ideal. Then S/I is said to be *Cohen-Macaulay* provided that

$$\mathrm{pd}(S/I) = \mathrm{codim}(S/I)$$

It should be noted that the traditional definition of the Cohen-Macaulay property concerns the *depth* of an ideal (which is the maximal length of a regular sequence contained in the module), and it states that S/I is Cohen-Macaulay provided that

$$\mathrm{depth}(S/I) = \dim(S/I).$$

However, in the case of the polynomial ring S, the following theorem justifies the use of the previous definition.

6.2.3 Theorem (Auslander-Buchsbaum Formula)**.** *Let $S = k[x_1, \ldots, x_n]$. Then for an ideal $I \subset S$, we have the following relationship between the depth of S/I and S, namely that*

$$\mathrm{pd}(S/I) + depth(S/I) = depth(S).$$

In the case where S/I is Cohen-Macaulay, the Auslander-Buchsbaum Formula reduces to

$$\mathrm{pd}(S/I) = \dim(S) - \dim(S/I) = \mathrm{codim}(S/I)$$

and hence in a polynomial ring the Cohen-Macaulay property is equivalent to the definition provided above.

The Cohen-Macaulay property is a very heavily studied property. Although the definition of this property has its roots in homological algebra, Cohen-Macaulay rings have many applications in other areas of mathematics such as algebraic combinatorics. For a thorough introduction to the theory of Cohen-Macaulay rings, we refer the reader to the book of Bruns and Herzog (see [1]).

Now that we have the definition of the Cohen-Macaulay property, we are able to state the result of Corso and Nagel concerning the special fiber ring corresponding to the edge ideal of a given Ferrers graph.

6.2.4 Proposition. *Let $\boldsymbol{X} = \{x_1, \ldots, x_n\}$ and $\boldsymbol{Y} = \{y_1, \ldots, y_m\}$ be distinct sets of variables. Set $S = k[\boldsymbol{X}, \boldsymbol{Y}]$, where k is an arbitrary field, and let I_λ be the edge ideal corresponding to a Ferrers graph G with associated tableaux $\boldsymbol{T}$ and $\boldsymbol{T}'$ and partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s, 1, \ldots, 1)$. Then the special fiber ring $\mathcal{F}(I_\lambda)$ of I_λ has the following properties:*

(i) $\mathcal{F}(I_\lambda)$ *is a Cohen-Macaulay normal domain of dimension* $n + m - 1$*; and*

(ii) $\mathcal{F}(I_\lambda)$ *is the ladder determinantal ring* $k[\boldsymbol{T}]/I_2(\boldsymbol{T}')$.

It should be noted that since edge ideals are generated in one degree, the special fiber ring is also isomorphic to the toric ring of the graph. In particular, this says that the toric ring of the graph is isomorphic to the ladder determinantal ring $k[\mathbf{T}]/I_2(\mathbf{T}')$.

6.2.5 Example. Consider the Ferrers graph with partition $\lambda = (4, 3, 2, 1)$ depicted via the following tableau.

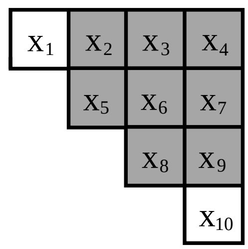


Figure 6.5: The Ferrers tableau with partition $\lambda = (4, 3, 2, 1)$

Then the toric ring is described by $k[x_1, x_2, \ldots, x_{10}]/I$ where

$$I = (x_2x_6 - x_3x_5, x_2x_7 - x_4x_5, x_3x_7 - x_4x_6, x_3x_9 - x_4x_8, x_6x_9 - x_7x_8).$$

In the next chapter we would like to consider square-free monomial ideals that are related to the defining toric ideal of the toric ring corresponding to a given Ferrers graph. The hope of studying these monomial ideals is to generate information back to the original toric ideals, whose minimal generators correspond to the (2×2)-minors of the given Ferrers tableau. Additionally, we saw in Chapter 2 that square-free quadratic monomial ideals occur as edge ideals of simple graphs, so the information that we gather about these square-free quadratic monomial ideals will also generate information concerning the edge ideals of the corresponding class of simple graphs.

7 Initial Ideals Associated to Ferrers Graphs

Given a Ferrers graph G with associated tableaux $\mathbf{T}$ and $\mathbf{T}'$, we want to study a monomial ideal that is related to $I_2(\mathbf{T}')$, the defining ideal of the toric ring. Since we will be looking at the (2×2)-minors of a given tableau, we will require from now on that the Ferrers tableau $\mathbf{T}$ have outer border with thickness greater than or equal to 2. This means that from now on we will require the defining partition of the Ferrers graph $\mathbf{T}$ to resemble $\lambda = (\lambda_2, \lambda_2, \lambda_3, \ldots, \lambda_n)$ where $\lambda_n \geq 2$.

The monomial ideals that we wish to study occur as *initial ideals* of $I_2(\mathbf{T})$. To proceed we will need the following definitions as in the book of Miller and Sturmfels (see [13]).

7.0.6 Definition. Let $S = k[x_1, \ldots, x_n]$.

(i) A *term order* $<$ is a total order on the monomials of S satisfying the following two conditions.

(a) $\mathbf{x}^{\mathbf{b}} < \mathbf{x}^{\mathbf{c}}$ if and only if $\mathbf{x}^{\mathbf{a}+\mathbf{b}} < \mathbf{x}^{\mathbf{a}+\mathbf{c}}$; and

(b) $1 < \mathbf{x}^{\mathbf{a}}$ for all nonunit monomials $\mathbf{x}^{\mathbf{a}} \in S$.

Unless otherwise noted, the chosen term order will satisfy $x_1 > x_2 > \cdots > x_n$.

(ii) Given a polynomial $f = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, the monomial $\mathbf{x}^{\mathbf{a}}$ that is largest under the given term order $<$ among those with nonzero coefficient determines the *leading term*, i.e. $\mathrm{lt}_<(f) = c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$.

(iii) If $I \subset S$ is an ideal, then the *initial ideal* of I is

$$\mathrm{in}_<(I) = (\mathrm{lt}_<(f) \mid f \in I).$$

The *lexicographic term order*, denoted by $<_{\mathrm{lex}}$, specifies that for two monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ of the same degree, $\mathbf{x}^{\mathbf{a}} >_{\mathrm{lex}} \mathbf{x}^{\mathbf{b}}$ provided that the leftmost nonzero entry of the vector $\mathbf{a} - \mathbf{b}$ is positive.

On the other hand, the *reverse lexicographic term order*, denoted by $<_{\mathrm{revlex}}$, specifies that for two monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ of the same degree, $\mathbf{x}^{\mathbf{a}} >_{\mathrm{revlex}} \mathbf{x}^{\mathbf{b}}$ provided that the rightmost nonzero entry of the vector $\mathbf{a} - \mathbf{b}$ is negative.

7.0.7 Example. Let $S = k[x_1, x_2, x_3]$. Then $x_1^2 x_3^3 >_{\mathrm{lex}} x_2^4 x_3$, but $x_2^4 x_3 >_{\mathrm{revlex}} x_1^2 x_3^3$.

In general, we should note that for a given term order, $<$,

$$\text{in}_<(f_1, \ldots, f_n) \neq (\text{lt}_<(f_1), \ldots, \text{lt}_<(f_n))$$

as illustrated by the following example.

7.0.8 Example. Consider $I = (x^2, xy + y^2) \subset k[x, y]$ in the reverse lexicographic term order. Set $f = x^2$ and $g = xy + y^2$. Then

$$yf - xg = -xy^2 \in I.$$

Moreover, this implies that

$$-xy^2 + yg = y^3 \in I$$

and hence $y^3 \in \text{in}_{\text{revlex}}(I)$. However, $y^3 \notin (\text{lt}_{\text{revlex}}(f), \text{lt}_{rlex}(g)) = (x^2, xy)$. Actually, one can check that $\text{in}_{\text{revlex}}(I) = (x^2, xy, y^3)$.

We are interested in the cases where $\text{in}_<(f_1, \ldots, f_n) = (\text{lt}_<(f_1), \ldots, \text{lt}_<(f_n))$, and with this in mind we have the following definition.

