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BOSONIZATION VS. SUPERSYMMETRY

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ABSTRACT OF DISSERTATION

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A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy in the
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By
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We study the conjectured equivalence between the $O(3)$ Gross-Neveu model and the supersymmetric sine-Gordon model under a naive application of the bosonization rules.

We start with a review of the equivalence between sine-Gordon model and the massive Thirring model. We study the models by perturbation theory and then determine the equivalence. We find that the dependence of the identifications on the couplings can change according to the definition of the vector current. With the operator identifications of the special case corresponding to a free fermionic theory, known as the bosonization rules, we describe the equivalence between the massless Thirring model and the model of a compactified free boson field.

For the massless Thirring model, or equivalently the $O(2)$ Gross-Neveu model, we study the conservation laws for the vector current and the axial current by employing a generalized point-splitting method which allows a one-parameter family of definitions of the vector current. With this parameter, we can make contact with different approaches that can be found in the literature; these approaches differ mainly because of the specific definition of the current that was used. We also find the Sugawara form of the stress-energy tensor and its commutation relations. Further, we rewrite the identifications between sine-Gordon and Thirring models in our generalized framework.

For the $O(3)$ Gross-Neveu model, we extend our point-splitting method to determine the exact expression for the supercurrent. Using this current, we compute the superalgebra which determines three quantum components of the stress-energy tensor. With an Ansatz for the undetermined component, we find the trace anomaly and the first beta-function coefficient. The central charge which can be computed without using our point-splitting method is independent of the coupling constant, in fact, it is always zero.

For the supersymmetric sine-Gordon model, we review its supersymmetry in the context of models derived from a scalar multiplet in two dimensions. We then obtain the central charge and discover an extra term that was missing in the original derivation. We also analyze how normal ordering modifies the central charge.

Finally, we discuss the conjectured equivalence of the $O(3)$ Gross-Neveu model and the supersymmetric sine-Gordon model under the naive application of the bosonization rules. Comparing our results of the central charges and the supercurrents for these models, we find that they disagree; consequently the models should be generically inequivalent. We also conclude that the naive application of the bosonization rules at the Lagrangian level does not always lead to an equivalent theory.

KEYWORDS: Quantum Field Theory, Supersymmetry, Bosonization, Anomalies, Point-splitting Regularization

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BOSONIZATION VS. SUPERSYMMETRY

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Table of Contents

List of Figures

Chapter One

Introduction

Bosonization is usually understood as a technique for "translating" purely fermionic models into purely bosonic models. The phenomenon of Bose-Fermi equivalence has a long history. In condensed matter, it was introduced by Bloch in studying the energy loss of charged particles traveling through a metal. He argued that the properties of the fermion systems could be described by (bosonic) plasmons [4], [5]. In particle physics, the idea came from Skyrme when he proposed a simple non-linear (bosonic) field theory to describe strongly interacting particles [43], [44]. However, it was not until Coleman's paper about the equivalence between the sine-Gordon model and the massive Thirring model that bosonization became a powerful technique for understanding quantum field theories [8]. (See reference [48] for reprints of many of the classic papers related to this subject.) In two-dimensional space-time, bosonization is in some way expected, since spin and statistics are matters of convention.

It is natural to ask how general the phenomenon of Bose-Fermi equivalence is, at least in the context of relativistic two-dimensional field theories. Most known examples are fairly straightforward generalizations of the equivalence between the sine-Gordon and Thirring models [25], [57], [23]. (See also reprints in [48].) In some of these cases, the bosonization rules are exactly given by equation (2.31). However, there is at least one example in the literature that is not such a straightforward generalization. This is Witten's conjecture of an equivalence between the $O(3)$ Gross-Neveu model and the supersymmetric sine-Gordon model under naive application of the bosonization rules [56]. In this case, the situation is a little bit different because a partial bosonization is used (only two of the three Majorana fermions are bosonized).

Supersymmetry is a continuous symmetry that links fermions and bosons [3], [54], [45]. It was originally introduced in the early seventies as a symmetry of the two-dimensional world-sheet in string theory [37], [38], [20], [21]. In two dimensions, the simplest supersymmetric models are derived from a scalar multiplet that contains a Majorana spinor and a scalar boson [14], [12], [26]. Therefore, the supercharge relates the Majorana field with the boson field. As a special case of these models we have the supersymmetric sine-Gordon model.

It is interesting to ask if bosonization and supersymmetry are compatible, since the total number of elementary bosonic and fermionic degrees of freedom are not maintained in the process of partial bosonization. That is the case in Witten's conjecture.

Our main goal in this work is to test the proposed equivalence by looking at the central charges of the models. Since these charges are protected from quantum corrections by supersymmetry, if the equivalence is correct the central charges should agree.

This work is organized as follows. In chapter 2, we review the equivalence between the sine-Gordon model and the massive Thirring model. We study both models by perturbation theory. By comparing the corresponding perturbation series, we then identify the fermion bilinears with bosonic operators and find the correspondence between the couplings of the two models. Our treatment is slightly different from Coleman's [8], because we do not require a specific definition of the vector current as a product of two fields at the same point. Consequently, our identifications can depend differently on the couplings because of the current definition. In all cases, the free massive Dirac field theory is equivalent to the $\beta = 2\sqrt{\pi}$ sine-Gordon model. For this special equivalence, we write down the identifications, or bosonization rules, and use them to describe the equivalence between the massless Thirring model and the model of a compactified free boson field, as an example of the naive application of the bosonization rules.

In chapter 3, we introduce the $O(N)$ Gross-Neveu model and study the particular case of $N = 2$ that is equivalent to the massless Thirring model. We focus our analysis on the conservation laws for the vector current and the axial vector current at the quantum level. As an intermediate step, we introduce a point-splitting method to deal with the divergences of field products at the same space-time point. Our approach has a free parameter which allows us to make contact with different ways that the current has been defined in the literature. We then find the Sugawara form of the stress-energy tensor and the commutation relations between the components of this tensor. We verify that these results agree with those in the current algebra literature [49], [47], [7]. We also establish the identifications between sine-Gordon and Thirring models in our framework and discuss the constraints that our free parameter has for physical solutions. In Appendix B, we give more technical details of our approach and examples of use of our point-splitting method.

In chapter 4, we introduce the $O(3)$ Gross-Neveu model and study a special classical current that becomes supersymmetric when quantum anomalies are included. Generalizing our approach for the $O(2)$ Gross-Neveu model, we obtain the exact expression for these current anomalies, including their dependence on the coupling constant. Using the modified current, we then compute the superalgebra which allows us to determine three quantum components of the stress-energy tensor. By making an Ansatz for the undetermined component of this tensor, we obtain the trace anomaly and the first β function coefficient which agrees with that found by loop calculations. We also find that the central charge of the model does not dependent on the coupling constant and does not require our point-splitting method for computing it. Further details concerning all of these calculations are shown in Appendix B.

In chapter 5, we review the supersymmetry of the general Lagrangian derived from a scalar multiplet in two dimensions. We compute the supercurrents, the supercharges and the components of the stress-energy tensor. Then we obtain the superalgebra, discovering in the process an extra term in the central charge that was missing in the original derivation [58]. Also, we discuss how normal ordering affects this result and analyze it for a particular case, the supersymmetric sine-Gordon model. Appendix C deals with more detailed calculations of these results and with a discussion of our normal ordering prescription.

In chapter 6, we review the proposed equivalence between the $O(3)$ Gross-Neveu model and supersymmetric sine-Gordon model under a naive application of the bosonization rules. Our treatment differs from Witten's [56], because we use a normal ordering identity (6.1) instead of a trigonometric identify to establish the proposed equivalence. In either case, the procedure establishes the correspondence between the couplings of the two models. We then discuss some consistency issues of the naive application of the bosonization rules in respect to this correspondence. With the results of the previous chapters, we first compare the central charges of the models and find that they disagree in general. As an independent argument for the validity of the proposed equivalence, we bosonize the $O(3)$ Gross-Neveu supercurrent to compare it with that of the supersymmetric sine-Gordon model, finding again a disagreement. Conversely, this last result shows that the fermionization of the supercurrent of the supersymmetric sine-Gordon model cannot be used to generate the supersymmetry transformations in the $O(3)$ Gross-Neveu model, as was used in references [2], [32], [40]. Finally, based on our comparisons, we conclude that the models are generically inequivalent. We also conclude that the naive application of the bosonization rules to determine a partially bosonized theory does not in this case lead to an equivalent theory.

Chapter Two

Bosonization

2.1 Sine-Gordon model

The sine-Gordon model (SGM) is a renormalizable field theory of a single scalar *boson* field ϕ in one space and one time dimension. The classical field equation of this model (2.2) has been extensively studied. In the nineteenth century, the differential geometers considered it in connection with the theory of pseudospherical surfaces (surfaces of constant negative curvature) [13], [31], [35]. Later it was regarded as an example of a wave equation that exhibits soliton solutions (time-independent solutions of finite energy) and it was also used in the description of some physical phenomena (see the review paper [42]). In particle physics, this model was introduced by Skyrme [43], [44]. He proposed it as a simple model of a nonlinear field theory that could describe the strongly interacting particles, mesons and baryons. Subsequently, the first attempts to investigate the quantum properties of the model, as a scalar field theory in two dimensions with nonderivative interactions, were made by using the properties of the classical theory as a starting point [17], [9]. Coleman then solved the model using perturbation theory and normal ordering as we will do in this section [8].

The Lagrangian for the SGM is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha_0}{\beta^2}\cos(\beta\phi) + \gamma_0, \tag{2.1}$$

where α_0, β and γ_0 are real parameters. β and α_0 can be considered positive, since we can redefine the sign of ϕ and shift ϕ by π/β, respectively. Classically, the physical meanings of these parameters are: β measures the strength of the interactions between small oscillations about the minimum ($\phi = 0$), α_0 is the "squared mass" associated with the spectrum of these oscillations and γ_0 is an adjustable parameter for the value of the minimum energy.

The classical field equation derived from $\mathcal{L}$ is

$$\partial_\mu\partial^\mu\phi + \frac{\alpha_0}{\beta}\sin(\beta\phi) = 0. \tag{2.2}$$

The most straightforward way to study this model is by means of a perturbation expansion in powers of α_0. Also a normal-ordering prescription is needed to remove all the ultraviolet divergences that occur in any order of this expansion.

For simplicity, we will compute the terms present in the expansion of the vacuum amplitude, namely,

$$\langle 0|T \exp\Big(-i\int d^2x \mathcal{H}_I\Big)|0\rangle, \tag{2.3}$$

where $|0\rangle$ is the free theory vacuum and T denotes that each term in the expansion is time-ordered. In our case, the interaction Hamiltonian is given by

$$\mathcal{H}_I = -:\frac{\alpha_R}{\beta^2}\cos(\beta\phi) + \gamma_R:, \tag{2.4}$$

where :: is the normal-ordering operation defined by the mass m (see Appendix C) and α_R and γ_R are renormalized parameters (see [8]).

As can been seen below, it is more instructive to compute the vacuum expectation value of the products of exponentials, since

$$:\cos(\beta\phi):=\frac{1}{2}\Big(:e^{i\beta\phi}:+:e^{-i\beta\phi}:\Big). \tag{2.5}$$

Using (C.9), we obtain

$$\langle 0|T\prod_{i=1}^{n}:e^{i\beta_i\phi(x_i)}:|0\rangle = e^{-\sum_{i<j}\beta_i\beta_j\Delta^{(+)}(x_i-x_j,m)}, \tag{2.6}$$

where $\Delta^{(+)}$ is defined in (C.1). This result is ill-defined in the zero-mass limit because of the function $\Delta^{(+)}$. However, we can regularize it by using the identity (see Appendix C)

$$:e^{i\beta\phi}:\equiv:e^{i\beta\phi}:_m=\Big(\frac{\mu^2}{m^2}\Big)^{\beta^2/8\pi}:e^{i\beta\phi}:_\mu, \tag{2.7}$$

where we have added a subscript to indicate that the normal-orderings are defined by different masses m and μ. Now we define the massless free scalar field theory by considering that μ is the actual mass of the free field ϕ (strictly it is the infrared cutoff). Therefore we take (2.6) to be normal-ordered with some free parameter m and then use the above identity to express it in terms of the mass μ. After substituting (C.1), the final result is

$$\langle 0|T\prod_{i=1}^{n}:e^{i\beta_i\phi(x_i)}:|0\rangle = \Big(\frac{\mu^2}{m^2}\Big)^{\sum\beta_i^2/8\pi}\prod_{i<j}(-\mu^2(x_i-x_j)^2)^{\beta_i\beta_j/4\pi}. \tag{2.8}$$

Thus this expectation value vanishes as μ goes to zero, unless

$$\sum_{i=1}^{n} \beta_i = 0. \tag{2.9}$$

This equation represents a superselection rule and implies a natural definition of charge in a theory of real bosonic fields.

In the zero-mass limit, the nonzero terms in the perturbation expansion for SGM are those with equal numbers of $:e^{i\beta\phi}:$'s and $:e^{-i\beta\phi}:$'s. Thus we have

$$\langle 0|T\prod_{i=1}^{n} :e^{i\beta\phi(x_i)}::e^{-i\beta\phi(y_i)}: |0\rangle = \frac{\prod_{i>j}\Big((x_i-x_j)^2(y_i-y_j)^2m^4\Big)^{\frac{\beta^2}{4\pi}}}{\prod_{i,j}\Big(-(x_i-y_j)^2m^2\Big)^{\frac{\beta^2}{4\pi}}}. \tag{2.10}$$

With this result, one can in principle compute the vacuum amplitude (2.3) to all the orders in perturbation theory.

From the commutation relations, we compute

$$[\partial_\nu\phi(x), :\exp(\pm i\beta\phi(y)):] = \pm\beta D_\nu(x-y) :\exp(\pm i\beta\phi(y)): . \tag{2.11}$$

Finally, we summarize some of the properties of the SGM

1. Renormalizable: All divergences that occur in any order of perturbation theory in powers of α_0 can be removed by normal ordering.

2. Integrable: The classical model has an infinite number of conservation laws and at the quantum level these laws allow one to obtain the exact S-matrix of the theory [1].

3. For $\beta^2 > 8\pi$, the energy per unit volume is unbounded below and therefore the model has no ground state [8].

4. For $\beta = 0$, the theory is a free massive boson field theory.

In the section 2.3, we will see some other properties by using the equivalence between this model and the massive Thirring model.

2.2 Thirring model

The Thirring model (TM) is a field theory of a single *spin*-1/2 *fermion* field with a current-current interaction in one space and one time dimension. This model for the massless case was originally introduced by Thirring [51]. He showed that the model is exactly soluble and found the eigenstates of the Hamiltonian. Glaser then solved the field equations in terms of the free massless Dirac field and obtained an operator solution [16]. Even though both authors dealt with mathematical and formal aspects of the model, there were some discrepancies, mainly, in the dependence of the results on the coupling constant (see next chapter). These discrepancies came from the eigenvalue equation that needs to be defined as a limit of a non-singular equation and this can be done in many different ways [16], [50]. Later, Johnson reconsidered the model by looking at the definition of the field products at coincident space-time points that appear in the current definition and in the field equations [27]. Thus he was able to solve a system of coupled equations for the time-ordered Green's functions. With Johnson's definition, Scarf and Wess found an operator solution [39], but their choice for the representation of the Hilbert space makes to difficult to check that the resulting n-point functions are positive definite. Subsequently, Klaiber constructed a complete operator solution of the model in a well-defined Hilbert space, so that the positive definiteness condition is fulfilled [28]. He found a two-parameter family of solutions for the massless TM in which the other solutions in the literature can be obtained. Also he discussed the properties of his solutions, such as Lorentz transformations, locality, cluster decomposition and infrared behavior. Finally, there are other approaches in solving TM, for example, Dell'Antonio, Frishman and Zwanziger made use of the properties of the currents, namely, conservation laws and commutation relations [11] and Hagen included an external field coupled to the current in finding the general spin-1/2 solution [24]. We will make more comments about the last three papers in the next chapter in which we will derive some known results in our own distinct approach.

Let us now study the massive TM using an approach similar to that in Coleman's paper about the perturbative equivalence between this model and the SGM [8]. In contrast to Coleman, we will not make a specific choice in Klaiber's parameters by demanding a particular commutator between the current and the Dirac field (see equation (2.14)).

First, the Lagrangian for the massive TM is given by

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - 2gj^\mu j_\mu - m_0\bar{\psi}\psi, \tag{2.12}$$

where ψ is a Dirac field, g is a free parameter (the coupling constant), m_0 is a real parameter (not be identified with the mass of any one-particle state) and $j^\mu = \bar{\psi}\gamma^\mu\psi$.

The classical field equation is

$$i\gamma^\mu\partial_\mu\psi - 4gj^\mu\gamma_\mu\psi - m_0\psi = 0. \tag{2.13}$$

(Notice that the quantum version of this equation and $\mathcal{L}$ will require a definition of the field products at coincident points.)

Since we are following Coleman's procedure, we use Klaiber's results for TM [28]. We will keep Klaiber's notation as much as possible, but we will add a subscript K in Klaiber's parameters α and β to avoid confusion with the SGM's parameters.

Klaiber's solution for the massless TM has two free parameters α_K and β_K. They are related to spin-statistics (since this is a matter of convention in two-dimensional space-time) and to the specific definition of the current (see next chapter for a discussion about that). Thus the first condition that we use to determine Klaiber's parameters is the equal-time commutation relation between the current and the Fermi field, namely,

$$[j^\mu(x), \psi(y)] = -\Big(c_1\eta^{\mu 0} + c_2\epsilon^{\mu 0}\gamma^5\Big)\delta(x-y)\psi(x). \tag{2.14}$$

where $c_1 = c_1(g)$ and $c_2 = c_2(g)$ depend on the actual definition of current ($c_1(0) = c_2(0) = 1$, the free Dirac field theory). In Klaiber's paper, these coefficients are given by

$$c_1 = 1 - \frac{\alpha_K}{\sqrt{\pi}}, \qquad c_2 = 1 - \frac{\beta_K}{\sqrt{\pi}}. \tag{2.15}$$

However, the overall normalization here is conventional, since we are free to redefine the current through some function of g, $j^\mu \to f(g)j^\mu$. Therefore physically different theories are distinguished by the ratio of these coefficients (not by the actual values)

$$C = C(g) \equiv \frac{c_2}{c_1} = \frac{1 - \beta_K/\sqrt{\pi}}{1 - \alpha_K/\sqrt{\pi}}. \tag{2.16}$$

Our second condition is to choose the spin-statistics of the solution which is given by $|1/2 + \lambda/2\pi|$, where

$$\lambda = \alpha_K\beta_K - \sqrt{\pi}(\alpha_K + \beta_K). \tag{2.17}$$

Then the spin-1/2 solution corresponds to the choice $\lambda = 0$.

