"THE THEORETICAL BACKGROUND OF IMPACT FACTORS FOR HIGHWAY BRIDGES"

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The purpose of this discussion is to show what impact is, what causes it, and the factors involved in the theoretical determination of impact factors. The purpose will not be to drive the AASHO impact formula (which cannot be done theoretically) or to derive some other formula.

Basically, the problem is one of dynamics, and hence of applying Newton's second law of motion, \( F = \frac{W}{g} a \). As is well known, the relationships between acceleration, velocity and displacements are: \( v = \frac{dy}{dt} \), \( a = \frac{dv}{dt} = \frac{d^2y}{dt^2} \).

Impact may be defined as the increase in static loading stresses or deformations due to movement of the loads, i.e. due to dynamic considerations. The simplest illustration of this is the case where a weight is placed on a spring. It may be assumed that the spring has a spring constant \( k \) defined as the force required to compress it a unit distance. When the weight \( W \) is placed on the spring slowly, the deflection of the spring will be \( y = \frac{W}{k} \). Now, let \( W \) be supposed that the weight \( W \) was held just in contact with the extended spring, and then suddenly released. The equations of dynamics may be written, using the freebody:

\[
(W - ky) = \frac{W}{g} \frac{d^2y}{dt^2}
\]

This may be rewritten as

\[
\frac{W}{g} \frac{d^2y}{dt^2} + ky = W
\]

\[
\frac{d^2y}{dt^2} + \frac{kg}{W} y = g
\]

Let \( \omega^2 = \frac{kg}{W} \) which gives \( \omega \) the dimensions of a frequency. Then the solution of the differential equation is:

\[
y = A \sin \omega t + B \cos \omega t + \frac{W}{k}
\]

The constants of integration may be evaluated from the initial conditions that at \( t = 0 \), \( y = 0 \) and also \( \frac{dy}{dt} = 0 \). The first of these conditions gives that \( B = -\frac{W}{k} \), and the second shows that \( A = 0 \). Thus the solution finally becomes

\[
y = -\frac{W}{k} \cos \omega t + \frac{W}{k}
\]
The maximum value of this is \( y = \frac{W}{k} \) or twice the static deflection. In terms of stress in the spring, this is also twice the maximum stress. One can then say that the impact factor for this case is 2. If the weight were dropped onto the spring from a height \( h \) the velocity of the weight would be equal to \( \sqrt{2gh} \) at the time \( t = 0 \) and the solution would be

\[
y = \frac{\sqrt{Wh}}{k} \sin \omega t - \frac{W}{k} \cos \omega t + \frac{W}{k}
\]

A similar analysis applies to a simply supported beam, also. If the beam is assumed to be weightless, and the load \( W \) is applied at the center of the beam, the deflection at the center will be \( \frac{WL^3}{48EI} \). The spring constant of the beam may therefore be said to be \( k = \frac{48EI}{L^3} \) pounds per inch. Using the same analysis as for the weight on the spring, the impact factor may again be shown to be equal to 2 for the case in which the weight is placed just in contact with the beam and then released.

However, this is not the actual case of highway bridges. There, the weight of the beam is also present. If the weight of the beam is uniformly distributed, it is the same as if \( \frac{17}{35} \) of the beam weight were concentrated at the center with the rest of the beam weightless. The following analysis is then possible:

Let \( y \) be the deflection of the center of the beam from the dead load position, and \( W' \) be \( \frac{17}{35} \) of the weight of the beam, with \( W \) again being the applied load. Then, assuming \( W \) has fallen a distance \( h \) onto the center of the beam

\[(W - ky) = \frac{W + W'}{g} \frac{d^2y}{dt^2}\]

Letting \( \omega^2 = \frac{kg}{W + W'} \) this may be rewritten as

\[
\frac{d^2y}{dt^2} + \omega^2y = \frac{gw}{W + W'}
\]

and the solution for this is

\[
y = \frac{W}{W + W'} \sqrt{\frac{(2W + W')h}{k^2}} \sin \omega t - \frac{W}{k} \cos \omega t + \frac{W}{k}
\]

If the weight \( W \) were released from a position just in contact with the center of the beam, i.e. \( h = 0 \), the impact factor would again obviously be 2.

This analysis is only correct, as far as the highway bridge is concerned, for the case of a truck being dropped onto the bridge, and even then, the action of the truck springs is neglected. It does, however, represent to some extent the problem of the vehicle falling into potholes or going over obstructions. The height of drop or the part of the weight of the vehicle to be assumed as dropping can only be judged from tests. In addition, there is the fact that with the full AASHO loading on the bridge, all of it will not be affected by pavement roughness in the same way at the same time.

