2011

THE APPLICATION OF DISCONTINUOUS GALKERIN FINITE ELEMENT TIME-DOMAIN METHOD IN THE DESIGN, SIMULATION AND ANALYSIS OF MODERN RADIO FREQUENCY SYSTEMS

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Recommended Citation
Zhao, Bo, "THE APPLICATION OF DISCONTINUOUS GALKERIN FINITE ELEMENT TIME-DOMAIN METHOD IN THE DESIGN, SIMULATION AND ANALYSIS OF MODERN RADIO FREQUENCY SYSTEMS" (2011). University of Kentucky Doctoral Dissertations. 186.
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ABSTRACT OF DISSERTATION

Bo Zhao

The Graduate School
University of Kentucky
2011
THE APPLICATION OF DISCONTINUOUS GALERIN FINITE ELEMENT TIME-DOMAIN METHOD IN THE DESIGN, SIMULATION AND ANALYSIS OF MODERN RADIO FREQUENCY SYSTEMS

ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering in the College of Engineering at the University of Kentucky

By

Bo Zhao

Lexington, Kentucky

Director: Dr. Stephen D. Gedney, Professor of Electrical Engineering

Lexington, Kentucky

2011

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ABSTRACT OF DISSERTATION

THE APPLICATION OF DISCONTINUOUS GALERKIN FINITE ELEMENT TIME-DOMAIN METHOD IN THE DESIGN, SIMULATION AND ANALYSIS OF MODERN RADIO FREQUENCY SYSTEMS

The discontinuous Galerkin finite element time-domain (DGFETD) method has been successfully applied to the solution of the coupled curl Maxwell’s equations. In this dissertation, important extensions to the DGFETD method are provided, including the ability to model lumped circuit elements and the ability to model thin-wire structures within a discrete DGFETD solution. To this end, a hybrid DGFETD/SPICE formulation is proposed for high-frequency circuit simulation, and a hybrid DGFETD/Thin-wire formulation is proposed for modeling thin-wire structures within a three-dimensional problem space. To aid in the efficient modeling of open-region structures, a Complex Frequency Shifted-Perfectly Matched Layer (CFS-PML) absorbing medium is applied to the DGFETD method for the first time. An efficient CFS-PML method that reduces the computational complexity and improves accuracy as compared to previous PML formulations is proposed. The methods have been successfully implemented, and a number of test cases are provided that validate the proposed methods. The proposed hybrid formulations and the new CFS-PML formulation dramatically enhances the ability of the DGFETD method to be efficiently applied to simulate complex, state of the art radio frequency systems.

KEYWORDS: Discontinuous Galerkin, Finite-Element Time-Domain, Circuit Modeling, Thin Wire Modeling, Perfectly Matched Layer

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July 27 2011
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July 27, 2011
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ACKNOWLEDGMENTS

The following dissertation, while an independent work, holds far more than the culmination of years of study. These pages also reflect the relationships with many generous and inspiring people I have met since beginning my graduate work. The list is long, but I cherish each contribution to my development as a Ph.D. student:

To my advisor Dr. Stephen D. Gedney, a gracious mentor who demonstrates that rigorous scholarship can and must be accessible to everyone, that social change is central to intellectual work and, as such, scholars have a responsibility to use the privileges of academia to imagine and create a better world.

To my committee members Dr. Robert Adams, Dr. William Smith and Dr. Changyou Wang for their encouraging words, thoughtful criticism, and time and attention during busy semesters.

To Dr. John C. Young, who provided significant help and support on my work.

To my other professors for showing me by example and through challenging coursework how to think, teach, and teach others to think: Dr. Caicheng Lu, Dr. Eides Michael.

To my other colleagues for sharing their enthusiasm for and comments on my work: Dr. Xin Xu and Dr. Zhiyong Zeng

In addition to the technical and instrumental assistance above, I received equally important assistance from family. My wife, Xiaofei Wang, provided on-going support throughout the dissertation process. My parents, Chang Song Zhao and Yu Liu, instilled in me, from an early age, the desire and skills to obtain a Ph.D.
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Chapter 1. Introduction

The subject of computational electromagnetics (CEM) has been under development for several decades. During these years, both scientific researchers and engineers put considerable effort in developing applicable electromagnetic (EM) theoretical frameworks. With these foundations established, a large number of numerical EM solvers, packages and simulation tools have been developed. With the support of modern high performance computing system, these tools have been widely applied in both academic communities and industries. For example, several real-world electromagnetic problems like scattering, radiation, wave-guiding and EM compatibility/interference analysis of modern radio frequency systems are not analytically calculable for the multitude of irregular geometries and various types of devices designed and used. The inability to derive closed form solutions of Maxwell's equations under various constitutive relations of media and boundary conditions is overcome by CEM tools. This makes CEM an important field in the design, and modeling of antenna, radar, satellite and other such communication systems, nano-photonic devices and high speed silicon electronics, medical imaging, cell-phone design, among other applications.

As a consequence, a large number of traditional physical experiment-based designing can be replaced by CEM simulations. Hence the cost of performing research based on repeated experiments as well as prototyping microwave devices and systems had been dramatically simplified. Moreover, with the accuracy and calculation efficiency of CEM simulations being improved constantly, the procedure of optimization is much easier and faster compared with traditional methods. These significant contributions have made
CEM play a very important role in both fundamental theoretical study and the advanced technical applications.

More recently, a number of branches of CEM have emerged [1-3]. Generally speaking, CEM methods can be divided into integral equation and differential equation based schemes. The methods are further distinguished by being posed in either the time or frequency-domain. As this subject developed, several factors have become the key parameters that have been used to verify and test the performance a CEM algorithm. They are:

1. Level of accuracy;
2. Computation cost (including time consuming and memory usage);
3. Problem types that can be handled.

It is obvious to realize that a good CEM algorithm should be able to provide high levels of accuracy with low memory cost, fast in calculation and be able to solve multiple types of problems as well. Due to the limitations of the computing system as well as the delimitations of the algorithm itself, it is difficult for a CEM algorithm to achieve high ratings in all three standards no matter how well the optimization based on computer science techniques is performed on the implementation and the execution. For example, in the Method of Moment (MoM) and Finite Element Method (FEM), a lower spatial discretization error can be achieved by creating a finer geometrical mesh, this can provide a better representation of the original object and hence will give a more accurate result like scattered fields, Radar-Cross Section (RCS) and current distributions. But doing this will enlarge the problem size by increasing the number of unknowns and hence will lead to a higher memory cost as well as a longer solving time [1]. It is also well known that a
finer spatial discretization in Finite Difference Time Domain (FDTD) will give us a smaller phase error and this can be achieved by using smaller size of the lattice grids and shorter time step. But doing this will again require larger computing resource and more time steps to run to get the result within the same time period [2]. With these limitations being realized, researchers have put efforts to study the method to enhance the accuracy of the CEM algorithms with a lowest possible running cost. For example, in order to provide a better representation of the fields without using finer geometrical models or dramatically increasing the number of unknowns, high order basis functions are derived and applied when discretizing the fields [4]. To speed up the computation, parallel algorithms were implemented and applied to CEM techniques such as FDTD and Fast Multipole Method (FMM) [5, 6]. At the mean time, the process of developing and studying new algorithms and solutions to handle more complicated problems which can not be solved by current CEM solvers has never stopped.

A more recent and viable technique that can be used is a novel algorithm referred to as the Discontinuous Galerkin Finite-Element Time-Domain (DGFETD) method [7]. This algorithm is a variant of the Discontinuous Galerkin methods [8]. Discontinuous Galerkin methods are a class of finite element methods that employ piecewise continuous basis and testing functions. The methods are characterized as being high-order accurate, able to model complex geometries, efficient, stable, and highly parallel. Discontinuous Galerkin Time-Domain (DGTD) methods have more recently been employed for the solution of Maxwell’s equations [9-11]. Unlike commercially available electromagnetic software, the DGFETD method is a high-order, finite-element based solution of coupled curl Maxwell's equations. Since it is based on a finite-element discretization, it can be
accurately and efficiently applied to model very complex structures with arbitrary geometries and complex materials (e.g., anisotropic and/or dispersive materials). Also, since it is based on the solution of the time-dependent Maxwell's equations, either linear or non-linear materials can be treated. Because the method is a high-order method (this includes high-order spatial discretization and time discretization), the overall dimensionality of the problem can be significantly reduced. This also allows for controllable accuracy. Furthermore, the DGFETD method is based on a domain-decomposition of the finite-element mesh. Consequently, it is a highly-parallel algorithm and is easily adaptable to take full advantage of modern computing systems. As a result, the DGFETD method is a very robust method that can be hybridized with many other solvers such as SPICE, to simulate complex problems such as the full model of modern radio frequency systems. In summary, the salient features of the DGFETD method are [12]:

1. Different with traditional FETD method, which formulate the global problem into a single linear system, the DGFETD decomposes the global problem domain into sub-domains and solves Maxwell’s equations within each local sub-domain implicitly. By properly handing the communications among the sub-domains, DGFETD is applying a differential scheme to solve the global problem explicitly. In contrast with FETD, which always involves filling and inversing a full size system matrix, DGFETD breaks the full size system matrix into much smaller matrix blocks associated to each sub-domain. Doing this will avoid the inverse of a large system matrix and hence can significantly enhance the solution efficiency;

2. With two independent DOF sets defined on either side of a sub-domain interface,
this algorithm is highly parallel. This will allow problems of large scale to be divided into multiple sub-domains and simulated on parallel computing systems. HPC techniques such as OpenMP, MPI and GPU protocols can be easily applied to DGFETD solver;

3. With a finite element discretization of the object, a fully unstructured mesh can be obtained and complex geometries can be accurately modeled with curvilinear polyhedral. This will offer a good modeling of the geometry. Local h-refinement can also be supported;

4. By using high-order vector element basis functions, both non-physical spurious solutions and the necessity to project field onto quadrature points can be avoided. It will also allow for local p-refinement of basis (basis associated with sub-topology rather than entire cell), which dramatically increases the flexibility to model the fields with high accuracy. With high-order basis and test functions applied, fewer degrees of freedom (DOF) are required per wavelength. Hence the memory cost for this algorithm will be relatively low;

5. With sub-domains being established, each sub-domain can support a local time stepping scheme. The time evolvement of the solution can be obtained through proper synchronizations between each sub-domains. Such t-refinement can significantly reduce the computation time for problems modeled by large range of various finite element sizes;

6. Absorbing Boundary Conditions (ABC) and Perfectly Matched Layers (PML) are naturally implemented.
In this dissertation, the discontinuous Galerkin finite element time-domain (DGFETD) method proposed for solving coupled curl Maxwell’s equations is detailed in Chapter 2. Based on such full-wave solver, three modeling approaches, including circuit modeling, thin-wire modeling and the complex frequency shifted (CFS) perfectly matched layer (PML) modeling are developed to extend the DGFETD application, which are presented in Chapters 3, 4, 5 respectively. With the proposed hybrid formulations, DGFETD has been implemented to simulate interactively with SPICE and thin-wire solvers. It can also incorporate the differential equations for solving the auxiliary fields in the CFS-PML medium. These extensions dramatically enhance the computing capability of the DGFETD full-wave solver in the simulation of complicated radio frequency systems. Various testing models are created and simulated to validate the proposed modeling approaches. Excellent agreement has been found between the test results and the reference data, which indicates a valid, efficient and accurate hybrid full-wave solver.

The major contributions of this work can be outlined in the following. First of all, it shows in theory how to incorporate the circuit elements, thin-wire coupling and the CFS-PML media effects into the Maxwell’s equations that is solved through the DGFETD method. Second, it utilizes a port model to handle the coupling between the DGFETD-EM and the auxiliary solvers such as SPICE, thin-wire, and CFS-PML auxiliary differential equation solvers. The port model will keep the DGFETD-EM field solver and the auxiliary solvers independent from each other and only hybridized through the port. Doing this will dramatically increase the flexibility of hybrid modeling approach. For example, in the proposed methods, the circuit port and thin-wire port regions can be defined independent of the field mesh, which will allow arbitrarily oriented and located
circuit elements as well as thin-wires to be properly handled by the hybrid solver. In addition, by using the high-order basis functions in the DGFETD-EM field and the auxiliary solvers, high order accuracy of the coupling can be achieved through the port coupling matrices. Finally, by using high-order time integration schemes such as Runge-Kutta scheme, the time-domain coupling between the EM and the auxiliary solvers can also achieve high-order accuracy.
Chapter 2. The DGFETD Method for Solving the Electromagnetic Fields

In this chapter, the DGFETD method for solving the electromagnetic fields is presented. This method is based on a finite-element discretization of Maxwell’s curl equations [13]. Rather than a point-based sampling, both the electric and magnetic fields are expanded via hierarchical Nedelec curl-conforming mixed-order basis functions [14, 15]. Similar to the DGTD method, tangential field continuity is weakly constrained across shared boundaries. Due to the properties of curl-conforming vector basis functions, only basis functions associated with topologies on a sub-domain boundary (i.e., edges and faces) have a non-zero tangential projection. Thus, only local basis functions are shared across domain boundaries. Furthermore, the use of hierarchal curl-conforming basis functions allows for local hp-refinement of the discretization. Another advantage of the DGFETD formulation is that sub-domains are not restricted to single cells. Rather, an arbitrary partitioning of the domain can be employed.
2.1. DGFETD-EM Formulation

Consider the electromagnetic fields in a domain $\Omega$ bound by $\partial \Omega$ radiated by a pair of volume electric and magnetic current densities. The global domain $\Omega$ can be spatially decomposed into a set of sub-domains $V_i$ bound by $\partial V_i$, as shown in Figure 1, where $E_i, H_i$ are the electric and magnetic fields within $V_i$. If the total number of $V_i$ is $N$, then $\Omega = \bigcup_{i=1}^{N} V_i$. $J_{i,imp}^V$ and $M_{i,imp}^V$ are impressed volume current density sources inside of $V_i$.

$E_{i}^-, H_{i}^-$ are the electric and magnetic fields on $\partial V_i$ just inside of $V_i$ ($\partial V_i^-$), and $E_{i}^+, H_{i}^+$ are the fields on $\partial V_i$ just outside of $V_i$ ($\partial V_i^+$). $\hat{n}$ is the outward unit normal vector with respect to $\partial V_i$. Within each sub-domain, the electric and magnetic fields must satisfy Maxwell’s curl equations:

\[
\frac{\partial}{\partial t} \vec{E} + \nabla \times \vec{H} + \vec{M}_{i,imp}^V = 0 , \tag{1}
\]

\[
\frac{\partial}{\partial t} \vec{H} - \nabla \times \vec{E} + \vec{J}_{i,imp}^V = 0 , \tag{2}
\]
where $\overline{\mu}$ and $\overline{\varepsilon}$ are permeability and permittivity tensors. For simplicity, a lossless medium is assumed. Applying Galerkin method, the inner product of the coupled curl equations with a set of test functions is performed:

$$
\int_{V_i} \vec{T}^h \cdot \left( \frac{\partial}{\partial t} \overline{\mu} \cdot \vec{H} + \nabla \times \vec{E} + \vec{M}^\text{imp}_{V_i} \right) dv = 0 ,
$$

(3)

$$
\int_{V_i} \vec{T}^e \cdot \left( \frac{\partial}{\partial t} \overline{\varepsilon} \cdot \vec{E} - \nabla \times \vec{H} + \vec{J}^\text{imp}_{V_i} \right) dv = 0 ,
$$

(4)

where $\vec{T}^h$ and $\vec{T}^e$ are testing functions that span the $\vec{H}$- and $\vec{E}$-field function spaces, respectively. The test vectors are assumed to be time-independent. Consequently, (3) and (4) can be re-written as:

$$
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^h \cdot \overline{\mu} \cdot \vec{H} dv = -\int_{V_i} \vec{T}^h \cdot \nabla \times \vec{E} dv - \int_{V_i} \vec{T}^h \cdot \vec{M}^\text{imp}_{V_i} dv ,
$$

(5)

$$
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^e \cdot \overline{\varepsilon} \cdot \vec{E} dv = \int_{V_i} \vec{T}^e \cdot \nabla \times \vec{H} dv - \int_{V_i} \vec{T}^e \cdot \vec{J}^\text{imp}_{V_i} dv .
$$

(6)

From the vector identity:

$$
\int_{V_i} \vec{A} \cdot \nabla \times \vec{B} dv = \int_{V_i} \vec{B} \cdot \nabla \times \vec{A} dv + \oint_{\partial V_i} \vec{A} \cdot \hat{n} \times \vec{B} \cdot ds ,
$$

(7)

where $\vec{A}^\cdot, \vec{B}^\cdot$ are the vector fields defined on the surface $\partial V_i$. Substitute (7) into (5) and (6) respectively, we obtain:

$$
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^h \cdot \overline{\mu} \cdot \vec{H} dv = -\int_{V_i} \vec{E} \cdot \nabla \times \vec{T}^h dv - \oint_{\partial V_i} \vec{T}^h \cdot \hat{n} \times \vec{E}^- ds - \int_{V_i} \vec{T}^h \cdot \vec{M}^\text{imp}_{V_i} dv ,
$$

(8)

$$
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^e \cdot \overline{\varepsilon} \cdot \vec{E} dv = \int_{V_i} \vec{H} \cdot \nabla \times \vec{T}^e dv + \oint_{\partial V_i} \vec{T}^e \cdot \hat{n} \times \vec{H}^- - \int_{V_i} \vec{T}^e \cdot \vec{J}^\text{imp}_{V_i} dv ,
$$

(9)
where the volume integrals are only operating and contributing to the DOFs that are local to \( V_i \). The surface integrals will be operating with both the local and neighboring tangential fields based on the boundary conditions defined on \( \partial V_i \). It should be noted that the approach to evaluate such tangential \( \vec{E}^- \) and \( \vec{H}^- \) is not necessarily unique. Since in a discontinuous Galerkin scheme, two completely independent basis function and DOF sets are being defined on both sides of \( \partial V_i \). Different approaches have been studied and applied to constrain the continuity of the tangential boundary fields. The first, which is also the originally derived formulation for DGFETD-EM, is referred as the central flux method \([7]\). Other approaches such as the upwind flux method and interior penalty method are also available in \([16, 17]\) respectively.

### 2.2. Central Flux Formulation

Central flux formulation is based on a first-order boundary condition, which directly applies the following source free boundary conditions into the Maxwell’s curl equations.

\[
\hat{n} \times (\vec{E}^- - \vec{E}^+) \bigg|_{\partial V_i} = 0 ,
\]

\[
\hat{n} \times (\vec{H}^- - \vec{H}^+) \bigg|_{\partial V_i} = 0 .
\]

Based on the boundary conditions defined above, a general relation between the local and neighboring tangential fields on the boundary can be derived as the following. First, multiply both sides of (10) by a factor of \( c_1^- \), thus (10) becomes:

\[
0 = \hat{n} \times (c_1^- \vec{E}^- - c_1^+ \vec{E}^+) \bigg|_{\partial V_i} .
\]

Then subtract \( \hat{n} \times c_2^- \vec{E}^- \bigg|_{\partial V} \) to both sides of (12), we obtain:
\[ -\hat{n} \times c_2^e \vec{E}^- \bigg|_{\partial V_i} = -\hat{n} \times c_2^e \vec{E}^- \bigg|_{\partial V_i} + \hat{n} \times \left( c_1^e \vec{E}^+ - c_2^e \vec{E}^+ \right) \bigg|_{\partial V_i} . \]  

Finally, divide both side with the factor \(-c_2^e\), we have:

\[ \hat{n} \times \vec{E}^- \bigg|_{\partial V_i} = \hat{n} \times \left( \frac{c_2^e - c_1^e}{c_2} \vec{E}^- + \frac{c_1^e}{c_2} \vec{E}^+ \right) \bigg|_{\partial V_i} , \]

where \(c_1^e\) and \(c_2^e\) are weighting coefficients that can be variously chosen. Similar to the magnetic field, we have:

\[ \hat{n} \times \vec{H}^- \bigg|_{\partial V_i} = \hat{n} \times \left( \frac{c_2^h - c_1^h}{c_2^h} \vec{H}^- + \frac{c_1^h}{c_2^h} \vec{H}^+ \right) \bigg|_{\partial V_i} . \]

It should be noted that although one can arbitrarily chose the values of \(c_1^e, c_2^e\) and \(c_1^h, c_2^h\), but all choices will still satisfy the boundary conditions defined in (10) and (11).

For a central flux method, the following coefficients are being chosen:

\[ c_1^e = 2, \quad c_2^e = 1 \]
\[ c_1^h = 2, \quad c_2^h = 1 \]

Then (14) and (15) becomes:

\[ \hat{n} \times \vec{E}^- \bigg|_{\partial V_i} = \frac{1}{2} \hat{n} \times \left( \vec{E}^- + \vec{E}^+ \right) \bigg|_{\partial V_i} , \]

\[ \hat{n} \times \vec{H}^- \bigg|_{\partial V_i} = \frac{1}{2} \hat{n} \times \left( \vec{H}^- + \vec{H}^+ \right) \bigg|_{\partial V_i} . \]

This can be interpreted as the local tangential fields on the boundary \(\partial V_i^-\) are the average of the local and neighboring tangential boundary fields. From this, (8) and (9) are rewritten as:
\[
\frac{\partial}{\partial t} \int_{V_i} \hat{T}^h \cdot \vec{\mu} \cdot \vec{H} dv = -\int_{V_i} \vec{E} \cdot \nabla \times \hat{T}^h dv - \int_{V_i} \hat{T}^h \cdot \vec{M}_{\psi}^{imp} dv - \frac{1}{2} \oint_{\partial V_i} \hat{T}^h \cdot \hat{n} \times \vec{E}^- ds + \frac{1}{2} \oint_{\partial V_i} \hat{T}^h \cdot \hat{n} \times \vec{E}^+ ds,
\]
(19)

\[
\frac{\partial}{\partial t} \int_{V_i} \hat{T}^e \cdot \vec{E} dv = \int_{V_i} \vec{H} \cdot \nabla \times \hat{T}^e dv - \int_{V_i} \hat{T}^e \cdot \vec{J}_{\psi}^{imp} dv + \frac{1}{2} \oint_{\partial V_i} \hat{T}^e \cdot \hat{n} \times \vec{H}^- ds + \frac{1}{2} \oint_{\partial V_i} \hat{T}^e \cdot \hat{n} \times \vec{H}^+ ds,
\]
(20)

From the vector identity:
\[
\oint_{\partial V_i} \vec{A} \cdot \hat{n} \times \vec{B}^- ds = \int_{V_i} \vec{A} \cdot \nabla \times \vec{B} dv - \int_{V_i} \vec{B} \cdot \nabla \times \vec{A} dv,
\]
(21)

The surface integral of \(\frac{1}{2} \hat{T}^h \cdot \hat{n} \times \vec{E}^-\) in (19) can be written back into the volume integrals:
\[
\frac{1}{2} \oint_{\partial V_i} \hat{T}^h \cdot \hat{n} \times \vec{E}^- ds = \frac{1}{2} \int_{V_i} \hat{T}^h \cdot \nabla \times \vec{E} dv - \frac{1}{2} \int_{V_i} \vec{E} \cdot \nabla \times \hat{T}^h dv.
\]
(22)

Similarly
\[
\frac{1}{2} \oint_{\partial V_i} \hat{T}^e \cdot \hat{n} \times \vec{H}^- ds = \frac{1}{2} \int_{V_i} \hat{T}^e \cdot \nabla \times \vec{H} dv - \frac{1}{2} \int_{V_i} \vec{H} \cdot \nabla \times \hat{T}^e dv.
\]
(23)

Apply (22) and (23) back to (19) and (20) respectively, a central flux formulation for DGFETD-EM can be finally expressed as:

\[
\frac{\partial}{\partial t} \int_{V_i} \hat{T}^h \cdot \vec{\mu} \cdot \vec{H} dv = -\frac{1}{2} \int_{V_i} \left( \hat{T}^h \cdot \nabla \times \vec{E} + \vec{E} \cdot \nabla \times \hat{T}^h \right) dv - \frac{1}{2} \oint_{\partial V_i} \hat{T}^h \cdot \hat{n} \times \vec{E}^- ds - \int_{V_i} \hat{T}^h \cdot \vec{M}_{\psi}^{imp} dv,
\]
(24)

\[
\frac{\partial}{\partial t} \int_{V_i} \hat{T}^e \cdot \vec{E} dv = \frac{1}{2} \int_{V_i} \left( \hat{T}^e \cdot \nabla \times \vec{H} + \vec{H} \cdot \nabla \times \hat{T}^e \right) dv + \frac{1}{2} \oint_{\partial V_i} \hat{T}^e \cdot \hat{n} \times \vec{H}^- ds - \int_{V_i} \hat{T}^e \cdot \vec{J}_{\psi}^{imp} dv.
\]
(25)
2.3. Upwind Flux Formulation

The upwind flux formulation is derived from a pair of second order boundary conditions:

\[
\hat{n} \times \left( \vec{E}^- - \vec{E}^+ \right) \bigg|_{\partial V_i} - \hat{n} \times \left( \vec{Z}^- \vec{H}^- - \vec{Z}^+ \vec{H}^+ \right) \bigg|_{\partial V_i} = 0, \tag{26}
\]
\[
\hat{n} \times \left( \vec{H}^- - \vec{H}^+ \right) \bigg|_{\partial V_i} + \hat{n} \times \left( \vec{Y}^- \vec{E}^- - \vec{Y}^+ \vec{E}^+ \right) \bigg|_{\partial V_i} = 0. \tag{27}
\]

This is again assuming a source free boundary. Similar to the derivation shown in (12)-(15), the upwind flux formulation can also be expressed with chosen coefficients as being:

\[
\hat{n} \times \vec{E}^- \bigg|_{\partial V_i} = \hat{n} \times \left( \frac{c_1^E - c_1^h}{c_2^E} \vec{E}^- + \frac{c_1^h}{c_2^E} \vec{E}^+ \right) \bigg|_{\partial V_i} + \frac{c_1^E}{c_2^E} \hat{n} \times \left( \vec{Z}^- \vec{H}^- - \vec{Z}^+ \vec{H}^+ \right) \bigg|_{\partial V_i}, \tag{28}
\]
\[
\hat{n} \times \vec{H}^- \bigg|_{\partial V_i} = \hat{n} \times \left( \frac{c_1^h - c_1^E}{c_2^h} \vec{H}^- + \frac{c_1^h}{c_2^h} \vec{H}^+ \right) \bigg|_{\partial V_i} - \frac{c_1^E}{c_2^E} \hat{n} \times \left( \vec{Y}^- \vec{E}^- - \vec{Y}^+ \vec{E}^+ \right) \bigg|_{\partial V_i}. \tag{29}
\]

Ideally, the tangential projections of \( \vec{E}^-, \vec{E}^+ \) and \( \vec{H}^-, \vec{H}^+ \) are continuous on \( \partial V_i \), however, as mentioned before, there will be small discontinuities due to the separation of basis functions and DOFs on the interface as well as the discretization over time. Therefore the subtracting terms appeared in (28) and (29) will not be zero and will act as a penalty terms in order to better suppress such discontinuities.