Solving for α_K and β_K for a given C, we obtain two different solutions

$$\frac{\alpha_K}{\sqrt{\pi}} = 1 \pm \frac{1}{\sqrt{C}}, \qquad \frac{\beta_K}{\sqrt{\pi}} = 1 \pm \sqrt{C}, \tag{2.18}$$

respectively. (In fact, the two solutions are perturbatively equivalent, see equation (2.21) or reference [28].) Coleman's choice for C is

$$C = \frac{1}{1 + 4g/\pi}. \tag{2.19}$$

More specifically, $c_1 = 1$ and $c_2 = C$ in (2.14). This choice allows us to normalize the conserved charge associated with the current as the fermion number (independent of the coupling constant g), so that field fermion number is minus one (like the model of free spinors).

Now let us find the perturbation series in powers of m_0 for the massive TM. First, we define

$$\sigma_\pm \equiv \frac{1}{2}\bar{\psi}(1 \pm \gamma^5)\psi. \tag{2.20}$$

Strictly this definition and the definition of the current should include a renormalization of the divergence due to taking the product of two fields at the same space-time point, but we refer the reader to the original articles [28] and [8] for a complete derivation of the n-point functions given below, especially because Klaiber's formulas are very lengthy. Since the massless TM is chirally invariant, the nonzero terms in the perturbation expansion are those with an equal number of σ_+'s and σ_-'s.

Therefore, with Klaiber's formulas, we obtain

$$\langle 0|T\prod_{i=1}^{n}\sigma_+(x_i)\sigma_-(y_i)|0\rangle = \Big(\frac{1}{2}\Big)^{2n} \frac{\prod_{i>j}\Big((x_i - x_j)^2(y_i - y_j)^2 M^4\Big)^{1+b/\pi}}{\prod_{i,j}\Big(-(x_i - y_j)^2 M^2\Big)^{1+b/\pi}}, \tag{2.21}$$

where M is a parameter in which all renormalization constants and numerical factors have been absorbed and the Klaiber's parameter b is given by

$$1 + \frac{b}{\pi} \equiv \Big(1 - \frac{\beta_K}{\sqrt{\pi}}\Big)^2 = C. \tag{2.22}$$

Notice that we differ from Coleman's result, because we have not chosen a specific value of b, i.e. C. The derivation of (2.21) does not require any choice of Klaiber's parameters

α_K and β_K, so that it is even possible for the field ψ to have fractional spin-statistics (in which case the equation (2.17) would not be set to zero and then the above equation would not be equal to C).

From (2.20) and the general commutation relation between the current and the Dirac field (see [28]), we compute the following commutator

$$[j^\mu(x), \sigma_\pm(y)] = \mp 2c_2 \epsilon^{\mu\nu} D_\nu(x-y)\sigma_\pm(y). \tag{2.23}$$

Finally, we summarize some of the properties of TM

1. Renormalizable: All divergences that occur in any order of perturbation theory in powers of m_0 can be removed by rescaling the fields, $\psi \to Z^{1/2}\psi_r$, where Z is a cutoff-dependent constant.

2. Integrable: The classical model has an infinite number of conservation laws and at the quantum level these laws allow one to obtain the exact S-matrix of the theory [1].

3. For $m_0 = 0$, the TM has a conformal symmetry at the classical and the quantum levels. Its central charge is equal to one [15]. In particular, this model is scale invariant.

In the next section, we will see some other properties by using the equivalence between this model and the SGM.

2.3 The equivalence

A long time ago, Coleman found an equivalence between the SGM and the massive TM [8]. He derived this result by comparing the corresponding perturbation series of these models and by making identifications between fermion bilinears and bosonic operators, as well as identifications between the couplings in the two models. Mandelstam then constructed the Fermi fields as non-local functions of the boson field [30]. In this section, we review Coleman's procedure in finding this equivalence.

Let us compare, term-by-term, the perturbation series in m_0 for TM (2.21) with the perturbation series in α_0 for SGM (2.10). On the right hand sides of these equations, we have the same expression if we identify the couplings and the renormalization parameters as follows

$$\frac{\beta^2}{4\pi} = 1 + \frac{b}{\pi} = C, \tag{2.24}$$

$$M^2 = m^2. \tag{2.25}$$

Now let us extend the identification to the operators on the left hand sides of the equations and claim that

$$-\sigma_{\pm} = \frac{1}{2} :\exp(\pm i\beta\phi): . \tag{2.26}$$

With this last result, we conclude that

$$\bar{\psi}\psi = A_B :\cos(\beta\phi):, \tag{2.27}$$

$$\bar{\psi}\gamma^5\psi = iA_B :\sin(\beta\phi):, \tag{2.28}$$

where A_B is a free constant that depends on the normal ordering prescription. This can be easily seen by equation (2.7), since we can change our normal ordering according to other mass and therefore use a different constant in the above identifications.

To complete our identifications, we compare equations (2.11) with (2.23) and because of (2.26), we get that

$$j^{\mu}(x) = -2\frac{c_2}{\beta}\epsilon^{\mu\nu}\partial_{\nu}\phi(x). \tag{2.29}$$

Using (2.19), we reproduce Coleman's identifications

$$\begin{aligned} \frac{\beta^2}{4\pi} &= \frac{1}{1 + 4g/\pi}, \\ -m_0\bar{\psi}(x)\psi(x) &= \frac{\alpha}{\beta^2} :\cos(\beta\phi(x)):, \\ j^{\mu}(x) &= -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\phi(x). \end{aligned} \tag{2.30}$$

Notice that our identifications are more general because they contain any definition of the current for TM, which determines what choice of c_1 and c_2 should be taken in our results. With Coleman's identifications and the properties of the models, we have the following features in the equivalence

1. Large g is small β and vice versa. Thus the strong-coupling TM can be approximated by the weak-coupling SGM. This conclusion can be interpreted differently if one choses another value of C.

2. If $\beta^2 < 8\pi$, the SGM is equivalent to the charge-zero sector of the massive TM (in Coleman's choice, $g > -\pi/8$).

3. For the special case $\beta^2 = 4\pi$, the SGM is equivalent to the charge-zero sector of a free massive spin-1/2 fermion field theory. This conclusion is independent of the choice of C, since $C(0) = 1$.

4. For $g \to \infty$, the massive TM is equivalent to a free massive boson field theory. This limit of g can change with other choices of C.

5. The soliton of SGM is nothing other than the fundamental fermion of TM (see [8], [30]). This feature suggests that the equivalence extends nonperturbatively.

2.4 Bosonization rules

As has been described in the previous section, the free massive Dirac field theory is equivalent to the $\beta = 2\sqrt{\pi}$ SGM. For this special case, we have from (2.27), (2.28) and (2.29) that the identifications are

$$\begin{aligned}
\bar{\psi} i \partial\!\!\!/ \psi &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\
j^\mu = \bar{\psi}\gamma^\mu\psi &= -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi, \\
j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi &= -\frac{1}{\sqrt{\pi}}\partial^\mu\phi, \\
\bar{\psi}\psi &= A_B :\cos(2\sqrt{\pi}\phi):, \\
\bar{\psi}\gamma^5\psi &= iA_B :\sin(2\sqrt{\pi}\phi): .
\end{aligned} \tag{2.31}$$

We have also identified the kinetic terms and the third of these equations follows from (2.29) and a property of the γ matrices (A.10). These identifications are called bosonization rules and they are usually used to relate fermionic models with bosonic ones [48], [1]. The rules have been also generalized to field theories with non-abelian symmetries [57].

As an example, let us apply these rules to the Lagrangian of the massless TM. Then (2.12) for $m_0 = 0$ becomes

$$\mathcal{L} = \frac{1}{2}\Big(1 + \frac{4g}{\pi}\Big)\partial_\mu\phi\partial^\mu\phi, \tag{2.32}$$

when the rules (2.31) are substituted.

Now let us compare this result with the Lagrangian of a compactified free boson field

$$\mathcal{L} = \frac{R^2}{4\pi\alpha'}\partial_\mu\varphi\partial^\mu\varphi, \tag{2.33}$$

where the boson field has the periodicity $\varphi \cong \varphi + 2\pi R$, R is the radius of the compact dimension and α' is the square of the fundamental string length scale, which is of order the Planck length.

Thus, we conclude that the massless TM is equivalent to the model of a compactified free boson field, where the compactification radius R and the coupling constant g are identified by

$$R^2 = 2\pi\alpha'\Big(1 + \frac{4g}{\pi}\Big). \tag{2.34}$$

We will refer to this way of trying to relate two theories at the Lagrangian level as a naive application of the bosonization rules (2.31). In the last chapter, we will discuss the validity of such an approach.

Chapter Three

$O(2)$ Gross-Neveu Model

3.1 $O(N)$ Gross-Neveu model

The $O(N)$ Gross-Neveu model (GNM) is a renormalizable field theory of an N-component Majorana Fermi field, transforming in the fundamental representation of the orthogonal group $O(N)$, with a quartic self-interaction. This model was first introduced by Nambu and Jona-Lasinio in four dimensions as a dynamical model of elementary particles in which nucleons and mesons are derived from a fundamental spinor field [33]. In two dimensions, Gross and Neveu studied the large N limit and performed an expansion in powers of $1/N$ to all orders in the coupling constant [22]. Thus, they found that the model displays dynamical symmetry breakdown that generates, in the resulting theory, a fermion mass and dimensional transmutation, i.e the conversion of a dimensionless coupling constant into a mass scale parameter. All these properties follow from the fact that the $O(N)$ GNM is an asymptotically free theory. Subsequently, other features of the model have been found. In a semiclassical analysis, the particle spectrum of GNM reveals a very rich structure [10]. The integrability of the classical model due to the existence of an infinite number of conservation laws [34] (which survive quantization) has been used to compute the exact S-matrix of the theory [59], [60].

The Lagrangian for the model is

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + g(\bar{\psi}_i \psi_i)^2, \tag{3.1}$$

where ψ_i is a Majorana spinor, g is a dimensionless coupling constant and repeated indices are summed over. A fermion mass term is forbidden by the discrete chiral symmetry

$$\psi_i \to \gamma^5 \psi_i. \tag{3.2}$$

If we use Dirac spinors in place of Majorana ones, the symmetry group will be $U(N)$. In either case, the symmetry that suffers spontaneous breakdown is the discrete chiral symmetry.

In this chapter, we study the case of $N = 2$ which is equivalent to the massless TM. In our approach, we introduce a point-splitting method and define the current by fermion fields taken at the same time. Then the current conservation laws leaves us with a free parameter that allows us to relate to other current definitions that have been used in the TM literature. We also obtain the Sugawara form of the stress-energy tensor and the identifications between the massive TM and the SGM in our approach. In the next chapter, we generalize our approach to study the $O(3)$ GNM supercurrent and the trace anomaly.

3.2 $O(2)$ Gross-Neveu model

The classical Lagrangian for the $O(2)$ GNM is

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}_1 \partial\!\!\!/ \psi_1 + \bar{\psi}_2 \partial\!\!\!/ \psi_2) + g(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2)^2, \tag{3.3}$$

which gives the classical field equations

$$\partial\!\!\!/ \psi_1 = 4ig\bar{\psi}_2\psi_2\psi_1, \qquad \partial\!\!\!/ \psi_2 = 4ig\bar{\psi}_1\psi_1\psi_2, \tag{3.4}$$

where ψ_1 and ψ_2 are Majorana spinors.

It is well known that the above Lagrangian becomes that for the massless TM, equation (2.12) with $m_0 = 0$, when the Fierz transformation (A.25) is used and the two Majorana spinors are expressed as one Dirac spinor. Therefore, these two models are the same and all properties and results of the massless TM discussed in the previous chapter hold for the $O(2)$ GNM.

In the Majorana-Weyl representation, $\mathcal{L}$ can be written in terms of chiral components as (see Appendix B)

$$\mathcal{L} = i(\psi_{1R}\partial_+\psi_{1R} + \psi_{2R}\partial_+\psi_{2R} + \psi_{1L}\partial_-\psi_{1L} + \psi_{2L}\partial_-\psi_{2L}) + 8g\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R}, \tag{3.5}$$

and so can the field equations

$$\begin{aligned} \partial_+\psi_{1R} &= 4ig\psi_{2L}\psi_{2R}\psi_{1L}, & \partial_+\psi_{2R} &= 4ig\psi_{1L}\psi_{1R}\psi_{2L}, \\ \partial_-\psi_{1L} &= -4ig\psi_{2L}\psi_{2R}\psi_{1R}, & \partial_-\psi_{2L} &= -4ig\psi_{1L}\psi_{1R}\psi_{2R}. \end{aligned} \tag{3.6}$$

Now let us define the vector current by a point-splitting method for space separations as (see Appendix B)

$$\begin{aligned} J_\mu &\equiv \frac{1}{2}G_{\mu\nu}\Big(\bar{\psi}(x+\epsilon)\gamma^\nu\psi(x) - \gamma^\nu\psi(x)\bar{\psi}(x-\epsilon)\Big) \\ &= \frac{1}{4}G_{\mu\nu}\Big(\bar{\psi}_1(x+\epsilon)\gamma^\nu\psi_1(x) - \bar{\psi}_1(x)\gamma^\nu\psi_1(x-\epsilon) + (1\to 2)\Big) \\ &\quad +\frac{i}{4}G_{\mu\nu}\Big(\bar{\psi}_1(x+\epsilon)\gamma^\nu\psi_2(x) - \bar{\psi}_2(x)\gamma^\nu\psi_1(x-\epsilon) - (1\leftrightarrow 2)\Big), \end{aligned} \tag{3.7}$$

where ψ is a Dirac spinor and $G_{\mu\nu} \equiv G_{\mu\nu}(g)$ is some symmetric tensor to be determined. The role of $G_{\mu\nu}$ is to "restore" J^μ as a Lorentz vector due to our point-splitting method. In this definition and in the following equations, only the space coordinate is explicitly written.

We also consider the following quantum field equations

$$\begin{aligned} \partial\!\!\!/\psi_1(x) &= 2g\Big(J_\mu(x+\epsilon)\gamma^\mu\psi_2(x) + \gamma^\mu\psi_2(x)J_\mu(x-\epsilon)\Big), \\ \partial\!\!\!/\psi_2(x) &= -2g\Big(J_\mu(x+\epsilon)\gamma^\mu\psi_1(x) + \gamma^\mu\psi_1(x)J_\mu(x-\epsilon)\Big). \end{aligned} \tag{3.8}$$

These equations are motivated by Klaiber's field equations [28]. However, we have a different definition of the current.

In his paper, Klaiber did not write down a Lagrangian for the model. Thus we consider the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}_1\partial\!\!\!/\psi_1 + \bar{\psi}_2\partial\!\!\!/\psi_2) - 2gG_{\mu\nu}j^\mu j^\nu, \tag{3.9}$$

where the composite operator $j^\mu j^\nu$ is also defined by the point-splitting method for space separations

$$j^\mu j^\nu \equiv \frac{1}{2}\Big(j^\mu(x+\epsilon)j^\nu(x) + j^\nu(x)j^\mu(x-\epsilon)\Big). \tag{3.10}$$

The agreement of the Euler-Lagrange equations with Klaiber's field equations is our motivation for choosing (3.9), since it tells us how the point-splitting method should be taken in the field equations.

In the Majorana-Weyl representation, we compute the composite operators and rewrite the Lagrangian as (see Appendix B)

$$\begin{aligned} \mathcal{L} &= i(1+\alpha G_{++})(\psi_{1R}\partial_+\psi_{1R} + \psi_{2R}\partial_+\psi_{2R}) - i\alpha G_{++}(\psi_{1R}\partial_-\psi_{1R} + \psi_{2R}\partial_-\psi_{2R}) \\ &\quad +i(1+\alpha G_{--})(\psi_{1L}\partial_-\psi_{1L} + \psi_{2L}\partial_-\psi_{2L}) - i\alpha G_{--}(\psi_{1L}\partial_+\psi_{1L} + \psi_{2L}\partial_+\psi_{2L}) \\ &\quad +4\pi\alpha G_{+-}\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R}, \end{aligned} \tag{3.11}$$

and the field equations as

$$\begin{aligned}
(1+\alpha G_{++})\partial_+\psi_{1R} &= 2\pi i\alpha G_{+-}\psi_{2L}\psi_{2R}\psi_{1L} + \alpha G_{++}\partial_-\psi_{1R},\\
(1+\alpha G_{++})\partial_+\psi_{2R} &= 2\pi i\alpha G_{+-}\psi_{1L}\psi_{1R}\psi_{2L} + \alpha G_{++}\partial_-\psi_{2R},\\
(1+\alpha G_{--})\partial_-\psi_{1L} &= -2\pi i\alpha G_{-+}\psi_{2L}\psi_{2R}\psi_{1R} + \alpha G_{--}\partial_+\psi_{1L},\\
(1+\alpha G_{--})\partial_-\psi_{2L} &= -2\pi i\alpha G_{-+}\psi_{1L}\psi_{1R}\psi_{2R} + \alpha G_{--}\partial_+\psi_{2L},
\end{aligned} \tag{3.12}$$

where $\alpha = 4g/\pi$ and $G_{\mu\nu}$ is written in its light-cone components.

3.3 The current conservation laws

In light-cone coordinates, the components of the vector current are

$$\begin{aligned}
j^+ &\equiv j^0 + j^1 = 2i\psi_{1R}\psi_{2R},\\
j^- &\equiv j^0 - j^1 = 2i\psi_{1L}\psi_{2L}.
\end{aligned} \tag{3.13}$$

At classical level, they are conserved

$$\partial_+ j^+ = 0, \qquad \partial_- j^- = 0. \tag{3.14}$$

These results follow from the field equations (3.6) and $\psi_{iR}^2 = \psi_{iL}^2 = 0$. In usual coordinates, these equations represent the conservation laws for the vector current and the axial vector current.

