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Absolutely Pure Modules

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ABSTRACT OF DISSERTATION

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A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy in the
College of Arts and Sciences
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By

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Absolutely pure modules act in ways similar to injective modules. Therefore, throughout this document we investigate many of these properties of absolutely pure modules which are modelled after those similar properties of injective modules. The results we develop can be broken into three categories: localizations, covers and derived functors.

We form $S^{-1}M$, an $S^{-1}R$ module, for any $R-$module M. We state and prove some known results about localizations. Using these known techniques and properties of localizations, we arrive at conditions on the ring R which make an absolutely pure $S^{-1}R-$module into an absolutely pure $R-$module. We then show that under certain conditions, if A is an absolutely pure $R-$module, then $S^{-1}A$ will be an absolutely pure $S^{-1}R-$module.

Also, we define conditions on the ring R which guarantee that the class of absolutely pure modules will be covering. These include R being left coherent, which we show implies a number of other necessary properties.

We also develop derived functors similar to Ext_R^n (whose development uses injective modules). We call these functors Axt_R^n, prove they are well defined, and develop many of their properties. Then we define natural maps between $\mathrm{Axt}^n(M, N)$ and $\mathrm{Ext}^n(M, N)$ and discuss what conditions on M and N guarantee that these maps are isomorphisms.

Keywords: absolutely pure, modules, covers, localizations, derived functors

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RULES FOR THE USE OF DISSERTATIONS

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Chapter 1

Introduction

A submodule $A \subset B$ is ***pure*** if $\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, B/A)$ is surjective for all finitely presented $R-$modules M. Equivalently, E' is a pure submodule of E if the canonical map $1 \otimes v : F \otimes E' \longrightarrow F \otimes E$ is an injection for all right modules F [15].

A module A is ***absolutely pure*** if it is pure in every module that contains it. Equivalently, if $\mathrm{Ext}^1_R(M, A) = 0$ for all finitely presented $R-$modules M. To see this look at the exact sequence

$$\mathrm{Hom}_R(M, E) \xrightarrow{\theta} \mathrm{Hom}_R(M, E/A) \longrightarrow \mathrm{Ext}^1_R(M, A) \longrightarrow 0 \, .$$

where $A \subset E$ and E is injective and recall that A is absolutely pure if and only if it is a pure submodule of some injective module.

Many properties have previously been proven about absolutely pure modules and their relationship to injective modules. An in-depth look at the properties of absolutely pure modules useful to this document can be found in [14] and [15]. We discuss some of these properties and how they relate to our results in the next chapter. Also, for a review of modules (in particular injective modules) see [8].

In Chapter 2, we identify many necessary properties used to prove our main results. We also look at techniques used to prove some of these properties and discuss how we use similar techniques throughout the document.

Chapter 3 defines the localization of modules namely, $S^{-1}M$ for any $R-$module M. Then we state and prove some known results about localizations that will be useful in proving our two main results. We show that every free $S^{-1}R-$module "comes from" a free $R-$module. Similarly, every finitely presented $S^{-1}R-$module "comes from" a finitely presented $R-$module. Using these ideas, we answer the question of whether an absolutely pure $S^{-1}R-$module "comes from" an absolutely pure $R-$module. More specifically, we see that if R is right coherent and A is an absolutely pure $R-$module then $S^{-1}A$ is an absolutely pure $S^{-1}R-$module.

A class of modules is said to be ***covering*** if for any module M and there is a morphism $C \longrightarrow M$, with C in our class, such that the diagram

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[dr] & \\ C \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram for all C' in our class. If $C' = C$ and $C \longrightarrow M$ is the original map $C \longrightarrow M$, then we require that it can be completed only by automorphisms of C. If the automorphism condition is not satisfied, we say the class is ***precovering***. Chapter 4 defines what it means for a class of left $R-$modules to be coresolving and notes that for the class of absolutely pure modules to be coresolving, R must be left coherent. We then use a result of El Bashir [3], along with the necessary condition that R be left coherent, to show that the class of absolutely pure modules is covering.

In [8], we can see how the derived functors Ext_R^n are developed. We use similar techniques in Chapter 5 to develop the corresponding functors using absolutely pure modules. We first show that these functors Axt_R^n are well-defined and then define the natural maps from $\operatorname{Axt}_R^n(M, N) \longrightarrow \operatorname{Ext}_R^n(M, N)$. We give four equivalent conditions which make these maps isomorphisms for all n, M, and N. These conditions answer the "global" question of when the maps will be isomorphisms. We then go on to address the "local" question, i.e. we show that the map $\operatorname{Axt}^1(M, -) \longrightarrow \operatorname{Ext}^1(M, -)$ is an isomorphism for all N if and only if $\operatorname{Ext}^1(M, A) = 0$ for all absolutely pure modules A.

Chapter 2

Preliminaries

Unless otherwise stated, module will always mean left module.

In this work we consider the notion of absolutely pure modules. We note that Garkusha and Generalov [9] considered these modules but from a more categorical viewpoint (suggested by Auslander [2]).

We note that absolutely pure modules are also studied with the terminology FP-injective (FP for finitely presented). For example see Stenström [16] and Jain [12]. Krause [13] considered related notions but from a completely categorical viewpoint.

Recall, in Chapter 1, we defined an absolutely pure module A as one in which $\mathrm{Ext}^1_R(M, A) = 0$ for all finitely presented $R-$modules M. Megibben [15] gave two more equivalent ways to define an absolutely pure module. The first requires that $\mathrm{Ext}^1_R(R/I, A) = 0$ for all finitely generated left ideals I of R. The second is the following:

Proposition 2.1. *An R-module A is absolutely pure if and only if every diagram*

$$\begin{tikzcd} P' \arrow[r, hook] \arrow[d] & P \arrow[dl, dotted] \\ A & \end{tikzcd}$$

with P' finitely generated and P projective can be completed to a commutative diagram.

We note that in the above definition we can assume that P' is a submodule of P and that P is actually free. Then using the exact sequence

$$\mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(P', A) \longrightarrow \mathrm{Ext}^1_R(F/P', A) \longrightarrow 0$$

with F free and P' a finitely generated submodule of F, we see that being able to complete the diagram is equivalent to

$$\mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(P', A)$$

being a surjection. (i.e. $\mathrm{Ext}^1_R(F/P', A) = 0$.)

We use this definition to prove Theorem 3.20 about absolutely pure localizations and also Proposition 4.2 (which is necessary to find our absolutely pure precover).

It is not true in general that a submodule of an absolutely pure module is absolutely pure. However, there are conditions on the submodule which guarantee that it is absolutely pure.

Proposition 2.2. *[14] If B is a pure submodule of an absolutely pure module A, then B is absolutely pure.*

There are questions that arise naturally about classes of modules. Two of these are whether the direct sum of elements of the class remains in the class and whether the direct limit of elements in the class also remains in the class. The direct sum of absolutely pure modules is always absolutely pure.

Proposition 2.3. *[14] Let $(A_i)_{i\in I}$ be a family of left $R-$modules and $\bigoplus_{i\in I} A_i$ be their direct sum. Then $\bigoplus_{i\in I} A_i$ is absolutely pure if and only if A_i is absolutely pure for each i in I.*

It is not true in general that for any ring R the direct limit of absolutely pure modules is absolutely pure. However, if we restrict R to be a right coherent ring, then the condition is true. A ring R is ***right coherent*** if every finitely generated submodule of a finitely presented right module is finitely presented. If R is commutative, we say R is coherent.

Proposition 2.4. *R is left coherent if and only if the direct limit of absolutely pure modules is absolutely pure.*

Stenström [16] (using the terminology FP-injective for absolutely pure) proved Proposition 2.4.

Propositions 2.2, 2.3 and 2.4 are useful in proving that the class of absolutely pure modules is covering if R is coherent. Proposition 2.2 helps us show that if R is coherent, then the quotient of an absolutely pure module by a pure submodule is absolutely pure. We then take the direct limit of these quotients and by Proposition 2.4, this remains absolutely pure. Finally, we form a set of absolutely pure modules through which we can factor all maps $A \longrightarrow M$, with A an absolutely pure module, through. We then form the direct sum of these modules. By Proposition 2.3, this is still absolutely pure.

In [14], it was shown that a direct summand of an absolutely pure module is absolutely pure. Now, since an injective module is a direct summand of every module that contains it, we have the following:

Proposition 2.5. *All injective modules are absolutely pure.*

The converse is not necessarily the case. It was shown by Megibben [15], that we have the following more restrictive result:

Theorem 2.6. *A ring R is left Noetherian if and only if every absolutely pure $R-$module is injective.*

This follows from the fact that a ring is Noetherian if and only if the direct sum of injective modules is injective. If this were not the case in our ring, then we could form a direct sum of injective modules which is not injective but which is absolutely pure [14].

Theorem 2.6 allows us to show that some of our results about localizations of absolutely pure modules imply a known result about localizations of injective modules which has a much different proof. Theorem 2.6 is also a necessary condition for us to find an answer to the "global" question of when the natural maps between derived functors $\mathrm{Axt}^n(M,N)$ and $\mathrm{Ext}^n(M,N)$ are isomorphisms.

Chapter 3

Localizations of Absolutely Pure Modules

Numerous results about localizations are known. For completeness, we state and prove some of these known results below. For a more in-depth treatment of localizations, the reader is referred to an algebra text, for example [8] or [4].

In this chapter R will always be a commutative ring and $S \subset R$ will be a mutliplicative set. We can form the ring of fractions $S^{-1}R$.

Localizations of injective modules have been studied, see [5]. In particular, we know that if R is commutative and noetherian, then for any injective $R-$module E, $S^{-1}E$ is an injective $S^{-1}R-$module. Similar to this property of injective localizations, we answer two main questions: If A is an absolutely pure $S^{-1}R-$module, then is A an absolutely pure $R-$module? Conversely, if A is an absolutely pure $R-$module, then is $S^{-1}A$ an absolutely pure $S^{-1}R-$module? If not, then are there conditions we can place on the ring which make these true?

Definition 3.1. *([8], pg. 44) Let* S *be a multiplicative subset of* R*; that is,* $1 \in S$ *and* S

is closed under multiplication. Then the localization of R with respect to S, denoted by $S^{-1}R$, is the set of all equivalence classes (a, s) with $a \in R$, $s \in S$ under the equivalence relation $(a, s) \sim (b, t)$ if there is an $s' \in S$ such that $(at - bs)s' = 0$. We also write a pair (a, s) as a/s.

We now define addition and multiplication on $S^{-1}R$ by

$$a/s + b/t = (at + bs)/st$$

$$(a/s)(b/t) = ab/st.$$

These operations are well-defined and $S^{-1}R$ is then a commutative ring with identity.

We can also form $S^{-1}M$ (an $S^{-1}R-$module), for any $R-$module M.

Definition 3.2. *[8] Let $S \subset R$ be a multiplicative set and M be an $R-$module. Then the localization of M with respect to S, denoted $S^{-1}M$ is defined as for $S^{-1}R$. The group $S^{-1}M$ is abelian under addition and is an $S^{-1}R-$module via*

$$(a/s) \cdot (x/t) = ax/st.$$

We ask whether $S^{-1}A$ is an absolutely pure $S^{-1}R-$module when A is an absolutely pure $R-$module. Using facts about injective modules, we believe this is only true under certain conditions. For, in general, $S^{-1}E$ is an injective $S^{-1}R-$module when E is an injective $R-$module, only when R is noetherian [5].

First, we need some known results about localizations.