For an upwind flux formulation, it is actually setting:

\[
c_1^E = 1, \quad c_2^E = \frac{Y^+}{Y^- + Y^+}, \tag{30}
\]
\[
c_1^h = 1, \quad c_2^h = \frac{Y^+}{Z^- + Z^+}.
\]
where \( Z^- \) and \( Z^+ \) are the characteristic impedance, and \( Y^- \) and \( Y^+ \) are the characteristic admittance defined in \( \partial V^- \) and \( \partial V^+ \), respectively. Then the local tangential boundary fields can be evaluated as:

\[
\hat{n} \times \vec{E} \bigg|_{\partial V} = \hat{n} \times \left( \frac{Y^-}{Y} \vec{E}^- + \frac{Y^+}{Y} \vec{E}^+ \right) + \frac{Y^+}{Y} \hat{n} \times \hat{n} \times \left( Z^- \vec{H}^- - Z^+ \vec{H}^+ \right) \bigg|_{\partial V},
\]

\[
\hat{n} \times \vec{H} \bigg|_{\partial V} = \hat{n} \times \left( \frac{Z^-}{Z} \vec{H}^- + \frac{Z^+}{Z} \vec{H}^+ \right) - \frac{Z^+}{Z} \hat{n} \times \hat{n} \times \left( Y^- \vec{E}^- - Y^+ \vec{E}^+ \right) \bigg|_{\partial V}.
\]

Substitute (31) and (32) back to (8) and (9), and apply (22) and (23) again, one can obtain the upwind flux formulation for DGFETD-EM as:

\[
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^h \cdot \vec{E} \cdot \vec{H} \, dv = -\frac{1}{2} \int_{V_i} \left( (\vec{T}^h \cdot \nabla \times \vec{E} + \vec{E} \cdot \nabla \times \vec{T}^h) \right) \, dv \\
- \oint_{\partial V_i} \vec{T}^h \cdot \hat{n} \times \left( \frac{Y^-}{2Y} \vec{E}^- + \frac{Y^+}{Y} \vec{E}^+ \right) \, ds \\
- \frac{Y^+}{Y} \oint_{\partial V_i} \vec{T}^h \cdot \hat{n} \times \left( Z^- \vec{H}^- - Z^+ \vec{H}^+ \right) \, ds - \int_{V_i} \vec{T}^h \cdot \vec{M}_{i,mp} \, dv,
\]

\[
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^e \cdot \vec{E} \cdot \vec{E} \, dv = \frac{1}{2} \int_{V_i} \left( (\vec{T}^e \cdot \nabla \times \vec{H} + \vec{H} \cdot \nabla \times \vec{T}^e) \right) \, dv \\
+ \oint_{\partial V_i} \vec{T}^e \cdot \hat{n} \times \left( \frac{Z^-}{2Z} \vec{H}^- + \frac{Z^+}{Z} \vec{H}^+ \right) \, ds \\
- \frac{Z^+}{Z} \oint_{\partial V_i} \vec{T}^e \cdot \hat{n} \times \left( Y^- \vec{E}^- - Y^+ \vec{E}^+ \right) \, ds - \int_{V_i} \vec{T}^e \cdot \vec{J}_{i,mp} \, dv.
\]
2.4. Discretization

To reduce the proposed formulation into a linear system, the electric and magnetic fields are expanded using hierarchical curl-conforming vector basis functions weighted by unknown time-dependent coefficients. The test functions are chosen to be identical to the basis functions. Only basis local to the faces have tangential projections. Consequently, only basis and test functions local to the faces contribute to the surface integrals over $\partial V_i$. Then, for a central flux formulation, (24) and (25) can be written in a discrete form as [7]:

$$M_{hh}^\prime h^\prime = -S_{he} e - F_{he}^e e^+ - T_{v}^{hm} m_v, \quad (35)$$

$$M_{ee}^\prime e^\prime = S_{eh}^e h + F_{eh}^e h^+ - T_{v}^{ej} j_v. \quad (36)$$

The discretized form for the upwind flux formulation can be expressed as:

$$M_{hh}^\mu h^\prime = -S_{he} e - F_{he}^h \left( \frac{Y^- - Y^+}{2Y} e^- + \frac{Y^+}{Y} e^+ \right) - G_{hh}^h \left( Z^- h - Z^+ h^+ \right) - T_{v}^{hm} m_v, \quad (37)$$

$$M_{ee}^\mu e^\prime = S_{eh}^h h + F_{eh}^h \left( \frac{Z^- - Z^+}{2Z} h^- + \frac{Z^+}{Z} h^+ \right) - G_{ee}^h \left( Y^- e - Y^+ e^+ \right) - T_{v}^{ej} j_v. \quad (38)$$

2.5. Summary

In this chapter, the DGFETD-EM formulation was proposed. Both the central flux and upwind flux formulations can be applied to constrain the continuity of the tangential fields at the interfaces between sub-domains.
Chapter 3. Circuit Component Modeling within DGFETD-EM Solver

Over the past several decades, circuit simulation tools such as SPICE [16] have had a significant impact on circuit design methodology. Such tools have been widely applied in both industry and academic communities and nearly every electronic design automation (EDA) software package will integrate SPICE functionalities for circuit system simulations. Efforts have been made to improve SPICE performance. Equivalent circuit modeling has also received great interest and a significant work has been devoted to formulate compact, yet complete, models of both passive and active devices of various kinds. With these efforts, SPICE has become the most popular tools in the field of circuit system design.

As modern manufacturing capability matures, a successive circuit system design always includes the following features: multifunctional, higher operating frequency and larger integration scale. To achieve these goals, several major challenges arise. First of all, multifunctional system requires the circuit components to be mounted together with many other types of devices, such as sensors and complex material blocks. The EM response of these devices can hardly be obtained via equivalent circuit modeling approach. Especially when dispersive, anisotropic and inhomogeneous materials or complex shaped metal structures are included in the circuit system design, simulating the global EM effects of such devices through an equivalent circuit modeling approach can be prohibitive. On the other hand, when the circuit is working at higher frequency, especially at microwave regime, the radiation effects from the circuit itself will bring dramatic distortion on the signal transporting and processing. Finally, larger integration scale also requires smaller space between circuit elements. Thus phenomena such as
crosstalk and packaging effects will play dominant roles on the circuit behavior which will bring poor simulation accuracy due to the lack of EM effect prediction. These challenges have made researchers and engineers realized that the electromagnetic responses of the circuit system can no longer been neglected and a more physical description at a relative “micro-level” is indeed necessary to capture such EM effects. Due to this, a modern successive circuit design must take the electromagnetic compatibility and interference (EMC/EMI) problems into consideration. Such topic have received increased interest during recent years [17]. Basically, a full-wave solution of Maxwell’s equations can provide a deeper insight into the complex configured circuit network behavior. Such a solution is comprised of spatial distribution of the electromagnetic field and its time evolvement, which are more fundamental parameters in contrast with the device port voltages and currents. Many EM effects, such as radiation, diffraction, multiple scattering and coupling, creeping waves and resonance can be modeled and described with sufficient details by the EM fields. Therefore, the computational electromagnetic (CEM) techniques have received great attention to analyze the EMC/EMI problems of modern circuit systems.

An early application of coupling a circuit model within a CEM simulation was presented by Horng [18], in which the circuit elements are directly coupled into the EM solver, in such case, the problems are often dealt with by producing various quasi-static circuit models or frequency-domain mathematical equivalent macro-models of the interconnect structures.

Later on, the finite element method (FEM) [3] has been applied to analyze microwave circuits with both linear and nonlinear devices [19-21]. While these frequency-domain
models can be simple to use, they are still difficult and expensive to produce sufficient accuracy when the result is converted and checked in time-domain. This is especially true when the nonlinear circuit components are included in the analysis [19-21].

An alternative to frequency-domain circuit modeling is to extend the widely used time-domain EM solvers to include lumped circuit elements. Transient approaches such as finite-difference time-domain (FDTD) [2], finite-element time-domain (FETD) [3] have inherent advantages especially when the circuit under investigation contains large number of circuit components with complex and non-linear voltage-current relations. They can also perform characterization over a broad frequency band with a single computation. Another significant advantage of transient solvers is the relative simplicity when interfacing with other circuit solvers such as SPICE. With proper interfacing, the problem can be conveniently partitioned into full-wave and circuit parts. The circuit part of the problem is handled by a general, widely used and trusted simulator such as SPICE, for which there are abundant model libraries for semiconductor devices, logical gates, and integrated circuits. In contrast with the frequency-domain EM solvers, the iterative algorithm of both EM and SPICE transient solvers allows the capability of adding more ports to the structure without significantly increasing computational cost and complexity.

A significant amount of work has been published on the topic of a transient hybrid EM/Circuit analysis using the finite-difference time-domain (FDTD) method. For simple linear elements the lumped element equations can be integrated directly within the FDTD discretization via local material approximations [22, 23]. For more general elements including non-linear devices, they can be modeled by a general circuit simulator such as SPICE [24-26].
The finite-element time-domain (FETD) method has also been coupled to hybrid EM/Circuit analysis. Similar to FDTD based methods, early studies on the incorporation of lumped circuit elements into FETD mainly focused on directly stamping their voltage-current relationships into the primary finite element matrices. The “direct stamping” approach is straightforward, but it is limited to cases that involve only linear and passive elements such as resistors, capacitors, and inductors of which the voltage-current relationships are governed by simple equations. Direct stamping also lacks the flexibility to include complex configured circuit networks, such as mixed RLC networks and active devices.

To extend the functionalities of such hybrid EM/Circuit solvers, the “equivalent source” approach was applied [27-29] in which the circuits is modeled as an equivalent Norton current source with an internal capacitive admittance at the lumped circuit port, the voltage-current relationships can be obtained by a direct call of SPICE solver [27-29].

As the research for hybrid EM/Circuit analysis continues, modeling of lumped circuit elements has been well documented in FDTD and FETD, yet their disadvantages were also revealed to us. For example, FDTD suffers from dispersion error and the difficulties of modeling complex geometries. FETD provides better accuracy and robust modeling of geometries but it is quite expensive in computation since a global linear system needs to be solved at each time iteration. Due to this, the effort of applying new transient EM solvers has never stopped. Recently DGFETD has been considered to hybridize with circuit solvers by Dosopoulos et al. [30]. Following the logic for FDTD and FETD research, simple circuit elements are first directly stamped into the DGFETD formulation. In Dosopoulos’s work [30], the lumped element are treated as isotropic material when
incorporated with the DGFETD formulation, and leap-frog scheme is applied when performing the time evolvement. In this work, simple circuit elements are also directly incorporated into the DGFETD formulation. However, it is treated as an anisotropic material block, which can be modeled by a surface or volume circuit port. Also a fourth-order Runge-Kutta scheme is used to evaluate the time derivative, which leads to a high-order accuracy for the time integration. Furthermore, a DGFETD-EM/SPICE interface is introduced here which can conveniently handle the coupling between the DGFETD-EM and SPICE software. The rest of the chapter is organized as follows: In Section 3.1, a general circuit port is designed to model lumped circuit elements. A direct stamping formulation for simple circuit elements such as resistors, capacitors and inductors and a hybrid DGFETD-EM/SPICE formulation are presented in Section 3.2. In addition, the technique to interface the DGFETD-EM and SPICE solvers is presented in Section 3.3. The hybrid DGFETD/Circuit analysis software is validated in Section 3.4 through the simulation of a number of examples including a microstrip transmission line mounted with various linear and non-linear lumped circuits.
3.1. Circuit Port Design

Lumped circuit models can be conveniently coupled to Maxwell’s equations through the use of a hybrid DGFETD-EM/Circuit formulation. It is assumed that the circuit elements are small relative to the smallest wavelength of interest such that the circuit can be approximated as lumped elements. With a circuit port being properly designed, the proposed hybrid formulation couples the field and circuit solvers together. Each port of the lumped circuit is driven by a time dependent port voltage which is computed from the electric field of the DGFETD-EM solver. The lumped circuit solver computes the port current which is coupled back into the DGFETD-EM solver as an electric current density source. The details of the coupling are demonstrated in the following.

Figure 2. Circuit port modeled by a rectangular volume
Initially, consider a volume circuit port defined in Figure 2, in which the shape of the port is restricted to be a rectangular volume with a cross sectional area of $A$. By restricting the port region to be rectangular, it puts more weight on the geometric definition. However, it relaxes any restrictions on the mesh, which can be arbitrary within this region. The circuit port presumably connects two conductors separated by a distance $h$, separating the conductors, as illustrated in Figure 2. A unit vector $\hat{p}$ is used to define the direction of the circuit port from low-potential to high-potential. The port voltage is defined via quasi-static approximation:

$$ V \approx -\frac{1}{A} \int_{V_c} \bar{E} \cdot \hat{p} dv. $$

(39)

The quasi-static approximation assumes that the electric field is locally conservative within the volume port. The port voltage is coupled to the circuit solver by acting as a time dependent voltage source. The circuit solver is then used to solve for the current that flows through the voltage source. This is the port current $I$ shown in Figure 2. The electric current will be represented as a volume electric current density $\bar{J}_c$ that couples back into the DGFETD-EM field formulation as a source term:

$$ \bar{J}_c = \hat{p} \frac{I}{A}. $$

(40)

Again, quasi-static approximation is assumed so that the electric current density $\bar{J}_c$ can be evaluated by evenly distributing the port electric current $I$ over the port region.
Next, consider the surface port. For simplicity, we will assume that the surface defining the port is rectangular. For example, the surface port is represented by the shaded region in Figure 3. It is discretized via the faces of the polyhedral mesh, which can be quadrilateral or triangular in shape. An example triangular mesh is illustrated in Figure 3. The voltage difference across such defined surface port can be expressed as:

\[ V \approx -\frac{1}{W} \int_{S_c} \hat{E} \cdot \hat{P} ds. \]  

Again the approximately equal indicates the quasi-static approximation of the local electric field. The electric current \( I \) obtained from the circuit solver can then be transferred to surface electric current density through:

\[ \vec{J}_c = \hat{p} \frac{I}{W}, \]  

where \( W \) is the width of the circuit port and \( \vec{J}_c \) is now a surface current density.
3.2. Hybrid DGFETD-EM/Circuit Formulation

The focus of this section is the coupling of the lumped circuit to the DGFETD-EM solver. Consider the Maxwell’s curl equations in the vicinity of a circuit port, this can be shown as:

\[
\frac{\partial}{\partial t} \bar{\mu} \cdot \bar{H} = -\nabla \times \bar{E}, \quad (43)
\]

\[
\frac{\partial}{\partial t} \bar{\varepsilon} \cdot \bar{E} = \nabla \times \bar{H} - \bar{J}_c, \quad (44)
\]

where \(\bar{J}_c\) represents the electric current density induced by the circuit port which is a function of the local electric field intensity. It is noted that \(\bar{J}_c\) can be either a volume current density (volume circuit port) or a surface current density (surface current port). The curl equations are cast into a weak form following the standard DGFETD-EM procedure as:

\[
\frac{\partial}{\partial t} \int_{V_i} \bar{T}^h \cdot \bar{\mu} \cdot \bar{H} dv = -\frac{1}{2} \int_{V_i} \left( \bar{T}^h \cdot \nabla \times \bar{E} + \bar{E} \cdot \nabla \times \bar{T}^h \right) dv - \frac{1}{2} \int_{\partial V_i} \bar{T}^h \cdot \hat{n} \times \bar{E}^* ds, \quad (45)
\]

\[
\frac{\partial}{\partial t} \int_{V_i} \bar{T}^e \cdot \bar{\varepsilon} \cdot \bar{E} dv = \frac{1}{2} \int_{V_i} \left( \bar{T}^e \cdot \nabla \times \bar{H} + \bar{H} \cdot \nabla \times \bar{T}^e \right) dv + \frac{1}{2} \int_{\partial V_i} \bar{T}^e \cdot \hat{n} \times \bar{H}^* ds - \int_{V_i} \bar{T}^e \cdot \bar{J}_c^e dv, \quad (46)
\]

where \(V_i\) represents the \(i^{th}\) local sub-domain volume which can be fully or partially included in the circuit port region. A central flux formulation is chosen here for simplicity. This assumes a volume circuit port. In the event that \(\bar{J}_c\) is a surface current density, (46) is expressed as:
\[
\frac{\partial}{\partial t} \int_{V_i} \bar{T}^e \cdot \bar{\varepsilon} \cdot \bar{E} dv = \frac{1}{2} \int_{V_i} \left( \bar{T}^e \cdot \nabla \times \bar{H} + \bar{H} \cdot \nabla \times \bar{T}^e \right) dv + \frac{1}{2} \oint_{\partial V_i} \bar{T}^e \cdot \hat{n} \times \bar{H}^+ ds \\
- \frac{1}{2} \int_{\partial V_i} \bar{T}^e \cdot \bar{J}_{S_c} ds
\]

where \( \partial V_i \) is either fully or partially touching the surface circuit port. Expanding the fields and test vectors using local vector finite elements, the equations are written in discrete form as:

\[
M_{\mu}^{hh} \bar{h}^' = -S^{he} \bar{e} - F^{he} \bar{e}^',
\]

\[
M_{e}^{se} \bar{e}^' = S^{eh} \bar{h} + F^{eh} \bar{h}^+ - C^{ej} \bar{J}_c,
\]

where the \( C^{ej} \) matrix will completely handle the transformation of \( \bar{E} \)-field within the circuit port to port voltage, and couple the feedback port electric current density \( \bar{J}_c \) back into the field equation. \( C^{ej} \) is defined as “circuit coupling matrix” which couples the field and circuit systems.

Using this method of coupling circuit elements to the DGFETD-EM solver, several common circuit components can be formulated in the following sections.
3.2.1. Lumped Resistor

Consider the case when the circuit component is a single resistor with resistance \( R \). A volume voltage port is assumed. The voltage-current relationship is defined by Ohm's law:

\[
I(t) = \frac{V(t)}{R}.
\]

(50)

Define \( A \) as the cross section area of the volume port, \( h \) as the height of the volume port. Also define the port axis as \( \hat{p} \), along which the \( \vec{E} \)-field is integrated. Then from (39) and (40) the related conductive electric current density in such specific port can be shown as:

\[
\vec{J}_R(t) = \left( \frac{1}{AR} \hat{p} \cdot \vec{E}(t) \right) \hat{p}.
\]

(51)

Eqs. (51) can be rewritten in a tensor form as being:

\[
\vec{J}_R(t) = \frac{1}{AR} \vec{q} \cdot \vec{E}(t),
\]

(52)

where \( \vec{q} \) is determined by the port. At this point we restrict the port axis for the voltage and current to be the same. Thus \( \vec{q} \) has no rotational terms. Plug in this \( \vec{J}_R \) source, the circuit coupling matrix can be calculated as

\[
\begin{align*}
C_{ee}^R &= \int_{V} \vec{T}^e \cdot \vec{J}_R \, dv = \frac{1}{AR} \int_{V} \vec{T}^e \cdot \vec{q} \cdot \vec{E}(t) \, dv. \\
&= \frac{1}{AR} \int_{V} \vec{T}^e \cdot \vec{q} \cdot \vec{E}(t) \, dv.
\end{align*}
\]

(53)

More specifically for the discrete DGFETD algorithm, each entry in the matrix can be calculated as:

\[
\left[ C_{ee}^R \right]_{m,n} = \frac{1}{AR} \int_{V} \vec{f}_m \cdot \vec{q} \cdot \vec{f}_n \, dv,
\]

(54)
where $\tilde{f}_{m,n}$ is the $m^{th}$ and $n^{th}$ vector finite element basis. It is observed a lumped resistor behaves as an anisotropic lossy material block in the DGFETD-EM formulation with the conductivity tensor defined as:

$$\bar{\sigma} = \frac{1}{AR} \bar{\rho}. \quad (55)$$

Reflected in the discrete form, this can then be expressed as:

$$M_{\mu}^{bh} h' = -S^{he} e - F^{he} e^+, \quad (56)$$

$$M_{\epsilon}^{ge} e' = S^{eh} h + F^{eh} h^+ - C^{se}_R e. \quad (57)$$
3.2.2. Lumped Capacitor

For a capacitor, the voltage-current relation is defined as:

\[ I(t) = C \frac{\partial V(t)}{\partial t}. \]  

(58)

From (39) and (40), the displacement electric current density resulting from the lumped capacitor can be expressed as:

\[ \tilde{J}_c(t) = \frac{C}{A} \bar{p} \cdot \frac{\partial \tilde{E}(t)}{\partial t}, \]

(59)

where \( \bar{p} \) is the port tensor as defined in (52). The circuit coupling matrix for a capacitive volume port can be calculated as:

\[ C_{ee}^c = \int_v \tilde{r}^e \cdot \tilde{J}_c \, dv = \frac{C}{A} \frac{\partial}{\partial t} \int_v \tilde{r}^e \cdot \bar{p} \cdot \tilde{E} \, dv. \]

(60)

For the discrete DGFETD representation, each entry of the \( C_{ee}^c \) matrix is represented as:

\[ \left[ C_{ee}^c \right]_{m,n} = \frac{C}{A} \int_{v_i} \tilde{f}_m \cdot \bar{p} \cdot \tilde{f}_n \, dv. \]

(61)

Since this \( C_{ee}^c \) matrix is operating on the time derivative of the \( \tilde{E} \)-field, then it behaves as an anisotropic dielectric material block with the permittivity tensor defined as:

\[ \bar{\varepsilon} = \frac{C}{A} \bar{p}. \]

(62)

In the discrete DGFETD-EM formulation, this can be reflected as an additional term in the matrices that operates on \( e' \)

\[ M^{hh'}_{\mu} h' = -S^{he} e - F^{he} e^+, \]

(63)

\[ \left( M_{ee}^{re} + C_{ee}^c \right) e' = S^{eh} h + F^{eh} h^+. \]  

(64)
3.2.3. Lumped Inductor

For an inductor, the voltage-current relationship is:

\[ V(t) = L \frac{\partial I(t)}{\partial t}. \]  \hspace{1cm} (65)

Define an “inductivity” for the inductive volume port as:

\[ \mu = L \frac{A}{h}. \]  \hspace{1cm} (66)

Define the port tensor \( \bar{p} \), then the relationship between the \( \bar{E} \)-field and related electric current density can be derived from (39) and (40) as:

\[ \frac{\partial \bar{J}_L(t)}{\partial t} = \frac{1}{AL} \bar{p} \cdot \bar{E}(t). \]  \hspace{1cm} (67)

Plug in this \( \bar{J}_L \) current density source, we obtain the circuit coupling matrix as

\[ C^e_L = \int_{V_i} \bar{T}^r \cdot \bar{J}_L \, dv. \]  \hspace{1cm} (68)

For each entry

\[ \left[ C^e_L \right]_{m,n} = \int_{V_i} \bar{J}_m \cdot \bar{p} \cdot \bar{j}_n \, dv. \]  \hspace{1cm} (69)

And an auxiliary differential equation is needed to calculate the \( \bar{J}_L \).

\[ \frac{\partial \bar{J}_L(t)}{\partial t} = \frac{1}{AL} \bar{p} \cdot \bar{E}(t). \]  \hspace{1cm} (70)

In discrete DGFETD-EM formulation, this can be shown as:
\[ M^{\text{hh}}_{\mu} h' = -S^{\text{he}} e - F^{\text{he}} e^+, \quad (71) \]

\[ M^{\text{ee}}_{\varepsilon} e' = S^{\text{eh}} h + F^{\text{eh}} h^+ - C^{\text{ej}} L j_L, \quad (72) \]

\[ j_L' = \frac{1}{AL} e. \quad (73) \]

It is realized that an auxiliary variable will be needed to represent the time-accumulated electric current density.
3.2.4. Lumped SPICE Circuit Port