At the quantum level, these conservation laws are still valid. Using our definition of the current (3.7), the conservation of J^μ implies

$$\partial_\mu J^\mu = 2(G_{-+}\partial_+ j^+ + G_{++}\partial_- j^+ + G_{+-}\partial_- j^- + G_{--}\partial_+ j^-) = 0. \tag{3.15}$$

For the axial vector current $J^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi = \epsilon^{\mu\nu}J_\mu$, we have

$$\partial_\mu J^{5\mu} = 2(G_{-+}\partial_+ j^+ - G_{++}\partial_- j^+ - G_{+-}\partial_- j^- + G_{--}\partial_+ j^-) = 0, \tag{3.16}$$

Solving these equations for $\partial_+ j^+$ and $\partial_- j^-$ in terms of $\partial_- j^+$ and $\partial_+ j^-$, we obtain the quantum version of (3.14)

$$\begin{aligned}
G_{-+}\partial_+ j^+ + G_{--}\partial_+ j^- &= 0,\\
G_{+-}\partial_- j^- + G_{++}\partial_- j^+ &= 0.
\end{aligned} \tag{3.17}$$

We compute $\partial_+ j^+$ and $\partial_- j^-$ by using the field equations (3.12) and the point-splitting method (see Appendix B)

$$\begin{aligned}(1+\alpha G_{++})\partial_+ j^+ &= \alpha G_{+-}(\partial_- j^- - \partial_+ j^-) + \alpha G_{++}\partial_- j^+, \\ (1+\alpha G_{--})\partial_- j^- &= \alpha G_{-+}(\partial_+ j^+ - \partial_- j^+) + \alpha G_{--}\partial_+ j^-. \end{aligned} \tag{3.18}$$

Solving again for $\partial_+ j^+$ and $\partial_- j^-$, we find

$$\begin{aligned} &\Big(1+\alpha(G_{++}+G_{--})+\alpha^2(G_{++}G_{--}-G_{-+}G_{+-})\Big)\partial_+ j^+ \\ &-\alpha\Big(G_{++}+\alpha(G_{++}G_{--}-G_{-+}G_{+-})\Big)\partial_- j^+ + \alpha G_{+-}\partial_+ j^- = 0, \\ &\Big(1+\alpha(G_{--}+G_{++})+\alpha^2(G_{--}G_{++}-G_{+-}G_{-+})\Big)\partial_- j^- \\ &-\alpha\Big(G_{--}+\alpha(G_{--}G_{++}-G_{+-}G_{-+})\Big)\partial_+ j^- + \alpha G_{-+}\partial_- j^+ = 0. \end{aligned} \tag{3.19}$$

Comparing these results with (3.17) and using $G_{-+} = G_{+-}$ we obtain

$$\begin{aligned} G_{--} &= G_{++}, \\ \alpha(G_{++}^2 - G_{+-}^2) + G_{++} &= 0. \end{aligned} \tag{3.20}$$

In usual coordinates, these equations reduce to

$$\begin{aligned} G_{01} &= G_{10} = 0, \\ G_{11} &= \frac{-G_{00}}{1+\alpha G_{00}}. \end{aligned} \tag{3.21}$$

Thus, the conservation laws leave us with one of the components of $G_{\mu\nu}$ undetermined. Furthermore, $G_{\mu\nu}$ should reduce to the metric in the free case, so that G_{00} should tend to η_{00} when $\alpha \to 0$. Notice that G_{11} will go directly to η_{11} in that limit by the above result. In other words, we have the condition $G_{00}(0) = 1$.

This freedom in choosing G_{00} can be related to the many different ways that the current can be defined in the massless TM as mentioned in the section 2.2. For example, Schwinger defined it by taking a spacelike limit [41] and Johnson did it by a symmetrical limit (averaging spacelike and timelike limits) [27]. As can be seen below, we can identify their results with ours for some particular value of G_{00}. This situation of the current definition is still present in a more general two-dimensional model including bosons [46].

With our definition of the current (3.7), we can infer the current commutators by using the usual canonical commutation relations. Therefore, the equal-time commutator between the current and the Dirac field is (see Appendix B)

$$[J^\mu(x), \psi(y)] = G^\mu{}_\nu [j^\nu(x), \psi(y)] = -(G^{\mu 0} - G^{\mu 1}\gamma^5)\delta(x-y)\psi(x), \qquad (3.22)$$

and the nonzero relation between currents is

$$[J^0(x), J^1(y)] = \frac{i}{\pi} G\delta'(x-y). \qquad (3.23)$$

where $G \equiv \det(G_{\mu\nu}) = G_{00}G_{11} - G_{01}G_{10}$. By the above discussion, the coefficients of these commutators are not completely determined (since G_{00} is not). This situation had been already known in the construction of the general solution of the massless TM, since there is a free parameter when the specific definition of the current is not taken. (See [24], [11], [28].) In the reference [24], there are initially two parameters from a generalization of Schwinger's current definition, but Lorentz invariance constraints only one of them. In the reference [11], the authors solve the model by using the current algebra, i.e. by considering the above commutators as the starting point (assigning a coefficient for each one), in that way they do not need an actual definition of the current. The Ward identities for the Green's functions of the model only restrict two coefficients out of three of these commutators. In that sense this approach is similar to ours, since $G_{\mu\nu}$ has three components and the current conservation equations only restrict two of them, but our calculations are simpler than computing Green's functions. References [11] and [28] even have another free parameter that is related to spin-statistics. (See section 2.2 for reference [28].) We do not have that parameter because we assume, from the start, that our fields are spin-1/2 fermions.

Finally, with the relation (3.23) we find the following nonzero commutators for products of currents which will be useful in the next section

$$\begin{aligned}
[J^0(x)J^0(x), J^1(y)J^1(y)] &= \frac{2i}{\pi} G\delta'(x-y)(J^0(x)J^1(y) + J^1(y)J^0(x)), \\
[J^0(x)J^0(x), J^0(y)J^1(y)] &= \frac{2i}{\pi} G\delta'(x-y)J^0(x)J^0(y), \\
[J^1(x)J^1(x), J^0(y)J^1(y)] &= \frac{2i}{\pi} G\delta'(x-y)J^1(x)J^1(y), \\
[J^0(x)J^1(x), J^0(y)J^1(y)] &= \frac{i}{\pi} G\delta'(x-y)(J^0(x)J^1(y) + J^0(y)J^1(x)).
\end{aligned} \qquad (3.24)$$

3.4 The stress-energy tensor

Classically, the Majorana-Weyl components of the canonical stress-energy tensor are given by

$$\begin{aligned}
T^{00} &= \frac{i}{2}(\psi_{1L}\partial_1\psi_{1L} + \psi_{2L}\partial_1\psi_{2L} - \psi_{1R}\partial_1\psi_{1R} - \psi_{2R}\partial_1\psi_{2R}) - 2\pi\alpha\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R}, \\
T^{01} &= -\frac{i}{2}(\psi_{1L}\partial_1\psi_{1L} + \psi_{2L}\partial_1\psi_{2L} + \psi_{1R}\partial_1\psi_{1R} + \psi_{2R}\partial_1\psi_{2R}), \\
T^{10} &= -\frac{i}{2}(\psi_{1L}\partial_0\psi_{1L} + \psi_{2L}\partial_0\psi_{2L} - \psi_{1R}\partial_0\psi_{1R} - \psi_{2R}\partial_0\psi_{2R}), \qquad (3.25) \\
T^{11} &= \frac{i}{2}(\psi_{1L}\partial_0\psi_{1L} + \psi_{2L}\partial_0\psi_{2L} + \psi_{1R}\partial_0\psi_{1R} + \psi_{2R}\partial_0\psi_{2R}) + 2\pi\alpha\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R}.
\end{aligned}$$

Using the classical field equations (3.6), it is easy to show the traceless and the symmetric properties of this tensor, i.e.

$$T^{11} = T^{00}, \qquad T^{01} = T^{10}, \tag{3.26}$$

respectively. Also its conservation can be shown in a similar way (taking $\psi_{iR}^2 = \psi_{iL}^2 = 0$ into account)

$$\partial_\mu T^{\mu\nu} = \partial_\nu T^{\mu\nu} = 0. \tag{3.27}$$

Before we try to guess the quantum $T^{\mu\nu}$, we notice that T^{01} can be written as (see Appendix B)

$$T^{01} = \frac{\pi}{2}\{j^0, j^1\}. \tag{3.28}$$

Thus, by analogy we make the following Ansatz for the respective quantum T^{01} component

$$T^{01} = T^{10} = E\{J^0, J^1\}, \tag{3.29}$$

where E is a normalization coefficient to be determined.

Now let us compute $\partial_1 T^{10}$ to find the T^{00} component by using the conservation laws of the vector and the axial vector currents

$$\begin{aligned}
\partial_1 T^{10} &= E\{\partial_1 J^0, J^1\} + E\{J^0, \partial_1 J^1\} \\
&= -E\{\partial_0 J^1, J^1\} - E\{J^0, \partial_0 J^0\}. \qquad (3.30)
\end{aligned}$$

Therefore, the conservation of stress-energy tensor follows if

$$T^{00} = E((J^0)^2 + (J^1)^2). \tag{3.31}$$

Similarly, we find the T^{11} component by computing $\partial_0 T^{01}$

$$T^{11} = E((J^0)^2 + (J^1)^2). \tag{3.32}$$

To complete our derivation of the quantum stress-energy tensor, we use

$$i[T^{00}(x), T^{01}(y)] = \delta'(x-y)(T^{00}(x) + T^{00}(y)), \tag{3.33}$$

as our normalization condition. (This commutator and those given below are required for Lorentz covariance of a theory with fields of spin ≤ 1, see [6].) From (3.24), this calculation is very straightforward and implies that

$$E = -\frac{\pi}{2G}. \tag{3.34}$$

Summarizing these results, we have

$$T^{\mu\nu} = -\frac{\pi}{2G}\Big(\{J^\mu, J^\nu\} - \eta^{\mu\nu} J^\rho J_\rho\Big). \tag{3.35}$$

Thus, our quantum stress-energy tensor is symmetric (by construction) and traceless (as a by-product of its conservation). This way of writing $T^{\mu\nu}$ in terms of vector currents had been heuristically established in the construction of field theories based on currents as dynamical variables. (See [49], [47], [7].) It is usually called the Sugawara form of the stress-energy tensor and can be used in any dimension. For example, reference [11] considers $T^{\mu\nu}$ in this form for finding the TM solution.

Again from (3.24), we compute all the commutators between the components of $T^{\mu\nu}$ for equal times

$$\begin{aligned} i[T^{00}(x), T^{00}(y)] &= \delta'(x-y)(T^{01}(x) + T^{01}(y)), \\ i[T^{00}(x), T^{01}(y)] &= \delta'(x-y)(T^{00}(x) + T^{00}(y)), \\ i[T^{01}(x), T^{01}(y)] &= \delta'(x-y)(T^{01}(x) + T^{01}(y)). \end{aligned} \tag{3.36}$$

These commutation relations agree with the above references.

Finally, we can rewrite these components of $T^{\mu\nu}$ in terms of derivatives as (see Appendix B)

$$\begin{aligned} T^{00} = T^{11} &= -\frac{i}{4}\Big(\frac{G_{00}^2 + G_{11}^2}{G}\Big)(\psi_{1L}\partial_1\psi_{1L} + \psi_{2L}\partial_1\psi_{2L} - \psi_{1R}\partial_1\psi_{1R} - \psi_{2R}\partial_1\psi_{2R}) \\ &\quad +\pi\Big(\frac{G_{00}^2 - G_{11}^2}{G}\Big)\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R}, \\ T^{01} = T^{10} &= -\frac{i}{2}(\psi_{iL}\partial_1\psi_{iL} + \psi_{2L}\partial_1\psi_{2L} + \psi_{iR}\partial_1\psi_{iR} + \psi_{2R}\partial_1\psi_{2R}). \end{aligned} \tag{3.37}$$

As expected, the classical components can be obtained from these expressions when α is small.

3.5 Bosonization of the model

It is well known the equivalence between the massive TM and the SGM [8], [30]. We would like to establish the identification rules in our framework. For that, we follow the procedure of section 2.3, i.e. as Coleman did it in his original paper [8].

First we need to find Klaiber's parameters, α_K and β_K. To do that, we compare the commutators (2.14) and (3.22), with c_1 and c_2 given by (2.15), and obtain

$$-\frac{G_{11}}{G_{00}} = \frac{1-\beta_K/\sqrt{\pi}}{1-\alpha_K/\sqrt{\pi}}. \tag{3.38}$$

(Again for generality, we use a ratio comparison.) Using this result and the condition (2.17) for $\lambda = 0$ (the spin-1/2 solution), we find two different solutions

$$\frac{\alpha_K}{\sqrt{\pi}} = 1 \pm \sqrt{-\frac{G_{00}}{G_{11}}}, \qquad \frac{\beta_K}{\sqrt{\pi}} = 1 \pm \sqrt{-\frac{G_{11}}{G_{00}}}, \tag{3.39}$$

respectively. (See comment below equation (2.18).) Finally, Klaiber's parameter b is given by

$$1+\frac{b}{\pi} \equiv \left(1-\frac{\beta_K}{\sqrt{\pi}}\right)^2 = -\frac{G_{11}}{G_{00}}. \tag{3.40}$$

Now making the same identifications between the perturbation series (2.21) and (2.10), we conclude that

$$\frac{\beta^2}{4\pi} = 1+\frac{b}{\pi} = -\frac{G_{11}}{G_{00}}. \tag{3.41}$$

And from (3.21), we get

$$\frac{\beta^2}{4\pi} = \frac{1}{1+\alpha G_{00}}. \tag{3.42}$$

For Coleman's calculations, a direct inspection of the commutators (2.14) and (3.22) gives $G_{00} = 1$ and $G_{11} = -1/(1+\alpha)$ (see equation (2.19)). Notice that this choice of G_{00} and G_{11} perfectly agrees with the condition (3.21). Also, (3.42) exactly reduces to Coleman's identification between the coupling constants of the models. Moreover, this choice of G_{00} and G_{11} can be clearly seen in Mandelstam's current definition for his bosonization approach [30], since it is for equal time (like ours with these particular values of G_{00} and G_{11}). Therefore, Coleman's choice corresponds to $G_{00} = 1$ in our approach.

To complete our identifications, we compute the general commutator between the current and $\sigma_\pm$ (see Appendix B)

$$[J^\mu(x), \sigma_\pm(y)] = G^\mu{}_\nu [j^\nu(x), \sigma_\pm(y)] = \pm 2 G_{11} \epsilon^{\mu\nu} D_\nu(x-y)\sigma_\pm(y). \tag{3.43}$$

Then, we compare this commutator with (2.11) and use the fact that $\sigma_\pm$ are identified with $:\exp(\pm i\beta\phi):$ to find

$$\begin{aligned} J^\mu(x) &= \frac{2}{\beta} G_{11} \epsilon^{\mu\nu} \partial_\nu \phi(x) \\ &= -\sqrt{\frac{-G}{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi(x) \\ &= \frac{-G_{00}}{\sqrt{\pi(1+\alpha G_{00})}} \epsilon^{\mu\nu} \partial_\nu \phi(x), \end{aligned} \tag{3.44}$$

where we have used (3.21) and (3.42). Again, with $G_{00} = 1$ we reproduce Coleman's result.

With these results, it is then possible to have slightly different identification rules from Coleman's according to the choice of G_{00} (or, equivalently, according to the current definition of TM as has been discussed in the previous chapter). But notice that for the free fermion case ($\alpha = 0$) all these results are the same (they do not depend on the choice of G_{00}, since $G_{00}(0) = 1$), so that the equivalence is always between the free massive Dirac field theory and the SGM for the value of $\beta = 2\sqrt{\pi}$.

Based on the equation (3.42), we conclude that $\alpha G_{00} > -1$ for a physical solution (since $\beta^2 > 0$). Also, the ratio G_{11}/G_{00} should be negative in this case. For the massive TM, we can get a stronger constraint if we use Coleman's bound ($\beta^2 < 8\pi$, see property 3 in the section 2.1), because now $\alpha G_{00} > -1/2$ for physical solutions (see Figure 3.1). For Coleman's case, these conclusions follow with $G_{00} = 1$.

Finally, we use Johnson's case as an example [27]. In our language, Johnson's choice corresponds to $G_{00} = 1/(1-\alpha/2)$, so that the above conclusions for physical solutions are $-2 < \alpha < 2$ for the massless model and $-2/3 < \alpha < 2$ for the massive case.

Therefore, taking a specific choice of G_{00} implies some range of the coupling constant or, equivalently, the definition of current constraints the allowed values of the coupling constant.

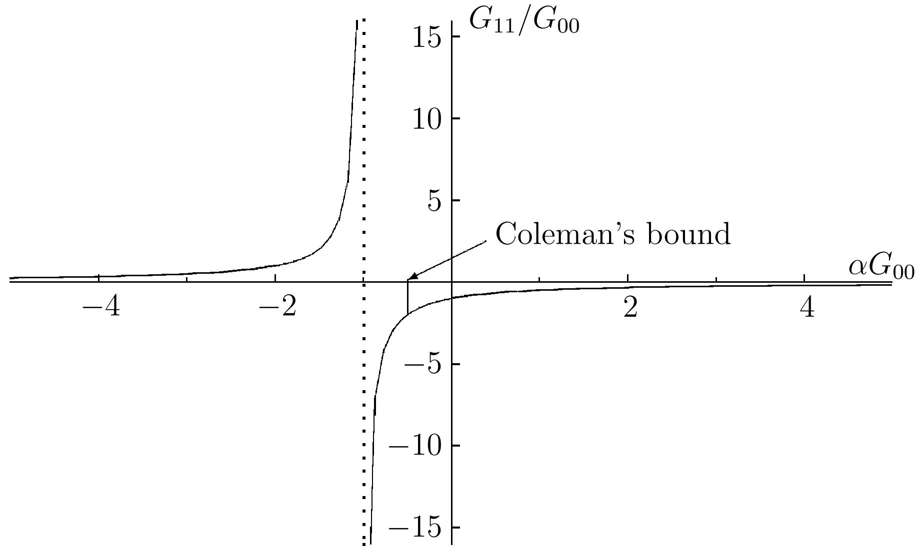

Figure 3.1: Graph for general G_{11}/G_{00} ratio.

Chapter Four

$O(3)$ Gross-Neveu Model

4.1 The model

The classical Lagrangian for the $O(3)$ GNM is

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + g\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j, \tag{4.1}$$

where ψ_i is a Majorana spinor and $i, j = 1, 2, 3$. Thus, the classical field equations are

$$\partial\!\!\!/ \psi_i = 4ig\bar{\psi}_j\psi_j\psi_i. \tag{4.2}$$

In the Majorana-Weyl representation, $\mathcal{L}$ can be written in terms of chiral components of the spinors as

$$\mathcal{L} = i\psi_{iR}\partial_+\psi_{iR} + i\psi_{iL}\partial_-\psi_{iL} + 4g\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}, \tag{4.3}$$

and so can the field equations

$$\begin{aligned} \partial_+\psi_{iR} &= 4ig\psi_{jL}\psi_{jR}\psi_{iL}, \\ \partial_-\psi_{iL} &= -4ig\psi_{jL}\psi_{jR}\psi_{iR}, \end{aligned} \tag{4.4}$$

where $j \neq i$.