7.0.9 Definition. Let $I = (f_1, \ldots, f_r) \subset S$. The set $\{f_1, \ldots, f_r\}$ of generators of I constitutes a *Gröbner basis* with respect to the term order $<$ if the leading terms of $f_1, \ldots, f_r$ generate the initial ideal of I, i.e. if

$$\text{in}_<(I) = (\text{lt}_<(f_1), \ldots, \text{lt}_<(f_r)).$$

We would like to state a criterion for determining whether a given set of polynomials is indeed a Gröbner basis. In order to introduce this criterion, which involves pairwise examination of the generators, we must first introduce the division algorithm used in the situation of more than one variable.

7.0.10 Theorem (Multivariable Division Algorithm)**.** *Let $S = k[x_1, \ldots, x_n]$. Fix a term order $<$ on the monomials in S, and let $F = (f_1, \ldots, f_s)$ be an ordered s-tuple of polynomials in S. Then every $f \in S$ can be written as*

$$f = a_1 f_1 + \cdots + a_s f_s + r$$

where $a_i, r \in S$ for $i \in \{1, \ldots, s\}$, and either $r = 0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any $\text{lt}(f_i)$ for $i \in \{1, \ldots, s\}$.

We now want to consider an example showing the implementation of the multivariate division algorithm. Further examples of this division algorithm can be found in the book of Cox, Little, and O'Shea (see [4]).

7.0.11 Example. Let us consider the polynomials $f = xy^2 + 1$, $g_1 = xy + 1$, and $g_2 = y + 1$ in the lexicographic term order. We would like to divide f by g_1 and g_2. To begin, we notice that $\mathrm{lt}(f) = xy^2$, and $\mathrm{lt}(g_1) = xy$ clearly divides this. Thus the first step of the multivariate division algorithm produces

$$xy^2 + 1 = y(xy + 1) - y + 1.$$

Now we consider the polynomial $f_1 = -y + 1$. Then $\mathrm{lt}(f_1) = -y$, and $\mathrm{lt}(g_2) = y$ divides this. Then the second step of the division algorithm provides

$$xy^2 + 1 = y(xy + 1) - (y + 1) + 2.$$

If we consider the polynomial $f_2 = 2$, we see that $\mathrm{lt}(f_2) = 2$. However, neither $\mathrm{lt}(g_1) = xy$ nor $\mathrm{lt}(g_2) = y$ divide 2. Hence we conclude that our remainder is 2.

This multivariate division algorithm leads to the following criterion for determining whether a given set is indeed a Gröbner basis.

7.0.12 Theorem (Buchberger's Criterion)**.** *Let I be a polynomial ideal in $S = k[x_1, \ldots, x_n]$. Then a basis $G = \{g_1, \ldots, g_t\}$ for I is a Gröbner basis for I if and only if for all pairs $i \neq j$, the remainder on division of*

$$S(g_i, g_j) = \frac{\mathrm{lcm}\{\mathrm{lt}(g_i), \mathrm{lt}(g_j)\}}{\mathrm{lt}(g_i)} g_i - \frac{\mathrm{lcm}\{\mathrm{lt}(g_i), \mathrm{lt}(g_j)\}}{\mathrm{lt}(g_j)} g_j$$

by G (listed in some order) is zero.

This algorithm and its underlying reliance on the multivariate division algorithm illustrates the importance of the term order considered. We will see this explicitly in the following example from the book of Cox, Little, and O'Shea (see [4]).

7.0.13 Example. Let $I = (y - x^2, z - x^3) \subset k[x, y, z]$. Consider the set of minimal generators $G = \{y - x^2, z - x^3\}$. We wish to determine whether or not G is a Gröbner basis by using Buchberger's Criterion.

(i) First let us consider the lexicographic term order with $y > z > x$. Then

$$\begin{aligned} s(y-x^2, z-x^3) &= \frac{yz}{y}(y-x^2) - \frac{yz}{y}(z-x^3) \\ &= -zx^2 + yx^3. \end{aligned}$$

Furthermore, the multivariate division algorithm provides

$$-zx^2 + yx^3 = x^3(y-x^2) + (-x^2)(z-x^3)$$

and hence by Buchberger's criterion, G is a Gröbner basis for this term order.

(ii) Now let us consider the lexicographic term order with $x > y > z$. Then

$$\begin{aligned} s(y-x^2, z-x^3) &= \frac{x^3}{-x^2}(y-x^2) - \frac{x^3}{-x^3}(z-x^3) \\ &= -xy + z. \end{aligned}$$

However, neither $\mathrm{lt}(y-x^2) = -x^2$ nor $\mathrm{lt}(z-x^3) = -x^3$ divides $\mathrm{lt}(-xy+z) = -xy$. Hence, we conclude from Buchberger's criterion that G is not a Gröbner basis for this term order.

At this point we would like to consider the initial ideal of the defining ideal of the toric ring introduced in Chapter 6 and studied by Corso and Nagel in [3] in the reverse lexicographic term order.

7.1 The Reverse Lexicographic Term Order

In this section we will prove that the toric generators of $I_2(\mathbf{T})$ are a Gröbner basis with respect to the reverse lexicographic term order. The main tool we will use in this section is from liaison theory and is known as a *basic double link*. The following definition is from the book of Migliore (see [12]), where you will also find an introduction to liaison theory.

7.1.1 Definition. Let $0 \neq J \subset I \subset S$ be homogeneous ideals such that

$$\operatorname{codim} I = \operatorname{codim} J + 1$$

and S/J is Cohen-Macaulay. Let $f \in S$ be a homogeneous element of degree d such that $J : f = J$. Then the ideal

$$\widetilde{I} := fI + J \tag{7.1}$$

is called a *basic double link* of I.

Furthermore, if we are given that $\widetilde{I}$ is a basic double link of I as in (7.1), we get the following relationship between the ideals $\widetilde{I}$ and I

$$\operatorname{codim} \widetilde{I} = \operatorname{codim} I.$$

7.1.2 Proposition. *Consider the Ferrers tableau* $\boldsymbol{T}$ *that resembles a* $2 \times n$ *matrix, i.e. the partition of the tableau is* $\lambda = (n, n)$*, where* $n \geq 3$*. The Ferrers tableau can be depicted as*

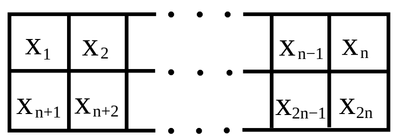


Figure 7.1: The Ferrers tableau with partition $\lambda = (n, n)$

Set

$$I_T = (\operatorname{lt}_{\text{revlex}}(t_i) \mid t_i \text{ is a minimal generator of } I_2(\boldsymbol{T}))$$

and let $\boldsymbol{T}_r$ *be the subtableau of* $\boldsymbol{T}$ *formed by removing the cell containing* x_{n+1}*. Then*

(i) $I_T = x_{n+1}(x_2, \ldots, x_n) + I_{T_r}$ *is a basic double link with* $\operatorname{codim} I_T = n - 1$*;*

(ii) $P(S/I_T, t) = \dfrac{1 + (n-1)t}{(1-t)^{n+1}}$*; and*

(iii) $\{t_i \mid t_i$ *is a minimal generator of* $I_2(\boldsymbol{T})\}$ *is a Gröbner basis in the reverse lexicographic term order.*

Proof.

(i) We proceed by induction on n. For $n = 3$, we have

$$(x_2x_4, x_3x_4, x_3x_5) = x_4(x_2, x_3) + (x_3x_5)$$

and it is clear that (x_2x_4, x_3x_4, x_3x_5) is a basic double link of (x_2, x_3). Consequently, $\operatorname{codim}(x_2x_4, x_3x_4, x_3x_5) = 2$. Assume true for a Ferrers graph with partition $\lambda = (n-1, n-1)$. By the induction hypothesis and basic double linkage, S/I_{T_r} is Cohen-Macaulay. Also

$$\operatorname{codim}(x_2, \ldots, x_n) = n - 1 \quad \text{and} \quad \operatorname{codim}(I_{T_r}) = \operatorname{codim}(I_{T_{(n-1,n-1)}}) = n - 2.$$

Hence, $I_T = x_{n+1}(x_2, \ldots, x_n) + I_{T_r}$ is a basic double link and consequently, $\operatorname{codim} I_T = \operatorname{codim}(x_2, \ldots, x_n) = n - 1$.