When the circuit port contains general interconnected circuit elements, it will be non-trivial to map the circuit behavior to a specific material property, since the voltage-current relationship will no longer be governed by simple equations. Instead, a SPICE network solver can be applied to obtain the voltage-current relation within the lumped circuit port. A SPICE solution is based on a Modified Nodal Analysis (MNA). The MNA formulation is based on a nodal analysis, which enforces Kirchoff’s current law at every non-reference node in the network. The unknowns of the problem are the non-reference node voltages and the currents through each voltage source. The latter is what distinguishes a MNA from a standard nodal analysis. In such case, the circuit coupling matrix can be expressed as:

\[
C_{SPICE}^{ee} = \int_{V_c} \tilde{T} \cdot \frac{1}{A} M_{\text{MNA}} \left( \int_{V_c} \hat{p} \cdot \tilde{E} dv, t \right) \hat{p} dv, \tag{74}
\]

where \(\tilde{T}, \tilde{E}\) are the test vector and basis vector, respectively, \(A\) is the cross sectional area of the volume port and \(\hat{p}\) is the unit vector specifying the port direction. The circuit simulator will take the port voltage as an input and calculate the port current internally as the output. The \(C_{SPICE}^{ee}\) matrix is actually combining the voltage probe calculation and \(\bar{J}_c\) excitation together. If the port is a surface circuit port, then

\[
C_{SPICE}^{ee} = \int_{V_c} \tilde{T} \cdot \frac{1}{W} M_{\text{MNA}} \left( \int_{S_c} \hat{p} \cdot \tilde{E} ds, t \right) \hat{p} ds, \tag{75}
\]

where \(W\) is the width of the surface port and MNA is the circuit simulation function. For a single port system, one can re-write the circuit coupling matrix in (75) in an operator form:
\( C_{\text{SPICE}}^{\text{ee}} = P_{\text{ext}}^{\text{ei}} \cdot Q_{\text{MNA}}^{\text{iv}} \cdot P_{\text{prb}}^{\text{ve}}, \) (76)

where \( P_{\text{prb}}^{\text{ve}} \) is the export matrix (voltage probe matrix) with dimension \( 1 \times n \) where \( n \) is the number of \( \vec{E} \)-DOFs within the circuit port. \( P_{\text{prb}}^{\text{ve}} \) will project the \( \vec{E} \)-field within the circuit port onto the port axis \( \hat{p} \) and integrate along the port axis to obtain a port voltage. It is defined as:

\[
\left[ P_{\text{prb}}^{\text{ve}} \right]_{m,1} = \int_{V_i} \hat{f}_n \cdot \hat{p} dv ,
\] (77)

where \( \hat{f}_n \) is the \( n^{th} \) basis function of the local sub-domain \( V_i \). \( Q_{\text{MNA}}^{\text{iv}} \) is the circuit simulator that will perform the MNA and transforms the port voltage to the port current. \( Q_{\text{MNA}}^{\text{iv}} \) is a 1 by 1 matrix. Reciprocally, \( P_{\text{ext}}^{\text{ei}} \) is the import matrix (\( J_c \) excitation matrix) with dimension \( n \times 1 \). It will inject the electric current density as an equivalent source that will radiate back to the DGFETD-EM field solver. It is defined as:

\[
\left[ P_{\text{ext}}^{\text{ei}} \right]_{1,n} = \int_{V_i} \vec{p} \cdot \vec{f}_n dv .
\] (78)

It is observed that:

\[
P_{\text{ext}}^{\text{ei}} = \left( P_{\text{prb}}^{\text{ve}} \right)^T .
\] (79)

The advantage of introducing the \( C_{\text{SPICE}}^{\text{ee}} \) matrix into DGFETD-EM field formulation is that it is operating on the DOFs for \( \vec{E} \)-field within the port. This will update the voltage input within the RK scheme, so that the feedback current will match the time as RK performs the time evolution, which will not raise extra stability controls on the RK scheme. In discrete form, this can be shown as:
\[ M^{hh'}_{hh'} = -S^{he} e - F^{he} e', \quad (80) \]

\[ M^{ee}_{ee'} = S^{eh} h + F^{eh} h' + C^{ee}_{SPICE} e. \quad (81) \]

As an example, consider the network shown in Figure 4, the nodal equations are written to enforce KCL such that the net current flowing out of each node sums to zero. In this particular case, there are three non-reference nodes. KCL for each node is written as:

\[ V_1 - V_2 + I_{V_1} = 0, \quad (82) \]
\[ V_2 - V_1 + \frac{V_2}{R_1} + \frac{V_2}{R_2} + \frac{V_2 - V_3}{R_3} = 0, \quad (83) \]
\[ V_3 - V_2 + I_{V_2} = 0. \quad (84) \]

The voltage constraints are added in:

\[ V_1 = V_{s_1}, \quad (85) \]
\[ V_3 = V_{s_2}. \quad (86) \]
Finally, (82)-(86) provide five equations for the five unknowns. This can be expressed in matrix form as:

\[
\begin{pmatrix}
\frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 & 0 \\
-\frac{1}{R_1} & \frac{1}{R_2} + \frac{1}{R_3} & -\frac{1}{R_3} & 0 & 0 \\
0 & -\frac{1}{R_3} & \frac{1}{R_3} & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
I_{v_1} \\
I_{v_2}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
V_s_1 \\
V_s_2
\end{pmatrix},
\]

which is a symmetric matrix that can be used to solve for the node voltages and the source currents.

In the hybrid model, the independent voltage sources can either be internal to the lumped circuit network, or be represented via a port voltage computed during the DGFETD-EM field simulation. In this case, the port current will couple back into the DGFETD-EM field solver. To this end, the MNA system of equations can be expressed in a more general form as:

\[
\begin{pmatrix}
Y & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
v_{sp} \\
i_v
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
v_s
\end{pmatrix},
\]

where \(Y\) represents the admittance matrix that couples the node voltages, \(B\) is the matrix block that ties the node voltage to the voltage source branch currents, \(v_s\) represents the forcing vector which is the voltage calculated from each circuit port, \(v_{sp}\) represent the vector of node voltages and \(i_v\) represents the vector of independent voltage source currents.
When the circuit contains reactive elements, the MNA also adds inductor currents to the vector of unknowns. The currents are constrained to the node voltages via the constitutive relation $V = L(\frac{dI}{dt})$. When solving non-linear circuits, the non-linearities are represented via non-linear expansions. Thus, in general, the MNA leads to an expression of the form:

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{sp} \\ i_v \end{pmatrix} + \begin{pmatrix} Y & B^T \end{pmatrix} \begin{pmatrix} x_{sp} \\ i_v \end{pmatrix} + \begin{pmatrix} i(x_{sp}) \\ 0 \end{pmatrix} = \begin{pmatrix} -i_s \\ v_s \end{pmatrix},$$

(89)

where $x_{sp}$ represents the vector of unknown node voltages and inductor currents, $\dot{x}_{sp} = \frac{dx_{sp}}{dt}$, $i(x_{sp})$ represents a vector of non-linear currents which are a function of $x_{sp}$, $i_s$ represents the contribution from independent current sources within the network, and $i_v$ represents the vector of independent voltage source currents, including the currents that flows through the DGFETD circuit ports.
3.3. Time Domain Coupling of the Hybrid DGFETD-EM/Circuit Solvers

There are specific logistics involved in interfacing the DGFETD-EM and the SPICE software. The objective was to keep the two software packages separate, and have them interface at each time step by passing the port voltages to SPICE, and then allowing the SPICE library to compute the port electric current at each time step, and finally let the port electric current density radiate back to the DGFETD-EM field system. Figure 5 provides an illustration of such exchange of information.

![Figure 5. Interactions between DGFETD-EM and SPICE Solvers](image)

In order to allow DGFETD-EM field solver and SPICE transient solver work interactively through the circuit port, it is necessary to perform the SPICE transient analysis step by step. As demonstrated in Figure 5, the red surface is the surface circuit port defined in the DGFETD mesh. The circuit structure it is representing is a series connected RC circuit. The SPICE configuration commands are included inside the red
frame in which the green colored commands will be updated by the data obtained from DGFETD-EM solver as well as the circuit history data at each time step. A subtle point that needs to be emphasized is that when reactive components are included in the circuit, the SPICE model will have to record its historical state, such that when computing the next time step the initial conditions of reactive elements can be properly set. For example when performing a transient analysis from 0ns to 60ns, normal SPICE simulation will perform a single transient analysis from 0ns to 60ns. However, when interacting with DGFETD-EM solver, the simulation is divided into 60 time steps, thus at each time step (1ns), it is necessary to:

1. Calculate $V$ from the line integral of the $\vec{E}$-field through the circuit port;

2. $V$ will be set as the input of the SPICE simulation through piece-wise linear voltage source (“VPWL”). Reset the SPICE configuration commands by updating the VPWL source voltage and the initial condition of the circuit, including all historical circuit status parameters such as, node voltages, current flows through all inductors and all DC values of the sources;

3. SPICE takes $V$ and all initial parameters as inputs and perform transient analysis from $t$ to $t + k_i \Delta t$;

4. $I_c(t + k_i \Delta t)$ is computed via SPICE and is returned to the DGFETD-EM routine

5. The port electric current $I_c$ is radiated back to the DGFETD-EM field solver by converting it into a port electric current density $\vec{J}_c$;

6. At the current time $t + k_i \Delta t$, record all necessary data that is relative to the current circuit status as the initial condition for the next time step calculation.
3.4. Validation

In this section, the hybrid DGFETD-EM/Circuit solver is validated. A 50 ohm microstrip transmission line is used as the base model, on which various types of circuit structures will be mounted. As shown in Figure 6 (a) and (b), the 50 ohm microstrip transmission line has a width of 1.5 mm, a length of 24 mm and a thickness of 0.035 mm. Such thin metal is mounted on a substrate plane with the width of 6 mm and the length of 30 mm. Its thickness is set to be 0.761 mm and the relative permittivity and permeability of the substrate are set to be 4.2 and 1.0, respectively.

The whole domain is discretized with finite hexes as shown in Figure 6 (c), of which the visibility of the mesh for free-space is turned off. The 3D problem domain is terminated with 4 cells thick PMLs on five sides of the rectangular domain (excluding the ground plane), as illustrated by the shaded regions of Figure 6 (a) and (b). The bottom side of the box is set to be PEC which is representing the ground plane of the model.

Various configurations of the circuit loads are chosen to terminate such microstrip transmission line model. Initially, the direct stamping approach is validated through a parallel RLC block terminating the microstrip line. Next arbitrary SPICE circuit loads are put at the end and the middle of the microstrip line, respectively. Later on, an array of SPICE circuit ports is mounted with the microstrip line. Finally, a non-linear diode circuit is applied. Details of the validated results are presented in the following sections.
Figure 6. Microstrip line on a substrate
(a) top view; (b) side view; (c) meshed models
3.4.1. Two Linear Loads at Both Ends

In this section, lumped circuit elements are configured at both ends of the transmission line. As shown in Figure 7 and Figure 8, in which the circuit loads can be modeled by either volume ports or surface ports. The circuit model of the loads can be variously specified at the circuit solver part.

Figure 7. Microstrip line terminated by two volume circuit loads

Figure 8. Microstrip line terminated by two surface circuit loads
3.4.1.1. Validation of Direct Stamping Approach

The validation test case is the volume circuit port terminating the microstrip line. The circuit model of the load is a parallel RLC circuit with the resistance of 50 ohm, inductance of 8 nH and capacitance of 2 pF, as shown in Figure 9. In this simulation, the direct stamping approach, as presented in (50)-(73), is applied and validated. To this end, the circuit coupling matrices, including the resistive coupling matrix shown in (57), the capacitive coupling matrix shown in (64) and the inductive coupling matrix shown in (72), were directly calculated within DGFETD-EM solver. The excitation is configured at the beginning terminal of the microstrip line, as shown in Figure 7 and Figure 8. A volume or a surface excitation port can be applied. The source is modeled by an equivalent Norton current source, with the resistance set as 50 ohm in order to match the characteristic impedance of microstrip line. The current magnitude is set to be 0.01 A so that a 0.5 V voltage can be obtained and will be used to excite the microstrip line. The time-domain signature of the source is a Gaussian pulse, with $t_w$ set as 0.2 ns and $t_0$ is set to be $5t_w$. The bandwidth of the source is 1.59 GHz. To validate the simulation, the scattering parameter $S_{11}$ is investigated. To calculate $S_{11}$, the time-domain voltage signal $V(t)$ at the terminal of the microstrip line is sampled. Another simulation is also performed by terminating the microstrip line with a matched 50 ohm resistor, so that a non-reflected signal $V_r(t)$ is obtained at the terminal and can be used as the reference signal. The $S_{11}$ parameter can then be calculated as:

$$S_{11} = \frac{\tilde{V}(f) - \tilde{V}_r(f)}{\tilde{V}_r(f)},$$

(90)
where $\tilde{V}(f)$ and $\tilde{V}_r(f)$ are the Fourier transforms of the time-domain signals $V(t)$ and $V_r(t)$, respectively.

Figure 10 shows the magnitude of the $S_{11}$ parameter. These results are compared with analytical prediction assuming an ideal 50 ohm line. The results agree extremely well and hence the direct stamping approach for DGFETD-EM/Circuit simulator has been verified.

![Figure 9. Parallel RLC circuit](image)

![Figure 10. Magnitude of S11 parameter (parallel RLC)](image)
3.4.1.2. Validation of SPICE Circuit Port

Next, consider the 50 ohm microstrip line in Figure 6 terminated by arbitrary loads. The DGFETD-EM mesh is the same as Figure 6 (c). The distinction is that the terminating load is now modeled by a SPICE circuit port, which can be a volume or a surface. Various circuit models can be configured at the SPICE simulator side. Three examples were picked up to validate the hybrid DGFETD-EM/SPICE solver.

![Diagram of SPICE circuit models](image)

Figure 11. SPICE circuit models. (a) Series RLC; (b) Series R Parallel LC; (c) RLC network

As shown in Figure 11 (a)-(c), series RLC, series R parallel LC and complex RLC network are configured to the SPICE circuit port respectively. Each of them is simulated within the hybrid DGFETD-EM/SPICE solver. The same excitation configuration is applied and $S_{11}$ is investigated via the similar approach demonstrated in the previous test case. Both 0th-order and 1st-order hierarchical curl conforming finite element hex basis are applied in the DGFETD-EM simulation. On the SPICE solver part, SPICE libraries are revised to synchronize with the RK4 scheme as DGFETD-EM performs the time evolution. The results are demonstrated in the following.
**Series RLC Load**

Initially, consider a load that is a series RLC circuit consisting of a 50 ohm resistor, an 8 nH inductor, and a 2 pF capacitor, as shown in Figure 11 (a). The time-domain loaded signal \( V(t) \) and the reference signal \( V_r(t) \) are shown in Figure 12 (a). The Fourier transforms of the time-domain signals are illustrated in Figure 12 (b). From the frequency-domain data obtained, the scattering parameter \( S_{11} \) of the microstrip line terminated via this load is calculated via (90) and the magnitude and the phase advance are illustrated in Figure 13 (a) and (b) respectively. The results are also compared to an analytical result assuming an ideal 50 ohm transmission line. The results compare quite well in both magnitude and phase. From Figure 13 (b), it is observed that surface port achieves better accuracy in the phase of \( S_{11} \). Also both \( H(0) \) and \( H(1) \) basis functions are applied in the DGFETD-EM field solver. From the results shown in Figure 13, the solution is already converged as the basis function order increases from 0\(^{th}\)-order to 1\(^{st}\)-order.
Figure 12. Waveform (a) and FFT (b) of loaded/reference signal (Series RLC)

Figure 13. S11 magnitude (a) and phase (b) compared with exact (Series RLC)
**Series R Parallel LC Load**

Next, consider a load that consist a series R connected to a parallel LC circuit with resistor of 50 ohm, inductor of 8 nH, and capacitor of 2 pF, as shown in Figure 11 (b). The time-domain and frequency-domain data of the DGFETD/SPICE simulations is illustrated in Figure 14. The magnitude and phase of the scattering parameter $S_{11}$ of the microstrip line terminated via this load is illustrated in Figure 15. Again both surface and volume circuit ports are applied in the simulation respectively. And $H(0)$ and $H(1)$ basis functions are applied in the field solver. All the four simulated results are summarized together and compared to an analytical result assuming an ideal 50 ohm transmission line. Again excellent agreement is observed from the results. From Figure 15, It is observed that surface port achieves better accuracy in the prediction of $S_{11}$. 
Figure 14. Waveform (a) and FFT (b) of loaded/reference signal (Series R, Parallel LC)

Figure 15. S11 magnitude (a) phase (b) compared with exact (Series R, Parallel LC)
Complex RLC Network Load

Lastly, consider a load that is a more complex network of resistors, capacitors and inductors, as illustrated in Figure 11 (c). This was simulated using the hybrid DGFETD/SPICE solver. The time-domain and frequency-domain data is shown in Figure 16. The scattering parameter $S_{11}$ of the microstrip line terminated with this load is illustrated in Figure 17. The results are also compared to results simulated by the Agilent Advanced Design System (ADS) software. The results compare quite well in magnitude. There are some disagreements in phase. This is due to the edge effects of microstrip line. Surface ports are still showing better accuracy than those generated via volume ports.
Figure 16. Waveform (a) and FFT (b) of loaded/reference signal (RLC Network)

Figure 17. S11 magnitude (a) and phase (b) compared with ADS result (RLC Network)
3.4.2. Three Linear Loads in the Middle and Both Ends

Next, consider the 50 ohm microstrip line loaded with a series connected surface port in the middle, as illustrated in Figure 18. The line is again excited via a 50 ohm Norton source and is terminated by a 50 ohm matching load. The same time-domain signature of the excitation is applied here, which is a Gaussian pulse with a 0.2 ns half pulse width. Both terminating circuit ports are modeled via surface. The centered surface load can be configured with a circuit model, which will be set via SPICE. The line, substrate and mesh dimensions are the same as what is shown in Figure 6.

Figure 18. Microstrip line mounted with three surface circuit loads at the middle and both ends
**Series RLC load**

The results of the DGFETD/SPICE simulations for the series RLC circuit shown in Figure 11 (a) are illustrated in Figure 19. The scattering parameter $S_{11}$ of the microstrip line terminated via this load is illustrated in Figure 20. The results are also compared to an analytical result assuming an ideal 50 ohm transmission line.

![Figure 19](image1.png) **Figure 19.** Waveform (a) and FFT (b) of loaded/reference signal (Series RLC)

![Figure 20](image2.png) **Figure 20.** S11 magnitude (a) and phase (b) compared with exact (Series RLC)
**Series R, Parallel LC load**

For the circuit that contains a resistor R in series with a parallel LC, as shown in Figure 11 (b). The results of the DGFETD/SPICE simulations are illustrated in Figure 21. The scattering parameter $S_{11}$ of the microstrip line terminated via this load is illustrated in Figure 22. The results are also compared to an analytical result assuming an ideal 50 ohm transmission line.

![Waveform (a) and FFT (b) of loaded/reference signal (Series R, Parallel LC)](image1.png)

**Figure 21.** Waveform (a) and FFT (b) of loaded/reference signal (Series R, Parallel LC)

![S11 magnitude (a) and phase (b) compared with exact (Series R, Parallel LC)](image2.png)

**Figure 22.** S11 magnitude (a) and phase (b) compared with exact (Series R, Parallel LC)
3.4.3. Series-Shunt Connected Linear Load Array

After the validations of several basic circuit element configurations, a more challenging circuit model is applied as the test case in this section. As shown in Figure 23, the microstrip line now is mounted with a series-shunt connected surface load array. The equivalent circuit model of such test case is illustrated in Figure 24. In which the parallel surface load consists a pair of series connected 10 ohm resistor and a 5 nH inductor, the perpendicular surface load consists a pair of series connected 10 ohm resistor and a 2 pF capacitor. Four pairs of such series-shunt connected loads are mounted in the microstrip line, as shown in Figure 23. And the beginning terminal of the microstrip line is again modeled with a matched Norton source. Hence there are, including the source part, nine surface loads configured in this test case.

Figure 23. Microstrip line mounted with a series-shunt surface circuit load array

Figure 24. Circuit model of series-shunt connected RLC array
The same surface excitations and time-domain source signatures are used here, the problem are simulated by the DGFETD-EM/SPICE solver. The time-domain signals and frequency-domain conversions are illustrated in Figure 25. And the scattering parameter $S_{11}$ is investigated in Figure 26. It is observed that the good agreements are still obtained and the improvement of using high-order basis functions in the DGFETD-EM solver side has also been observed.

![Figure 25. Waveform (a) and FFT (b) of loaded/reference signal (RLC Array)](image)

![Figure 26. S11 magnitude (a) and phase (b) compared with exact (RLC Array)](image)
3.4.4. Non-linear Diode Detector Circuit

Finally, consider a non-linear circuit consist a diode detector terminating the microstrip line shown in Figure 6. Figure 27 illustrates the mesh, with a series connected surface port modeling a series diode, and a shunt connected volume port modeling a parallel RC load. The SPICE model for this circuit is illustrated in Figure 28. Different than the previous test cases, the microstrip line was excited with a modulated Gaussian pulse in this simulation, with the carrier frequency set as 2 GHz, and the Gaussian pulse had a 10 ns half pulse width. The diode rectifies the signal, and the parallel RC circuit performs a charge and hold of the signal, such that the output approximates the envelope of the modulated pulse. To validate the DGFETD-EM/SPICE solver in this particular test case, the simulation was also performed in PSPICE software as a comparison.

![Microstrip line terminated with a series surface load and a shunt volume load](image1)

**Figure 27. Microstrip line terminated with a series surface load and a shunt volume load**

![Circuit model of the diode signal detector](image2)

**Figure 28. Circuit model of the diode signal detector**
Figure 29 illustrates the modulated waveform of the DGFETD-EM/SPICE and PSPICE line voltages at the beginning terminal of the microstrip line. It should be noted that PSPICE takes the exact same signal waveform as what has been applied in DGFETD-EM. This is done by exporting the time-domain excitation data from the DGFETD-EM solver and importing the data file directly into PSPICE as a piece-wise linear voltage source. Figure 30 illustrates the output voltage across the parallel RC load with the recovered envelop of the signal computed using the DGFETD-EM/SPICE solver and PSPICE. The direct PSPICE and the DGFETD-EM/SPICE simulations compare very well. The small difference of the output signal is caused by edge effects.

![Figure 29. Voltage signal at the beginning of the transmission line](image)

![Figure 30. Voltage signal across the RC load at the end of the transmission line](image)
3.5. Summary

In this chapter, a circuit port that interfaces the full-wave (DGFETD-EM) and circuit (SPICE) solvers was introduced. The circuit port can be modeled by either a volumetric region or a surface embedded within the field mesh. The volume port region does not have to be defined by a single finite element, but can be defined by a group of finite elements. The port geometry is fixed to be rectangular and can be chosen independent of the field mesh. A hybrid formulation to incorporate passive lumped circuit elements such as resistors, capacitors and inductors has been derived. Simple voltage-current relations can be coupled into the DGFETD-EM framework by using the direct stamping approach. For complex configured circuits and non-linear devices, a hybrid DGFETD-EM/SPICE solver was developed, which dramatically broadens the types of devices that can be incorporated with DGFETD-EM solver. To validate the proposed approach, various test cases including both linear and non-linear devices, single and multiple circuit ports, were simulated by the hybrid DGFETD-EM/Circuit solver. The simulation results show very good agreement with a reference results obtained either from analytical predications or a third party, highly trusted simulator. Both the direct stamping approach and the SPICE interfacing functionalities have been validated.
Chapter 4. Thin-Wire Modeling within DGFETD-EM Solver

Thin-wires are often essential components of electromagnetic compatibility and interference (EMC/EMI) and antenna problems. They are also increasingly important to the full-wave simulation of high frequency circuit systems. Examples of thin-wire structures can be wire antennas of various shapes, wire bonds for microwave circuits, probe feeds of wave guides, and so on. Many modern communication, radar and medical instrument designs always require multiple sub-systems to be mounted within the same package and often with a large integration scale. Various components with different properties of the system are put together with very small separation. EM effects, such as radiation, edge and aperture diffraction, multiple scattering and coupling, creeping waves and resonance, will play a dominant role when the system is operating at high frequencies. Due to this, the risk of incompatibility among discrete components, such as crosstalk, is dramatically increased. For example, a typical EMI effect is the desensitization of a receiver, in which a strong disturbing signal transmitted by a nearby radiator is coupled and activates the automatic gain control of the receiver such that weaker desired signals are not properly processed. More catastrophic situations can occur if the impacted components are the critical parts of the system, such as the power supplying or clocking devices. In addition, sensitive circuitry such as signal and control lines can act as receiving antennas, and have unwanted signals induced on them which override the desired information. Because of these and other EM effects, the accurate prediction of the field coupling between thin-wires and other devices becomes an important part of the EMC/EMI analysis of such near-located components.
The research of thin-wire modeling can be generally categorized into three phases. Early studies are mainly focused on the theoretical derivations of the relationships between the Maxwell’s equations which describe the EM field behaviors and the telegrapher’s equations which govern the variations of the line voltages and currents. Such works can be found in [31-34]. With these efforts, the mathematical models for thin-wire radiating, scattering and coupling effects have been well documented. Later on, as CEM techniques developed, numerical solutions of Maxwell’s equations can be obtained with high accuracy and efficiency. Researchers began to tackle the challenge of how to model the thin-wire EM effects using the method of moments [35-39]. With these works, the EM effects of the thin-wire with both straight and curved shapes have been well studied and can be accurately predicted with low computational cost. More recently, as the capability of numerical analysis technique enhances, the concept of hybrid simulations, which incorporate devices of various properties that are simulated altogether within a single hybrid simulator engine, has become popular. A more general terminology for this concept is the multi-physics simulation. Inspired by this concept, the interest of EMC/EMI analysis of the thin-wire no longer only focuses on the prediction of the EM effects of a single or multiple thin-wires. It also requires the capability to accurately capture the interactive coupling between the thin-wires and many other devices, which may include complex materials and even nonlinear electrical properties. Such problems can be difficult to model in method of moments due to the complexity of the problem.

