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Adams operations on the Burnside ring from power operations

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Adams operations on the Burnside ring from power operations

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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2024

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ABSTRACT OF DISSERTATION

Adams operations on the Burnside ring from power operations

Topology furnishes us with many commutative rings associated to finite groups. These include the complex representation ring, the Burnside ring, and the G-equivariant K-theory of a space. Often, these admit additional structure in the form of natural operations on the ring, such as power operations, symmetric powers, and Adams operations. We will discuss two ways of constructing Adams operations. The goal of this work is to understand these in the case of the Burnside ring.

KEYWORDS: Burnside Ring, Power Operations, Adams Operations

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April 26, 2024

Adams operations on the Burnside ring from power operations

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To my closest friends, family, supporters, and the residents of POT 702.
I couldn't have done it without you.

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Chapter 1 Introduction

1.1 Setting and Motivation

Let G be a finite group and let $A(G)$ be the Burnside ring of G. This is the Grothendieck ring of isomorphism classes of finite G-sets and is a $\mathbb{Z}$-module of rank equal to the number of isomorphism classes of transitive G-sets. The power operation is a multiplicative map

$$P_n : A(G) \to A(G \times \Sigma_n)$$

induced by taking a G-set X to the n-fold Cartesian product $X^{\times n}$ where Σ_n reorders components and G acts diagonally. The Burnside ring is a pre-λ-ring, that is to say it has a notion of symmetric powers satisfying certain identities. This provides sufficient data to construct an Adams operation, which is an additive map

$$\psi_n : A(G) \to A(G).$$

Let $R(G)$ be the complex representation ring of G. As with the Burnside ring, $R(G)$ has a power operation that is a multiplicative map induced by taking a G-representation V to the n-fold tensor power $V^{\otimes n}$ with the usual $G \times \Sigma_n$-action. The representation ring is also a pre-λ-ring with symmetric powers that give rise to an Adams operation that is a ring map. However, these two operations are closely related in the case of the representation ring, as the n^{th} Adams operation on $R(G)$ factors through the power operation:

$$\begin{tikzcd} R(G) \arrow[r, "P_n"] \arrow[rrr, bend right, "\psi_n"'] & R(G \times \Sigma_n) \arrow[r, "\cong"] & R(G) \otimes_{\mathbb{Z}} R(\Sigma_n) \arrow[r, "\mathrm{id} \otimes ev"] & R(G). \end{tikzcd}$$

Such a factorization has appeared in other settings as well. For instance, Ando's construction for Adams operations on Morava E-theory are defined through the power operation.

1.2 Desirable Conditions for a factorization

Our goal is to produce a similar factorization for Burnside rings. However, to ensure such a factorization has content and more closely mimics other settings, we place various restrictions on it.

Definition 1.2.1. We denote $\alpha_{G,n} : A(G \times \Sigma_n) \to A(G)$ any map that factors ψ_n through the power operation: $\alpha_{G,n} \circ P_n = \psi_n$. We will write α_n when G is clear from context. Further, it is desirable that $\alpha_{G,n}$ satisfies the following conditions:

1. $\alpha_{G,n}$ is additive and factors through the transfer ideal.

2. $\alpha_{G,n}$ is natural with respect to restriction, i.e for $H \leq G$, we have $\mathrm{Res}^G_H \circ \alpha_{G,n} = \alpha_{H,n} \circ \mathrm{Res}^G_H$.

3. The following diagram commutes, where $\alpha_{e,n}$ is considered to be a map from $A(\Sigma_n) \cong A(e \times \Sigma_n)$ to $A(e) \cong \mathbb{Z}$.

$$\begin{tikzcd}
A(G) \otimes A(\Sigma_n) \arrow[r, "1\otimes\alpha_{e,n}"] \arrow[d] & A(G) \otimes \mathbb{Z} \arrow[d, "\cong"] \\
A(G \times \Sigma_n) \arrow[r, "\alpha_{G,n}"] & A(G)
\end{tikzcd}$$

Note that $\alpha_{G,n}$ is not assumed to be unique, and in general, may fail to be unique. In particular, transitively stabilized basis elements which do not appear in the image of the power operation may be mapped freely, though conditions (2) and (3) place restrictions on this.

1.3 Current Results

One may ask if the existence of an α_n satisfying these conditions is guaranteed for all groups and power operations, but this is not true. In particularly, we construct an explicit counterexample which shows it is not possible to construct an $\alpha_{\Sigma_4,3}$ satisfying conditions (1) and (3).

There are large classes of well behaved groups which may provide refuge from the complications occurring for Σ_4, the first of which we consider is the setting of abelian groups. Abelian groups provide a clear description for the decomposition of $(G/H)^n$ as a $G \times \Sigma_n$-set understood through the work of Bonventre, Guillou, and Stapleton. Furthermore, a corollary to the work of Gay, Morris, and Morris simplifies the computation of the Adams operation, greatly reducing the work necessary to show a proposed α_n factors ψ_n.

Theorem 1.3.1. *For abelian groups and* $n \geq 1$, *there exists an additive map* α_n *such that* $\alpha_n \circ P_n = \psi_n$ *satisfying all of our desired properties when* $n = p$ *is a prime.*

In addition to abelian groups, for p prime, we may consider the setting of p-groups and the p^{th} power and Adams operation. These groups provide significant differences in the structure of $(G/H)^p$ as a $G \times \Sigma_p$-set, however, the computation of the Adams operations remains simple compared to the more general setting.

Theorem 1.3.2. *For* p *prime and* G *a* p*-group, there exists an additive map* α_p *such that* $\alpha_p \circ P_p = \psi_p$ *satisfying properties* (1) *and* (3)

Despite the fact a factorization satisfying some of the conditions can be described, restriction is more difficult to understand when G is not assumed to be abelian. As such, proving a constructed α_p is natural with respect to restriction remains unclear.

Chapter 2 Background

We begin with an overview of the Burnside ring, the ring of marks, the representation ring and its relations to the Burnside ring, as well as operations associated to these rings. This material can be found in a number of sources, such as Tom Dieck's Transformation Groups[11, Chapter 1], Bouc's Handbook of Algebra article [4], and Boorman's S-operations in Representation Theory [3, Chapter 1].

2.1 The Burnside Ring

Fix a finite group G. First, we consider the groupoid of finite G-sets, which we will denote G-set. This category admits two symmetric monoidal structures, one is given by disjoint union, and the other by Cartesian product.

Definition 2.1.1. The Burnside semiring, denoted $A^+(G)$, is the commutative semiring structure on the set of isomorphism classes of finite G-sets in which addition is induced by disjoint union and multiplication induced by Cartesian product. The Burnside ring $A(G)$ is the Grothendieck completion of $A^+(G)$ with respect to disjoint union.

Remark 2.1.2. The Grothendieck completion of $A^+(G)$ may be constructed by considering formal differences of isomorphism classes of G-sets of the form $[X]-[Y]$, and then setting $[X]-[Y]$ equal to $[A]-[B]$ when $[X]+[B]=[A]+[Y]$ in $A^+(G)$. We refer to such formal differences of isomorphism classes of G-sets as 'virtual' G-sets.

Additionally, it should be noted that the concept of a G-action is not well-defined on virtual G-sets, as there is not a canonical method to pick a representative for the classes under Grothendieck completion. However, when considering an additive map out of $A(G)$, it is sufficient to understand its image on elements of the form $[G/H] \in A^+(G)$.

Example 2.1.3. Consider when $G = e$ is the trivial group. The isomorphism classes of finite e-sets are given by cardinality. As each finite set is a disjoint union of singletons, we see that $A^+(e) \cong \mathbb{N}$. The Grothendieck completion of $\mathbb{N}$ as a semiring gives $A(e) \cong \mathbb{Z}$.