Proposition 3.3. *([8], Proposition 2.2.4) Let $S \subset R$ be a multiplicative set. If $f : M \longrightarrow N$ is an $R-$module homomorphism, then $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$, defined by*

$(S^{-1}f)\left(\frac{x}{s}\right) = \frac{f(x)}{s}$, *is an* $S^{-1}R-$*module homomorphism.*

Proof. We show that $S^{-1}f$ is well defined. Suppose $\frac{x}{s} = \frac{y}{t}$, i.e., there exists an $s' \in S$ such that $(xt - ys)s' = 0$. Now $(S^{-1}f)\left(\frac{x}{s}\right) = \frac{f(x)}{s}$ and $(S^{-1}f)\left(\frac{y}{t}\right) = \frac{f(y)}{t}$. We have

$$\begin{aligned}
(f(x)t - f(y)s)s' &= (f(xt) - f(ys))s' \\
&= f(xt - ys)s' \\
&= f((xt - ys)s') \\
&= f(0) \\
&= 0.
\end{aligned}$$

Therefore, $\frac{f(x)}{s} = \frac{f(y)}{t}$ and the function is well defined. It is easy to check that it is a homomorphism. □

Proposition 3.4. *([8], Proposition 2.2.4) If* $M' \xrightarrow{f} M \xrightarrow{g} M''$ *is exact, then*

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact.

Proof. Let $\frac{x}{s} \in S^{-1}M'$. Now

$$\begin{aligned}
S^{-1}g(S^{-1}f)\left(\frac{x}{s}\right) &= S^{-1}g\left(\frac{f(x)}{s}\right) \\
&= \frac{g(f(x))}{s} = \frac{0}{s}
\end{aligned}$$

since the original sequence was exact. So $\operatorname{im}(S^{-1}f) \subset \ker(S^{-1}g)$.

Next, let $\frac{x}{s} \in \ker(S^{-1}g)$. This means $S^{-1}g\left(\frac{x}{s}\right) = \frac{g(x)}{s} = 0$. Then, by definition,

$tg(x) = 0$ for some $t \in S$, i.e. $g(tx) = 0$. Since $\ker(g) = \operatorname{im}(f)$, we have a y such that $f(y) = tx$. Hence, $S^{-1}f\left(\frac{y}{ts}\right) = \frac{f(y)}{ts} = \frac{tx}{ts}$ and $\frac{x}{s} \in \operatorname{im}(S^{-1}f)$. Therefore, $\ker(S^{-1}g) = \operatorname{im}(S^{-1}f)$ and the sequence is exact. $\square$

Using Proposition 3.4 and a particular natural exact sequence, we can show the following:

Remark 3.5. *([8], Proposition 2.2.4) If $N \subset M$ are $R-$modules, then $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.*

Proof. Consider the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow \frac{M}{N} \longrightarrow 0$. By Proposition 3.4, the sequence $0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$ is exact. Hence $S^{-1}M/S^{-1}N \cong S^{-1}(M/N)$. $\square$

We need the following proposition to determine whether $S^{-1}N \cong N$ as $S^{-1}R-$ modules.

Proposition 3.6. *([4], §2.7) Let M be an $R-$module. The canonical homomorphism $\phi : M \longrightarrow S^{-1}M$, which maps $x \mapsto \frac{x}{1}$, is an isomorphism if and only if for every $s \in S$ and $x \in M$ there is a unique $y \in M$ such that $sy = x$.*

Proof. Suppose for every $s \in S$ and $x \in M$ there is a unique $y \in M$ such that $sy = x$. Suppose $\phi(x) = \phi(\overline{x})$. Then $\frac{x}{1} = \frac{\overline{x}}{1}$ and there exists an $s' \in S$ such that $(x - \overline{x})s' = 0$. This gives $xs' - \overline{x}s' = 0$. So $xs' = \overline{x}s'$ and $x = \overline{x}$.

Let $\frac{x}{s} \in S^{-1}M$. There exists a unique $y \in M$ such that $sy = x$. Then we have $\phi(y) = \frac{y}{1}$ and $x - sy = 0$, so $\frac{y}{1} = \frac{x}{s}$. Thus, $\phi(y) = \frac{x}{s}$ and ϕ is a surjection. Hence, ϕ is an isomorphism.

Conversely, suppose ϕ is an isomorphism. Let $x \in M$ and $s \in S$. Then since ϕ is onto, there exists a $\overline{y} \in M$ such that $\phi(\overline{y}) = \frac{\overline{y}}{1} = \frac{x}{s}$. This leads to an $s' \in S$ such that

$(x - s\overline{y})s' = 0$. Then $xs' - s\overline{y}s' = 0$. Thus $\dfrac{x}{1} = \dfrac{s\overline{y}}{1}$ and $x = s\overline{y}$. Now, $\overline{y}$ is unique since ϕ is an isomorphism. $\square$

The conditions in Proposition 3.6 are equivalent to requiring the function $M \xrightarrow{s\cdot -} M$ that maps x to sx for each $x \in M$ be an isomorphism for every $s \in S$.

Proposition 3.7. *([4], Proposition 3 of §2.2) Given M and N, define a function*

$$\sigma : Hom_R(M, N) \longrightarrow Hom_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

If the sets $Hom_R(M, N)$ and $Hom_{S^{-1}R}(S^{-1}M, S^{-1}N)$ are made into $R-$modules in the natural fashion, then the function σ is $R-$linear.

Proof. Define the well-defined function $\sigma(f)(x) = S^{-1}f\left(\dfrac{x}{1}\right)$ where

$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$$

via $S^{-1}f\left(\dfrac{x}{s}\right) = \dfrac{f(x)}{s}$. Let $x \in M$. Then

$$\begin{aligned}
\sigma(rf + \overline{r}g)(x) &= S^{-1}(rf + \overline{r}g)\left(\frac{x}{1}\right) \\
&= (rS^{-1}f + \overline{r}S^{-1}g)\left(\frac{x}{1}\right) \\
&= rS^{-1}f\left(\frac{x}{1}\right) + \overline{r}S^{-1}g\left(\frac{x}{1}\right) \\
&= \frac{rf(x)}{1} + \frac{\overline{r}g(x)}{1} \\
&= r\sigma(f)(x) + \overline{r}\sigma(g)(x) \\
&= [r\sigma(f) + \overline{r}\sigma(g)](x).
\end{aligned}$$

So σ is an $R-$linear homomorphism. $\square$

Let N be an $S^{-1}R-$module. Now using the homomorphism $\phi : R \longrightarrow S^{-1}R$ from Proposition 3.6, we can make N into an $R-$module.

Remark 3.8. *$S^{-1}N \cong N$ as $S^{-1}R-$modules*

Proof. Define $\psi : N \longrightarrow S^{-1}N$ via $x \mapsto \dfrac{x}{1}$. By Proposition 3.6, to see this is an isomorphism we only need to show that for every $s \in S$ and $x \in N$, there is a unique $y \in N$ such that $sy = x$. Choose an $s \in S$ and an $x \in N$. Since N is an $S^{-1}R-$module, we know that $\dfrac{1}{s} \cdot x \in N$, so let $y = \dfrac{1}{s} \cdot x$. Now $sy = s\left(\dfrac{1}{s} \cdot x\right) = x$ as desired. $\square$

When M is such that $M \longrightarrow S^{-1}M$ is an isomorphism, we usually say $S^{-1}M = M$ (but this is an "abuse of language").

Proposition 3.9. *$M \longrightarrow S^{-1}M$ is "universal" in the sense that if N is any $S^{-1}R-$module and if $f : M \longrightarrow N$ is $R-$linear, then*

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "f"'] & S^{-1}M \arrow[d, dotted] \\ & N \end{tikzcd}$$

can be made into a commutative diagram in a unique fashion by an $S^{-1}R-$linear map $S^{-1}M \longrightarrow N$.

Proof. Define $g : S^{-1}M \longrightarrow N$ via $g\left(\dfrac{x}{s}\right) = \dfrac{1}{s}f(x)$, for all $\dfrac{x}{s} \in S^{-1}M$. This makes sense because N is an $S^{-1}R-$module and $f(x) \in N$. Suppose that $\dfrac{x}{s} = \dfrac{x'}{s'}$, so there exists a $u \in S$ such that $u(s'x - sx') = 0$. Then $g\left(\dfrac{x}{s}\right) = \dfrac{1}{s}f(x)$ and $g\left(\dfrac{x'}{s'}\right) = \dfrac{1}{s'}f(x')$. Now take $\dfrac{x}{s}$ and $\dfrac{x'}{s'}$ in $S^{-1}M$, then

$$\begin{aligned} g\left(\frac{x}{s} + \frac{x'}{s'}\right) &= g\left(\frac{s'x + sx'}{ss'}\right) \\ &= \frac{1}{ss'}f(s'x + sx') \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{ss'}(f(s'x) + f(sx')) \\
&= \frac{1}{ss'}s'f(x) + \frac{1}{ss'}sf(x') \\
&= \frac{1}{s}f(x) + \frac{1}{s'}f(x') \\
&= g\left(\frac{x}{s}\right) + g\left(\frac{x'}{s'}\right).
\end{aligned}$$

Hence, g is linear. It is easy to see that g makes the diagram commutative, as desired.

To show that g is unique, suppose you can also complete the diagram with h. Let $x \in M$, then $g\left(\frac{x}{1}\right) = f(x) = h\left(\frac{x}{1}\right)$ since both diagrams

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "f"'] & S^{-1}M \arrow[d, dotted, "g"] \\ & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r] \arrow[dr, "f"'] & S^{-1}M \arrow[d, dotted, "h"] \\ & N \end{tikzcd}$$

are commutative. Let $y \in S^{-1}M$, then $y = \dfrac{x}{s}$ and

$$g(y) = g\left(\frac{x}{s}\right) = \frac{1}{s}f(x) = \frac{1}{s}h\left(\frac{x}{1}\right) = h\left(\frac{x}{s}\right) = h(y).$$

Therefore, the map g is unique. $\square$

Let N be an $S^{-1}R-$module. Using Remark 3.8, N is also an $R-$module. So now if M is any $R-$module, we can prove that $S^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(M, N)$.

Proposition 3.10. *Let N be an $S^{-1}R-$module. If M is any $R-$module,*

$$S^{-1}Hom_R(M, N) \cong Hom_R(M, N).$$

Proof. We know by Remark 3.8, that N is also an $R-$module and $N \cong S^{-1}N$. By Proposition 3.6, we need only show that for every $s \in S$ and $\phi \in \mathrm{Hom}_R(M, N)$ there is a unique $\overline{\phi} \in \mathrm{Hom}_R(M, N)$ such that $s\overline{\phi} = \phi$.

Let $s \in S$, $\phi \in \mathrm{Hom}_R(M, N)$. Now for all $x \in M$, $\phi(x) \in N$, then since $N \cong S^{-1}N$, there exists a unique $y \in N$ such that $sy = \phi(x)$. Define $\overline{\phi} : M \longrightarrow N$ via $\overline{\phi}(x) = y$ where y is as above. Then $s\overline{\phi}(x) = sy = \phi(x)$ and $\overline{\phi}$ is well defined by the uniqueness of y. Therefore, $s\overline{\phi} = \phi$. Now

$$\begin{aligned} s\overline{\phi}(x_1 + x_2) &= \phi(x_1 + x_2) \\ &= \phi(x_1) + \phi(x_2) \\ &= s\overline{\phi}(x_1) + s\overline{\phi}(x_2) \\ &= s(\overline{\phi}(x_1) + \overline{\phi}(x_2)). \end{aligned}$$

So $\overline{\phi}(x_1 + x_2) = \overline{\phi}(x_1) + \overline{\phi}(x_2)$. Also

$$\begin{aligned} s\overline{\phi}(rx) &= \phi(rx) \\ &= r\phi(x) \\ &= r(s\overline{\phi}(x)) \\ &= s(r\overline{\phi}(x)) \end{aligned}$$

since R is commutative, so $\overline{\phi}(rx) = r\overline{\phi}(x)$. Therefore, $\overline{\phi}$ is $R-$linear and thus, $\overline{\phi} \in \mathrm{Hom}_R(M, N)$. Suppose also that $\psi : M \longrightarrow N$ is such that $s\psi = \phi$. Then for all x $s\psi(x) = \phi(x) = s\overline{\phi}(x)$ and so $\psi(x) = \overline{\phi}(x)$. Hence, $\overline{\phi}$ is unique as desired. $\square$

A similar argument shows that $S^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(M, N)$ when M is an $S^{-1}R-$module and N is an $R-$module.