Other CEM techniques, such as the finite difference time-domain method (FDTD) [2] and the finite element method (FEM) [3] began to receive attention on how to incorporate
thin-wires in the EM simulations. In these methods, a three dimensional (3D) field mesh is always required. The ability to model thin-wire structures, that have features that are small relative to the global cell size in an efficient and accurate manner is critical. In general, a refined 3D mesh that conforms to the wire surface can be used to model small cylindrical wires. However, the cost can be prohibitive if the wire radius is very small relative to the global dimensions of the problem. Therefore, the development of accurate models that can characterize the physical effects of the thin-wire without increasing the resolution of the mesh is desirable.

There are two major stages for the research of thin-wire modeling in FDTD or FEM frameworks. The first modeling approach is to snap the thin-wire axis to the FDTD stencil or the FEM edge elements. Works, such as [40-44] are mainly focusing on building the snapped thin-wire models. The proposed approaches established the foundation for incorporating thin-wire models to FDTD or FEM schemes. However, the requirement for snapped thin-wires has limited the flexibility of simulating arbitrarily located and oriented thin-wires. Namely when the wire is slightly shifted or rotated, one has to either modify the field mesh in order to capture the wire position changes or to modify the shape and orientation of the wire to fit the mesh. In addition, when curved wires are included, FDTD thin-wire models have to rely on a stair-cased discretization, which often introduces large errors. FEM thin-wire models have to carefully treat the field mesh in order to build the edge elements on which the thin-wire is snapped. Due to such limit and inconvenience, efforts listed in [45-47] have been made to allow arbitrarily oriented thin-wires to be simulated in FDTD and FEM frameworks. Approaches to handle curved wire segments were also provided in these works. There are additional
logistics needed to handle arbitrarily oriented and curved thin-wires in the field mesh. But such approaches have significantly improved the flexibility of simulating thin-wires within FDTD and FEM frameworks.

In this paper, the thin-wire modeling technique proposed by Edelvik et al. [47] is extended to the Discontinuous Galerkin Finite-Element Time-Domain (DGFETD) method. A sub-cell model is proposed that can represent cables, wire antennas and transmission lines. The one dimensional wire is modeled as a transmission line. A one dimensional DGFETD formulation is used to solve the 1D telegrapher’s equations. The coupling between the 1D wire system and the 3D field system is performed through direct field coupling. That is, the local electric field will couple to the thin-wire, which acts as a distributed series scattered voltage source on the transmission line. Reciprocally, the currents on the wire will radiate back into the 3D field simulator as a current density source. The coupling between the two domains is implemented in manner similar to that proposed in [47], leading to a symmetric formulation. The geometry of the wires can be chosen independent of the 3D field mesh. Thus the proposed hybrid modeling approach supports wires that are arbitrarily located and oriented within the 3D mesh. This increases the modeling flexibility dramatically. In addition, a method to treat curved wire segments is also provided. The rest of the paper is organized as follows. In Section 4.1, a thin-wire port which can handle the field to voltage and current to field conversions is first designed. Then the 1D DGFETD formulations for the telegrapher’s equations are derived in Section 4.2. The hybrid DGFETD-EM/ThinWire formulation is proposed in Section 4.3. The treatment of curved wire segments is presented in Section 4.4. Finally, the
hybrid DGFETD-EM/ThinWire solver is validated using two test cases: including a thin-wire dipole antenna and a thin-wire loop antenna.

4.1. Thin-Wire Port Design

Prior to the derivation of the hybrid DGFETD-EM/ThinWire formulation, it is necessary to properly address the interactions between the wire and field systems. To this end, a thin-wire port with the shape of a center-hollowed cylinder is applied here to process the coupling. As shown in Figure 31, consider a discretized thin-wire segment with the length of \( h \). Each segment is treated as a single conducting transmission line and it does not have to be straight and can be curved. The axis of the wire is defined by the parametric coordinate \( p \) and has a unit vector in a 3D coordinate frame of \( \hat{p} \). The thin-wire has a radius of \( a \) which is assumed to be much smaller than its length. The radial electric field and the azimuthal magnetic field near the wire are also assumed to have a \( 1/r \) spatial dependence near the wire (for \( r > a \)). Finally, a cylindrical zone of influence...
is defined about the wire \((a < r < r_c)\) within which the near fields of the wire will be modeled.

![Diagram](image)

**Figure 32. Overlaps between the thin-wire port and 3D finite elements**

A more detailed illustration of the geometrical coupling between the thin-wire port volume and the 3D finite element mesh is demonstrated in Figure 32, in which the red cylindrical beam is representing the 1D meshed thin-wire segment. The thick yellow cylinder is the thin-wire port volume and the blue box meshed by 27 hexes is representing the 3D mesh of the space surrounding the thin-wire beam. It is noted that the thin-wire mesh is a separate mesh and can pass arbitrarily through the 3D field mesh.

To find the appropriate non-zero overlaps, some additional processing is needed. This is being done by placing the two meshes in a common octree. The overlap is found by searching through cells that lie in the same neighborhood of the octree group. This reduces the searching to a linear operation.

A critical reason of why the thin-wire port is designed as a much thicker cylinder compared to the thickness of the wire itself is to achieve the goal of arbitrary positioning of the thin-wire. Only when the coupling is performed within the thick thin-wire port
volume, multiple 3D finite elements will be included. Hence there will be sufficient number of DOFs modeling the near field around the wire. When the alignment of the wire is modified, the near field modeling will be “insensitive” to the wire position, since approximately the same number of DOFs is used to model the near field. If the thin-wire port cylinder is too thin, then only the DOFs of the hexes that are just containing the wire will be used to model the near field. When the wire is re-positioned, the near field model will highly rely on the wire orientation because there is not sufficient number of DOFs included, and in such case the coupling is very “sensitive” to the wire position. Consequently, to achieve an accurate model of the field-wire coupling with arbitrarily positioned wires, one has to apply a sufficiently thick thin-wire port volume.

In general, the function of a port is to perform the forward and backward transformations of two sets of parameters which couple between two different solvers, respectively. The thin-wire port designed above will perform the appropriate coupling. In more detail, this can be described as: compute a scattered voltage $V_{\text{scat}}$ excited on the 1D thin-wire transmission line from the surrounding $\vec{E}$ -field within the port obtained from the DGFETD-EM field solver. $V_{\text{scat}}$ will drive the wire as a distributed series voltage source. The scattered voltage is computed as:

$$V_{\text{scat}} = - \int_{V_{\text{tw}}} \vec{E} \cdot g(r) r \cdot \hat{p} dv,$$

(91)

where $V_{\text{tw}}$ is the cylindrical thin-wire port volume defined over the radius $a < r < r_o$. $g(r)$ is a weighting function over the radial direction of the cylindrical thin-wire port. A convenient choice for $g(r)$ as suggested by Edelvik, et al. [47], for straight wires is:
where $r_o$ is the radius of influence zone of the thin-wire. Typically, $r_o \sim d$, where $d$ is the average cell radius of the 3D DGFETD-EM field mesh in the vicinity of the thin-wire.

It should be noted that such weighting function $g(r)$ is not necessarily unique, as long as the following condition is satisfied:

$$2\pi \int_{r \geq a} g(r) r dr = 1,$$  \hspace{1cm} (93)

the energy conservation will be preserved during the coupling. Some examples of $g(r)$ expressed in (92) are plotted in Figure 33.
From Figure 33, it is realized that $g(r)$ governs the shape of the $\mathbf{J}_c$ distribution along the radial direction of the thin-wire port and it has an impact of the accuracy of the field-wire coupling.

Reciprocally, the electric current $I$ flowing through the wire will be smeared out through the cylindrical coupling zone and hence is transformed into a volumetric electric current density $\mathbf{J}_c$, which will radiate back to the DGFETD-EM field system as a source. The relation between the total port electric current $I$ and the density $\mathbf{J}_c$ can be expressed as:

$$I = \int_{vw} \mathbf{J}_c \cdot g(r) r \cdot \hat{p} dv.$$ (94)

Therefore $\mathbf{J}_c$ in the cylindrical coordinate can be calculated as:

$$\mathbf{J}_c(r, \phi, p) = I(p) g(r) \hat{p},$$ (95)

where, $r$ is the local coordinate of the radial direction, $\phi$ is the rotation angle and $p$ is the local coordinate of the axial direction of the cylindrical thin-wire port.

In summary, the coupling between the DGFETD-EM solution and the thin-wire is defined via (91)-(95). How this impacts the global formulation is the topic of the following sections.
4.2. DGFETD-ThinWire Formulation

Consider the 1D domain $W$ bound by $\partial W$ representing the thin-wire which supports line voltages and currents $V$ and $I$, respectively. $V$ and $I$ are assumed to satisfy the telegrapher’s equations which are to be solved via a 1D DGFETD method. The wire is spatially decomposed into $N$ sub-domains. The $i^{th}$ sub-domain is represented by a curve $C_i$, as shown in Figure 34. The curve is bound by two vertices $p_a$ and $p_b$. Hence $W = \bigcup_{i=1}^{N} C_i$. Assuming a lossless wire, then within each sub-domain, the line voltages and currents must satisfy the coupled telegrapher’s equations:

\[
C_w \frac{\partial V}{\partial t} = - \frac{\partial I}{\partial p} + I_{imp}^{\text{imp}}, \tag{96}
\]

\[
L_w \frac{\partial I}{\partial t} = - \frac{\partial V}{\partial p} + V_{imp}^{\text{imp}}, \tag{97}
\]

where $C_w, L_w$, are the per-unit length capacitance and inductance, respectively. Holland and Simpson [40] approximate the per-unit length inductance of an isolated thin-wire as:

\[
L_w = \frac{\mu}{2\pi} \log \frac{r_a + a}{2a}, \tag{98}
\]
where \((r_u + a)/2\) is the chosen since it is the average distance from the thin-wire to the surrounding electric fields [47]. The per-unit length capacitance is then expressed as [40]:

\[
C_w = \frac{\varepsilon \mu}{L_w}.
\]  

These values were chosen based on the assumed \(1/r\) variation of the local fields. For the source terms in (96) and (97), \(V_{C_i}^{\text{imp}}\) is the impressed shunt voltage source. \(I_{C_i}^{\text{imp}}\) is the impressed series current source. Applying Galerkin method, the inner product of (96) and (97) is then performed with a set of test functions over the support of the sub-domain, leading to:

\[
C_w \frac{\partial}{\partial t} \int_{c_i} T^V \cdot V dp = -\int_{c_i} T^V \cdot \frac{\partial I}{\partial p} dp + \int_{c_i} T^V \cdot I_{C_i}^{\text{imp}} dp, \tag{100}
\]

\[
L_w \frac{\partial}{\partial t} \int_{c_i} T^I \cdot I dp = -\int_{c_i} T^I \cdot \frac{\partial V}{\partial p} dp + \int_{c_i} T^I \cdot V_{C_i}^{\text{imp}} dp, \tag{101}
\]

where the test function \(T^V\) shares the same function space as \(V\) and \(T^I\) shares the same function space as \(I\). It can be proved that the following identity holds:

\[
\int_c A \cdot \frac{\partial B}{\partial p} dp = A \cdot B \bigg|_{p_a}^{p_b} - \int_c B \cdot \frac{\partial A}{\partial p} dp. \tag{102}
\]

Therefore, the integration of \(A \cdot \frac{\partial B}{\partial p}\) over the curve \(C\) will be impacted by the boundary values of \(A \cdot B^-\). If we define the two bounding vertices of the curve \(C\) as \(p_a\) and \(p_b\), as shown in Figure 34, then the boundary term appeared in (102), can be evaluated as:

\[
A \cdot B^- \bigg|_{p_a}^{p_b} = A(p_b) B^-(p_b) - A(p_a) B^-(p_a). \tag{103}
\]

Apply (102) to (100) and (101) respectively, we obtain:
Central Flux Formulation

For simplicity, consider the central flux formulation to evaluate the boundary terms $V^{-\mid_{p_o}}$ and $I^{-\mid_{p_o}}$ in (104) and (105), respectively, according to the boundary conditions defined at the bounding points:

\[
\left( I^{-} - I^{+} \right) \mid_{p_o} = 0, \quad (106)
\]

\[
\left( V^{-} - V^{+} \right) \mid_{p_o} = 0. \quad (107)
\]

Based on (106) and (107), one can obtain the following relationships:

\[
I^{-\mid_{p_o}} = \frac{1}{2} \left( I^{-} + I^{+} \right) \mid_{p_o}, \quad (108)
\]

\[
V^{-\mid_{p_o}} = \frac{1}{2} \left( V^{-} + V^{+} \right) \mid_{p_o}, \quad (109)
\]

where $I^{-}, V^{-}$ are the current and voltage inside of $C_i$ and $I^{+}, V^{+}$ are the current and voltage just outside. Substitute (108) and (109) into (104) and (105) respectively, we obtain:

\[
C_w \frac{\partial}{\partial t} \int T^V \cdot V dp = \int I \cdot \frac{\partial T^V}{\partial p} dp - \frac{1}{2} T^V \cdot \left( I^{-} + I^{+} \right) \mid_{p_o} + \int T^V \cdot I_i^{\text{imp}} dp, \quad (110)
\]

\[
L_w \frac{\partial}{\partial t} \int T^I \cdot I dp = \int V \cdot \frac{\partial T^I}{\partial p} dp - \frac{1}{2} T^I \cdot \left( V^{-} + V^{+} \right) \mid_{p_o} + \int T^I \cdot V_i^{\text{imp}} dp. \quad (111)
\]
Re-apply the identity shown in (102) as the following:

\[ A \cdot B \Big|_{p_b}^{p_a} = \int_c A \cdot \frac{\partial B}{\partial p} \, dp + \int_c B \cdot \frac{\partial A}{\partial p} \, dp. \]  

(112)

Then the boundary terms in (110) and (111) can be re-written as being:

\[ \frac{1}{2} T^V I \bigg|_{p_b}^{p_a} = \frac{1}{2} \int_{c_i} T^V \cdot \frac{\partial I}{\partial p} \, dp + \frac{1}{2} \int_{c_i} I \cdot \frac{\partial T^V}{\partial p} \, dp, \]  

(113)

\[ \frac{1}{2} T^I V \bigg|_{p_b}^{p_a} = \frac{1}{2} \int_{c_i} T^I \cdot \frac{\partial V}{\partial p} \, dp + \frac{1}{2} \int_{c_i} V \cdot \frac{\partial T^I}{\partial p} \, dp. \]  

(114)

Applying them back to (110) and (111) respectively, a final DGFETD-ThinWire central flux formulation can be obtained.

\[ C_w \frac{\partial}{\partial t} \int_{c_i} T^V \cdot V \, dp = -\frac{1}{2} \int_{c_i} \left( T^V \cdot \frac{\partial I}{\partial p} - I \cdot \frac{\partial T^V}{\partial p} \right) dp + \int_{c_i} T^V \cdot I_{p_b} \, dp - \frac{1}{2} T^V \cdot I_{p_a} \bigg|_{p_b}^{p_a}, \]  

(115)

\[ L_w \frac{\partial}{\partial t} \int_{c_i} T^I \cdot I \, dp = -\frac{1}{2} \int_{c_i} \left( T^I \cdot \frac{\partial V}{\partial p} - V \cdot \frac{\partial T^I}{\partial p} \right) dp + \int_{c_i} T^I \cdot V_{p_b} \, dp - \frac{1}{2} T^I \cdot V_{p_a} \bigg|_{p_b}^{p_a}. \]  

(116)

**Basis Functions**

Each segment supports a local basis function expansion, which is a set of 1D scalar functions complete to order \( N \). Thus, there are \( N + 2 \) basis functions per segment. In the local coordinates, the 0\(^{th}\)-order basis functions are defined as:

\[ \Phi_0^{ID}(L_1) = 1 - L_1, \quad \Phi_1^{ID}(L_1) = L_1, \]  

(117)

where \( L_1 \) is the local coordinate which has a range of (0,1). For \( N^{th} \) order, there are \( N \) additional basis functions being defined:

\[ \Phi_{n+2}^{ID}(L_1) = (L_1)(1-L_1)P_n(1-2L_1); \quad n = 0..N-1, \]  

(118)
and $P_n$ is the $n^{th}$ order Legendre polynomial. Some examples of such basis functions are shown in Figure 35.

![Figure 35. High order one-dimensional basis function](image)

Given this basis function space, we expand the line voltage and current as:

$$ V(p, t) = \sum_{n=0}^{N} \Phi_n^{1D} \left( \frac{p}{h} \right) v_n(t), \quad I(p, t) = \sum_{n=0}^{N} \Phi_n^{1D} \left( \frac{p}{h} \right) i_n(t), $$

(119)

where $v_n$ and $i_n$ are the unknown coefficients to be solved for. $p$ is the local coordinate from $0 \sim h$, $h$ is the length of the thin-wire segment. This gives a polynomial complete basis up to order $N+1$. It is noted that only two basis functions contribute to the basis at the end points. That is, for the $0^{th}$-order basis: $\Phi_0^{1D}(0) = 1$, and $\Phi_1^{1D}(1) = 1$, all the other basis equal to 0 at $L_a = 0$ or $L_a = 1$, respectively. Consequently, only these two basis contribute to the boundary terms.
**Discretized Equations**

To this end, the line voltage and current are expanded using the proposed 1D basis functions. The test functions are expanded using the same set of functions. This leads to a discrete linear system of equations for each sub-domain. Using the central flux formulation of (115) and (116), one writes the first order differential equations as:

\[ M^v_i v' = -S^v_i + T^v_i i_{imp} - F^v_i i^p |_{p_u}, \]  

(120)

\[ M^i_v i' = -S^i_v v + T^i_v v_{imp} - F^i_v v^+ |_{p_u}, \]  

(121)

where \( i^p \) and \( v^+ \) are neighboring current and voltage DOFs, respectively. The \( T \) matrices represent the source operators. To this end, the DGFETD-ThinWire formulation for solving the telegrapher’s equations has been completed.
4.3. Hybrid DGFETD-EM/ThinWire Formulation

In order to perform the coupling between the thin-wire and field systems, one needs to apply a thin-wire port to exchange the wire and field data, this has been illustrated in Section 4.1. In this section, the details of the hybrid DGFETD-EM/ThinWire formulation will be demonstrated.

First, consider the coupling from DGFETD-EM field to the thin-wire equations, which can be expressed as:

\[
C_w \frac{\partial V}{\partial t} = -\frac{\partial I}{\partial p} + I_{imp}^t, \tag{122}
\]

\[
L_w \frac{\partial I}{\partial t} = -\frac{\partial V}{\partial p} + V_{imp}^t - \frac{\partial V_{scat}}{\partial p}. \tag{123}
\]

The scattered voltage \( V_{scat} \) appearing on the right-hand-side of (97) is acting as an additional series voltage source on the transmission line. From (91), it is realized that such \( V_{scat} \) term actually couples the DGFETD-EM field to the transmission line equations since \( \vec{E} \) is the driving field local to the thin-wire port provided by the DGFETD-EM field solver. And the derivative of \( V_{scat} \) over the parametric coordinate \( p \) on the 1D wire can be expressed as:

\[
\frac{\partial V_{scat}}{\partial p} = -\vec{E}(p) \cdot \hat{p}, \tag{124}
\]

where \( \vec{E}(p) \) is the electric field intensity and \( \hat{p} \) is the unit direction vector of the wire, both are functions of the local coordinate \( p \). Eqs. (124) illustrates that at a particular point \( p \) of the thin-wire, the \( \vec{E} \)-field at such point will act as a series voltage source that will drive the wire. Thus such scattered voltage governs the coupling from the field to the
wire. Following the steps of the DGFETD-ThinWire formulation, the modified telegrapher’s equations with the field-to-wire coupling term added can then be shown as:

\[
C_w \frac{\partial}{\partial t} \int_{c_i} T^v \cdot V dp = -\frac{1}{2} \int_{c_i} \left( T^v \frac{\partial I}{\partial p} - I \frac{\partial T^v}{\partial p} \right) dp \\
+ \int_{c_i} T^v \cdot I^{imp} dp - \frac{1}{2} T^v \cdot I^p|_{p_a},
\]

(125)

\[
L_w \frac{\partial}{\partial t} \int_{c_i} T^t \cdot I dp = -\frac{1}{2} \int_{c_i} \left( T^t \frac{\partial V}{\partial p} - V \frac{\partial T^t}{\partial p} \right) dp \\
+ \int_{c_i} T^t \cdot V^{imp} dp - \frac{1}{2} T^t \cdot V^t|_{p_a} + \int_{c_i} T^t \left( \vec{E} \cdot \hat{p} \right) dp.
\]

(126)

Next and reciprocally, consider the coupling from the thin-wire current to the DGFETD-EM field system. Such transmission line current will couple back into the local DGFETD-EM field equations by converting it into an electric current density source. In DGFETD-EM field formulation, this can be shown as:

\[
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^h \cdot \vec{H} dv = -\frac{1}{2} \int_{V_i} \left( \vec{T}^h \cdot \nabla \times \vec{E} + \vec{E} \cdot \nabla \times \vec{T}^h \right) dv - \frac{1}{2} \oint_{\partial V_i} \vec{T}^h \cdot \hat{n} \times \vec{E}^* ds,
\]

(127)

\[
\frac{\partial}{\partial t} \int_{V_i} \vec{T}^e \cdot \vec{E} dv = \frac{1}{2} \int_{V_i} \left( \vec{T}^e \cdot \nabla \times \vec{H} + \vec{H} \cdot \nabla \times \vec{T}^e \right) dv + \frac{1}{2} \oint_{\partial V_i} \vec{T}^e \cdot \hat{n} \times \vec{H}^* ds \\
- \int_{V_i} \vec{T}^e \cdot \vec{J} c dv.
\]

(128)

where an additional source term of electric current density \( \vec{J}_c \) has been added to (128).

**Discretized Equations**

For the non-coupling terms shown in eqs. (125), (126) and (127), (128), the discretization of the unknowns to be solved will follow the methods that have already been demonstrated in DGFETD-EM and DGFETD-ThinWire formulations, respectively. For the thin-wire coupling terms however, since the coupling between the field and wire...
are being computed through the thin-wire port, then some distinct operations need to be addressed. According to (95), we’ve shown the approach to convert the line current into the volume current density. And from (91), it is also clear of how to convert the surrounding $\vec{E}$-field within the thin-wire port into a scattered voltage. Nevertheless, based on (119), it has been realized that the line current and voltage on the thin-wire have been expanded through a set of 1D scalar basis functions and the DGFETD-EM fields are also expanded with a set of 3D finite element vector basis functions. Therefore, to express the converting relations shown in (91) and (95) more specifically, one needs to re-write them in an operator form which will provide more details of the computation.

Due to the application of the thin-wire port model, the reaction of the electric field with the thin-wire is not restricted to the thin-wire surface. Rather, the reaction occurs over the cylindrical volume of the thin-wire port. With this being realized, the coupling term in (126) should be re-written such that the reaction occurs over the cylindrical volume of the thin-wire port:

$$\int_{C_i} T' \cdot (\vec{E} \cdot \hat{p}) dp \approx \int_{V_i^{\text{TW}}} T' g(r) r (\vec{E} \cdot \hat{p}) dv,$$

(129)

where $C_i$ is the $i^{th}$ segment of the thin-wire mesh, $V_i^{\text{TW}}$ is the center-hollowed cylindrical volume that defines the $i^{th}$ thin-wire port associated with $C_i$. $T'$ shares the same 1D scalar basis functions as $I$ and can be expressed as:

$$T' (r, \phi, p) = \sum_{n=0}^{N_3} \Phi_n^{1D} \left( \frac{p}{r} \right).$$

(130)

$\vec{E}$ is expanded via $N_3$ 3D finite element vector basis
\[ \vec{E}(r, \phi, p) = \sum_{n=0}^{N_{3D}} \vec{F}^{3D}_n(r, \phi, p) \vec{e}_n. \] (131)

It should be noted that when the thin-wire port is applied, both the 1D and 3D basis functions have to be calculated with respect to the thin-wire port, which assumes a localized cylindrical coordinate system as the reference coordinate frame. Necessary coordinate transformations have to be performed so that the 1D and 3D basis can be mapped and coupled within the same coordinate system. As shown in Figure 32, there will be multiple 3D finite elements included in the thin-wire port. Assuming this number is \( P \), then the total number of the \( \vec{E} \)-field DOFs included in the thin-wire port is \( P \times N_{3D} \). Finally the coupling term appeared in (126) can be discretized in the following form:

\[ \vec{v}_{\text{scat}} = \mathbf{C}_{EM}^{\text{ie}} \mathbf{e}, \] (132)

where \( \vec{v}_{\text{scat}} \) is the DOF vector describing the series line voltage and it will drive the thin-wire transmission line by contributing to the DOFs of the line current. \( \mathbf{C}_{EM}^{\text{ie}} \) is the coupling matrix from the \( \vec{E} \)-field to the thin-wire. It has the dimension of \( N_{1D} \) by \( P \times N_{3D} \), \( \mathbf{e} \) is the \( \vec{E} \)-field DOF vector that is associated within the thin wire-port. For each entry in \( \mathbf{C}_{EM}^{\text{ie}} \):

\[ \left[ \mathbf{C}_{EM}^{\text{ie}} \right]_{m,n} = \int_0^h \int_e \int_0^{2\pi} \Phi_m \left( \frac{P}{h} \right) g(r) r \left[ \vec{F}^{3D}_n(r, \phi, p) \cdot \hat{p} \right] d\phi dr dp . \] (133)

Next, consider the reciprocal operation: the coupling from the line current on the thin-wire to the DGFETD-EM field system. Since the same thin-wire port is applied to such type of coupling, the operator can be expressed reciprocally as:
\[ e_{\text{scat}} = C^{ei} i = \left( C^{ie}_{EM} \right)^T i, \]  

(134)

where \( e_{\text{scat}} \) is the DOF vector for the scattered \( \vec{E} \)-field from the line current on the thin-wire. To this end, both the forward and backward coupling terms appear in (126) and (128), respectively, have been discretized with the thin-wire port being applied. The final discretized hybrid DGFETD-EM/ThinWire central flux formulation can be written as:

\[ M^\nu_C v' = -S^{\nu i} i + T^{\nu i} i_{\text{imp}} - F^{\nu i} i \bigg|_{p_b} + p_b, \]  

(135)

\[ M^\nu_L i' = -S^{\nu v} v + T^{\nu v} v_{\text{imp}} - F^{\nu v} v \bigg|_{p_b} + C^{ie}_{EM} e, \]  

(136)

\[ M^{he}_h h' = -S^{he} e - F^{he} e, \]  

(137)

\[ M^{eh}_e e' = S^{eh} h + F^{eh} h - C^{ei}_{TW} i. \]  

(138)

This set of equations are assuming that the excitation is put on the thin-wire, it should be noted that the excitations can be put either on the telegrapher’s equations as impressed voltage and current sources or on the Maxwell’s equations as electric and magnetic current density sources. Different source configurations will depend on the problem to be solved.
4.4. Time Domain Coupling of the Hybrid DGFETD-EM/ThinWire Solvers

The coupled sets of equations are solved using a common time integration scheme. The time stepping of the 1D DGFETD-ThinWire problem is coupled to the 3D DGFETD-EM field problem. The DGFETD-EM field solver is expected to have local time stepping based on a root time step $\Delta t$. The same root time step is used for both the 1D and the 3D DGFETD solvers. A high order solution algorithm is to be used. Typically, a high order Runge-Kutta (RK) scheme is employed [48]. Locally, the time step of the local 1D problem is anticipated to be the same of the local 3D problem.