(ii) From the basic double link shown in (i), we get the exact sequence

$$0 \longrightarrow S/I_{T_r}(-1) \longrightarrow S/(x_2, \ldots, x_n)(-1) \bigoplus S/I_{T_r} \longrightarrow S/I_T \longrightarrow 0.$$

We proceed by induction on n. For $n = 3$, $I_T = (x_2x_4, x_3x_4, x_3x_5)$ and

$$P(S/(x_2x_4, x_3x_4, x_3x_5), t) = \frac{1 + 2t}{(1-t)^3}.$$

Assume true for a Ferrers graph with partition $\lambda = (n-1, n-1)$. Then since the Hilbert function adds along exact sequences, we get

$$H_{S/I_T}(j) = H_{S/(x_2,\ldots,x_n)}(j-1) + H_{S/I_{T_r}}(j) - H_{S/I_{T_r}}(j-1)$$

which yields the following relationship among Hilbert Series.

$$P(S/I_T, t) = tP(S/(x_2, \ldots, x_n), t) + (1-t)P(S/I_{T_r}, t)$$

However, $S/(x_2, \ldots, x_n) \cong k[x_1, x_{n+1}, \ldots, x_{2n}]$ and hence

$$P(S/I_T, t) = t\left(\frac{1}{(1-t)^{n+1}}\right) + (1-t)P(S/I_{T_r}, t).$$

Furthermore,

$$P(S/I_T, t) = t\left(\frac{1}{(1-t)^{n+1}}\right) + (1-t)\left[\frac{1}{(1-t)^2}P\left(k[\mathbf{T}_{n-1}]/I_{T_{n-1}}, t\right)\right]$$

because S/I_{T_r} is isomorphic to a polynomial ring in 2 variables over $k[\mathbf{T}_{n-1}]/I_{T_{n-1}}$. Hence,

$$P(S/I_T, t) = \frac{t}{(1-t)^{n+1}} + (1-t)\left[\frac{1}{(1-t)^2}\frac{1+(n-2)t}{(1-t)^n}\right] = \frac{1+(n-1)t}{(1-t)^{n+1}}.$$

(iii) Corso and Nagel proved in [3] that in the case of the $2 \times n$ Ferrers tableau,

$$P(S/I_2(\mathbf{T}), t) = \frac{1+(n-1)t}{(1-t)^{n+1}}.$$

Furthermore, since $I_T \subset \text{in}_{\text{revlex}}(I_2(\mathbf{T}))$ and $\dim(I_T) = \dim(I_2(\mathbf{T}))$ it follows that $I_T = \text{in}_{\text{revlex}}(I_2(\mathbf{T}))$.

□

Using the case of the Ferrers tableau with partition $\lambda = (n, n)$ as our foundation, we would like to extend this result to the Ferrers tableau with outer border thickness at least two. In particular, we would like to show that the toric generators of the defining ideal for the toric ring form a Gröbner basis.

7.1.3 Theorem. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$ with $n \geq 3$, and $\lambda_n \geq 2$. Furthermore, let $\boldsymbol{T'}$ be the subtableau of $\boldsymbol{T}$ formed by deleting the $\lambda_n - 1$ rightmost columns and the n^{th}-row of $\boldsymbol{T}$, and let N be the subtableau of $\boldsymbol{T}$ formed by considering the top $n-1$ rows of the rightmost $\lambda_n - 1$ columns. If we additionally let x_r be the leftmost entry in the n^{th}-row of $\boldsymbol{T}$ and $\boldsymbol{T}_r$ be the subtableau of $\boldsymbol{T}$ formed by deleting the box containing x_r, then*

$$I_T = x_r(I_{T'} + I_1(N)) + I_{T_r}$$

is a basic double link with

$$\text{codim}(I_T) = \left(\sum_{j=2}^{n} \lambda_j\right) - n + 1.$$

Before providing the proof of this theorem, let us first illustrate the components of the tableau, namely $\mathbf{T'}$ and N as they relate to the original tableau $\mathbf{T}$.

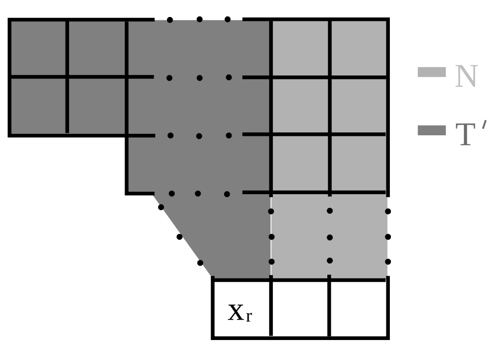


Figure 7.2: The subtableaux components occurring in the basic double link

Proof. Proceed by induction on the addition of boxes to a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2)$. Consider adding 2 boxes to the tableau. Then

$$I_T = x_r(I_{T_{2\times(\lambda_2-1)}} + (x_{\lambda_2}, x_{2\lambda_2})) + I_{T_{2\times\lambda_2}}. \tag{7.2}$$

However, Proposition (7.1.2) provides that $S/I_{T_{2\times\lambda_2}}$ is Cohen-Macaulay with

$$\begin{aligned}\operatorname{codim}(I_{T_{2\times(\lambda_2-1)}} + (x_{\lambda_2}, x_{2\lambda_2})) &= \operatorname{codim}(I_{T_{2\times(\lambda_2-1)}}) + \operatorname{codim}((x_{\lambda_2}, x_{2\lambda_2})) \\ &= (\lambda_2 - 2) + 2 \\ &= \lambda_2\end{aligned}$$

and

$$\operatorname{codim}(I_{T_{2\times\lambda_2}}) = \lambda_2 - 1.$$

Hence, (7.2) is indeed a basic double link. Consequently,

$$\operatorname{codim}(I_T) = \operatorname{codim}(I_{T_{2\times(\lambda_2)}} + (x_{\lambda_2}, x_{2\lambda_2})) = \lambda_2.$$

Assume true for the addition of $m-1$ boxes to the tableau $\mathbf{T}_{(\lambda_2,\lambda_2)}$. Consider the equality

$$I_T = x_r(I_{T'} + I_1(N)) + I_{T_r}. \tag{7.3}$$

By the induction hypothesis and the consequences of basic double linkage, S/I_{T_r} is Cohen-Macaulay. We must check that $\operatorname{codim}(I_{T'}+I_1(N)) = \operatorname{codim}(I_{T_r})+1$. However, from the induction hypothesis we have

$$\begin{aligned}\operatorname{codim}(I_{T_r}) &= \left(\sum_{j=2}^{n-1} \lambda_j\right) + \lambda_n - 1 - n + 1 \\ &= \left(\sum_{j=2}^{n-1} \lambda_j\right) + \lambda_n - n \\ \operatorname{codim}(I_{T'}) &= \left(\sum_{j=2}^{n-1} \lambda_j - (\lambda_n - 1)\right) - (n-1) + 1 \ . \\ &= \left(\sum_{j=2}^{n-1} \lambda_j\right) - (n-2)(\lambda_n - 1) + 2 \\ \operatorname{codim}(I_1(N)) &= (n-1)(\lambda_n - 1) \\ &= \lambda_n n - n - \lambda_n + 1\end{aligned}$$

Since the minimal generating sets of $I_{T'}$ and $I_1(N)$ are disjoint, it follows that

$$\operatorname{codim}(I_{T'} + I_1(N)) = \operatorname{codim}(I_{T'}) + \operatorname{codim}(I_1(N)) = \left(\sum_{j=2}^{n-1} \lambda_j\right) + \lambda_n - n + 1.$$

Hence, $\operatorname{codim}(I_{T'} + I_1(N)) = \operatorname{codim}(I_{T_r}) + 1$, and thus (7.3) is a basic double link. As a consequence,

$$\operatorname{codim}(I_T) = \operatorname{codim}(I_{T'} + I_1(N)) = \left(\sum_{j=2}^{n} \lambda_j\right) - n + 1.$$

□

7.1.4 Proposition. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $(\lambda_2, \lambda_2, \dots, \lambda_n)$ where $n \geq 2$ and $\lambda_n \geq 2$. Since $\dim(S/I_{\boldsymbol{T}}) = \lambda_1 + n - 1$, the Hilbert series can be written as*

$$P(S/I_{\boldsymbol{T}}, t) = \frac{p_{\boldsymbol{T}}(t)}{(1-t)^{\lambda_1 + n - 1}}$$

and

$$p_{\boldsymbol{T}}(T) = p_{\boldsymbol{T}_r}(t) + t p_{\boldsymbol{T}'}(t)$$

for $n \geq 3$.