Recall, in Proposition 3.7, we define a function from

$$\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

Now, by Proposition 3.10 if N is an $S^{-1}R-$module,

$$\mathrm{Hom}_R(M,N) \cong S^{-1}\mathrm{Hom}_R(M,N).$$

So a natural question is: What is the map

$$S^{-1}\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)?$$

Proposition 3.11. *([4], Proposition 19 of §2.8) For any two $R-$modules M and N there is a natural $S^{-1}R$ linear map*

$$S^{-1}Hom_R(M,N) \longrightarrow Hom_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

Proof. Define a map from $S^{-1}\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ via $\frac{\phi}{s} \mapsto \frac{1}{s}\phi$ with $s \in S$ and $\phi \in \mathrm{Hom}_R(M,N)$, where $\frac{1}{s}\phi\left(\frac{x}{t}\right) = \frac{\phi(x)}{st}$, for any $\frac{x}{t} \in S^{-1}M$.

Suppose $\frac{\phi}{s} = \frac{\overline{\phi}}{\overline{s}}$ for $s, \overline{s} \in S$ and $\phi, \overline{\phi} \in \mathrm{Hom}_R(M,N)$. Then there exists an $s' \in S$ such that $(\overline{s}\phi - s\overline{\phi})s' = 0$. Take $\frac{x}{t} \in S^{-1}M$. We want to show that $\frac{1}{s}\phi\left(\frac{x}{t}\right) = \frac{1}{\overline{s}}\overline{\phi}\left(\frac{x}{t}\right)$, i.e. $\frac{\phi(x)}{st} = \frac{\overline{\phi}(x)}{\overline{s}t}$, i.e. for some $u \in S$, $(\overline{s}t\phi(x) - st\overline{\phi}(s))u = 0$. But $(\overline{s}\phi - s\overline{\phi})s' = 0$. So, since R is commutative, $(\overline{s}t\phi - st\overline{\phi})s' = 0$. Therefore $(\overline{s}t\phi(x) - st\overline{\phi}(x))s' = 0$ and the map is well-defined.

Next, we need to show that $\frac{1}{s}\phi \in \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$.

$$\begin{aligned}
\frac{1}{s}\phi\left(\frac{x_1}{t_1}+\frac{x_2}{t_2}\right) &= s\phi\left(\frac{t_2x_1+t_1x_2}{t_1t_2}\right) \\
&= \frac{\phi(t_2x_1+t_1x_2)}{st_1t_2} \\
&= \frac{t_2\phi(x_1)+t_2\phi(x_2)}{st_1t_2} \\
&= \frac{\phi(x_1)}{st_1}+\frac{\phi(x_2}{st_2} \\
&= \frac{1}{s}\phi\left(\frac{x_1}{t_1}\right)+\frac{1}{s}\phi\left(\frac{x_2}{t_2}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{s}\phi\left(\left(\frac{r}{t}\right)\left(\frac{x}{s'}\right)\right) &= \frac{1}{s}\phi\left(\frac{rx}{ts'}\right) \\
&= \frac{\phi(rx)}{sts'} \\
&= \frac{r\phi(x)}{sts'} \\
&= \left(\frac{r}{t}\right)\frac{\phi(x)}{ss'} \\
&= \left(\frac{r}{t}\right)\frac{1}{s}\phi\left(\frac{x}{s'}\right).
\end{aligned}$$

Hence $\frac{1}{s}\phi \in \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$. It is clear that this map is $S^{-1}R$ linear. $\square$

Using similar techniques, it can be shown that if

$$S^{-1}\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

is an isomorphism when $M = M_1$ and when $M = M_2$ (for two $R-$modules M_1 and M_2), then it is an isomorphism when $M = M_1 \oplus M_2$.

What we need to show our results about absolutely pure localizations are certain conditions under which the map above is an isomorphism. First, we have an example of when the map is not an isomorphism.

Example 3.12. Let $R = \mathbb{Z}$ and $S = \{n | n \in \mathbb{Z}, n \neq 0\}$. Let $M = \mathcal{Q}$, $N = \mathbb{Z}$. Now $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{Q}, \mathbb{Z})$ only contains the zero map, so $S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathcal{Q}, \mathbb{Z})$ has only 1 element. But $\mathrm{Hom}_{\mathcal{Q}}(\mathcal{Q}, \mathcal{Q})$ has an infinite number of maps, namely $\phi : \mathcal{Q} \longrightarrow \mathcal{Q}$ where $\phi(1) = r$ for an $r \in \mathcal{Q}$. Hence the two groups cannot be isomorphic.

We do have the following more restrictive result:

Remark 3.13. *If N is an $S^{-1}R-$module, then $S^{-1}Hom(M, N) \longrightarrow Hom_{S^{-1}R}(S^{-1}M, S^{-1}N)$ is an isomorphism if $M = R$.*

Proof. Suppose $M = R$. Then $\mathrm{Hom}_R(R, N) \cong N$ and

$$\mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}N) \cong S^{-1}N \cong S^{-1}\mathrm{Hom}(R, N).$$

$\square$

Definition 3.14. *Let $S \subset R$ be multiplicative, (R a commutative ring). An $S^{-1}R-$module N is said to come from an $R-$module M if $N \cong S^{-1}M$.*

A problem is to show that a given $S^{-1}R-$module N with a special property comes from an $R-$module M with the same property.

Proposition 3.15. *Every free $S^{-1}R-$module comes from a free $R-$module.*

Proof. Suppose N is a free $S^{-1}R-$module with base $(y_i)_{i\in I}$. Let M be the free module with base $(x_i)_{i\in I}$. Let $\frac{x}{s} \in S^{-1}M$. Then $x \in M$, so $x = \sum_{i\in I} r_i x_i$. Now

$$\frac{x}{s} = \frac{1}{s}\left(\frac{x}{1}\right) = \frac{1}{s}\left(\frac{\sum r_i x_i}{1}\right) = \frac{\sum r_i x_i}{s} = \sum \frac{r_i x_i}{s} = \sum \frac{r_i}{s}\left(\frac{x_i}{1}\right).$$

Hence, $S^{-1}M$ is free with base $\left(\frac{x_i}{1}\right)_{i\in I}$.

Define $\phi : S^{-1}M \longrightarrow N$ via $\phi\left(\frac{x_i}{1}\right) = y_i$. This is clearly a well-defined homomorphism. Now define $\psi : N \longrightarrow S^{-1}M$ via $\psi(y_i) = \frac{x_i}{1}$. This too is a well-defined homomorphism. Let $\frac{x}{s} \in S^{-1}M$, so $\frac{x}{x} = \sum\left(\frac{r_i}{s}\right)\left(\frac{x_i}{1}\right)$. Now $\psi\circ\phi : S^{-1}M \longrightarrow S^{-1}M$ and

$$\begin{aligned}
\psi\circ\phi\left(\frac{x}{s}\right) &= \psi\left(\phi\left(\sum\left(\frac{r_i}{x}\right)\left(\frac{x_i}{1}\right)\right)\right)\\
&= \psi\left(\sum\left(\frac{r_i}{s}\right)\phi\left(\frac{x_i}{1}\right)\right)\\
&= \psi\left(\sum\frac{r_i}{s}y_i\right)\\
&= \sum\frac{r_i}{s}\psi(y_i)\\
&= \sum\frac{r_i}{s}\left(\frac{x_i}{1}\right)\\
&= \frac{x}{s}.
\end{aligned}$$

Let $y = \sum r_i y_i \in N$. We have $\phi\circ\psi : N \longrightarrow N$ with

$$\begin{aligned}
\phi\circ\psi(y) &= \phi\circ\psi\left(\sum r_i y_i\right)\\
&= \phi\left(\sum r_i\psi(y_i)\right)\\
&= \phi\left(\sum r_i\left(\frac{x_i}{1}\right)\right)\\
&= \sum r_i\phi\left(\frac{x_i}{1}\right)\\
&= \sum r_i y_i\\
&= y.
\end{aligned}$$

Therefore, ψ and ϕ are inverses of one another and $S^{-1}M \cong N$. $\square$

A similar argument shows that a free $S^{-1}R-$module with a finite base having n

elements comes from a free $R-$module having a base with n elements.

Let $g : S^{-1}R^m \longrightarrow S^{-1}R^n$ be an $S^{-1}R-$linear map for some $m, n \geq 1$. It is not true in general that there is an $R-$linear map $f : R^m \longrightarrow R^n$, such that $S^{-1}f = g$, but we do have the following theorem:

Theorem 3.16. *If $g : S^{-1}R^m \longrightarrow S^{-1}R^n$ is an $S^{-1}R-$linear map, then there is an $s \in S$ such that for some $R-$linear map $f : R^m \longrightarrow R^n$, $S^{-1}f = sg$.*

Proof. Assume g maps $\dfrac{e_i}{1} \mapsto \dfrac{(j_{i,1}, i_{i,2}, ...j_{i,n})}{s_i}$ where $e_1, ..., e_m$ is the standard base of R^m. Define $s = s_1 \cdots s_m$ and $\overline{s_i} = s_1 \cdots s_{i-1}s_{i+1} \cdots s_m$. Now let $f : R^m \longrightarrow R^n$ via $e_i \mapsto \overline{s_i} \cdot (j_{i,1}, i_{i,2}, ...j_{i,n})$. Clearly f is $R-$linear, since R is commutative and any map defined by choosing $f(e_i)$, $i = 1, ..., m$ and extending $R-$linearly is $R-$linear. Now

$$\begin{aligned} S^{-1}f\left(\frac{e_i}{1}\right) &= \frac{f(e_i)}{1} \\ &= \frac{\overline{s_i}(j_{i,1}, i_{i,2}, ...j_{i,n})}{1} \\ &= \frac{s_i}{s_i} \cdot \frac{\overline{s_i}(j_{i,1}, i_{i,2}, ...j_{i,n})}{1} \\ &= \frac{s \cdot (j_{i,1}, i_{i,2}, ...j_{i,n})}{s_i} \\ &= \frac{s}{1}\left(\frac{(j_{i,1}, i_{i,2}, ...j_{i,n})}{s_i}\right) \\ &= sg\left(\frac{e_i}{1}\right). \end{aligned}$$

So $S^{-1}f = sg$ $\square$

Using the same notation as in Theorem 3.16, we get the following two corollaries:

Corollary 3.17. *$coker(sg) = coker(g)$.*

Proof. Let $\dfrac{(b_1, ..., b_n)}{t} \in \operatorname{im}(g)$. Then there exists an $\dfrac{(a_1, ..., a_m)}{\overline{t}} \in S^{-1}R^m$ such that

$g\left(\frac{(a_1,...,a_m)}{\bar{t}}\right) = \frac{(b_1,...,b_n)}{t}$. Also,

$$sg\left(\frac{(a_1,...,a_m)}{s\bar{t}}\right) = g\left(\frac{(a_1,...,a_m)}{\bar{t}}\right) = \frac{(b_1,...,b_n)}{t},$$

since g is $S^{-1}R-$linear, so $\frac{(b_1,...,b_n)}{t} \in \text{im}(sg)$.

Let $\frac{(b_1,...,b_n)}{t} \in \text{im}(sg)$, then there exists an $\frac{(a_1,...,a_m)}{\bar{t}} \in S^{-1}R^m$ such that $sg\left(\frac{(a_1,...,a_m)}{\bar{t}}\right) = \frac{(b_1,...,b_n)}{t}$. Now $\frac{(sa_1,...,sa_m)}{\bar{t}} \in S^{-1}R^m$ and

$$g\left(\frac{(sa_1,...,sa_m)}{\bar{t}}\right) = g\left(\frac{s}{1}\frac{(a_1,...,a_m)}{\bar{t}}\right) = sg\left(\frac{(a_1,...,a_m)}{\bar{t}}\right) = \frac{(b_1,...,b_n)}{t}.$$

So $\frac{(b_1,...,b_n)}{t} \in \text{im}(g)$.