As each finite G-set decomposes into a disjoint union of transitive G-sets and the Burnside ring is built out of the groupoid of finite G-sets, it suffices to understand the additive and multiplicative operations on finite transitive G-sets. These are in bijection with conjugacy classes of subgroups of G, and so there is a canonical additive basis for the Burnside ring, giving the following decomposition as an abelian group. Boorman provides elementary proofs for this decomposition [3, Chapter 1]. We define $\mathrm{SubConj}(G)$ to be the set of conjugacy classes of subgroups of G. There is an isomorphism of abelian groups

$$A(G) \cong \bigoplus_{[H]\in \mathrm{SubConj}(G)} \mathbb{Z}\{[G/H]\}.$$

Furthermore, as these elements are a basis for the additive structure, the multiplicative structure is determined by the pairwise products. Since every G-set decomposes into a collection of transitive orbits, there is necessarily a decomposition for the product of two transitive G-sets into basis elements. Let H and K be two subgroups of G, then the product of G-sets $G/H \times G/K$ decomposes as follows:

$$G/H \times G/K \cong \coprod_{HgK \in H\backslash G/K} G/(H \cap gKg^{-1}).$$

Remark 2.1.4. When $G = K$ in the double coset formula, we see G/G is the multiplicative unit of $A(G)$, as there is only one double coset for HgG and the intersection gives H, so $G/H \times G/G \cong G/H$. For this reason, we will often refer to G/G as 1 or $*$.

Example 2.1.5. Consider $G = C_p$, for p prime. In this case, there are two transitive G-sets, $1 = C_p/C_p$ and C_p/e. Using the canonical basis, we see $A(C_p) \cong \mathbb{Z}\{1, w\}$ as an abelian group, where $w := C_p/e$. The multiplicative structure must still be determined, but the only calculation required is $C_p/e \times C_p/e = w^2$, and as w is a free G-set of size p, one can see $w^2 = pw$. We then quotient by this relation to see the following presentation:

$$A(C_p) \cong \mathbb{Z}[w]/\langle w^2 - pw\rangle.$$

Observe that $A(G)$ can always be computed as a quotient of a polynomial ring on $|\,\mathrm{SubConj}(G)| - 1$ many variables representing the transitive G-sets G/H for each $H \neq G$. In general, this will require the calculation of $\binom{|\,\mathrm{SubConj}(G)|}{2}$ relations, though products involving G/G or G/e are simple to compute.

Example 2.1.6. For $G = \Sigma_3$, the symmetric group on three elements, we omit the calculations involving the double coset formula, but give the following presentation for $A(\Sigma_3)$, where $x := \Sigma_3/A_3, y := \Sigma_3/C_2, z := \Sigma_3/e$:

$$A(\Sigma_3) \cong \mathbb{Z}[x, y, z]/\langle x^2 = 2x, y^2 = y + z, z^2 = 6z, xy = z, xz = 2z, yz = 3z\rangle,$$

Now, given a G-set, we can define an H-set for $H \leq G$ by restricting the action, which induces a map of rings. More generally, this can be defined along any group homomorphism.

Definition 2.1.7. For any group homomorphism $\varphi : H \to G$, there is an induced map of commutative rings

$$\mathrm{Res}^G_H : A(G) \xrightarrow{\qquad\qquad} A(H)$$

$$X \longmapsto \mathrm{Res}^G_H = X \text{ with } hx = \varphi(h)x,$$

referred to as a restriction map.

As it relies on precomposition with the action map, restriction is contravariant. We may ask for a covariant map, this is the transfer. It will not result in a ring map. However, it will produce an additive map.

Definition 2.1.8. For any group homomorphism $\varphi : H \to G$, there is an induced map of abelian groups

$$\mathrm{Tr}_H^G : A(H) \xrightarrow{\hspace{6cm}} A(G)$$

$$X \longmapsto G \times_H X := G \times X / \langle (g\varphi(h), x) \sim (g, hx) \rangle,$$

referred to as either the induction or transfer map.

Note that restrictions and transfers are defined along all group homomorphisms. When clarification is necessary, we may use the notation φ^* or Res_φ for restriction along φ and φ_* or Tr_φ for the transfer along φ. Furthermore, the restriction of a transfer has a particularly nice decomposition.

Definition 2.1.9. Fix $H \leq G$, then for $g \in G$, there is an isomorphism $c_g : A(H) \to A(gHg^{-1})$ induced by conjugation. This follows from the fact $A(-)$ is a functor, hence it preserves isomorphisms.

Proposition 2.1.10. *[11, 6.1.7](Double Coset Formula) For subgroups $H, K \leq G$, we have the following relation:*

$$\mathrm{Res}_H^G \mathrm{Tr}_H^G = \bigoplus_{KgH \in K \backslash G / H} \mathrm{Tr}_{K \cap gHg^{-1}}^K c_g \mathrm{Res}_H^{gKg^{-1}} \cap H.$$

Example 2.1.11. Let $G = \Sigma_3$ and $H = C_3$ with $\varphi : H \hookrightarrow G$ given by the inclusion of the three cycle subgroup. As we have the descriptions of both Burnside rings as polynomial rings, we need only determine the images of the additive generators. The map is then guaranteed to respect the appropriate quotients, and furthermore, as Res is a ring map, 1 will go to 1. Note that this is not true for the transfer map, as it is not multiplicative. Again, we omit the calculation, but include the final result for each map below.

Table 2.1: Restriction and Transfer between Σ_3 and C_3

$A(\Sigma_3)$	$\mathrm{Res}_{C_3}^{\Sigma_3}$	$A(C_3)$	$\mathrm{Tr}_{C_3}^{\Sigma_3}$
x	2	1	x
y	w	w	$3z$
z	$2w$		

Now, it would be helpful to understand elements of the Burnside ring for groups with particular structure. While the abelian group structure of $A(G)$ is easily classified, it will be helpful to simplify the data associated to subgroups of products. For this reason, we will review Goursat's Lemma.

2.1.1 Goursat's Lemma

As finite transitive G-sets are in correspondence with the conjugacy classes of subgroups, it would be helpful to understand the subgroup structure of a product group. However, given groups G, G', it is important to note that the subgroups of $G \times G'$ need not be of the form $H \times K$ for $H \leq G$ and $K \leq G'$. The diagonal subgroup of $C_2 \times C_2$ provides a simple example of a subgroup that is not a product of two subgroups. However, Goursat's lemma provides a way to understand the subgroups of $G \times G'$ in terms of subgroups in G and G' equipped with extra data. Let $H \leq G \times G'$ be a subgroup of the product, then we can construct the following information from H:

$$\begin{tikzcd}
& \ker(p_1|_H) \arrow[d] \arrow[dr, hook] & \\
\ker(p_2|_H) \arrow[r] \arrow[dr, hook] & H \leq G \times G' \arrow[r, "p_2"] \arrow[d, "p_1"] & \operatorname{im}(p_2|_H) \leq G' \\
& \operatorname{im}(p_1|_H) \leq G &
\end{tikzcd}$$

Notice that we can consider $\ker(p_1|_H)$ as a subgroup of G', and more specifically $\operatorname{im}(p_2|_H)$, as all elements are of the form (e, h') with $h' \in \operatorname{im}(p_2|_H)$. Similarly, $\ker(p_2|_H)$ may be viewed as a subgroup of $\operatorname{im}(p_1|_H)$. Even further, these are normal subgroups, as conjugation preserves the form (e, h') or (h, e) respectively.

Lemma 2.1.12. *[8] Let G, G' be groups, and let H be a subgroup of $G \times G'$ with projections $p_1 : G \times G' \to G$ and $p_2 : G \times G' \to G'$. Then we have an isomorphism of the quotients:*

$$\operatorname{im}(p_1|_H)/\ker(p_2|_H) \cong \operatorname{im}(p_2|_H)/\ker(p_1|_H)$$

This allows us to construct a bijection between subgroups $H \leq G \times G'$ and triples of the form (N, N', f) with N normal in G, N' normal in G' and f an isomorphism of G/N onto G'/N'.

While this will allow us to extract data, the multiplicative structure of $A(G)$ can be quite complex. We will make use of the ring of marks to better understand the Burnside ring.

2.1.2 The ring of marks

The ring of marks is a commutative ring which will simplify some of our necessary calculations. We will review some basic properties of it before also considering the representation ring, ring of class functions, and how all of these relate.