Now, let us apply the procedure of section 3.2 to construct a Lagrangian which we will consider as the quantum one. First, we use the Fierz transformation (A.22) to rewrite $\mathcal{L}$ in (4.1) as

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + gj^\mu{}_{ij}j_{\mu ij}, \tag{4.5}$$

where $j^\mu{}_{ij} = \bar{\psi}_i\gamma^\mu\psi_j$. Notice that $j^\mu{}_{ji} = -j^\mu{}_{ij}$ (by a Majorana bilinear identity in (A.21)).

Due to singularities of the composite operators $j^\mu{}_{ij}$ and $j^\mu{}_{ij}j_{\mu ij}$, we define them by a point-splitting method for space separations as (see Appendix B)

$$j^\mu{}_{ij} \equiv \frac{1}{2}\Big(\bar{\psi}_i(x+\epsilon)\gamma^\mu\psi_j(x) - \bar{\psi}_j(x)\gamma^\mu\psi_i(x-\epsilon)\Big), \tag{4.6}$$

$$j^\mu{}_{ij}j^\nu{}_{ij} \equiv \frac{1}{2}\Big(j^\mu{}_{ij}(x+\epsilon)j^\nu{}_{ij}(x) + j^\nu{}_{ij}(x)j^\mu{}_{ij}(x-\epsilon)\Big), \tag{4.7}$$

where the space coordinate is only written down. Then, we assume the following quantum Lagrangian

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_i \partial\!\!\!/ \psi_i + gG_{\mu\nu}j^{\mu}{}_{ij}j^{\nu}{}_{ij}, \tag{4.8}$$

where $G_{\mu\nu} \equiv G_{\mu\nu}(g)$ is some symmetric tensor to be determined.

Thus, our quantum field equations follow from Euler-Lagrange equations

$$\partial\!\!\!/ \psi_i(x) = 2igG_{\mu\nu}\Big(j^{\nu}{}_{ij}(x+\epsilon)\gamma^{\mu}\psi_j(x) + \gamma^{\mu}\psi_j(x)j^{\nu}{}_{ij}(x-\epsilon)\Big). \tag{4.9}$$

Again this is our motivation for choosing (4.8), since the point-splitting method in the field equations follows from the Lagrangian.

In the Majorana-Weyl representation, the Lagrangian becomes (see Appendix B)

$$\begin{aligned} \mathcal{L} &= i(1+2\alpha G_{++})\psi_{iR}\partial_+\psi_{iR} - 2i\alpha G_{++}\psi_{iR}\partial_-\psi_{iR} \\ &\quad + i(1+2\alpha G_{--})\psi_{iL}\partial_-\psi_{iL} - 2i\alpha G_{--}\psi_{iL}\partial_+\psi_{iL} \\ &\quad + 2\pi\alpha G_{-+}\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}, \end{aligned} \tag{4.10}$$

where $\alpha = 4g/\pi$, $i \neq j$ and $G_{\mu\nu}$ is written in its light-cone components. Finally, the field equations (4.9) become

$$\begin{aligned} (1+2\alpha G_{++})\partial_+\psi_{iR} &= 2\pi i\alpha G_{+-}\psi_{jL}\psi_{jR}\psi_{iL} + 2\alpha G_{++}\partial_-\psi_{iR}, \\ (1+2\alpha G_{--})\partial_-\psi_{iL} &= -2\pi i\alpha G_{-+}\psi_{jL}\psi_{jR}\psi_{iR} + 2\alpha G_{--}\partial_+\psi_{iL}. \end{aligned} \tag{4.11}$$

4.2 A special conservation law

Following [56], let us define the following fermionic currents which will later become the supercurrents

$$\begin{aligned} j^+ &\equiv \psi_{1R}\psi_{2R}\psi_{3R}, \\ j^- &\equiv \psi_{1L}\psi_{2L}\psi_{3L}. \end{aligned} \tag{4.12}$$

Classically, they are conserved

$$\partial_+ j^+ = 0, \qquad \partial_- j^- = 0. \tag{4.13}$$

These results follow from the field equations (4.4) and $\psi_{iR}^2 = \psi_{iL}^2 = 0$.

Strictly, j^+ and j^- form the lower and upper components of the spinor current J^μ, i.e.

$$J^0 = \begin{pmatrix} -j^- \\ j^+ \end{pmatrix}, \qquad J^1 = \begin{pmatrix} j^- \\ j^+ \end{pmatrix}, \tag{4.14}$$

so that (4.13) is implemented in

$$\partial_\mu J^\mu = 0. \tag{4.15}$$

At the quantum level, these conservation laws are modified by anomalies. Then, we claim that the proper spinor current is given by (like the vector current in the $O(2)$ GNM, see section 3.2)

$$S_\mu \equiv E G_{\mu\nu} J^\nu, \tag{4.16}$$

where $E = E(g)$ is a normalization coefficient.

From dimensional and symmetry considerations, the modifications must take the form [56]

$$\partial_\mu S^\mu + \partial_\mu \mathcal{R}^\mu = 0, \tag{4.17}$$

where

$$\mathcal{R}_\mu = E H_{\mu\nu} R^\nu, \tag{4.18}$$

$$R^0 = \begin{pmatrix} -r^+ \\ r^- \end{pmatrix}, \qquad R^1 = \begin{pmatrix} -r^+ \\ -r^- \end{pmatrix}, \tag{4.19}$$

$$\begin{aligned} r^+ &= \psi_{1R}\psi_{2R}\psi_{3L} + \psi_{3R}\psi_{1R}\psi_{2L} + \psi_{2R}\psi_{3R}\psi_{1L}, \\ r^- &= \psi_{1L}\psi_{2L}\psi_{3R} + \psi_{3L}\psi_{1L}\psi_{2R} + \psi_{2L}\psi_{3L}\psi_{1R}, \end{aligned} \tag{4.20}$$

and $H_{\mu\nu} \equiv H_{\mu\nu}(g)$ is some tensor to be determined. In spinor components, equation (4.17) is written as

$$\begin{aligned} G_{-+}\partial_+ j^+ + G_{++}\partial_- j^+ + H_{--}\partial_+ r^- + H_{+-}\partial_- r^- &= 0, \\ G_{--}\partial_+ j^- + G_{+-}\partial_- j^- + H_{-+}\partial_+ r^+ + H_{++}\partial_- r^+ &= 0, \end{aligned} \tag{4.21}$$

Now, let us compute $\partial_+ j^+$ by using the field equations (4.11) and the point-splitting method to determine $G_{\mu\nu}$ and $H_{\mu\nu}$ (see Appendix B)

$$\begin{aligned} (1 + 2\alpha G_{++})\partial_+ j^+ = \; & \alpha G_{+-}\Big(\partial_-(\psi_{1L}\psi_{2L})\psi_{3R} + \partial_-(\psi_{3L}\psi_{1L})\psi_{2R} + \partial_-(\psi_{2L}\psi_{3L})\psi_{1R} \\ & -\partial_+ r^- + \psi_{1L}\psi_{2L}\partial_+\psi_{3R} + \psi_{3L}\psi_{1L}\partial_+\psi_{2R} + \psi_{2L}\psi_{3L}\partial_+\psi_{1R}\Big) \\ & +2\alpha G_{++}\partial_- j^+. \end{aligned} \tag{4.22}$$

Similarly, we work out the non-total-derivative terms in the above expression (See Appendix B) and obtain

$$
\begin{aligned}
&(1+2\alpha G_{--})\Big(\partial_-(\psi_{1L}\psi_{2L})\psi_{3R}+\partial_-(\psi_{3L}\psi_{1L})\psi_{2R}+\partial_-(\psi_{2L}\psi_{3L})\psi_{1R}\Big)= \\
&\quad=\frac{\alpha}{2}G_{-+}\Big(4\partial_+j^++\partial_+r^-+\psi_{1L}\psi_{2L}\partial_+\psi_{3R}+\psi_{3L}\psi_{1L}\partial_+\psi_{2R}+\psi_{2L}\psi_{3L}\partial_+\psi_{1R} \\
&\quad-4\partial_-j^+-2\partial_-r^-+\partial_-(\psi_{1L}\psi_{2L})\psi_{3R}+\partial_-(\psi_{3L}\psi_{1L})\psi_{2R}+\partial_-(\psi_{2L}\psi_{3L})\psi_{1R}\Big) \\
&\quad+2\alpha G_{--}\Big(\partial_+r^--\psi_{1L}\psi_{2L}\partial_+\psi_{3R}-\psi_{3L}\psi_{1L}\partial_+\psi_{2R}-\psi_{2L}\psi_{3L}\partial_+\psi_{1R}\Big),
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
&(1+2\alpha G_{++})\Big(\psi_{1L}\psi_{2L}\partial_+\psi_{3R}+\psi_{3L}\psi_{1L}\partial_+\psi_{2R}+\psi_{2L}\psi_{3L}\partial_+\psi_{1R}\Big)= \\
&\quad=\frac{\alpha}{2}G_{+-}\Big(-\partial_+r^--\psi_{1L}\psi_{2L}\partial_+\psi_{3R}-\psi_{3L}\psi_{1L}\partial_+\psi_{2R}-\psi_{2L}\psi_{3L}\partial_+\psi_{1R} \\
&\quad+2\partial_-r^--\partial_-(\psi_{1L}\psi_{2L})\psi_{3R}-\partial_-(\psi_{3L}\psi_{1L})\psi_{2R}-\partial_-(\psi_{2L}\psi_{3L})\psi_{1R}\Big) \\
&\quad+2\alpha G_{++}\Big(\partial_-r^--\partial_-(\psi_{1L}\psi_{2L})\psi_{3R}-\partial_-(\psi_{3L}\psi_{1L})\psi_{2R}-\partial_-(\psi_{2L}\psi_{3L})\psi_{1R}\Big).
\end{aligned}
\tag{4.24}
$$

Solving (4.22)-(4.24) in favor of total-derivative terms, we find

$$
\begin{aligned}
&\Big(2+\alpha(8G_{++}+G_{+-}-G_{-+}+4G_{--}) \\
&+2\alpha^2(G_{++}(4G_{++}+G_{+-}-G_{-+}+4G_{--})-2G_{+-}G_{-+})\Big)\partial_+j^+ \\
&-2\alpha\Big(2G_{++}+\alpha(G_{++}(4G_{++}+G_{+-}-G_{-+}+4G_{--})-2G_{+-}G_{-+})\Big)\partial_-j^+ \\
&+2\alpha G_{+-}(1+\alpha(2G_{++}+G_{+-}-G_{-+})\partial_+r^- \\
&-2\alpha^2G_{+-}(2G_{++}+G_{+-}-G_{-+})\partial_-r^- \;=\; 0.
\end{aligned}
\tag{4.25}
$$

A similar expression for ∂_-j^- is obtained by switching $+$ and $-$.

Comparing these results with (4.21) and using $G_{-+}=G_{+-}$, we find

$$
\begin{aligned}
\frac{-2\alpha\Big(G_{++}+\alpha(2G_{++}(G_{++}+G_{--})-G_{+-}^2)\Big)}{1+2\alpha(2G_{++}+G_{--})+2\alpha^2(2G_{++}(G_{++}+G_{--})-G_{+-}^2)} &= \frac{G_{++}}{G_{+-}}, \\
\frac{-2\alpha\Big(G_{--}+\alpha(2G_{--}(G_{--}+G_{++})-G_{+-}^2)\Big)}{1+2\alpha(2G_{--}+G_{++})+2\alpha^2(2G_{--}(G_{--}+G_{++})-G_{+-}^2)} &= \frac{G_{--}}{G_{+-}},
\end{aligned}
\tag{4.26}
$$

and also $H_{\mu\nu}$ gets completely determined by $G_{\mu\nu}$

$$
\begin{aligned}
\frac{H_{--}}{G_{+-}} &= \frac{\alpha G_{+-}(1+2\alpha G_{++})}{1+2\alpha(2G_{++}+G_{--})+2\alpha^2(2G_{++}(G_{++}+G_{--})-G_{+-}^2)}, \\
\frac{H_{+-}}{G_{+-}} &= \frac{-2\alpha^2G_{+-}G_{++}}{1+2\alpha(2G_{++}+G_{--})+2\alpha^2(2G_{++}(G_{++}+G_{--})-G_{+-}^2)}, \\
\frac{H_{-+}}{G_{+-}} &= \frac{-2\alpha^2G_{+-}G_{--}}{1+2\alpha(2G_{--}+G_{++})+2\alpha^2(2G_{--}(G_{--}+G_{++})-G_{+-}^2)}, \\
\frac{H_{++}}{G_{+-}} &= \frac{\alpha G_{+-}(1+2\alpha G_{--})}{1+2\alpha(2G_{--}+G_{++})+2\alpha^2(2G_{--}(G_{--}+G_{++})-G_{+-}^2)}.
\end{aligned}
\tag{4.27}
$$

For simplicity, we take the case $G_{--} = G_{++}$ and find

$$\frac{G_{++}}{G_{+-}} = \frac{-2\alpha\Big(G_{++} + \alpha(4G_{++}^2 - G_{+-}^2)\Big)}{1 + 6\alpha G_{++} + 2\alpha^2(4G_{++}^2 - G_{+-}^2)}, \tag{4.28}$$

and

$$\begin{aligned} H_{++} = H_{--} &= \frac{\alpha G_{+-}^2(1 + 2\alpha G_{++})}{1 + 6\alpha G_{++} + 2\alpha^2(4G_{++}^2 - G_{+-}^2)}, \\ H_{+-} = H_{-+} &= \frac{-2\alpha^2 G_{+-}^2 G_{++}}{1 + 6\alpha G_{++} + 2\alpha^2(4G_{++}^2 - G_{+-}^2)}. \end{aligned} \tag{4.29}$$

In usual coordinates, equation (4.28) reduces to

$$\frac{G_{00}}{G_{11}} = \frac{-4(1 + \alpha(G_{00} + G_{11}))}{4 + 8\alpha(G_{00} + G_{11}) + \alpha^2(3G_{00}^2 + 10G_{00}G_{11} + 3G_{11}^2)}, \tag{4.30}$$

thus solving for G_{11}, the usual components of $G_{\mu\nu}$ are

$$G_{01} = G_{10} = 0, \tag{4.31}$$

$$G_{11} = \frac{-2 - 6\alpha G_{00} - 5\alpha^2 G_{00}^2 + 2\sqrt{1 + 2\alpha G_{00} + 3\alpha^2 G_{00}^2 + 6\alpha^3 G_{00}^3 + 4\alpha^4 G_{00}^4}}{4\alpha + 3\alpha^2 G_{00}},$$

where we have chosen the solution of G_{11} that tends to $-G_{00}$ when $\alpha \to 0$, since $G_{\mu\nu}$ should become the metric in the free case. (See section 3.3.) Therefore, we obtain the components of $H_{\mu\nu}$

$$\begin{aligned} H_{01} &= H_{10} = 0, \\ H_{00} &= \frac{\alpha}{4}\frac{G_{00}(G_{00} - G_{11})}{1 + \alpha(G_{00} + G_{11})}, \\ H_{11} &= \frac{\alpha}{4}G_{00}(G_{00} - G_{11}), \end{aligned} \tag{4.32}$$

where we have used (4.30) to simplify these expressions.

4.3 The superalgebra

Let us define the modified current of S^μ which is quantumly conserved by (4.17)

$$\mathcal{J}_\mu \equiv S_\mu + \mathcal{R}_\mu = E(G_{\mu\nu}J^\nu + H_{\mu\nu}R^\nu). \tag{4.33}$$

The spinor charge associated with this current is defined in the usual way

$$Q \equiv \int dx\, \mathcal{J}^0 = E\int dx\, (G^{0\mu}J_\mu + H^{0\mu}R_\mu). \tag{4.34}$$

or in components, $Q = (-Q^- \; Q^+)^T$,

$$\begin{aligned} Q^+ &= E\int dx\,(G_{00} j^+ + H_{00} r^-), \\ Q^- &= E\int dx\,(G_{00} j^- + H_{00} r^+), \end{aligned} \tag{4.35}$$

where we have employed the results of the previous section, (4.31) and (4.32).

Using contractions (see Appendix B), we compute the algebra made by these charges

$$\begin{aligned} (Q^+)^2 &= \frac{E^2}{2}(G_{00} - H_{00})\int dx\left(\frac{i}{2\pi}(G_{00}+H_{00})\psi_{iR}\partial_1\psi_{iR} + H_{00}\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}\right), \\ (Q^-)^2 &= \frac{E^2}{2}(G_{00} - H_{00})\int dx\left(\frac{-i}{2\pi}(G_{00}+H_{00})\psi_{iL}\partial_1\psi_{iL} + H_{00}\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}\right), \\ \{Q^+, Q^-\} &= 0. \end{aligned} \tag{4.36}$$

(We have omitted a c-number in the first two equations.) For this algebra to be supersymmetric, we need the integrals to be proportional to the conserved charges associated with translations,

$$P^\pm = P^0 \pm P^1 = \int dx\,T^{00} \pm \int dx\,T^{01}, \tag{4.37}$$

respectively. Therefore we read off

$$\begin{aligned} T^{00} &= \frac{1}{4\pi}E^2(H_{00}^2 - G_{00}^2)\Big(\frac{i}{2}(\psi_{iL}\partial_1\psi_{iL} - \psi_{iR}\partial_1\psi_{iR}) - 2\pi\frac{H_{00}}{G_{00}+H_{00}}\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}\Big), \\ T^{01} &= \frac{1}{4\pi}E^2(H_{00}^2 - G_{00}^2)\Big(\frac{-i}{2}(\psi_{iL}\partial_1\psi_{iL} + \psi_{iR}\partial_1\psi_{iR})\Big). \end{aligned} \tag{4.38}$$

These components of $T^{\mu\nu}$ are expected to be the quantum ones. (Notice that the expressions in the big parentheses agree with the classical components if the limit of small α is taken, see equations (4.31) and (4.32).) E can be found through the proper normalization of $T^{\mu\nu}$ and H_{00} will encode the trace anomaly, $T^\mu_{\ \mu} \neq 0$ (see next section). The conservation of this stress-energy tensor follows from supersymmetry.

Moreover, the third equation in (4.36) says that the central charge for the model is zero (independent of the coupling constant). The exact form of H_{00} is just important for the trace anomaly, but not for the central charge. Notice that the third equation in (4.36) can then be computed without using the point-splitting method.