Hence, $\text{im}(sg) = \text{im}(g)$, and thus, $\text{coker}(g) = \text{coker}(sg)$. □

Corollary 3.18. *If g and e_i are as in Theorem 3.16, then for some $s \in S$, $sg\left(\frac{e_i}{1}\right) = \frac{a_i}{1}$ for $1 \leq i \leq m$ and some $a_i \in R^n$.*

Proof. Suppose $g\left(\frac{e_i}{1}\right) = \frac{b_i}{s_i}$. Then let $s = s_1 \cdots s_m$ and $a_i = s_1 \cdots s_{i-1}s_{i+1} \cdots s_m b_i$. Then

$$\begin{aligned} sg\left(\frac{e_i}{1}\right) &= \frac{s}{1}\left(\frac{b_i}{s_i}\right) \\ &= \frac{s_1 \cdots s_m \cdot b_i}{s_i} \\ &= \frac{s_1 \cdots s_{i-1}s_{i+1} \cdots s_m b_i}{1} \\ &= \frac{a_i}{1}. \end{aligned}$$

□

Similar to Proposition 3.15 about free modules, using Theorem 3.16 we get that every finitely presented $S^{-1}R-$module comes from a finitely presented $R-$module.

Proposition 3.19. *If P is a finitely presented $S^{-1}R-$module, then there is a finitely presented $R-$module Q such that $S^{-1}Q \cong P$.*

Proof. P is finitely presented, so the sequence $S^{-1}R^m \longrightarrow S^{-1}R^n \longrightarrow P \longrightarrow 0$ is exact. Use the map $S^{-1}R^m \longrightarrow S^{-1}R^n$ to get a map $R^m \longrightarrow R^n$, as in Theorem 3.16. Now, there is a Q so that $R^m \longrightarrow R^n \longrightarrow Q \longrightarrow 0$ is exact. This gives $S^{-1}R^m \longrightarrow S^{-1}R^n \longrightarrow S^{-1}Q \longrightarrow 0$ exact, by Proposition 3.4. Also

$$\begin{tikzcd}
S^{-1}R^m \arrow[r] \arrow[d, "\cong"] & S^{-1}R^n \arrow[r] \arrow[d, "\cong"] & P \arrow[r] \arrow[d] & 0 \\
S^{-1}R^m \arrow[r] & S^{-1}R^n \arrow[r] & S^{-1}Q \arrow[r] & 0
\end{tikzcd}$$

is commutative. Since P and $S^{-1}Q$ are both cokernels of $S^{-1}R^m \longrightarrow S^{-1}R^n$, we have the induced map $P \longrightarrow S^{-1}Q$. Hence $P \cong S^{-1}Q$. □

Theorem 3.20. *If A is an absolutely pure $S^{-1}R-$module, then A is an absolutely pure $R-$module.*

Proof. By Proposition 2.1, we need to show that

$$\begin{tikzcd}
T \arrow[d, "f"] & \subset & R^n \arrow[dll, dotted] \\
A & &
\end{tikzcd}$$

can be completed to a commutative diagram when T is a finitely generated submodule of any free module R^n. If T is a finitely generated $R-$module, then $S^{-1}T$ is a finitely generated $S^{-1}R-$module and $S^{-1}T \subset S^{-1}R^n$ with $S^{-1}R^n$ a free module. Therefore, by

Proposition 2.1, since A is an absolutely pure $S^{-1}R-$module,

$$\begin{tikzcd}
S^{-1}T \arrow[d] & \subset & S^{-1}R^n \arrow[dll, dotted] \\
A & &
\end{tikzcd}$$

can be completed to a commutative diagram.

Suppose we have

$$\begin{tikzcd}
T \arrow[d, "f"] & \subset & R^n \arrow[dll, dotted] \\
A & &
\end{tikzcd}$$

By Proposition 3.9, we know that $T \longrightarrow S^{-1}T$ is "universal", so we can find a linear map g from $S^{-1}T \longrightarrow A$ that makes

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & S^{-1}T \arrow[dl, dotted, "g"] \\
A &
\end{tikzcd}$$

commutative. Now we have the following diagram

$$\begin{tikzcd}
T \arrow[d] \arrow[dd, bend right, "f"'] & \subset & R^n \arrow[d] \\
S^{-1}T \arrow[rr, hook] \arrow[d, "g"] & & S^{-1}R^n \arrow[dll, dotted, "h"] \\
A & &
\end{tikzcd}$$

where g is our completion above.

Since A is an absolutely pure $S^{-1}R-$module, the map g can be extended by h. It is easy to see then that

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "f"'] & R^n \arrow[dl, dotted, "x \mapsto \frac{x}{1} \mapsto h(\frac{x}{1})"] \\ A & \end{tikzcd}$$

is commutative. So A is an absolutely pure $R-$module. $\square$

We can now see that every absolutely pure $S^{-1}R-$module is of the form $S^{-1}A$ for an absolutely pure $R-$module A. For let B be an absolutely pure $S^{-1}R-$module. Then by Theorem 3.20, B is an absolutely pure $R-$module. But $S^{-1}B \cong B$ as $S^{-1}R-$modules. Therefore, $B = S^{-1}B$, where B is an absolutely pure $R-$module, as desired.

Suppose we have again that A is an $S^{-1}R-$module. Theorem 3.20 says that if A is an absolutely pure $S^{-1}R-$module, then A is an absolutely pure $R-$module. The question now is: if A is an absolutely pure $R-$module, then is $S^{-1}A$ an absolutely pure $S^{-1}R-$module?

This is true only for certain rings R.

Theorem 3.21. *If A is an absolutely pure $R-$module and R is coherent, then $S^{-1}A$ is an absolutely pure $S^{-1}R$ module.*

Proof. Suppose A is an absolutely pure $R-$module. If we want to show that $S^{-1}A$ is an absolutely pure $S^{-1}R-$module. Again by Proposition 2.1, we need to be able to complete

$$\begin{tikzcd} T \arrow[r, phantom, "\subset"] \arrow[d] & S^{-1}R^n \arrow[dl, dotted] \\ S^{-1}A & \end{tikzcd}$$

to a commutative diagram where T is finitely generated.

If $T \subset S^{-1}R^n$ is finitely generated, then $T \cong S^{-1}U$ for a finitely generated submodule

$U \subset R^n$ (as $R-$modules). So we have the diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & R^n \arrow[d] \\ T = S^{-1}U \arrow[r, hook] \arrow[d] & S^{-1}R^n \\ S^{-1}A & \end{tikzcd}$$

Since R is coherent, every finitely generated submodule of a free module is finitely presented. Hence, U is finitely presented. So by definition, there is an exact sequence

$$R^n \longrightarrow R^m \longrightarrow U \longrightarrow 0.$$

The sequence

$$0 \longrightarrow \mathrm{Hom}(U, A) \longrightarrow \mathrm{Hom}(R^m, A) \longrightarrow \mathrm{Hom}(R^n, A)$$

is also exact (see Theorem 5.1). Since S^{-1} is exact, by Proposition 3.3, we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S^{-1}\mathrm{Hom}(U, A) \arrow[r] \arrow[d] & S^{-1}\mathrm{Hom}(R^m, A) \arrow[r] \arrow[d, "\cong"] & S^{-1}\mathrm{Hom}(R^n, A) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathrm{Hom}(S^{-1}U, S^{-1}A) \arrow[r] & \mathrm{Hom}(S^{-1}R^m, S^{-1}A) \arrow[r] & \mathrm{Hom}(S^{-1}R^n, S^{-1}A). \end{tikzcd}$$

Therefore, the map

$$S^{-1}\mathrm{Hom}_R(U, A) \longrightarrow \mathrm{Hom}_{S^{-1}R}(S^{-1}U, S^{-1}A)$$

is an isomorphism.

So now we use Theorem 5.1 to see that $\mathrm{Hom}_R(R^n, A) \longrightarrow \mathrm{Hom}_R(U, A) \longrightarrow 0$ is exact (since A is absolutely pure). But then

$$S^{-1}\mathrm{Hom}_R(R^n, A) \longrightarrow S^{-1}\mathrm{Hom}_R(U, A) \longrightarrow 0$$

is also exact. But, using our isomorphism, we get

$$\mathrm{Hom}(S^{-1}R^n, S^{-1}A) \longrightarrow \mathrm{Hom}(S^{-1}U, S^{-1}A) \longrightarrow 0$$

exact. Hence $S^{-1}A$ is an absolutely pure $S^{-1}R-$module. $\square$

If R is commutative and noetherian, then it is known that for any injective module E, $S^{-1}E$ is injective as an $S^{-1}R-$module [5]. Recall, from Theorem 2.6, that if R is commutative and noetherian, then E being injective implies that it is absolutely pure. So our result implies this known result. (The usual proof is very different from this one.)

In general there are many examples of $R-$modules with $M \neq 0$, but $S^{-1}M = 0$. So it is easy to find modules A that are not absolutely pure but such that $S^{-1}A$ is an absolutely pure $S^{-1}R-$module.

Chapter 4

Absolutely Pure Covers

It is known that a ring R is left Noetherian if and only if every left $R-$module M has an injective cover and if and only if every left $R-$module M has an injective precover (see Theorem 5.4.1 of [8]). So the questions we ask are: What conditions on R imply that every left $R-$module M has an absolutely pure precover and what conditions on R imply that every left $R-$module M has an absolutely pure cover?

Definition 4.1. *A class $\mathcal{A}$ of left $R-$modules is said to be coresolving if $E \in \mathcal{A}$ for all injective modules E and if given an exact sequence of left R modules*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

$A'' \in \mathcal{A}$ whenever $A', A \in \mathcal{A}$.

We now have the following proposition that gives a condition on the ring R that will guarantee that the class $\mathcal{A}$ of absolutely pure modules is coresolving.

Proposition 4.2. *If R is left coherent, then the class $\mathcal{A}$ of absolutely pure modules is coresolving.*

Proof. Suppose R is left coherent and that $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence with $A', A \in \mathcal{A}$. Let $S \subset P$ be a finitely generated submodule of the finitely generated projective module P and let $S \longrightarrow A''$ be linear. By Proposition 2.1, we want to show that $S \longrightarrow A''$ can be extended to $P \longrightarrow A''$. This will make A'' absolutely pure and $\mathcal{A}$ coresolving. This means we want to complete the diagram

$$\begin{tikzcd}
S \arrow[r, phantom, "\subset"] \arrow[d] & P \arrow[dl, dotted] \\
A'' &
\end{tikzcd}$$

to a commutative diagram.

Since R is left coherent, if we have $P' \longrightarrow S \longrightarrow 0$ exact with P' a finitely generated projective module, then $T = \ker(P' \longrightarrow S)$ is also finitely generated. Also $P'/T \cong S$. So this gives

$$\begin{tikzcd}
P' \arrow[r] \arrow[d, dotted] & \frac{P'}{T} \arrow[r, "\cong"] & S \arrow[d] \\
A \arrow[rr] & & A''
\end{tikzcd}$$

Since P' is projective, there is a linear $P' \longrightarrow A$ making the diagram commutative. But then if we suppose that $A' \subset A$, we see that $P' \longrightarrow A$ maps T into A'. So we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since A' is absolutely pure, $T \longrightarrow A'$ can be extended to a linear map $P' \longrightarrow A'$.

$$\begin{tikzcd}
T \arrow[r, phantom, "\subset"] \arrow[d] & P' \arrow[dl, dotted] \\
A' &
\end{tikzcd}$$

Thus, we have two maps $P' \longrightarrow A$ namely, the original map $P' \longrightarrow A$ and now $P' \longrightarrow A' \longrightarrow A$. The difference of the two maps will be 0 on T. Thus it will induce a map $P'/T \longrightarrow A$. Since $P'/T \cong S$, this gives a linear $S \longrightarrow A$ so that

$$\begin{tikzcd} & S \arrow[dl] \arrow[d] & \\ A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

is commutative. Then, since A is absolutely pure, $S \longrightarrow A$ can be extended to $P \longrightarrow A$.

$$\begin{tikzcd} S \arrow[d] & \subset & P \arrow[dll, dotted] \\ A & & \end{tikzcd}$$

The composition $P \longrightarrow A \longrightarrow A''$ gives the desired extension $P \longrightarrow A''$. So A'' is absolutely pure. □

In the next proposition we see that if the class of absolutely pure modules is coresolving, then the quotient by a pure submodule of an absolutely pure module will be absolutely pure. This is a necessary condition to find our absolutely pure precover.