Definition 2.1.13. The ring of marks, denoted $\mathrm{Marks}(G)$, can be defined as $\prod_{[H]} \mathbb{Z}\{[H]\}$, or equivalently as $\mathrm{Fun}(\mathrm{SubConj}(G), \mathbb{Z})$. There is a canonical map of commutative rings denoted χ from $A(G)$ to $\mathrm{Marks}(G)$, often referred to as the marks homomorphism or character map:

$$\chi : A(G) \longrightarrow \mathrm{Marks}(G)$$

$$X \longmapsto ([H] \mapsto |X^H|).$$

Example 2.1.14. Assume $G = C_p$. As $A(C_p) \cong \mathbb{Z}[w]/\langle w^2 - pw\rangle$ and χ is a commutative ring homomorphism, the map is determined by the value $\chi(w)$. If we write $\mathrm{Marks}(C_p) \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{C_p\}$, then the image of w will be $(p, 0)$, as C_p/e has p e-fixed points and zero C_p-fixed points. Hence, the image is generated by $(1, 1)$ and $(p, 0)$. Note that this is not a surjection, and in almost all cases, χ will not be a surjection.

Proposition 2.1.15. *[6, Theorem 1, Page 238] The marks homomorphism* $\chi : A(G) \to \mathrm{Marks}(G)$ *is an injection, and as they are the same rank as* $\mathbb{Z}$*-modules,* $\chi \otimes \mathbb{Q}$ *is an isomorphism.*

Remark 2.1.16. If we consider the bases for $A(G)$ and $\mathrm{Marks}(G)$, the bijection between transitive G-sets and conjugacy classes of subgroups shows these are free $\mathbb{Z}$-modules of the same rank. Thus, we see that if χ is an injection, it is immediately a rational isomorphism. The proposition can be proven by partially ordering the basis by subconjugacy and writing the associated matrix in increasing rank, giving a triangular matrix with no zeroes on the diagonal (as $\chi(G/H)[H]$ is always non-zero).

Due to its ring structure as a finite product of the integers, it is often simpler to do computation on $A(G)$ within $\mathrm{Marks}(G)$ and then lift along the section to determine what the value on G-sets is. As χ is rationally an isomorphism, the section can be calculated by inverting the matrix calculated in the remark. Additionally, there are relations to determine when a function in $\mathrm{Marks}(G)$ is within the image of the Burnside ring.

Remark 2.1.17. As $\mathrm{Marks}(G)$ is the ring of $\mathbb{Z}$-valued functions on $\mathrm{SubConj}(G)$, one can replace $\mathbb{Z}$ by another commutative ring, commonly denoted $\mathrm{Marks}(G, R) := \mathrm{Fun}(\mathrm{Conj}(G), R)$ when relevant. Following this, χ is still defined as above, making use of the fact $\mathbb{Z}$ is the initial commutative ring.

2.2 The representation ring and its relationship to the Burnside ring

As with the Burnside ring, we first consider the groupoid of finite dimensional complex G-representations, which we will denote GRep. Recall that a G-representation is a complex vector space equipped with a G-action compatible with scaling and addition. This category admits two compatible symmetric monoidal structures, notably direct sum and tensor product.

Definition 2.2.1. The complex representation semiring, denoted $R^+(G)$, is the commutative semiring structure on the set of isomorphism classes of finite dimensional complex G-representations in which addition is induced by direct sum and multiplication induced by tensor product. The complex representation ring $R(G)$, which we will further refer to as the representation ring, is the Grothendieck completion of $R^+(G)$ with respect to direct sum.

Remark 2.2.2. As before, the Grothendieck completion may be constructed using formal differences of isomorphism classes of representations, which we will refer to as virtual representations. Similarly, if a representation V is referred to as non-virtual, it is an element of $R^+(G)$ and representable by the class $[V] - [0]$.

Definition 2.2.3. A subrepresentation of a representation V is a subspace which is stabilized by the G-action. By stabilized, we mean that if ρ is our representation, a subspace U is stabilized if $\rho(g)(U) = U$ for every $g \in G$. Similarly, a complex G-representation is said to be irreducible if it contains no proper non-trivial subrepresentation. The representation ring $R(G)$ has a canonical basis given by isomorphism classes of irreducible representations, of which there are always finitely many.

Example 2.2.4. Consider when $G = e$ is the trivial group. Similar to G-sets, the irreducible representations of e are the 0-dimensional vector space and a fixed 1-dimensional vector space. As any higher dimensional representation splits as a sum of 1-dimensional representations, we see that $R^+(e) \cong \mathbb{N}$ and $R(e) \cong \mathbb{Z}$.

As with the G-set G/G in the Burnside ring, we may always consider a 1-dimensional representation with trivial G-action. This acts as the multiplicative unit of the ring $R(G)$, and as such, we may refer to it as 1.

Example 2.2.5. For $G = C_3$, there are two non-trivial 1-dimensional representations of C_3, which can both be given by having a generator act as multiplication by a 3rd root of unity. These two representations will both be squares of one another, giving us the following presentation:

$$R(C_3) \cong \mathbb{Z}[x]/\langle x^3 - 1\rangle.$$

As with the Burnside ring, we may make use of the interaction between group homomorphisms and action maps to obtain maps between representation rings.

Definition 2.2.6. For any group homomorphism $\varphi : H \to G$, there is an induced map of commutative rings, referred to as a restriction map:

$$\mathrm{Res}_H^G : R(G) \longrightarrow R(H)$$

$$V \longmapsto \mathrm{Res}_H^G(V) = V \text{ with } hv = \varphi(h)v.$$

Definition 2.2.7. The 'regular' representation of G is denoted $\mathbb{C}\{G\}$ and is the complex vector space with basis given by elements of G, using the action of G on itself by multiplication. It always contains at least one copy of every irreducible representation of G, notably showing that the number of irreducibles is at most the order of the group.

Definition 2.2.8. As with the Burnside ring, there is an induced map of abelian groups for any group homomorphism $\varphi : H \to G$, referred to as an induction or transfer map:

$$\mathrm{Tr}_H^G : R(H) \longrightarrow R(G)$$

$$V \longmapsto \mathbb{C}\{G\} \otimes_H V := \mathbb{C}\{G\} \otimes V / \langle (g\varphi(h), v) \sim (g, hv) \rangle.$$

Note that as before, restriction and transfer are defined along all group homomorphisms.

Definition 2.2.9. There is a map of commutative rings from the Burnside ring to the representation ring, which we will refer to with L for the 'linearization' of the G-set:

$$L : A(G) \longrightarrow R(G)$$

$$X \longmapsto \mathbb{C}\{X\}.$$

Example 2.2.10. Consider $G = C_3$, then $A(C_3) \cong \mathbb{Z}[w]/\langle w^2 - 3w \rangle$ and L depends only on the image of w. As $L(w)$ is given by G acting on $\mathbb{C}\{G/e\}$, we will obtain the regular representation, given by the element $x^2 + x + 1$. Thus, the image is generated by $\langle 1, x^2 + x + 1 \rangle$, which notably gives that L is not surjective.

Remark 2.2.11. The linearization map L is not necessarily injective or surjective, but its image will always contain 1 and the regular representation, as it is a map of commutative rings and $\mathbb{C}\{G/e\}$ is the regular representation.

Definition 2.2.12. Let $\mathrm{Cl}(G) := \mathrm{Fun}(\mathrm{Conj}(G), \mathbb{C})$ be the ring of class functions on G, where $\mathrm{Conj}(G)$ is the collection of conjugacy classes of elements of G. Then χ, the character map, is defined as follows, as well as ι, which is induced by subgroup inclusion.

$$\chi : R(G) \longrightarrow \mathrm{Cl}(G) \qquad\qquad \iota : \mathrm{Marks}(G) \longrightarrow \mathrm{Cl}(G)$$

$$G \curvearrowright V \longmapsto f : [g] \to \mathrm{Tr}(g) \qquad\qquad f \longmapsto \iota(f)([g]) = f(\langle g \rangle)$$

While χ is injective, ι need not be.

We denote both the character map and the marks homomorphism with χ as they are both defined by considering fixed point data. We will subscript them with the domain if clarification is necessary.

Proposition 2.2.13. *The maps L, χ, ι form the following commutative diagram.*

$$\begin{tikzcd}
& A(G) \arrow[dl, "L"'] \arrow[dr, hook, "\chi"] & \\
R(G) \arrow[dr, hook, "\chi"] & & \mathrm{Marks}(G) \arrow[dl, "\iota"'] \\
& \mathrm{Cl}(G) &
\end{tikzcd}$$

While we omit the full proof, the result relies on the fact that $\dim(\mathbb{C}\{X\}^{\langle g \rangle}) = |X^{\langle g \rangle}|$ *for non-virtual G-sets. As all of these objects are free $\mathbb{Z}$-modules and the maps are commutative ring homomorphisms, it is sufficient to see it commutes on basis elements.*

2.3 Power Operations

Now, we move onto operations which are defined for all of the above rings, the first of which is the power operations. These occur commonly are induced by maps of G-sets and G-representations for the Burnside ring and representation ring. As before, these are not necessarily a ring maps. Rymer provides further details on the power operations for the Burnside ring [10, Section 2].