To illustrate the time-domain coupling between the field and wire solvers, re-write (135) and (136) in a compact operator form:

$$y_{TW}^\prime = TW(y_{TW}^\prime, y_{EM}) = A_{TW}y_{TW} + B_{TW}y_{TW}^+ + T_{TW}y_{TW}^{imp} + C_{EM}y_{EM},$$

where

$$y_{TW}^\prime = \begin{pmatrix} v' \\ i' \end{pmatrix}, \quad y_{TW} = \begin{pmatrix} v \\ i \end{pmatrix}, \quad y_{TW}^+ = \begin{pmatrix} v^+ \\ i^+ \end{pmatrix}, \quad y_{TW}^{imp} = \begin{pmatrix} v^{imp} \\ i^{imp} \end{pmatrix}, \quad y_{EM} = \begin{pmatrix} h \\ e \end{pmatrix},$$

and

$$A_{TW} = \begin{pmatrix} 0 & -M_{L}^{vy}v \\ -M_{L}^{iy}i & 0 \end{pmatrix},$$

$$B_{TW} = \begin{pmatrix} 0 & -M_{C}^{vy}v \\ -M_{C}^{i}i & 0 \end{pmatrix},$$

$$T_{TW} = \begin{pmatrix} 0 & M_{C}^{vy}v \\ M_{L}^{i}i & 0 \end{pmatrix},$$

$$C_{EM} = \begin{pmatrix} 0 & 0 \\ 0 & M_{L}^{i}C_{EM} \end{pmatrix}.$$
Similarly, for the DGFETD-EM field solver, we can re-write (137) and (138) in the compact form as being:

\[
y_{EM} = EM(y_{EM}, y_{TW}) = A_{EM}y_{EM} + B_{EM}y_{EM}^+ + C_{TW}y_{TW},
\]

(145)

where

\[
y_{EM} = \begin{pmatrix} h' \\ e' \end{pmatrix}, \quad y_{EM} = \begin{pmatrix} h \\ e \end{pmatrix}, \quad y_{EM}^+ = \begin{pmatrix} h^+ \\ e^+ \end{pmatrix}, \quad y_{TW} = \begin{pmatrix} v \\ i \end{pmatrix},
\]

(146)

and

\[
A_{TW} = \begin{pmatrix} 0 & -M_{\mu}^{hh}S_{eh} \\ M_{ee}^{ee}S_{eh} & 0 \end{pmatrix},
\]

(147)

\[
B_{TW} = \begin{pmatrix} 0 & -M_{\mu}^{hh}F_{he} \\ M_{ee}^{ee}F_{eh} & 0 \end{pmatrix},
\]

(148)

\[
C_{EM} = \begin{pmatrix} 0 & 0 \\ 0 & M_{ee}^{-1}C_{TW} \end{pmatrix}.
\]

(149)

With the DGFETD compact operators on both the thin-wire and the EM fields being defined, one can present the time-domain coupling between the two independent transient solvers as the following:

\[
k_{1EM} = \Delta t \cdot EM(y_{EM}^{nEM}, y_{nEM}) \quad k_{1TW} = \Delta t \cdot TW(y_{nTW}^{TW}, y_{nEM})
\]

\[
k_{2EM} = \Delta t \cdot EM(y_{nEM}^{nEM} + \frac{1}{2}k_{1EM}, y_{nEM}^{EM} + \frac{1}{2}k_{1EM}) \quad k_{2TW} = \Delta t \cdot TW(y_{nTW}^{TW} + \frac{1}{2}k_{1TW}, y_{nEM}^{EM} + \frac{1}{2}k_{1EM})
\]

\[
k_{3EM} = \Delta t \cdot EM(y_{nEM}^{nEM} + \frac{1}{2}k_{2EM}, y_{nEM}^{EM} + \frac{1}{2}k_{2EM}) \quad k_{3TW} = \Delta t \cdot TW(y_{nTW}^{TW} + \frac{1}{2}k_{2TW}, y_{nEM}^{EM} + \frac{1}{2}k_{2EM})
\]

\[
k_{4EM} = \Delta t \cdot EM(y_{nEM}^{nEM} + k_{3EM}, y_{nEM}^{EM} + k_{3EM}) \quad k_{4TW} = \Delta t \cdot TW(y_{nTW}^{TW} + k_{3TW}, y_{nEM}^{EM} + k_{3EM})
\]

\[
y_{n+1} = y_{nEM}^{TW} + (k_{1EM} + 2k_{2EM} + 2k_{3EM} + k_{4EM}) / 6 \quad y_{n+1} = y_{nEM}^{TW} + (k_{1EM} + 2k_{2EM} + 2k_{3EM} + k_{4EM}) / 6
\]

This is assuming that an RK4 scheme is being applied. As observed in the above algorithm, at each slope calculation: \(k_1, k_2, k_3, k_4\), the two independent solvers exchange
the solution data and perform the coupling operation. Doing this is necessary to guarantee that the high-order accurate time integration can be maintained on both time-domain solvers. Of course, one may realize that if RK4 is used for both 1D and 3D solvers, then it can be implemented as a single RK4 with a combined operator on both 1D and 3D solutions. However, doing this will restrict the 1D and 3D solvers to use the same time-domain scheme. In the case when different time-domain schemes are applied to 1D and 3D solvers, e.g. RK2 for 1D and RK4 for 3D, one has to allow the two to solve for their time evolvement independently and synchronize them through additional process. Thus, the proposed approach of time-domain coupling between the 1D and 3D solvers has already considered a mixed time-integration scheme, or equivalently, a local time stepping scheme.
4.5. Additional Processing for Curved Thin-Wire Structures

The previous discussion assumed a straight wire. Bent wires can be similarly treated. However, the overlapping connection must be accounted for when computing the field wire coupling. There are three major deformations when the calculation encounters the bent region, the volume integration, the 1D basis function and the unit direction vector. These deformations will be demonstrated in more detail in the following sections. To begin with, the approaches to calculate some basic geometrical parameters are first presented, which can be prepared for later use. Consider the junction between two bent wires, as illustrated in Figure 36. If \( g(r) \) is to maintain a constant radial distance \( r_o \) from the wire axis, the wire path must be deformed. This is done in an intuitive manner such that in the overlapping region the wire is deformed onto a circular arc of radius \( r_o \) as illustrated in Figure 36. Let \( \vec{r}_i \) represent the position vector of the \( i^{th} \) node and then the unit direction vector of the \( i^{th} \) wire segment \( C_i \) can be calculated as:
\[ \hat{p}_i = \frac{\vec{r}_{i+1} - \vec{r}_i}{|\vec{r}_{i+1} - \vec{r}_i|}. \] (150)

Then, we can define the unit normal to the plane containing both wires to be:

\[ \hat{n}_{i,j-1} = \frac{\hat{p}_i \times \hat{p}_{i-1}}{|\hat{p}_i \times \hat{p}_{i-1}|} \] (151)

Of course, if \( \hat{p}_i \times \hat{p}_{i-1} = 0 \), then the wires are linear, and there is no bend. As illustrated in Figure 36, there are two cylindrical faces of radius \( r_o \) that intersect at a point \( \vec{r}_c \). The unit tangent vectors to these two faces in the plane containing the two wires are:

\[ \hat{t}_i = \frac{\hat{n} \times \hat{p}_i}{|\hat{n} \times \hat{p}_i|}. \] (152)

The sector angle \( \psi \) between the two cylindrical faces is then computed as:

\[ \cos \psi = \hat{t}_i^b \cdot \hat{t}_i^a. \] (153)

Note that the superscript \( a \) or \( b \) indicate if the face is with the junction of the previous wire or the next wire, respectively. The corner intersecting the two cylindrical faces \( \vec{r}_c \) is then computed via the simple relationship:

\[ \vec{r}_c = \vec{r}_i - \frac{r_o}{\cos (\psi / 2)} \frac{\hat{t}_i^b + \hat{t}_i^a}{|\hat{t}_i^b + \hat{t}_i^a|}. \] (154)

Finally, the points at which the wire intersects the cylindrical surfaces are:

\[ \vec{r}_i^{b} = \vec{r}_c + \hat{t}_i^b r_o, \quad \vec{r}_i^{a} = \vec{r}_c + \hat{t}_i^a r_o. \] (155)

and we can compute the intersecting points in axial coordinates as:

\[ p_i^{b} = p_{i-1} + |\vec{r}_{i-1}^{b} - \vec{r}_i^{b}|, \quad p_i^{a} = p_{i} + |\vec{r}_i^{a} - \vec{r}_i^{b}|. \] (156)
Deformation of the Integration over the Thin Wire Port Volume:

The integral over the volume of influence of the field for the straight wire regions is now expressed as:

\[
\int_{p_0}^{p_i} \int_{a}^{b} \int_{0}^{2\pi} g(r) r d\phi dr dp = p_i^{b} - p_i^{-1}.
\]  

(157)

Figure 37. Circular cross section of the sector region of the bent wire

The remaining volume region is over the curved cylindrical section. This integration is performed as follows. The volume region has a circular cross section, as illustrated in Figure 37. The differential volume of this cross section is simply \( r d\phi dr \). The third dimension is over the sector angle \( \phi \). The jacobian of integration along this direction is a function of the sector radius. Consider the integration point \( r' \) in Figure 37. The radius of the sector integration is the projection onto the vertical axis. Consequently, the differential length of the sector integral is \((r_0 + r \cos \phi) d\phi\). Finally, the volume integral over the sector for wire \( C_{r-1} \) is:

\[
\int_{0}^{\psi} \int_{r_0}^{r_0} \int_{0}^{2\pi} g(r) r (r_0 + r \cos \phi) d\phi dr dp = \int_{0}^{\psi} r_0 dr \int_{0}^{r_0} \int_{0}^{2\pi} g(r) r d\phi dr = \frac{\psi r_0}{2},
\]  

(158)
which is the arc length. Then, for the complete wire segment $C_i$, the volume integral can be obtained as:

$$\int_{\rho_i}^{\rho_i} \int_{a}^{2\pi} g(r) rd\phi dp + \int_{0}^{\psi} \int_{a}^{2\pi} g(r) r(r_{\phi} + r \cos \phi) d\phi drd\phi = p_{i+1} - p_{i}^{a} + \frac{\psi r_{0}}{2} \quad (159)$$

which is the summation of the contributions from both straight and the bent regions.

**Deformation of the 1D Basis Function:**

Besides the deformation of the volume integration over the bent region, the 1D basis function defined on the bent wire segment also needs to be deformed. This is because the total length of the meshed wire segment is modified due to the curved connection. The new length of the bent wire segment can now be calculated as:

$$h' = p_{i}^{b} - p_{i}^{a} + r_{\phi} (\psi_{a} + \psi_{b})/2 \quad (160)$$

This is the sum of the length of the straight part and the lengths of the curved parts at both ends of the wire segment. It should be noted that $\psi_{a}$ and $\psi_{b}$ are two independent angles, which are determined by the connections of $C_{i-1}, C_{i}$ and $C_{i}, C_{i+1}$ wire segments, respectively. The modified 1D basis function defined on the curved wire can be expressed as:

$$B(p) = \sum_{n=0}^{N} \Phi_{p}^{1D} \left( \frac{p}{h'} \right), \quad (161)$$

where the local coordinate $p$ with respect to the wire segment $C_i$ can be evaluated as:

$$p = \begin{cases} r_{\phi} & \text{bent region } a \ (0 \leq \phi \leq \psi_{a} / 2) \\ p & \text{straight region } \left( p_{a}^{b} < p < p_{b}^{b} \right) \\ p_{i}^{b} + r_{\phi} & \text{bent region } b \ (0 \leq \phi \leq \psi_{b} / 2) \end{cases} \quad (162)$$
Deformation of the Unit Direction Vector:

Another major deformed parameter is the unit direction vector $\hat{p}$. When entered into the bent region, such unit direction vector is rotated according to the curving rate of the bent region, as shown in Figure 38.

To properly calculate the rotated unit direction vector, Euler angles are applied to calculate the rotation tensor between two Cartesian coordinate systems. As shown in Figure 39, the rotations of an objective coordinate system (red marked) with respect to the reference coordinate system (blue marked) can be measured through three angles. Each angle shown in the figure can be calculated via the following:

\[
\begin{align*}
\alpha &= \arccos \left( -\frac{Z_2}{\sqrt{1 - Z_3^2}} \right) \\
\beta &= \arccos (Z_3) \\
\gamma &= \arccos \left( -\frac{Y_3}{\sqrt{1 - Z_3^2}} \right)
\end{align*}
\]  

(163)
where $Z_3$ is the projection from the objective $z$-axis to the reference $z$-axis. $Z_2$ is the projection from the objective $z$-axis to the reference $y$-axis and $Y_3$ is the projection from the objective $y$-axis to the reference $z$-axis. In our particular case, the wire segment is always put into a global Cartesian coordinate system, assuming it’s defined through three axis $(\hat{x}, \hat{y}, \hat{z})$. Within each wire segment, it defines its own local coordinate system as $(\hat{x}', \hat{y}', \hat{z}')$. If we constrain:

$$
\hat{z}' = \hat{p}_i, \quad \hat{y}' = \hat{t}_i^{a,b}, \quad \hat{x}' = \hat{y}' \times \hat{z}',
$$

(164)

where $\hat{p}_i$ is the unit direction vector for the straight part of the wire, $\hat{t}_i^{a,b}$ represents the tangential vectors calculated by (152) from both previous wire and next wire, respectively. To simplify the notations, we take $\hat{t}_i^b$ as an example to illustrate the rotation. Then a new coordinate system local to the wire axis has been defined. And we can always obtain the projections between the local wire coordinate system and the global coordinate system by computing:
\[
Z_2 = \hat{p}_i \cdot \hat{y}, \quad Z_3 = \hat{p}_i \cdot \hat{z}, \quad \mathbf{Y}^h_i = \hat{r}^h_i \cdot \hat{y}.
\] (165)

Substituting these projection values into (163), one can get a set of rotation angles \((\alpha_i, \beta_i, \gamma_i)\) between the global coordinate system and local wire coordinate system. The rotation tensor can then be calculated as:

\[
\bar{\mathbf{T}}_i = \begin{pmatrix}
   a_{11} & a_{12} & a_{13} \\
   a_{21} & a_{22} & a_{23} \\
   a_{31} & a_{32} & a_{33}
\end{pmatrix},
\] (166)

where

\[
\begin{align*}
   a_{11} &= \cos \alpha_i \cos \gamma_i - \sin \alpha_i \sin \gamma_i \cos \beta_i, \\
   a_{21} &= -\cos \alpha_i \sin \gamma_i - \sin \alpha_i \cos \gamma_i \cos \beta_i, \\
   a_{31} &= \sin \alpha_i \sin \beta_i, \\
   a_{12} &= \sin \alpha_i \cos \gamma_i + \cos \alpha_i \sin \gamma_i \cos \beta_i, \\
   a_{22} &= -\sin \alpha_i \sin \gamma_i + \cos \alpha_i \cos \gamma_i \cos \beta_i, \\
   a_{32} &= \cos \alpha_i \sin \beta_i, \\
   a_{13} &= \sin \gamma_i \sin \beta_i, \\
   a_{23} &= \cos \gamma_i \sin \beta_i, \\
   a_{33} &= \cos \beta_i,
\end{align*}
\] (167)

It is obvious that:

\[
\hat{p}_i = \bar{\mathbf{T}}_i^{-1} \cdot \hat{z}.
\] (168)

Now when the wire connects to a neighbor at a bend (in this case it is bending to the \(p_{i+1}\) direction), it is actually performing a second rotation with respect to its local coordinate system, with the rotation angle specified as:

\[
\alpha_2 = \gamma_1, \quad \beta_2 = \varphi, \quad \gamma_2 = 0.
\] (169)

Then the rotation tensor \(\bar{\mathbf{T}}^h_i\) of the unit direction vector \(\hat{p}^h_i\) at angle \(\varphi\) can then be calculated as:
\[
\tilde{\tau}_i^b = \begin{pmatrix}
\cos \alpha_2 & \sin \alpha_2 & 0 \\
-\cos \beta_2 \sin \alpha_2 & \cos \beta_2 \cos \alpha_2 & \sin \beta_2 \\
\sin \beta_2 \sin \alpha_2 & -\sin \beta_2 \cos \alpha_2 & \cos \beta_2
\end{pmatrix}.
\] (170)

And the unit direction vector in the global coordinate system can be calculated as being:

\[
\hat{p}_i^b = \tilde{\tau}_i^{-1} \cdot (\tilde{\tau}_i^b)^{-1} \cdot \hat{z}.
\] (171)

In the case when \( \hat{t}_i^a \) is used, the only change is to set the second set of rotation angles as being:

\[
\alpha_2 = \gamma_1 + \pi, \quad \beta_2 = \frac{\psi_a}{2} - \phi, \quad \gamma_2 = 0.
\] (172)

And then:

\[
\hat{p}_i^a = \tilde{\tau}_i^{-1} \cdot (\tilde{\tau}_i^a)^{-1} \cdot \hat{z}.
\] (173)

Note that such \( \hat{p}_i^a \) and \( \hat{p}_i^b \) are no longer a constant vector, they are a function of \( \phi \) within the bent region.

At this point, all the necessary deformations have been considered. The generalized computation of the coupling between the bent wire at both ends and the field will be expressed as:

\[
\left[ C_{EM}^{ie} \right]_{m,n} = \int_0^{\frac{V_o}{2}} \int_0^{\frac{\psi_b}{2}} \int_0^{2\pi} \Phi_{m} \left( \frac{r_o \phi}{h'} \right) g \left( r \right) r \left( r_o + r \cos \phi \right) \left[ \tilde{F}_{n}^{3D} \cdot \hat{p}_i^a \right] d\phi dr d\phi
\]
\[
+ \int_{e}^{^o} \int_{e}^{p} \int_0^{2\pi} \Phi_{m} \left( \frac{p}{h'} \right) g \left( r \right) r \left[ \tilde{F}_{n}^{3D} \cdot \hat{p}_i^b \right] d\phi dr dp
\]
\[
+ \int_0^{\frac{V_o}{2}} \int_0^{\frac{\psi_b}{2}} \int_0^{2\pi} \Phi_{m} \left( \frac{p}{h'} \right) g \left( r \right) r \left( r_o + r \cos \phi \right) \left[ \tilde{F}_{n}^{3D} \cdot \hat{p}_i^b \right] d\phi dr d\phi.
\] (174)
4.6. Validation

Both the 1D DGFETD-ThinWire solver and the 3D DGFETD-EM solver have been implemented. The two solvers were hybridized with the approach demonstrated in the previous sections so that the field and wire couplings can be properly handled. In this section, the DGFETD-ThinWire solver will be first validated by simulating the time-domain line voltage and current signals. The hybrid DGFETD-EM/ThinWire solver will then be validated. The examples used are a straight dipole antenna and a loop antenna. Both are simulated in the transmitting mode and the input impedance and admittance are calculated using the hybrid DGFETD-EM/ThinWire solver and compared to the results obtained from method of moments.

4.6.1. Validation of the DGFETD-ThinWire Solver

Consider a straight wire with a current excitation in the middle. Both the voltage and current signals are sampled such that the wave impedance, resonating frequencies can be examined to verify the solver. It should be noted that at this point, the radiation effects from the wire are not considered. Thus the 1D thin-wire is a completely closed system without any coupling to the EM field system.

4.6.1.1. Voltage/Current Propagating on a Straight Thin-Wire

Initially, consider a straight thin-wire, terminated with open circuits at both ends, with the length of 10 meters. The wire is discretized with 100 segments with each beam length of 0.1m. The per-unit length inductance of the wire is set to be 647.17nH, the per-unit length capacitance is set as 17.177pF. Then the characteristic impedance of this wire can be calculated as \( Z_0 = \sqrt{\frac{L}{C}} \), which is 194.1035 ohm in this case. The speed of signal propagation can also be obtained as: \( v_p = \sqrt{\frac{1}{LC}} \), which is 2.999e8 m/s. A current
excitation is put at the center cell of the thin-wire mesh. The time-domain signature of the excitation is specified as a Gaussian pulse with pulse half width $t_w$ set as 1 ns and pulse delay $t_0 = 5t_w$. The -3dB bandwidth of the pulse is 318.3 MHz. The line voltages and currents are expanded via a 4$^{\text{th}}$ order 1D line basis. Such problem is simulated by DGFETD-ThinWire solver and the time-domain voltage and current signals are sampled at the 60$^{\text{th}}$ cell of the thin-wire, which is 10 cells away from the excitation cell.

\[ \text{Figure 40. Current (a) and voltage (b) sampled at the 60}^{\text{th}} \text{ segment of the thin-wire.} \]

Figure 40 shows the time-domain signals of the current and voltage measured at the center of the 60$^{\text{th}}$ cell of the thin-wire. From the result, we can interpret that the first peak of the waveform in Figure 40 (a) is the initial current signal that comes from the excitation cell (50th) that passed through the probing cell (60th). The following peak is the reflected signal that finishes the round trip at the end of the wire. Since the wire is terminated with an open circuit, the current is completely reflected with a reflection coefficient of -1. The third peak shown in Figure 40 (a) is the reflected signal that comes from the other end of the wire. Similar interpretation can be used to explain the voltage signal, the only distinction is that the reflected voltage at the open circuit end will not
change its sign, thus the second peak of Figure 40 (b) keeps the same sign as the first peak.

According to the simulated data, we can measure that the peak value of the voltage is 49.55 mV and the peak value of the current is 0.2551 mA. Thus the characteristic impedance of the wire calculated from the measurements is 49.55 mV / 0.2551 mA = 194.238 ohm, which is in very good agreement with what has been predicted analytically (194.1035 ohm). The time delay between two voltage peaks is measured as (35.36 - 8.339) = 27.021 ns, the distance from the cell center of the 60th edge segment to the nearing end of the thin-wire is 0.1*40+0.05=4.05 m, hence the total travel distance between two peaks is 4.05 * 2 = 8.1 m, then the speed of the signal propagation can be calculated as 8.1 m / 27.021 ns = 2.997e8 m/s, which is again matching what has been calculated analytically (2.999e8 m/s).

![Figure 41. Current (a) and voltage (b) distribution over the thin wire (at t=13.34ns)](image)

Next, investigate the space distributions of the current and voltage. Figure 41 shows the snap shot of the current and voltage distributions over the entire thin-wire at the 1000th time step (t=13.34 ns). Since it is a current excitation in the middle, it is logical
that the current are flowing to the same direction on both sides of the excitation, thus the
two peaks of the current signal should have the same sign, which is observed in Figure 41 (a). The current excitation in the middle will raise a higher potential on one side, while on
the other side, it will draw a lower potential in order to force the charge to move. Therefore the voltages propagating to the both ends of the wire should have the opposite
sign, which is again illustrated in Figure 41 (b).

4.6.1.2. Voltage/Current Resonating on a Straight Thin Wire

In this test case, the voltage and current resonances on the thin-wire will be simulated
using DGFETD-ThinWire solver. The length of the wire in this simulation is 41 m. It is
meshed with 41 segments with each beam length of 1 m. Again the wire is connected to
nothing at both ends which leads to an open circuit boundary condition. The same per-
unit length inductance (647.17 nH) and capacitance (17.177 pF) as used in the previous
test case are applied here. In addition, a per-unit conductance is set as 3.34e-5 siemens,
which leads to a lossy wire. The current excitation is again applied and is put in the
middle of the wire (21$^{st}$ cell). Differential Gaussian pulse is set as the time signature with
\( t_w \) set as 30 ns and \( t_0 = 5t_w \).