Proposition 4.3. *The class $\mathcal{A}$ of absolutely pure left $R-$modules is coresolving if and only if for every $A \in \mathcal{A}$ and for every pure submodule S of A, A/S is also absolutely pure.*

Proof. Suppose $\mathcal{A}$ is coresolving. Let $A \in \mathcal{A}$ and $S \subset A$ be a pure submodule of A. Now $0 \longrightarrow S \longrightarrow A \longrightarrow \dfrac{A}{S} \longrightarrow 0$ is an exact sequence. Since $S \subset A$ is a pure submodule of an absolutely pure module A, by Proposition 2.2, S is absolutely pure. Therefore, since $\mathcal{A}$ is coresolving and $S, A \in \mathcal{A}$, $A/S \in \mathcal{A}$, as desired.

Conversely, suppose that for every $A \in \mathcal{A}$ and $S \subset A$ a pure submodule of A, A/S

is absolutely pure. Also suppose $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact, with A' and A absolutely pure.

We wish to show that A'' is absolutely pure.

Since A' is absolutely pure, so is $\mathrm{im}(A' \longrightarrow A) \subset A$. So $\mathrm{im}(A' \longrightarrow A) \subset A$ is pure in A. Now

$$\mathrm{coker}(A' \longrightarrow A) \cong A/\mathrm{im}(A' \longrightarrow A) \cong A'',$$

with $\mathrm{im}(A' \longrightarrow A) \subset A$ pure. So by assumption $A/\mathrm{im}(A' \longrightarrow A)$, and hence A'', is absolutely pure. Therefore, since $E \in \mathcal{A}$ for all injective modules E, we have that $\mathcal{A}$ is coresolving. $\square$

Let R be a ring and $\mathcal{F}$ be a class of $R-$modules. Then, for any $R-$module M, a morphism $\phi : C \to M$, where $C \in \mathcal{F}$, is called an $\mathcal{F}$-cover of M if

(i) any diagram with $C' \in \mathcal{F}$

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[dr] & \\ C \arrow[r, "\phi"'] & M \end{tikzcd}$$

can be completed to a commutative diagram and

(ii) the diagram

$$\begin{tikzcd} C \arrow[d, dotted] \arrow[dr] & \\ C \arrow[r, "\phi"'] & M \end{tikzcd}$$

can be completed only by automorphisms of C.

If $\phi : C \to M$ satisfies only (1), then it is called an $\mathcal{F}-$precover.

Note that to get an $\mathcal{F}-$precover one is tempted to form $\bigoplus_{F\in\mathcal{F}} F = G$ then use the evaluation map $G^{Hom(G,M)} \longrightarrow M$. Assuming $\mathcal{F}$ is closed under direct sums, this will be a precover. The problem is that, in general, $\mathcal{F}$ is a class and not a set and so it is not legitimate to form G.

If $\mathcal{A}$ is the class of absolutely pure modules then an $\mathcal{A}-$(pre)cover is just called an absolutely pure (pre)cover.

First we will discuss some results about absolutely pure precovers and covers. Then we will show that the class of absolutely pure modules is covering under certain conditions on the ring R.

Proposition 4.4. *If M_1 and M_2 have absolutely pure precovers, then so does $M_1 \bigoplus M_2$.*

Proof. Let $\psi_1 : A_1 \longrightarrow M_1$ and $\psi_2 : A_2 \longrightarrow M_2$ be absolutely pure precovers of M_1 and M_2 respectively. We claim that $\psi_1 \bigoplus \psi_2 : A_1 \bigoplus A_2 \longrightarrow M_1 \bigoplus M_2$ is an absolutely pure precover. First note that $A_1 \bigoplus A_2$ is absolutely pure, since the direct sum of absolutely pure modules is absolutely pure. Now let C' be absolutely pure. To show that $\psi_1 \bigoplus \psi_2 : A_1 \bigoplus A_2 \longrightarrow M_1 \bigoplus M_2$ is a precover we must show that we can complete the following

$$\begin{tikzcd}
C' \arrow[d, dotted] \arrow[dr] & \\
A_1 \bigoplus A_2 \arrow[r, "\psi_1 \bigoplus \psi_2"'] & M_1 \bigoplus M_2
\end{tikzcd}$$

to a commutative diagram. This is easy to see by considerng the commutative diagram

$$\begin{tikzcd}
 & & C' \arrow[dll, bend right, ""'] \arrow[dl, dotted] \arrow[d, dotted] \arrow[dr, dotted] \arrow[drr, bend left] \arrow[dd, bend right] & & \\
 & A_1 \arrow[r] \arrow[dd] & A_1 \oplus A_2 \arrow[d] & A_2 \arrow[l] \arrow[dd] & \\
 & & M_1 \oplus M_2 \arrow[dl] \arrow[dr] & & \\
 & M_1 & & M_2 &
\end{tikzcd}$$

$\square$

Suppose that M has an injective precover $E \longrightarrow M$. Since E is also absolutely pure, $E \longrightarrow M$ could possibly be an absolutely pure precover.

Let C' be absolutely pure. We wish to show that

$$\begin{tikzcd}
C' \arrow[d, dotted] \arrow[dr] & \\
E \arrow[r] & M
\end{tikzcd}$$

can be completed to a commutative diagram. First note that if $E(C')$ to be the injective envelope of C', and if $C' \longrightarrow M$ can be extended to a map $E(C') \longrightarrow M$, then since ψ is an injective precover, we can complete

$$\begin{tikzcd}
E(C') \arrow[d, dotted] \arrow[dr] & \\
E \arrow[r] & M
\end{tikzcd}$$

to a commutative diagram. This means if we could factor the map $C' \longrightarrow M$ through

$E(C')$, then we could complete the diagram

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[ddr] & \\ E(C') \arrow[d, dotted] \arrow[dr, dotted] & \\ E \arrow[r] & M \end{tikzcd},$$

to a commutative diagram and have $E \longrightarrow M$ an absolutely pure precover.

This can be restated as the following theorem:

Theorem 4.5. *If $E \longrightarrow M$ is an injective precover, then $E \longrightarrow M$ is an absolutely pure precover if an only if for every absolutely pure module A and map $A \longrightarrow M$, there is some factorization $A \longrightarrow \overline{E} \longrightarrow M$ of $A \longrightarrow M$, where $\overline{E}$ is an injective module.*

Proof. Suppose $E \longrightarrow M$ is an absolutely pure precover and A is absolutely pure. Since $E \longrightarrow M$ is an absolutely pure precover,

$$\begin{tikzcd} A \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram. Hence there is a factorization

$$A \longrightarrow \overline{E} \longrightarrow M$$

of $A \longrightarrow M$ with $\overline{E}$ injective, namely $\overline{E} = E$.

Conversely, let C' be absolutely pure. We wish to show that

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram.

By our assumption, there exists an injective module $\overline{E}$ such that $C' \longrightarrow \overline{E} \longrightarrow M$ is a factorization of $C' \longrightarrow M$. Now since $E \longrightarrow M$ is an injective precover there is a map $\overline{E} \longrightarrow E$ that completes

$$\begin{tikzcd} \overline{E} \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

to a commutative diagram. So we have the commutative diagram

$$\begin{tikzcd} C' \arrow[d] \arrow[ddr] & \\ \overline{E} \arrow[d] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

which gives us the completion map $C' \longrightarrow E$ as desired. $\square$

Using the same ideas we get at the next theorem:

Theorem 4.6. *If M has an absolutely pure precover $A \longrightarrow M$ and an injective precover $E \longrightarrow M$ then, in fact, $E \longrightarrow M$ is an absolutely pure precover if and only if*

$$\begin{tikzcd} A \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram.

Proof. If $E \longrightarrow M$ is an absolutely pure precover, then by definition

$$\begin{tikzcd} A \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram.

Conversely, suppose

$$\begin{tikzcd} A \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram. Let C' be absolutely pure. Since $A \longrightarrow M$ is an absolutely pure precover

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[dr] & \\ A \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram. So composing the following two maps

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[dr] & \\ A \arrow[r] \arrow[d] & M \\ E \arrow[ur] & \end{tikzcd}$$

we can complete

$$\begin{tikzcd} C' \arrow[d, dotted] \arrow[dr] & \\ E \arrow[r] & M \end{tikzcd} .$$

to a commutative diagram. Therefore $E \longrightarrow M$ is an absolutely pure precover. $\square$

If $\mathcal{A}$ is a class of left $R-$modules closed under direct sums and if we want to find an $\mathcal{A}-$precover of M, it is tempting to consider

$$A^{(Hom(A,M))} \longrightarrow M$$

(where the map is $(a_f)_f \mapsto \sum f(a_f)$, with $f \in \mathrm{Hom}(A, M))$ and $A \in \mathcal{A}$, since for any $\phi : A \longrightarrow M$ the diagram

$$\begin{tikzcd} & A \arrow[dl, dotted] \arrow[d, "\phi"] \\ A^{(Hom(A,M))} \arrow[r] & M \end{tikzcd}$$

can be completed to a commutative diagram. (We will see the proof of this in Lemma 4.7). Then we could argue

$$\bigoplus_{A \in \mathcal{A}} A^{(Hom(A,M))} \longrightarrow M$$

is an $\mathcal{A}-$precover if $\bigoplus\limits_{A \in \mathcal{A}} A^{(Hom(A,M))}$ "makes sense". The problem is that this might not be a set, since $\mathcal{A}$ is not necessarily a set. So our goal is to modify this construction so that we do get a set. First, we prove the following lemma.

Lemma 4.7. *If $\mathcal{B} \subset \mathcal{A}$, for some set $\mathcal{B}$, is such that any $A \longrightarrow M$, with $A \in \mathcal{A}$, can be factored $A \longrightarrow B \xrightarrow{f} M$ for some $B \in \mathcal{B}$, then M has an $\mathcal{A}$-precover.*

Proof. Claim: $\bigoplus\limits_{B \in \mathcal{B}} B^{(Hom(B,M))} \longrightarrow M$ is an $\mathcal{A}$-precover.
Let $A \in \mathcal{A}$ and $g : A \longrightarrow M$. Then g can be factored into $A \longrightarrow B' \longrightarrow M$. This gives

the diagram

$$\begin{tikzcd}
 & A \arrow[d] \arrow[dd, bend left, "g"] \arrow[ddl, dotted] \\
 & B' \arrow[d, "f"] \\
\bigoplus_{B\in\mathcal{B}} B^{(Hom(B,M))} \arrow[r] & M
\end{tikzcd}$$

Now

$$\begin{tikzcd}
 & B' \arrow[d] \arrow[dl, dotted, "\psi"'] \\
\bigoplus_{B\in\mathcal{B}} B^{(Hom(B,M))} \arrow[r] & M
\end{tikzcd}$$

can be completed to a commutative diagram via the map $\psi : B \longrightarrow B^{(Hom(B,M))}$ (where $\psi(b) = (b_f, 0, 0, ...)$, with $b_f = b$ and where, by abuse of notation, we are letting b_f be the component corresponding to $f \in \mathrm{Hom}(B, M)$).

$$\psi(a+b)_f = a+b = \psi(a)_f + \psi(b)_f$$

$$\psi(a+b)_h = 0 = \psi(a)_h + \psi(b)_h \qquad \text{if} h \neq f$$

Therefore, ψ is a linear map.

To show the diagram is commutative, let $b \in B$. Then we have the maps in the following diagram

$$\begin{tikzcd}
 & b \arrow[dl, mapsto] \arrow[drr, mapsto] & & \\
(b_f, 0, ...) \arrow[rr, mapsto] & & f(b) + 0 + ... = & f(b)
\end{tikzcd}$$

Hence ψ completes the diagram commutatively.