4.4 The stress-energy tensor

We consider the commutator between T^{00} and T^{01}, equation (3.33), as our normalization condition for the stress-energy tensor. Thus, we infer from the section 3.4 that E should be

$$E = \sqrt{4\pi(H_{00}^2 - G_{00}^2)^{-1}}. \tag{4.39}$$

Now, we make the following Ansatz by analogy to the form of the classical stress-energy tensor

$$\begin{aligned} T^{00} &= \frac{i}{2}(\psi_{iL}\partial_1\psi_{iL} - \psi_{iR}\partial_1\psi_{iR}) - \frac{1}{2}(F+h)\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}, \qquad (4.40) \\ T^{11} &= \frac{i}{2}\frac{1}{1+4\alpha G_{++}}(\psi_{iL}\partial_0\psi_{iL} + \psi_{iR}\partial_0\psi_{iR}) + \frac{1}{2}(F+h)\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}. \end{aligned}$$

where F is chosen such that

$$T^{\mu}_{\ \mu} = -h\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}. \tag{4.41}$$

We need to make an Ansatz, because the superalgebra allows us to find only two components of $T^{\mu\nu}$. (The symmetric property determines T^{10}.) In the Ansatz, T^{00} and T^{11} contribute equally in the anomaly (4.41). From (4.38) and the field equations (4.11), we get that

$$F = 4\pi\frac{H_{00}}{G_{00}}, \quad h = -\frac{4\pi}{G_{00}}\frac{H_{00}^2}{G_{00}+H_{00}}. \tag{4.42}$$

With (4.32) and (4.31),

$$\begin{aligned} h &= \frac{-\pi\alpha^2(G_{00}-G_{11})^2}{(1+\alpha(G_{00}+G_{11}))(4+\alpha(5G_{00}+3G_{11}))} \qquad (4.43) \\ &\approx -\pi\alpha^2 G_{00}^2 + \frac{\pi}{2}\alpha^3 G_{00}^3 + \frac{3}{4}\pi\alpha^4 G_{00}^4 + O(\alpha^5) \qquad (4.44) \end{aligned}$$

The trace anomaly can also be computed with

$$T^{\mu}_{\ \mu} = \beta\frac{\partial\mathcal{L}}{\partial g}, \tag{4.45}$$

where $\mathcal{L}$ is the quantum effective action, but for finding the leading term we just need to consider (4.3) instead [36]. Therefore, we conclude the first β function coefficient, which is independent of the renormalization scheme, is

$$\beta = \frac{\pi}{4}\alpha^2, \tag{4.46}$$

where we have used equations (4.41), (4.44) and the fact that $G_{00} = 1 + O(\alpha)$. This result agrees completely with that of loop calculation. For higher order terms, the agreement goes through a specific choice of G_{00} that will be related to the renormalization scheme. (See [55], [19], [18], [29], [53] and [52].)

Chapter Five

Supersymmetric Sine-Gordon Model

5.1 Supersymmetric Lagrangians for scalar multiplets

Let us review the supersymmetry of the general Lagrangian derived from a scalar multiplet in two dimensions (see [14], [12] and [26]). This Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}V^2(\phi) - \frac{1}{2}\partial_\phi V(\phi)\bar{\psi}\psi, \tag{5.1}$$

where ϕ is a scalar boson field, ψ is a Majorana spinor and $V(\phi)$ is an arbitrary function of ϕ.

Under the infinitesimal supersymmetry transformations

$$\begin{aligned} \delta\phi &= \bar{\varepsilon}\psi, \\ \delta\psi &= -(i\partial\!\!\!/\phi + V(\phi))\varepsilon, \end{aligned} \tag{5.2}$$

where ε is a Majorana constant spinor, the variation of $\mathcal{L}$ is a total derivative $\bar{\varepsilon}\partial_\mu\mathcal{K}^\mu$ with

$$\mathcal{K}^\mu = \frac{1}{2}\gamma^\mu(\partial\!\!\!/\phi - iV(\phi))\psi. \tag{5.3}$$

Using Noether's theorem, we find the supersymmetry current is

$$J^\mu = (\partial\!\!\!/\phi + iV(\phi))\gamma^\mu\psi = (\partial^\mu\phi + \epsilon^{\mu\nu}\partial_\nu\phi\gamma^5 + iV(\phi)\gamma^\mu)\psi, \tag{5.4}$$

where we have used a property of the γ matrices (A.10).

With the classical field equations

$$\begin{aligned} \partial^\mu\partial_\mu\phi + V(\phi)\partial_\phi V(\phi) + \frac{1}{2}\partial_\phi^2 V(\phi)\bar{\psi}\psi &= 0, \\ \partial\!\!\!/\psi + i\partial_\phi V(\phi)\psi &= 0, \end{aligned} \tag{5.5}$$

and $(\bar{\psi}\psi)\psi = 0$, we can check the conservation of this supercurrent

$$\partial_\mu J^\mu = 0. \tag{5.6}$$

In the Majorana-Weyl representation, the field equations are given by

$$\begin{aligned} 4\partial_+\partial_-\phi + V(\phi)\partial_\phi V(\phi) - i\partial_\phi^2 V(\phi)\psi_L\psi_R &= 0, \\ \partial_+\psi_R - \frac{1}{2}\partial_\phi V(\phi)\psi_L &= 0, \\ \partial_-\psi_L + \frac{1}{2}\partial_\phi V(\phi)\psi_R &= 0, \end{aligned} \tag{5.7}$$

and the components of J^μ are

$$J^0 = \begin{pmatrix} 2(\partial_+\phi)\psi_L + V(\phi)\psi_R \\ 2(\partial_-\phi)\psi_R - V(\phi)\psi_L \end{pmatrix}, \qquad J^1 = \begin{pmatrix} -2(\partial_+\phi)\psi_L + V(\phi)\psi_R \\ 2(\partial_-\phi)\psi_R + V(\phi)\psi_L \end{pmatrix}. \tag{5.8}$$

For the supersymmetric sine-Gordon model (SSGM), we would like to treat $V(\phi)$ as a normal-ordered composite operator, whose normal ordering amounts to a multiplicative renormalization; in particular, $V(\phi) = A_0 \sin(\beta\phi) = A_R :\sin(\beta\phi):$ (see Appendix C). In this case, the supersymmetry of the model also holds with normal-ordered expressions.

5.2 The stress-energy tensor

The components of the canonical stress-energy tensor are computed in the usual way, so that we obtain

$$\begin{aligned} T^{00} &= \frac{1}{2}\Big((\partial^0\phi)^2 + (\partial^1\phi)^2 - i\bar\psi\gamma^1\partial_1\psi + V^2(\phi) + \partial_\phi V(\phi)\bar\psi\psi\Big), \\ T^{01} &= (\partial^0\phi)(\partial^1\phi) + \frac{i}{2}\bar\psi\gamma^0\partial^1\psi, \\ T^{10} &= (\partial^1\phi)(\partial^0\phi) + \frac{i}{2}\bar\psi\gamma^1\partial^0\psi, \\ T^{11} &= \frac{1}{2}\Big((\partial^0\phi)^2 + (\partial^1\phi)^2 + i\bar\psi\gamma^0\partial_0\psi - V^2(\phi) - \partial_\phi V(\phi)\bar\psi\psi\Big). \end{aligned} \tag{5.9}$$

In the Majorana-Weyl representation, they are written as

$$\begin{aligned} T^{00} &= \frac{1}{2}\Big((\partial_0\phi)^2 + (\partial_1\phi)^2 + i\psi_L\partial_1\psi_L - i\psi_R\partial_1\psi_R + V^2(\phi) - 2i\partial_\phi V(\phi)\psi_L\psi_R\Big), \\ T^{01} &= -(\partial_0\phi)(\partial_1\phi) - \frac{i}{2}\psi_L\partial_1\psi_L - \frac{i}{2}\psi_R\partial_1\psi_R, \\ T^{10} &= -(\partial_1\phi)(\partial_0\phi) - \frac{i}{2}\psi_L\partial_0\psi_L + \frac{i}{2}\psi_R\partial_0\psi_R, \\ T^{11} &= \frac{1}{2}\Big((\partial_0\phi)^2 + (\partial_1\phi)^2 + i\psi_L\partial_0\psi_L + i\psi_R\partial_0\psi_R - V^2(\phi) + 2i\partial_\phi V(\phi)\psi_L\psi_R\Big). \end{aligned} \tag{5.10}$$

Using the field equations (5.5), one can check that

$$T^{01} = T^{10}, \qquad \partial_\mu T^{\mu\nu} = \partial_\nu T^{\mu\nu} = 0, \tag{5.11}$$

i.e. the stress-energy tensor is symmetric and conserved.

Since its trace is non zero

$$T^{\mu}_{\ \mu} = V^2(\phi) - i\partial_\phi V(\phi)\psi_L\psi_R, \tag{5.12}$$

the model is not scale invariant.

Finally, the conserved charges associated with translations are given by

$$P^0 \equiv \int dx\, T^{00}, \qquad P^1 \equiv \int dx\, T^{01}. \tag{5.13}$$

Let us define for future reference

$$P^{\pm} \equiv P^0 \pm P^1, \tag{5.14}$$

which are the conserved charges in light-cone coordinates.

5.3 Supercharges

The spinor supercharge that generates the supersymmetry transformations (5.2) is defined by

$$Q \equiv \int dx\, J^0. \tag{5.15}$$

With (5.4), we find

$$Q = \int dx\, \Big((\partial_0\phi)\psi - (\partial_1\phi)\gamma^5\psi + iV(\phi)\gamma^0\psi\Big). \tag{5.16}$$

Then the Majorana-Weyl components of $Q = (Q^- \ Q^+)^T$ (the supercharges) are

$$\begin{aligned} Q^+ &= \int dx\, \Big((\partial_0\phi - \partial_1\phi)\psi_R - V(\phi)\psi_L\Big), \\ Q^- &= \int dx\, \Big((\partial_0\phi + \partial_1\phi)\psi_L + V(\phi)\psi_R\Big), \end{aligned} \tag{5.17}$$

as found in reference [58].

Using the canonical commutation relations for boson fields and the contractions for fermion fields, we can check that the supercharges satisfy the following supersymmetry algebra (see Appendix C)

$$\begin{aligned} (Q^{\pm})^2 &= P^{\pm}, \\ \{Q^+, Q^-\} &= Z, \end{aligned} \tag{5.18}$$

where $P^{\pm}$ is given by (5.14) and

$$Z = -\int dx \left(2V(\phi) + \frac{1}{4\pi}\partial_\phi^2 V(\phi)\right)\partial_1\phi. \tag{5.19}$$

The second term of this central charge, which is a total derivative, is missing in reference [58] (see Appendix C). The central charge gets modified further if normal ordering of $V(\phi)$ is taken into account (see Appendix C)

$$Z = -\int dx \left(2 :V(\phi)\partial_1\phi: + \frac{1}{2\pi} :\partial_\phi^2 V(\phi)\partial_1\phi: \right). \tag{5.20}$$

With this result, the SSGM, $V(\phi) = A\sin(\beta\phi)$, has zero central charge at $\beta^2 = 4\pi$. (Notice that without normal ordering the zero value would occur at $\beta^2 = 8\pi$.) Due to supersymmetry, there are no other quantum corrections to the central charge; equation (5.20) is then exact.

It is very interesting that the zero central charge for SSGM is at the same value of β where the SGM is equivalent to the free massive Dirac field theory. However, we do not have any physical reason for why this agreement happens. In next chapter, we will compare the central charge (5.20) with that for $O(3)$ GNM to analyze the equivalence between the models.

Chapter Six

About the equivalence

6.1 Application of the bosonization rules

The conjecture of an equivalence between the $O(3)$ GNM and the SSGM was first introduced by Witten, when he studied some properties of the $O(N)$ GNM for $N = 3$ and $N = 4$ [56]. Witten's conjecture is based on the observation that by applying the standard bosonization rules (2.31) to two of the three Majorana spinors in the $O(3)$ GNM Lagrangian (4.1), one obtains a Lagrangian that is closely related to the SSGM. In fact, for the special case $g = -\pi/2$, one obtains precisely the SSGM Lagrangian (5.1) with $V(\phi) = A_0 \sin(\beta\phi)$ and $\beta = 2\sqrt{\pi}$.

Let us repeat Witten's procedure in detail to establish this equivalence. We start by applying the basic rules (2.31) to the GNM Lagrangian (4.1) and consider the identity

$$:\cos(2\sqrt{\pi}\phi):^2 =: \sin(2\sqrt{\pi}\phi):^2 \; . \tag{6.1}$$

The proof of this identity is given in Appendix C. It should be contrasted with the trigonometric identity $\sin^2\varphi + \cos^2\varphi = 1$ used in references [56], [2]. Note that applying the rules (2.31) to the second Fierz transformation in (A.25) implies (6.1), i.e. our identity (6.1) agrees completely with that Fierz transformation, as it should for the consistency of the bosonization prescription.

Therefore, the GNM Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{i}{2}\bar{\psi}_3 \partial\!\!\!/ \psi_3 + 4gA_B^2 :\sin(2\sqrt{\pi}\phi):^2 + 4gA_B :\cos(2\sqrt{\pi}\phi): \bar{\psi}_3\psi_3. \tag{6.2}$$

Comparing with (5.1), we have that

$$V(\phi) = A_0 \sin(2\sqrt{\pi}\phi), \qquad g = -\pi/2, \tag{6.3}$$

where $A_R = 2\sqrt{\pi}A_B$. Our value of g differs from that of references [56], [2], where $g = \pi/2$, because of the identity (6.1). Thus we find at least the possibility of an identification between the values $g = -\pi/2$ for GNM and $\beta = 2\sqrt{\pi}$ for SSGM.

Witten also argues that it is possible to extend the equivalence to other values of β and g by rescaling ϕ and using a normal-ordered boson identity that is the counterpart of a Fierz transformation. Let us review his rescaling procedure. Before comparing (6.2) with (5.1) in the previous discussion, we rescale the Lagrangian (6.2) with

$$2\sqrt{\pi}\phi \to \beta\phi, \tag{6.4}$$

and use the following identity to renormalize properly the kinetic term

$$A_B^2 :\sin(2\sqrt{\pi}\phi(x)):^2 = \frac{1}{2\pi}\partial^\mu\phi(x)\partial_\mu\phi(x) + \text{c number}. \tag{6.5}$$

(See its proof in Appendix C.) Notice that this identity also agrees with the Fierz transformations in (A.25) under a naive application of the bosonization rules. The c number is present in the quantum Fierz transformation because of the divergence due to taking the field products at the same space-time point.

Thus, the Lagrangian (6.2) is rewritten as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{i}{2}\bar{\psi}_3\partial\!\!\!/\psi_3 + \left(4g + \pi\left(1 - \frac{4\pi}{\beta^2}\right)\right)A_B^2 :\sin(\beta\phi):^2 + 4gA_B :\cos(\beta\phi): \bar{\psi}_3\psi_3. \tag{6.6}$$

Comparing with (5.1), we now obtain that

$$V(\phi) = A_0 \sin(\beta\phi) \tag{6.7}$$

where

$$A_R = -\frac{8g}{\beta}A_B, \tag{6.8}$$

and the identification of couplings is given by

$$\frac{\beta^2}{4\pi} = \frac{1 - 8(g/\pi)^2}{1 + 4g/\pi}. \tag{6.9}$$

(This last equation is not written down in [56]. We use β in place of Witten's γ.) The map from positive beta to g is one-to-two, with the two branches taking values in the intervals $(-\infty, -\pi/\sqrt{8}]$ and $(-\pi/4, \pi/\sqrt{8}]$. Based on this information, it is suspicious how the equivalence is obtained, since the bosonization rules, as stated in (2.31), relate $g = 0$ with $\beta = 2\sqrt{\pi}$, but their direct application implies $g = -\pi/2$ with $\beta = 2\sqrt{\pi}$, and the rescaling says that these two values of g belong to disjoint intervals. Therefore, the bosonization rules are being applied in an interval that is disconnected from their original domain of validity. It is interesting to notice that the identification of couplings (6.9) differs from Coleman's (2.30) by the g^2 term in the numerator, however there is not any physical significance since the identifications belong to different model equivalences.

Later references discuss the equivalence from other points of view. Aratyn and Damgaard investigated the invariance of the $O(3)$ GNM Lagrangian under the purely fermionic form of the supersymmetry transformations (fermionization) [2]. They claimed that the invariance holds by requiring a highly non-trivial interpretation of the fermionic counterpart of a trivial mathematical identity for the boson field. Also the supersymmetry algebra, as realized on the fields, is not manifest in their approach. Nakawaki constructed a fermionic representation of the superalgebra on the fields, but he could not identify the resulting fermionic model with the $O(3)$ GNM [32]. Schaposnik and Trobo analyzed the supersymmetry for the $O(3)$ GNM by employing the path-integral approach to bosonization and claimed its equivalence to the SSGM, but they still required Witten's rescaling procedure in order to compare the models in general [40].

Now let us study this equivalence from the central-charge point of view. The third equation in (4.36) tells us that the central charge for the $O(3)$ GNM is always zero, while the equation (5.20) says that for the SSGM it is only zero at $\beta = 2\sqrt{\pi}$. Based on this information and the fact that the central charge is protected by supersymmetry, we conclude the equivalence can only be valid between the $O(3)$ GNM at $g = -\pi/2$ and the SSGM at $\beta = 2\sqrt{\pi}$, at least under a naive use of bosonization rules. (Of course, we omit the trivial equivalence between the free cases in both models.) In the next section, we will discuss more this proposed equivalence by looking at the supercurrents.

6.2 Supercurrents

In this section, we would like to find another independent argument for the validity of the proposed equivalence. Let us compare the supercurrents of the models by bosonization. First, we rewrite our original current J^μ and its anomaly R^μ for the $O(3)$ GNM in terms of two-component spinors (see section 4.2)

$$\begin{aligned} J^\mu &= \frac{1}{2i}\Big((\bar{\psi}\gamma^\mu\gamma^5\psi) + (\bar{\psi}\gamma^\mu\psi)\gamma^5\Big)\psi_3, \\ R^\mu &= \frac{1}{2i}\Big(- (\bar{\psi}\gamma^\mu\gamma^5\psi) + (\bar{\psi}_1\gamma^\mu\psi)\gamma^5 + 2(\bar{\psi}\gamma^5\psi)\gamma^\mu\Big)\psi_3, \end{aligned} \tag{6.10}$$

where ψ is a Dirac spinor that includes two of the original three Majorana spinors and ψ_3 is the other Majorana spinor. Therefore, the modified current of J^μ (the GNM supercurrent $\mathcal{J}^\mu$) is written as

$$\mathcal{J}^\mu = \frac{E(g)}{2i}\Big((1-f(g))(\bar{\psi}\gamma^\mu\gamma^5\psi) + (1+f(g))(\bar{\psi}\gamma^\mu\psi)\gamma^5 + 2f(g)(\bar{\psi}\gamma^5\psi)\gamma^\mu\Big)\psi_3. \quad (6.11)$$

For simplicity, in this result we have assumed $G_{\mu\nu} = \eta_{\mu\nu}$ and $H_{\mu\nu} = f(g)\eta_{\mu\nu}$ in equation (4.33). This assumption avoids our point-splitting method, but it implies some formal definition of the product of two fields at the same space-time point. In reference [56], the GNM supercurrent is discussed in this form.