![Figure 42. Input voltage in time-domain (a) and frequency-domain (b).](image)
Figure 43. Measured current in time-domain (a) and frequency-domain (b)

Figure 44. Input impedance calculated from the input voltage and measured current
(a) real part; (b) imaginary part;

The series current excitation in the middle of the wire can be converted to a series input voltage by multiplying the current magnitude with the length of the exciting cell. Figure 42 and Figure 43 show both the time-domain signals and the frequency-domain distributions of the input voltage and the measured current at the center of the thin-wire respectively. From the result shown in Figure 43 (a), it is observed that the current flowing along the wire is resonating and due to the loss added, the magnitude of the current will decay over time and will eventually damp out. With this time-domain current
signal, we can obtain its frequency-domain distribution by performing a Fourier transform. The input impedance of this resonating wire is calculated over the 0-30 MHz frequency range. The input impedance can be calculated as:

\[ Z_{in}(f) = \frac{V_{in}(f)}{I(f)}, \]

(175)

where \( V_{in}(f) \) and \( I(f) \) are the Fourier transform of the sampled time-domain input voltage and measured current, respectively. The real and imaginary parts of the input impedance are calculated and shown in Figure 44 (a) and (b), respectively. Here we verify the resonate frequencies of the current by measuring the frequencies at which the real(Z) goes to peak value or the imag(Z) falls to zero, for a 41 m wire, its resonate frequency can also be predicted analytically. The measured and analytical results are compared in Table 1

<table>
<thead>
<tr>
<th>Wavelength</th>
<th>( \lambda )</th>
<th>( 2\lambda )</th>
<th>( 3\lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency Analytical (MHz)</td>
<td>7.312312</td>
<td>14.624624</td>
<td>21.936936</td>
</tr>
<tr>
<td>Frequency Measured (MHz)</td>
<td>7.311</td>
<td>14.620</td>
<td>21.99</td>
</tr>
</tbody>
</table>

From Table 1, the measured resonant frequencies have good agreement with the results predicted analytically, which indicates a valid simulation.

The voltage and current behaviors on the thin-wire, including propagation, attenuation and resonance have been simulated by the DGFETD-ThinWire solver. The simulated results match the analytical predictions quite well. Thus the proposed DGFETD-ThinWire solver has been validated.
4.6.2. Dipole Antenna

![Dipole Antenna](image)

**Figure 45. Dipole antenna**
(a) model; (b) mesh

In this section, the input impedance and admittance of a thin-wire dipole antenna is examined. Figure 45 (a) shows the model of a straight thin-wire dipole antenna along the x-direction, in which the 41 m antenna is discretized into 41 beams with each having a beam length of 1 m. The antenna is also put into a rectangular finite element box, representing the problem domain to be simulated by the DGFETD-EM field solver. Figure 45 (b) illustrates the 3D field mesh, in which the box size is $51 \times 11 \times 11$ m and is meshed with $51 \times 11 \times 11$ cubic hexes with the edge length of 1 m. Figure 46 shows the cross sectional view of the thin-wire and its coupling zone, which demonstrates the geometrical overlaps between a single 1D meshed thin-wire segment and its surrounding 3D meshed hex elements. In such case, eighteen hexes are included within the coupling zone. The radius of the wire is set as 1 cm and the radius of the cylindrical coupling zone is set to be 2 m. To excite the antenna, a series current excitation is put in the middle of the wire. The current had a differentiated Gaussian time signature with $t_w = 30$ ns and $t_0 = 5t_w$. To simulate an unbounded medium, all six boundaries of the 3D mesh were terminated with perfectly matched layer (PML) absorbing boundary layers.
As shown in Figure 46, the wire is first put right in the middle of the centered hexes. This position is referenced as the origin of the wire. The wire is then shifted in the axial direction by a distance of $x$, as shown in Figure 47 (a). In the test cases, two $x$ values 0.2 m and 0.5 m are chosen as examples. Next the wire is diagonally shifted from its origin with a distance of $d$, as shown in Figure 47 (b). Again two examples (0.2 m and 0.5 m) are chosen in the test cases. Finally, the wire is rotated. This is done by first rotating the wire about $z$ axis with an angle of $\phi$, and then about $y$ axis with an angle of $\theta$, as illustrated in Figure 47 (c). Two sets of angles ($\phi = \theta = 5^\circ$ and $\phi = \theta = 9^\circ$) are specified in the test cases. Consequently, there are seven independent, different positioned dipole antennas to be simulated. Each of the positioned antennas has its own 1D thin-wire mesh which is different from others. But all of these test cases are using the same 3D field mesh, which is the $51 \times 11 \times 11$ hex meshed box.
The seven problems configured above are simulated by the hybrid DGFETD-EM/ThinWire solver with H(0) basis functions applied to both the 1D and 3D solvers. The currents are measured in the middle of the wire for each case. Both the time-domain current signal and the frequency-domain distribution are illustrated in Figure 48 (a) and (b) respectively. As observed, the current on the wire will radiate into the free-space which is modeled by DGFETD-EM field solver. Due to the radiation, the magnitude of the wire current will decay over time until completely damped out. The Fourier transform of the time-domain current will show the characteristic feature of the antenna and from which the input impedance $Z_{in}$ can be calculated by (175). The input admittance can be easily obtained as:

$$Y_{in}(f) = \frac{1}{Z_{in}(f)}.$$  \hspace{1cm} (176)

The calculated input impedance and admittance over the frequency band (1-20MHz) are shown in Figure 49 and Figure 50, respectively. Both the real part and imaginary part are presented. The simulated results are also compared to that obtained from method of moments. From Figure 49, the input impedance of the dipole antenna is generally in good...
agreement with the reference result. There are some disagreements between the simulated and referenced results on the peak frequencies. This is mainly due to the approximation of the per-unit length inductance and capacitance. A convergence study can be performed to obtain a set of optimal parameters. From Figure 50, excellent agreement of the input admittance between the simulated and referenced results can be observed in both real and imaginary parts. In addition, the seven simulations with different positioned dipole antennas provide nearly the same result, indicating the solver is very “insensitive” to the wire position. Consequently, the hybrid DGFETD-EM/ThinWire solver for arbitrarily located and oriented straight thin-wires has been validated.
Figure 48. Measured current in time-domain (a) and frequency-domain (b) (H0 basis)

Figure 49. Input impedance calculated from the input voltage and measured current (H0 basis) (a): real part; (b): imaginary part

Figure 50. Input admittance calculated from the input voltage and measured current (H0 basis) (a): real part; (b): imaginary part
In the above simulations, H(0) basis are used in both the DGFETD-EM and DGFETD-ThinWire solver to expand the unknowns. In the following set of simulations, H(1) basis are used for both the 1D and 3D solvers. The currents measured in the middle of the wire are illustrated in Figure 51 (a) and (b) respectively. The calculated input impedance and admittance over the frequency band (1-20MHz) are shown in Figure 52 and Figure 53, respectively. Both the real part and imaginary part are presented. The simulated results are also compared to that obtained from method of moments. Better agreement has been found for the H(1) basis functions compared with those simulated with H(0) basis.
Figure 51. Measured current in time-domain (a) and frequency-domain (b) (H1 basis)

Figure 52. Input impedance calculated from the input voltage and measured current. (H1 basis)
(a): real part; (b): imaginary part

Figure 53. Input admittance calculated from the input voltage and measured current. (H1 basis)
(a): real part; (b): imaginary part
4.6.3. Loop Antenna

Figure 54. Loop antenna
(a) model; (b) mesh

In this section, the input impedance and admittance of a thin-wire loop antenna is examined. The reason for choosing a loop antenna is to mainly test the field-wire coupling for curved wire joints. For a loop wire, each of the meshed wire segments will have two curved regions at both ends as it connects to the two neighboring wire segments. Figure 54 (a) shows the model of a circular thin-wire loop antenna in a finite element box, in which the diameter of the loop is 1 m and it is discretized into 50 beams with each beam length of 6.279 cm. The antenna is also put into a rectangular finite element box, representing the problem domain to be simulated by the DGFETD-EM field solver. Figure 54 (b) illustrates the 3D field mesh, in which the box size is 1.625 m×1.625 m×0.6875 m and it is meshed with 26×26×11 cubic hexes with the edge length of 0.0625 m. Figure 55 shows the cross sectional view of the thin-wire and its coupling zone, which demonstrates the geometrical overlaps between a single 1D meshed thin-wire segment and its surrounding 3D meshed hex elements. In such case, eighteen hexes are included within the coupling zone. The radius of the wire is set as 1.05 mm and the radius of the cylindrical coupling zone is set to be 13 cm. To excite the antenna, a series current excitation is put on one segment of the wire. The current had a
differentiated Gaussian pulse time-signature with $t_w = 1.25$ ns and $t_0 = 5t_w$. To simulate an unbounded free space, all six of the exterior boundaries of the 3D mesh were terminated with PML boundary layers.

![Figure 55. Cross sectional view of the overlaps between thin-wire port and hex elements](image)

As shown in Figure 54, the loop is first put right in the middle of the box. This position is referenced as the origin of the loop. Then the arbitrarily located and oriented loop wire models are established. To do this, the loop is first shifted in the vertical direction by a distance of $z$, as shown in Figure 56 (a). In the test cases, two $z$ values 3 cm and 5 cm are picked up as examples. Next the loop is diagonally shifted from its origin with a distance of $d$, as shown in Figure 56 (b). Again two examples (3 cm and 5 cm) are chosen in the test cases. Finally, the loop is rotated. This is being done by rotating the loop about $x$ axis with an angle of $\phi$, and then about $y$ axis with an angle of $\theta$, as illustrated in Figure 56 (c). Two sets of angles ($\phi = \theta = 3^\circ$ and $\phi = \theta = 5^\circ$) are specified in the test cases. Consequently, there are seven independent, different positioned loop antennas to be simulated. Each of the positioned antennas has its own 1D thin-wire mesh which is different from others. But all of these test cases are using the same 3D field mesh, which is the $26 \times 26 \times 11$ hex meshed box.
Figure 56. Moving the loop wire
(a) vertical shifting; (b) diagonal shifting; (c) rotation

The seven problems configured above are simulated by the hybrid DGFETD-EM/ThinWire solver with H(0) basis applied to both 1D and 3D solvers. The currents are measured in the middle of the excitation cell for each case. The input current is converted to a series voltage input by multiplying the current with the length of the excitation cell. Both the time-domain input voltage and its frequency-domain distribution are illustrated in Figure 57 (a) and (b) respectively, in which a differentiated Gaussian pulse is observed in the time-domain and the main energy of the pulse resides within 0-600MHz in the frequency-domain. The current flowing through the excitation cell is also sampled over time and is illustrated in Figure 58 (a). As observed, the current on the loop will radiate to the free-space which is modeled by DGFETD-EM field solver. Due to the radiation, the magnitude of the loop current will decay over time until completely damped out. The Fourier transform of the time-domain current shown in Figure 58 (b) will reflect the characteristic feature of the loop antenna and from which the input impedance and admittance can be calculated by (175) and (176) respectively. The calculated input impedance and admittance over the frequency band (80-450 MHz) are shown in Figure
59 and Figure 60, respectively. Both the real part and imaginary part are presented. The simulated results are also compared to that obtained from method of moments. From Figure 59, the input impedance of the loop antenna is generally in agreement with the reference results. There are some disagreements between the simulated and referenced results on the peak frequencies. This is mainly due to the approximation of the per-unit length parameters of the thin-wire. Since (98) and (99) are based on a straight thin-wire. From Figure 60, excellent agreement of the input admittance between the simulated and referenced results can be observed in both real and imaginary parts. In addition, the seven simulations with different positioned loop antennas provide nearly the same result, indicating the solver is again “insensitive” to the wire position. Consequently, the hybrid DGFETD-EM/ThinWire solver for arbitrarily located and oriented curved thin-wires has been validated.

![Figure 57](image)

**Figure 57. Input voltage in time-domain (a) and frequency-domain (b)**
Figure 58. Measured current in time-domain (a) and frequency-domain (b).

Figure 59. Input impedance calculated from the input voltage and measured current:
(a): real part; (b): imaginary part.

Figure 60. Input admittance calculated from the input voltage and measured current:
(a): real part; (b): imaginary part.
4.7. Summary

A hybrid thin-wire model has been implemented within the DGFETD method for solving concurrently the time-dependent Maxwell’s equations and the telegrapher’s equations. A thin-wire port was designed to properly handle the coupling between the DGFETD-EM and DGFETD-ThinWire solvers. The hybrid solver maintains high-order accuracy in both field modeling and time integration. With proper calculation of the geometrical overlaps, the 1D thin-wire mesh can be chosen independent of the 3D field mesh, which allows arbitrarily located and oriented thin-wires to be simulated within the DGFETD-EM framework without re-constraining the 3D mesh. In addition, a method to treat curvilinear thin-wires is also provided, in which three deformations in the bent region of the wire junction were demonstrated.

To validate the proposed thin-wire model, the 1D DGFETD-ThinWire solver was first tested. Good agreement was found when comparing the simulated results with analytical predicated data. The hybrid solver was also validated through the simulation of arbitrarily positioned thin-wire dipole and loop antennas. The input impedance and admittance were examined with the results obtained from the method of moments. Again excellent agreement was found, validating hybrid solver.
Chapter 5. Complex Frequency Shifted Perfect Matched Layer for DGFETD-EM Solver

The Perfectly Matched Layer (PML) absorbing boundary condition [50] has revolutionized the termination of unbounded domains for differential equation based solvers such as the Finite Difference Time Domain (FDTD) and Finite Element Time Domain (FETD). The significant advantage of the PML absorbing media is that it provides a mesh truncation algorithm that is independent of frequency, wave polarization, and angle of incidence, and has extremely small reflection errors. It has also been shown that the PML is inherently “material-independent”, and can terminate domains with inhomogeneous, dispersive, and non-linear materials [51-53].

Berenger’s original method is now referred to as the “split-field” PML. Other variants of the PML have also been introduced, and are now referred to as the un-split, or anisotropic PML (APML) [51, 54], and the stretched coordinate PML [55]. While each of these techniques offers different mathematical representations of the PML, the formulations will lead to equivalent reflection properties [56]. As a consequence, more recent research on improving the performance of the PML has focused on modifying the choice of the constitutive parameters [57-61].

The most accurate and robust choice for the constitutive parameters currently in use is the Complex-Frequency Shifted (CFS) PML parameters [56, 57, 59-61]. An efficient implementation of the CFS-PML for FDTD methods is Roden's stretched coordinate formulation implemented with the discrete recursive-convolution method [61]. The discrete recursive-convolution method is second-order accurate and is efficient in terms of memory and computational cost.
With the proper choice of scaling the constitutive parameters, this CFS-PML provides excellent absorption of both propagating and evanescent waves, and the PML boundary can be placed extremely close to the device under test [56, 59]. The method can also be applied to terminate domains with arbitrary media without any specialization of the implementation [56],[61].

The present focus is on an implementation of the CFS-PML within the Discontinuous Galerkin Method [10, 11, 13]. However, in order to utilize the CFS-PML constitutive parameters, a formulation that allows high-order implementations of the CFS-PML must be developed. To this end, an alternate form of the CFS-PML expressed via an Auxiliary Differential Equation (ADE) form will be followed. It has been found that the ADE CFS-PML developed by Gedney and Zhao [62] can be applied to high-order solution methods.

5.1. A General ADE Formulation of the CFS-PML

Consider the time-harmonic Maxwell’s equations in a lossless, source-free media:

\[ j \omega \mu \hat{H} = -\nabla \times \hat{E}, \]  
\[ j \omega \epsilon \hat{E} = \nabla \times \hat{H}. \]  

The curl operator can be expressed in a curvilinear coordinate frame as:

\[ \nabla \times \hat{E} = \sum_{\gamma=1}^{3} \hat{a}^\gamma \times \frac{\partial}{\partial u^\gamma} \hat{E}, \]

where, \( u^\gamma \) are the general curvilinear coordinates, and \( \hat{a}^\gamma \) are the reciprocal unitary vectors [63].
In the perfectly matched layer medium region, the PML material parameters can be expressed as an equivalent complex-stretched coordinate frame [55]. The curl within the curvilinear-complex-stretched coordinate frame is then expressed as:

\[
\vec{\nabla} \times \vec{E} = \sum_{v=1}^{3} \frac{1}{s_v} \vec{a}^v \times \frac{\partial}{\partial u^v} \vec{E},
\]

(180)

where \( s_v \) are the stretched coordinate metric coefficients. These coefficients are typically chosen via the CFS-PML parameters [10-12]:

\[
s_v = \kappa_v + \frac{\sigma_v}{a_v + j\omega \epsilon},
\]

(181)

where \( a_v, \kappa_v \), and \( \sigma_v \) are assumed to be positive real, and can be one dimensional functions of \( u^v \). Typically, the PML region is identified via a normal to the interface separating the PML region and the working volume.

It is assumed that the surface defining the PML is separable and defined via a single local coordinate value. The unit normal would be defined by a reciprocal vector, for example \( \vec{a}^v \). In the PML region, \( s_v \) would be expressed as in (181). If it is a non-overlapped region, the remaining \( s_v \) would be 1. In overlapped PML regions (i.e., corner regions), each overlap region is defined by the normal vector \( \vec{a}^v \), and each would have a separate value for \( s_v \). Combining (180) with (177), Faraday’s law can be expressed within the PML region as:

\[
-j \omega \mu_0 \vec{E} \times \vec{B} = \vec{\nabla} \times \vec{E} = \sum_{v=1}^{3} \frac{1}{s_v} \vec{a}^v \times \frac{\partial}{\partial u^v} \vec{E}.
\]

(182)

Transforming this into the time-domain leads to a convolution between the inverse of the stretched coordinate parameters and the partial derivatives [61]. Alternatively, an
auxiliary variable can be introduced which is constrained via the introduction of the appropriate ADE based on the identity [62, 64]:

\[
\frac{1}{s_v} = \frac{1}{\kappa_v} - \frac{1}{\kappa_v B_v},
\]

where

\[
\frac{1}{B_v} = \frac{\sigma_v}{(\kappa_v(j\omega\varepsilon_v + a_v) + \sigma_v)},
\]

Combining (183), the derivative on the right-hand-side of (177) can be expressed as:

\[
\frac{1}{s_v} \tilde{a}^v \times \frac{\partial}{\partial u^v} \tilde{E} = \frac{1}{\kappa_v} \tilde{a}^v \times \frac{\partial}{\partial u^v} \tilde{E} + \frac{1}{\kappa_v} \tilde{a}^v \times \frac{\partial}{\partial u^v} \tilde{Q}_v^E,
\]

where the new auxiliary parameter \( \tilde{Q}_v^E \) is defined as:

\[
\tilde{Q}_v^E = -\frac{1}{B_v} \tilde{E}.
\]

Applying (184), (186) is transformed into the time-domain, leading to the Auxiliary Differential Equation (ADE):

\[
\kappa_v \varepsilon_v \frac{\partial}{\partial t} \tilde{Q}_v^E + (\kappa_v a_v + \sigma_v) \tilde{Q}_v^E = -\sigma_v \tilde{E}.
\]

In conclusion, Faraday’s law in the PML can be rewritten as:

\[
-\frac{\partial}{\partial t} \vec{\mu} \cdot \vec{H} = \nabla \times \vec{E} + \sum_v \frac{1}{\kappa_v} \tilde{a}^v \times \frac{\partial}{\partial u^v} \tilde{Q}_v^E,
\]

where, the sum is over each of the PML regions \( v \), and

\[
\nabla \times \vec{E} = \sum_{v=1} \frac{1}{\kappa_v} \tilde{a}^v \times \frac{\partial}{\partial u^v} \tilde{E},
\]

which represents the curl with a purely real coordinate stretching, and each \( \tilde{Q}_v^E \) satisfies the differential equation expressed in (187). We can similarly express Ampere’s law as:
\[
\frac{\partial}{\partial t} \vec{E} \cdot \vec{E} = \nabla \times \vec{H} + \sum \frac{1}{\kappa_v} \vec{a}_v \times \frac{\partial}{\partial u^v} \vec{Q}_v^\mu ,
\]  

(190)

where the \( \vec{Q}_v^\mu \) satisfy the differential equation:

\[
\kappa_v \varepsilon_0 \frac{\partial}{\partial t} \vec{Q}_v^\mu + (\kappa_v \alpha_v + \sigma_v) \vec{Q}_v^\mu = -\sigma_v \vec{H} .
\]  

(191)

5.2. Application to the DGFETD-EM Method

In this section, the ADE-CFS-PML representation of Maxwell's equations is applied to a Discontinuous Galerkin Finite Element Time Domain (DGFETD) formulation [7]. The DGFETD problem domain is spatially decomposed into sub-domains, where each sub-domain supports an independent finite element mesh. Within each sub-domain, a Galerkin FEM formulation of Maxwell's coupled curl equations is posed. Neighboring sub-domains then couple through their shared boundaries by weakly constraining the continuity of the tangential fields across shared boundaries. A high-order time integration, such as Runge-Kutta (RK) methods, is applied leading to high-order time dependent solutions [7].

At first, a non-overlapping PML is assumed. Thus there is only one set of PML parameters being defined. Initially, the inner-product of a test vector with Faraday’s law is performed:

\[
\frac{\partial}{\partial t} \left[ \int_{v_i} \vec{T}_h^v \cdot \vec{E} \right] dv = -\int_{v_i} \nabla \times \vec{E} dv - \int_{v_i} \vec{T}_h^v \cdot \vec{a}_v \times \frac{\partial}{\partial u^v} \vec{Q}_v^\mu dv .
\]  

(192)

And for Ampere’s law:

\[
\frac{\partial}{\partial t} \left[ \int_{v_i} \vec{T}_e^v \cdot \vec{E} \right] dv = \int_{v_i} \nabla \times \vec{H} dv + \int_{v_i} \vec{T}_e^v \cdot \vec{a}_v \times \frac{\partial}{\partial u^v} \vec{Q}_v^\mu dv ,
\]  

(193)
where $V_i$ is the sub-domain volume that resides in the PML region. The field intensities $\vec{E}$ and $\vec{H}$ and the test-vectors $\vec{T}^e$ and $\vec{T}^h$ are discretized via H(p)-curl conforming basis functions [4]. The auxiliary fields $\vec{Q}^E_v$ and $\vec{Q}^H_v$ are using the same function spaces as $\vec{E}$ and $\vec{H}$ respectively.

The first task is to project the $\vec{T}^h \cdot \vec{\nabla} \times \vec{E}$ term, which involves the stretched coordinates, into the S-matrix and the F-matrix terms as done in a standard DGFETD discretization. Initially, from Corollary 4.2 of Appendix:

$$\int_{V_i} \vec{T}^h \cdot \vec{\nabla} \times \vec{E} dv = \int_{V_i} \vec{E} \cdot \vec{\nabla} \times \vec{T}^h dv - \int_{V_i} \vec{\nabla} \cdot \vec{T}^h \times \vec{E} dv. \quad (194)$$

Then, from Corollary 4.4 of Appendix:

$$\int_{V_i} \vec{T}^h \cdot \vec{\nabla} \times \vec{E} dv = \int_{V_i} \vec{E} \cdot \vec{\nabla} \times \vec{T}^h dv + \int_{\partial V_i} \vec{T}^h \cdot \hat{n} \times \vec{E}^- ds, \quad (195)$$

where

$$\hat{n} = \frac{1}{\kappa} \vec{a}^v \quad (196)$$

This can be interpreted that only the face that is perpendicular to $\vec{a}^v$ will have a non-zero contribution to the local fields. We must assume that the electric field is tangentially continuous across the surface $\partial V_i$. Thus, if we let $\vec{E}^-$ be the electric field on $\partial V_i$ just interior to $V_i$, and $\vec{E}^+$ be the electric field on $\partial V_i$ just exterior to $V_i$, we can express the boundary condition:

$$\hat{n} \times (\vec{E}^- - \vec{E}^+) = 0. \quad (197)$$

This can similarly be written as:
\[ \hat{n} \times \hat{E} = \frac{1}{2} \hat{n} \times \left( \hat{E}^+ + \hat{E}^- \right). \]  

Inserting this back into (195) leads to:
\[
\int_{V_i} \tilde{T}^h \cdot \tilde{\nabla} \times \tilde{E} dv = \int_{V_i} \tilde{E} \cdot \tilde{\nabla} \times \tilde{T}^h dv + \frac{1}{2} \int_{\partial V_i} \tilde{T}^h \cdot \hat{n} \times \left( \tilde{E}^+ + \tilde{E}^- \right) ds .
\]  

Then, applying the inverse transform of (195), this becomes:
\[
\int_{V_i} \tilde{T}^h \cdot \tilde{\nabla} \times \tilde{E} dv = \frac{1}{2} \int_{V_i} \left( \tilde{E} \cdot \tilde{\nabla} \times \tilde{T}^h + \tilde{T}^h \cdot \tilde{\nabla} \times \tilde{E} \right) dv + \frac{1}{2} \int_{\partial V_i} \tilde{T}^h \cdot \hat{n} \times \tilde{E}^+ ds ,
\]  

which is the classic transformation used based on the conformal central flux method.

