SL_k-Tilings and Paths in $\mathbb{Z}^k$

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SL_k-Tilings and Paths in $\mathbb{Z}^k$

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

SL_k-Tilings and Paths in $\mathbb{Z}^k$

An SL_k-frieze is a bi-infinite array of integers where adjacent entries satisfy a certain diamond rule. SL_2-friezes were introduced and studied by Conway and Coxeter. Later, these were generalized to infinite matrix-like structures called tilings as well as higher values of k. A recent paper by Short showed a bijection between bi-infinite paths of reduced rationals in the Farey graph and SL_2-tilings. We extend this result to higher k by constructing a bijection between SL_k-tilings and certain pairs of bi-infinite strips of vectors in $\mathbb{Z}^k$ called paths. The key ingredient in the proof is the relation to Plücker friezes and Grassmannian cluster algebras. As an application, we obtain results about periodicity, duality, and positivity for tilings.

KEYWORDS: Combinatorics, Representation Theory, Algebra, Grassmannians, Friezes, Tilings

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Chapter 1 Introduction

A *frieze* is a bi-infinite offset array of entries satisfying certain properties. Friezes were first introduced and studdied by Conway and Coxeter in 1970's [5][6]. They proved various facts about periodicity and symmetry of friezes and, in particular, that there is a bijection between SL_2-friezes over positive integers, often called *Conway-Coxeter Friezes*, and triangulations of polygons. Later, the discovery of cluster algebras in 2000's [12] has created a newfound interest in friezes. In particular, cluster algebras of type A are closely tied to triangulations of polygons. Moreover, additive categorification of cluster algebras yields another important relation between friezes and representation theory of quivers, first shown in [4]. Subsequent connections were also discovered between frieze patterns and Farey graphs, cross-ratios, and continued fractions, see the survey [13] and references therein. From this foundation came a series of generalizations of SL_2-friezes.

The first generalization is in terms of k, which can be also thought of as dimension. While Conway and Coxeter defined friezes specifically for $k = 2$, the definition was then expanded to SL_k-friezes. These were found to be related to linear difference equations and the combinatorial Gale transform [14].

Another direction of study comes from considering friezes with non-integer entries. The Grassmannian $\mathrm{Gr}(k, n)$ is a projective variety via the Plücker embedding, with homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(k, n)]$, which is one of the first and most well-known examples of cluster algebras [16]. It was shown by Baur, Faber, Gratz, Serhiyenko, and Todorov that one may use certain Plücker coordinates as entries in an SL_k-frieze and the Plücker relations that they satisfy yield the desired diamond rule, the defining property of a frieze [2]. This lead to a classification of all (finite) SL_k-friezes as Plücker friezes applied to a particular element of the Grassmannian. Later in [10], a generalization of SL_k friezes were studied in relation to juggling functions and positroid varieties, certain important subvarieties of the Grassmannian. In addition, recently [9] extend the work on triangulations of subpologyons and friezes beyond the two-dimensional case.

In another direction, friezes were further generalized to SL_k-tilings, infinite arrays $\mathcal{M} = (m_{ij})$ with $i, j \in \mathbb{Z}$ where the determinant of every $k \times k$ submatrix equals 1, by Bergeron and Reutenauer [3]. These tilings are called tame if every larger adjacent minor has a determinant of 0. They showed how one could construct a tiling from a frieze by rotating the frieze 45° clockwise and extending infinitely. In their exploration of tame SL_k-tilings they show that tilings may be represented using their so-called linearization data. In the case of two-dimensional tilings, Short [17] relates this linearization data to paths, and proves a bijection between SL_2-tilings and paths in the Farey graph. Further restrictions of this map give geometric interpretations for positive and periodic SL_2-tilings as well as tilings from SL_2-friezes. This bijection was later extended to the 3D Farey graph by considering tilings over Eisenstein integers [11], as well as entries in the field $\mathbb{Z}/n\mathbb{Z}$ [18].

In summary, there has been a lot of work in the case $k = 2$ as well as some more recent developments in the case of SL_k freizes for higher values of k. In this paper, we study SL_k tilings for $k \geq 2$, as these objects are much less understood. We consider them from the perspective of linear algebra and obtain a generalization of Short's result [17]. Since we lack a connection to the geometry of the Farey graph, we introduce a new algebraic notion of a path in $\mathbb{Z}^k$. Let $\gamma = \{\gamma_i\}_{i\in\mathbb{Z}}$ be a bi-infinite strip of k-column vectors $\gamma_i \in \mathbb{Z}^k$ with the property that the matrix $(\gamma_i, \ldots, \gamma_{i+k-1})$ whose columns are k consecutive entries of γ is an element of $SL_k(\mathbb{Z})$. We denote the set of all such strips $\mathcal{P}_k$ and we call γ a *k-path*. Additionally, we define the notion of multiplication of a path by a matrix $A \in SL_k(\mathbb{Z})$ as

$$A\gamma = (\cdots, A\gamma_1, A\gamma_2, A\gamma_3, \cdots).$$

With this, we are able to prove our main result.

Theorem 1.0.1. The map Φ given by

$$\begin{aligned}\Phi : (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) &\to \mathbb{SL}_k \\ (\gamma, \delta) &\mapsto \mathcal{M} = (m_{i\,j})_{i,j\in\mathbb{Z}},\end{aligned}$$

where $m_{i\,j} = \det(\gamma_i, \ldots, \gamma_{i+k-2}, \delta_j)$ is a bijection between tame SL_k-tilings and pairs of paths modulo the action by $SL_k(\mathbb{Z})$.

The proof relies heavily on the connection between SL_k friezes and Plücker coordinates. We are also able to obtain a number of other correspondences similar to those of Short. The first is about tilings which result specifically from friezes.

Theorem 1.0.2. The restriction of the map Φ given by

$$\begin{aligned}\Phi_\iota : \mathcal{P}_k/SL_k(\mathbb{Z}) &\to \mathbb{FR}_k \\ \gamma &\mapsto \mathcal{M} = (m_{i\,j})_{i,j\in\mathbb{Z}},\end{aligned}$$

where $m_{i\,j} = \det(\gamma_i, \ldots, \gamma_{i+k-2}, \gamma_j)$ is a bijection between tame SL_k-tilings from SL_k-friezes and equivalence classes of paths.

We also prove that pairs of periodic paths are in bijection with SL_k-tilings which have the corresponding periods on their rows and columns. On the other hand, Short's result about positivity relies heavily on the geometry of the Farey graph by considering paths which move clockwise. It is not clear how to interpret this geometric notion in our setting. However, in the specific case of tilings from friezes, we can look at the quiddity sequence. Rather than relying on the geometry of the Farey graph or triangulations of polygons, we appeal to the structure of $\mathbb{C}[\mathrm{Gr}(k,n)]$ and Plücker relations. This construction allows us to prove that tilings from positive friezes are in bijection with paths with positive quiddity sequences for certain values of n and k, such as most cases where $n \leq 8$. This proof requires a case-by-case analysis for different choices of k and n. Moreover, this correspondence fails for higher values of n. Hence, for general k and n, one needs to develop another notion in this higher-dimensional setting that captures positivity.

Finally, our bijection reinterprets the dual tiling of Bergeron and Reutenauer [3] and further justifies calling this operation a dual. We define a diagonalization operation on a path γ written as $\widetilde{\gamma}$. This leads to the following result.

Theorem 1.0.3. The dual tiling $\Phi(\gamma, \delta)^* = \Phi(A\widetilde{\gamma}, \widetilde{\delta})$ for a fixed $A \in SL_k(\mathbb{Z})$ up to a shift of indices.

In particular, since the diagonalization operation is an involution, one can easily deduce the same for the dual, whereas the original proof of this in [3] relied on some complex calculations.

This paper is organized as follows. In Chapter 1, we discuss the key definitions and past results relating to friezes, tilings, and Plücker coordinates. In Chapter 2 we define the main bijection between paris of paths and SL_k tilings and prove its validity. In Chapter 3 we introduce various restrictions of our bijection and show its connection to duality. In Chapter 4 we complete the necessary calculations to prove the partial positivity results.

1.1 Friezes

We begin by defining a structure first studied by Conway and Coxeter in the case $k = 2$ [5][6] and then extended for higher k by Bergeron and Reutenaur [3].

Definition 1.1.1. An SL_k*-frieze* is an array of offset bi-infinite rows of integers consisting of $k-1$ rows of zeros at the top and bottom, a row of ones below and above them, respectively, and in between $w \geq 1$ rows of integers satisfying the following properties. For an example, see Figure 1.1.

1. Every $k \times k$ diamond of neighboring entries has determinant 1 when considered as a $k \times k$ matrix formed by a 45° clockwise rotation.

2. Every $(k+1) \times (k+1)$ diamond has determinant 0 when considered as a matrix.

We call w the *width* of the frieze. We say the frieze is *positive* if all the entries between the rows of ones are positive.

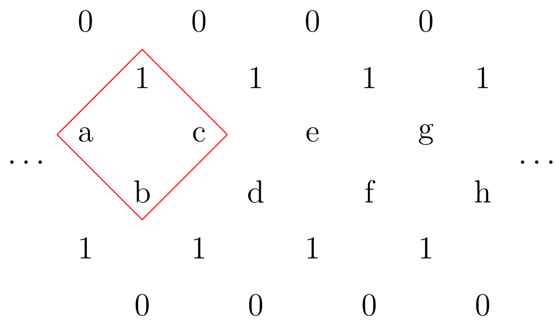

Figure 1.1: An SL_2-frieze of width 2. Every 2×2 diamond must have determinant 1, for example $\begin{vmatrix} a & 1 \\ b & c \end{vmatrix} = ac - b = 1$.

We will also consider friezes with infinite rows.

Definition 1.1.2. An *infinite SL_k-frieze* is an array of offset bi-infinite rows of integers consisting of $k-1$ rows of zeros at the top followed first by a row of ones and then infinitely many rows of integers satisfying properties (1) and (2) of Definition 1.1.1. We say an infinite frieze is *positive* if all the entries below the row of ones are positive.

Note that a non-infinite SL_k frieze may be extended to an infinite frieze with periodic rows. When referring to infinite friezes specifically, we will explicitly use the phrase "infinite friezes".

For the following definitions, fix $k, n \in \mathbb{Z}_{>0}$ with $k < n$. Let $\mathrm{Gr}(k, n)$ denote the Grassmannian of k planes in $\mathbb{R}^n$. That is, elements of $\mathrm{Gr}(k, n)$ are k-linear subspaces of $\mathbb{R}^n$ which can be represented by $k \times n$ matrices of full rank. It is known that $\mathrm{Gr}(k, n)$ is a projective variety identified with the image of the Plücker embedding, with homogenous coordinate ring

$$\mathcal{A}(k, n) = \mathbb{C}[\mathrm{Gr}(k, n)].$$

Moreover, it was shown by Scott [16] that $\mathcal{A}(k,n)$ is a cluster algebra where Plücker coordinates are cluster variables. Let I be a k-tuple with entries in $[n]$ where we allow for repeated entries. Such a tuple gives rise to a *Plücker coordinate* defined as follows.

Definition 1.1.3. A *Plücker coordinate* p_I with $I = (i_1, \dots, i_k)$ is a map from $\mathrm{Gr}(k,n)$ into $\mathbb{R}$,

$$\begin{array}{rccc} p_I : & \mathrm{Gr}(k,n) & \to & \mathbb{R} \\ & A & \mapsto & \det(a_{i_1}, \dots, a_{i_k}), \end{array}$$

where $A = (a_1, \dots, a_n)$ is a $k \times n$ matrix.

Note that if there exists $p \neq q$ with $i_p = i_q$ then $p_I = 0$. Additionally, if $(i_1, \dots, i_k) = \pi(j_1, \dots, j_m)$ where π is a permutation on an ordered k-tuple, then $p_{(i_1,\dots,i_k)} = \mathrm{sign}(\pi) p_{(j_1,\dots,j_m)}$. Note that we want the indices of Plücker coordinates to be ordered and elements of $[n]$. Thus, we introduce the following notation. Given a k-set, possibly a multiset, $\{i_1, \dots, i_k\}$, let $j_m \equiv i_m \pmod n$ with $1 \leq j_m \leq n$ for all $m \in [k]$. Then we write $o(i_1, \dots, i_k) := (j_{\ell_1}, \dots, j_{\ell_k})$ where $\{j_1, \dots, j_k\} = \{j_{\ell_1}, \dots, j_{\ell_k}\}$ and $j_{\ell_1} \leq \cdots \leq j_{\ell_k}$. Hence, $o(I)$ is obtained from a set I by first reducing the entries mod n and then reordering them in increasing order.

Let $I = \{i_1, \dots, i_{k-1}\}$ and $J = \{j_0, \dots, j_k\}$. The Plücker coordinates satisfy the *Plücker relations*

$$\sum_{\ell=0}^{k} (-1)^{\ell} p_{o(I)j_\ell} \cdot p_{o(J \setminus j_\ell)} = 0, \tag{1.1}$$

where $o(I)j_p$ denotes the ordered tuple obtained by adjoining j_p at the end of $o(I)$. We will frequently refer to certain kinds of Plücker coordinates whose indices are at least partially consecutive.

Definition 1.1.4. Fix a k-set I such that $I = [r]^k = \{r, r+1, \dots, r+k-1\}$. Then the Plücker coordinate $p_{o(I)}$ is called *consecutive*. If instead $o(I)$ is a consecutive $(k-1)$-tuple and $m \in [n] \setminus I$, we call the Plücker coordinate $p_{o(I \cup \{m\})}$ *almost consecutive.*

We now define a familiar structure which makes use of Plücker coordinates in place of integers.

Definition 1.1.5. The *Plücker frieze* of type (k,n) denoted by $\mathcal{F}_{(k,n)}$ is a $\mathbb{Z} \times [n+k-1]$ array with entries given by the map

$$(r,m) \mapsto p_{o\left([r]^{k-1}, m'\right)},$$

where $m' = m + r - 1$. For an example, see Figure 1.2.

$$
\begin{array}{cccccccccc}
 & 0 & & 0 & & 0 & & 0 & & \\
 & & p_{12} & & p_{23} & & p_{34} & & p_{45} & \\
\cdots & p_{25} & & p_{13} & & p_{24} & & p_{35} & & \cdots \\
 & & p_{35} & & p_{14} & & p_{25} & & p_{13} & \\
 & p_{34} & & p_{45} & & p_{15} & & p_{12} & & \\
 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Figure 1.2: The Plücker frieze $\mathcal{F}_{(2,5)}$ of type $(2,5)$. When applied to a matrix satisfying the assumptions in Theorem 1.1.6, the consecutive Plücker coordinates in rows 2 and 5 become all 1's and the remaining entries become a frieze as in Figure 1.1.

Baur, Faber, Gratz, Serhiyenko and Todorov have shown that $\mathcal{F}_{(k,n)}$ satisfies properties (1) and (2) of Definition 1.1.1. Hence, it is a frieze with entries in $\mathcal{A}(k,n)$ rather than integers [2]. We wish to discuss their relationship to traditional friezes over the integers. Let $\mathcal{F}_{(k,n)}(A)$ denote the array of numbers $p_I(A)$ resulting from applying each Plücker coordinate in the frieze $\mathcal{F}_{(k,n)}$ to the matrix $A \in \mathrm{Gr}(k,n)$.