Proof. Since $I_{\mathbf{T}} = x_r(I_{\mathbf{T}'} + I_1(N)) + I_{\mathbf{T}_r}$ is a basic double link, we get the following exact sequence

$$0 \longrightarrow S/I_{\mathbf{T}_r}(-1) \longrightarrow S/(I_{\mathbf{T}'} + I_1(N))(-1) \oplus S/I_{\mathbf{T}_r} \longrightarrow S/I_{\mathbf{T}} \longrightarrow 0.$$

Since Hilbert functions add along exact sequences, we get the following relationship among Hilbert functions

$$H_{S/I_{\mathbf{T}}}(j) = H_{S/(I_{\mathbf{T}'}+I_1(N))}(j-1) + H_{S/I_{\mathbf{T}}}(j) - H_{S/I_{\mathbf{T}_r}}(j-1)$$

which provides the following relationship among Hilbert series

$$P(S/I_{\mathbf{T}}, t) = tP(S/(I_{\mathbf{T}'} + I_1(N)), t) + (1-t)P(S/I_{\mathbf{T}_r}, t). \tag{7.4}$$

Since $\dim(S/I_{\mathbf{T}}) = \lambda_1 + n - 1$, we can write

$$P(S/I_{\mathbf{T}}, t) = \frac{p_{\mathbf{T}}(t)}{(1-t)^{\lambda_1+n-1}}.$$

Then (7.4) can be written as

$$\frac{p_{\mathbf{T}}(t)}{(1-t)^{\lambda_1+n-1}} = tP(S/(I_{\mathbf{T}'}+I_1(N)),t)+(1-t)P(S/I_{\mathbf{T}_r},t). \tag{7.5}$$

However,

$$\begin{aligned} P(S/I_{\mathbf{T}_r},t) &= \frac{1}{1-t}P(k[\mathbf{T}_r]/I_{\mathbf{T}_r}) \\ &= \left(\frac{1}{1-t}\right)\left(\frac{p_{\mathbf{T}_r}(t)}{(1-t)^{\lambda_1+n-1}}\right) \\ &= \frac{p_{\mathbf{T}_r}}{(1-t)^{\lambda_1+n}}. \end{aligned}$$

Using this, (7.5) becomes

$$\frac{p_{\mathbf{T}}(t)}{(1-t)^{\lambda_1+n-1}} = tP(S/(I_{\mathbf{T}'}+I_1(N)),t)+\frac{p_{\mathbf{T}_r}(t)}{(1-t)^{\lambda_1+n-1}}. \tag{7.6}$$

Notice that $S/(I_{\mathbf{T}'}+I_1(N))$ is isomorphic to a polynomial ring in λ_n variables over $k[\mathbf{T}']/I_{\mathbf{T}'}$. Furthermore,

$$\dim(k[\mathbf{T}']/I_{\mathbf{T}'}) = (\lambda_1-\lambda_n+1)+(n-1)-1 = \lambda_1+n-\lambda_n-1$$

and

$$\begin{aligned} P(S/(I_{\mathbf{T}'}+I_1(N)),t) &= \frac{1}{(1-t)^{\lambda_n}}P(k[\mathbf{T}']/I_{\mathbf{T}'},t) \\ &= \left(\frac{1}{(1-t)^{\lambda_n}}\right)\left(\frac{p_{\mathbf{T}'}(t)}{(1-t)^{\lambda_1+n-\lambda_n-1}}\right) \\ &= \frac{p_{\mathbf{T}'}(t)}{(1-t)^{\lambda_1+n-1}}. \end{aligned}$$

Substituting this into (7.6) provides

$$\frac{p_{\mathbf{T}}(t)}{(1-t)^{\lambda_1+n-1}} = t\left(\frac{p_{\mathbf{T}'}(t)}{(1-t)^{\lambda_1+n-1}}\right)+\frac{p_{\mathbf{T}_r}(t)}{(1-t)^{\lambda_1+n-1}}.$$

Therefore,

$$p_{\mathbf{T}}(t) = tp_{\mathbf{T}'}(t)+p_{\mathbf{T}_r}(t).$$

□

7.1.5 Corollary. *Let* $\boldsymbol{T}$ *be a Ferrers tableau with partition* $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$ *where*

$n \geq 2$. Then

$$I_{\boldsymbol{T}} = \text{in}_{\text{revlex}}(I_2(\boldsymbol{T})).$$

In particular, the set of minimal generators of $I_2(\boldsymbol{T})$ forms a Gröbner basis in the reverse lexicographic term order.

Proof. We begin by noting that $I_{\mathbf{T}} \subset \text{in}_{\text{revlex}}(I_2(\mathbf{T}))$ and $\dim(I_{\mathbf{T}}) = \dim(\text{in}_{\text{revlex}}(I_2(\mathbf{T}))$. We wish to show that

$$p_{\mathbf{T}}(t) = p_{\text{in}_{\text{revlex}}(I_2(\mathbf{T}))}(t).$$

We will induct on the addition of boxes to the Ferrers tableau $\mathbf{T}_{(\lambda_2,\lambda_2)}$. We begin by adding 2 boxes to $\mathbf{T}_{(\lambda_2,\lambda_2)}$. Then Proposition (7.1.2) provides

$$p_{\mathbf{T}_r}(t) = p_{\text{in}_{\text{revlex}}(\mathbf{T}_r)}(t) \quad \text{and} \quad p_{\mathbf{T}'}(t) = p_{\text{in}_{rlex}(\mathbf{T}')}(t)$$

and hence $p_{\mathbf{T}}(t) = p_{\text{in}_{\text{revlex}}(I_2(\mathbf{T}))}(t)$. Assume the result is true for the addition of $m-1$ boxes to the Ferrers tableau $\mathbf{T}_{(\lambda_2,\lambda_2)}$. Then by the induction hypothesis

$$p_{\mathbf{T}_r}(t) = p_{\text{in}_{\text{revlex}}(\mathbf{T}_r)}(t) \quad \text{and} \quad p_{\mathbf{T}'}(t) = p_{\text{in}_{rlex}(\mathbf{T}')}(t)$$

and therefore, $I_{\mathbf{T}} = \text{in}_{\text{revlex}}(I_2(\mathbf{T}))$. □

One of the goals of studying initial ideals of $I_2(\mathbf{T})$ is to generate information about the toric ring $k[\mathbf{T}]/I_2(\mathbf{T})$. In particular, we would like to show that the toric ring is *level*, i.e. that the last module in the minimal free resolution for $k[\mathbf{T}]/I_2(\mathbf{T})$ has only one degree shift. The method we will use is to find a level initial ideal of $I_2(\mathbf{T})$. For a term order $<$, we have the following relationship among Betti numbers of $k[\mathbf{T}]/I_2(\mathbf{T})$ and $k[\mathbf{T}]/\text{in}_<(I_2(\mathbf{T}))$,

$$\beta_{i,j}\left(k[\mathbf{T}]/I_2(\mathbf{T})\right) \leq \beta_{i,j}\left(k[\mathbf{T}]/\text{in}_<(I_2(\mathbf{T}))\right).$$

Thus, if we show that an initial ideal of $I_2(\mathbf{T})$ is level, then we would also have shown that the original toric ideal $I_2(\mathbf{T})$ is level. However, the following example demonstrates that the reverse lexicographic term order will not be the right term order to choose to show that $I_2(\mathbf{T})$ is level using this method.