Applying the bosonization rules (2.31) to this current, we obtain

$$\begin{aligned}\mathcal{J}^\mu \quad = \quad & \frac{iE(g)}{2\sqrt{\pi}}(1-f(g))\Big(\partial^\mu\phi + \big(\frac{1+f(g)}{1-f(g)}\big)\epsilon^{\mu\nu}\partial_\nu\phi\gamma^5 \\ & -2i\sqrt{\pi}\big(\frac{f(g)}{1-f(g)}\big)A_B :\sin(2\sqrt{\pi}\phi): \gamma^\mu\Big)\psi_3. \quad (6.12)\end{aligned}$$

Comparing this result to the equation (5.4) with $V(\phi) = A_R :\sin(2\sqrt{\pi}\phi):$, we find that $f(g)$ should be zero at $g = -\pi/2$ if the equivalence holds. Also, we need that $A_R = 0$ for SSGM at $\beta = 2\sqrt{\pi}$, which implies that the equivalence is with respect to the free supersymmetric case (not to the SSGM). Assuming this assertion is correct, we would conclude that the $O(3)$ GNM for $g = -\pi/2$ would be equivalent to the model of free Majorana spinors (because of the trivial equivalence between the free models). Also notice if the rescaling argument is correct to use (see previous section), we can compare for all values of g and β and conclude that they are in general not equivalent, since the coefficients in the respective derivatives of ϕ cannot be matched, except for $f(g) = 0$. In respect to this problem, Witten claimed that the SSGM supercurrent (5.4) is modified by an anomaly to include a term $(\partial_\mu\phi)\psi$, therefore the identification of the supercurrents would follow by bosonization. (He did not give any reasons for that claim.) However, the presence of such a term would modify the supersymmetric algebra, for example the central charge would change which contradicts that this charge is protected by supersymmetry.

As has been seen, the application of the bosonization rules to the GNM supercurrent does not lead to the SSGM supercurrent (or vice versa). Therefore the fermionization of the SSGM supercurrent (5.4) cannot be used to generate supersymmetry transformations in the $O(3)$ GNM, as was done in references [2], [32], [40].

Finally, based on our discussion about the comparisons of the central charges and the supercurrents, we conclude that the $O(3)$ GNM and the SSGM are generically inequivalent. We also conclude that the naive application of the bosonization rules to determine a partially bosonized theory does not in this case lead to an equivalent theory.

Appendix A

Notation

A.1 Two-dimensional Minkowski metric

The metric and antisymmetric Levi-Civita tensor are

$$\begin{aligned} \eta^{00} = -\eta^{11} &= 1, \\ \epsilon_{01} = -\epsilon^{01} &= 1. \end{aligned} \tag{A.1}$$

They satisfy the relations

$$\begin{aligned} \eta^{\mu\nu}\eta_{\nu\mu} &= \eta^{\mu}_{\mu} = 2, \\ \epsilon^{\mu\nu}\epsilon_{\nu\rho} &= \epsilon_{\rho\nu}\epsilon^{\nu\mu} = \eta^{\mu}_{\rho}, \\ \epsilon^{\mu\nu}\epsilon^{\rho\sigma} &= \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}. \end{aligned} \tag{A.2}$$

In light-cone coordinates, we define

$$x^{+} \equiv x^{0} + x^{1}, \qquad x^{-} \equiv x^{0} - x^{1}, \tag{A.3}$$

whose metric and antisymmetric tensor are

$$\begin{aligned} \eta_{++} = \eta_{--} = 0, &\qquad \eta_{+-} = \eta_{-+} = 1/2, \\ \eta^{++} = \eta^{--} = 0, &\qquad \eta^{+-} = \eta^{-+} = 2, \\ \epsilon_{-+} = -\epsilon_{+-} = 1/2, &\qquad \epsilon^{+-} = -\epsilon^{-+} = 2. \end{aligned} \tag{A.4}$$

Moreover, we have

$$\partial_{+} = \frac{\partial}{\partial x^{+}} \equiv \frac{1}{2}(\partial_0 + \partial_1), \qquad \partial_{-} = \frac{\partial}{\partial x^{-}} \equiv \frac{1}{2}(\partial_0 - \partial_1). \tag{A.5}$$

The light-cone components of a second rank tensor $G_{\mu\nu}$ are given in terms of the usual components by

$$\begin{aligned} G^{++} &= G^{00} + G^{10} + G^{01} + G^{11}, \\ G^{+-} &= G^{00} + G^{10} - G^{01} - G^{11}, \\ G^{-+} &= G^{00} - G^{10} + G^{01} - G^{11}, \\ G^{--} &= G^{00} - G^{10} - G^{01} + G^{11}. \end{aligned} \tag{A.6}$$

A.2 Two-dimensional Dirac algebra

The γ matrices obey the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv 2\eta^{\mu\nu} I, \tag{A.7}$$

where I is the 2×2 unit matrix.

Their hermitian conjugates are defined by

$$(\gamma^\mu)^\dagger \equiv \gamma^0 \gamma^\mu \gamma^0 = \gamma_\mu, \tag{A.8}$$

therefore, γ^0 is hermitian and γ^1 is antihermitian in our conventions.

We define

$$\gamma^5 = \gamma_5 \equiv \frac{1}{2}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu. \tag{A.9}$$

Some useful properties of the γ matrices are

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= 2I, \\
\{\gamma^5, \gamma^\mu\} &= 0, \\
(\gamma^5)^2 &= I, \\
(\gamma^5)^\dagger &= \gamma^5, \\
\gamma^\mu \gamma^5 &= \epsilon^{\mu\nu}\gamma_\nu, \\
\gamma^\mu \gamma^\nu &= \eta^{\mu\nu} I - \epsilon^{\mu\nu}\gamma^5, \\
[\gamma^\mu, \gamma^\nu] &= -2\epsilon^{\mu\nu}\gamma^5.
\end{aligned} \tag{A.10}$$

The charge conjugation is defined by

$$C^{-1}\gamma^\mu C \equiv -(\gamma^\mu)^T, \tag{A.11}$$

with properties

$$C^T = -C, \qquad C^{-1} = C^\dagger = -C^*. \tag{A.12}$$

Moreover, we have the contraction identities

$$\begin{aligned}
\gamma^\mu \gamma^\nu \gamma_\mu &= 0, \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 2\gamma^\rho \gamma^\nu, \\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= 2\gamma^\nu \gamma^\rho \gamma^\sigma - 2\gamma^\sigma \gamma^\rho \gamma^\nu,
\end{aligned} \tag{A.13}$$

and the traces

$$\begin{aligned}
\mathrm{tr}(\text{odd } \gamma^\mu) &= 0, \\
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 2\eta^{\mu\nu}, \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 2\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu\rho}\eta^{\nu\sigma} + 2\eta^{\mu\sigma}\eta^{\nu\rho}, \\
\mathrm{tr}(\gamma^5 \text{odd } \gamma^\mu) &= \mathrm{tr}(\gamma^5) = 0, \\
\mathrm{tr}((\gamma^5)^2) &= 2, \\
\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) &= -2\epsilon^{\mu\nu}, \\
\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -2\eta^{\mu\nu}\epsilon^{\rho\sigma} - 2\eta^{\rho\sigma}\epsilon^{\mu\nu}.
\end{aligned} \tag{A.14}$$

In the Weyl representation, the γ matrices are written as

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.15}$$

In the Majorana representation,

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma^5 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.16}$$

And in the Majorana-Weyl representation,

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \gamma^5 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.17}$$

where σ_i are the Pauli matrices.

In light-cone coordinates, the γ matrices are given by

$$\gamma^+ \equiv \gamma^0 + \gamma^1, \qquad \gamma^- \equiv \gamma^0 - \gamma^1, \tag{A.18}$$

A.3 Spinors in two dimensions

We use

$$\begin{aligned}
\psi(x) &= \psi(x^0, x^1), \\
\bar{\psi} &= \psi^\dagger\gamma^0, \\
\psi^C &= C\bar{\psi}^T,
\end{aligned} \tag{A.19}$$

where we choose $C = -\gamma^0$.

In the Majorana-Weyl representation, a Majorana spinor is written in components as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{A.20}$$

with $\psi_L^\dagger = \psi_L$ and $\psi_R^\dagger = \psi_R$.

Whenever we consider expressions involving more than one spinor, we must remember that the spinor components anticommute. Using this fact we can show the following Majorana bilinear identities

$$\begin{aligned} \bar{\psi}\chi &= \bar{\chi}\psi, \\ \bar{\psi}\gamma^\mu\chi &= -\bar{\chi}\gamma^\mu\psi, \\ \bar{\psi}\gamma^5\chi &= -\bar{\chi}\gamma^5\psi, \\ \bar{\psi}\gamma^\mu\gamma^\nu\chi &= \bar{\chi}\gamma^\nu\gamma^\mu\psi, \\ \bar{\psi}\gamma^\mu\gamma^5\chi &= -\bar{\chi}\gamma^\mu\gamma^5\psi. \end{aligned} \tag{A.21}$$

Using the completeness of the matrix set I, γ^μ and γ^5, we can find the classical Fierz transformations. For Majorana spinors, some of these transformations are

$$\begin{aligned} (\bar{\psi}\gamma^\mu\chi)(\bar{\psi}\gamma_\mu\chi) &= (\bar{\psi}\psi)(\bar{\chi}\chi) \\ (\bar{\psi}\gamma^5\chi)^2 &= \frac{1}{2}(\bar{\psi}\psi)(\bar{\chi}\chi) \\ (\bar{\psi}\chi)^2 &= -\frac{1}{2}(\bar{\psi}\psi)(\bar{\chi}\chi) \end{aligned} \tag{A.22}$$

Finally, a Dirac spinor is expressed in terms of two Majorana spinors as

$$\psi_D = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2). \tag{A.23}$$

Consequently, the Dirac bilinears can be written as

$$\begin{aligned} \bar{\psi}_D\psi_D &= \frac{1}{2}(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2), \\ \bar{\psi}_D\gamma^\mu\psi_D &= i\bar{\psi}_1\gamma^\mu\psi_2, \\ \bar{\psi}_D\gamma^5\psi_D &= i\bar{\psi}_1\gamma^5\psi_2, \\ \bar{\psi}_D\gamma^\mu\gamma^\nu\psi_D &= \frac{\eta^{\mu\nu}}{2}(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) - i\epsilon^{\mu\nu}\bar{\psi}_1\gamma^5\psi_2, \\ \bar{\psi}_D\gamma^\mu\gamma^5\psi_D &= i\bar{\psi}_1\gamma^\mu\gamma^5\psi_2. \end{aligned} \tag{A.24}$$

And the first two Fierz transformations in (A.22) become

$$
\begin{aligned}
(\bar{\psi}_D \gamma^\mu \psi_D)(\bar{\psi}_D \gamma_\mu \psi_D) &= -2(\bar{\psi}_D \psi_D)^2, \\
(\bar{\psi}_D \gamma^5 \psi_D)^2 &= -(\bar{\psi}_D \psi_D)^2.
\end{aligned} \tag{A.25}
$$

Appendix B

Calculations for GNM

B.1 Fermi fields

The contraction of free massless Dirac fields is given by the Wightman function (see [28], [1])

$$\begin{aligned} S^{(+)}(\xi = y - x) &\equiv \overbrace{\psi(y)\bar{\psi}}(x) \\ &= \frac{i}{(2\pi)^2}\int d^2k\, \frac{k^\mu \gamma_\mu}{k^2}\, e^{-ik\cdot\xi} \\ &= \frac{1}{2\pi}\int \frac{dk^1}{2k^0}\, k^\mu \gamma_\mu\, e^{-ik\cdot\xi - k^0\epsilon}, \end{aligned} \tag{B.1}$$

where $\epsilon > 0$ is a UV regularization.

We define

$$\begin{aligned} C^\mu(\xi) &\equiv \frac{1}{2}\mathrm{tr}(\gamma^\mu S^{(+)}(\xi)) \\ &= \frac{1}{4\pi}\int dk^1\, \mathrm{sgn}(k^\mu) e^{-ik\cdot\xi - k^0\epsilon}, \end{aligned} \tag{B.2}$$

so that

$$S^{(+)}(\xi) = C^\mu(\xi)\gamma_\mu. \tag{B.3}$$

In the Majorana-Weyl representation, we have

$$\begin{aligned} C^+(\xi = y - x) &= \overbrace{\psi_R(y)\psi_R}(x) = -\frac{i}{2\pi}\Big(\frac{1}{\xi^- - i\epsilon}\Big), \\ C^-(\xi = y - x) &= \overbrace{\psi_L(y)\psi_L}(x) = -\frac{i}{2\pi}\Big(\frac{1}{\xi^+ - i\epsilon}\Big), \\ \overbrace{\psi_R(y)\psi_L}(x) &= \overbrace{\psi_L(y)\psi_R}(x) = 0, \end{aligned} \tag{B.4}$$

where ψ_R and ψ_L are the chiral components of a Majorana spinor.

They have the following properties

$$\begin{aligned} C^\pm(\xi) + C^\pm(-\xi) &= \delta(\xi^\mp), \\ (C^\pm(\xi))^2 &= \frac{i}{2\pi}(\partial_\mp C^\pm(\xi)), \\ (C^\pm(-\xi))^2 &= -\frac{i}{2\pi}(\partial_\mp C^\pm(-\xi)), \\ (C^\pm(\xi))^2 - (C^\pm(-\xi))^2 &= \frac{i}{2\pi}\delta'(\xi^\mp). \end{aligned} \tag{B.5}$$

For equal-time situations (from here onward x denotes just the space coordinate), we have

$$\begin{aligned}
C^{\pm}(-x) &= C^{\mp}(x), \\
C^{\pm}(x)+C^{\pm}(-x) &= \delta(x), \\
(C^{\pm}(x))^2 &= \mp\frac{i}{2\pi}(\partial_x C^{\pm}(x)), \\
(C^{\pm}(x))^2-(C^{\pm}(-x))^2 &= \mp\frac{i}{2\pi}\delta'(x), \\
C^{-}(x)C^{+}(x)-C^{-}(-x)C^{+}(-x) &= 0, \\
(C^{\pm}(x))^3 &= -\frac{1}{8\pi^2}(\partial_x^2 C^{\pm}(x)), \\
(C^{\pm}(x))^3+(C^{\pm}(-x))^3 &= -\frac{1}{8\pi^2}\delta''(x), \\
(C^{\pm}(x))^2C^{\mp}(x)+(C^{\pm}(-x))^2C^{\mp}(-x) &= -\frac{1}{8\pi^2}\frac{\delta'(x)}{x}.
\end{aligned} \tag{B.6}$$

Using these functions C^+ and C^-, we can "derive" the equal-time canonical anticommutation relations for the Majorana-Weyl spinor components

$$\begin{aligned}
\{\psi_R(x),\psi_R(y)\} &= C^+(x-y)+C^+(y-x)=\delta(x-y), \\
\{\psi_L(x),\psi_L(y)\} &= C^-(x-y)+C^-(y-x)=\delta(x-y), \\
\{\psi_R(x),\psi_L(y)\} &= 0,
\end{aligned} \tag{B.7}$$

where we have used Wick's theorem in an intermediate step. Now these results imply the canonical anticommutators for Dirac spinors

$$\begin{aligned}
\{\psi_a(x),\psi_b^\dagger(y)\} &= \frac{1}{2}\Big(\{\psi_{1a}(x),\psi_{1b}(y)\}+\{\psi_{2a}(x),\psi_{2b}(y)\}\Big)=\delta_{ab}\delta(x-y), \\
\{\psi_a(x),\psi_b(y)\} &= \{\psi_a^\dagger(x),\psi_b^\dagger(y)\}=0,
\end{aligned} \tag{B.8}$$

where $a=b=L,R$.

Similarly, we can compute other equal-time commutators of these components and then realize these relations for Dirac spinors. For example, we can find the equal-time current commutators through (again, Wick's theorem is used)

$$\begin{aligned}
[\psi_{1R}(x)\psi_{2R}(x),\psi_{1R}(y)\psi_{2R}(y)] &= -(C^+(x-y))^2+(C^+(y-x))^2 \\
&= \frac{i}{2\pi}\delta'(x-y), \\
[\psi_{1L}(x)\psi_{2L}(x),\psi_{1L}(y)\psi_{2L}(y)] &= -\frac{i}{2\pi}\delta'(x-y), \\
[\psi_{1R}(x)\psi_{2R}(x),\psi_{1L}(y)\psi_{2L}(y)] &= 0,
\end{aligned} \tag{B.9}$$

and since

$$j^0 = i(\psi_{1R}\psi_{2R} + \psi_{1L}\psi_{2L}), \qquad j^1 = i(\psi_{1R}\psi_{2R} - \psi_{1L}\psi_{2L}), \tag{B.10}$$

we obtain that the nonzero commutator is

$$[j^0(x), j^1(y)] = -\frac{i}{\pi}\delta'(x-y), \tag{B.11}$$

where the right-hand side is a Schwinger term [41].

For the commutator between the current and the Dirac field, we use $(i \neq j)$

$$\begin{aligned}
[\psi_{iR}(x)\psi_{jR}(x), \psi_{iR}(y)] &= -C^+(x-y)\psi_{jR}(x) - C^+(y-x)\psi_{jR}(x) \\
&= -\delta(x-y)\psi_{jR}(x), \\
[\psi_{iL}(x)\psi_{jL}(x), \psi_{iL}(y)] &= -\delta(x-y)\psi_{jL}(x), \\
[\psi_{iR}(x)\psi_{jR}(x), \psi_{iL}(y)] &= [\psi_{iL}(x)\psi_{jL}(x), \psi_{iR}(y)] = 0,
\end{aligned} \tag{B.12}$$

to obtain

$$[j^\mu(x), \psi(y)] = -(\eta^{\mu 0} + \epsilon^{\mu 0}\gamma^5)\delta(x-y)\psi(x). \tag{B.13}$$

Now, we want to determine the anticommutators for products of Majorana components at the same point. To accomplish that, we employ a point-splitting method for space separations (see next section). For example, we define

$$\begin{aligned}
\{\psi_{iR}(x)\psi_{jR}(x), \psi_{iR}(x)\} &\equiv \lim_{\epsilon\to 0} \psi_{iR}(x+\epsilon)\psi_{jR}(x+\epsilon)\psi_{iR}(x) \\
&\qquad + \psi_{iR}(x)\psi_{iR}(x-\epsilon)\psi_{jR}(x-\epsilon) \\
&= \lim_{\epsilon\to 0} -C^+(\epsilon)\psi_{jR}(x+\epsilon) + C^+(\epsilon)\psi_{jR}(x-\epsilon) \\
&= -\frac{i}{\pi}\partial_1\psi_{jR}(x),
\end{aligned} \tag{B.14}$$

where we have used Wick's theorem in the second step and a Taylor expansion to get the final result. For the left-handed Majorana components, we work out a similar calculation to obtain

$$\{\psi_{iL}(x)\psi_{jL}(x), \psi_{iL}(x)\} = \frac{i}{\pi}\partial_1\psi_{jL}(x). \tag{B.15}$$

These equations can be summarized in the following anticommutator between currents and the Dirac field,

$$\{j^1 + \gamma^5 j^0, \psi\} = -\frac{2i}{\pi}\partial_1\psi. \tag{B.16}$$

For the free case, we can combine the Dirac equation and the above result to have a covariant expression

$$\{j^\mu + \gamma^5 \epsilon^{\mu\nu} j_\nu, \psi\} = \frac{2i}{\pi} \partial^\mu \psi. \tag{B.17}$$

So that $j^\mu + \gamma^5 \epsilon^{\mu\nu} j_\nu$ generates space-time translations. This equation (B.17) had been found before by inspection of the matrix elements of the free theory (see [47]). It is easy to check that the equation (B.17) can also be derived from $[P_\mu, \psi(x)] = -i\partial_\mu \psi(x)$, where $P^\mu = \int dx\, T^{0\mu}(x)$ and $T^{\mu\nu}$ is given by (B.22). Notice that the calculation involved in the equations (B.14) and (B.15) is the same as that in finding the quantum field equations (3.12) and (4.11) of the $O(2)$ and $O(3)$ GNM, respectively.