Theorem 1.1.6. [2, Theorem 3.1] Let A be a $k \times n$ matrix with entries in $\mathbb{Z}$ with the property that each consecutive Plücker coordinate has a value 1 when applied to A. Then $\mathcal{F}_{(k,n)}(A)$ is an SL_k-frieze.

Of additional use to us is that we can also represent an arbitrary frieze as a Plücker frieze for a particular Grassmannian.

Remark 1.1.7. [2, Remark 3.7] Any SL_k-frieze F of width w over $\mathbb{Z}$ can be embedded into $\mathrm{Gr}(k,n)$ where $n = w + k + 1$ as a point which can be represented by a matrix M_F whose consecutive $k \times k$ minors are ones. Moreover, $\mathcal{F}_{(k,n)}(M_F) = F$.

We will also need the following result about particular matrices formed by Plücker coordinates. For $m_i \in [n]$, we use the notation $[m_1, m_2]$ for the closed cyclic interval $\{m_1, m_1+1, m_1+2, \ldots, m_2\}$ where the elements are considered modulo n. We define open and half-open intervals similarly.

Definition 1.1.8. Let $r, s \in [n]$ and $\underline{m} = (m_1, \ldots, m_s)$ with $m_i \in [n]$. We define the $(s \times s)$ matrix

$$
A_{\underline{m};r} := (a_{i\,j})_{1 \le i,j \le s},
$$

where $a_{i\,j} := p_{o\left([r+i-1]^{k-1}, m_j\right)}$ for $1 \le i, j \le s$.

Proposition 1.1.9. [2, Proposition 3.5] Let $r \in [n]$, $s \in [k]$, and $\underline{m} = (m_1, \ldots, m_s)$ with $m_i \in [n]$ for all i satisfying the following conditions.

(c1) $\underline{m}$ is ordered cyclically modulo n.

(c2) We have $r + k - 2 \notin [m_1, m_s)$.

Then

$$\det\left(A_{\underline{m};r}\right) = \left[\prod_{\ell=0}^{s-2} p_{o\left([r+\ell]^k\right)}\right] \cdot p_{o\left([r+s-1]^{k-s}, m_1, \ldots, m_s\right)}.$$

Example 1.1.10. Suppose in the case of $\mathrm{Gr}(3,8)$ we have $r = 1$, $s = 3$, and $\underline{m} = (3,4,5)$. Then we have

$$A_{\underline{m};r} = \begin{pmatrix} p_{1\,2\,3} & p_{1\,2\,4} & p_{1\,2\,5} \\ p_{2\,3\,3} & p_{2\,3\,4} & p_{2\,3\,5} \\ p_{3\,4\,3} & p_{3\,4\,4} & p_{3\,4\,5} \end{pmatrix} = \begin{pmatrix} p_{1\,2\,3} & p_{1\,2\,4} & p_{1\,2\,5} \\ 0 & p_{2\,3\,4} & p_{2\,3\,5} \\ 0 & 0 & p_{3\,4\,5} \end{pmatrix}.$$

By Proposition 1.1.9, this determinant is

$$\det\left(A_{\underline{m};r}\right) = \left[\prod_{\ell=0}^{1} p_{o([1+\ell]^3)}\right] \cdot p_{o([3]^0,3,4,5)} = p_{1\,2\,3}p_{2\,3\,4}p_{3\,4\,5}.$$

In the set-up of Theorem 1.1.6, Plücker coordinates with consecutive entries will be sent to 1 when applied to the matrices A. Thus, we have

$$\left[\prod_{\ell=0}^{s-2} p_{o\left([r+\ell]^k\right)}(A)\right] = 1,$$

and the determinant in Proposition 1.1.9 is then given by

$$\det\left(A_{\underline{m};r}\right) = p_{o\left([r+s-1]^{k-s}, m_1, \ldots, m_s\right)}(A).$$

1.2 Tilings

Mainly, we are going to focus not on friezes, but on more general structures called *tilings* introduced and studied by Bergeron and Reutenauer [3].

Definition 1.2.1. A *tiling* $\mathcal{M} = (m_{i\,j})_{i,j\in\mathbb{Z}}$ with values $m_{i\,j} \in \mathbb{Z}$ is an infinite array. We denote by $M_{i\,j}$ the adjacent $k \times k$ sub-matrix of $\mathcal{M}$

$$M_{i\,j} = \mathcal{M}_{\{i,\ldots,i+k-1\},\{j,\ldots,j+k-1\}}.$$

We say that $\mathcal{M}$ is an SL_k*-tiling* if $M_{i\,j} \in SL_k(\mathbb{Z})$ for all $i, j \in \mathbb{Z}$. In this case we say that $\mathcal{M}$ satisfies the SL_k*-property*. We say that an SL_k-tiling $\mathcal{M}$ is *tame* if every adjacent $(k+1) \times (k+1)$ sub-matrix of $\mathcal{M}$ has determinant 0. We denote by $\mathbb{SL}_k$ the set of all tame SL_k-tilings.

Remark 1.2.2. Condition (2) of Definition 1.1.1 is not always required in defining friezes. It is also called *tameness*. In this paper we only consider tame friezes and tilings, but *wild* tilings, those not satisfying this condition, have been studied by Cuntz [8].

Friezes correspond to a special type of periodic tilings described as follows.

Definition 1.2.3. [3, p. 266] Let F be an SL_k-frieze. Then we denote by $\mathcal{M}_F$ the tiling constructed from F using the following process. Let the rows of a frieze become a falling diagonal of a tiling by rotating the frieze 45° clockwise as in Figure 1.3. We may fill in the rest of the tiling through a skew extension in both directions as follows. Let $a_{i\,j}$ be an entry of the rotated frieze, then define $a_{i\,j+w+k+1} = (-1)^{k-1}a_{i\,j}$ where w is the width of the frieze.

$$\begin{array}{ccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 1 & b & c & 1 & 0 & -1 & -b & -c & \cdots \\
\cdots & -1 & 0 & 1 & d & e & 1 & 0 & -1 & -d & \cdots \\
\cdots & -g & -1 & 0 & 1 & f & g & 1 & 0 & -1 & \cdots \\
\cdots & -h & -i & -1 & 0 & 1 & h & i & 1 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Figure 1.3: Tiling resulting from rotating and extending the frieze in Figure 1.1.

Another important structure in our study of tilings is k-column vectors. Our goal is to construct a bijection between SL_k-tilings and pairs of sequences of vectors satisfying certain properties.

Definition 1.2.4. Let $\gamma = \{\gamma_i\}_{i\in\mathbb{Z}}$ be a bi-infinite strip of k-column vectors $\gamma_i \in \mathbb{Z}^k$ with the property that the matrix $(\gamma_i, \ldots, \gamma_{i+k-1})$, whose columns are k consecutive entries of γ, is an element of $SL_k(\mathbb{Z})$. We denote the set of all such strips $\mathcal{P}_k$ and we call γ a *path.*

The structure of these paths gives rise to additional information in how one set of adjacent vectors relates to the next one. We encode the information in terms of J matrices, which record how one column of γ relates to the preceding k columns.

Definition 1.2.5. We define the matrix $J_n^k \in SL_k(\mathbb{Z})$ for $n \in \mathbb{Z}$ as follows

$$J_n^k = \left[\begin{array}{c|c} 0\cdots 0 & (-1)^{k-1} \\ \hline & j_{n\,2} \\ I_{k-1} & \vdots \\ & j_{n\,k} \end{array}\right]$$

where I_{k-1} is the identity matrix of size $k-1$. We may also refer to the entry in the top right corner as $J_{n\,1}$ for consistency of notation. When clear, we omit the superscript k and write simply j_n. We use the term J *matrices* when referring to matrices of this form in general.

Observe that multiplying a matrix $A = (a_1, \ldots, a_k)$ on the right by a matrix J_n results in applying the following steps to A.

1. Delete the first column a_1 of A.
2. Shift all remaining columns to the left by one.
3. Append a new final column a_1' which is a linear combination of the columns of A, namely
$$a_1' = (-1)^{k-1}a_1 + j_{n\,2}a_2 + \cdots + j_{n\,k}a_k.$$

The result is the matrix

$$AJ_n = (a_2, \ldots, a_k, a_1').$$

Recall that $J_n \in SL_k(\mathbb{Z})$. Thus, if $A \in SL_k(\mathbb{Z})$, then so is AJ_n. In fact, for any matrix $A' \in SL_k(\mathbb{Z})$ where the first $k-1$ columns of A' are $a_2, \ldots, a_k$, there exists a unique J_n such that $AJ_n = A'$.

We later show that for any general $B \in SL_k(\mathbb{Z})$, there is a sequence of J matrices which allow us to transform A into B via right multiplication. Thus, these J matrices serve as transitions between k consecutive columns of a path. More precisely, to a path $\gamma \in \mathcal{P}_k$, we associate a collection of J matrices $\{J_i\}_{i\in\mathbb{Z}}$ called *transition matrices* such that

$$(\gamma_i, \dots, \gamma_{i+k-1})J_i = (\gamma_{i+1}, \dots, \gamma_{i+k}).$$

On the other hand, Bergeron and Reutenauer show that all rows (respectively columns) of a tiling $\mathcal{M} \in \mathbb{SL}_k$ may be written as a linear combination of the previous k rows (respectively columns) with a coefficient of $(-1)^{k-1}$ on the first [3, Lemma 2]. In particular, they state that

$$\text{Row}_{i+k} = (-1)^{k-1}\text{Row}_i + j_{i\,2}\text{Row}_{i+1} + \cdots + j_{i\,k}\text{Row}_{i+k-1}$$

$$\text{Col}_{i+k} = (-1)^{k-1}\text{Col}_i + j'_{i\,2}\text{Col}_{i+1} + \cdots + j'_{i\,k}\text{Col}_{i+k-1},$$

where Row_i (respectively Col_i) refers to the i-th row (respectively column) of $\mathcal{M}$. They refer to the other coefficients, $\{j_{i\,2}, j_{i\,3}, \dots, j_{i\,k}\}$ and $\{j'_{i\,2}, j'_{i\,3}, \dots, j'_{i\,k}\}$, as the linearization data, and these, in turn, correspond directly with the last column of the J matrix if we take the first coefficient to be the $(-1)^{k-1}$ term. They prove the following result.

Proposition 1.2.6. [3, Proposition 3] The mapping

$$\begin{aligned} \xi : \mathbb{SL}_k &\to SL_k(\mathbb{Z}) \times \left(\mathbb{Z}^{1\times(k-1)}\right)^{\mathbb{Z}} \times \left(\mathbb{Z}^{(k-1)\times 1}\right)^{\mathbb{Z}} \\ \mathcal{M} &\mapsto (M_{1\,1}, \lambda, \mu), \end{aligned}$$

which associates to a tame SL_k-tiling its linearization data λ and μ of the rows and columns respectively and a central matrix $M_{1\,1} = \mathcal{M}_{[k],[k]}$, is a bijection.

This demonstrates that J matrices also serve as transitions between rows and columns of SL_k-tilings. We recall a few additional definitions due to Bergeron and Reutenaur [3], starting with the notion of periodicity in tilings and paths.

Definition 1.2.7. Let $m \in \mathbb{Z}_{>0}$. We say a path γ is *p-periodic* if it has the property that $\gamma_i = \gamma_{i+p}$ for all $i \in \mathbb{Z}$. We denote by $(\mathcal{P}_k)_p$ the set of all p-periodic paths. Similarly, a sequence of J matrices $\{J_i\}_{i\in\mathbb{Z}}$ is *p-periodic* if they have the property that $J_i = J_{i+p}$ for all $i \in \mathbb{Z}$. We say a tiling $\mathcal{M}$ is *p-row periodic* (respectively *p-column periodic*) if $m_{i\,j} = m_{i+p\,j}$ (respectively $m_{i\,j} = m_{i,j+p}$) for all $i, j \in \mathbb{Z}$. We say that a tiling $\mathcal{M}$ is $(p \times q)$*-periodic* if

$$m_{i\,j} = m_{i+p\,j} = m_{i\,j+q} = m_{i+p\,j+q}$$

for all $i, j \in \mathbb{Z}$.

Bergeron and Reutenauer introduce an interesting operation on tilings called duality which we recall below. Later, we will present an alternative interpretation of this by applying our results.

Definition 1.2.8. [3, Equation 10] The *p-derived tiling* of a tiling $\mathcal{M}$, denoted $\partial_p\mathcal{M}$ is given by

$$\partial_p\mathcal{M} := \left(M_{i\,j}^{(p)}\right)_{i,j\in\mathbb{Z}}$$

where $M_{i\,j}^{(p)}$ is the adjacent $p \times p$ minor of $\mathcal{M}$ with upper-right corner $m_{i\,j}$. When $p = k-1$, we call $\partial_{k-1}\mathcal{M}$ the *dual* of $\mathcal{M}$ and we write $\mathcal{M}^*$.

The following proposition shows that the terminology of the dual is justified.

Proposition 1.2.9. [3, Proposition 6] The dual of a tame SL_k-tiling is a tame SL_k-tiling. Moreover, $(\mathcal{M}^*)^*$ and $\mathcal{M}$ coincide up to translation.

1.3 SL_2-tilings

SL_2-tilings were studied in detail by Short where he related them to the combinatorics of the Farey graph [17]. We recall the main results below.

Definition 1.3.1. A *Farey graph* is a graph with vertices in $\mathbb{Q} \cup \infty$. Two reduced rationals $\frac{a}{b}$ and $\frac{b}{c}$ are connected by an edge if $ad - bc = \pm 1$. In this case, we take $\infty = \frac{1}{0}$.

Short constructs a bijection between certain pairs of structures which he also calls paths, though his definition differs slightly.

Definition 1.3.2. [17] A *path of reduced rationals* is a bi-infinite sequence of reduced rationals $\gamma = \left\{\frac{a_i}{b_i}\right\}_{i\in\mathbb{Z}}$ which satisfy the property that

$$a_i b_{i+1} - a_{i+1} b_i = 1.$$

We denote the set of all such paths as $\mathcal{P}\mathbb{Q}$. Note that elements of $\mathcal{P}\mathbb{Q}$ are paths in the graph theoretic sense in the Farey graph.

We take a moment to draw an important distinction between $\mathcal{P}\mathbb{Q}$ and $\mathcal{P}_2$ as defined in Definition 1.2.4. Observe that for paths $\gamma \in \mathcal{P}_2$, there is a difference between columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} -a \\ -b \end{pmatrix}$ of γ which is lacking in the Farey graph. Both of these represent the same reduced rational $\frac{a}{b} = \frac{-a}{-b}$. Note that while replacing one with the other changes the sign on the determinant of the two consecutive columns of γ, and thus matters in the definition of path $\gamma \in \mathcal{P}_2$, the sign does not matter when thought of as a path in the Farey graph. Hence, Short must identify $\mathcal{M}$ and $-\mathcal{M}$, the tiling obtained by negating all entries of $\mathcal{M}$.

Short constructs a bijection between $SL_2(\mathbb{Z})$-tilings and paths in $\mathcal{P}\mathbb{Q}$. In aid of this, he defines the following map.

The notation $(\mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q})/SL_2(\mathbb{Z})$ denotes the set of conjugacy classes consisting of pairs of paths γ, δ up to multiplying all columns in γ and δ by the same matrix $A \in SL_k(\mathbb{Z})$.