2.3.1 Power Operations

Definition 2.3.1. There is a multiplicative map $P_n : A^+(G) \to A^+(G \times \Sigma_n)$ called the power operation,

$$P_n : A^+(G) \longrightarrow A^+(G \times \Sigma_n)$$

$$X \longmapsto X^{\times n},$$

where an element $(g, \sigma) \in G \times \Sigma_n$ acts on a tuple of the form $(x_i) \in X^{\times n}$ by $(g, \sigma)(x_i) = (gx_{\sigma(i)})$.

Given two G-sets X and Y, the power operation interacts with addition as follows, where the transfer is along the inclusion map. This will be referred to as the coproduct formula when relevant.

$$P_n(X+Y) \cong \sum_{i+j=n} \mathrm{Tr}_{G\times\Sigma_i\times\Sigma_j}^{G\times\Sigma_n}(X^i \times Y^j)$$

Definition 2.3.2. For representation rings, the power operation is defined analogously as $P_n : R^+(G) \to R^+(G\times\Sigma_n)$,

$$\begin{tikzcd} P_n : R^+(G) \arrow[r] & R^+(G\times\Sigma_n) \\ V \arrow[r, mapsto] & V^{\otimes n}, \end{tikzcd}$$

where an element $(g,\sigma) \in G\times\Sigma_n$ acts on a tuple of the form $(v_i) \in V^{\otimes n}$ by $(g,\sigma)(v_i) = (gv_{\sigma(i)})$.

Similarly, given two G-representations V and W, the power operation interacts with addition as follows:

$$P_n(V+W) \cong \sum_{i+j=n} \mathrm{Tr}_{G\times\Sigma_i\times\Sigma_j}^{G\times\Sigma_n}(V^{\otimes i} \times W^{\otimes j}),$$

where the transfer is along the inclusion map.

Remark 2.3.3. We can adjust the additive formula to extend these operations to virtual G-sets and virtual G-representations through a small modification. If A and B are G-sets or G-representations, then the power operation on virtual elements is given by the following formula

$$P_n(A-B) \cong \sum_{i+j=n} \mathrm{Tr}_{G\times\Sigma_i\times\Sigma_j}^{G\times\Sigma_n}(P_i(A) \times P_j(-B)),$$

where the transfer is along inclusion as before.

It remains to give a formula for $P_n(-B)$. We can make use of the fact that $P_n(0) = 0$, as well as $\mathrm{Tr}_{G\times\Sigma_n}^{G\times\Sigma_n} = \mathrm{id}_{A(G\times\Sigma_n)}$. By separating terms and solving, we obtain an inductive formula for $P_n(-B)$:

$$0 = P_n(B-B) \cong \sum_{i+j=n} \mathrm{Tr}_{G\times\Sigma_i\times\Sigma_j}^{G\times\Sigma_n}(P_i(B) \times P_j(-B))$$

$$P_n(-B) \cong -P_n(B) - \sum_{\substack{i+j=n \\ i,j\neq n}} \mathrm{Tr}_{G\times\Sigma_i\times\Sigma_j}^{G\times\Sigma_n}(P_i(B) \times P_j(-B)).$$

As this allows us to calculate the power operation on a formal sum or difference, we may extend these operations to $A(G)$ and $R(G)$ respectively.

2.3.2 The Transfer Ideal and Restrictions

While the power operation is not additive, we may quotient by a certain ideal to make a composite which is. In this case, we should note that transfer of the form $\mathrm{Tr}^{G\times\Sigma_n}_{G\times\Sigma_n\times\Sigma_0}$ and $\mathrm{Tr}^{G\times\Sigma_n}_{G\times\Sigma_0\times\Sigma_n}$ may be identified with the identity map. Consider the following expansion of our sum for power operations:

$$P_n(A-B) \cong \sum_{i+j=n} \mathrm{Tr}^{G\times\Sigma_n}_{G\times\Sigma_i\times\Sigma_j}(P_i(A)\times P_j(-B))$$

$$P_n(A-B) \cong P_n(A) + P_n(B) + \sum_{\substack{i+j=n\\ i,j\neq 0}} \mathrm{Tr}^{G\times\Sigma_n}_{G\times\Sigma_i\times\Sigma_j}(P_i(A)\times P_j(-B)).$$

Definition 2.3.4. We say that $H \leq G\times\Sigma_n$ is 'transitive' if the projection to Σ_n gives a transitive subgroup, i.e one which acts transitively on the set of n elements. Following this, a $G\times\Sigma_n$-set is 'transitively stabilized' if its stabilizer is transitive. The transfer ideal of the $A(G\times\Sigma_n)$ is generated by basis elements which are not transitively stabilized:

$$I_{\mathrm{tr}} := \langle [G\times\Sigma_n/H] | H \text{ is not transitive}\rangle$$

Note that the quotient by this ideal will be a map of free $\mathbb{Z}$-modules, as it acts as identity on all basis elements which are transitively stabilized while sending those which are not to 0. Hence, it suffices to describe its effect on an additive basis. Furthermore, I_{tr} is minimal among ideals I in $A(G\times\Sigma_n)$ and natural in G with the property that $q_I \circ P_n$ is additive. We will refer to operations that factor through the power operation modulo the transfer ideal as the 'additive' power operations.

Proposition 2.3.5. *The power operations P_n are natural with respect to restriction maps, in the sense that the following diagrams commute:*

$$\begin{tikzcd}
A(G) \arrow[r, "P_n"] \arrow[d, "\mathrm{Res}^G_H"] & A(G\times\Sigma_n) \arrow[d, "\mathrm{Res}^{G\times\Sigma_n}_{H\times\Sigma_n}"] & R(G) \arrow[r, "P_n"] \arrow[d, "\mathrm{Res}^G_H"] & R(G\times\Sigma_n) \arrow[d, "\mathrm{Res}^{G\times\Sigma_n}_{H\times\Sigma_n}"] \\
A(H) \arrow[r, "P_n"] & A(H\times\Sigma_n) & R(H) \arrow[r, "P_n"] & R(H\times\Sigma_n).
\end{tikzcd}$$

Remark 2.3.6. The power operation also commutes with linearization, which in turn respects restriction, so one may construct a commutative cube of maps.

While the above commutative diagrams explain that power operations and restriction interact nicely, they do not describe how a G-set explicitly decomposes into basis elements of $A(G\times\Sigma_n)$ or $A(H\times\Sigma_n)$. In general, this is a difficult problem, but in the case G is the trivial group, it is reasonably easy to determine.

2.3.3 Understanding P_n for $G = e$

We will compute P_n for when G is the trivial group. While this is will be simple in comparison to the general case, computations of various transfers will still be necessary. Afterwards, we will quotient by the transfer ideal to show the composite is additive.

Example 2.3.7. Consider $G = e$, so the power operation lands in $A(\Sigma_n)$. As isomorphism classes of e-sets are given by cardinality, we will write $\underline{k}$ for a set of size $k \in \mathbb{Z}_{\geq 0}$. Let the elements of the set be the numbers 1 through k, where e acts trivially on all elements. Then, an element of $P_n(\underline{k})$ is an n-tuple with each entry a number from 1 to k.

We can make use of the coproduct formula to determine the basis decomposition. By induction on the number of entries in the coproduct, we obtain the following isomorphism:

$$P_n(\underline{k}) \cong \coprod_{i_1+\ldots+i_k=n} \mathrm{Tr}^{\Sigma_n}_{\prod_{j=1}^{k}(\Sigma_{i_j})} \left(\prod_{j=1}^{k} (*_j^{i_j}) \right)$$

Here, we write $*_j$ for each element of $\underline{k}$, as they are trivial e sets when considered individually, but in general, the powers on each piece of the product would be dependent on the composition in the index of the summand.