Therefore, the diagram

$$\begin{tikzcd}
& A \arrow[d] \arrow[dd, bend left, "g"] \arrow[ddl, dotted] \\
& B \arrow[d, "f"] \\
\bigoplus\limits_{B\in\mathcal{B}} B^{(Hom(B,M))} \arrow[r] & M
\end{tikzcd}$$

is commutative. Thus, $\bigoplus\limits_{B\subset\mathcal{B}} B^{Hom(B,M)} \longrightarrow M$ is an $\mathcal{A}$-precover. $\square$

Using Lemma 4.7, we see that if we can find a set $\mathcal{B}$ such that every map $A \longrightarrow M$ with A absolutely pure can be factored through some $B \in \mathcal{B}$, then we will have shown that the class of absolutely pure modules is precovering.

In order to do this, we will use a deep result of Robert El Bashir, namely Theorem 5 of [3]. Loosely, this result says that if M is a sufficiently big module and $L \subset M$ a submodule such that M/L is sufficiently small, then L contains a nonzero pure submodule of M. We will use this result in the following way:

Suppose we want to prove that any module M has an absolutely pure precover $A \longrightarrow M$. If we have $A \longrightarrow M$ and $B \longrightarrow M$ and you want a $C \longrightarrow M$ (with A, B, and C absolutely pure), so that $A \longrightarrow M$ and $B \longrightarrow M$ factor through $C \longrightarrow M$, we can just take $C = A \bigoplus B$ and then take the sum of the maps $A \longrightarrow M$ and $B \longrightarrow M$ to get $C = A \bigoplus B \longrightarrow M$.

So we are tempted to take all morphisms $A \longrightarrow M$ with A an absolutely pure module and then form the sum $\bigoplus A \longrightarrow M$ to get a precover. But we have the usual problem of having $\bigoplus A$ a set. Here is where we can use El Bashir's result to get a set of $A \longrightarrow M$'s so that $\bigoplus A \longrightarrow M$ will be a precover in a special situation.

More precisely, El Bashir [3] says that given R and given a cardinal λ, there is a cardinal κ so that if $|M| \geq \kappa$ and $|M/L| \leq \lambda$ then L contains a nonzero pure submodule of M.

The following lemma takes care of not having a set.

Lemma 4.8. *Let R be left coherent and let $|M| = \lambda$ for an $R-$module M. Let κ be as in El Bashir's result. Then any map $A \longrightarrow M$ with A absolutely pure can be factored through an absolutely pure module B with $|B| < \kappa$.*

Proof. Take any map $A \longrightarrow M$, with A sufficiently large and let K be the kernel. Then A/K is sufficiently small, since $|A/K| \leq |M|$. So by [3], K has a nonzero submodule L that is pure in A. Now since R is left coherent, the set of absolutely pure modules is coresolving, by Lemma 4.2. So by Proposition 4.3, A/L is absolutely pure. This module may still be too large.

If so, repeating the process, we have a map $A/L \longrightarrow M$. If A/L is still sufficiently large, then take K_1/L to be the kernel of $A/L \longrightarrow M$. Again, A/K_1 is sufficiently small, so K_1/L has a nonzero submodule L_1/K_1 that is pure in A/L. So $\frac{A}{K_1}/\frac{L_1}{K_1} = A/L_1$ is absolutely pure. But again, this may be too large.

Continuing the process we ultimately arrive at $\varinjlim A/L_i$. This is absolutely pure since R is left coherent, by Proposition 2.4, and is sufficiently small, namely $\left|\varinjlim \dfrac{A}{L_i}\right| \leq \kappa$. $\square$

Using Lemma 4.8, we form a set from which we will can make an absolutely pure precover. This set is like the one described in Lemma 4.7 in that we will be able to factor every map $A \longrightarrow M$, with A absolutely pure, through some element B of our set.

Theorem 4.9. *If R is coherent, every left $R-$module M has an absolutely pure precover.*

Proof. Choose any $R-$module M. Take any set X of cardinality κ, where κ is the cardinal in El Bashir's Theorem [3]. Form the set of all subsets of X. For each of these subsets, find all the binary operations on them. This is simply the set of functions from the cross product into itself, so it remains a set.

Now on this new collection $\bigcup_{\overline{B} \subset X} \{\overline{B}, *\} = \overline{\mathcal{B}}$ find all the scalar multiplications, which are functions from the cross product into itself. This again remains a set $\bigcup_{(\overline{B}, *) \in \overline{\mathcal{B}}} \{(\overline{B}, *, \cdot)\}$.

Some of these form modules, so choose from these the modules that are absolutely pure modules. This forms a set $\mathcal{B}$. Therefore $\bigoplus_{B \in \mathcal{B}} B^{(Hom(B,M))}$ is a well defined set.

Claim: $\bigoplus B^{(Hom(B,M))} \longrightarrow M$ is an absolutely pure precover.

Take any map $A \longrightarrow M$ with A absolutely pure. We wish to complete the following

$$\begin{tikzcd}
A \arrow[d, dotted] \arrow[dr] & \\
\bigoplus_{B \in \mathcal{B}} B^{(Hom(B,M))} \arrow[r] & M
\end{tikzcd}$$

to a commutative diagram.

By Lemma 4.8, $A \longrightarrow M$ can be factored through an absolutely pure module that is isomorphic to one in the above set. Hence, by Lemma 4.7, the above diagram can be completed to a commutative diagram. Therefore, $\bigoplus B^{(Hom(B,M))} \longrightarrow M$ is an absolutely pure precover $\square$

It is known that if $\mathcal{F}$ is a class of modules which is closed under taking direct limits, then if a module M has an $\mathcal{F}-$precover it has an $\mathcal{F}-$cover [8]. If R is right coherent, then the class $\mathcal{A}$ of absolutely pure left $R-$modules is closed under direct limits. Hence, in fact, every M has an absolutely pure cover when R is coherent. It is an open question whether R must be coherent in order for this to happen.

Chapter 5

Derived Functors using Absolutely Pure Resolutions

Throughout this chapter we define derived functors using absolutely pure resolutions and show that these functors are well defined. Using these functors and the functors obtained from injective resolutions we get a natural map between them. We will discuss conditions on the modules and the ring R that give certain properties about these natural maps.

An injective resolution of a left $R-$module M is an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

where the E^n's are injective modules. This implies that if we apply the functor $\operatorname{Hom}(-, E)$ to this exact sequence we still get an exact sequence if E is injective.

We want to argue that we get an analogous sequence using absolutely pure modules instead of injective modules. We need the result that every module over any ring has an absolutely pure preenvelope (see D. Adams [1]). This fact can also be deduced from El Bashir's Theorem [3].

Theorem 5.1. *Given an $R-$module N there is an exact sequence*

$$0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

with the A^i absolutely pure, which remains exact if we apply any functor $Hom(-, A)$ where A is absolutely pure (such a sequence will be called an absolutely pure resolution of N).

Proof. Given an $R-$module N, take A^o to be an absolutely pure preenvelope of N. Since A^o and A are absolutely pure, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & A^o \arrow[dl, dotted] \\
& A &
\end{tikzcd}$$

so $\mathrm{Hom}(A^o, A) \to \mathrm{Hom}(N, A) \to 0$ is exact.

Now we have the exact sequence $0 \to N \to A^o \to C^1 \to 0$. Take A^1 to be an absolutely pure preenvelope of C^1 which gives the commutative diagram

$$\begin{tikzcd}
& & & C^1 \arrow[dr] \arrow[ddl] & \\
0 \arrow[r] & N \arrow[r] & A^o \arrow[ur] \arrow[rr] \arrow[d] & & A^1 \arrow[dll, dotted] \\
& & A & &
\end{tikzcd}$$

So $\mathrm{Hom}(A^1, A) \to \mathrm{Hom}(A^o, A) \to \mathrm{Hom}(N, A) \to 0$ is exact. Continuing this procedure gives

$$\cdots \to \mathrm{Hom}(A^2, A) \to \mathrm{Hom}(A^1, A) \to \mathrm{Hom}(A^o, A) \to \mathrm{Hom}(N, A) \to 0$$

an exact sequence. Therefore, $0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$ is an absolutely pure resolution and $\mathrm{Hom}(-, A)$ is exact. $\square$

By the method described above, we know that we can get an absolutely pure resolution $0 \longrightarrow N \to A^0 \to A^1 \to$ for any $R-$module N. Using a similar argument to that for injective modules, we can show that this complex is unique up to homotopy. This leads us to new derived functors, which are well defined. We will call these $\mathrm{Axt}^n_R(M, N)$ (using Axt for absolutely pure, instead of Ext, as used for injective modules.)

Theorem 5.2. *The $Axt^n_R(M, N)$ are well defined.*

Proof. Take two different absolutely pure resolutions and a map $\phi \in \mathrm{Hom}(N, \overline{N})$. We need to show that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\pi"] \arrow[d, "\phi"] & A^o \arrow[r, "d_1"] \arrow[d, dotted, "\phi_o"] & A^1 \arrow[r, "d_2"] \arrow[d, dotted, "\phi_1"'] & \cdots \arrow[r] & A^n \arrow[r, "d_{n+1}"] \arrow[d, dotted, "\phi_n"'] & A^{n+1} \arrow[r] \arrow[d, dotted, "\phi_{n+1}"] & \cdots \\
0 \arrow[r] & \overline{N} \arrow[r, "\pi"] & \overline{A^o} \arrow[r, "\overline{d_1}"] & \overline{A^1} \arrow[r, "\overline{d_2}"] & \cdots \arrow[r] & \overline{A^n} \arrow[r, "\overline{d_{n+1}}"] & \overline{A^{n+1}} \arrow[r] & \cdots
\end{tikzcd}$$

and that the associated map of absolutely pure resolutions is unique up to homotopy.

A^o is a preenvelope, so there exists a $\phi_o : A^0 \to \overline{A^o}$ which makes the following diagram commutative.

$$\begin{tikzcd}
N \arrow[r] \arrow[d, "\phi"'] & A^o \arrow[ddl, dotted, "\phi_o"] \\
\overline{N} \arrow[d] & \\
\overline{A^o} &
\end{tikzcd}$$

Next, use ϕ_o to find ϕ_1. We have the following two commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\phi"] & A^o \arrow[r] \arrow[d, "\phi_o"] & C^1 \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \overline{N} \arrow[r] & \overline{A^o} \arrow[r] & \overline{C^1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & C^1 \arrow[r] \arrow[d, "\psi"] & A^1 \arrow[d, dotted, "\phi_1"] \\
0 \arrow[r] & \overline{C^1} \arrow[r] & \overline{A^1}
\end{tikzcd}$$

and A^1 is a preenvelope of C^1 so there exists a $\phi_1 : A^1 \to \overline{A^1}$.

Then assume that $\phi_o, \ldots, \phi_{n-1}$ are defined. Complete the following diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\phi"] & A^o \arrow[r] \arrow[d, "\phi_o"] & A^1 \arrow[r] \arrow[d, "\phi_1"'] & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"'] & C^n \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \overline{N} \arrow[r] & \overline{A^o} \arrow[r] & \overline{A^1} \arrow[r] & \cdots \arrow[r] & \overline{A^{n-1}} \arrow[r] & \overline{C^n} \arrow[r] & 0
\end{tikzcd}$$

to get a ψ which makes this commutative and since A^n is a preenvelope of C^n we have a $\phi_n : A^n \to \overline{A^n}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^n \arrow[r] \arrow[d, "\psi"] & A^n \arrow[d, dotted, "\phi_n"] \\
0 \arrow[r] & \overline{C^n} \arrow[r] & \overline{A^n}
\end{tikzcd}$$

commutative. This tells us that we can complete the diagram.