Definition 1.3.3. [17] Let $\gamma = \left\{ \frac{a_i}{b_i} \right\}_{i \in \mathbb{Z}}$ and $\delta = \left\{ \frac{c_j}{d_j} \right\}_{j \in \mathbb{Z}}$ be paths in $\mathcal{P}\mathbb{Q}$. Define the map $\Phi\mathbb{Q}$ as follows.

$$\begin{aligned} \Phi\mathbb{Q} : (\mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q})/SL_2(\mathbb{Z}) &\to \mathbb{SL}_2/\pm \\ (\gamma, \delta) &\mapsto (m_{i\,j})_{i,j \in \mathbb{Z}} \end{aligned}$$

where $\mathbb{SL}_2/\pm$ is the set of all tilings where we identify $\mathcal{M} = -\mathcal{M}$, and $m_{i\,j} = a_i d_j - b_i c_j$.

Theorem 1.3.4. [17, Theorem 1.1] The map $\Phi\mathbb{Q}$ in Definition 1.3.3 is a bijection between pairs of paths modulo $SL_2(\mathbb{Z})$ and the set of all SL_2-tilings up to a global change of sign $\mathbb{SL}_2/\pm$.

In this paper, we obtain a generalization of Short's result for all $k \geq 2$. Moreover, Short's use of the geometry of the Farey graph allows him to establish further bijections by placing certain restrictions on paths. While Short's main result and several others are extended here, it is not clear how to generalize all of them as we lack the connection to geometry.

Chapter 2 The Bijection

In this chapter we define a map Φ from pairs of paths to SL_k-tilings and then prove that it is a bijection.

2.1 Defining Φ

We first introduce a map $\widetilde{\Phi}$. We will later show that this map is well-defined.

Definition 2.1.1. We define a map $\widetilde{\Phi}$ from a pair of paths to a tiling as follows. Fix a pair $(\gamma, \delta) \in \mathcal{P}_k \times \mathcal{P}_k$.

$$\begin{aligned} \widetilde{\Phi} : (\mathcal{P}_k \times \mathcal{P}_k) &\to \mathbb{SL}_k \\ (\gamma, \delta) &\mapsto (m_{i\,j})_{i,j \in \mathbb{Z}}, \end{aligned}$$

where

$$m_{i\,j} = \det\left(\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+k-2}, \delta_j\right).$$

Thus, the entries $m_{i\,j}$ of the tiling come from taking the determinant of $k-2$ consecutive columns of γ and a single column of δ.

Example 2.1.2. Suppose we have a pair of paths $\gamma, \delta \in \mathbb{Z}^3$ defined as follows:

$$\gamma = (\cdots, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \cdots) = \begin{pmatrix} & 1 & 0 & 0 & 1 & \\ \cdots & 0 & 1 & 0 & 5 & \cdots \\ & 0 & 0 & 1 & 2 & \end{pmatrix}$$

and

$$\delta = (\cdots, \delta_1, \delta_2, \delta_3, \cdots) = \begin{pmatrix} & 1 & 1 & 1 & \\ \cdots & 1 & 2 & 3 & \cdots \\ & 1 & 3 & 6 & \end{pmatrix}.$$

To derive a specific entry of the tiling $\tilde{\Phi}$, say $m_{1\,2}$, we take two entries of γ starting at γ_1 and append δ_2. The resulting determinant gives the entry:

$$m_{1\,2} = \det(\gamma_1, \gamma_2, \delta_2) = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = 3.$$

We may assemble the whole tiling in this fashion. For example,

$$M = \begin{array}{ccccc} \ddots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 3 & 6 & \cdots \\ \cdots & 1 & 1 & 1 & \cdots \\ \cdots & -4 & -3 & -2 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{array}.$$

Note that it is not clear from the definition that the array $\widetilde{\Phi}(\gamma, \delta)$ satisfies the SL_k-property. We will show this later. For now, we need some preliminary results. Let A be a $k \times k$ matrix. We define the multiplication of A by $\gamma \in \mathcal{P}_k$, denoted by $A\gamma$, by

$$A\gamma = (\cdots, A\gamma_1, A\gamma_2, \ldots).$$

Lemma 2.1.3. The map $\widetilde{\Phi}$ is invariant under multiplication by $A \in SL_k(\mathbb{Z})$, that is $\widetilde{\Phi}(\gamma, \delta) = \widetilde{\Phi}(A\gamma, A\delta)$.

Proof. Fix $(\gamma, \delta) \in (\mathcal{P}_k \times \mathcal{P}_k)$. Let $m_{i\,j}$ be the entry of the image $\mathcal{M}$ of $\widetilde{\Phi}(\gamma, \delta)$. Then $\widetilde{\Phi}(A\gamma, A\delta)$ has image $m'_{i\,j}$ where

$$\begin{aligned} m'_{i\,j} &= \det(A\gamma_i, \ldots, A\gamma_{i+k-2}, A\delta_j) \\ &= \det(A(\gamma_i, \gamma_{i+1}, \cdots, \gamma_{i+k-2}, \delta_j)) \\ &= \det(A) m_{i\,j} \\ &= m_{i\,j}. \end{aligned}$$

$\square$

Let $\pi : (\mathcal{P}_k \times \mathcal{P}_k) \to (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$ be the quotient map. By Lemma 2.1.3, $\widetilde{\Phi}(A\gamma, A\delta) = \widetilde{\Phi}(\gamma, \delta)$, so $\widetilde{\Phi}$ induces a map Φ from $(\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$ to $\mathbb{SL}_k$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{P}_k \times \mathcal{P}_k & \xrightarrow{\widetilde{\Phi}} & \mathbb{SL}_k \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \Phi} & \\ (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) & & \end{array}$$

We will focus on the map Φ and prove that it is a bijection. First, we make an observation about J matrices.

Lemma 2.1.4. The J matrices generate $SL_k(\mathbb{Z})$.

Proof. It is a classical result in linear algebra that shear matrices generate the special linear group, see for example [1]. Thus, it suffices to show that J matrices generate shear matrices $S_{i\,j}(\lambda)$ which consist of the identity matrix with a single nonzero entry λ in position $(i\,j)$ where $i \neq j$.

Fix a shear matrix $S_{i\,j}(\lambda)$. Recall the structure of a J-matrix from Definition 1.2.5. Consider $(J_n)^{j-1}$ where J_n is the J matrix with $j_{n\,\ell} = 0$ for all $\ell \in \{2, \ldots, k\}$. This matrix $(J_n)^{j-1}$ has columns

$$(e_j, \ldots, e_k, (-1)^{k-1}e_1, \ldots, (-1)^{k-1}e_{j-1}).$$

Then, multiply by the J matrix J_m with a single nontrivial j-entry in the last column where

$$j_{m,i'} = \begin{cases} \lambda & \text{if } i > j \\ (-1)^{k-1}\lambda & \text{if } i < j \end{cases}$$

where $i' \equiv (i - j + 1) \pmod{k}$ and $j_{m\,\ell} = 0$ for $\ell \neq i'$. The resulting matrix $J_n^{j-1}J_m$ is of the form

$$J_n^{j-1}J_m = \left(e_{j+1}, \ldots, e_k, (-1)^{k-1}e_1, \ldots, (-1)^{k-1}e_{j-1}, (-1)^{k-1}(\lambda e_i + e_j)\right).$$

Then multiply by $(J_n)^{k-j}$ to obtain $(-1)^{k-1}S_{i\,j}(\lambda)$. Finally, to get the desired $S_{i\,j}(\lambda)$, multiply by $(J_n)^k$. □

From this, we develop the following technique. Consider $(\gamma, \delta) \in (\mathcal{P}_k \times \mathcal{P}_k)$. We construct a periodic path with m consecutive entries from γ and n consecutive entries from δ with $m, n \geq k$. Without loss of generality, we may select a labeling such that the desired elements of γ and δ are $\{\gamma_1, \ldots, \gamma_m\}$ and $\{\delta_1, \ldots, \delta_n\}$ respectively. By Lemma 2.1.4, there is a sequence of J matrices $\{J_i\}_{i \in [\ell+k]}$ for some $\ell \in \mathbb{Z}$ such that

$$(\gamma_{m-k+1}, \ldots, \gamma_m)J_1 \cdots J_{\ell+k} = (\delta_1, \ldots, \delta_k).$$

Let λ_i with $i \in [\ell]$ be the last column of the product

$$(\gamma_{m-k+1}, \ldots, \gamma_m)J_1 \cdots J_i.$$

Similarly, there is a sequence of J matrices $\{J'_i\}_{i \in [q+k]}$ for some $q \in \mathbb{Z}$ such that

$$(\delta_{n-k+1}, \ldots, \delta_n)J'_1 \cdots J'_{q+k} = (-1)^{k-1}(\gamma_1, \ldots, \gamma_k).$$

Let μ_i with $i \in [q]$ be the last column of the product

$$(\delta_{n-k+1}, \dots, \delta_n) J'_1 \cdots J'_i.$$

We obtain a matrix A with columns

$$A = (\gamma_1, \dots, \gamma_m, \lambda_1, \dots, \lambda_\ell, \delta_1, \dots, \delta_n, \mu_1, \dots, \mu_q). \tag{2.1}$$

Note that, by construction, A satisfies the properties of Theorem 1.1.6, that is all consecutive Plücker coordinates of A are 1.

Definition 2.1.5. Fix $m, n \geq k$. We define a map

$$\varphi_{m,n} : (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) \to \mathcal{P}_k/SL_k(\mathbb{Z})$$

as follows. Given $(\gamma, \delta) \in (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$, let A be constructed as in Equation (2.1). Then we define a new path $\varphi_{m,n}(\gamma, \delta)$ by extending A periodically as follows.

$$\left(\varphi_{m,n}(\gamma, \delta)\right)_i = (-1)^{r(k-1)} A_{i'}$$

where $i' = i + r \cdot (m + \ell + n + q)$ When obvious, we will omit the subscripts m, n and write $\varphi(\gamma, \delta)$.

We provide an example of the construction of the path $\varphi(\gamma, \delta)$ below.

Example 2.1.6. Suppose we have a pair of paths $\gamma, \delta \in \mathbb{Z}^2$ defined as follows

$$\gamma = \begin{pmatrix} \cdots & 0 & -1 & -2 & -1 & \cdots \\ & 1 & 1 & 1 & 0 & \cdots \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} \cdots & -4 & 1 & 2 & \cdots \\ & 3 & -1 & -1 & \cdots \end{pmatrix}.$$

We wish to construct a path $\varphi_{3,2}(\gamma, \delta)$. We start with the first three columns of γ

$$\begin{pmatrix} 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Our goal is to use this starting place to create a path which reaches the first two columns of δ, i.e.

$$\begin{pmatrix} 0 & -1 & -2 & \cdots & -4 & 1 \\ 1 & 1 & 1 & \cdots & 3 & -1 \end{pmatrix}.$$

We may do this by constructing an appropriate sequence of J matrices which generate

$$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -4 & 1 \\ 3 & -1 \end{pmatrix}.$$

In this case, we obtain a single vector $\lambda = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. We do the same in order to return back to the initial entry of γ scaled by -1. In this case, the sequence of vectors μ is $\mu = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. The resulting path has the form

$$\begin{aligned}\varphi_{3,2}(\gamma,\delta) &= (\cdots, \gamma_1, \gamma_2, \gamma_3, \lambda_1, \delta_1, \delta_2, \mu_1, -\gamma_1, -\gamma_2, -\gamma_3, \cdots) \\ &= \begin{pmatrix} \cdots & 0 & -1 & -2 & 3 & -4 & 1 & -1 & 0 & 1 & 2 & \cdots \\ & 1 & 1 & 1 & -2 & 3 & -1 & 2 & -1 & -1 & -1 & \end{pmatrix}.\end{aligned}$$

From this partial data about γ and δ, we will be able to extract certain partial information about their image $\mathcal{M} = \Phi(\gamma, \delta)$ and then expand this to a variety of conclusions about the whole image $\mathcal{M}$.

Lemma 2.1.7. Let $\mathcal{M} := \Phi(\gamma, \delta)$ be an array and let $M = \mathcal{M}_{\{1,\ldots,m\},\{1,\ldots,n\}}$ be an adjacent $m \times n$ sub-matrix in $\mathcal{M}$ with $m, n \geq k$. Then there exists a tame SL_k-tiling $\mathcal{M}'$ from a frieze which has the property that the adjacent $m \times n$ sub-matrix

$$M' = \mathcal{M}'_{\{i,\ldots,i+m-1\},\{j,\ldots,j+n-1\}}$$

is equal to M for some i and j.

Proof. For paths γ and δ and $m+k-2, n \geq k$, use Equation (2.1) to construct a matrix A and a path $\varphi(\gamma, \delta)$ as in Definition 2.1.5. By Theorem 1.1.6, $\mathcal{F}_{(k,n+\ell+m+q)}(A)$ is a tame SL_k-frieze. We may extend this frieze to a tame SL_k-tiling $\mathcal{M}' = \mathcal{M}_{\mathcal{F}_{(k,n+\ell+m+q)}(A)}$ as in Definition 1.2.3. Let M' be a submatrix of $\mathcal{M}'$ of size $m \times n$ with upper-left entry $p_{1,\ldots,k-1,m+\ell+1}(A)$. By construction, this is given by the determinant of the matrix $(\gamma_1, \cdots, \gamma_{k-1}, \delta_1)$, and is therefore equal to $m_{1,1}$ in $\mathcal{M}$. The same holds for all other entries of M'. $\square$

This finally allows us to show that Φ, and hence $\widetilde{\Phi}$, are well-defined.

Proposition 2.1.8. The map $\Phi : (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) \to \mathbb{SL}_k$ is well-defined.

Proof. Let $(\gamma, \delta) \in (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$ and $\mathcal{M} = \Phi(\gamma, \delta)$. We wish to show that $\mathcal{M} \in \mathbb{SL}_k$. By Lemma 2.1.7, we can map $M = \mathcal{M}_{\{1,\ldots,m\},\{1,\ldots,n\}}$ to a portion of a tame SL_k-tiling $\mathcal{M}'$. Observe that in Equation (2.1), our choice of indexing for γ and δ was arbitrary. Therefore, we may adjust our labels on γ and δ such that the upper left corner of M may be any entry in $\mathcal{M}$. Therefore, since $\mathcal{M}' \in \mathbb{SL}_k$, we have that, in particular, every adjacent $k \times k$ sub-matrix is in $SL_k(\mathbb{Z})$ and every adjacent $(k+1) \times (k+1)$ adjacent sub-matrix has determinant 0. Thus, $\mathcal{M}$ is in $\mathbb{SL}_k$. $\square$

2.2 Transition Matrices

Next, we present a series of results which will allow us to compare the transition matrices of tilings and paths. First, we give a notation for the transition matrices in a tiling.

Definition 2.2.1. Given a tiling $\mathcal{M}$ we write H_i for the horizontal transition matrix of $\mathcal{M}$ which transitions from column $i+k-1$ to $i+k$, i.e.

$$M_{i\,j}H_j = M_{i\,j+1}.$$

Similarly, we write V_i for the vertical transition matrix which transitions from row $i+k-1$ to $i+k$, i.e.

$$M^T_{i\,j}V_i = M^T_{i+1\,j}.$$

With this notation in mind, we make the following observation. Let $\mathcal{M} = \mathcal{M}_F$ be a tiling from a frieze F with entries in some integral domain. We may index $\mathcal{M}$ such that $M_{0\,0} := M$ is of the following form.

$$M = \begin{pmatrix} 1 & m_{1\,1} & m_{1\,2} & \cdots & m_{1\,(k-1)} \\ 0 & 1 & m_{2\,2} & \cdots & m_{2\,(k-1)} \\ & \ddots & \ddots & \cdots & \vdots \\ & & 0 & 1 & m_{(k-1)\,(k-1)} \\ \mathbf{0} & & & 0 & 1 \end{pmatrix}.$$

In order to transition from M to the k-th column of $\mathcal{M}$, we use a horizontal transition matrix which takes the form of a J matrix. In particular,

$$M \begin{pmatrix} (-1)^{k-1} \\ j_2 \\ \vdots \\ j_{k-p+1} \\ \vdots \\ j_k \end{pmatrix} = \begin{pmatrix} m_{1\,k} \\ m_{2\,k} \\ \vdots \\ m_{k-p+1\,k} \\ \vdots \\ m_{k\,k} \end{pmatrix}, \tag{2.2}$$

for $p \in [k]$ where $((-1)^{k-1}, j_2, \ldots, j_{k-p+1}, \ldots, j_k)^T$ is the final column of the horizontal transition matrix. This gives rise to the following lemma.