To determine the basis decomposition, we need to determine what these transfers correspond to. This is simpler than it may appear, because after using the definition of the transfer, the product of fixed sets is still a fixed set, so we can simplify as follows:

$$\mathrm{Tr}^{\Sigma_n}_{\prod_{j=1}^{k}(\Sigma_{i_j})} \left(\prod_{j=1}^{k} (*_j^{i_j}) \right) \cong \Sigma_n \times_{\prod_{j=1}^{k}(\Sigma_{i_j})} \left(\prod_{j=1}^{k} (*_j^{i_j}) \right) \cong \Sigma_n \times_{\prod_{j=1}^{k}(\Sigma_{i_j})} * \cong \Sigma_n / \prod_{j=1}^{k} (\Sigma_{i_j}).$$

Hence, after taking the sum, we get the complete decomposition for the power operation. If we had instead considered $-\underline{k}$, we would obtained $P_n(-1)$ times this answer. It should be noted that there are repeat stabilizers throughout the sum.

$$P_n(\underline{k}) \cong \coprod_{i_1+\ldots+i_k=n} \left(\Sigma_n / \prod_{j=1}^{k} (\Sigma_{i_j}) \right)$$

Now, while this is an explicit answer, we can obtain a clearer answer by making use of the ring of marks. In general, it's difficult to write a power operation on the ring of marks, but here, we can determine the image by calculating the mark of $\underline{k}^{\times n}$ directly. If H is a subgroup of $e \times \Sigma_n$, we can consider how H acts on the set of numbers 1 through n. This decomposes into a collection of transitive orbits $\coprod_i \underline{n}_i$. This will give us the following formula:

$$\begin{tikzcd}
\underline{k} \in A(e) \arrow[r, mapsto, "P_n"] \arrow[d, mapsto, "\chi"] & \underline{k}^{\times n} \in A(\Sigma_n) \arrow[d, mapsto, "\chi"] \\
(f : [e] \mapsto k) \arrow[r, mapsto, dashed, "P_n"] & P_n(f)[H] = \prod_i k^{n_i}.
\end{tikzcd}$$

Remark 2.3.8. The power operations factor through $A(G \wr \Sigma_n)$ and $R(G \wr \Sigma_n)$. This is called the 'total' power operation and denoted $\mathbb{P}_n : A(G) \to A(G \wr \Sigma_n)$ and $\mathbb{P}_n : R(G) \to R(G \wr \Sigma_n)$, which is induced by the action of $G \wr \Sigma_n$ on the n-fold product of the set or n-fold tensor power of the vector space. As $G \times \Sigma_n$ includes into $G \wr \Sigma_n$ through the diagonal map $G \to G^n$, there is a restriction map $\mathrm{Res}^{G \wr \Sigma_n}_{G \times \Sigma_n}$, giving the desired factorization. As with the power operation P_n, this is a multiplicative operation, but the quotient by a minimal 'transfer' ideal will make the composite additive.

Example 2.3.9. Let $G = e$ as before. In the prior example, we gave an explicit decomposition for the nth power operation of any finite e-set. Each stabilizer was given as a product of symmetric groups at most as large as Σ_n, and up to order only one such product will contain a copy of Σ_n, notably the indices of the sum which correspond to the trivial composition with $i_\ell = n$ and $i_{m \neq \ell} = 0$. This will be the only transitively stabilized basis element in the decomposition. Hence, we obtain the following image after taking the quotient by the transfer ideal:

$$\begin{tikzcd}
A(e) \arrow[r, "P_n"] & A(\Sigma_n) \arrow[r, "/I_{\mathrm{tr}}"] & A(\Sigma_n)/I_{\mathrm{tr}} \\
\underline{k} \arrow[r, mapsto] & \coprod_{i_1+\ldots+i_k=n} \left(\Sigma_n / \prod_{j=1}^{k} (\Sigma_{i_j}) \right) \arrow[r] & k\Sigma_n/\Sigma_n.
\end{tikzcd}$$

As mentioned earlier, after the taking the quotient by the transfer ideal, the composite with the power operation is a map of commutative rings.

2.4 The Adams Operation

Many natural operations can be built out of the power operations by considering a map from the target of the power operation back to the source of the power operation. We will use this idea to define a pre-λ-structure on the Burnside ring. First, we recall what a λ-ring and a pre-λ-ring are. These definitions follow along with those given by Gay, Morris, and Morris, which draw from Knutson [9, Chapter 1]. Let R be a torsion-free commutative ring and $\mathbb{N}$ the set of non-negative integers.

2.4.1 Construction via pre-λ-rings

As the representation ring is constructed using vector spaces, some of the natural operations on vector spaces extend to the representation ring, notably that of symmetric powers and exterior powers. Similarly, symmetric powers of sets extend to the

Burnside ring. These operations are closely related and together form a category of rings, although we will not consider the categorical structure.

Definition 2.4.1. A *pre-λ-ring* is a ring R with operations $\lambda_n, \psi_n, \beta_n : R \to R, n \in \mathbb{N}$. These operations must satisfy a collection of identities found below [7, Section 2]. Here, we should interpret β_n as analogous to the symmetric powers, while λ_n is similar to the exterior powers. The operations $\lambda_0, \beta_0, \psi_0$ must be the constant map to 1, and $\lambda_1, \beta_1, \psi_1$ are id_R. The operations λ_n and β_n are not additive or multiplicative, while the Adams operation ψ_n is required to be additive and may be multiplicative.

$$
\begin{aligned}
0 &= \sum_{i=0}^{n} (-1)^i \lambda_i \beta_{n-i} \\
n\beta_n &= \sum_{i=0}^{n-1} \beta_i \psi_{n-i} \\
(-1)^{-1} n\lambda_n &= \sum_{i=0}^{n-1} (-1)^i \lambda_i \psi_{n-i}
\end{aligned}
$$

Remark 2.4.2. These identities allow us to recover all three operations from any one of them as long as R is torsion-free. As both the Burnside ring and the representation ring are torsion-free, we need only define one of the collections of operations and we may use the identities to induce a pre-λ-ring structure.

In the case of $R = A(G)$ or $R = R(G)$, we obtain β_n from the power operation as follows. If we quotient the power operation by the Σ_n action, it gives a G-set or G-representation respectively, referred to as the n^{th} symmetric power, or Symm^n. The symmetric powers satisfy conditions on β_n such as $\beta_0 = 1$ and $\beta_1 = \mathrm{id}_R$, thus allowing us to inductively define ψ_n and λ_n using the identities.

Furthermore, for the Burnside ring, Bouc and Rökaeus provide a direct construction for the operations λ_n in terms of G-sets, allowing for an alternative construction for the Adams operation [5, Theorem 1.1]. This induces a pre-λ-ring structure identical to the one obtained by making use of symmetric powers.

These methods give us a recursive formula for the operations ψ_n in terms of the symmetric powers or alternating powers and lower Adams operations. However, both of these methods require us to calculate all lower Adams operations, and as such a non-recursive formula would be preferred. In the case of the representation ring, we see there is a more direct method.

Proposition 2.4.3. *[1, Proposition 2.5] The n^{th} Adams operation on the representation ring factors through the power operation, where the middle isomorphism is the Kunneth isomorphism of the representation ring and lastly we evaluate the character of a representation of Σ_n at the conjugacy class of the long cycle:*

$$\begin{tikzcd}
R(G) \arrow[r, "P_n"] \arrow[rrr, bend right, "\psi_n"'] & R(G \times \Sigma_n) \arrow[r, "\cong"] & R(G) \otimes_{\mathbb{Z}} R(\Sigma_n) \arrow[r, "id \otimes ev"] & R(G).
\end{tikzcd}$$

While such a factorization for the Adams operation on the Burnside ring would be ideal, the Burnside ring does not have a Kunneth isomorphism and also $A(\Sigma_n)/I_{\mathrm{Tr}}$ is quite large. However, Gay, Morris, and Morris describe an elegant formula for the Adams operations that makes use of the ring of marks.

Theorem 2.4.4. *[7, Theorem 4.2] For $H, K \leq G$, and r_d equal to the number of orbits of size d when K acts on G/H, we have the following equality.*

$$\chi(\psi_n([G/H]))[K] = \sum_{d|n} dr_d.$$

As the ring of marks is rationally isomorphic to the Burnside ring and both are free $\mathbb{Z}$-modules, we need only invert a triangular matrix to use this to calculate the Adams operation on the Burnside ring.