We now argue uniqueness up to homotopy. This means that given the following diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook, "d_0"] \arrow[d, "id"'] & A^0 \arrow[r, "d_1"] \arrow[d, shift right, "\overline{\phi_0}"'] \arrow[d, shift left, "\phi_0"] \arrow[dl, dotted, "s_{-1}"'] & A^1 \arrow[r] \arrow[d, shift right, "\overline{\phi_1}"'] \arrow[d, shift left, "\phi_1"] \arrow[dl, dotted, "s_0"'] & \cdots \arrow[r, "d_n"] & A^n \arrow[r, "d_{n+1}"] \arrow[d, shift right, "\overline{\phi_n}"'] \arrow[d, shift left, "\phi_n"] \arrow[dl, dotted, "s_{n-1}"'] & A^{n+1} \arrow[r] \arrow[d, shift right, "\overline{\phi_{n+1}}"'] \arrow[d, shift left, "\phi_{n+1}"] \arrow[dl, dotted, "s_n"'] & \cdots \\
0 \arrow[r] & N \arrow[r, hook, "\overline{d_0}"'] & \overline{A^0} \arrow[r, "\overline{d_1}"'] & \overline{A^1} \arrow[r] & \cdots \arrow[r, "\overline{d_n}"'] & \overline{A^n} \arrow[r, "\overline{d_{n+1}}"'] & \overline{A^{n+1}} \arrow[r] & \cdots
\end{tikzcd}$$

we can find $s_0, \ldots, s_n, \ldots$, with $s_n : A^{n+1} \to \overline{A^n}$, such that $\phi_n - \overline{\phi_n} = \overline{d_n} \circ s_{n-1} + s_n \circ d_{n+1}$, where $s_{-1} = 0$.

We know that $\phi_0 \circ d_0 = \overline{d_0} = \overline{\phi_o} \circ d_0$, so $(\phi_0 - \overline{\phi_0})d_0 = 0$. Therefore we have the diagram

$$\begin{tikzcd}
N \arrow[r, "d_0"] & A^0 \arrow[r] \arrow[d, "\phi_0 - \overline{\phi_0}"'] & \frac{A^0}{N} \arrow[r, hook] \arrow[d] & A^1 \arrow[dl, dotted] \\
& \overline{A^0} \arrow[r] & \frac{\overline{A^0}}{N} &
\end{tikzcd}$$

which can be completed since A^1 is an absolutely pure preenvelope. Call this map s_0, which gives us $\phi_0 - \overline{\phi_0} = s_0 \circ d_1$.

The next step is to create an s_1 which will complete the following diagram commutatively.

$$\begin{tikzcd}
N \arrow[r] & A^0 \arrow[r, "d_1"] \arrow[d, "\phi_0 - \overline{\phi_0}"'] & A^1 \arrow[r, "d_2"] \arrow[d, "\phi_1 - \overline{\phi_1}"'] \arrow[dl, "s_0"'] & A^2 \arrow[dl, dotted] \\
& \overline{A^0} \arrow[r, "\overline{d_1}"'] & \overline{A^1} \arrow[r, "\overline{d_2}"'] & \overline{A^2}
\end{tikzcd}$$

We need a map which is 0 on A^0. Namely, let s_1 be the map which completes the following diagram

$$\begin{tikzcd}
\dfrac{A^1}{\mathrm{im}(A^0 \to A^1)} \arrow[r, hook, "d_2"] \arrow[d, "\phi_1 - \overline{\phi_1} - \overline{d_1} s_0"'] & A^2 \arrow[dl, dotted, "s_1"] \\
\overline{A^1} &
\end{tikzcd}$$

Therefore we have that

$$\begin{aligned}
(\phi_1 - \overline{\phi_1} - \overline{d_1} s_0) d_1 &= (\phi_1 - \overline{\phi_1}) d_1 - (\overline{d_1} s_0) d_1 \\
&= (\phi_1 - \overline{\phi_1}) d_1 - \overline{d_1}(\phi_0 - \overline{\phi_0}) \\
&= 0,
\end{aligned}$$

as desired.

Now suppose that $s_0, \ldots, s_{n-1}$ are determined. Define s_n as the completion of the following diagram

$$\begin{tikzcd}
\dfrac{A^n}{\mathrm{im}(A^{n-1} \to A^n)} \arrow[r, hook, "d_{n+1}"] \arrow[d, "\phi_n - \overline{\phi_n} - \overline{d_n} s_n"'] & A^{n+1} \arrow[dl, dotted, "s_n"] \\
\overline{A^n} &
\end{tikzcd}$$

This gives the commutative diagram

$$\begin{tikzcd}
 & A^{n-1} \arrow[r, "d_n"] \arrow[dl, "S_{n-2}"'] \arrow[d, "\phi_{n-1} - \overline{\phi_{n-1}}"'] & A^n \arrow[r, "d_{n+1}"] \arrow[dl, "S_{n-1}"'] \arrow[d, "\phi_n - \overline{\phi_n}"'] & A^{n+1} \arrow[dl, dotted, "S_n"] \\
\overline{A^{n-2}} \arrow[r, "\overline{d_{n-1}}"'] & \overline{A^{n-1}} \arrow[r, "\overline{d_n}"'] & \overline{A^n} &
\end{tikzcd}$$

Now, as desired, we have

$$\begin{aligned}(\phi_n - \overline{\phi_n} - \overline{d_n}s_{n-1})d_n &= (\phi_n - \overline{\phi_n})d_n - \overline{d_n}(s_{n-1}d_n) \\ &= (\phi_n - \overline{\phi_n})d_n - \overline{d_n}(\phi_{n-1} - \overline{\phi_{n-1}} - \overline{d_{n-1}}s_{n-1}) \\ &= (\phi_n - \overline{\phi_n})d_n - \overline{d_n}(\phi_{n-1} - \overline{\phi_{n-1}}) + \overline{d_n d_{n-1}}s_{n-1} \\ &= 0 + 0,\end{aligned}$$

since our diagram has exact rows. Then the similar argument for that of injective modules gives that the process of proving the choice of maps and then of absolutely pure resolutions is unique up to homotopy. □

Computing the homology groups of this absolutely pure resolution gives a well defined derived functor which we will call $\mathrm{Axt}^n(M, N)$.

Suppose that we have an absolutely pure resolution

$$0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots$$

of N. Suppose further that we have an injective resolution

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

of N. We would like to complete the following diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d, dotted] & A^1 \arrow[r] \arrow[d, dotted] & {} \\ 0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & {} \end{tikzcd}$$

to a commutative diagram. Define ϕ_0 as the completion of the diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & A^0 \arrow[dl, dotted, "\phi_0"] \\ E^0 & \end{tikzcd}$$

which we have since E^0 is injective. Now define ϕ_1 as the completion of

$$\begin{tikzcd} A^0 \arrow[r] \arrow[d] & A^1 \arrow[ddl, dotted, "\phi_1"] \\ E^0 \arrow[d] & \\ E^1 & \end{tikzcd}$$

which we know exists again, since E^1 is injective. Continuing in this manner and using the same technique from the proof of Theorem 5.2, we can complete the diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d, "\phi_0"] & A^1 \arrow[r] \arrow[d, "\phi_1"] & {} \\ 0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & {} \end{tikzcd}$$

to a commutative diagram uniquely, up to homotopy.

Now applying $\operatorname{Hom}(M, -)$ to the diagram

$$\begin{tikzcd} 0 \arrow[r] & A^0 \arrow[r] \arrow[d, "\phi_0"] & A^1 \arrow[r] \arrow[d, "\phi_1"] & {} \\ 0 \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & {} \end{tikzcd}$$

gives natural maps $\operatorname{Axt}^n_R(M, N) \longrightarrow \operatorname{Ext}^n_R(M, N)$ for all $n \geq 0$.

Using these natural maps, the questions that we can ask are for example: when are

$$\mathrm{Axt}_R^n(M,N) \longrightarrow \mathrm{Ext}_R^n(M,N)$$

isomorphisms for all $n \geq 0$?, when are they 0?

Proposition 5.3. *$Axt^0(M,N) \cong Hom(M,N)$*

Proof. Given the absolutely pure resolution

$$0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

of N, we know that the cohomology groups of the complex

$$0 \longrightarrow \mathrm{Hom}(M,A^0) \longrightarrow \mathrm{Hom}(M,A^1) \longrightarrow \cdots$$

give us the groups $\mathrm{Axt}_R^n(M,N)$.

So $\mathrm{Axt}_R^0(M,N)$ is the kernel of $\mathrm{Hom}(M,A^0) \longrightarrow \mathrm{Hom}(M,A^1)$. But the functor $\mathrm{Hom}(M,-)$ is left exact. So the exactness of

$$0 \longrightarrow N \longrightarrow A^0 \longrightarrow A^1$$

give an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(M,A^0) \longrightarrow \mathrm{Hom}(M,A^1).$$

Now we see that $\mathrm{Hom}(M,N)$ is isomorphic to the kernel of this map, that is $\mathrm{Hom}(M,N) \cong \mathrm{Axt}_R^0(M,N)$, as desired.

$\square$

Using Proposition 5.3, it is easy to see that $\mathrm{Axt}^0(M,N) \longrightarrow \mathrm{Ext}^0(M,N)$ is always an isomorphism, since $\mathrm{Axt}^0(M,N) \cong \mathrm{Hom}(M,N) \cong \mathrm{Ext}^0(M,N)$.

The following result is a"global" answer to when Axt $\longrightarrow$ Ext is an isomorphism.

Theorem 5.4. *The following conditions are equivalent:*

(i) *R is left noetherian.*

(ii) *$Axt^n(M,N) \longrightarrow Ext^n(M,N)$ is an isomorphism for all n, M, and N.*

(iii) *$Axt^1(M,N) \longrightarrow Ext^1(M,N)$ is an isomorphism for all M and N.*

Proof. (i) $\Rightarrow$ (ii) since if R is left noetherian, the class of absolutely pure modules is equal to the class of injective modules.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$(i) Let $N = A$ be absolutely pure. Then our absolutely pure resolution looks like

$$0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

(i.e. $A^1 = 0$).

Hence, $\mathrm{Axt}^1(M,A) = 0$. So by the assumption $\mathrm{Ext}^1(M,A) \cong \mathrm{Axt}^1(M,A) = 0$ for all M, A is also injective. Therefore, all absolutely pure modules are injective and using Theorem 2.6, R is left noetherian.

$\square$

Given a left $R-$module M and a family $(N_i)_{i\in I}$ of left $R-$modules, there is a natural map $\bigoplus_{i\in I} \mathrm{Hom}(M,N_i) \longrightarrow \mathrm{Hom}(M, \bigoplus_{i\in I} N_i)$.

This is not an isomorphism in general. For example, let I be infinite and each $N_i \neq 0$. Then if $M = \bigoplus\limits_{i \in I} N_i$, we see that id_M is not in the image of

$$\bigoplus_{i \in I} \mathrm{Hom}(M, N_i) \longrightarrow \mathrm{Hom}(M, \bigoplus_{i \in I} N_i)$$

But it is an isomorphism if M is finitely generated.

Now for any $j \in I$, the embedding $N_j \longrightarrow \bigoplus\limits_{i \in I} N_i$ also gives maps $\mathrm{Ext}^n(M, N_j) \longrightarrow \mathrm{Ext}^n(M, \bigoplus\limits_{i \in I} N_i)$. But these give us a map

$$\bigoplus_{i \in I} \mathrm{Ext}^n(M, N_i) \longrightarrow \mathrm{Ext}^n(M, \bigoplus_{i \in I} N_i)$$

and if $n = 0$ this is the map above with Hom.