Next, we apply a similar transformation based on the last term on the right-hand-side of (192). From Corollary 4.1 of Appendix:
\[
\int_{V_i} \tilde{T}^h \cdot \frac{1}{\kappa_v} \tilde{a}^\nu \times \frac{\partial}{\partial \nu} \tilde{Q}^E dv = \int_{V_i} \tilde{Q}^E \cdot \frac{1}{\kappa_v} \tilde{a}^\nu \times \frac{\partial}{\partial \nu} \tilde{T}^h dv
- \int_{V_i} \frac{1}{\sqrt{g}} \frac{1}{\kappa_v} \frac{\partial}{\partial \nu} \left( \tilde{a}^\nu \cdot \tilde{T}^h \times \tilde{Q}^E \sqrt{g} \right) dv .
\]  

Then, applying Corollary 4.4 of Appendix:
\[
\int_{V_i} \tilde{T}^h \cdot \frac{1}{\kappa_v} \tilde{a}^\nu \times \frac{\partial}{\partial \nu} \tilde{Q}^E dv = \int_{V_i} \tilde{Q}^E \cdot \frac{1}{\kappa_v} \tilde{a}^\nu \times \frac{\partial}{\partial \nu} \tilde{T}^h dv + \int_{\partial V_i} \tilde{T}^h \cdot \hat{n} \times \tilde{Q}^E ds ,
\]  

where a closed surface integral is used, since the integrand is zero on all boundaries that are transverse to \( \tilde{a}^\nu \). Given the relationship in (186), since \( \tilde{E} \) must be tangentially continuous across \( \partial V_i \), then \( \tilde{Q}^E \) must also be tangentially continuous. Applying a conformal central flux type relationship, the following relationship on \( \partial V_i \) must be true:
\[
\hat{n} \times \tilde{Q}^E = \hat{n} \times \frac{\tilde{Q}^E + \tilde{Q}^E}{2} .
\]  

Then, following a similar progression as (199) and (200), (202) can be expressed as:
\[
\begin{align*}
\left( \int_{V_i} \tilde{T}_h \cdot \frac{1}{\kappa_v} \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{Q}_v^E dv + \frac{1}{2} \int_{V_i} \frac{1}{\kappa_v} \left( \tilde{T}_h \cdot \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{Q}_v^E + \tilde{Q}_v^E \cdot \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{T}_h \right) dv \right) \\
- \frac{1}{2} \int_{\partial V_i} \tilde{T}_h \cdot \hat{n} \times \tilde{Q}_v^E ds 
\end{align*}
\]  

(204)

In summary, (200) and (204) are applied within (192), leading to the DGFETD formulation of the Maxwell’s equations in the CFS-PML region:

\[
\begin{align*}
\frac{\partial}{\partial t} \int_{V_i} \tilde{T}_h \cdot \vec{a}^v \cdot \vec{H} dv = & - \frac{1}{2} \int_{V_i} \left( \tilde{E}_v \cdot \nabla \times \tilde{T}_h + \tilde{T}_h \cdot \nabla \times \tilde{E}_v \right) dv \\
- & \frac{1}{2} \sum_v \int_{V_i} \frac{1}{\kappa_v} \left( \tilde{T}_h \cdot \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{Q}_v^E + \tilde{Q}_v^E \cdot \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{T}_h \right) dv \\
- & \frac{1}{2} \int_{\partial V_i} \tilde{T}_h \cdot \hat{n} \times \tilde{E}_v^E ds - \frac{1}{2} \sum_v \int_{\partial V_i} \tilde{T}_h \cdot \hat{n} \times \tilde{Q}_v^E ds 
\end{align*}
\]

(205)

A dual expression can be derived for Ampère’s law:

\[
\begin{align*}
\frac{\partial}{\partial t} \int_{V_i} \tilde{T}_e \cdot \vec{E} \cdot \vec{E} dv = & \frac{1}{2} \int_{V_i} \left( \tilde{H}_v \cdot \nabla \times \tilde{T}_e + \tilde{T}_e \cdot \nabla \times \tilde{H}_v \right) dv \\
+ & \frac{1}{2} \sum_v \int_{V_i} \frac{1}{\kappa_v} \left( \tilde{T}_e \cdot \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{Q}_v^H + \tilde{Q}_v^H \cdot \vec{a}^v \times \frac{\partial}{\partial u^v} \tilde{T}_e \right) dv \\
+ & \frac{1}{2} \int_{\partial V_i} \tilde{T}_e \cdot \hat{n} \times \tilde{H}_v^E ds + \frac{1}{2} \sum_v \int_{\partial V_i} \tilde{T}_e \cdot \hat{n} \times \tilde{Q}_v^H ds 
\end{align*}
\]

(206)

where \( \tilde{T}_e \) shares the same curl-conforming function space as \( \tilde{E} \).

Finally, we apply the DGFETD method to solve for the auxiliary fields \( \tilde{Q}_v^E \) and \( \tilde{Q}_v^H \) described in (187) and (191) respectively. Performing the inner product, we obtain the weak forms of the auxiliary differential equations (187) and (191) as:

\[
\begin{align*}
\varepsilon_\circ \frac{\partial}{\partial t} \int_{V_i} \tilde{T}_e \cdot \kappa_v \tilde{Q}_v^E dv + \int_{V_i} \tilde{T}_e \cdot (\kappa_v \alpha_v + \sigma_v) \tilde{Q}_v^E dv = - \int_{\partial V_i} \tilde{T}_e \cdot \sigma_v \tilde{E} dv, \\
(207)
\end{align*}
\]

\[
\begin{align*}
\varepsilon_\circ \frac{\partial}{\partial t} \int_{V_i} \tilde{T}_h \cdot \kappa_v \tilde{Q}_v^H dv + \int_{V_i} \tilde{T}_h \cdot (\kappa_v \alpha_v + \sigma_v) \tilde{Q}_v^H dv = - \int_{\partial V_i} \tilde{T}_h \cdot \sigma_v \tilde{H} dv, \\
(208)
\end{align*}
\]
where, $V_i$ is the sub-domain volume that resides in the CFS-PML region, $\tilde{Q}_v^E$ and $\tilde{Q}_v^H$ represent the function spaces for the auxiliary field. $\tilde{T}^e$ and $\tilde{T}^h$ are their test function spaces. Both share the same function space as $\tilde{E}$ and $\tilde{H}$.

The field intensities and the test vectors $\tilde{T}^h$ and $\tilde{T}^e$ are discretized via H(p)-curl conforming basis functions [4]. The auxiliary vectors are also expanded via H(p)-curl conforming basis functions. It is noted that due to the partial derivative, the basis functions representing the auxiliary fields that are parallel to $\tilde{a}^\nu$ or are constant with respect to $\tilde{u}^\nu$ are neglected from the auxiliary vector function space, only those who are perpendicular to $\tilde{a}^\nu$ will have non-zero contributions after the partial derivative along $\tilde{u}^\nu$.

With this choice of field and test spaces, (205)-(208) can be re-cast in a discrete form as:

$$M_{\mu}^{hh}h' = -S^e e - \sum \mathcal{Q}^{ho} q^e - F^{he} e^+ - F^{hq} q^{e^+}, \quad (209)$$

$$M_{\epsilon}^{ee} e' = S^{eh} h + \sum \mathcal{Q}^{eq} q^h + F^{eh} h^+ + F^{eq} q^{h^+}, \quad (210)$$

$$P_{\alpha} q^{e^+} + P_{\alpha} q^{e^+} = P_{\sigma} e, \quad (211)$$

$$P_{\alpha} q^{h^+} + P_{\alpha} q^{h^+} = P_{\sigma} h, \quad (212)$$

where the matrices are defined with one-to-one correspondence with the volume and surface integrals in (205)-(208) with the vector field and test functions expanded into their respective function spaces, $e$ and $h$ are the coefficient vectors representing the discrete $\tilde{H}$ - and $\tilde{E}$ -fields, $e^+$ and $h^+$ are the unknown coefficient vectors of the neighboring subdomains, $q^h$ and $q^e$ are the coefficient vectors of unknowns representing the auxiliary fields in the PML regions, and $q^{h^+}$ and $q^{e^+}$ are the auxiliary
coefficient vectors of the neighboring sub-domains. It is noted that \((F^{\text{he}})^T = F^{\text{eh}}\), and this has been exploited in (210). It is further anticipated that on a single face, \(F^{\text{he}} = F^{\text{hy}}\), and \(F^{\text{eh}} = F^{\text{eq}}\).

It is assumed that the constitutive parameters of the PML are constant over a sub-domain. Note that this is also necessary for the proof’s in Corollary 4.3 and 4.4. Consequently, the CFS-PML parameters can be extracted out of the integrations, thus, leading to:

\[
\varepsilon_j \kappa_v \frac{\partial}{\partial t} \int T^e \cdot \tilde{Q}_v^E dv + (\kappa_v a_v + \sigma_v) \int T^e \cdot \tilde{Q}_v^E dv = -\sigma_v \int T^e \cdot \tilde{E} dv, \quad (213)
\]

\[
\varepsilon_j \kappa_v \frac{\partial}{\partial t} \int T^h \cdot \tilde{Q}_v^H dv + (\kappa_v a_v + \sigma_v) \int T^h \cdot \tilde{Q}_v^H dv = -\sigma_v \int T^h \cdot \tilde{H} dv. \quad (214)
\]

5.3. Efficiencies in Matrix Storage

5.3.1. S, Q, and F-matrices

If the sub-domains are limited to a single cell, then using the formulation presented in (209)-(212), the formulation can be derived such that only a single matrix must be stored per subdomain – namely a single M-matrix. The role of this section is to explore how this can be done. Consider first the S-matrix, computed via the integral:

\[
S^{\text{he}}_{j,i} = \int \left[ \frac{1}{2} \tilde{T}_j^h \cdot \nabla \times \tilde{E}_i + \frac{1}{2} \tilde{E}_i \cdot \nabla \times \tilde{T}_j^h \right] dv. \quad (215)
\]

From (179), the following is observed:

\[
\int \tilde{T}_j^h \cdot \nabla \times \tilde{E}_i dv = \int \tilde{T}_j^h \cdot \sum_{k=1}^3 \dot{a}_k \frac{\partial}{\partial u_k} \tilde{E}_i \sqrt{g} du^1 du^2 du^3. \quad (216)
\]

Then, expanding the test vector and the field vector via covariant projects, leads to:
\[
\int \sum_{m=1}^{3} \mathbf{\hat{a}}_{m}^{\nu} \cdot t_{m}^{i} \cdot \sum_{k=1}^{3} \mathbf{\hat{a}}_{k}^{\nu} \times \sum_{n=1}^{3} \mathbf{\hat{a}}_{n}^{\nu} \frac{\partial}{\partial u^{k}} e_{n}^{i} \sqrt{g} du'^{1} du'^{2} du'^{3} = \\
\int \frac{1}{\sqrt{g}} \sum_{k=1}^{3} t_{k}^{i} \cdot \left( \frac{\partial}{\partial u^{k+1}} e_{k+1}^{i} - \frac{\partial}{\partial u^{k-1}} e_{k-1}^{i} \right) \sqrt{g} du'^{1} du'^{2} du'^{3},
\] (217)

where, \( k+1 \) and \( k-1 \) are performed using modulo-3 type arithmetic. To derive this expression, it was realized that the triple scalar products of the reciprocal unitary vectors are only non-zero when \( i \neq j \neq k \), and are equal to \( \pm 1/\sqrt{g} \) based on the permutation of \( i,j,k \). Then, from (217), it is observed that the Jacobians cancel, leading to the result:

\[
\int \mathbf{T}^{h} \cdot \nabla \times \mathbf{E} \cdot dv = \int \sum_{k=1}^{3} t_{k}^{i} \cdot \left( \frac{1}{\kappa_{k+1}} \frac{\partial}{\partial u^{k+1}} e_{k+1}^{i} - \frac{1}{\kappa_{k-1}} \frac{\partial}{\partial u^{k-1}} e_{k-1}^{i} \right) du'^{1} du'^{2} du'^{3}.
\] (218)

Consequently, this integral is a function of the local coordinates only! Similarly, in the stretched coordinate frame:

\[
\int \mathbf{T}^{h} \cdot \nabla \times \mathbf{E} \cdot dv = \int \sum_{k=1}^{3} t_{k}^{i} \cdot \left( \frac{1}{\kappa_{k+1}} \frac{\partial}{\partial u^{k+1}} e_{k+1}^{i} - \frac{1}{\kappa_{k-1}} \frac{\partial}{\partial u^{k-1}} e_{k-1}^{i} \right) du'^{1} du'^{2} du'^{3},
\] (219)

which again is a function of the local coordinate only. We can similarly pose for the Q-matrix entries:

\[
\int \mathbf{T}^{h} \cdot \mathbf{Q}^{\nu}_{\nu} dv = \int \mathbf{T}^{h} \cdot \frac{1}{\kappa_{\nu}} \mathbf{\hat{a}}^{\nu} \times \frac{\partial}{\partial u^{\nu}} \mathbf{\hat{f}}_{\nu} dv = \\
\int \frac{1}{\kappa_{\nu}} \frac{1}{\sqrt{g}} \left( \frac{\partial}{\partial u^{\nu}} \mathbf{\hat{f}}_{\nu} - \frac{\partial}{\partial u^{\nu}} \mathbf{\hat{f}}_{\nu}^{\nu} \right) \sqrt{g} du'^{1} du'^{2} du'^{3},
\] (220)

which is also a function of the local coordinates only.

Next, we turn our attention to the F-matrices. Consider the integration over a single face. From, (205) the face integral is expressed as:

\[
\int \mathbf{T}^{h} \cdot \mathbf{\hat{n}} \times \mathbf{\hat{E}}^{\nu} ds = \int \sum_{m=1}^{2} t_{m}^{i} \mathbf{\hat{a}}_{m}^{\nu} \cdot \mathbf{\hat{n}} \times \sum_{n=1}^{2} e_{n}^{i} \mathbf{\hat{a}}_{n}^{\nu} \sqrt{g_{2D}} du'^{1} du'^{2},
\] (221)
where $\sqrt{g_{2D}}$ is the two-dimensional surface Jacobian. Recognizing that on a two-dimensional surface:

$$\hat{a}^n \cdot \hat{a}^1 \times \hat{a}^2 = \frac{1}{\sqrt{g_{2D}}}.$$  \hfill (222)

Then, (221) can be written as:

$$\int_{\partial V} \tilde{T}^h_j \cdot \hat{n} \times \tilde{E}^+ \, ds = \int_{\partial V} \sum_{n=1}^{2} \left(e_{n}^{l} t_{n+1}^{l} - e_{n+1}^{l} t_{n}^{l}\right) u^1 du^2.$$  \hfill (223)

Again, the F-matrix integral can be expressed in local coordinates only.

The consequence of having the S, Q, and F-matrices a function of the local coordinates only, is that these matrices are independent of the geometrical cell map! As a consequence, these matrices can potentially only be calculated once and cached and then re-used for every system matrix. Any subdomain can then use these matrices by applying a permutation of the basis function ordering and a sign-flip based on the basis function orientation. This can potentially lead to a tremendous savings in memory.

5.3.2. Dirichlet Boundary Conditions

One issue that clouds the use of matrix-footprints is the presence of Dirichlet-type boundary conditions. Since Dirichlet boundary conditions constrain a degree of freedom to a fixed value, such boundary conditions reduce the degrees of freedom. As a consequence, these boundary conditions change the dimensions of a matrix. Since any random edge or face can lie on a Dirichlet boundary, there are hundreds of possible permutations. This over complicates the use of a matrix footprint.

Since the S, Q, and F-matrices are only ever used in matrix-vector products, one possible solution is to maintain the full matrices, and then when permuting and signing the product vector and/or solution vector, zero out the appropriate elements in the column
vectors prior to (and after) the multiplication. The problem with this approach is that prior to and after every product with these matrices we will be forced to gather the data into a temporary vector and scatter it back out to a solution vector.

An alternative approach is the incorporate Dirichlet boundary conditions into the F-matrix. For example, on the surface of a perfectly electrical conducting object, one enforces the boundary condition:

$$\hat{n} \times \vec{E} \bigg|_{s_{pec}} = 0.$$  \hspace{1cm} (224)

This can alternatively be enforced on the subdomain boundary as:

$$\frac{1}{2} \hat{n} \times \vec{E} \bigg|_{s_{pec}} = -\frac{1}{2} \hat{n} \times \vec{E}^+ \bigg|_{s_{pec}},$$  \hspace{1cm} (225)

where, $\vec{E}^+$ is the field exterior to the domain. Similarly, on a PMC boundary, we can apply the boundary condition:

$$\frac{1}{2} \hat{n} \times \vec{H} \bigg|_{s_{pec}} = -\frac{1}{2} \hat{n} \times \vec{H}^+ \bigg|_{s_{pec}}.$$  \hspace{1cm} (226)

Consequently, when multiplying with the F-matrix, the degree of freedom (DOF) representing $\vec{E}^+$ on the PEC boundary (or $\vec{H}^+$ on the PMC boundary) points back to the local DOF which is then negated prior to the matrix-vector product. The disadvantage of this approach is that it increases the overall number of DOF’s. However, the significant advantage is that the magnetic field and the electric field DOFs will have the same number of degrees of freedom independent of the presence of Dirichlet boundaries. Thus, only a single footprint for the S, Q, and F-matrices will be needed. Furthermore, a direct product of the S matrix can be used. However, the F-matrix product will require a gather-scatter of the product and result vectors, respectively. Another significant advantage is
that the $\mathbf{M}^e$ and $\mathbf{M}^h$ matrices are the same order. And if the material is isotropic and homogeneous, then these matrices will be identical, and are only distinguished by a scalar weight. As a consequence, only 1 matrix would need to be stored per subdomain! This presents a significant savings in memory. Roughly, a factor of 4 outside the PML region, and upwards of a factor of 12 inside the PML media. Since the PML typically dominates the overall mesh, this presents a savings in memory of nearly and order of magnitude. It is also anticipated that this will also significantly save CPU time since the foot-printed matrices will remain in cache, and only the $\mathbf{M}$-matrix will have to be pulled into cache for each subdomain. Note, that if mixed elements are used (e.g., tetrahedron and Prism elements), then two such footprints of the $\mathbf{F}$ and $\mathbf{S}$-matrices will exist.

**5.3.3. Some Practical Issues**

In order to best take advantage of a single reference matrix, it is realized that there should be a canonical ordering of the edges, faces and volume basis functions. That is, every cell should order them in identical fashion. For the edge basis, each edge will be polarized. That is, a canonical edge will be identified. However, the edge ordering will be based on the cell that defined the edge. Thus, there can be a sign-flip of the basis. In general, the $\mathbf{H}(0)$-curl edge basis can require a sign flip. However, the odd-order Gradient basis do not, whereas the even-order Gradient basis do.

Face basis functions present a bit more of a challenge, since, there is more than a polarization. Rather, there is a rotation. For quad faces, this can present a permutation and a sign flip. Furthermore, the rotational space for triangular faces (starting with the $\mathbf{H}(1)$-curl function space), only two of three-possible functions are chosen (since the third
function is dependent). How this maps from a single footprint matrix to a randomly ordered face will have to be worked through carefully.

For the F-matrices, one approach is rather than store the F-matrix as a single sparse matrix (with a footprint), one could have a single F-matrix per face. In this way, each face reaction would be treated as a separate matrix-vector product. In the PML region, the F-matrix is scaled by $1/\kappa_v$ for faces with normals directed along the PML surface normal. This would simplify the product with the F-matrix.

### 5.3.4. S-matrices in the PML Region

Another issue to consider is that in the PML, the S-matrix will have a different scaling based on the type of overlap. As a consequence, we could have upwards to 27 different S-matrices as references. Applying symmetry this could hopefully be reduced to 7 if the $\kappa_v$ are constant. However, if the $\kappa_v$ are scaled from layer to layer, then in the overlap region, the S-matrix will have to be different in each layer. This presents an additional challenge. One solution to this is to pose an alternate DGFETD formulation. Returning to (185), this expression can also be written as:

$$\frac{1}{s_v} \frac{\partial}{\partial u^v} \bar{a}^v \times \bar{E} = \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{E} + \left( \frac{1}{\kappa_v} - 1 \right) \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{E} + \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{Q}^v. \quad (227)$$

Then, inserting this within Faraday’s law and applying the typical DGFETD transformations leads to the weak form expression:
\[
\frac{\partial}{\partial t} \int_{V_v} \bar{T}^h \cdot \bar{\mu} \cdot \bar{H} dv =
\]
\[
- \frac{1}{2} \int_{V_v} \left( \bar{E} \cdot \nabla \times \bar{T}^h + \bar{T}^h \cdot \nabla \times \bar{E} \right) dv
\]
\[
- \frac{1}{2} \left( \frac{1}{\kappa_v} - 1 \right) \sum_{v} \int_{V_v} \left( \bar{T}^h \cdot \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{E} + \bar{E} \cdot \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{T}^h \right) dv.
\]
\[
- \frac{1}{2} \sum_{v} \int_{V_v} \frac{1}{\kappa_v} \left( \bar{T}^h \cdot \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{Q}_v + \bar{Q}_v^E \cdot \bar{a}^v \times \frac{\partial}{\partial u^v} \bar{T}^h \right) dv
\]
\[
- \frac{1}{2} \int_{\partial V_v} \bar{T}^h \cdot \hat{n} \times \bar{E} ds - \frac{1}{2} \sum_{v} \int_{\partial V_v} \bar{T}^h \cdot \hat{n} \times \bar{Q}_v^E ds.
\]

Note that in this form, the curl operators are the standard curl-operators. Thus, the S-matrix will be identical in the PML region as it is in the working volume. And the stretching operator on the \( \bar{E} \) field will be the same as that operates on \( \bar{Q}_v^E \). Then this can be expressed in a linear operator form similar to (209) as:

\[
M^{hh}_\mu h' = -S^{he} e - \sum_{v} Q^{he}_v \left[ \left( \frac{1}{\kappa_v} - 1 \right) e + q^e \right] - F^{he} \left( e^+ + q^{e+} \right).
\]

In this way, only a single S-matrix is needed, and \( Q^{he} \) is used on both \( e \) and \( q^e \). And only a single matrix product with \( Q^{he} \) would actually have to be performed.

The advantage of the proposed approach is that it significantly reduces the overall complexity of the algorithm, since the only change of the difference operator in the PML region is the introduction of the Q matrix. The rest of the operator remains exactly the same. It would appear that this would be the preferred method of implementation. Note that a dual form is realized for the weak form of Ampere’s law.
5.4. Mesh Extrusion

In the process of product design and research validation, there is a common case in which the user does not want to change the mesh but does want to enlarge the problem domain or attach different thickness of PML layers. In order to do this, the user has to invest significant time on re-meshing the problem again and again. To alleviate this, an automated mesh extrusion was developed. The extrusion routine assumes the exterior boundary to be rectangular. The mesh extrusion routine can extrude the original mesh along any or all of the 6 (x-,y-,z-) directions with any specified number of layers and thickness. The overlapped corner region is properly handled. Consequently, the user can perform multiple runs without having to re-mesh.
5.5. Validation

5.5.1. 1D Parallel Plate Waveguide

The first test model is a parallel plate waveguide with a volume excitation set in the middle. The model is shown in Figure 61, in which the waveguide is meshed with $61 \times 1 \times 1$ hexes, with two vertical walls set as the perfectly magnetic conducting (PMC) boundary. The two PML interfaces locate at -1.55 m and 1.55 m, respectively. To obtain the optimal absorbing rate for a desired reflection error, $\sigma$ is evaluated as [51]:

$$\sigma_{opt} = \frac{-\log(err)(m+1)}{2d\eta_r}$$  \hspace{1cm} (230)

where $err$ is the desired reflection error, $m$ is the scaling polynomial order, $d$ is the PML depth and $\eta_r$ is the relative wave impedance of the host media. The simulation configuration is illustrated in Table 2.
Table 2. Configuration of the simulation for 1D PPWG

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size (m)</td>
<td>6.1<em>0.1</em>0.1</td>
</tr>
<tr>
<td>Mesh</td>
<td>61<em>1</em>1 hexes</td>
</tr>
<tr>
<td>Basis</td>
<td>H(1) curl-conforming hex basis</td>
</tr>
<tr>
<td>PML</td>
<td>15 cells thick PMLs attached to X+, X- sides</td>
</tr>
<tr>
<td></td>
<td>( \kappa = 3, \sigma = \sigma_{opt}, a = 0.2, m = 1 ) for CFS-PML</td>
</tr>
<tr>
<td>Excitation</td>
<td>J current volume excitation (0,0,1) with amplitude of 1.0</td>
</tr>
<tr>
<td>Source</td>
<td>Sinusoid source with freq = 300MHz</td>
</tr>
<tr>
<td>Notes</td>
<td>PMC wall defined on the two side walls</td>
</tr>
</tbody>
</table>

Figure 62. Snapshot of field within the 1D PPWG region
(a) without PML, (b) with PML

Figure 62 (a) shows the simulated results when the CFS-PML is turned off. From the wave form it is observed that the wave is completely bounced back as it hits both side of the PEC walls. As a comparison, the results with the CFS-PML turned on is illustrated in Figure 62 (b), in which the sinusoid wave propagates towards both ends of the waveguide, as it enters into the PML region, the wave is clearly decayed and squeezed, which indicates that the PML media is absorbing the incoming wave as \( \kappa \) and \( \sigma \) being properly set.
Next, a more systematic study of the reflection error of the CFS-PML is presented. The parallel plate waveguide geometry is used again for this purpose. To extract the reflection error for the PML, a reference parallel plate waveguide is used that is sufficiently long so that the simulation would cease before reflections from the terminating boundary wall would return. The simulation configuration is illustrated in Table 3. When applying the PML to FDTD applications, the PML is not actually perfectly matched since the discrete electric and magnetic fields are staggered in both space and time [51]. As a consequence, the PML constitutive parameters must be spatially scaled to avoid large reflection errors. In the DGFETD formulation, the discrete electric and magnetic fields are co-located in both space and time. Thus, the PML is matched in the discrete space. Discretization errors will still lead to reflection error. However, spatial scaling is not as imperative, and thinner PML layers can be used. The reflection error due to the PML as a function of the PML conductivity is presented in Figure 63. The simulations assume other PML parameters with a constant profile. Cases are presented where the PML was 2 hexahedral cells thick. H0-H4 curl-conforming basis functions were employed. The exact reflection error is also illustrated in the plot, where

\[ R(\text{Exact}) = e^{-2d\sigma\eta/(m+1)} \]  

(231)

where, \( d \) is the thickness of the PML slab (in meters), \( \sigma \) is the normal PML conductivity, \( \eta \) is the free-space wave impedance, and \( m \) is the polynomial scaling factor of \( \sigma \). The PML reflection error is dominated by the reflection error at the back PEC wall (namely, due to a round trip of the wave through the PML) for sufficiently small values of sigma. As sigma becomes sufficiently large, the reflection error levels off due to discretization error of the fields. From Figure 63, it is observed that increasing the basis order
dramatically improves the reflection error. However, since in this case only propagating waves exists in the problem domain, thus CFS-PML shows no advantage on the convergence rate when compared with APML.