Lemma 2.2.2. Let $\mathcal{M} = \mathcal{M}_F$ be a tiling from a frieze F and let $((-1)^{k-1}, j_2, \cdots, j_{k-p+1}, \cdots, j_k)^T$ be the final column of the horizontal transition matrix H_0. Then $j_{k-p+1} = (-1)^{p-1}M_p$ where M_p is the p-th adjacent minor of $\mathcal{M}_{[k],[k]}$ whose bottom right corner is $m_{k\,k}$.

Proof. We proceed by strong induction. Take as a base case $p = 1$. By Equation (2.2), clearly $j_k = m_{k\,k}$. Additionally, the 1 minor is $|m_{k\,k}| = m_{k\,k}$. Thus, $j_k = (-1)^{1-1} m_{k\,k} = m_{k\,k}$, as desired, and the base case holds.

Suppose the statement holds for j_{k-i+1} where $i \in [p-1]$ with $p \leq k$. Consider the case j_{k-p+1}. By Equation (2.2), we see that

$$j_{k-p+1} = m_{(k-p+1)\,k} - \sum_{i=1}^{p-1} j_{k-i+1} m_{(k-p+1)\,(k-i)}. \tag{2.3}$$

The minor M_p has the form

$$M_p = \begin{vmatrix} m_{(k-p+1)\,(k-p+1)} & m_{(k-p+1)\,(k-p+2)} & & \cdots & m_{(k-p+1)\,k} \\ 1 & m_{(k-p+2)\,(k-p+2)} & & \cdots & m_{(k-p+2)\,k} \\ & \ddots & & \cdots & \vdots \\ & & 1 & m_{(k-1)\,(k-1)} & m_{(k-1)\,k} \\ \mathbf{0} & & & 1 & m_{k\,k} \end{vmatrix}.$$

We compute the determinant by going across the top row. Denote by r_q in the determinant of the $(p-1)$-st minor associated with the term $m_{(k-p+1)\,(k-p+q)}$ where $q < p$ from the top row. This minor r_p is of the form

$$r_q = \left|\begin{array}{ccc|ccc} 1 & & * & & & \\ & \ddots & & & * & \\ \mathbf{0} & & 1 & & & \\ \hline & & & m_{(k-p+q+1)\,(k-p+q+1)} & \cdots & m_{(k-p+q+1)\,k} \\ & \mathbf{0} & & 1 & \cdots & m_{(k-p+q+2)\,k} \\ & & & & \ddots & \vdots \\ & & & & 1 & m_{k\,k} \end{array}\right|.$$

This determinant is given entirely by the determinant of the lower-right block of size $p - q$, which, by the inductive hypothesis, is $(-1)^{p-q-1} j_{k-p+1+q}$. Thus, each of the terms in the expression for M_p are given by

$$(-1)^{q-1} m_{(k-p+1)\,(k-p+q)} r_q = (-1)^{q-1} m_{(k-p+1)\,(k-p+q)} (-1)^{p-q-1} j_{k-p+1+q} = (-1)^p j_{k-p+1+q} m_{(k-p+1)\,(k-p+q)}.$$

When $q = p$, r_p is given by $(-1)^{p-1} m_{(k-p+1)\,k}$ times the identity matrix. Thus, we have that the minor M_p is given by

$$M_p = (-1)^{p-1} m_{(k-p+1)\,k} + (-1)^p \sum_{i=1}^{p-1} j_{k-i+1} m_{(k-p+1)\,(k-i)}.$$

Multiplying Equation (2.3) by $(-1)^{p-1}$ gives us that

$$(-1)^{p-1}j_{k-p+1} = (-1)^{p-1}m_{(k-p+1)\,k} + (-1)^p \sum_{i=1}^{p-1} j_{k-i+1}m_{(k-p+1)\,(k-i)} = M_p,$$

as desired. □

We generalize this result to give us the entries of all J matrices in a Plücker frieze.

Proposition 2.2.3. Let $\mathcal{M} = \mathcal{M}_{\mathcal{F}(k,n)}$ such that $m_{1\,1} = p_{o([1]^{k-1}[1])}$.

(a) The entry $j_{p\,q+1}$ of H_p is given by

$$j_{p\,q+1} = (-1)^{k-q-1}p_{o([p]^q,[p+q+1]^{k-q})}.$$

(b) The entry $j_{p\,q+1}$ of V_p is given by

$$j_{p\,q+1} = (-1)^{k-q-1}p_{o([p+q-1]^{k-q},[p+k]^q)}.$$

Proof. We begin by proving part (a). The horizontal transition matrix H_{p+k-1} is a transition between the adjacent submatrix

$$A_{[p+k-1]^k;p} = \begin{pmatrix} p_{o([p]^{k-1},p+k-1)} & p_{o([p]^{k-1},p+k)} & \cdots & p_{o([p]^{k-1},p+2k-2)} \\ p_{o([p+1]^{k-1},p+k-1)} & p_{o([p+1]^{k-1},p+k)} & \cdots & p_{o([p+1]^{k-1},p+2k-2)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{o([p+k-1]^{k-1},p+k-1)} & p_{o([p+k-1]^{k-1},p+k)} & \cdots & p_{o([p+k-1]^{k-1},p+2k-2)} \end{pmatrix}$$

as in Definition 1.1.8 and the next column of $\mathcal{M}$. By Lemma 2.2.2, this means $j_{p+k-1\,q+1}$ is equal to $(-1)^{k-q-1}$ times the adjacent $(k-q)$-minor of $A_{[p+k]^k;p}$ aligned with the bottom-right corner. This corresponds to the matrix $A_{[p+k+q]^{k-q};p+q}$. By Proposition 1.1.9, we have that

$$\det(A_{[p+k+q]^{k-q};p+q}) = p_{o([(p+q)+(k-q)-1]^{k-(k-q)},[p+k+q]^{k-q})} = p_{o([p+k-1]^q,[p+k+q]^{k-q})}.$$

Thus, we have

$$j_{k+p-1\,q+1} = (-1)^{k-q-1}p_{o([p+k-1]^q,[p+k+q]^{k-q})}.$$

Setting p to $p-k+1$, we obtain the desired result.

Part (b) follows similarly by using M^T. The corresponding matrices are now of the form $A_{[p+q-2]^{k-q};p+q}$. □

We show an example of how the formulas in parts (a) and (b) of Proposition 2.2.3 are related.

Example 2.2.4. We may first use the formula in Proposition 2.2.3 part (a) to construct the sequence of the final column of the J matrices of a Plücker frieze, hence the horizontal transition matrices of the corresponding tiling. For example, if we take $k = 4$, we have the following sequence starting with $p = 1$:

$$\left(\dots, \begin{pmatrix} -1 \\ p_{1345} \\ -p_{1245} \\ p_{1235} \end{pmatrix}, \begin{pmatrix} -1 \\ p_{2456} \\ -p_{2356} \\ p_{2346} \end{pmatrix}, \begin{pmatrix} -1 \\ p_{3567} \\ -p_{3467} \\ p_{3457} \end{pmatrix}, \begin{pmatrix} -1 \\ p_{4678} \\ -p_{4578} \\ p_{4568} \end{pmatrix}, \dots \right).$$

We may similarly use the formula from Proposition 2.2.3 part (b) to construct the sequence for vertical transition matrices starting with $p = 1$:

$$\left(\dots, \begin{pmatrix} -1 \\ p_{1235} \\ -p_{2356} \\ p_{3567} \end{pmatrix}, \begin{pmatrix} -1 \\ p_{2346} \\ -p_{3467} \\ p_{4678} \end{pmatrix}, \begin{pmatrix} -1 \\ p_{3457} \\ -p_{4578} \\ p_{5789} \end{pmatrix}, \begin{pmatrix} -1 \\ p_{4568} \\ -p_{5689} \\ p_{68910} \end{pmatrix}, \dots \right).$$

Note that column entries along the ascending diagonals of one sequence form the columns of the other.

With this example as guidance, we define an operation and notation on paths which alters their J matrices.

Definition 2.2.5. Given a path $\gamma \in SL_k(\mathbb{Z})$ with J matrices $\{J_i\}_{i\in\mathbb{Z}}$, define a new path $\widetilde{\gamma} \in SL_k(\mathbb{Z})$ such that $\widetilde{\gamma}_{[k]} = (\gamma_{[k]})^T$ and the J matrices of $\widetilde{\gamma}$ written $\{\widetilde{J_i}\}_{i\in\mathbb{Z}}$ have entries in the final column given by

$$\widetilde{j_{i\,q}} = \begin{cases} j_{i\,1} = (-1)^{k-1} & q = 1 \\ (-1)^k j_{i+q-2\,k-q+2} & \text{else} \end{cases}$$

for $i \in \mathbb{Z}$ and $q \in [k]$.

Remark 2.2.6. Note that $\widetilde{(\widetilde{J_i})}$ from the J matrices of $\widetilde{\widetilde{\gamma}}$ equals J_i from the J matrices of γ up to a shift in indexing by $k-2$, i.e.

$$\widetilde{(\widetilde{J_i})}_i = J_{i+k-2}$$

for all $i \in \mathbb{Z}$.

Next, we obtain results about the transition matrices of tilings.

Lemma 2.2.7. The horizontal transition matrices of the tiling $\mathcal{M} := \Phi(\gamma, \delta)$ are equal to the J matrices of δ.

Proof. Let $\gamma, \delta \in (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$. Without loss of generality, we may select our γ such that $(\gamma_1, \gamma_2, \ldots, \gamma_k) = I_k$. We want to show that the horizontal transition matrices of $\mathcal{M}$ match up with the J matrices of δ. It suffices to show that $M_{1\,1}J_1 = M_{1\,2}$ where J_1 is the corresponding transition matrix of δ, i.e. $J_1 = H_1$. In particular, we want to show that

$$\sum_{i=1}^{k} j_{1\,i} m_{\ell\,i} = m_{\ell\,k+1}$$

for all $\ell \in [k]$ where $j_{1\,i}$ is the i-th entry in the last column of J_1. For the case of the first row, we note that

$$m_{1\,i} = \det(\gamma_1, \cdots, \gamma_{n-1}, \delta_i) = \delta_{i\,k},$$

hence the conclusion follows from the definition of J_1. For the remaining rows, observe that the entry $m_{\ell+1\,i}$ is given by the determinant of the matrix

$$\left(\begin{array}{c|c|c} \mathbf{0} & P & \begin{matrix}\delta_{i\,1}\\ \delta_{i\,2}\\ \vdots \\ \delta_{i\,\ell}\end{matrix} \\ \hline I_{k-\ell} & \multicolumn{2}{c}{*} \end{array}\right),$$

where the $\ell \times (\ell-1)$ matrix P consists of the upper ℓ entries of the matrix $(\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_{k+\ell-1})$. We may write this determinant as

$$m_{\ell+1\,i} = (-1)^{k-\ell} \sum_{n=1}^{\ell} a_n \delta_{i\,n}, \tag{2.4}$$

where a_n is some fixed value based on P used in taking the determinant of the upper right block. Note that P, and hence a_n, does not depend on i. Thus, we have

$$\begin{aligned}\sum_{i=1}^{k} j_{1\,i} m_{\ell+1\,i} &= (-1)^{k-\ell} \sum_{i=1}^{k} j_{1\,i} \sum_{n=1}^{\ell} a_n \delta_{i\,n} \\ &= (-1)^{k-\ell} \sum_{n=1}^{\ell} a_n \sum_{i=1}^{k} j_{1\,i} \delta_{i\,n}.\end{aligned}$$

By definition of J_1, we may rewrite the final sum as $\delta_{k+1\,n}$. Thus, we have

$$\sum_{i=1}^{k} j_{1\,i} m_{\ell+1\,i} = (-1)^{k-\ell} \sum_{n=1}^{\ell} a_n \delta_{k+1\,n} = m_{\ell+1\,k+1},$$

for all $\ell \in [k-1]$ where the last equality follows from Equation (2.4). □

Lemma 2.2.8. The vertical transition matrices of $\Phi(\gamma, \delta)$ correspond to the J matrices of $\widetilde{\gamma}$ as in Definition 2.2.5.

Proof. Let $\mathcal{M} := \Phi(\gamma, \delta)$. Recall that by Lemma 2.1.7, we may construct an SL_k-tiling $\mathcal{M}'$ from a single path $\varphi(\gamma, \delta)$ with contains an adjacent $m \times n$ matrix $\mathcal{M}'_{\{1,\ldots,m\},\{i,\ldots,i+n-1\}}$ equal to $\mathcal{M}_{[1]^m,[1]^n}$ where $m, n > k$. By construction, the path $\varphi(\gamma, \delta)$ has as its first J matrix J_1 the first J matrix of γ. By Lemma 2.2.7, this means that the horizontal transition matrices starting at $\mathcal{M}'_{[k],[k]} = M'_{1\,1}$ are the J matrices of γ. Note that the matrix $\mathcal{M}'_{[k],[k]}$ need not lie in $\mathcal{M}$. Since $\mathcal{M}'$ is a tiling from a frieze, by Proposition 2.2.3 part (b), the vertical transition matrix V'_1 has entries given by

$$j_{1\,q+1} = (-1)^{k-q-1} p_{o([q]^{k-q},[k+1]^q)}$$

where the Plücker coordinate is applied to A where A is one period of $\varphi(\gamma, \delta)$. Consider the entries of the matrix $\widetilde{J_1}$ of $\widetilde{\gamma}$. By Defintion 2.2.5 and and Proposition 2.2.3, these correspond to the entries of the final column of the J matrix given by

$$\widetilde{j_{1\,q+1}} = (-1)^k j_{q\,k-q+1} = (-1)^k j_{q\,k-q+1} = (-1)^{k-q-1} p_{o([q]^{k-q}[k+1]^q}, $$

applied to A. Thus, by taking m, n large enough, the vertical transition matrix V'_1 equals the first J matrix $\widetilde{J_1}$ of $\widetilde{\gamma}$. Now, we need to show that V_1, the vertical transition matrix of $\mathcal{M}$, equals V'_1. This follows by construction, since $M_{1\,1} = M'_{1\,1}$ and $M_{2\,1} = M'_{2\,1}$. Thus, $\widetilde{J_1} = V_1$, as desired. Since our choice of starting position for $\mathcal{M}'$ was arbitrary, this holds for all vertical transition matrices and all J matrices of γ. □

We combine the above lemmas to obtain the following crucial result.