The problem of repeated blows on the bridge is also an important one. It is
possible to visualize a hole in the pavement near the center of the span, with a line of trucks passing over the bridge. Let the total live plus dead load be $W$, uniformly distributed over the length of the bridge. Letting $W' = \frac{17}{35}W$ be the equivalent weight concentrated at the center of the bridge, the system is again equivalent to a load on a spring. The blows from the vehicles falling into the hole can be idealized as a disturbing force $P \sin \sigma t$ which is a periodic force with a maximum value $P$ and a frequency $f = \frac{\sigma}{2\pi}$ cycles per second. The equation of equilibrium then becomes:

$$W' \frac{d^2y}{dt^2} - \frac{8}{\text{kg}} \frac{dy}{dt} = P \sin \sigma t - ky$$

Setting $- = \omega^2$ this may be rewritten as

$$W' \frac{d^2y}{dt^2} + \omega^2y = \frac{Pg}{W'} \sin \sigma t$$

The solution of this is

$$y = A \sin \omega t + B \cos \omega t + \frac{Pg}{\omega^2 - \sigma^2} (\frac{1}{W'} \sin \sigma t)$$

It is now possible to see what will happen if $\omega^2$ approaches $\sigma^2$ in value. The displacement $y$ becomes very large which is equivalent to saying that the impact becomes very large. This condition is known as resonance. It is the same phenomenon that caused the collapse of the first Tacoma Narrows Bridge, except that there it was wind that provided the pulsating force.

![Diagram](image)

Fortunately, the problem is not so acute as it would seem from this analysis, since there is a lot of damping in the bridge which will reduce the magnitude of the oscillations. Also, it would take a very bad combination of circumstances to get full resonance, though for some bridges $\frac{17}{35}W$ will be close to $\sigma$ at reasonable vehicle speeds, remembering that $k = \frac{48EI}{L^3}$.

At this point it would seem that a smooth pavement would be the answer to the impact problem. There are, however, two other factors which cause vibrations
of bridges. The first of these is the motion of the vehicle body on its springs when the vehicle first comes onto the bridge. This motion, however, is completely random, and furthermore, subject to heavy damping from the shock absorbers. It is therefore of doubtful importance. The other is of great interest, and may be of great importance. It is the actual vibration of the beam under the load, taken as a continuous system, the way it actually is.

Starting with the well known relation

\[
EI \frac{d^2y}{dx^2} = M, \text{ the differential equation of the elastic curve,}
\]

\[
KI \frac{d^3y}{dx^3} = \frac{dM}{dx} = V, \text{ the shear, and then}
\]

\[
EI \frac{d^4y}{dx^4} = \frac{dV}{dx} = p, \text{ the load per unit length.}
\]

This load consists of several parts. The part due to motion of the dead load is

\[
g \frac{1}{g} \frac{d^2y}{dt^2} \quad \text{where \( w \) is dead load per unit length and partial derivatives must be used to distinguish between differentiation with respect to \( x \) and with respect to \( t \).}
\]

The static dead load deflection is assumed to be 0, i.e. the beams are cambered for dead load. The live load can be expressed as a function of location and time, i.e. live load = \( F(x, t) \). Then the part of the load due to the live load and its motion is \( F(x, t) \left(1 - \frac{1}{g} \frac{d^2y}{g dt^2}\right)\) where the total derivative must be used since the motion of the live load takes place with respect to both location and time. (The dead load can only move up and down, while the live load also moves horizontally).

The fourth order partial differential equation thus becomes:

\[
EI \frac{\delta^4y}{\delta x^4} = -w \frac{\delta^2y}{g \delta t^2} + F(x, t) \left[1 - \frac{1}{g} \frac{d^2y}{g dt^2}\right]
\]

The most interesting factor here is \( \frac{\delta^2y}{g \delta t^2} \). This can be broken down into:

\[
\frac{\delta^2y}{\delta x \delta t} \left(-\frac{dx^2}{dt^2}\right) + \frac{\delta^2y}{\delta x \delta t} \frac{dx}{dt} + \frac{\delta^2y}{\delta x^2} \frac{dx}{dt} + \frac{\delta^2y}{\delta x^2} \frac{dt}{dt}
\]

In physical terms, \( \frac{dx}{dt} \) is the velocity of the load, \( \frac{\delta^2y}{\delta x^2} \) is the curvature of the beam, so that the first term represents the centrifugal acceleration of the load. The second term represents the Coriolis acceleration of the load. The third term is the product of the slope of the beam and the acceleration of the load in the horizontal direction and is zero if the load crosses the beam with a constant speed; and the fourth term represents the vertical acceleration of the beam under the load.

This equation cannot be solved directly no matter what form \( F(x, t) \) is given, let alone if it is to represent a series of concentrated loads. However, a numerical solution is possible, and has been carried out for several concentrated loads in tandem, by changing the differential equation into finite difference form, and applying the concentrated loads to the several segments into which the beam is divided in succession. Using an IBM 650, moment diagrams and deflection curves were drawn. These gave evidence that for smoothly rolling loads, resonance could occur, both in first mode vibrations as well as in second mode vibrations when two loads were on the beam simultaneously. The second mode vibrations
springs

are particularly interesting since they involve small deflections, probably not easily felt by persons on the bridge, but sharp curvatures and thus high stresses. Fortunately, second mode resonance occurs only within a very narrow range of velocities of the loads for any one beam. Resonance in the first mode is more likely, especially for short bridges, which will be affected as one load after another passes over them in a heavy stream of traffic. This is similar to the resonance due to the vehicles falling into a hole.

Finally, it is possible to think of the combination of the effects due to pavement roughness and the smoothly rolling load. Since the smoothly rolling load effects are not linear, however, so that the frequency of the beam is not constant, superposition does not apply, and theoretical analysis becomes too difficult for presently known methods of solution.