7.1.6 Example. Consider the Ferrers tableau given by the partition $\lambda = (3,3,2)$ depicted as

x_1	x_2	x_3
x_4	x_5	x_6
	x_7	x_8

Figure 7.3: The Ferrers tableau with partition $\lambda = (3, 3, 2)$

Then $\text{in}_{\text{revlex}}(I_2(\mathbf{T})) = (x_2x_4, x_3x_4, x_3x_5, x_3x_7, x_6x_7)$, and the Betti diagram from Macaulay 2 (see [8]) for $\text{in}_{\text{revlex}}(I_2(\mathbf{T}))$ is given by

Total :	1	5	6	2
0 :	1	–	–	–
1 :	–	5	5	1
2 :	–	–	1	1

We see from this Betti diagram that $\text{in}_{\text{revlex}}(I_2(\mathbf{T}))$ is not level. In particular, the last module in a minimal free resolution for $S/\text{in}_{\text{revlex}}(I_2(\mathbf{T}))$ has shifts in coarsely graded degrees 4 and 5.

In the following section we will examine a modification of the reverse lexicographic term order in hopes that it will produce a level initial ideal.

7.2 The Diagonal Term Order

In [2] Conca, Hoşten, and Thomas posed the question of when a given term order produces an initial ideal with the same Betti numbers as the original ideal, i.e. for which classes of ideals is there a term order $<$ such that

$$\beta_{i,j}(S/I) = \beta_{i,j}(S/\text{in}_<(I)). \tag{7.7}$$

In particular, they looked at the class of ideals occurring as $(n-1)$-minors of $n \times n$ matrices. They determined that when considering a modification of the reverse lexicographic term order, called the *diagonal order*, the Betti numbers of the initial ideal correspond with that of the original ideal generated by the $(n-1)$-minors of $n \times n$ matrices. Since Ferrers tableaux are portions of full matrices and we are considering minors of these tableaux, we will also consider the initial ideal obtained in the diagonal term order for $I_2(\mathbf{T})$.

It should be noted that for an arbitrary ideal I, there is not necessarily an initial ideal satisfying equation (7.7) above. Actually, for a given ideal there are only a finite number of initial ideals, so this property can be checked manually with a computer.

7.2.1 Definition. Consider the Ferrers tableau given by the partition

$$\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n) \quad \text{where} \quad n \geq 2 \quad \text{and} \quad \lambda_n \geq 2.$$

Just as in the case of a square matrix, we can distinguish the main diagonal of the tableaux.

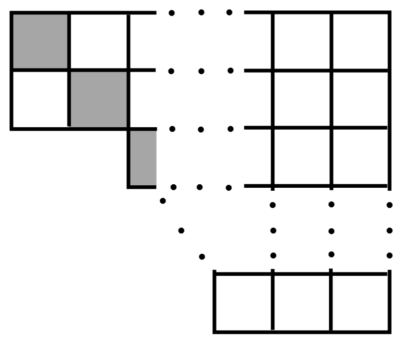


Figure 7.4: The main diagonal of a Ferrers tableau

Then the *diagonal term order* is defined as a modification of the reverse lexicographic term order where the main diagonal entries are smallest.

Previously, in the reverse lexicographic term order, the leading term of each toric generator corresponded to the anti-diagonal of the corresponding (2×2)-minor. In the case of the diagonal term order, we modify this by requiring that no leading term of a toric generator can hit the main diagonal of the Ferrers tableau. If the leading term of the toric generator in the reverse lexicographic term order hits the main diagonal of the Ferrers tableau, we will let the main diagonal of the corresponding (2×2)-minor be the leading term of the toric generator in the diagonal term order.

7.2.2 Example. Consider the Ferrers tableau with partition $\lambda = (3, 3, 2)$ considered in the reverse lexicographic order in Example (7.1.6). Then the Ferrers tableau is pictured as

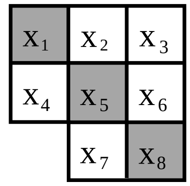


Figure 7.5: The Ferrers tableau $\mathbf{T}_{(3,3,2)}$ with the main diagonal highlighted

Then

$$I_2(\mathbf{T}) = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5, x_2x_8 - x_3x_7, x_5x_8 - x_6x_7)$$

and

$$\begin{aligned} \mathrm{in}_{\mathrm{diag}}(x_1x_5 - x_2x_4) &= x_2x_4 \\ \mathrm{in}_{\mathrm{diag}}(x_1x_6 - x_3x_4) &= x_3x_4 \\ \mathrm{in}_{\mathrm{diag}}(x_2x_6 - x_3x_5) &= x_2x_6 \\ \mathrm{in}_{\mathrm{diag}}(x_2x_8 - x_3x_7) &= x_3x_7 \\ \mathrm{in}_{\mathrm{diag}}(x_5x_8 - x_6x_7) &= x_6x_7. \end{aligned}$$

7.2.3 Proposition. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$, $n \geq 2$, and $\lambda_n \geq 2$. Additionally, we assume that*

$$\lambda_i \leq \lambda_1 - i + 2 \quad \textit{for} \quad i \in \{2, \ldots, n\}.$$

Then the set $\{t_1, t_2, \ldots, t_p\}$ of minimal toric generators of $I_2(\boldsymbol{T})$ is a Gröbner basis in the diagonal term order, i.e.

$$(\mathrm{lt}_{\mathrm{diag}}(t_1), \mathrm{lt}_{\mathrm{diag}}(t_2), \ldots, \mathrm{lt}_{\mathrm{diag}}(t_p)) = \mathrm{in}_{\mathrm{diag}}(I_2(\boldsymbol{T})).$$

Proof. We will proceed using Buchberger's Criterion (7.0.12). We will assume that if $i < j$. Furthermore, if $i < j$, then t_i is located relatively above t_j; or if they are located on the same rows, t_i is to the left of t_j. Recall that the s-pair of t_i and t_j is given by

$$s(t_i, t_j) = \frac{\mathrm{lcm}(t_i, t_j)}{\mathrm{lt}(t_i)} t_i - \frac{\mathrm{lcm}(t_i, t_j)}{\mathrm{lt}(t_j)} t_j.$$

We will let $t_i = m_i - a_i$ where m_i denotes the main diagonal, and a_i denotes the anti-diagonal of the minor corresponding to the toric generator t_i.

Case (i): Assume $\mathrm{lt}(t_i) = m_i$ and $\mathrm{lt}(t_j) = m_j$. This corresponds to the situation when the anti-diagonals of both t_i and t_j contain an entry of the tableau's main diagonal.

Case (a): Assume $\mathrm{lcm}(t_i, t_j) = m_i m_j$. Then

$$\begin{aligned} s(t_k, t_j) &= m_j t_i - m_i t_j \\ &= m_i a_j - m_j a_i. \end{aligned}$$

Since the anti-diagonals of both t_i and t_j hit the main diagonal and $a_j <_{\text{diag}} a_i$, it follows that

$$\mathrm{lt}(m_i a_j - m_j a_i) = m_j a_i.$$

Then since $\mathrm{lt}(t_j) = m_j$, the multivariate division algorithm provides

$$\begin{aligned} s(t_i, t_j) &= -a_i(m_j - a_j) - a_i a_j + m_i a_j \\ &= -a_i(m_j - a_j) + a_j(m_i - a_i). \end{aligned}$$

Case (b): Assume $\mathrm{lcm}(t_i, t_j) = x_i x x_j$ where $m_i = x_i x$ and $m_j = x_j x$. Then

$$\begin{aligned} s(t_i, t_j) &= x_j t_i - x_i t_j \\ &= x_i a_j - x_j a_i. \end{aligned}$$

Case (1): Assume $a_i = y_i y$ and $a_j = y_j y$. Then

$$s(t_i, t_j) = y_j y x_i - y_i y x_j = y(x_i y_j - x_j y_i).$$

However, there exists a k such that $t_k = x_i y_j - x_j y_i$.