Example 2.4.5. Let $G = \Sigma_4$. The following table gives the 3rd Adams operation on $A(\Sigma_4)$, calculated using the ring of marks formula. This is done by finding the orbit decomposition for each subgroup K acting on each coset Σ_4/H, then using the above theorem to give a function in the ring of marks, and finding which element in the Burnside ring maps to this by inverting the matrix associated to χ.

Table 2.2: Computing the image of ψ_3 on $A(\Sigma_4)$

$A(\Sigma_4)$ Basis Element	ψ_3 Image
Σ_4/e	$-3\Sigma_4/e + 12\Sigma_4/A_3$
$\Sigma_4/\Sigma_2 = \Sigma_4/\langle(12)\rangle$	$1\Sigma_4/e - 5\Sigma_4/\Sigma_2 + 3\Sigma_4/A_3 + 6\Sigma_4/A_4$
$\Sigma_4/\langle(12)(34)\rangle$	$-2\Sigma_4/e + 1\Sigma_4/\langle(12)(34)\rangle + 6\Sigma_4/A_3$
$\Sigma_4/C_4 = \Sigma_4/\langle(1234)\rangle$	$-1\Sigma_4/e + 1\Sigma_4/C_4 + 3\Sigma_4/A_3$
$\Sigma_4/\langle(12)(34), (13)(24)\rangle$	$3\Sigma_4/A_4$
$\Sigma_4/\langle(12), (34)\rangle$	$2\Sigma_4/e - 6\Sigma_4/\Sigma_2 + 1\Sigma_4/\langle(12), (34)\rangle + 6\Sigma_4/\Sigma_3$
$\Sigma_4/D_8 = \Sigma_4/\langle(1234), (13)\rangle$	$1\Sigma_4/\langle(12)(34), (13)(24)\rangle - 2\Sigma_4/D_8 + 3\Sigma_4/\Sigma_4$
$\Sigma_4/A_3 = \Sigma_4/\langle(123)\rangle$	$-1\Sigma_4/e + 4\Sigma_4/A_3$
$\Sigma_4/\Sigma_3 = \Sigma_4/\langle(123), (12)\rangle$	$1\Sigma_4/e - 3\Sigma_4/\Sigma_2 + 4\Sigma_4/\Sigma_3$
$\Sigma_4/A_4 = \Sigma_4/\langle(123), (12)(34)\rangle$	Σ_4/A_4
Σ_4/Σ_4	Σ_4/Σ_4

While the result above simplifies the computation of ψ_3, it's not as direct as in the representation theory example, and additionally, doesn't give a factorization through the power operation. If we return to our earlier example with $G = e$, we find some motivation that such a factorization through P_n can exist, as below.

Motivated by all of this, our goal is to construct a map

$$\alpha_{G,n} : A(G \times \Sigma_n) \to A(G)$$

such that $\psi_n = \alpha_{G,n} \circ P_n$. When G is clear from context, we may abuse notation and write α_n for $\alpha_{G,n}$.

Chapter 3 Constraints on factorization

There are a few properties we will require of any α_n such that $\alpha_n \circ P_n = \psi_n$. They are not strictly necessary, however they do hold in the representation theory case, and in fact, stronger conditions are met within representation theory.

3.1 Desirable Conditions

Definition 3.1.1. We denote $\alpha_{G,n} : A(G \times \Sigma_n) \to A(G)$ any map that factors ψ_n through the power operation: $\alpha_{G,n} \circ P_n = \psi_n$. Further, it is desirable that $\alpha_{G,n}$ satisfies the following conditions:

1. $\alpha_{G,n}$ is additive and factors through the transfer ideal.
2. $\alpha_{G,n}$ is natural with respect to restriction, i.e for $H \leq G$, we have $\operatorname{Res}^G_H \circ \alpha_{G,n} = \alpha_{H,n} \circ \operatorname{Res}^G_H$.
3. The following diagram commutes, where $\alpha_{e,n}$ is considered to be a map from $A(\Sigma_n) \cong A(e \times \Sigma_n)$ to $A(e) \cong \mathbb{Z}$. In general, we will write α_n when G is clear from context.

$$\begin{tikzcd}
A(G) \otimes A(\Sigma_n) \arrow[r, "1\otimes\alpha_{e,n}"] \arrow[d] & A(G) \otimes \mathbb{Z} \arrow[d, "\cong"] \\
A(G \times \Sigma_n) \arrow[r, "\alpha_{G,n}"] & A(G)
\end{tikzcd}$$

Note that $\alpha_{G,n}$ is not assumed to be unique, and in general, may fail to be unique.

Remark 3.1.2. While the conditions (1) and (2) are reasonable, one may wonder why condition (3) is desirable. In short, ideally α_n would be an $A(G)$-module map, however this seems to be too much to ask. This weaker condition is motivated by the following commutative diagram which can be drawn in the representation theory case, since α_n for the representation ring is an $R(G)$-module map, where a is the action of $R(G)$ on an $R(G)$-module. For the Burnside ring, we ask only the outermost equivalent square to commute.

$$\begin{tikzcd}
R(G) \otimes R(\Sigma_n) \arrow[r, "1\otimes\alpha_{e,n}"] \arrow[d, "1\otimes\operatorname{Res}^e_{\Sigma_n}"] & R(G) \otimes \mathbb{Z} \arrow[d] \\
R(G) \otimes R(G \times \Sigma_n) \arrow[r, "1\otimes\alpha_{G,n}"] \arrow[d, "a"] & R(G) \otimes R(G) \arrow[d, "a"] \\
R(G \times \Sigma_n) \arrow[r, "\alpha_{G,n}"] & R(G)
\end{tikzcd}$$

3.2 Factorization in the case $G = e$

Example 3.2.1. In the case of $G = e$, we can factor ψ_n through the power operation as follows. By using the ring of marks formula, we know that any additive map α_n to complete the factorization must send Σ_n/Σ_n to $\underline{1}$. Furthermore, factoring through the transfer ideal requires all non-transitive basis elements must be sent to 0. To ensure that α_n is defined on all of $A(\Sigma_n)$, we need also must define its image on $e \times \Sigma_n$-sets which are not in the image of the power operation, notably for Σ_n/T with T any transitive subgroup. These will be sent to the given image, and then all images are defined and the two possible images for Σ_n/Σ_n agree. We then have the factorization $\psi_n = \alpha_n \circ P_n$.

$$\begin{tikzcd}
A(e) \arrow[r, "P_n"] & A(\Sigma_n) \arrow[r, "\alpha_n"] & A(e) \\
\underline{k} \arrow[r] & \displaystyle\sum_{\sum_{j=1}^{k} i_j = n} \left(\Sigma_n / \prod_{j=1}^{k} (\Sigma_{i_j}) \right) \arrow[r] & \underline{k} \\
 & \Sigma_n/T \arrow[r] & \underline{|\Sigma_n/T|}
\end{tikzcd}$$

This example hints at some future structure we'll make use of to show such a factorization for various types of groups. In the abelian case, we can even give an explicit decomposition into transitively stabilized basis elements for the power operation, but in general, we make use of Goursat's lemma data to capture how the Σ_n action interacts with the G action.

Chapter 4 A Counterexample to the general setting

While a well-behaved factorization of the Adams operation through the power operation exists for the representation ring and we can clearly write one for $A(e)$, there's no guarantee one exists for all groups. As such, one might ask what barriers exist to a well-behaved factorization. In this section, we consider where challenges may arise, then show this problem arises in the case of $G = \Sigma_4$.