To find the anticommutator between the products of two right-handed Majorana components, we again use a point-splitting method

$$\begin{aligned}
&\{\psi_{1R}(x)\psi_{2R}(x), \psi_{1R}(x)\psi_{2R}(x)\} \equiv \\
&\quad = \lim_{\epsilon\to 0} \psi_{1R}(x+\epsilon)\psi_{2R}(x+\epsilon)\psi_{1R}(x)\psi_{2R}(x) + \psi_{1R}(x)\psi_{2R}(x)\psi_{1R}(x-\epsilon)\psi_{2R}(x-\epsilon) \\
&\quad = \lim_{\epsilon\to 0} -C^+(\epsilon) :\psi_{1R}(x+\epsilon)\psi_{1R}(x) + \psi_{2R}(x+\epsilon)\psi_{2R}(x): -(C^+(\epsilon))^2 \\
&\qquad\qquad -C^+(\epsilon) :\psi_{1R}(x)\psi_{1R}(x-\epsilon) + \psi_{2R}(x)\psi_{2R}(x-\epsilon): -(C^+(\epsilon))^2 \\
&\quad = \frac{i}{\pi}\Big(\psi_{1R}(x)\partial_1\psi_{1R}(x) + \psi_{2R}(x)\partial_1\psi_{2R}(x)\Big) + \text{c number}.
\end{aligned} \tag{B.18}$$

Similarly, we have for the left-handed Majorana components

$$\{\psi_{1L}(x)\psi_{2L}(x), \psi_{1L}(x)\psi_{2L}(x)\} = -\frac{i}{\pi}\Big(\psi_{1L}(x)\partial_1\psi_{1L}(x) + \psi_{2L}(x)\partial_1\psi_{2L}(x)\Big) + \text{c number}. \tag{B.19}$$

With these results, we can write the anticommutators between the currents at the same point as

$$\begin{aligned}
\{j^0, j^1\} &= -\frac{i}{\pi}\Big(\psi_{1L}\partial_1\psi_{1L} + \psi_{2L}\partial_1\psi_{2L} + \psi_{1R}\partial_1\psi_{1R} + \psi_{2R}\partial_1\psi_{2R}\Big), \\
\{j^0, j^0\} &= \frac{i}{\pi}\Big(\psi_{1L}\partial_1\psi_{1L} + \psi_{2L}\partial_1\psi_{2L} - \psi_{1R}\partial_1\psi_{1R} - \psi_{2R}\partial_1\psi_{2R}\Big) \\
&\quad -4\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R} + \text{c number}, \\
\{j^1, j^1\} &= \frac{i}{\pi}\Big(\psi_{1L}\partial_1\psi_{1L} + \psi_{2L}\partial_1\psi_{2L} - \psi_{1R}\partial_1\psi_{1R} - \psi_{2R}\partial_1\psi_{2R}\Big) \\
&\quad +4\psi_{1L}\psi_{2L}\psi_{1R}\psi_{2R} + \text{c number}.
\end{aligned} \tag{B.20}$$

Combining with the Dirac equation, we can rewrite the above results in a covariant form

$$\{j^\mu, j^\nu\} = \frac{i}{2\pi}\Big(\bar{\psi}\gamma^\mu(\partial^\nu\psi) + \bar{\psi}\gamma^\nu(\partial^\mu\psi) - (\partial^\nu\bar{\psi})\gamma^\mu\psi - (\partial^\mu\bar{\psi})\gamma^\nu\psi\Big) - 2\eta^{\mu\nu}(\bar{\psi}\psi)^2 \tag{B.21}$$

Now, it is easy to show that the stress-energy tensor for the free case can be expressed in terms of currents as (see [49], [47], [7])

$$T^{\mu\nu} = \frac{\pi}{2}\Big(\{j^\mu, j^\nu\} - \eta^{\mu\nu} j_\rho j^\rho\Big) \tag{B.22}$$

We have also used the equations (B.18) and (B.19) to obtain the Lagrangian (3.11) of the $O(2)$ GNM.

For the $O(3)$ GNM, we have that in the Majorana-Weyl representation, ${j^\mu}_{ij}$ can be expressed through the chiral components as

$$\begin{aligned} {j^\mu}_{ii} &= 0, \\ {j^0}_{ij} &= \psi_{iR}\psi_{jR} + \psi_{iL}\psi_{jL}, \\ {j^1}_{ij} &= \psi_{iR}\psi_{jR} - \psi_{iL}\psi_{jL}, \end{aligned} \tag{B.23}$$

where $i \neq j$. And in light-cone coordinates, we get

$${j^+}_{ij} = {j^0}_{ij} + {j^1}_{ij} = 2\psi_{iR}\psi_{jR}, \qquad {j^-}_{ij} = {j^0}_{ij} - {j^1}_{ij} = 2\psi_{iL}\psi_{jL}. \tag{B.24}$$

Now, the equations (B.18) and (B.19) imply that (no sum over any index and $i \neq j$)

$$\begin{aligned} j^0_{ij} j^1_{ij} &= \frac{i}{2\pi}\Big(\psi_{iL}\partial_1\psi_{iL} + \psi_{jL}\partial_1\psi_{jL} + \psi_{iR}\partial_1\psi_{iR} + \psi_{jR}\partial_1\psi_{jR}\Big), \\ j^0_{ij} j^0_{ij} &= -\frac{i}{2\pi}\Big(\psi_{iL}\partial_1\psi_{iL} + \psi_{jL}\partial_1\psi_{jL} - \psi_{iR}\partial_1\psi_{iR} - \psi_{jR}\partial_1\psi_{jR}\Big) \\ &\quad + 2\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR} + \text{c number}, \\ j^1_{ij} j^1_{ij} &= -\frac{i}{2\pi}\Big(\psi_{iL}\partial_1\psi_{iL} + \psi_{jL}\partial_1\psi_{jL} - \psi_{iR}\partial_1\psi_{iR} - \psi_{jR}\partial_1\psi_{jR}\Big) \\ &\quad - 2\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR} + \text{c number}. \end{aligned} \tag{B.25}$$

And the Lagrangian of the model (4.10) follows by plugging these composite operators into (4.8). (Notice that there is a sign difference with the above anticommutators because the vector current is defined with an i.)

B.2 Current calculations

In this section, we will give a fuller derivation of the conservation laws for the currents in the $O(2)$ and $O(3)$ GNM. Especially, we want to define our point-splitting method when we encounter singularities in the spinor products due to the substitution of the field equations.

First, we define the product of two local operators O_1 and O_2 in the limit when they approach one another as

$$O_1(x)O_2(x) \equiv \lim_{\epsilon\to 0}\frac{1}{4}\Big(O_1(x+\epsilon)O_2(x) \pm O_2(x)O_1(x-\epsilon) \\ +O_1(x)O_2(x+\epsilon) \pm O_2(x-\epsilon)O_1(x)\Big), \qquad \text{(B.26)}$$

where the sign is chosen according to whether the operators commute or anticommute at different points. Moreover, we consider the separation between the operators ϵ is only in space, so that this definition requires four terms because we fix one of these operators at some point when the other approaches from the right in some particular operator order and from the left in the opposite order. This change of the operator order allows us to cancel out algebraically the singularities. (Notice the final result is zero when the product is just a c number, for example, $\psi_R^2(x)$.)

When the operators are the same and commute, the above definition needs only two terms, since the second line on the right-hand side gives the same result as the first one. For example, the equation (B.18) and the composite operators in the Lagrangians were computed in this way. Also, our definition of the vector current in (3.7) is done in this way by taking advantage of a Majorana-spinor property (A.21).

In equations (B.14) - (B.16), we do not apply this definition because we consider that the vector current is acting on the field, so that the field is fixed at some point while the current approaches. Notice that this way of taking the point-splitting method is suggested by our quantum field equations (3.8) and (4.9).

For the $O(2)$ GNM, we have by the field equations (3.12) that

$$(1+\alpha G_{++})\partial_+(\psi_{1R}\psi_{2R}) = 2\pi i\alpha G_{+-}((\psi_{2L}\psi_{2R}\psi_{1L})\psi_{2R} + \psi_{1R}(\psi_{1L}\psi_{1R}\psi_{2L})) \\ +\alpha G_{++}\partial_-(\psi_{1R}\psi_{2R}). \qquad \text{(B.27)}$$

We evaluate the four-spinor products in general ($i \neq j$) by using the above point-splitting method with Wick's theorem and Taylor expanding as follows (no sum over i)

$$(\psi_{iL}\psi_{iR}\psi_{jL})\psi_{iR} =$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0} \frac{1}{4}\Big(\psi_{iL}(x+\epsilon)\psi_{iR}(x+\epsilon)\psi_{jL}(x+\epsilon)\psi_{iR}(x) \\
&\qquad -\psi_{iR}(x)\psi_{iL}(x-\epsilon)\psi_{iR}(x-\epsilon)\psi_{jL}(x-\epsilon) \\
&\qquad +\psi_{iL}(x)\psi_{iR}(x)\psi_{jL}(x)\psi_{iR}(x+\epsilon) - \psi_{iR}(x-\epsilon)\psi_{iL}(x)\psi_{iR}(x)\psi_{jL}(x)\Big) \\
&= \lim_{\epsilon\to 0} \frac{1}{4}\Big(- C^{+}(\epsilon)\psi_{iL}(x+\epsilon)\psi_{jL}(x+\epsilon) + C^{+}(\epsilon)\psi_{iL}(x-\epsilon)\psi_{jL}(x-\epsilon) \qquad \text{(B.28)} \\
&\qquad -C^{+}(-\epsilon)\psi_{iL}(x)\psi_{jL}(x) + C^{+}(-\epsilon)\psi_{iL}(x)\psi_{jL}(x)\Big) \\
&= \frac{i}{4\pi}\partial_1(\psi_{jL}\psi_{iL}).
\end{aligned}
$$

(Notice that we consider these four spinors as two operators because three of them come together from the field equations.) We make a similar calculation for

$$
\psi_{iR}(\psi_{iL}\psi_{iR}\psi_{jL}) = \frac{i}{4\pi}\partial_1(\psi_{iL}\psi_{jL}). \qquad \text{(B.29)}
$$

Therefore, we find that

$$
(1+\alpha G_{++})\partial_+(\psi_{1R}\psi_{2R}) = \alpha G_{+-}(\partial_-(\psi_{1L}\psi_{2L}) - \partial_+(\psi_{1L}\psi_{2L})) + \alpha G_{++}\partial_-(\psi_{1R}\psi_{2R}). \quad \text{(B.30)}
$$

For $\partial_-(\psi_{1L}\psi_{2L})$, an identical calculation can be done where the expressions are the above with the interchange of $(R \leftrightarrow L)$ and $(+ \leftrightarrow -)$.

For the $O(3)$ GNM, we will work out $(\partial_+\psi_{1R})\psi_{2R}\psi_{3R}$ explicitly. The results for $\psi_{1R}(\partial_+\psi_{2R})\psi_{3R}$ and $\psi_{1R}\psi_{2R}(\partial_+\psi_{3R})$ follow with a proper interchange of the indices. Then the field equation (4.11) implies

$$
\begin{aligned}
(1+2\alpha G_{++})(\partial_+\psi_{1R})\psi_{2R}\psi_{3R} \;&=\; 2\pi i\alpha G_{+-}((\psi_{2L}\psi_{2R}\psi_{1L})\psi_{2R}\psi_{3R} \qquad \text{(B.31)} \\
&\quad +(\psi_{3L}\psi_{3R}\psi_{1L})\psi_{2R}\psi_{3R}) + 2\alpha G_{++}(\partial_-\psi_{1R})\psi_{2R}\psi_{3R}.
\end{aligned}
$$

Again we solve the five-spinor products by using the point-splitting method (B.26), but now we take the average of all the combinations of using a two-point splitting for three operators, i.e.

$$
O_1O_2O_3 \equiv \frac{1}{3}\Big([O_1O_2]O_3 + O_1[O_2O_3] \pm [O_1O_3]O_2\Big). \qquad \text{(B.32)}
$$

where the brackets indicate which pair of operators is computed first. Here each term is solved by Wick's theorem and Taylor expansion as before. This definition guarantees associativity of the operator products.

For $(\psi_{2L}\psi_{2R}\psi_{1L})\psi_{2R}\psi_{3R}$, the respective terms on the right-hand side of (B.32) are given by

$$\begin{aligned} (\psi_{2L}\psi_{2R}\psi_{1L})[\psi_{2R}\psi_{3R}] &= \frac{i}{4\pi}\Big(\partial_1(\psi_{1L}\psi_{2L})\psi_{3R} - \psi_{1L}\psi_{2L}(\partial_1\psi_{3R})\Big), \\ [(\psi_{2L}\psi_{2R}\psi_{1L})\psi_{2R}]\psi_{3R} &= \frac{i}{4\pi}\partial_1(\psi_{1L}\psi_{2L})\psi_{3R}, \\ [(\psi_{2L}\psi_{2R}\psi_{1L})\psi_{3R}]\psi_{2R} &= \frac{i}{4\pi}\partial_1(\psi_{2L}\psi_{1L}\psi_{3R}). \end{aligned} \tag{B.33}$$

Therefore, we have that

$$(\psi_{2L}\psi_{2R}\psi_{1L})\psi_{2R}\psi_{3R} = \frac{i}{4\pi}\partial_1(\psi_{1L}\psi_{2L})\psi_{3R}. \tag{B.34}$$

The second five-spinor product of (B.31) is solved in the same fashion

$$(\psi_{3L}\psi_{3R}\psi_{1L})\psi_{2R}\psi_{3R} = \frac{i}{4\pi}\partial_1(\psi_{3L}\psi_{1L})\psi_{2R}. \tag{B.35}$$

And from these results, (B.31) is written as

$$\begin{aligned} (1+2\alpha G_{++})(\partial_+\psi_{1R})\psi_{2R}\psi_{3R} &= \frac{\alpha}{2}G_{+-}\Big(\partial_-(\psi_{1L}\psi_{2L})\psi_{3R} + \partial_-(\psi_{3L}\psi_{1L})\psi_{2R} \\ &\quad -\partial_+(\psi_{1L}\psi_{2L})\psi_{3R} - \partial_+(\psi_{3L}\psi_{1L})\psi_{2R}\Big) \\ &\quad +2\alpha G_{++}(\partial_-\psi_{1R})\psi_{2R}\psi_{3R}. \end{aligned} \tag{B.36}$$

Finally, we find that

$$\begin{aligned} (1+2\alpha G_{++})\partial_+(\psi_{1R}\psi_{2R}\psi_{3R}) &= \alpha G_{+-}(\partial_-(\psi_{1L}\psi_{2L})\psi_{3R} + \partial_-(\psi_{3L}\psi_{1L})\psi_{2R} \\ &\quad +\partial_-(\psi_{2L}\psi_{3L})\psi_{1R} - \partial_+(\psi_{1L}\psi_{2L})\psi_{3R} \\ &\quad -\partial_+(\psi_{3L}\psi_{1L})\psi_{2R} - \partial_+(\psi_{2L}\psi_{3L})\psi_{1R}) \\ &\quad +2\alpha G_{++}\partial_-(\psi_{1R}\psi_{2R}\psi_{3R}). \end{aligned} \tag{B.37}$$

For equations (4.23) and (4.24), we apply the same approach to compute the five-spinor products. First, we find that

$$\begin{aligned} (1+2\alpha G_{--})(\partial_-\psi_{1L})\psi_{2L}\psi_{3R} &= \frac{\alpha}{2}G_{-+}\Big(\partial_+(\psi_{1R}\psi_{2R})\psi_{3R} + \partial_+(\psi_{3L}\psi_{1R})\psi_{2L} \\ &\quad -\partial_-(\psi_{1R}\psi_{2R})\psi_{3R} - \partial_-(\psi_{3L}\psi_{1R})\psi_{2L}\Big) \\ &\quad +2\alpha G_{--}(\partial_+\psi_{1L})\psi_{2L}\psi_{3R}, \end{aligned} \tag{B.38}$$

and

$$
\begin{aligned}
(1+2\alpha G_{++})(\partial_+\psi_{1R})\psi_{2L}\psi_{3L} &= \frac{\alpha}{2}G_{+-}\Big(\partial_-(\psi_{1L}\psi_{2R})\psi_{3L}+\partial_-(\psi_{3R}\psi_{1L})\psi_{2L} \\
&\qquad -\partial_+(\psi_{1L}\psi_{2R})\psi_{3L}-\partial_+(\psi_{3R}\psi_{1L})\psi_{2L}\Big) \\
&\quad +2\alpha G_{++}(\partial_-\psi_{1R})\psi_{2L}\psi_{3L}. \qquad \text{(B.39)}
\end{aligned}
$$

Then the equations (4.23) and (4.24) follow with a proper interchange of indices in the above results, respectively.

For $\partial_-(\psi_{1L}\psi_{2L}\psi_{3L})$, the calculation procedure is completely identical and the final results are the above with the interchange of $(R \leftrightarrow L)$ and $(+ \leftrightarrow -)$.

B.3 Superalgebra calculations

In this section, we will show the equal-time anticommutators of three-spinor products that are needed for deriving the supersymmetry algebra of the $O(3)$ GNM. The procedure is the same as that for the above current commutators, i.e. we find the anticommutators by using the functions $C^{\pm}$ and Wick's theorem. (Normal ordering should be understood on the right-hand side of the following equations.) We assume that these functions $C^{\pm}$ are valid, since this model is asymptotically free [22].