Case (2): Assume $a_i = y_{i_1} y_{i_2}$ and $a_j = y_{j_1} y_{j_2}$. Then since a_i and a_j both contain an entry of the tableau's main diagonal and $a_j <_{\text{diag}} a_i$, it follows that $\mathrm{lt}(a_j x_i - a_i x_j) = a_j x_i = y_{j_1} y_{j_2} x_i$. Furthermore, there exists a k such that $t_k = x_i y_{j_2} - y_{i_1} z$ and $\mathrm{lt}(t_k) = x_i y_{j_2}$. Then

$$\begin{aligned} s(t_i, t_j) &= y_{j_1}(x_i y_{j_2} - y_{i_1} z) + y_{i_1} y_{j_1} z - y_{i_1} y_{i_2} x_j \\ &= y_{j_1}(x_i y_{j_2} - y_{i_1} z) - y_{i_1}(y_{i_2} x_j - y_{j_1} z). \end{aligned}$$

Moreover, there exists an ℓ such that $t_\ell = y_{i_2} x_j - y_{j_1} z$.

Case (ii): Assume $\mathrm{lt}(t_i) = a_i$ and $\mathrm{lt}(t_j) = a_j$. Then we have the following two subcases.

Case (a): Assume $\mathrm{lcm}(t_i, t_j) = a_i a_j$. Then

$$s(t_i, t_j) = a_j t_i - a_i t_j = a_j m_i - a_i m_j.$$

Furthermore, $\mathrm{lt}(a_j m_i - a_i m_j) = a_j m_i$ and $\mathrm{lt}(t_j) = a_j$. Hence, the multivariate division algorithm provides

$$\begin{aligned} s(t_i, t_j) &= -m_i(m_j - a_j) + m_i m_j - a_i m_j \\ &= -m_i(m_j - a_j) + m_j(m_i - a_i). \end{aligned}$$

Case (b): Assume $\mathrm{lcm}(t_i, t_j) = y_i y y_j$ where $a_i = y_i y$ and $a_j = y_j y$. Then

$$s(t_i, t_j) = y_j t_i - y_i t_j = y_j m_i - y_i m_j.$$

Case (1): Assume $m_i = x_i x$ and $m_j = x_j x$. Then

$$s(t_i, t_j) = y_j x_i x - y_i x_j x = x(x_i y_j - x_j y_i)$$

and there exists a k such that $t_k = x_i y_j - x_j y_i$.

Case (2): Assume $m_i = x_{i_1} x_{i_2}$ and $m_j = x_{j_1} x_{j_2}$. Then

$$\mathrm{lt}(y_j m_i - y_i m_j) = y_j m_i = y_j x_{i_1} x_{i_2}.$$

Moreover, there exists a k such that $t_k = z x_{j_2} - y_j x_{i_1}$ and $\mathrm{lt}(t_k) = y_j x_{i_1}$. Then

$$\begin{aligned} s(t_i, t_j) &= -x_{i-2}(z x_{j_2} - y_j x_{i_1}) + x_{i_2} z x_{j_2} - y_i x_{j_i} x_{j_2} \\ &= -x_{i-2}(z x_{j_2} - y_j x_{i_1}) + x_{j_2}(z x_{i_2} - y_i x_{j_1}). \end{aligned}$$

Furthermore, there exists an ℓ such that $t_\ell = z x_{i_2} - y_i x_{j_1}$.

Case (iii): Assume $\mathrm{lt}(t_i) = a_i$ and $\mathrm{lt}(t_j) = m_j$.

Case (a): Assume $\mathrm{lcm}(t_i, t_j) = a_i m_j$. Then

$$s(t_i, t_j) = m_j t_i - a_i t_j = m_i m_j - 2a_i m_j + a_i a_j.$$

Furthermore, $\mathrm{lt}(m_i m_j - 2a_i m_j + a_i a_j) = 2a_i m_j$ and since $\mathrm{lt}(t_i) = a_i$

we obtain

$$\begin{aligned} s(t_i, t_j) &= m_i m_j - 2a_i m_j + a_i a_j \\ &= 2m_j(m_i - a_i) - m_i m_j + a_i a_j. \end{aligned}$$

Case (1): Assume $\text{lt}(-m_i m_j + a_i a_j) = m_i m_j$. Since $\text{lt}(t_j) = m_j$, the s-pair becomes

$$\begin{aligned} s(t_i, t_j) &= 2m_j(m_i - a_i) - m_i m_j + a_i a_j \\ &= 2m_j(m_i - a_i) - m_i(m_j - a_j) - a_j m_i + a_i a_j \\ &= 2m_j(m_i - a_i) - m_i(m_j - a_j) - a_j(m_i - a_i). \end{aligned}$$

Case (2): Assume $\text{lt}(-m_i m_j + a_i a_j) = a_i a_j$. Since $\text{lt}(t_i) = a_i$, the multivariate division algorithm provides

$$\begin{aligned} s(t_i, t_j) &= 2m_j(m_i - a_i) - m_i m_j + a_i a_j \\ &= 2m_j(m_i - a_i) - a_j(m_i - a_i) + a_j m_i - m_i m_j \\ &= 2m_j(m_i - a_i) - a_j(m_i - a_i) - m_i(m_j - a_j). \end{aligned}$$

Case (b): Assume $\text{lcm}(t_i, t_j) = y_i z x_j$ where $a_i = y_i z$ and $m_j = x_j z$. Then

$$\begin{aligned} s(t_i, t_j) &= x_j t_i - y_i t_j \\ &= x_j m_i - 2x_j y_i z + a_j y_i. \end{aligned}$$

Moreover, $\text{lt}(x_j m_i - 2x_j y_i z + a_j y_i) = 2x_j y_i z$. Since $\text{lt}(t_i) = a_i = y_i z$, the s-pair becomes

$$\begin{aligned} s(t_i, t_j) &= x_j m_i - 2x_j y_i z + a_j y_i \\ &= 2x_j(m_i - y_i z) - x_j m_i + a_j y_i. \end{aligned}$$

In this case, $m_i = x_i w$ and $a_j = y_j w$ and hence

$$\begin{aligned} s(t_i, t_j) &= 2x_j(m_i - y_i z) - x_i x_j w + y_i y_j w \\ &= 2x_j(m_i - y_i z) - w(x_i x_j - y_i y_j). \end{aligned}$$

Furthermore, there exists a k such that $t_k = x_i x_j - y_i y_j$.

□

The condition in the above proposition that

$$\lambda_i \leq \lambda_1 - i + 2 \quad \text{for} \quad i \in \{2, \ldots, n\}$$

means that in any row of the given Ferrers tableau we are not allowed to have more than one entry to the left of the main diagonal entry. We will see shortly that if we extend the possible entries of a Ferrers tableau too far to the left of the main diagonal, the minimal toric generators of $I_2(\mathbf{T})$ will not form a Gröbner basis.

Since the minimal toric generators of $I_2(\mathbf{T})$ form a Gröbner basis in this case, it follows that $\text{in}_{\text{diag}}(I_2(\mathbf{T}))$ is square-free. In [16], Sturmfels has the following result concerning a square-free Gröbner basis of a toric ideal.

7.2.4 Proposition. *Let I be a prime ideal that has binomial minimal generators. Suppose that for some term order $<$ on $k[x_1, \ldots, x_n]$ the initial ideal, $\text{in}_<(I)$, is square-free, then $\text{in}_<(I)$ is Cohen-Macaulay.*

It should be mentioned that Sturmfels actually showed that I is normal. However, in [11], Hochster showed that if I is normal, then I is necessarily Cohen-Macaulay. The above proposition has the following consequence.

7.2.5 Corollary. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$, $n \geq 2$, and $\lambda_n \geq 2$. Additionally, we assume that*

$$\lambda_i \leq \lambda_1 - i + 2 \quad \textit{for} \quad i \in \{2, \ldots, n\}.$$

Then $\text{in}_{\text{diag}}(I_2(\boldsymbol{T}))$ is Cohen-Macaulay.

Proof. Apply Proposition (7.2.4) to Proposition (7.2.3). □

The following example demonstrates the weakness of the diagonal order with respect to the formation of a Gröbner basis from the minimal toric generators of $I_2(\mathbf{T})$.