Even if M is finitely generated, this map might not be an isomorphism for $n > 0$. The reason is that the direct sum of injective resolutions of the individual N_i will not necessarily be an injective resolution of $\bigoplus\limits_{i \in I} N_i$. So using the notation

$$0 \longrightarrow N \longrightarrow E^0(N) \longrightarrow E^1(N) \longrightarrow \cdots$$

for a minimal injective resolution of N, we get

$$0 \longrightarrow \bigoplus_{i \in I} N_i \longrightarrow \bigoplus_{i \in I} E^0(N_i) \longrightarrow \bigoplus_{i \in I} E^1(N_i) \longrightarrow \cdots .$$

This sequence is exact, but the terms $\bigoplus\limits_{i \in I} E^n(N_i)$ might not be injective. With these ideas we can prove the following theorem:

Theorem 5.5. *If M is finitely generated and $(N_i)_{i\in I}$ is any family, then the map*

$$\bigoplus_{i\in I} Axt^n(M, N_i) \longrightarrow Axt^n(M, \bigoplus_{i\in I} N_i)$$

is an isomorphism, for any $n \geq 0$.

Proof. Since M is finitely generated, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M, \bigoplus A_i^0) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(M, \bigoplus A_i^1) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \bigoplus \mathrm{Hom}(M, A_i^0) \arrow[r] & \bigoplus \mathrm{Hom}(M, A_i^1) \arrow[r] & \cdots
\end{tikzcd}$$

Computing the cohomology of a direct sum (i.e. finding $\mathrm{Axt}^n(M, \oplus N_i)$ is the same as taking the direct sum of the cohomology of

$$0 \longrightarrow \mathrm{Hom}(M, A_j^0) \longrightarrow \mathrm{Hom}(M, A_j^1) \longrightarrow \cdots$$

(i.e. finding $\oplus \mathrm{Axt}^n(M, N_j)$).

$\square$

From this, we can add the following condition to Theorem 5.4:

Theorem 5.6. *The following conditions are equivalent:*

(i) *R is left noetherian.*

(ii) *$Axt^n(M, N) \longrightarrow Ext^n(M, N)$ is an isomorphism for all n, M, and N.*

(iii) *$Axt^1(M, N) \longrightarrow Ext^1(M, N)$ is an isomorphism for all M and N.*

(iv) $\bigoplus_{i\in I} Ext^1(M,N_i) \longrightarrow Ext^1(M,\bigoplus_{i\in I} N_i)$ *is an isomorphism for all finitely generated* M *and any family* $(N_i)_{i\in I}$.

Proof. (iii) $\Rightarrow$ (iv) Assume $\mathrm{Axt}^1(M,N) \cong \mathrm{Ext}^1(M,N)$ for all M,N. Then

$$\begin{aligned}\bigoplus_{i\in I}\mathrm{Ext}^1(M,N_i) &\cong \bigoplus_{i\in I}\mathrm{Axt}^1(M,N_i) &(5.1)\\ &\cong \mathrm{Axt}^1(M,\bigoplus_{i\in I}N_i) &(5.2)\\ &\cong \mathrm{Ext}^1(M,\bigoplus_{i\in I}N_i) &(5.3)\end{aligned}$$

The isomorphism in line (5.1) and (5.3) is given by our assumption. Line (5.2) is an isomorphism by Theorem 5.4.

(iv) $\Rightarrow$ (i) Assume $\bigoplus_{i\in I}\mathrm{Ext}^1(M,N_i) \longrightarrow \mathrm{Ext}^1(M,\bigoplus_{i\in I}N_i)$ is an isomorphism for all finitely generated M and any family $(N_i)_{i\in I}$. Take $(N_i)_{i\in I}$ to be a family of injective modules. Then $\mathrm{Ext}^1(M,N_i)=0$ for all i. So

$$\mathrm{Ext}^1(M,\bigoplus_{i\in I}N_i) \cong \bigoplus_{i\in I}\mathrm{Ext}^1(M,N_i)=0$$

Hence, $\bigoplus_{i\in I}N_i$ is injective. Therefore, since the direct sum of injective modules is injective, R is left noetherian.

□

Our version of the "local" question is as follows: For which M is the functor $\mathrm{Axt}^1(M,-) \longrightarrow \mathrm{Ext}^1(M,-)$ an isomorphism of functors? (i.e. When is it true that for a given M, $\mathrm{Axt}^1(M,N) \longrightarrow \mathrm{Ext}^1(M,N)$ is an isomorphism for all N?) Then there is the analogous question for N. (i.e. For which N is $\mathrm{Axt}^1(-,N) \longrightarrow \mathrm{Ext}^1(-,N)$ an isomorphism of functors?) We will only consider the M question, i.e., M is fixed.

Note that to compute $\mathrm{Axt}^1(M,N)$ and $\mathrm{Ext}^1(M,N)$, we would need to consider $0 \longrightarrow N \longrightarrow \overline{A} \longrightarrow C \longrightarrow 0$, with $N \longrightarrow \overline{A}$ an absolutely pure preenvelope, and then $0 \longrightarrow N \longrightarrow E \longrightarrow D \longrightarrow 0$, with $N \longrightarrow E$ an injective preenvelope. This just means that $N \longrightarrow E$ is an injection with E injective. This leads to the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "="] & \overline{A} \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \arrow[r] & E \arrow[r] & D \arrow[r] & 0 \end{tikzcd}$$

From here, we can easily see that $N \longrightarrow \overline{A} \bigoplus E$ is an absolutely pure preenvelope. Just use the fact that the two diagrams

$$\begin{tikzcd} N \arrow[r] \arrow[d] & \overline{A} \arrow[dl, dotted] \\ \overline{A} & \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r] \arrow[d] & E \arrow[dl, dotted] \\ \overline{A} & \end{tikzcd}$$

can be completed to commutative diagrams. Therefore, if we add the two completion maps together, we will be able to complete

$$\begin{tikzcd} N \arrow[r] \arrow[d] & \overline{A} \bigoplus E \arrow[dl, dotted] \\ \overline{A} & \end{tikzcd}$$

to a commutative diagram.

Note that when we chose an absolutely pure preenvelope of N, we could have chosen it to be $\overline{A} \bigoplus E$, since $\overline{A}$ and E are both absolutely pure modules and the direct sum of absolutely pure modules is absolutely pure. This would make our commutative diagram look like

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "="] & A = \overline{A} \bigoplus E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

and so $A \longrightarrow E$ is a surjection. So, we see we can assume $A \longrightarrow E$ is a surjective map.

Proposition 5.7. *If $A \longrightarrow E$ is a surjection and $C \longrightarrow D$ is a surjection, then $ker(A \longrightarrow E) \longrightarrow ker(C \longrightarrow D)$ is an isomorphism.*

Proof. Let $x \in \ker(C \longrightarrow D)$. Then $x \in C$ implies there exists an $a \in A$, such that $a \mapsto x$. Now $x \in \ker(C \longrightarrow D)$, so $x \mapsto 0$ and the following diagram is commutative

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[d] \\
E \arrow[r] & D
\end{tikzcd}$$

Therefore, going clockwise $a \mapsto x \mapsto 0$ and the same happens going counterclockwise. This means $a \in \ker(A \longrightarrow E)$ and $\ker(A \longrightarrow E) \longrightarrow \ker(C \longrightarrow D)$ is a surjection. Hence, $\ker(A \longrightarrow E) \longrightarrow \ker(C \longrightarrow D)$ is an isomorphism. $\square$

We first show that $\text{Axt}^1(M,N) \longrightarrow \text{Ext}^1(M,N)$ is an injection for any ring R, but note that it is not always an isomorphism. Then we show that under certain conditions, namely if $\text{Ext}^1(M,A) = 0$ for all absolutely pure modules A, then $\text{Axt}^1(M,N) \longrightarrow \text{Ext}^1(M,N)$ is an isomorphism.

Proposition 5.8. *$Axt^1(M,N) \longrightarrow Ext^1(M,N)$ is injective for every ring.*

Proof. By the way we compute $\mathrm{Axt}^1(M,N)$ and $\mathrm{Ext}^1(M,N)$, if we show that whenever

$$\begin{tikzcd}
 & M \arrow[d] \arrow[ddl, dotted] \\
 & C \arrow[d] \\
E^0 \arrow[r] & D
\end{tikzcd}$$

can be completed to a commutative diagram, then

$$\begin{tikzcd}
 & M \arrow[d] \arrow[dl, dotted] \\
A^0 \arrow[r] & C
\end{tikzcd}$$

can also be completed, we will show that our map is an injection. To see this, look at the following commutative diagram with exact rows.

$$\begin{tikzcd}
 & & & M \arrow[d] \arrow[dl, dotted] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, "="] & A^0 \arrow[r] \arrow[d, hook] & C = \dfrac{A^0}{N} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & N \arrow[r] & E^0 \arrow[r] & D = \dfrac{E^0}{N} \arrow[r] & 0
\end{tikzcd}$$

From here, we see that the image of $M \longrightarrow E^0$ is contained inside of A^0 and so the diagram

$$\begin{tikzcd}
 & M \arrow[d] \arrow[dl, dotted] \\
A^0 \arrow[r] & C
\end{tikzcd}$$

can be completed as desired. □

Theorem 5.9. *For a given M, $Axt^1(M,N) \longrightarrow Ext^1(M,N)$ is an isomorphism for all*

N *if and only if* $Ext^1(M, A) = 0$ *for all absolutely pure* A*'s.*

Proof. Recall that for any M, $\mathrm{Axt}^1(M, A) = 0$ for all absolutely pure modules A. So if $\mathrm{Axt}^1(M, N) \longrightarrow \mathrm{Ext}^1(M, N)$ is an isomorphism for all N. Then for all absolutely pure modules A, $\mathrm{Ext}^1(M, A) \cong \mathrm{Axt}^1(M, A) = 0$, as desired.

Conversely, suppose that $\mathrm{Ext}^1(M, A) = 0$ for all absolutely pure modules A. Recall from Theorem 5.1, that when we compute an absolutely pure resolution of N, we begin with the short exact sequence

$$0 \longrightarrow N \longrightarrow A^0 \longrightarrow C^1 \longrightarrow 0.$$

From here we get two long exact sequences, which give the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(M, N) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(M, A) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^1(M, N) \arrow[r] & \mathrm{Ext}^1(M, A) = 0 \\
\mathrm{Hom}(M, N) \arrow[r] & \mathrm{Hom}(M, A) \arrow[r] & \mathrm{Axt}^1(M, N) \arrow[r] & \mathrm{Axt}^1(M, A) = 0
\end{tikzcd}$$

Therefore, $\mathrm{Axt}^1(M, N) \cong \mathrm{Ext}^1(M, N)$ for all N.

$\square$

Recall that a module A is absolutely pure if and only if $\mathrm{Ext}^1(M, A) = 0$ for all finitely presented M. It is easy to argue that there is a set of representatives of such M. So using the notation and terminology of Eklof and Trlifaj [6], we see that if $\mathcal{S}$ is such a set of finitely presented modules, then $S^\perp = \mathcal{A}$ where $\mathcal{A}$ is the class of absolutely pure modules. Then by a main result of Eklof and Trlifaj, we get that $(\mathcal{A}^\perp, \mathcal{A})$ is a complete (or in other words "has enough injectives and projectives") cotorsion theory (for more on this see [8], Section 7.4).

If we let $\mathcal{E}$ be the class of injective modules, then $\mathcal{E}^{\perp} = \mathcal{M}$ with $\mathcal{M}$ all modules. Then $(\mathcal{M}, \mathcal{E})$ is another complete cotorsion theory. Since $\mathcal{E} \subset \mathcal{A}$, the two theories are comparable. In this situation A. Iacob [11] has proved that there exist generalized Tate cohomology groups $\widehat{\mathrm{Ext}}^n(M, N)$ and Avramov-Martsinkovsky sequences which in our situation are

$$0 \longrightarrow \mathrm{Axt}^1(M, N) \longrightarrow \mathrm{Ext}^1(M, N) \longrightarrow \widehat{\mathrm{Ext}}^1(M, N) \longrightarrow \mathrm{Axt}^2(M, N) \longrightarrow \cdots$$

So we note then that when R is left noetherian each $\mathrm{Axt}^n(M, N) \longrightarrow \mathrm{Ext}^n(M, N)$, $n \geq 1$, is an isomorphism, by Theorem 5.4. We then get that $\widehat{\mathrm{Ext}}^n(M, N) = 0$ for all $n \geq 1$ if and only if R is left noetherian.

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