<table>
<thead>
<tr>
<th>Table 3. Configuration of the convergence study for 1D PPWG</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Size (m)</strong></td>
</tr>
<tr>
<td><strong>Mesh</strong></td>
</tr>
<tr>
<td><strong>Basis</strong></td>
</tr>
<tr>
<td><strong>PML</strong></td>
</tr>
<tr>
<td><strong>Excitation</strong></td>
</tr>
<tr>
<td><strong>Source</strong></td>
</tr>
</tbody>
</table>
| **Notes**    | $\kappa = 3$, $a = 0$, $m = 1$ for APML  
$\kappa = 3$, $a = 0.2$, $m = 1$ for CFS-PML  
sweeping $\sigma$ from 1 to 50,  
$\sigma_{opt}$ is calculated as 23.0259 for the reflection error $1e^{-4}$. |

![Figure 63. Reflection error due to the PML termination of the parallel plate waveguide versus the PML conductivity (m = 0). (a) APML [7]; (b) CFS-PML](image-url)
5.5.2. 2D Parallel Plate Waveguide

In this section, a 2D parallel plate waveguide with a volume source excited in the middle of the geometry is simulated. The model is shown in Figure 64. The 2D parallel plate is terminated with 4 PML walls. The simulation parameters are illustrated in Table 4.

Table 4. Configuration of the simulation for 2D PPWG

<table>
<thead>
<tr>
<th>Size (m)</th>
<th>6.1<em>6.1</em>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh</td>
<td>61<em>61</em>1 hexes</td>
</tr>
<tr>
<td>Basis</td>
<td>H(0) curl-conforming hex basis</td>
</tr>
<tr>
<td>PML</td>
<td>10 cells thick PMLs attached to X+, X-, Y+, Y- sides</td>
</tr>
<tr>
<td></td>
<td>$\kappa = 3, \sigma = 9.2, a = 0.02, m = 1$ for the CFS-PML</td>
</tr>
<tr>
<td>Excitation</td>
<td>J current volume excitation (0,0,1) with amplitude of 1.0</td>
</tr>
<tr>
<td>Source</td>
<td>Sinusoid source with freq = 300MHz</td>
</tr>
</tbody>
</table>
As observed in Figure 65, the field distributions in the 2D without/with the CFS-PML turned on are both demonstrated. Obvious reflections are observed in Figure 65 (a) when all PMLs are turned off. In contrast, when the CFS-PML is turned on, the PML media will absorb the incoming wave from any incident angles and the 2D waves are propagating just as in an open bounded domain.
5.5.3. 3D Dipole Radiation

![Figure 66. 3D Cube model](image)

(a) mesh; (b) probe plane.

In this section, a 3D box with a dipole volume source excited in the middle is simulated. The model is shown in Figure 66 (a). Such 3D box is terminated with 6 PML walls. The field is sampled on a face cut of the cube, as shown in Figure 66 (b). Simulation configuration is illustrated in Table 5.

<table>
<thead>
<tr>
<th>Table 5. Configuration of the simulation for 3D Cube</th>
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</thead>
<tbody>
<tr>
<td>Size (m)</td>
</tr>
<tr>
<td>Mesh</td>
</tr>
<tr>
<td>Basis</td>
</tr>
<tr>
<td>PML</td>
</tr>
<tr>
<td>Excitation</td>
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<tr>
<td>Source</td>
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</tbody>
</table>
As observed in Figure 67, the field distributions in the 2D face cut without/with the CFS-PML turned on are both demonstrated. Obvious reflections are observed in Figure 67 (a) when all PMLs are turned off. In contrast, when the CFS-PML is turned on, the PML media will absorb the incoming wave from any incident angles and the dipole fields are propagating just as in free-space.
5.5.4. Comparing CFS-PML with Non-CFS-PML

In this section, the DGFETD CFS-PML is compared with the APML. The test case is two dimensional 40 cm × 40 cm working volume excited with a TM$_z$ polarized current source at the center of the domain, as illustrated in Figure 68.

![Figure 68. 2D problem description](image)

The source is a differentiated Gaussian pulse with $t_w = 2$ ns and $t_0 = 5t_w$. The field is sampled 1.71 cm from the corner of the PML boundary, which is shown as point B in Figure 68. Four PML walls are attached to the problem domain. The problem is simulated with both CFS-PML and APML. The reflection error is calculated as:

$$Err(t) = 20 \log \left( \frac{|\bar{E}_r(t) - \bar{E}(t)|}{|\bar{E}_{r,max}(t)|} \right), \quad (232)$$

where the $\bar{E}(t)$ is the simulated the electric field intensity. $\bar{E}_r(t)$ is the reference time-domain signal. $\bar{E}_r(t)$ can be obtained by simulating a much larger problem, so that the reflected signal has not finish the round trip to reach point B. Thus $\bar{E}_r(t)$ is reflectionless.
signal and can be used as reference. The problem is simulated with H(1) curl conforming vector basis. The detailed simulation configuration is illustrated in Table 6.

| Table 6. Configuration of the simulation for 2D PPWG with both CPML and APML |
|---------------------------------|--------|
| Size (m)                        | 4.1*4.1*0.1 |
| Mesh                            | 41*41*1 hexes |
| Basis                           | H(1) curl-conforming hex basis |
| PML                             | 2, 4, 6 cells thick attached to X+, X-, Y+, Y- sides, respectively $\kappa=3, \sigma=\sigma_{\text{opt}}, a=0.2, m=2$ for the CFS-PML $\kappa=3, \sigma=\sigma_{\text{opt}}, m=2$ for the APML |
| Excitation                      | J current volume excitation (0,0,1) with amplitude of 1.0 |
| Source                          | differentiated Gaussian pulse with -3dB bandwidth 158MHz |
Figure 69. Time domain signal of the electric field intensity

Figure 70. Time domain reflection error of the electric field intensity
   (a) APML; (b) CPML
Figure 71. Comparison of CPML with APML on reflection error
(a) 2 layers; (b) 4 layers; (c) 6 layers
Both the CFS-PML and APML walls are applied to truncate the problem domain, respectively, and the test case is simulated via the DGFETD-EM solver. Figure 70 (a) shows the time-domain $E_z$ field sampled at point B. Figure 70 (b) provides a closer view of the $E_z$ field, where obvious reflections are observed for 2-layer cases due to the insufficient thickness of both the CFS-PML and the APML for H(1) basis. The time-domain reflection errors calculated via (232) for APML and CFS-PML with different thicknesses are summarized in Figure 70 (a) and (b) respectively. For comparison, the reflection errors of APML and CFS-PML with 2, 4 and 6 layers are also illustrated in Figure 71 (a)-(c), respectively. From the results, it is observed that CFS-PML provides much better absorption for the late time signals (>400 ns) for different thicknesses. This is validating that CFS-PML provides absorption of the evanescent waves and hence become advantageous to APML when the incoming waves contain evanescent modes.
5.6. Summary

A CFS-PML has been formulated and implemented. A pair of auxiliary differential equations (ADE) was formulated to solve for the auxiliary fields. The CFS-PML was then validated for 1D, 2D and 3D test cases. Among all test cases, it is observed that the waves are absorbed with very small reflections. The CFS-PML was also compared with Non-CFS-PML when simulating an open bound 2D problem. Better absorption of the evanescent waves is observed when using the CFS-PML. Certain features such as the footprint of the S, F and Q matrices were also found and proved.
Chapter 6. Conclusion

The discontinuous Galerkin finite element time-domain (DGFETD) method was detailed in Chapter 2 for solving coupled curl Maxwell’s equations. Both central flux and upwind flux formulations were presented. Three modeling approaches, including circuit modeling, thin-wire modeling and the complex frequency shifted (CFS) perfectly matched layer (PML) modeling were then developed in subsequent chapters to extend the DGFETD application.

In Chapter 3, a hybrid DGFETD-EM/Circuit solver was formulated and validated. A circuit port that interfaces the full-wave (DGFETD-EM) and circuit (SPICE) solvers was introduced. The circuit port can be modeled by either a volumetric region or a surface embedded within the field mesh. A hybrid formulation to incorporate passive lumped circuit elements such as resistors, capacitors and inductors was derived. Simple voltage-current relations were developed for the DGFETD-EM framework by using direct stamping approach. For more complex circuits and nonlinear devices, a hybrid DGFETD-EM/SPICE solver was developed. To validate the proposed approach, various test cases including both linear and nonlinear devices, single and multiple circuit ports, were simulated by the hybrid DGFETD-EM/Circuit solver. The simulation results showed very good agreement with a reference result.

In Chapter 4, a hybrid DGFETD-EM/ThinWire solver was introduced. The hybrid simulator can solve concurrently the time-dependent Maxwell’s equations and the telegrapher’s equations. A thin-wire port was designed to properly handle the coupling between the DGFETD-EM and DGFETD-ThinWire solvers. The hybrid solver maintains high-order accuracy in both field modeling and time integration. With proper calculation
of the geometrical overlaps, the 1D thin-wire mesh can be chosen independent of the 3D field mesh, which allows arbitrarily located and oriented thin-wires to be simulated within the DGFETD-EM framework without constraining the 3D mesh. In addition, a method to treat curvilinear thin-wires is also provided. The proposed thin-wire model was then validated through the simulation of arbitrarily positioned thin-wire dipole and loop antennas. The input impedance and admittance were examined with the results obtained from the method of moment. Excellent agreement was found, validating hybrid solver.

In Chapter 5, a CFS-PML was formulated and implemented within the DGFETD-EM solver. A set of auxiliary differential equations (ADE) was formulated to solve for the auxiliary fields. The CFS-PML was then validated for 1D, 2D and 3D test cases. Among all test cases, it is observed that the waves are absorbed with very small reflections. The CFS-PML was also compared with the traditional anisotropic PML (APML) when simulating an open bound 2D problem. Better absorption of the evanescent waves is observed when using the CFS-PML. Certain features such as the footprint of the S, F and Q matrices were also found and proved.

The major contributions of this work can be outlined in the following. First of all, it shows in theory how to incorporate circuit elements, thin-wires and CFS-PML media effects into the Maxwell’s equations that is solved via the DGFETD method. Second, it utilizes the port to handle the coupling between the DGFETD-EM and the auxiliary solvers such as SPICE, thin-wire, and CFS-PML auxiliary differential equation solvers. So that the DGFETD-EM field solver and the auxiliary solvers are kept independent from each other and only hybridized through the port. Doing this will dramatically increase the flexibility of hybrid modeling approach. For example, in the proposed methods, the
circuit port and thin-wire port regions can be defined independent of the field mesh, which will allow arbitrarily oriented and located circuit elements as well as thin-wires to be properly handled by the hybrid solver. In addition, by using the high-order basis functions in the DGFETD-EM field and the auxiliary solvers, high order accuracy of the coupling can be achieved through the port coupling matrices. And lastly, by using high-order time integration schemes such as Runge-Kutta scheme, the time-domain coupling between the EM and the auxiliary solvers can also achieve high-order accuracy.
Appendix

Corollary 4.1

The following identity involving the partial curl holds true:

$$\tilde{A} \cdot \tilde{a}^i \times \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \tilde{B} = \tilde{B} \cdot \tilde{a}^i \times \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \tilde{A} - \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} (\tilde{a}^i \cdot \tilde{A} \times \tilde{B} \sqrt{g}).$$  \hfill (233)

Proof: The proof is made by first assuming the corollary to be true. The vectors are initially expanded via their covariant projections:

$$\sum_{k=1}^{3} \tilde{a}^k a_k \cdot \tilde{a}^i \times \sum_{j=1}^{3} \tilde{a}^j \frac{1}{\kappa_j} \frac{\partial}{\partial u^j} b_j = \sum_{j=1}^{3} \tilde{a}^j b_j \cdot \tilde{a}^i \times \sum_{k=1}^{3} \tilde{a}^k \frac{1}{\kappa_k} \frac{\partial}{\partial u^k} a_k - \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left( \tilde{a}^i \cdot \left( \sum_{k=1}^{3} \tilde{a}^k a_k \right) \times \left( \sum_{j=1}^{3} \tilde{a}^j b_j \right) \sqrt{g} \right),$$  \hfill (234)

where, we have expanded:

$$\tilde{A} = \sum_{k=1}^{3} \tilde{a}^k a_k \text{, and } \tilde{B} = \sum_{j=1}^{3} \tilde{a}^j b_j,$$  \hfill (235)

$$a_{i-1} \tilde{a}^{i-1} \cdot \tilde{a}^i \times \tilde{a}^{i+1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i+1} + a_{i+1} \tilde{a}^{i+1} \cdot \tilde{a}^i \times \tilde{a}^{i-1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i-1}$$

$$= \tilde{a}^{i-1} b_{i-1} \cdot \tilde{a}^i \times \tilde{a}^{i+1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i+1} + \tilde{a}^{i+1} b_{i+1} \cdot \tilde{a}^i \times \tilde{a}^{i-1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i-1},$$  \hfill (236)

$$- \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left[ \tilde{a}^i \cdot \left( \tilde{a}^{i-1} a_{i-1} \times \tilde{a}^{i+1} b_{i+1} + \tilde{a}^{i+1} a_{i+1} \times \tilde{a}^{i-1} b_{i-1} \right) \sqrt{g} \right]$$

Next, we recognize that:

$$\tilde{a}^i \cdot \tilde{a}^j \times \tilde{a}^k = \begin{cases} \frac{1}{\sqrt{g}} \nu_{i,j,k}, & i \neq j \neq k \\ 0, & \text{else} \end{cases},$$  \hfill (237)

where $\nu_{i,j,k} = \pm 1$ is based on the cycle ordering of i,j,k. Thus, (236) can be expressed as:
\[ a_{i-1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i+1} = a_{i+1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i-1} = b_{i+1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i+1} - b_{i-1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i-1} \]

\[-\frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left[ -\frac{1}{\sqrt{g}} a_{i+1} b_{i+1} + \frac{1}{\sqrt{g}} a_{i+1} b_{i-1} \right] \sqrt{g} \]

Expanding this and cancelling terms, leads to:

\[ a_{i-1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i+1} - a_{i+1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i-1} = b_{i+1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i+1} - b_{i-1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i-1} \]

\[ + \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left[ (a_{i+1} b_{i+1} - a_{i+1} b_{i-1}) \right] \]

\[ = b_{i+1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i+1} - b_{i-1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i-1} \]

\[ + \frac{1}{\sqrt{g}} a_{i-1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i+1} + \frac{1}{\sqrt{g}} b_{i+1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i+1} \]

\[ - \frac{1}{\sqrt{g}} a_{i+1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i-1} - \frac{1}{\sqrt{g}} b_{i-1} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_{i-1} \]

\[ = a_{i-1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i+1} - a_{i+1} \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_{i-1} \]

The equality is thus proof that Corollary 4.1 is true.
Corollary 4.2

The following identity is true:
\[
\tilde{A} \cdot \tilde{\nabla} \times \tilde{B} = \tilde{B} \cdot \tilde{\nabla} \times \tilde{A} - \tilde{\nabla} \cdot \left( \tilde{A} \times \tilde{B} \right) .
\] (240)

Proof: The proof is performed in a similar manner as Corollary 4.1. The relation is first expanded into curvilinear coordinates:
\[
\tilde{A} \cdot \sum_{i=1}^{3} \tilde{a}^i \times \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \tilde{B} = \tilde{B} \cdot \sum_{i=1}^{3} \tilde{a}^i \times \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \tilde{A} - \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left( \tilde{a}^i \cdot \tilde{A} \times \tilde{B} \sqrt{g} \right) .
\] (241)

The vectors are expanded via their covariant projections:
\[
\sum_{k=1}^{3} \tilde{a}^k a_k \cdot \sum_{i=1}^{3} \tilde{a}^i \times \sum_{j=1}^{3} \tilde{a}^j \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_j = \sum_{i=1}^{3} \tilde{a}^i b_j \cdot \sum_{j=1}^{3} \tilde{a}^j \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} a_k - \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left( \tilde{a}^i \cdot \left( \sum_{k=1}^{3} \tilde{a}^k a_k \right) \times \left( \sum_{i=1}^{3} \tilde{a}^i b_i \right) \sqrt{g} \right) ,
\] (242)

where, we have also applied the identity:
\[
\tilde{\nabla} \times \tilde{B} = \sum_{i=1}^{3} \tilde{a}^i \times \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \tilde{B} = \sum_{i=1}^{3} \tilde{a}^i \times \sum_{j=1}^{3} \tilde{a}^j \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} b_j .
\] (243)

Next, we recognize that:
\[
\tilde{a}^i \cdot \tilde{a}^j \times \tilde{a}^k = \begin{cases} \frac{1}{\sqrt{g}} V_{i,j,k} , & i \neq j \neq k \\ 0 , & \text{else} \end{cases}
\] (244)

where \( V_{i,j,k} = \pm 1 \) is based on the cycle ordering of i,j,k. Thus, (242) can be expressed as:
\[
\frac{1}{\sqrt{g}} \sum_{k=1}^{3} a_k \left( \frac{1}{\kappa_{k+1}} \frac{\partial}{\partial u^{k+1}} b_{k-1} - \frac{1}{\kappa_{k-1}} \frac{\partial}{\partial u^{k-1}} b_{k+1} \right) = \frac{1}{\sqrt{g}} \sum_{j=1}^{3} b_j \left( \frac{1}{\kappa_{j+1}} \frac{\partial}{\partial u^{j+1}} a_{j-1} - \frac{1}{\kappa_{j-1}} \frac{\partial}{\partial u^{j-1}} a_{j+1} \right) - .
\] (245)
The differentiation in the second term is then performed via a product rule. Then, canceling terms, this leads to:

\[
\frac{1}{\sqrt{g}} \sum_{k=1}^{3} a_k \left( \frac{1}{\kappa_{k+1}} \frac{\partial}{\partial u_{k+1}^{k-1}} b_{k-1} - \frac{1}{\kappa_{k-1}} \frac{\partial}{\partial u_{k-1}^{k+1}} b_{k+1} \right) =
\]

\[
\frac{1}{\sqrt{g}} \sum_{k=1}^{3} a_k \left( \frac{1}{\kappa_{k+1}} \frac{\partial}{\partial u_{k+1}^{k-1}} b_{k-1} - \frac{1}{\kappa_{k-1}} \frac{\partial}{\partial u_{k-1}^{k+1}} b_{k+1} \right). \tag{246}
\]

This equality thus proves that the corollary posed in (240) is true.
Corollary 4.3

The following identity involving the partial curl holds true:

\[
\int \int \int_V \frac{1}{\sqrt{g}} \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left( \vec{a}_i \cdot \vec{F} \sqrt{g} \right) dv = \int_{S_i^l} \vec{F} \cdot \vec{n}_i^l ds + \int_{S_i^r} \vec{F} \cdot \vec{n}_i^r ds ,
\]

(247)

where \( S_i^l \) and \( S_i^r \) are the “left” and “right” boundaries with unit normal directed parallel to \( \vec{a}_i \), and \( \vec{n}_i^l \) and \( \vec{n}_i^r \) are the unit normal’s to \( S_i^l \) and \( S_i^r \) that are directed out of \( V \), and

\[ ds = \sqrt{g} du^{i+1} du^{i-1} \]

where \( \sqrt{g} \) is the surface Jacobian, and \( du^{i+1} \) and \( du^{i-1} \) are evaluated with modulo-base 3 arithmetic.

Proof: The reciprocal unitary vector \( \vec{a}_i \) is expanded as a function of the unitary vectors:

\[
\int \int \int_V \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left( \vec{a}_{i+1} \times \vec{a}_{i-1} \cdot \vec{F} \sqrt{g} \right) dv = \int \int \int_V \frac{1}{\kappa_i} \frac{\partial}{\partial u^i} \left( \vec{a}_{i+1} \times \vec{a}_{i-1} \cdot \vec{F} \right) du^{i+1} du^{i-1} ,
\]

(248)

If we assume that \( \kappa_i \) is a constant along \( u^i \), then this can be expressed as

\[
\frac{1}{\kappa_i} \int_{V} \frac{\partial}{\partial u^i} \left( \vec{a}_{i+1} \times \vec{a}_{i-1} \cdot \vec{F} \right) du^{i+1} du^{i-1}
\]

\[
= \frac{1}{\kappa_i} \int_{S_i^l} \vec{a}_{i+1} \times \vec{a}_{i-1} \cdot \vec{F} dv^{i+1} dv^{i-1} - \frac{1}{\kappa_i} \int_{S_i^r} \vec{a}_{i+1} \times \vec{a}_{i-1} \cdot \vec{F} dv^{i+1} dv^{i-1}
\]

\[
= \frac{1}{\kappa_i} \int \sqrt{g} \vec{a}_i \cdot \vec{F} dv^{i+1} dv^{i-1} - \frac{1}{\kappa_i} \int \sqrt{g} \vec{a}_i \cdot \vec{F} dv^{i+1} dv^{i-1}
\]

\[
= \frac{1}{\kappa_i} \int \vec{F} \cdot \vec{n}_i^l ds + \frac{1}{\kappa_i} \int \vec{F} \cdot \vec{n}_i^r ds
\]

(249)

which proves the corollary.
Corollary 4.4

Given a volume $V$ bound by a surface $S$, which is defined by a set of faces that are separable in the local curvilinear coordinate system, the following identity is true:

$$
\int_{V} \nabla \cdot \vec{F} dv = \iint_{S} \vec{F} \cdot d\vec{s},
$$

(250)

where, on face $v$, which is directed along the unitary axis $\hat{a}^v$,

$$
d\vec{s} = \frac{1}{\kappa_v} \hat{a}^v \sqrt{g_v} \cdot d\vec{s},
$$

(251)

where $\hat{a}^v$ is the unit normal to $S$ directed out of the volume and

$$
\sqrt{g_v} = |\vec{a}_{v+1} \times \vec{a}_{v-1}|,
$$

(252)

where the indices are evaluated via modulo-base three arithmetic.

Proof: The divergence on the left-hand-side of (250) is expressed in the local stretched curvilinear coordinates as:

$$
\int_{V} \nabla \cdot \vec{F} dv = \int_{V} \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{1}{\kappa_i} \frac{\partial}{\partial u_i} (\hat{a}^i \cdot \vec{F} \sqrt{g}) \sqrt{g} du^1 du^2 du^3. \tag{253}
$$

Consider the first term in the summation

$$
\int_{V} \frac{1}{\kappa_i} \frac{\partial}{\partial u_i} (\hat{a}^i \cdot \vec{F} \sqrt{g}) du^1 du^2 du^3. \tag{254}
$$

Applying Corollary 4.3, this is expressed as:

$$
\frac{1}{\kappa_i} \int_{V} \frac{\partial}{\partial u_i} \left(\vec{a}_i \cdot \vec{F} \right) du^1 du^2 du^3 = \frac{1}{\kappa_i} \int_{S_{\vec{a}_i}} \sqrt{g_i} \hat{a}^i \cdot \vec{F} du^2 du^3 - \frac{1}{\kappa_i} \int_{S_{\vec{a}_i}} \sqrt{g_i} \hat{a}^i \cdot \vec{F} du^2 du^3. \tag{255}
$$

Expanding all three terms on the right-hand-side of (253) via (255), and applying superposition then leads to:
\[
\int \vec{\nabla} \cdot \vec{F} \, dv = \iint \vec{F} \cdot d\vec{s},
\]

(256)

where, we assume that the faces are separable in the curvilinear coordinate system.
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Vita

The author of this dissertation was born in the city of Shenyang, P. R. China, July 2nd 1983. He finished his undergraduate study in Zhejiang University at Hangzhou, China and got his Bachelor’s degree in June 2006. During his undergraduate study, he was research assistant in Audio-Video Signal Processing Laboratory in Zhejiang University instructed by Professor Yong Cheng (March 2003-December 2004, Hangzhou, China). Later he attended the research project and became a research assistant at the Center for Computational Electromagnetism and State Key Laboratory of Millimeter Waves in Southeast University instructed by Professor Tie Jun Cui (September 2005-July 2006, Nanjing China). He received the award of “Outstanding Thesis for Bachelor’s Degree at Zhejiang University” at June 2006.

At August 2006, he came to University of Kentucky and continues his graduate study in the Department of Electrical and Computer Engineering. His advisor was Dr. Stephen D. Gedney. During his graduate study, he was well trained with modern computational electromagnetic techniques including Method of Moment, Finite Difference Time Domain Method and Finite Element Method. In addition, he also took independent research projects and become a core development member of the General Electromagnetic Framework. In addition, he becomes the student member of IEEE Antennas and Propagation Society since 2008, and received IEEE Antennas and Propagation Society Pre-Doctoral/Doctoral Research Awards at August 2009.
Journal Articles


Conferences