7.2.6 Example. Consider the Ferrers tableau with partition $\lambda = (5, 5, 5, 4)$ depicted below.

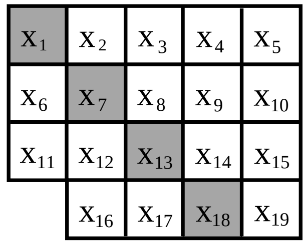


Figure 7.6: The Ferrers tableau $\mathbf{T}_{(5,5,5,4)}$ in the diagonal term order

Set

$$I_T := (\mathrm{lt}_{\mathrm{diag}}(t_i) \mid t_i \text{ is a minimal generator of } I_2(\mathbf{T})).$$

Then $\mathrm{pd}(S/I_T) = 11$, but $\mathrm{codim}(S/I_T) = 10$. This implies that S/I_T is not Cohen-Macaulay. Hence, Proposition (7.2.4) implies that the set of minimal toric generators of $I_2(\mathbf{T})$ does not form a Gröbner basis in the diagonal term order.

We notice that in the above example, we allowed the cells in a given row of the Ferrers tableau to venture to two cells past the main diagonal. It is for this reason that we have the restriction on the number of entries in each row of the Ferrers tableau in Proposition (7.2.3).

Let us revisit the Ferrers tableau examined in Example (7.1.6), but let us now consider it in the diagonal term order.

7.2.7 Example. Consider the Ferrers tableau with partition $\lambda = (3, 3, 2)$.

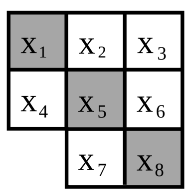


Figure 7.7: The Ferrers tableau $\mathbf{T}_{(3,3,2)}$ in the diagonal term order

Then

$$\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}_{(3,3,2)})) = (x_2x_4, x_3x_4, x_2x_6, x_3x_7, x_6x_7)$$

and the corresponding Betti diagram from Macaulay 2 (see [8]) is given by

Total :	1	5	5	1
0 :	1	–	–	–
1 :	–	5	5	–
2 :	–	–	–	1

In particular, we notice that $\text{in}_{\text{diag}}(I_2(\mathbf{T}_{(3,3,2)}))$ is level.

When considering Ferrers tableaux in the diagonal order where there are only two entries on the main diagonal, we get the following result concerning the corresponding initial ideals.

7.2.8 Proposition. *Let* $\boldsymbol{T}$ *be a Ferrers tableau with partition* $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$*,* $n \geq 2$*, and* $\lambda_n \geq 2$*. Additionally, we specify that for* $i \in \{3, \ldots, n\}$

$$\lambda_i \leq \lambda_1 - i.$$

Then $\text{in}_{\text{diag}}(I_2(\boldsymbol{T}))$ *is level with level shift given by*

$$\sum_{j=2}^{n} \lambda_j.$$

Proof. We will prove this claim in two steps. First we will show that the claim is true for a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2)$. Then, in the second step, we will prove the claim for the remaining cases by inducting on the addition of boxes to the Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2)$. To simplify our notation, we will set

$$I_T := \text{in}_{\text{diag}}(I_2(\mathbf{T})).$$

(i) Consider the Ferrers tableau with partition $\lambda = (2, 2)$. Then

$$\text{in}_{\text{diag}}(I_2(\mathbf{T}_{(2,2)})) = (x_2 x_3)$$

which is clearly level with level shift 2. Assume that the claim is true for Ferrers tableaux with partition $\lambda = (m-1, m-1)$. Consider the Ferrers tableau with partition $\lambda = (m, m)$ depicted below.

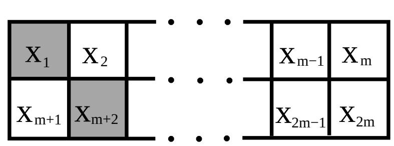


Figure 7.8: The Ferrers tableau $\mathbf{T}_{(m,m)}$ in the diagonal term order

Then $\text{in}_{\text{diag}}(I_2(\mathbf{T})) = \text{in}_{\text{revlex}}(I_2(\widetilde{\mathbf{T}}))$ where $\widetilde{\mathbf{T}}$ is given by the Ferrers tableau shown below.

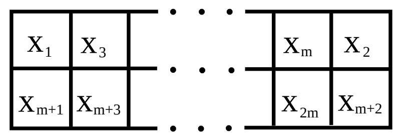


Figure 7.9: The Ferrers tableau $\widetilde{\mathbf{T}}$

Consequently, this demonstrates that it is equivalent to show the claim for the reverse lexicographic term order. Recall from Theorem (7.1.3) that we have the following relationship among subtableaux of $\mathbf{T}$ when considered in the reverse lexicographic term order.

$$I_T = x_{m+1}(x_2, \ldots, x_m) + I_{T_r}$$

Furthermore, since this is a basic double link, we get the following short exact sequence.

$$0 \longrightarrow S/I_{T_r}(-1) \longrightarrow S/(x_2, \ldots, x_m)(-1) \bigoplus S/I_{T_r} \longrightarrow S/I_T \longrightarrow 0$$

Moreover, from Theorem (7.1.3) and the Koszul resolution of $S/(x_2, \ldots, x_m)$ we also obtain that

$$\begin{aligned} \operatorname{pd}(S/(x_2, \ldots, x_m)) &= m-1 \\ \operatorname{pd}(S/I_{T_r}) &= (m-1)-2+1 = m-2. \end{aligned}$$

Additionally, $S/(x_2, \ldots, x_m)$ is level with level shift $m-1$, and the induction hypothesis provides that I_{T_r} is level with level shift $m-1$. Therefore, we conclude from the mapping cone construction that S/I_T is also level with level shift $(m-1)+1 = m$.

(ii) In this step, we will use the notation of Theorem (7.1.3) and proceed by inducting on the number of boxes added to the tableau with partition $\lambda = (\lambda_2, \lambda_2)$. Consider the addition of two boxes to such a tableau. Then we have the following relationship among Ferrers tableaux

$$I_T = x_r(I_{T_{(\lambda_2-1,\lambda_2-1)}} + (x_{\lambda_2}, x_{2\lambda_2})) + I_{T_r}$$

where $\mathbf{T}_r$ is the subtableau of $\mathbf{T}$ obtained by removing the cell containing x_r. Since $I_{T_r} \subset (I_{T_{(\lambda_2-1,\lambda_2-1)}} + (x_{\lambda_2}, x_{2\lambda_2}))$ and x_r does not divide a minimal generator of I_{T_r} we get the following exact sequence

$$0 \longrightarrow S/I_{T_r}(-1) \longrightarrow S/(I_{T_{(\lambda_2-1,\lambda_2-1)}}+(x_{\lambda_2},x_{2\lambda_2}))(-1) \bigoplus S/I_{T_r} \longrightarrow S/I_T \longrightarrow 0$$

Additionally,

$$\begin{aligned} \text{pd}(S/(I_{T_{(\lambda_2-1,\lambda_2-1)}} + (x_{\lambda_2},x_{2\lambda_2}))) &= (\lambda_2-2)+2=\lambda_2 \\ \text{pd}(S/I_{T_r}) &= \lambda_2 - 1 \end{aligned}$$

Furthermore, both $S/(I_{T_{(\lambda_2-1,\lambda_2-1)}} + (x_{\lambda_2},x_{2\lambda_2}))$ and S/I_{T_r} are level by the induction hypothesis. Moreover, the level shift of $S/(I_{T_{(\lambda_2-1,\lambda_2-1)}} + (x_{\lambda_2},x_{2\lambda_2}))$ is $(\lambda_2-1)+2=\lambda_2+1$, and the level shift of S/I_{T_r} is λ_2+1. Hence, S/I_T is also level with level shift $(\lambda_2+1)+1=\lambda_2+2$. Assume the claim holds true for the addition of $m-1$ boxes to the Ferrers tableau with partition $\lambda=(\lambda_2,\lambda_2)$. Then we get the following relationship among Ferrers tableaux.

$$I_T = x_r(I_{T'}+I_1(N)) + I_{T_r}$$

Moreover, since $I_{T_r} \subset (I_{T'}+I_1(N))$ and x_r does not divide a minimal generator of I_{T_r} we get the following exact sequence

$$0 \longrightarrow S/I_{T_r}(-1) \longrightarrow S/(I_{T'}+I_1(N))(-1) \bigoplus S/I_{T_r} \longrightarrow S/I_T \longrightarrow 0$$