Corollary 2.2.9. Let $\mathcal{M} = \Phi(\gamma, \delta)$ be an SL_k-tiling. The horizontal transition matrices of $\mathcal{M}$ correspond to the J matrices of δ and the vertical transition matrices of $\mathcal{M}$ correspond to the J matrices of $\widetilde{\gamma}$

Proof. This result follows from Lemmas 2.2.7 and 2.2.8. □

2.3 Defining Φ^{-1}

In order to demonstrate that Φ is a bijection, we construct a map Ψ from tilings to paths. To begin with, we construct a matrix C which captures a specific transformation of a path $\delta \in \mathcal{P}_k$.

Lemma 2.3.1. Consider the tiling $\mathcal{M} := \Phi(\gamma, \delta)$. Without loss of generality, we may assume $(\gamma_1, \gamma_2, \ldots, \gamma_k) = I_k$. Then for all $j \in \mathbb{Z}$

$$\begin{pmatrix} m_{1,j} \\ m_{2,j} \\ \vdots \\ m_{k,j} \end{pmatrix} = C\delta_j$$

where $C \in SL_k(\mathbb{Z})$ has the following form:

$$C = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline (-1)^{k-1} & & \mathbf{0} & 0 \\ & \ddots & & \vdots \\ * & & (-1)^{k-1} & 0 \end{array}\right].$$

In particular, the entries on the lower-left of the matrix C are determined uniquely by the J matrices of γ.

Proof. We construct the image of $\Phi(\gamma, \delta)$. Let J_n be the transition matrices for γ. Observe that, by construction, since $\gamma_i = e_i$ for all $i \in [k]$, we have

$$m_{1\,j} = \det(e_1, \ldots, e_{k-1}, \delta_j) = \det \left[\begin{array}{c|c} & \delta_{1\,j} \\ I_{k-1} & \vdots \\ & \delta_{k-1\,j} \\ \hline 0 \cdots 0 & \delta_{k\,j} \end{array}\right] = \delta_{k\,j} = C\delta_j,$$

for C as in the statement of the lemma. For $2 \le i \le k$, we have that

$$m_{i\,j} = \det \left[\begin{array}{c|c} \mathbf{0} & P \\ \hline I_{k-i+1} & * \end{array}\right] = \left((-1)^{i-1}\right)^{k-i+1} \det(P)$$

where P is the upper right $(i-1) \times (i-2)$ matrix of $\prod_{n=1}^{i-2} J_n$ appended with the upper $(i-1)$ entries of δ_j. Note that the determinant of P is a linear combination of the entries of the last column, which is the upper $i-1$ entries of δ_j. We focus on the coefficient of $\delta_{i-1\,j}$, which is given by the determinant of the upper left $(i-2)\times(i-2)$ matrix of P, call it Q.

Note that the product $\prod_{n=1}^{i-2} J_n$ may be written as

$$\prod_{n=1}^{i-2} J_n = (\gamma_{i-1}, \ldots, \gamma_{k+i-2}) = \left[\begin{array}{c|c} \mathbf{0} & Q \\ \hline I_{k-i+2} & * \end{array}\right].$$

Since each $J_n \in SL_k(\mathbb{Z})$, the determinant of this product is one. We may also calculate it in terms of Q as

$$1 = \det\left(\prod_{n=1}^{i-2} J_n\right) = \left((-1)^{i-2}\right)^{k-i+2} \det(Q).$$

This gives us that $\det(Q) = \left((-1)^{i}\right)^{k-i}$.

By definition of $\Phi(\gamma,\delta)$, the coordinate $m_{i\,j}$ is given by

$$m_{i\,j} = \det\left(\begin{array}{c|c|c} & Q & \begin{matrix}\delta_{1\,j}\\ \vdots \\ \delta_{i-2\,j}\end{matrix} \\ \cline{2-3} \mathbf{0} & ** & \delta_{i-i\,j} \\ \hline I_{k-i+1} & \multicolumn{2}{c}{*} \end{array}\right)$$

Hence, the coefficient of $\delta_{i-1\,j}$ in $m_{i\,j}$ is

$$\left((-1)^{i-1}\right)^{k-i+1} \det(Q) = \left((-1)^{i-1}\right)^{k-i+1} \left((-1)^{i}\right)^{k-i} = (-1)^{k-1}.$$

Thus, we have the desired equation

$$\begin{pmatrix} m_{1,j} \\ m_{2,j} \\ \vdots \\ m_{k,j} \end{pmatrix} = C\delta_j.$$

□

We use this to construct a map Ψ.

Definition 2.3.2. We define a map Ψ from a tiling to a pair of paths modulo $SL_k(\mathbb{Z})$. Let $\mathcal{M} \in \mathbb{SL}_k$ be a tiling. Then

$$\Psi : \mathbb{SL}_k \to (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$$
$$\mathcal{M} \to (\gamma, \delta),$$

where γ and δ are defined as follows. We define the path δ to be the horizontal strip from $\mathcal{M}$ that is k entries tall centered at $\mathcal{M}_{[1]^k[1]^k} := M$, which is to say that

$$\delta_i = \begin{pmatrix} m_{1\,i} \\ m_{2\,i} \\ \vdots \\ m_{k\,i} \end{pmatrix}.$$

To define γ we construct γ_i for $i \in [k]$ and its J matrices. In order to construct the J matrices, let γ' be a vertical strip of $\mathcal{M}$ with width k. That is,

$$\gamma'_i = \begin{pmatrix} m_{i\,1} \\ m_{i\,2} \\ \cdots \\ m_{i\,k} \end{pmatrix}.$$

Then the J matrices J_i of γ are given by the J matrices $\widetilde{J_{i-k+2}}$ of $\widetilde{\gamma'}$. For the entries $\gamma_{[k]}$, we set $(\gamma_1, \gamma_2, \ldots, \gamma_k) = C$ where C is as defined as in Lemma 2.3.1 from the matrices of $\widetilde{\gamma'}$.

Remark 2.3.3. By construction, under the map Ψ the matrix H_i of $\mathcal{M}$ equals the i-th J matrix of δ and the J matrices of γ are constructed from V_i according to Corollary 2.2.9.

Lemma 2.3.4. The map Ψ is well-defined.

Proof. Clearly, $\delta \in \mathcal{P}_k$, since each adjacent set of k consecutive columns forms an adjacent $k \times k$ submatrix of $\mathcal{M}$. Furthermore, $\gamma \in \mathcal{P}_k$ since C is in $SL_k(\mathbb{Z})$ and every other set of k adjacent columns $(\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+k-1})$ of γ form a matrix which can be written as a product of C and J matrices, which are all elements of $SL_k(\mathbb{Z})$. $\square$

We proceed to show that Φ and Ψ are inverses of each other.

Proposition 2.3.5. $\Phi \circ \Psi = \mathrm{id}_{\mathbb{SL}_k}$.

Proof. Let $\mathcal{M} \in \mathbb{SL}_k$. Let V_i and H_i be the transition matrices of $\mathcal{M}$. Then $\Psi(\mathcal{M}) = (\gamma, \delta)$. By Remark 2.3.3, H_i is equal to the i-th J matrix of δ and the i-th J matrix for γ is given by $\widetilde{V_{i+k-2}}$. Thus, by Corollary 2.2.9, $\Phi(\gamma, \delta)$ has as its i-th horizontal transition matrix H_i and as its i-th vertical transition matrix

$$\widetilde{\widetilde{V_{i+k-2}}} = V_i.$$

It therefore suffices to check that $M := M_{1\,1}$ is equal to $((\Phi \circ \Psi)(\mathcal{M}))_{[k],[k]} := M'$. Let $\Psi(\mathcal{M}) = (\gamma, \delta)$. By construction of Ψ, δ has the property that $M = (\delta_1, \delta_2, \ldots, \delta_k)$ and $(\gamma_1, \gamma_2, \ldots, \gamma_k) = C$. We may multiply (γ, δ) by C^{-1} without changing equivalence classes in $(\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$. Thus, $\Phi(\gamma, \delta) = \Phi(C^{-1}\gamma, C^{-1}\delta)$. Then $(\gamma_1, \gamma_2, \ldots, \gamma_k) = C^{-1}\gamma = I_k$. By Lemma 2.3.1, we have

$$M' = C\left(C^{-1}(\delta_1, \delta_2, \ldots, \delta_k)\right) = (\delta_1, \delta_2, \ldots, \delta_k) = M,$$

as desired. □

Proposition 2.3.6. $\Psi \circ \Phi = \mathrm{id}_{\mathcal{P}_k \times \mathcal{P}_k / SL_k(\mathbb{Z})}$.

Proof. Consider a pair of paths γ and δ. Let $(\gamma', \delta') = (\Psi \circ \Phi)(\gamma, \delta)$. By Corollary 2.2.9 and Remark 2.3.3, we know that the J matrices are the same for γ and γ' as well as δ and δ'. After multiplying both paths γ' and δ' by some matrix $A \in SL_k(\mathbb{Z})$, which does not change the equivalence class in $(\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$, we may assume that $\gamma = \gamma'$. Furthermore, we may set $(\gamma_1, \gamma_2, \ldots, \gamma_k) = I_k$. Since the J matrices are the equal, it suffices to show that

$$(\delta_1, \delta_2, \ldots, \delta_k) = (\delta'_1, \delta'_2, \ldots, \delta'_k).$$

Since $(\gamma, \delta') = (\Psi \circ \Phi)(\gamma, \delta)$, by Proposition 2.3.5, we may apply Φ to both sides to get

$$\Phi(\gamma, \delta) = \Phi(\gamma, \delta') := \mathcal{M}.$$

By Lemma 2.3.1, this implies that $C\delta_j = C'\delta'_j$ for all $j \in \mathbb{Z}$. Since C is uniquely determined by the J matrices of γ, we have $C = C'$ and, since $C \in SL_k(\mathbb{Z})$, this gives $\delta_j = \delta'_j$ for all $j \in \mathbb{Z}$.

□

With both directions proven, we get our main result.

Theorem 2.3.7. The map Φ given by

$$\begin{aligned}\Phi : (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) &\to \mathbb{SL}_k \\ (\gamma, \delta) &\mapsto \mathcal{M} = (m_{i\,j})_{i,j \in \mathbb{Z}},\end{aligned}$$

where $m_{i\,j} = \det(\gamma_i, \ldots, \gamma_{i+k-2}, \delta_j)$ is a bijection between tame SL_k-tilings and pairs of paths modulo the action by $SL_k(\mathbb{Z})$.

Proof. Follows from Proposition 2.3.5 and Proposition 2.3.6. □

By restricting to the case $k = 2$, we derive a result which is equivalent to the one produced by Short. In particular, the the two results are related as in the following commutative diagram

$$\begin{array}{ccc} (\mathcal{P}_2 \times \mathcal{P}_2)/SL_2(\mathbb{Z}) & \xrightarrow{\Phi} & \mathbb{SL}_2 \\ \big\downarrow \pi & & \big\downarrow \pi' \\ (\mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q})/SL_2(\mathbb{Z}) & \xrightarrow{\Phi\mathbb{Q}} & \mathbb{SL}_2/\pm \end{array},$$

where π is a quotient map which identifies $(\gamma, \delta) \sim (-\gamma, \delta)$.

This also gives us a nice description of the map ξ^{-1} from Proposition 1.2.6. Let us first define another map ρ which associates a path with its linearization data [3, Equation 7].

Definition 2.3.8. Define the function ρ as follows.

$$\begin{aligned}\rho : (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) &\to SL_k(\mathbb{Z}) \times \left(\mathbb{Z}^{1\times(k-1)}\right)^{\mathbb{Z}} \times \left(\mathbb{Z}^{1\times(k-1)}\right)^{\mathbb{Z}} \\ (\gamma, \delta) &\mapsto (\gamma_1, \ldots, \gamma_k)^T \times \{(j_{i\,2} \ldots, j_{i\,k})^T\}_{i \in \mathbb{Z}} \times \{(j'_{i\,2} \ldots, j'_{i\,k})^T\}_{i \in \mathbb{Z}}\end{aligned}$$

where the elements $j_{i\,j}$ and $j'_{i\,j}$ are given by the final columns of the J matrices of $\widetilde{\gamma}$ and δ respectively.

Since linearization data is in bijection with tilings by Proposition 1.2.6, we see that ρ is also a bijection, hence it is invertible. While ξ^{-1} is defined recursively, the map Φ is explicit. This makes the following result particularly noteworthy.

Corollary 2.3.9. The following diagram commutes.

$$\begin{array}{ccc} (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z}) & \xrightarrow{\quad\Phi\quad} & \mathbb{SL}_k \\ \big\downarrow{\scriptstyle \rho} & \nearrow{\scriptstyle \xi^{-1}} & \\ SL_k(\mathbb{Z}) \times \left(\mathbb{Z}^{1\times(k-1)}\right)^{\mathbb{Z}} \times \left(\mathbb{Z}^{(k-1)\times 1}\right)^{\mathbb{Z}} & & \end{array}$$

Chapter 3 Applications

3.1 Periodicity

Here we detail some consequences about periodicity of paths and tilings. Recall the notion of periodicity given in Definition 1.2.7.

Lemma 3.1.1. For a path γ, if the J matrices are m-periodic and $J_1 J_2 \cdots J_m = I_k$, then γ is m-periodic.

Proof. By definition

$$(\gamma_i, \ldots, \gamma_{i+k-1})\, J_i J_{i+1} \cdots J_{i+m-1} = (\gamma_{i+m}, \ldots, \gamma_{i+m+k-1})\,.$$

Thus, it suffices to show $J_i \cdots J_{i+m-1} = I_k$. Since $J_i = J_{i+m}$ for all $i \in \mathbb{Z}$, we may write the subscripts $\bmod m$. Thus,

$$J_i J_{i+1} \cdots J_{i+m-1} = J_i J_{i+1} \cdots J_m J_1 J_2 \cdots J_{i-1}$$

Note that $J_1 J_2 \cdots J_m = I_k$ by assumption. Therefore, we multiply

$$\begin{aligned} J_i J_{i+1} \cdots J_m J_1 J_2 \cdots J_{i-1} &= J_i J_{i+1} \cdots J_m J_1 J_2 \cdots J_{i-1} \left(J_i J_{i+1} \cdots J_m J_m^{-1} J_{m-1}^{-1} \cdots J_i^{-1} \right) \\ &= J_i J_{i+1} \cdots J_m \left(J_1 J_2 \cdots J_{i-1} J_i J_{i+1} \cdots J_m \right) J_m^{-1} J_{m-1}^{-1} \cdots J_i^{-1} \\ &= J_i J_{i+1} \cdots J_m J_m^{-1} J_{m-1}^{-1} \cdots J_i^{-1} \\ &= I_k. \end{aligned}$$

□

We may construct a similar result going from tilings to J matrices.

Lemma 3.1.2. If $\mathcal{M}$ is m-column periodic, then the J matrices of δ are m-periodic and $J_i J_{i+1} \cdots J_{i+m-1} = I_k$ for all $i \in \mathbb{Z}$.

Proof. We know that we may get from $M_{1\,i}$ to $M_{1\,i+m}$ by multiplying by $J_i J_{i+1} \cdots J_{i+m-1}$. Since $M_{1\,i} = M1\,i+m$, as $\mathcal{M}$ is m-column periodic, this product must be the identity I_k. Furthermore, since column i and column $i+m$ are the same for all $i \in \mathbb{Z}$, the same J matrix must be used to transition between them, so $J_i = J_{i+m}$ for all $i \in \mathbb{Z}$. □

This gives us the following proposition.

Proposition 3.1.3. Fix a path δ. The following are equivalent.