Hence, we first obtain (no sum over any index except l and all indices are different)

$$
\begin{aligned}
\{\psi_{1R}(x)\psi_{2R}(x)\psi_{3R}(x),\psi_{1R}(y)\psi_{2R}(y)\psi_{3R}(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{lR}(x)\psi_{lR}(y) \\
&\quad +\frac{1}{8\pi^2}\delta''(x-y), \\
\{\psi_{iR}(x)\psi_{jR}(x)\psi_{kR}(x),\psi_{iR}(y)\psi_{jL}(y)\psi_{kL}(y)\} &= \delta(x-y)\psi_{jL}(y)\psi_{kL}(y)\psi_{jR}(x)\psi_{kR}(x), \\
\{\psi_{iR}(x)\psi_{jL}(x)\psi_{kL}(x),\psi_{iR}(y)\psi_{jL}(y)\psi_{kL}(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{iR}(x)\psi_{iR}(y) \qquad \text{(B.40)} \\
&\quad +\frac{1}{8\pi^2}\frac{\delta'(x-y)}{x-y}, \\
\{\psi_{iR}(x)\psi_{jL}(x)\psi_{kL}(x),\psi_{iL}(y)\psi_{jR}(y)\psi_{kL}(y)\} &= -\delta(x-y)\psi_{jL}(x)\psi_{iL}(y)\psi_{jR}(y)\psi_{iR}(x).
\end{aligned}
$$

With these relations, we find the anticommutators from which the calculation of $(Q^+)^2$ follows (here repeated indices are summed over and $i \neq j$)

$$
\begin{aligned}
\{j^+(x),j^+(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{iR}(x)\psi_{iR}(y)+\frac{1}{8\pi^2}\delta''(x-y), \\
\{j^+(x),r^-(y)\} &= \frac{1}{2}\delta(x-y)\psi_{iL}(y)\psi_{jL}(y)\psi_{iR}(x)\psi_{jR}(x), \qquad \text{(B.41)}
\end{aligned}
$$

$$\{r^-(x), r^-(y)\} = -\frac{i}{2\pi}\delta'(x-y)\psi_{iR}(x)\psi_{iR}(y) + \frac{3}{8\pi^2}\frac{\delta'(x-y)}{x-y} -\delta(x-y)\psi_{iL}(x)\psi_{jL}(y)\psi_{iR}(y)\psi_{jR}(x).$$

Similarly, we work out

$$\begin{aligned}
\{\psi_{1L}(x)\psi_{2L}(x)\psi_{3L}(x), \psi_{1L}(y)\psi_{2L}(y)\psi_{3L}(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{lL}(x)\psi_{lL}(y) \\
&\quad +\frac{1}{8\pi^2}\delta''(x-y), \\
\{\psi_{iL}(x)\psi_{jL}(x)\psi_{kL}(x), \psi_{iL}(y)\psi_{jR}(y)\psi_{kR}(y)\} &= \delta(x-y)\psi_{jL}(x)\psi_{kL}(x)\psi_{jR}(y)\psi_{kR}(y), \\
\{\psi_{iL}(x)\psi_{jR}(x)\psi_{kR}(x), \psi_{iL}(y)\psi_{jR}(y)\psi_{kR}(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{iL}(x)\psi_{iL}(y) \\
&\quad +\frac{1}{8\pi^2}\frac{\delta'(x-y)}{x-y}, \\
\{\psi_{iL}(x)\psi_{jR}(x)\psi_{kR}(x), \psi_{iR}(y)\psi_{jL}(y)\psi_{kR}(y)\} &= -\delta(x-y)\psi_{iL}(x)\psi_{jL}(y)\psi_{iR}(y)\psi_{jR}(x),
\end{aligned} \tag{B.42}$$

to get the anticommutators needed for $(Q^-)^2$ calculation

$$\begin{aligned}
\{j^-(x), j^-(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{iL}(x)\psi_{iL}(y) + \frac{1}{8\pi^2}\delta''(x-y), \\
\{j^-(x), r^+(y)\} &= \frac{1}{2}\delta(x-y)\psi_{iL}(x)\psi_{jL}(x)\psi_{iR}(y)\psi_{jR}(y), \\
\{r^+(x), r^+(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{iL}(x)\psi_{iL}(y) + \frac{3}{8\pi^2}\frac{\delta'(x-y)}{x-y} \\
&\quad -\delta(x-y)\psi_{iL}(x)\psi_{jL}(y)\psi_{iR}(y)\psi_{jR}(x).
\end{aligned} \tag{B.43}$$

Finally, we repeat the procedure for

$$\begin{aligned}
\{\psi_{1R}(x)\psi_{2R}(x)\psi_{3R}(x), \psi_{1L}(y)\psi_{2L}(y)\psi_{3L}(y)\} &= 0, \\
\{\psi_{iR}(x)\psi_{jR}(x)\psi_{kR}(x), \psi_{iR}(y)\psi_{jR}(y)\psi_{kL}(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{kL}(y)\psi_{kR}(x), \\
\{\psi_{iL}(x)\psi_{jL}(x)\psi_{kR}(x), \psi_{iL}(y)\psi_{jL}(y)\psi_{kL}(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{kL}(y)\psi_{kR}(x), \\
\{\psi_{iL}(x)\psi_{jL}(x)\psi_{kR}(x), \psi_{iR}(y)\psi_{jR}(y)\psi_{kL}(y)\} &= 0, \\
\{\psi_{iL}(x)\psi_{jL}(x)\psi_{kR}(x), \psi_{iR}(y)\psi_{jL}(y)\psi_{kR}(y)\} &= 0,
\end{aligned} \tag{B.44}$$

which imply that

$$\begin{aligned}
\{j^+(x), j^-(y)\} &= 0, \\
\{j^+(x), r^+(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{iL}(y)\psi_{iR}(x), \\
\{r^-(x), j^-(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{iL}(y)\psi_{iR}(x), \\
\{r^-(x), r^+(y)\} &= 0.
\end{aligned} \tag{B.45}$$

Then the anticommutator $\{Q^+, Q^-\}$ in equation (4.36) is computed from the above results.

Appendix C

Calculations for SSGM

C.1 Boson fields

The contraction of free massless bosonic fields is given by the Wightman function with IR regularization μ (see [28], [1])

$$\begin{aligned}
\Delta^{(+)}(\xi = y - x, \mu) &\equiv \overbrace{\phi(y)\phi(x)} \\
&= \frac{1}{4\pi} \int \frac{dk^1}{|k^1|} \Big(e^{-ik\cdot\xi} - \theta(\mu e^{-\gamma} - |k^1|) \Big) \\
&= -\frac{1}{4\pi} \ln(-\mu^2 \xi^2 + i\varepsilon\xi^0) \\
&= -\frac{1}{4\pi} \ln(i\mu\xi^+ + \varepsilon)(i\mu\xi^- + \varepsilon) \\
&= -\frac{1}{4\pi} \ln \mu^2 |\xi^2| - \frac{i}{4} \mathrm{sgn}(\xi^0)\theta(\xi^2) \\
&= -\frac{1}{4\pi} \ln \mu^2 |\xi^2| - \frac{i}{8} \Big(\mathrm{sgn}(\xi^+) + \mathrm{sgn}(\xi^-) \Big),
\end{aligned} \tag{C.1}$$

where γ is the Euler constant.

This function appears as follows. The free massive boson field admits an expansion in terms of the continuum creation and annihilation operators, $a(k)$ and $a^\dagger(k)$,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dk^1}{\sqrt{2k^0}} \Big(a(k) e^{-ik\cdot x} + a^\dagger(k) e^{ik\cdot x} \Big), \tag{C.2}$$

where

$$[a(k), a^\dagger(p)] = \delta(k^1 - p^1), \quad \text{and} \quad k^0 = \sqrt{|k^1|^2 + m^2}. \tag{C.3}$$

Now we introduce a normal ordering of operators which corresponds to rearranging all the annihilation operators to the right (denoted by surrounding colons). Applying this rearrangement to the product of two boson fields, we find

$$\phi(x)\phi(y) =: \phi(x)\phi(y) : + \Delta^{(+)}(x - y, m). \tag{C.4}$$

For the massless case, $\Delta^{(+)}$ is given by (C.1). However, this normal ordering prescription does not tell us what μ is in (C.1), so that its ambiguity is indicated by the "mass" μ. Therefore, we can choose a different mass to define the theory.

With Wick's theorem and (C.4), we can extend the normal ordering prescription to more complicated situations. Thus, we can translate any arbitrary function of ϕ to a normal-ordered expression by

$$V(\phi(x)) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{1}{2}\Delta^{(+)}(0,\mu)\Big)^k :\partial_\phi^{2k} V(\phi(x)):, \tag{C.5}$$

where $\Delta^{(+)}(0,\mu) = -(1/4\pi)\ln(\mu/\Lambda)$ and Λ is a cutoff. Some particular cases are

$$\begin{aligned}
\exp(i\beta\phi(x)) &= \exp\Big(-\frac{\beta^2}{2}\Delta^{(+)}(0,\mu)\Big) :\exp(i\beta\phi(x)):, \\
\sin(\beta\phi(x)) &= \exp\Big(-\frac{\beta^2}{2}\Delta^{(+)}(0,\mu)\Big) :\sin(\beta\phi(x)):, \\
\cos(\beta\phi(x)) &= \exp\Big(-\frac{\beta^2}{2}\Delta^{(+)}(0,\mu)\Big) :\cos(\beta\phi(x)): .
\end{aligned} \tag{C.6}$$

Also, we can simplify a product of two normal-ordered expressions by

$$:F(\phi(x))::G(\phi(y)):= \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\Delta^{(+)}(x-y,\mu)\Big)^k :\partial_\phi^k F(\phi(x))\partial_\phi^k G(\phi(y)): . \tag{C.7}$$

For some special products, we have

$$\begin{aligned}
:\exp(i\beta\phi(x))::\exp(i\alpha\phi(y)): &= \exp\Big(-\beta\alpha\Delta^{(+)}(x-y,\mu)\Big) :\exp(i\beta\phi(x))\exp(i\alpha\phi(y)):, \\
:\cos(\beta\phi(x))::\cos(\alpha\phi(y)): &= \frac{1}{2}\exp\Big(-\beta\alpha\Delta^{(+)}(x-y,\mu)\Big) :\cos(\beta\phi(x)+\alpha\phi(y)): \\
&\quad +\frac{1}{2}\exp\Big(\beta\alpha\Delta^{(+)}(x-y,\mu)\Big) :\cos(\beta\phi(x)-\alpha\phi(y)):, \\
:\sin(\beta\phi(x))::\sin(\alpha\phi(y)): &= -\frac{1}{2}\exp\Big(-\beta\alpha\Delta^{(+)}(x-y,\mu)\Big) :\cos(\beta\phi(x)+\alpha\phi(y)): \\
&\quad +\frac{1}{2}\exp\Big(\beta\alpha\Delta^{(+)}(x-y,\mu)\Big) :\cos(\beta\phi(x)-\alpha\phi(y)): .
\end{aligned} \tag{C.8}$$

With these results, we can easily show that

$$\prod_i :\exp(i\beta_i\phi(x_i)): = \prod_{j>i}\exp\Big(-\beta_i\beta_j\Delta^{(+)}(x_i-x_j,\mu)\Big) :\prod_i \exp(i\beta_i\phi(x_i)):, \tag{C.9}$$

$$:\cos(\beta\phi(x)):^2 = :\sin(\beta\phi(x)):^2, \tag{C.10}$$

$$:\cos(\beta\phi(x)):^2 = \frac{\pi}{2\mu^2}\partial^\mu\phi(x)\partial_\mu\phi(x) + \text{c number}. \tag{C.11}$$

In deriving this last equation, we have used a Taylor expansion in the second equation of (C.8) with the explicit form of $\Delta^{(+)}$ in (C.1). We have also taken the limits to be symmetrical, i.e. we average over all separations between x and y before taking the limit $y \to x$ (see [36]). This last equation was originally derived by Witten as the counterpart of the first Fierz transformation in (A.25). This correspondence implies that our constant A_B in the bosonization rules (2.31) is related to the mass μ by $\mu^2 = \pi^2 A_B^2$.

Finally, we define the commutator function

$$\begin{aligned}
\Delta(\xi) &\equiv \Delta^{(+)}(\xi,\mu) - \Delta^{(+)}(-\xi,\mu) \\
&= -\frac{i}{2}\mathrm{sgn}(\xi^0)\theta(\xi^2) \\
&= -\frac{i}{4}\Big(\mathrm{sgn}(\xi^+) + \mathrm{sgn}(\xi^-)\Big).
\end{aligned} \tag{C.12}$$

Thus, we have

$$\begin{aligned}
[\phi(x),\phi(y)] &= \Delta(x-y), \\
[\phi(x),\partial_0\phi(y)] &= \frac{i}{2}\Big(\delta((x-y)^+) + \delta((x-y)^-)\Big), \\
[\phi(x),\partial_1\phi(y)] &= \frac{i}{2}\Big(\delta((x-y)^+) - \delta((x-y)^-)\Big), \\
[\partial_\mu\phi(x),\partial_\mu\phi(y)] &= \frac{i}{2}\Big(\delta'((x-y)^+) + \delta'((x-y)^-)\Big), \\
[\partial_0\phi(x),\partial_1\phi(y)] &= \frac{i}{2}\Big(\delta'((x-y)^+) - \delta'((x-y)^-)\Big).
\end{aligned} \tag{C.13}$$

(There is no sum in the fourth equation.)

Moreover, we can relate this $\Delta^{(+)}$ to the functions C_μ by (see equation (B.2))

$$\partial_\mu\Delta^{(+)}(\xi,\mu) = -iC_\mu(\xi), \tag{C.14}$$

where $\epsilon = \varepsilon/\mu$.

For equal-time situations (from here x and y stand for space coordinate), we can show the canonical commutation relations follow from the above equations in the same fashion as the spinors in the previous Appendix. Thus, the nonzero relations are

$$\begin{aligned}
[\phi(x),\partial_0\phi(y)] &= i\delta(x-y), \\
[\partial_0\phi(x),\partial_1\phi(y)] &= i\delta'(x-y).
\end{aligned} \tag{C.15}$$

Similarly, we can obtain the nonzero commutator between a function of ϕ and its derivative, i.e.

$$[V(\phi(x)),\partial_0\phi(y)] = i\delta(x-y)\partial_\phi V(\phi(x)). \tag{C.16}$$

C.2 Central charge calculation

Formally, we compute the commutators and the anticommutators of two operators A and B by using

$$\begin{aligned}
\{A(x)B(x), A(y)B(y)\} &= \frac{1}{2}\{A(x), A(y)\}\{B(x), B(y)\} \\
&\quad + \frac{1}{2}[A(x), A(y)][B(x), B(y)], \\
[A(x)B(x), A(y)B(y)] &= \frac{1}{2}[A(x), A(y)]\{B(x), B(y)\} \\
&\quad + \frac{1}{2}\{A(x), A(y)\}[B(x), B(y)].
\end{aligned} \tag{C.17}$$

For more operators, we use these equations repeatedly.

Our anticommutator $\{Q^+, Q^-\}$ differs from that in reference [58], because we have a nonzero extra term in the central charge that comes from the following components of the supercurrent,

$$\begin{aligned}
&[\partial_0\phi(x), V(\phi(y))][\psi_L(x), \psi_L(y)] = \\
&\quad = -i\delta(x-y)\partial_\phi V(\phi(y))[\psi_L(x), \psi_L(y)] \\
&\quad = -i(C^-(x-y) + C^-(y-x))(C^-(x-y) - C^-(y-x))\partial_\phi V(\phi(y)) \\
&\quad = \frac{1}{2\pi}\delta'(x-y)\partial_\phi V(\phi(y)),
\end{aligned} \tag{C.18}$$

where we have used Wick's theorem and the equations in (B.6) to combine the delta function with the contractions. The other contribution to this extra term is from

$$[V(\phi(x)), \partial_0\phi(y)][\psi_R(x), \psi_R(y)] = \frac{1}{2\pi}\delta'(x-y)\partial_\phi V(\phi(x)). \tag{C.19}$$

As can be seen in the first step of the former equation, these two contributions look like they are null after integration (the singularities of the spinors "cancel out"), but this conclusion is wrong because it does not take into account the finite contribution that appears when $y \to x$, due to the delta function in the integration. The cancellation is also not possible between these contributions, since left-handed and right-handed spinors do not have the same contractions (see (B.4)). Also notice that strictly only one of these contributions appears after integration (because of the delta prime), but together they give us a symmetric term respect to the interchange of x and y (the first of them goes with a minus in the anticommutator $\{Q^+, Q^-\}$).

If we consider a normal ordered expression in $V(\phi)$, extra care is needed in the first term in the central charge. To see this, we compute explicitly the two contributions for that term. The first contribution comes from

$$\begin{aligned}
&\{\partial_1\phi(x), :V(\phi(y)):\}\{\psi_L(x), \psi_L(y)\} = \\
&\quad = \delta(x-y)\{\partial_1\phi(x), :V(\phi(y)):\} \\
&\quad = \delta(x-y)\Big(2 :V(\phi(y))\partial_1\phi(x): + (\partial_1\Delta^+(x-y) + \partial_1\Delta^+(y-x)) :\partial_\phi V(\phi(y)): \Big) \\
&\quad = \delta(x-y)\Big(2 :V(\phi(y))\partial_1\phi(x): - i(C_1(x-y) - C_1(y-x)) :\partial_\phi V(\phi(y)): \Big) \\
&\quad = 2\delta(x-y) :V(\phi(y))\partial_1\phi(x): + \frac{1}{2\pi}\delta'(x-y) :\partial_\phi V(\phi(y)):,
\end{aligned} \tag{C.20}$$

where we have used Wick's theorem in the second step, equation (C.14) in the third one and $C^1(x) = (C^+(x) - C^-(x))/2$ to apply the same manipulation of (C.18) for obtaining the δ' term in the final result. And the second contribution is from

$$\begin{aligned}
&\{:V(\phi(x)):, \partial_1\phi(y)\}\{\psi_R(x), \psi_R(y)\} = \\
&\qquad 2\delta(x-y) :V(\phi(x))\partial_1\phi(y): - \frac{1}{2\pi}\delta'(x-y) :\partial_\phi V(\phi(x)): .
\end{aligned} \tag{C.21}$$

As before these two contributions form a symmetric term respect to $x \leftrightarrow y$. Notice that the first term in these results gives the original central charge of reference [58]. The normal ordering makes a difference through the second term and the central charge is again modified by a term that has a second derivative in $V(\phi)$.

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