4.1 Challenges for a factorization

In the non-abelian setting, there is a reasonable concept for building a counterexample. If α_n is going to be defined on $A(G \times \Sigma_n)$, then α_n must be well-defined for each basis member. However, when we calculate P_n for each basis member X of $A(G)$, there can be overlap in the basis decomposition in $A(G \times \Sigma_n)$. A mock example is provided below, where we would need to ensure $\alpha_n(Z)$ can be defined coherently:

$$\begin{tikzcd} A(G) \arrow[r, "P_n"] & A(G \times \Sigma_n) \arrow[r, dotted, "\alpha_n"] & A(G) \end{tikzcd}$$

$$\begin{tikzcd} \begin{array}{c} X \\ Y \end{array} \arrow[r] & \begin{array}{c} X^n \cong Z \coprod W \\ Y^n \cong 2Z \coprod \hat{W} \end{array} \arrow[r] & \begin{array}{c} \psi_n(X) = \alpha_n(X^n) \cong \alpha_n(Z) \coprod \alpha_n(W) \\ \psi_n(Y) = \alpha_n(Y^n) \cong 2\alpha_n(Z) \coprod \alpha_n(\hat{W}). \end{array} \end{tikzcd}$$

Any α_n would have satisfy both decompositions, so it creates a system of equations in $A(G)$. If this system is inconsistent, there's no way to construct an α_n with the desired properties, showing that it is not always possible.

4.2 A counterexample for $G = \Sigma_4$

For the remainder of this section, we will focus on $G = \Sigma_4$ and the third power operation P_3 on it. However, we will refer to Σ_4 as S_4 for clarity to distinguish from the symmetric group component of the power operation. First, we need to calculate the image of each basis element under the Adams operation, then decompose the images under the power operation, and lastly, determine if the resulting system can be solved. As we have already calculated ψ_3 on S_4 in Example 2.4.5, we may skip to determining decompositions.

For all $H \leq S_4$, we will only explicitly write down the basis decomposition of $P_3(S_4/H)$ into transitive components, any orbits which are not transitively stabilized will be included in a general form of NTO_H for non-transitive orbits of $P_3(S_4/H)$. As α_3 must factor through the transfer ideal, it necessarily sends all of these to 0, so their image is already known and doesn't affect the system.

4.2.1 Computing $P_3(S_4/e)$

The following computation will be necessary to compute $\alpha_{S_4,3}$, and much of the work holds for understanding P_3 in generality.

Lemma 4.2.1. *$P_3(S_4/e)$ has the following decomposition in $A(S_4 \times \Sigma_3)$:*

$$P_3(S_4/e) \cong [S_4 \times \Sigma_3/(e \times \Sigma_3)] \coprod 4[S_4 \times \Sigma_3/\langle(123),(123)\rangle] \coprod NTO_e$$

Proof. First, we may observe that the orbit generated by (e,e,e) is stabilized by $(e \times \Sigma_3)$. Note that any transitive subgroup of Σ_3 contains a 3-cycle, so we may assume up to conjugacy that any transitively stabilized orbit is stabilized by at least one element of the form $(g,(123))$ for some $g \in S_4$. Furthermore, we may choose each orbit to be generated by a tuple of the form (e, g_1, g_2) for some $g_1, g_2 \in S_4$, with $g_1 \neq g_2$, as this is either non-transitive or covered by the orbit (e,e,e). Thus, the following must be true:

$$(g,(123))(e,g_1,g_2) = (gg_2, g, gg_1) = (e, g_1, g_2)$$

This implies $g_1 = g, g_2 = g^2$ and $e = g^3$. Outside of $g = e$, this gives that every transitively stabilized orbit is generated by a tuple of the form (e,g,g^2) for some order 3 element in S_4. This gives four possible orbits, plus the trivial 'diagonal' orbit generated by (e,e,e). We can see that these orbits do not collapse, as any (k,σ) which gives $(k,\sigma)(e,g,g^2) = (e,\hat{g},\hat{g}^2)$ will lead to the conclusion that k is one of e, g, g^2 as either $ke = e, kg = e$ or $kg^2 = e$. Hence, as there are four eligible g up to reordering and all of which are conjugate to (123), we obtain the desired decomposition. ■

The above argument regarding the structure of tuples which generate a transitively stabilized orbit can be extended beyond $H = e$. Again, consider (eH, g_1H, g_2H) to be the generator of a transitively stabilized orbit, which is stabilized by at least one $(g,(123))$ for $g, g_1, g_2 \in S_4$, with $g_1H \neq g_2H$. We exclude the case of $eH = g_1H = g_2H$ as this is stabilized by a subgroup conjugate to $(H \times \Sigma_3)$. As before, the following equation holds:

$$(g,(123))(eH, g_1H, g_2H) = (gg_2H, gH, gg_1H) = (eH, g_1H, g_2H)$$

This implies $g_1H = gH$, $g_2H = gg_1H = g^2H$ and $gg_2H = eH = g^3H$, allowing us to conclude $g^3 \in H$. In a general setting, this is not sufficient to make any conclusions about g, but as $g \in S_4$ and the entries of the tuple are distinct, either $g^3 = e$ or g is order 4, as no higher order elements exist in S_4. If g is order 4, we conclude that $g \in H$, giving us the case where $eH = g_1H = g_2H$. Hence, the tuple generating the orbit can be assumed to be of the form (eH, gH, g^2H) for g an element of order 3, up to reordering.

It should be noted that this does not guarantee all such tuples generate distinct orbits. Notably, if two three cycles are conjugate by an element of H, the associated tuples will generate the same orbit. We will observe an example of this shortly.

4.2.2 Computing $P_3(S_4/S_2)$

As before, we must compute the decomposition of $P_3(S_4/S_2)$ into $S_4 \times \Sigma_3$ basis elements.

Lemma 4.2.2. *$P_3(S_4/S_2)$ has the following decomposition in $A(S_4 \times \Sigma_3)$:*

$$P_3(S_4/\langle(14)\rangle) \cong [S_4 \times \Sigma_3/(\langle(14)\rangle \times \Sigma_3)] \coprod [S_4 \times \Sigma_3/\langle(123),(123)\rangle] \\ \coprod 2[S_4 \times \Sigma_3/\langle((12),(12)),((14),(23))\rangle] \coprod NTO_{\langle(14)\rangle}.$$

Proof. We will take $\langle(14)\rangle$ as the conjugacy class representative of S_2 for calculation. As before, we first notice the tuple (eS_2, eS_2, eS_2) is stabilized by $S_2 \times \Sigma_3$, giving us the first desired orbit. Following this, our prior work gives us that we need only check the orbits of the tuples generated by three cycles, namely those of the form (eS_2, gS_2, g^2S_2) for $g = (123), (124), (134), (234)$.

If $g = (123)$, we first recognize this generates the same orbit as $g = (234)$, as acting by $((14), ())$ on $(eS_2, (123)S_2, (132)S_2)$ will give us $(eS_2, (234)S_2, (243)S_2)$, and notably, these are distinct representatives of the S_2 cosets, so these generated the same orbit. Hence, we need only consider the stabilizer when $g = (123)$. If some (k, σ) is going to stabilize $(eS_2, (123)S_2, (132)S_2)$, it must be true that k is one of the six representatives of the three cosets in the tuple, simply because $k * eS_2$ must remain one of the cosets in the tuple. This means it suffices to check $k = e, (14), (123), (132), (1423), (1432)$ and determine if these permute the entries of the tuple. If so, there is some $\sigma \in S_3$ which results in (k, σ) being in the stabilizer.

Table 4.1: Computing Stabilizer Elements for $(eS_2, (123)S_2, (132)S_2)$

$k \in A(S_4) * (eS_2, (123)S_2, (132)S_2)$	Associated $\sigma \in \Sigma_3$
$e(eS_2, (123)S_2, (132)S_2) = (eS_2, (123)S_2, (132)S_2)$	$\sigma = ()$
$(14)(eS_2, (123)S_2, (132)S_2) = (eS_2, (234)S_2, (243)S_2)$	Non-stabilizing
$(123)(eS_2, (123)S_2, (132)S_2) = ((123)S_2, (132)S_2, eS_2)$	$\sigma = (123)$
$(132)(eS_2, (123)S_2, (132)S_2) = ((132)S_2, eS_2, (123)S_2$	$\sigma = (132)$
$(1423)(eS_2, (123)S_2, (132)S_2) = ((1423)S_2, (1342)S_2, (24)S_2)$	Non-stabilizing
$(1432)(eS_2, (123)S_2, (132)S_2) = ((1432)S_2, (34)S_2, (1243)S_2)$	Non-stabilizing

Hence, we conclude the stabilizer is generated by $\langle(123), (123)\rangle$, giving us the second orbit in our desired decomposition.