1. The J matrices of δ are m-periodic and $J_1 J_2 \cdots J_m = I_k$.

2. The path δ is m-periodic.

3. The tiling $\mathcal{M} := \Phi(\gamma, \delta)$ for any path γ is m-column periodic.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 3.1.1, and the implication (3) $\Rightarrow$ (1) from Lemma 3.1.2 and Corollary 2.2.9. For the implication (2) $\Rightarrow$ (3), suppose δ is m-periodic. If δ repeats every m entries, the resulting determinants $m_{i\,j} = \det(\gamma_i, \ldots, \gamma_{i+k-1}, \delta_j)$ will also repeat every m entries going across the columns. Thus, the resulting $\mathcal{M}$ will be m-column periodic, as desired. $\square$

Remark 3.1.4. The dual of this proposition is also true. That is, the following are equivalent.

1. The J matrices of γ are m-periodic and $J_1 J_2 \cdots J_m = I_k$.

2. The path γ is m-periodic.

3. The tiling $\mathcal{M} := \Phi(\gamma, \delta)$ for any path δ is m-row periodic.

Together, these results give us the necessary and sufficient conditions for periodicity in a tiling as in Definition 1.2.7.

Corollary 3.1.5. The paths γ and δ are m and n periodic, respectively, if and only if $\mathcal{M} := \Phi(\gamma, \delta)$ is $(m \times n)$-periodic.

Proof. The forward direction follows from (2) $\Rightarrow$ (3) in Proposition 3.1.3 and Remark 3.1.4. Similarly, the backward direction follows from (3) $\Rightarrow$ (2). $\square$

3.2 Duality

Our goal is to show that the dual $\mathcal{M}^*$ of an SL_k-tiling $\mathcal{M} := \Phi(\gamma, \delta)$ defined in Definition 1.2.8 has a simple interpretation in terms of the map Φ. In particular, it is given by $\Phi(A\widetilde{\gamma}, \widetilde{\delta})$, i.e. a shift of conjugacy class for γ together with the tilde operator, up to some shift in indices. We begin by presenting several results about the dual as it relates to Plücker friezes and their J matrices. A special case of Theorem 3.2.2 for friezes was proven by Morier-Genoud, Ovsienko, Schwatz, and Tobachnikov [14] as well as Cordes and Roselle [7].

We first give a lemma which shows that the dual preserves J matrices up to the tilde operator.

Lemma 3.2.1. Let $\mathcal{M} = \Phi(\gamma, \delta)$ be an SL_k-tiling and let $\mathcal{M}^* = \Phi(\gamma^*, \delta^*)$ be its dual. Then the sequences of transition matrices for $\widetilde{\gamma}$ and $\widetilde{\delta}$ from Definition 2.2.5 coincide with those for γ^* and δ^* respectively up to a shift in indexing.

Proof. Let $M^* := \mathcal{M}^*_{[2k+1],[2k+1]}$. Note that M^* is entirely determined by $\mathcal{M}_{[3k-1],[3k-1]}$. Let $\mathcal{M}'$ be the tiling resulting from the path $\varphi_{3k-1,3k-1}(\gamma, \delta)$ from Definition 2.1.5. By construction,

$$\mathcal{M}_{[3k-1],[3k-1]} = \mathcal{M}'_{[3k-1],[i]^{3k-1}} = \begin{pmatrix} p_{[1]^{k-1}\,i}(A) & p_{[1]^{k-1}\,i+1} & \cdots & p_{[1]^{k-1}\,i+k-1} \\ p_{[2]^{k-1}\,i}(A) & p_{[2]^{k-1}\,i+1} & \cdots & p_{[2]^{k-1}\,i+k-1} \\ \vdots & \cdots & \ddots & \vdots \\ p_{[k]^{k-1}\,i}(A) & p_{[k]^{k-1}\,i+1} & \cdots & p_{[k]^{k-1}\,i+k-1} \end{pmatrix}.$$

Then, by Proposition 1.1.9,

$$\left(\mathcal{M}'_{[3k-1],[i]^{3k-1}}\right)^* = \begin{pmatrix} p_{[i]^{k-1}\,k-1}(A) & p_{[i+1]^{k-1}\,k-1} & \cdots & p_{[i+k-1]^{k-1}\,k-1} \\ p_{[i]^{k-1}\,k}(A) & p_{[i+1]^{k-1}\,k} & \cdots & p_{[i+k-1]^{k-1}\,k} \\ \vdots & \cdots & \ddots & \vdots \\ p_{[i]^{k-1}\,2k-2}(A) & p_{[i+1]^{k-1}\,2k-2} & \cdots & p_{[i+k-1]^{k-1}\,2k-2} \end{pmatrix}.$$

Thus, the first horizontal transition matrix for $\mathcal{M}^*$ is the i-th horizontal transition matrix for $(M')^*$ which transitions from the first k columns of $\left(\mathcal{M}'_{[3k-1],[i]^{3k-1}}\right)^*$ to columns 2 throuh $k+1$. But this is V'_{i+1}, the i-th vertical transition matrix for $\mathcal{M}'$. Similarly, the first vertical transition matrix of $\mathcal{M}^*$ is the $(k-1)$-st horizontal transition matrix of $\mathcal{M}'$, which is J_{k-1} of the path $\varphi(\gamma, \delta)$, i.e. J_{k-1} of γ. Thus, the dual operator preserves J matrices up to the tilde operator, as desired. ☐

It should be noted that, although the dual operator preserves information about the transition matrices, it does not necessarily preserve the information about the initial $k \times k$ adjacent submatrix. Thus, applying the dual to the tiling $\Phi(\gamma, \delta)$ may not preserve the conjugacy class, but it will preserve information about the sequence of J matrices.

Theorem 3.2.2. Let $\mathcal{M} = \Phi(\gamma, \delta)$ be an SL_k-tiling. Then its dual $\mathcal{M}^* = \Phi(A\widetilde{\gamma}, \widetilde{\delta})$ for some $A \in SL_k(\mathbb{Z})$.

Proof. By Lemma 3.2.1, the sequence of J matrices associated with $\widetilde{\gamma}$ correspond to the sequence of vertical transition matrices of $\mathcal{M}^*$. Similarly, the sequence of J matrices of $\widetilde{\delta}$ correspond to the sequence of horizontal transition matrices of $\mathcal{M}^*$.

Thus, this maintains the linearization data up to the tilde operator. Multiplying $\widetilde{\delta}$ by a matrix $A \in SL_k(\mathbb{Z})$ preserves the J matrices by Lemma 2.1.3. Doing so changes the central $k \times k$ adjacent submatrix such that its image under the dual has the correct central $k \times k$ adjacent submatrix. $\square$

As a corollary, we can easily recover the fact the dual operator has the desired duality property.

Corollary 3.2.3. For any SL_k-tiling $\mathcal{M}$, $(\mathcal{M}^*)^* = \mathcal{M}$ up to a shift.

Proof. Let $\mathcal{M} = \Phi(\gamma, \delta)$. For sufficiently large m, n, we can construct $\varphi_{m,n}(\gamma, \delta)$ as in the proof of Lemma 3.2.1 such that the resulting tiling $\mathcal{M}'$ has the property that $(M^*)^* = (\mathcal{M}'^*)^*_{[k]\,[k]} = (\mathcal{M}^*)^*_{[k]\,[k]}$. We may calculate $(M^*)^*$ using Plücker coordinates, i.e.

$$(M^*)^* = \begin{pmatrix} p_{o([k-1]^{k-1}, i+k-2)} & p_{o([k-1]^{k-1}, i+k-1)} & \cdots & p_{o([k-1]^{k-1}, i+2k-3)} \\ p_{o([k]^{k-1}, i+k-2)} & p_{o([k]^{k-1}, i+k-1)} & \cdots & p_{o([k]^{k-1}, i+2k-3)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{o([2k-2]^{k-1}, i+k-2)} & p_{o([2k-2]^{k-1}, i+k-1)} & \cdots & p_{o([2k-2]^{k-1}, i+2k-3)} \end{pmatrix},$$

which is just $\mathcal{M}'_{[k]\,[k]} = \mathcal{M}_{[k]\,[k]}$ shifted right and down by $k-2$. By Remark 2.2.6, this corresponds to the shift in the J matrices of $\widetilde{\widetilde{\gamma}}$ and $\widetilde{\widetilde{\delta}}$. Since this preserves a central adjacent $k \times k$ submatrix and the J matrices of γ and δ, the result follows. $\square$

3.3 Friezes

Our goal is to show that the bijection Φ has a simple restriction to infinite friezes, namely that friezes are in bijection with pairs of paths where both paths are identical up to a shift. We first define some notation to clarify this.

Definition 3.3.1. Define the inclusion function ι as follows.

$$\iota : \mathcal{P}_k / SL_k(\mathbb{Z}) \to (\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$$
$$(\gamma) \mapsto (\gamma, \gamma).$$

Note that the image of the map ι is a subset of $(\mathcal{P}_k \times \mathcal{P}_k)/SL_k(\mathbb{Z})$, so we may use it as a restriction of the domain of the map Φ. Additionally, note that ι is a bijection on its own image, so $\Phi \circ \iota$ is injective. We write Φ_ι for $\Phi \circ \iota$. We wish to show that this restriction on the domain corresponds to a restriction on the range to tilings resulting from infinite SL_k-friezes.

Definition 3.3.2. Let F be an infinite frieze. We define a tiling $\mathcal{M}_F$ as follows. Let the rows of F become the falling diagonals of a tiling by rotating the frieze 45° clockwise. Since F is infinite, the left half of the tiling is complete. Fix an element on the last known falling diagonal of the tiling, and let this be the upper-right corner of a $k \times k$ matrix $M_{1\,1}$, i.e. an entry with index $m_{1\,k}$ lies on the falling diagonal of zeroes with nothing known above it or to its right. Let H_i be the horizontal transition matrix which takes $M_{i-1\,i}$ to $M_{i\,i}$. We use these transition matrices to construct the right half of the tiling.

Remark 3.3.3. Let F be a finite frieze and F' its infinite extension. We want to verify that the tiling $\mathcal{M}_{F'}$ in Definition 3.3.2 coincides with $\mathcal{M}_F$ which results from Definition 1.2.3. Observe that both $\mathcal{M}_F$ and $\mathcal{M}_{F'}$ will have the same left half by construction. Given such a portion of an infinite tiling, there is a unique complete tiling which results by extending to the right along the falling diagonals [3, Proposition 7]. Both $\mathcal{M}_F$ and $\mathcal{M}_{F'}$ are valid tilings, and must therefore be the same.

It should be noted that the right half of the tiling $\mathcal{M}_F$, though recoverable as an infinite frieze, need not be a periodic extension of F as with finite friezes. It also need not have the same properties as F, such as positivity. However, since this part is uniquely determined by F, we can restrict ourselves to talking about the left part of F. Let $\mathbb{FR}_k$ denote the set of all SL_k-tilings resulting from infinite SL_k-friezes.

Theorem 3.3.4. The restriction of the map Φ given by

$$\begin{aligned} \Phi_\iota : \mathcal{P}_k/SL_k(\mathbb{Z}) &\to \mathbb{FR}_k \\ \gamma &\mapsto \mathcal{M} = (m_{i\,j})_{i,j\in\mathbb{Z}}, \end{aligned}$$

where $m_{i\,j} = \det(\gamma_i, \ldots, \gamma_{i+k-2}, \gamma_j)$ is a bijection between tame SL_k-tilings from SL_k-friezes and equivalence classes of paths.

Proof. We first show that the map is well-defined. Let $\gamma \in \mathcal{P}_k$ and let $\mathcal{M} = \Phi_\iota(\gamma)$. Observe that the entries $m_{i\,j}$ in $\mathcal{M}$ where $i \in \mathbb{Z}$ and $j \in [i]^{k-1}$ are all zeros since they are given by determinants of matrices of the form $(\gamma_i, \ldots, \gamma_{i+k-2}, \gamma_j)$ where γ_j is the same as one of the previous columns. These constitute $k-1$ falling diagonals of zero entries. The next falling diagonal consisting of entries of the form $m_{i\,i-1}$ is all ones, since these are given by determinants of matrices formed by adjacent columns in γ. Thus, the left half of $\mathcal{M}$ is recoverable as an infinite frieze F.

We now show that the map is a bijection. The map Φ_ι inherits injectivity from the injectivity of Φ and ι. For surjectivity, let F be an infinite frieze (possibly the result of an extension of a finite frieze) and $\mathcal{M} = \mathcal{M}_F$. Index the horizontal transition matrices H_i as in Definition 3.3.2. Note that for a conjugacy class in $\mathcal{P}_k/SL_k(\mathbb{Z})$, it is sufficient to construct the sequence of J matrices to describe its elements uniquely. Consider the tiling $\Phi_\iota(\gamma)$ where γ has J matrices H_i with $(\gamma_1, \gamma_2, \ldots, \gamma_k) = I_k$. We claim $\mathcal{M}_F = \Phi_\iota(\gamma)$ up to a shift. Consider a $(k+1) \times (k+1)$ adjacent submatrix of $\mathcal{M}_F$. Since this adjacent submatrix lies in a frieze, it may be realized as a part of a tiling from a finite frieze, hence represented with Plücker coordinates. It's horizontal transition matrix is a J matrix of γ and, since it is from a frieze, its vertical transition matrix is $\widetilde{J}_i$ for some i based on its vertical position in the tiling. Since our choice of submatrix was arbitrary, this holds for all indices $i \in \mathbb{Z}$. Thus, the linearization data of $\mathcal{M}_F$ corresponds to the J matrices of γ, proving the claim. □

Note that this means finite friezes become periodic tilings. Thus, we can make a further restriction. We denote by $\mathbb{FR}_{k,n}$ the set of all SL_k-tilings resulting from SL_k-friezes of width $w = n - k - 1$. Let $\mathcal{P}_{k,p}$ denote the set of p-periodic paths in $\mathcal{P}_k$.

Corollary 3.3.5. The restriction of Φ_ι to p-periodic paths

$$\Phi_\iota : \mathcal{P}_{k,p}/SL_k(\mathbb{Z}) \to \mathbb{FR}_{k,p+k+1}$$

is a bijection between p-periodic paths and SL_k-friezes of width $n - k - 1$.

Proof. The map is well-defined as a result of Proposition 3.1.3 and Theorem 3.3.4. Surjectivity follows from Remark 1.1.7, as the elements of the Grassmannian correspond exactly with paths γ. Injectivity is inherited from the injectivity of Φ_ι. □

Chapter 4 Positivity

4.1 Background

We have made several references to the work of Short. Short was able to prove several results about positive SL_2-tilings and friezes and their relations to paths using the geometry of the Farey graph [17, Theorem 1.4]. Lacking a connection to geometry, we cannot use the same methodology. Instead, we focus on studying positivity for friezes using Plücker coordinates.

Recall that all friezes can be realized as Plücker friezes evaluated at certain elements of the Grassmannian. We begin by defining a new class of Plücker coordinate which will play an important role in our discussion.

Definition 4.1.1. A Plücker coordinate of the form $p_{o([i]^{k+1}\setminus\{j\})}$ where $i \in [n]$ and $j \in [i]^{k-1}$ but $j \neq i$ and $j \neq i+k-1$, i.e. a Plücker coordinate which consists of two consecutive runs separated by a gap of size one, is called *semi-consecutive*.

Note that consecutive Plücker coordinates are not semi-consecutive. We will later make use of the following definitions and results due to Morier-Genoud, Ovsienko, Schwartz, and Tobachnikov [14]. Since their work deals with SL_k-friezes, we adapt it to fit the notion of tilings.