We further note that

$$\begin{aligned} \text{pd}(S/(I_{T'}+I_1(N))) &= \left[\left(\sum_{j=2}^{n-1}\lambda_j\right) - (n-2)\lambda_n\right] + (n-1)(\lambda_n-1) \\ &= \left(\sum_{j=2}^{n}\lambda_j\right) - n + 1 \\ \text{pd}(S/I_{T_r}) &= \left(\sum_{j=2}^{n}\lambda_j\right) - n \end{aligned}$$

Then the induction hypothesis implies that the level shift of $S/(I_{T'}+I_1(N))$ is given by

$$\left(\sum_{j=2}^{n-1}\lambda_j - (\lambda_n-1)\right) + (\lambda_n-1)(n-1) = \left(\sum_{j=2}^{n-1}\lambda_j\right) + \lambda_n - 1$$

and the level shift of S/I_{T_r} is also given by

$$\left(\sum_{j=2}^{n-1} \lambda_j\right) + \lambda_n - 1.$$

Therefore, from the mapping cone construction, we conclude that S/I_T is also level with level shift

$$\left[\left(\sum_{j=2}^{n-1} \lambda_j\right) + \lambda_n - 1\right] + 1 = \sum_{j=2}^{n} \lambda_j.$$

□

The added restriction of the number of boxes in each row of the Ferrers tableau in the above proposition, namely that

$$\lambda_i \leq \lambda_1 - i \quad \text{for } i \in \{3, \ldots, n\},$$

means that our tableau will only have 2 main diagonal entries. The same method used in the above proof cannot be used for the addition of a main diagonal entry. In particular, we do not get the equality

$$I_T = x_r(I_{T'} + I_1(N)) + I_{T_r}$$

when x_r is located on the main diagonal of the Ferrers tableau $\mathbf{T}$.

The above proposition provides the following result concerning the original toric ideal $I_2(\mathbf{T})$.

7.2.9 Corollary. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$, $n \geq 2$, and $\lambda_n \geq 2$. Additionally, we specify that for $i \in \{3, \ldots, n\}$*

$$\lambda_i \leq \lambda_1 - i.$$

Then $I_2(\boldsymbol{T})$ is level.

Proof. We have the following relationship among the Betti numbers of $I_2(\mathbf{T})$ and $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$

$$\beta_{i,j}(S/I_2(\mathbf{T})) \leq \beta_{i,j}(S/\,\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))).$$

Thus Proposition (7.2.8) provides that $I_2(\mathbf{T})$ is level. □

In the above corollary we have shown that the toric ring of a Ferrers graph is level when the Ferrers graph has only two main diagonal entries. We mentioned earlier that the method we used in the above proof of Proposition (7.2.8) fails upon the addition of a third main diagonal entry. However, from numerous examples, it seems likely that the initial ideal in the diagonal term order is level in all of the cases for which we have shown that the toric generators of $I_2(\mathbf{T})$ form a Gröbner basis, namely when no row of the Ferrers graph goes more than one entry past the main diagonal.

7.2.10 Conjecture. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$, $n \geq 2$, and $\lambda_n \geq 2$. Additionally, we assume that*

$$\lambda_i \leq \lambda_1 - i + 2 \quad \textit{for} \quad i \in \{2, \ldots, n\}.$$

Then both $\mathrm{in}_{\mathrm{diag}}(I_2(\boldsymbol{T}))$ and $I_2(\boldsymbol{T})$ are level.

It should be be noted that, for the above conjecture, if one were able to show that $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$ was level for the addition of a main diagonal entry to any row, then the addition of an entry one past the main diagonal in any row is a consequence of the relationship

$$I_T = x_r(I_{T'} + I_1(N)) + I_{T_r}$$

which holds true as long as x_r is not an entry of the main diagonal and remains at most one box to the left of the main diagonal of the given Ferrers tableau.

In [3], Corso and Nagel showed that $I_2(\mathbf{T})$ is Cohen-Macaulay. Furthermore, via the results of Sturmfels and Hochster shown in Proposition (7.2.4), we have shown that $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$ is also Cohen-Macaulay in certain cases. Then we make the following conjecture concerning the Cohen-Macaulay types of both $I_2(\mathbf{T})$ and $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$.

7.2.11 Conjecture. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_n)$, $n \geq 2$, and $\lambda_n \geq 2$. Additionally, we assume that*

$$\lambda_i \leq \lambda_1 - i + 2 \quad \textit{for} \quad i \in \{2, \ldots, n\}.$$

Then $\mathrm{in}_{\mathrm{diag}}(I_2(\boldsymbol{T}))$ and $I_2(\boldsymbol{T})$ share the same Cohen-Macaulay type.

In the above conjectures we have kept the restriction that

$$\lambda_i \leq \lambda_1 - i + 2 \quad \text{for} \quad i \in \{2, \ldots, n\}.$$

The reason for this restriction is illustrated in Example (7.2.6). It was in this example that we saw that if we ventured to expand to a Ferrers graph with two boxes to the left of the main diagonal we may end up in the situation where the toric generators of $I_2(\mathbf{T})$ do not form a Gröbner basis. However, it appears that as we approach the full rectangular Ferrers tableau, the set of toric generators of $I_2(\mathbf{T})$ again becomes a Gröbner basis, thus leading to the following conjecture.

7.2.12 Conjecture. *Let $\boldsymbol{T}$ be a Ferrers tableau with partition $\lambda = (\lambda_2, \lambda_2, \ldots, \lambda_2)$ where the Ferrers graph has at least 3 rows. Then the set $\{t_1, t_2, \ldots, t_p\}$ of minimal toric generators of $I_2(\boldsymbol{T})$ is a Gröbner basis in the diagonal term order, i.e.*

$$(\mathrm{lt}_{\mathrm{diag}}(t_1), \mathrm{lt}_{\mathrm{diag}}(t_2), \ldots, \mathrm{lt}_{\mathrm{diag}}(t_p)) = \mathrm{in}_{\mathrm{diag}}(I_2(\boldsymbol{T})).$$

In conclusion we would like to relate the study of the monomial ideals that we studied in this chapter to the study of edge ideals of simple graphs examined in earlier chapters. In Propositions (7.1.3) and (7.2.3) we showed that the toric generators of $I_2(\mathbf{T})$ form a Gröbner basis in both the reverse lexicographic order and the diagonal order for certain classes of Ferrers graphs. Additionally, we saw that both $\mathrm{in}_{\mathrm{revlex}}(I_2(\mathbf{T}))$ and $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$ were generated by square-free quadratic monomials. This implies that both $\mathrm{in}_{\mathrm{revlex}}(I_2(\mathbf{T}))$ and $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$ occur as edge ideals of simple graphs. Therefore, Propositions (7.1.3) and (7.2.3) are also statements about the edge ideals of the corresponding simple graphs, namely that the edge ideals of these simple graphs are Cohen-Macaulay. Furthermore, Proposition (7.2.8) states that the edge ideals of the simple graphs corresponding to the ideals $\mathrm{in}_{\mathrm{diag}}(I_2(\mathbf{T}))$ associated to certain Ferrers graphs are level.

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Vita

(i) Background

  (a) Date of Birth: January 23, 1982
  (b) Place of Birth: Natrona Heights, PA

(ii) Academic Degrees

  (a) B.S., Mathematics (Computer Science Minor), University of Pittsburgh at Johnstown, 2003.
  (b) M.A., Mathematics, University of Kentucky, 2005.

(iii) Professional Experience

  (a) Teaching Assistant, Mathematics Department, University of Kentucky, Fall 2003-Summer 2008.
  (b) Algebra Cubed Fellow, Mathematics Department, University of Kentucky, Fall 2006-Summer 2007.
  (c) Research Assistant under Dr. Uwe Nagel, University of Kentucky, Summer 2006.
  (d) Math Excel Leader, Mathematics Department, University of Kentucky, Fall 2004-Spring 2005.

(iv) Awards and Fellowships

  (a) College of Arts and Sciences Outstanding Teaching Assistant Award, University of Kentucky, Spring 2008.
  (b) Wimberly C. Royster Teaching Assistant Award, University of Kentucky, Spring 2006.
  (c) Fugate-Wells Fellowship, University of Kentucky, Fall 2004-Spring 2007.
  (d) Women in Underrepresented Areas Fellowship, University of Kentucky, Fall 2003-Summer 2004.