Here we drive the general Ward identity, equation (14) of our Letter, in the presence of a uniform external magnetic field \( B \) with conserved current \( J^\mu \). The momentum density and its conservation equation are modified as in equations (12) and (13). Taking two time derivatives of the retarded Green function \( G^{0i,0j}(x^\mu; x'^\mu) = i\theta(x^0 - x'^0) \langle [T^{0i}(x^\mu), T^{0j}(x'^\mu)] \rangle \) gives

\[
\begin{align*}
\partial_0 \partial'_0 G^{0i,0j} &= \partial_0 \partial'_0 G^{n\mu,m\mu} + B \gamma^i \partial_0 G^{n,0l} + B \gamma^j \partial_0 G^{0j,m} + B^2 \gamma^i \gamma^j G^{n,m} - B^2 \gamma_j \gamma^i G^{n,m} \\
- \frac{1}{2} (\partial_0 - \partial'_0) \left( \delta(x^0 - x'^0) G^{0i,0j} \right),
\end{align*}
\]

(S1)

where \( G^{a,b}(x^\mu; x'^\mu) = i\theta(x^0 - x'^0) \langle [V^a(x^\mu), V^b(x'^\mu)] \rangle \) are the retarded correlators. The index \( a \) can either represent one or two indices, depending on context: e.g., \( V^a \) can represent either the momentum density components \( T^{0i} \) or a current \( J^i \). Note that the first line in (S1) is a consequence of the conservation equation \( \partial_0 T^{\mu i} = B \gamma^i J^\mu \), while the second line comes from a contact term with an equal time correlator \( C^{a,b}(x^0; \vec{x}, \vec{x}') = i\langle [V^a(x^0, \vec{x}), V^b(x^0, \vec{x}')] \rangle \). There are four other terms

\[
\frac{1}{2} \delta(x^0 - x'^0) \left( \partial_0 C^{n\mu,0l} - \partial'_0 C^{0j,m\mu} + B \left( \epsilon^i_n C^{n,0l} - \epsilon^j_m C^{0j,m} \right) \right)
\]


Using time translation symmetry, we perform a Fourier transform \( \int d(x^0 - x'^0) e^{i\omega(x^0 - x'^0)} \) on (S1), followed by

\[
G^{a,b}(x^0 - x'^0, \vec{x}, \vec{x}') = \frac{1}{2\pi} \int e^{-i\omega(x^0 - x'^0)} \tilde{G}^{a,b}(q_0, \vec{x}, \vec{x}')
\]

using equation (3) and \( i[T^{\mu i}_B(\vec{x}), O(\vec{x}')] = \partial_0 O(\vec{x}') \delta(\vec{x} - \vec{x}'), \) where \( T_B \) is the momentum generator in the presence of the magnetic field given in equation (12). We evaluate the contact term in (S1) as \( i\omega \epsilon^j \left[ \epsilon^i_n \partial_0 \langle T^0m \rangle + c - B\rho \right] \delta^2(\vec{x} - \vec{x}'), \) where \( q_0 \equiv \omega \). The first term is incompatible with translation and rotation symmetries, and we discard it. \( c \) and \( \rho = \langle J^0 \rangle \) are the topological number density and charge density. The Ward identity becomes

\[
\omega^2 \tilde{G}^{ij,0l} - i\omega B \epsilon^j_n \tilde{G}^{n,0l} + i\omega B \epsilon^j_m \tilde{G}^{0j,m} + B^2 \gamma^i \gamma^j \tilde{G}^{n,m} = \partial_0 \partial'_0 \tilde{G}^{n,j,m\mu} - i\omega \epsilon^j \left[ c - B\rho \right] \delta^2(\vec{x} - \vec{x}').
\]

(S2)

Using space translation symmetry, we perform a Fourier transform \( \tilde{G}^{a,b}(\omega; \vec{x}, \vec{x}') = \int \frac{d^d q}{(2\pi)^d} e^{i\vec{q}(\vec{x} - \vec{x}')} \tilde{G}^{a,b}(\omega, \vec{q}) \). To extract useful information, we exploit rotational invariance to rewrite the retarded Green’s functions in the form

\[
\tilde{G}^{a,b} = \Pi^{a,b} + i\omega \epsilon^b q^a \Pi^q + (\epsilon^{bq} q^a + \epsilon^{aq} q^b) q_0 \Pi_{q} \frac{\partial}{\partial q_0},
\]

(S3)

where \( \Pi^{TT}, \Pi^{TT}, \Pi^{JT}, \Pi^{JJ} \) are four different types of form factors, and \( \eta, \zeta, \eta_H \) are the shear, bulk and Hall viscosities, respectively. Plugging them into (S2) and doing some algebra, we arrive at

\[
\delta^{ij} \left[ \omega^2 \Pi^{TT} + i\omega B (\Pi^{TT} + \Pi^{TT}) + B^2 \Pi^{JJ} + B^2 q^2 \Pi_{q}^{JJ} - i\omega B q^2 (\Pi^{TT} - \Pi^{TT}) \right]
\]

\[
+ \delta^{ij} \left[ i\omega B (\Pi^{TT} - \Pi^{TT}) + B^2 \Pi^{JJ} - i\omega B q^2 (\Pi^{TT} - \Pi^{TT}) \right]
\]

\[
+ q^{ij} \left[ \omega^2 \Pi_{q}^{JJ} + 2i\omega B (\Pi_{q}^{TT} - \Pi_{q}^{TT}) - B^2 \Pi_{q}^{JJ} \right] + (\epsilon^{a} q^a + \epsilon^{a} q^a) q_0 \left[ \omega^2 \Pi_{q}^{TT} - \frac{i\omega B}{2} (\Pi_{q}^{TT} - \Pi_{q}^{TT}) - B^2 \Pi_{q}^{JJ} \right]
\]

\[
= \epsilon^{ij} \left[ i\omega \left[ c + B\rho - q^2 \xi_H \right] - i\omega \delta^{ij} q^a \eta - i\omega q^2 \xi \right].
\]

We identify \( \Pi^{TT} = -i\omega \kappa, \Pi^{JJ} = -i\omega \alpha, \Pi^{JT} = -i\omega \sigma, \) where \( \kappa, \alpha, \sigma \) are thermal, thermoelectric, and electric conductivity tensors. Matching tensor structures in (S4), we find four independent Ward identities. The resulting equation appears as equation (14).