Definition 4.1.2. [14, Definition 4.1.1] Let F be a frieze. Consider the array resulting from rotating F 45° clockwise. Let a_i^j be the adjacent minor of size j whose bottom right corner is taken as $m_{i\,i}$ in the array. The *Gale dual of* F, denoted $F^{\mathcal{G}}$, is the frieze where the (i,j)-th entry in its array is given by $(a_{i-1}^{k-j+i-1})$. Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a frieze F. Then the *Gale dual of* $\mathcal{M}_F$, is given by $\mathcal{M}_F^{\mathcal{G}} := \mathcal{M}_{F^{\mathcal{G}}}$.

They prove that the Gale dual of a frieze is itself a frieze, so this notation is well-defined. We reformulate this notion here.

Theorem 4.1.3. [14] Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a frieze F of width $n-k+2$. Then $\mathcal{M}_F^{\mathcal{G}}$ also is a tame SL_{n-k}-tiling from a frieze of width $k+2$.

4.2 Paths where $k = 3$

When $k = 3$ we have a special result about the paths γ which result from positive tilings from friezes. First, we examine two specific elements of γ.

Lemma 4.2.1. Let $\gamma \in \mathcal{P}_k$ be a path with $(\gamma_1, \gamma_2, \gamma_3) = I_3$. If $\mathcal{M} := \Phi_\iota(\gamma)$ is a tiling from a positive frieze, then the entries of γ_4 and γ_0 alternate in sign. In particular, the first and last entries are positive and the middle entries are negative.

Proof. Since $(\gamma_0, \gamma_1, \gamma_2), (\gamma_2, \gamma_3, \gamma_4) \in SL_3(\mathbb{Z})$, γ_4 and γ_0 are of the form

$$\gamma_4 = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} c \\ d \\ 1 \end{pmatrix}.$$

By definition of Φ_ι, we have the following entries of $\mathcal{M}$ which lie in the first nontrivial row:

$$m_{1\,4} = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & a \\ 0 & 0 & b \end{pmatrix} = b, \qquad m_{3\,1} = \det \begin{pmatrix} 0 & 1 & 1 \\ 0 & a & 0 \\ 1 & b & 0 \end{pmatrix} = -a$$

and

$$m_{2\,0} = \det \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & d \\ 0 & 1 & 1 \end{pmatrix} = c, \qquad m_{0\,3} = \det \begin{pmatrix} c & 1 & 0 \\ d & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = -d.$$

Since $\mathcal{M}$ is positive, so are $b, c, -a$, and $-d$. □

We now examine arbitrary elements of γ.

Lemma 4.2.2. Let $\gamma \in \mathcal{P}_k$ be a non-periodic path with $(\gamma_1, \gamma_2, \gamma_3) = I_3$. Let $\gamma_i = (x_i, y_i, z_i)^T$ for i any integer not in $\{0, 1, 2, 3, 4\}$. If $\mathcal{M} := \Phi_\iota(\gamma)$ is a tiling from a positive frieze, then the following hold.

1. $x_i > 0$
2. $z_i > 0$
3. $y_i < 0$ if and only if $y_{i+1} < 0$.

If γ is m-periodic, the same holds for γ_i where i is not congruent to to $\{0, 1, 2, 3, 4\}$ mod m.

Proof. As in the proof of Lemma 4.2.1, the positivity of $m_{1\,i}$ and $m_{2\,i}$ give the positivity of x_i and z_i respectively. Furthermore, we have that

$$m_{i\,3} = \det \begin{pmatrix} x_i & x_{i+1} & 0 \\ y_i & y_{i+1} & 0 \\ z_i & z_{i+1} & 1 \end{pmatrix} = x_i y_{i+1} - x_{i+1} y_i,$$

$$m_{i\,1} = \begin{vmatrix} x_i & x_{i+1} & 1 \\ y_i & y_{i+1} & 0 \\ z_i & z_{i+1} & 0 \end{vmatrix} = y_i z_{i+1} - y_{i+1} z_i.$$

By the positivity of $\mathcal{M}$ as well as x_i and z_i, we have the following. If $y_i < 0$, then $m_{i\,1}$ gives us that

$$\frac{z_{i+1}}{z_i} < \frac{y_{i+1}}{y_i},$$

so $y_{i+1} < 0$. If $y_{i+1} < 0$, then $m_{i\,3}$ gives us that

$$\frac{x_i}{x_{i+1}} < \frac{y_i}{y_{i+1}},$$

so $y_i < 0$. □

This allows us to make a general statement about the path γ.

Theorem 4.2.3. Let $\gamma \in \mathcal{P}_k$ be a path with $(\gamma_1, \gamma_2, \gamma_3) = I_3$. If $\mathcal{M} := \Phi_\iota(\gamma)$ is a tiling from a positive infinite frieze, then the entries of γ_i alternate in sign for $i \in \mathbb{Z} \setminus [3]$. In particular, the first and last entries are positive, and the middle entry is negative. If $\mathcal{M} = \Phi_\iota(\gamma)$ is a tiling from a positive finite frieze with period m, then entries of γ_i alternate in sign for $i \not\equiv 1, 2, 3 \pmod m$.

Proof. We induct on i from above starting with $i = 4$ and below starting with $i = 0$. Lemma 4.2.1 gives the base cases, and Lemma 4.2.2 gives the inductive step. □

The converse of this is not generally true. Consider the following example:

$$\gamma = \left(\cdots, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}1\\-3\\2\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \cdots \right). \quad (4.1)$$

This gives the following tiling:

$$\Phi_\iota(\gamma) = \begin{array}{cccccccccccccc}
0 & 0 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & & & & & \\
 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & & & & \\
 & & 0 & 0 & 1 & \textcolor{red}{-1} & 0 & 2 & 1 & 0 & 0 & & & \\
 & & & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & & \\
 & & & & 0 & 0 & 1 & 1 & -1 & 2 & 1 & 0 & 0 & \\
 & & & & & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0,
\end{array}$$

which is clearly not a tiling from a positive frieze. Note the entry $m_{3\,5} = -1$ is given by

$$-1 = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -2 & -3 \\ 1 & 1 & 1 \end{vmatrix}.$$

4.3 J matrices

Nest, we want to study the relationship between friezes and entries of their J matrices. We must first show a result about the entries of J matrices which will allow us to prove certain facts about the entries of dual tilings. We have the following corollary to Proposition 2.2.3.

Corollary 4.3.1. All entries of J matrices of tilings $\mathcal{M}$ from Plücker friezes are semi-consecutive Plücker coordinates. In particular, all semi-consecutive Plücker coordinates appear as an entry in some J matrix of $\mathcal{M}$.

Before continuing, we make several observations about Plücker coordinates concerning the sizes of k and n.

Remark 4.3.2. We consider two cases.

(a) If $k \geq n-3$, then all almost consecutive Plücker coordinates p_I are either semi-consecutive or consecutive. With only three entries to exclude from $[n]$ to form I, there must either be a gap of one entry between sequences or all entries will be consecutive.

(b) If $k \leq 3$, then all semi-consecutive Plücker coordinates p_I are almost consecutive. With only three entries in I, at least one consecutive run will have a length of one.

For ease of notation in the following results, we define a structure using the final columns of J matrices.

Definition 4.3.3. For a tiling $\mathcal{M}_F \in \mathbb{SL}_k$, we call the vectors $\mathbf{q}_i = \begin{pmatrix} (-1)^{k-2} j_{i\,2} \\ \vdots \\ (-1)^p j_{i\,k-p} \\ \vdots \\ (-1)^0 j_{i\,k} \end{pmatrix}$,

where $j_{i\,\ell}$ are the elements from the final column of the i-th horizontal transition matrix of $\mathcal{M}_F$, the *quiddity vectors*. The sequence $(\mathbf{q}_i)_{i\in\mathbb{Z}}$ is called the *quiddity sequence* of $\mathcal{M}_F$. We say a quiddity sequence is *positive* if all of its entries are positive.

This extends the traditional notion of the quiddity sequence in the case where $k = 2$ to higher dimensions. Note that by Proposition 2.2.3 part (a), for Plücker friezes, the entries in these vectors correspond exactly to the semi-consecutive Plücker coordinates.

Lemma 4.3.4. Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a frieze F with $n - 3 \leq k$. Then $\mathcal{M}_F$ is positive if the quiddity sequence is positive.

Proof. Since the quiddity sequence is positive, this means all semi-consecutive Plücker coordinates are positive. By part (a) of Remark 4.3.2, all almost consecutive Plücker coordinates in this case are also semi-consecutive. Since all entries of $\mathcal{M}_{\mathcal{F}(k,n)}$ are given by almost consecutive Plücker coordinates, then all of them are also positive. □

Lemma 4.3.5. Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a positive frieze F with $k \leq 3$. Then the quiddity sequence is positive.

Proof. By part (b) of Remark 4.3.2, we know that the entries of the quiddity vectors are almost consecutive, hence are entries of $\mathcal{M}_F$. Since all entries of $\mathcal{M}_F$ are positive, so is the quiddity sequence. □

4.4 The Cases of $(4, 7)$ and $(5, 8)$

For some special pairs of (k, n), we can make additional arguments. It should be noted that, while the representation theory of algebras that provide categorification of the cluster structure on $\mathcal{A}_{k,n}$ does not feature directly in either of these arguments, it did serve as inspiration for the proofs. For both of the following special cases, we refer to the Plücker relations as given by Equation (1.1). We begin with the case where $k = 4$ and $n = 7$.

Lemma 4.4.1. Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a positive frieze F where $n \leq 7$ and $k = 4$. Then the quiddity sequence is positive.

Proof. Given that almost consecutive Plücker coordinates are positive, it suffices to show that the semi-consecutive Plücker coordinates are positive. It is only when $n \geq 6$ that Plücker coordinates of the form $p_{o(m\, m+1\, m+3\, m+4)}$ are semi-consecutive, but not almost consecutive. We show that these are positive. Consider the Plücker relation with $I = \{m, m+1, m+3\}$ and $J = \{m+1, m+2, m+3, m+4, m+5\}$. This gives the equation

$$\begin{aligned} 0 = p_{o(m\,m+1\,m+3)\,m+1}p_{o(m+2\,m+3\,m+4\,m+5)} - p_{o(m\,m+1\,m+3)\,m+2}p_{o(m+1\,m+3\,m+4\,m+5)} \\ + p_{o(m\,m+1\,m+3)\,m+3}p_{o(m+1\,m+2\,m+4\,m+5)} - p_{o(m\,m+1\,m+3)\,m+4}p_{o(m+1\,m+2\,m+3\,m+5)} \\ + p_{o(m\,m+1\,m+3)\,m+5}p_{o(m+1\,m+2\,m+3\,m+4)}. \end{aligned}$$

Recall that Plücker coordinates with repeated entries are 0 and consecutive Plücker coordinates are 1. Thus, we may simplify the equation.

$$0 = p_{o(m+1\,m+3\,m+4\,m+5)} - p_{o(m\,m+1\,m+3\,m+4)}p_{o(m+1\,m+2\,m+3\,m+5)} + p_{o(m\,m+1\,m+3\,m+5)}.$$

By assumption, $p_{o(m+1\,m+3\,m+4\,m+5)}$ and $p_{o(m+1\,m+2\,m+3\,m+5)}$ are positive since they are almost consecutive. Thus,

$$p_{o(m\,m+1\,m+3\,m+4)} > 0 \quad \text{if} \quad p_{o(m\,m+1\,m+3\,m+5)} \geq 0.$$

In the case of $n = 6$, $m+5 \equiv m-1 \pmod 6$, so the latter is almost consecutive, hence positive. In the case of $n = 7$, note that $m + 6 \equiv m - 1 \pmod 7$ and consider the Plücker relation with $I = \{m, m+1, m+3\}$ and $J = \{m, m+1, m+2, m+5, m+6\}$. Simplified as above, the resulting equation is

$$0 = -1 - p_{o(m\,m+1\,m+3\,m+5)} + p_{o(m\,m+1\,m+3\,m+6)}p_{o(m\,m+1\,m+2\,m+5)}.$$

Note that the final term is the product of almost consecutive Plücker coordinates, hence it is at least 1. Therefore, $p_{o(m\,m+1\,m+3\,m+5)} \geq 0$, as desired. □

For the case where $k = 5$ and $n = 8$, we first make claims regarding $k = 3$ and $n = 8$.

Remark 4.4.2. Note that $\mathbb{C}[\mathrm{Gr}(5, 8)] \cong \mathbb{C}[\mathrm{Gr}(3, 8)]$ with $p_I \mapsto p_{I^c}$ where $I^c = [8] \setminus I$.

Hence, we consider $\mathbb{C}[\mathrm{Gr}(3, 8)]$. For the following, we assume that consecutive Plücker coordinates $p_{o(m\,m+1\,m+2)} = 1$ and all Plücker coordinates are integers. Additionally, recall the Plücker relations as described in Equation (1.1).

Lemma 4.4.3. Suppose Plücker coordinates of the form $p_{o(m\,m+1\,m+3)}$ and $p_{o(m\,m+2\,m+3)}$ are positive for all $m \in \mathbb{N}$. Then Plücker coordinates of the form $p_{o(m\,m+2\,m+4)} \geq 0$ for all $m \in \mathbb{N}$. Furthermore, for a fixed $m \in \mathbb{N}$, $p_{o(m\,m+2\,m+4)} = 0$ if and only if $p_{o(m\,m+2\,m+3)} = p_{o(m+1\,m+2\,m+4)} = 1$.

Proof. Consider the Plücker relation where $I = \{m, m+2\}$ and $J = \{m+1, m+2, m+3, m+4\}$. This gives us the following equation:

$$0 = p_{o(m\,m+2)\,m+1}p_{o(m+2\,m+3\,m+4)} - p_{o(m\,m+2)\,m+2}p_{o(m+1\,m+3\,m+4)} + p_{o(m\,m+2)\,m+3}p_{o(m+1\,m+2\,m+4)} - p_{o(m\,m+2)\,m+4}p_{o(m+1\,m+2\,m+3)}.$$

Recall that consecutive Plücker coordinates are 1 and that Plücker coordinates with repeated indices are 0. We may therefore simplify the equation as

$$p_{o(m\,m+2\,m+4)} = -1 + p_{o(m\,m+2\,m+3)}p_{o(m+1\,m+2\,m+4)}.$$

By assumption, the second term on the right side is a positive integer, therefore $p_{o(m\,m+2\,m+4)} \geq 0$. Observe, additionally, that $p_{o(m\,m+2\,m+4)} = 0$ if and only if the second term on the right side is 1. Since both Plücker coordinates in the product are positive integers, this occurs if and only if both are 1. □

The same assumption also provides an additional result about a crucial orbit of Plücker coordinates.

Proposition 4.4.4. Consider $\mathrm{Gr}(3, 8)$. Suppose Plücker coordinates of the form $p_{o(m\,m+1\,m+3)}$ and $p_{o(m\,m+2\,m+3)}$ are positive for all $m \in \mathbb{N}$ and that, for some fixed $q \in \mathbb{N}$, either $p_{o(q\,q+1\,q+5)} \geq 0$ or $p_{o(q\,q+1\,q+4)} \geq 0$. Then $p_{o(m\,m+1\,m+5)} \geq 0$ and $p_{o(m\,m+1\,m+4)} \geq 0$ for all $m \in \mathbb{N}$.

Proof. We proceed to prove two claims from which we may derive the desired result.

Claim 1 If $p_{o(m\,m+1\,m+5)} \geq 0$, then $p_{o(m\,m+1\,m+4)} \geq 0$.

Claim 2 If $p_{o(m\,m+1\,m+4)} \geq 0$, then $p_{o(m\,m+4\,m+7)} \geq 0$.

Consider the Plücker relation where $I = \{m, m+1\}$ and $J = \{m+2, m+3, m+4, m+5\}$. This gives us the following equation:

$$0 = p_{o(m\,m+1)\,m+2}p_{o(m+3\,m+4\,m+5)} - p_{o(m\,m+1)\,m+3}p_{o(m+2\,m+4\,m+5)} + p_{o(m\,m+1)\,m+4}p_{o(m+2\,m+3\,m+5)} - p_{o(m\,m+1)\,m+5}p_{o(m+2\,m+3\,m+4)}.$$

Recall consecutive Plücker coordinates are 1. We may therefore simplify the equation as

$$p_{o(m\,m+1\,m+5)} = 1 - p_{o(m\,m+1\,m+3)}p_{o(m+2\,m+4\,m+5)} + p_{o(m\,m+1\,m+4)}p_{o(m+2\,m+3\,m+5)}.$$

By assumption, all terms other than $p_{o(m\,m+1\,m+5)}$ and $p_{o(m\,m+1\,m+4)}$ are positive. Thus, we have proven Claim 1.

Additionally, consider the Plücker relation where $I = \{m, m+4\}$ and $J = \{m+2, m+3, m+4, m+5\}$. This gives us the following equation:

$$0 = p_{o(m\,m+4)\,m+2}p_{o(m+3\,m+4)\,m+5} - p_{o(m\,m+4)\,m+3}p_{o(m+2\,m+4)\,m+5} \\ + p_{o(m\,m+4)\,m+4}p_{o(m+2\,m+3)\,m+5} - p_{o(m\,m+4)\,m+5}p_{o(m+2\,m+3)\,m+4}.$$

This simplifies as

$$p_{o(m\,m+4\,m+5)} = -p_{o(m\,m+2\,m+4)} + p_{o(m\,m+3\,m+4)}p_{o(m+2\,m+4\,m+5)}.$$

We may shift all indices by 4, recalling that we are in $\mathrm{Gr}(3,8)$, to get the equation

$$p_{o(m\,m+1\,m+4)} = -p_{o(m\,m+4\,m+6)} + p_{o(m\,m+4\,m+7)}p_{o(m\,m+1\,m+6)}.$$

Note that, by shifting indices, $p_{o(m\,m+1\,m+6)}$ is of the form $p_{o(m\,m+2\,m+3)}$ and $p_{o(m\,m+4\,m+6)}$ is of the form $p_{o(m\,m+2\,m+4)}$, hence the former is positive and the latter non-negative by assumption and Lemma 4.4.3 respectively. Thus, we have proven Claim 2.

Note that $p_{o(m\,m+4\,m+7)}$ is of the form $p_{o(m\,m+1\,m+5)}$ where the indices are shifted by -7. Suppose that $p_{o(q\,q+1\,q+5)} \geq 0$ for some $q \in \mathbb{N}$. By Claim 1, this implies $p_{o(q\,q+1\,q+4)} \geq 0$. By Claim 2, this implies $p_{o(q\,q+4,q+7)} \geq 0$ which in turn implies $p_{o(q-1\,q\,q+3)} \geq 0$ by Claim 1. One may iterate this process, concluding that $p_{o(m\,m+1\,m+5)} \geq 0$ and $p_{o(m\,m+1\,m+4)} \geq 0$ for all $m \in \mathbb{N}$. Alternatively, by starting from the assumption that $p_{o(q\,q+1\,q+4)} \geq 0$ for some $q \in \mathbb{N}$, Claim 2 implies $p_{o(q\,q+4,q+7)} \geq 0$ which in turn implies $p_{o(q-1\,q\,q+3)} \geq 0$ by Claim 1, and the iteration proceeds similarly. □

We now present a list of specific Plücker relations which we will use in the coming results.

Lemma 4.4.5. The following equations in $\mathrm{Gr}(3,8)$ follow from Equation (1.1). Each is presented with its sets I and J and simplified such that all Plücker coordinates with repeated entry are set to 0 and all consecutive Plücker coordinates are set to 1.

1. If $I = \{3, 7\}$ and $J = \{1, 2, 3, 4\}$, then

$$p_{3\,4\,7} = p_{2\,3\,7}p_{1\,3\,4} - p_{1\,3\,7}.$$

2. If $I = \{3, 7\}$ and $J = \{4, 5, 6, 7\}$, then

$$p_{367}p_{457} = p_{357}p_{467} - p_{347}.$$

3. If $I = \{3, 7\}$ and $J = \{2, 3, 4, 5\}$, then

$$p_{237} = p_{347}p_{235} - p_{357}.$$

4. If $I = \{3, 4\}$ and $J = \{2, 3, 6, 7\}$, then

$$p_{367} = p_{347}p_{236} - p_{346}p_{237}.$$

5. If $I = \{5, 7\}$ and $J = \{1, 3, 4, 5\}$, then

$$p_{157} = p_{357}p_{145} - p_{457}p_{135}.$$

6. If $I = \{1, 2\}$ and $J = \{2, 3, 4, 7\}$, then

$$p_{247} = p_{124}p_{237} - p_{127}.$$

7. If $I = \{2, 6\}$ and $J = \{3, 4, 5, 7\}$, then

$$p_{267} = -p_{236}p_{457} + p_{246}p_{357} - p_{256}p_{347}.$$

8. If $I = \{2, 6\}$ and $J = \{2, 4, 6, 7\}$, then

$$p_{267}p_{246} = p_{246}p_{247}.$$

9. If $I = \{4, 6\}$ and $J = \{2, 3, 4, 8\}$, then

$$p_{468} = p_{246}p_{348} - p_{346}p_{248}.$$

10. If $I = \{2, 4\}$ and $J = \{1, 2, 6, 8\}$, then

$$p_{124}p_{268} = p_{248}p_{126} - p_{246}.$$

For the following proposition, we examine Plücker coordinates of certain types in the case $\mathrm{Gr}(3, 8)$. We will use this result later to make claims about the corresponding coordinates in the case $\mathrm{Gr}(5, 8)$.

Proposition 4.4.6. Consider $\mathrm{Gr}(3, 8)$. Let consecutive Plücker coordinates be 1 and almost consecutive Plücker coordinates be positive. Let S be the collection of all Plücker coordinates of the form $p_{o(m\,m+1\,m+5)}$ and $p_{o(m+1\,m+2\,m+5)}$ where $m \in \mathbb{N}$ and R be the collection of all Plücker coordinates of the form $p_{o(m\,m+1\,m+3)}$ and $p_{o(m\,m+2\,m+3)}$ where $m \in \mathbb{N}$. Suppose all elements of R are positive and that there exists a non-positive element in S. Then all elements of R are 1 and all elements of S are 0.

Proof. We may, without loss of generality, pick a particular Plücker coordinate of each form from the collection R since all equations hold after a shift in indices. We set $m = 2$ and take the first case where $p_{2\,3\,7} \leq 0$. For the following result, we refer to the list of equations from Lemma 4.4.5.

Consider Relation (1). By assumption, $p_{2\,3\,7} \leq 0$ and $p_{1\,3\,4} > 0$. By Lemma 4.4.3, $p_{1\,3\,7} \geq 0$. Thus, $p_{3\,4\,7} \leq 0$. Consider Relation (2). By assumption, $p_{4\,5\,7}, p_{4\,6\,7} \geq 0$. By Lemma 4.4.3, $p_{3\,5\,7} \geq 0$. By the above, $p_{3\,4\,7} \leq 0$. Thus, $p_{3\,6\,7} \geq 0$. Note that $p_{3\,6\,7}$ is of the form $p_{o(n\,n+1\,n+5)}$, so we may apply Proposition 4.4.4. Thus, $p_{2\,3\,7} \geq 0$ and $p_{3\,4\,7} \geq 0$. Combining with above results, we conclude $p_{2\,3\,7} = p_{3\,4\,7} = 0$ as desired.

We now consider the second case where $p_{3\,4\,7} \leq 0$. Consider Relation (3). By Lemma 4.4.3, $p_{3\,5\,7} \geq 0$. Thus, $p_{2\,3\,7} \leq 0$. As above, Relation (2) and Proposition 4.4.4 give us the same result. Since each assumption leads to $p_{2\,3\,7} = p_{3\,4\,7} = 0$, the remainder of the proof is identical regardless of starting case.

We now proceed to simplify the remaining relations of Lemma 4.4.5 by replacing Plücker coordinates with known values. Relation (1) gives $p_{1\,3\,7} = 0$, hence $p_{1\,3\,8} = p_{1\,2\,7} = 1$ by Lemma 4.4.3. Relation (4) gives $p_{3\,6\,7} = 0$. Relation (2) gives $p_{3\,5\,7} = 0$, hence $p_{3\,5\,6} = p_{4\,5\,7} = 1$ by Lemma 4.4.3. Relation (5) gives $p_{1\,5\,7} = -p_{4\,5\,7}p_{1\,3\,5}$. Since all terms are non-negative by Lemma 4.4.3, this implies $p_{1\,5\,7} = p_{1\,3\,5} = 0$. Then $p_{5\,7\,8} = p_{1\,6\,7} = 1$ and $p_{1\,3\,4} = p_{2\,3\,5} = 1$ by Lemma 4.4.3. Relation (6) gives $p_{2\,4\,7} = -1$. Relation (7) gives $p_{2\,6\,7} = -p_{2\,3\,6}$. By Proposition 4.4.4, both are non-negative, so $p_{2\,6\,7} = p_{2\,3\,6} = 0$. Relation (8) gives $p_{2\,4\,6} = 0$, hence $p_{2\,4\,5} = p_{3\,4\,6} = 1$ by Lemma 4.4.3. Relation (9) gives $p_{4\,6\,8} = -p_{3\,4\,6}p_{2\,4\,8}$. Since all terms are non-negative by Lemma 4.4.3, and $p_{3\,4\,6} = 1$, this implies $p_{4\,6\,8} = p_{2\,4\,8} = 0$, hence $p_{4\,6\,7} = p_{5\,6\,8} = 1$ and $p_{2\,3\,8} = p_{1\,2\,4} = 1$ by Lemma 4.4.3. Relation (10) gives $p_{2\,6\,8} = 0$, hence, by Lemma 4.4.3, $p_{1\,6\,8} = p_{2\,7\,8} = 1$. We may now use the two claims used in the proof of Proposition 4.4.4 to demonstrate that all Plücker coordinates of the form $p_{o(n\,n+1\,n+5)}$ and $p_{o(n\,n+1\,n+4)}$ are 0. □

This leads us to a statement concerning the case of $(5, 8)$.

Proposition 4.4.7. Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a positive frieze F where $n = 8$ and $k = 5$. Then the quiddity sequence is positive.

Proof. Recall from Remark 4.4.2 that there is an isomorphism between Gr(3, 8) and Gr(5, 8). Consider the collections R and S from Proposition 4.4.6. Performing the complement operation p_{I^c} on elements of R yields all almost consecutive Plücker coordinates in Gr(5, 8) and performing it on elements of S yields all semi-consecutive Plücker coordinates. Since $\mathcal{M}_F$ is a tiling from a positive frieze, this implies that all entries in R are positive. Thus, by Proposition 4.4.6, any tiling without a strictly positive quiddity sequence comes from the frieze with only trivial entries and whose quiddity sequence is the sequence of zeros. □

4.5 Gale Dual

Using the Gale dual, we are able to extend our notions of positivity to even more cases. We first make an observation about the entries in the Gale dual. The entries of the Gale dual are consecutive minors of Plücker coordinates. So, by Lemma 2.2.2, semi-consecutive Plücker coordinates.

Additionally, we refer to the following result from Ovsienko.

Theorem 4.5.1. [15] Let $k = 2$ and $n \leq 9$. Then the Gale dual restricts to a bijection on positive friezes.

We can now state a partial positivity result for SL_k-tilings from friezes.

Theorem 4.5.2. Let $\mathcal{M}_F \in \mathbb{SL}_k$ be a tiling from a frieze F of type (k, n) satisfying one of the following conditions:

1. $k = 2$ and $n \leq 9$
2. $k = 3, 6$ and $n \leq 8$
3. $k = 4$ and $n \leq 7$
4. $k = 5$ and $n \leq 7$
5. $k = 5$ and $n = 8$ with the exception of the frieze of all ones and the quiddity vectors are all $(1, 0, 0, 0, 1)^T$

Then $\mathcal{M}_F$ is positive if and only if the quiddity sequence is positive.

Proof. We break this into cases.

Case (1): $k = 2$ The forward direction follows from Lemma 4.3.5. We show the backward direction. Suppose that the semi-consecutive Plücker coordinates are all positive. Then $\mathcal{M}_F^{\mathcal{G}}$ is a positive tiling. By Theorem 4.5.1, this means $\mathcal{M}_F$ is as well.

Case (2): $k = 4$ The forward direction follows from Lemma 4.4.1. The backward direction follows from Lemma 4.3.4.

Case (3): $k = 5$ The backward direction follows from Lemma 4.3.4. The forward direction follows from Proposition 4.4.7 when $n = 8$. For $n \leq 7$, suppose that the tiling is positive. Then, by Theorem 4.5.1, $\mathcal{M}_F^{\mathcal{G}}$ is a positive tiling where $k = 2$. This means that semi-consecutive Plücker coordinates of $\mathcal{M}_F$ are also positive.

Case (4): $k = 3$ The forward direction follows from Lemma 4.3.5. We prove the backward direction. For $n \leq 6$, this follows from Lemma 4.3.4. For $n = 7, 8$, suppose that the semi-consecutive Plücker coordinates are all positive. Then $\mathcal{M}_F^{\mathcal{G}}$ is a positive tiling where $n = 7$ and $k = 4$ or $n = 8$ and $k = 5$. By Cases (2) and (3), respectively, the resulting tilings have positive semi-consecutive Plücker coordinates with a single exception, hence their own Gale duals are positive. Thus, the initial tiling $\mathcal{M}_F$ is also positive. For the exception in Case (3), note that the Gale dual of the all ones tiling in $\mathrm{Gr}(3, 8)$ is not a positive tiling in $\mathrm{Gr}(5, 8)$ as the consecutive Plücker coordinates of the form $p_{o(n\, n+1\, n+5)} = p_{o(n\, n+1\, n+4)} = 0$. Thus, this case does not come into play here.

Case (5): $k = 6$ The backward direction follows from Lemma 4.3.4. We prove the forward direction. Suppose that the tiling is positive. Then $\mathcal{M}_F^{\mathcal{G}}$ is a positive tiling where $k = 2$. This means that the semi-consecutive Plücker coordinates of $\mathcal{M}_F$ are also positive.

□

As an application, we make a connection to the following conjecture by Cuntz.

Conjecture 4.5.3. [8, Conjecture 2.1] There are 26952 positive friezes with $k = 3$ and $n = 8$.

By Theorem 4.5.2, these friezes are in bijection with the positive friezes where $k = 5$ and $n = 8$ with a single exception in the $(5, 8)$ case not appearing in the $(3, 8)$ case. Thus, Conjecture 4.5.3 together with Theorem 4.5.2 imply that there are conjecturally 26951 positive friezes with $k = 5$ and $n = 8$.

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