Now we must consider the stabilizers of the orbits for $g = (124), (134)$. These will turn out to be similar cases, so below will only consider $g = (124)$. These two orbits are distinct, and the work in calculating the stabilizer will show this. As before, it is true that k must be one of the six representatives of the cosets $eS_2, (124)S_2$ and $(142)S_2$, so it suffices to check $k = e, (14), (124), (142), (24), (12)$.

We can first observe that $g = (134)$ necessarily generates a distinct orbit, as these six k values are also the only elements of G which ensure there is a eS_2 in the resulting tuple, which $(eS_2, (134)S_2, (143)S_2)$ would have to contain. Furthermore,

Table 4.2: Computing Stabilizer Elements for $(eS_2, (124)S_2, (142)S_2)$

$k \in A(S_4) * (eS_2, (124)S_2, (142)S_2)$	Associated $\sigma \in \Sigma_3$
$e(eS_2, (124)S_2, (142)S_2) = (eS_2, (124)S_2, (142)S_2)$	$\sigma = ()$
$(14)(eS_2, (124)S_2, (142)S_2) = (eS_2, (142)S_2, (124)S_2)$	$\sigma = (23)$
$(124)(eS_2, (124)S_2, (142)S_2) = ((124)S_2, (142)S_2, eS_2)$	$\sigma = (123)$
$(142)(eS_2, (124)S_2, (142)S_2) = ((142)S_2, eS_2, (124)S_2)$	$\sigma = (132)$
$(24)(eS_2, (124)S_2, (142)S_2) = ((124)S_2, eS_2, (142)S_2)$	$\sigma = (12)$
$(12)(eS_2, (124)S_2, (142)S_2) = ((142)S_2, (124)S_2, eS_2)$	$\sigma = (13)$

this stabilizer is generated by $((24), (12))$ and $((12), (13))$, which is conjugate to the desired $\langle((12), (12)), ((14), (23))\rangle$. We will find that $g = (134)$ similarly generates an orbit with a conjugate stabilizer. This gives us the desired sum of transitively stabilized orbits, and all other orbits are necessarily non-transitive. ■

4.2.3 Computing $P_3(S_4/S_3)$

In this subsection, we will compute the last necessary decomposition prior to showing $\alpha_{S_4,3}$ is inconsistent.

Lemma 4.2.3. *$P_3(S_4/S_3)$ has the following decomposition in $A(S_4 \times \Sigma_3)$:*

$$P_3(S_4/S_3) \cong [S_4 \times \Sigma_3/(S_3 \times \Sigma_3)] \coprod [S_4 \times \Sigma_3/\langle((12), (12)), ((14), (23))\rangle] \coprod NTO_{S_3}.$$

Proof. We will take $\langle(12), (23)\rangle$ as the conjugacy class representative of S_3. As before, we can first notice (eS_3, eS_3, eS_3) is stabilized by $S_3 \times \Sigma_3$, giving us the first desired orbit, and furthermore, this is the same orbit given by $g = (123)$ for the orbits of tuples generated by three cycles. Additionally, it will turn out that $g = (124), (134), (234)$ all generate the same orbit. We will consider the tuple $(eS_3, (124)S_3, (142)S_3)$ and determine its stabilizer.

As before, it suffices to check the action of every element in the cosets $eS_3, (124)S_2$ and $(142)S_3$. For brevity, we omit the full calculation of these actions, as it would require 24 rows. The stabilizer will be $\langle((12), (13)), ((14), (12))\rangle$, which is conjugate to the desired second orbit. This gives us the above decomposition into transitive orbits. ■

The third power operation of these three S_4-sets have overlap, so any α_3 which factors ψ_3 would need to give coherent images. If we view this as in the mock example, this is a system of three equations with five unknowns. We will need to make use of a final lemma to determine three of those unknowns.

Lemma 4.2.4. *For a given G, if α_3 exists and factors ψ_3 as $\alpha_3 \circ P_3 = \psi_3$ with our desired properties, it must satisfy the following for any $H \leq G$.*

$$\alpha_3([G \times \Sigma_3/(H \times \Sigma_3)]) = G/H$$

Proof. This follows immediately from property (3) by taking $[G/H] \in A(G)$ and $\Sigma_3/\Sigma_3 \in A(\Sigma_3)$. ■

4.2.4 $\alpha_{S_4,3}$ is inconsistent

Now, we can construct our system of equations and try to solve it, since it is now three equations with two unknowns. If α_3 exists and satisfies the desired conditions, then the following equations must hold.

$$\begin{aligned}
\alpha_3(P_3(S_4/e)) &= \alpha_3([S_4 \times \Sigma_3/(e \times \Sigma_3)]) + 4\alpha_3([S_4 \times \Sigma_3/\langle(123),(123)\rangle]) \\
\alpha_3(P_3(S_4/\langle(14)\rangle)) &= \alpha_3([S_4 \times \Sigma_3/(\langle(14)\rangle \times \Sigma_3)]) + \alpha_3([S_4 \times \Sigma_3/\langle(123),(123)\rangle]) \\
&\quad + 2\alpha_3([S_4 \times \Sigma_3/\langle((12),(12)),((14),(23))\rangle]) \\
\alpha_3(P_3(S_4/S_3)) &= \alpha_3([S_4 \times \Sigma_3/(S_3 \times \Sigma_3)]) + \alpha_3([S_4 \times \Sigma_3/\langle((12),(12)),((14),(23))\rangle])
\end{aligned}$$

We may first eliminate all of the variables associated to orbits generated by tuples of the form (eH, eH, eH) using the lemma. Following this, as we have assumed $\alpha_3 \circ P_3 = \psi_3$, we may apply our calculations of ψ_3 from the prior section. After some simplification, we arrive at this new system:

$$\begin{aligned}
-S_4/e + 3S_4/A_3 &= \alpha_3([S_4 \times \Sigma_3/\langle(123),(123)\rangle]) \\
S_4/e - 6S_4/\langle(14)\rangle + 3S_4/A_3 + 6S_4/A_4 &= \alpha_3([S_4 \times \Sigma_3/\langle(123),(123)\rangle]) \\
&\quad + 2\alpha_3([S_4 \times \Sigma_3/\langle((12),(12)),((14),(23))\rangle]) \\
S_4/e - 3S_4/\langle(14)\rangle + 3S_4/S_3 &= \alpha_3([S_4 \times \Sigma_3/\langle((12),(12)),((14),(23))\rangle])
\end{aligned}$$

We can use the two equations which are solved for their respective variables and check coherence on the middle equation. After simplification, this will result in the conclusion $S_4/A_4 = S_4/S_3$ in $A(S_4)$, but these are distinct basis elements, leading to a contradiction. As such, it is impossible to construct an α_3 factoring ψ_3 such that $\alpha_3 \circ P_3 = \psi_3$. It is of special note that this failure was not dependent on our original choice of $\alpha_{e,3}$, nor did it make use of condition (2). Lemma 4.2.4 forced the image of certain basis elements based condition (3) and the fact that Σ_3/Σ_3 is the only transitively stabilized basis element of $A(\Sigma_3)$ which appeared in the image of the power operation. For $\alpha_{e,3}$ to factor ψ_3, it must send Σ_3/Σ_3 to 1, resulting in the contradiction regardless of which $\alpha_{e,3}$ we choose.

Chapter 5 Factoring the Adams operation in abelian groups

The prior section demonstrates that a well-behaved factorization of the Adams operation through the power operation does not exist in general. Note that in our counterexample, G is nonabelian. In this section, we explore the setting of abelian groups, first providing a description of transitive components of $(G/H)^n$ for G abelian, and then using this to construct a factorization for all n. We show that the map α_n that we construct is natural when $n = p$ is a prime.

We begin by observing a corollary to Gay, Morris, and Morris' theorem:

Corollary 5.0.1. *For G an abelian group and $H, K \leq G$, the Adams operation on the ring of marks can be computed as:*

$$\chi(\psi_n([G/H]))[K] = \begin{cases} |G/H|, & \textit{if } (|K/K \cap H|) \textit{ divides } n \\ 0, & \textit{else.} \end{cases}$$

Proof. This follows from the original formula and the observation that all orbits will be isomorphic to $K/K \cap H$ as K-sets. Hence, if the size of $K/K \cap H$ divides n, the sum is concentrated in one term which resolves to $|G/H|$, or otherwise, the sum is empty. ■

Using this corollary, we can more easily verify if an α_n we have constructed properly factors ψ_n through P_n, since it will only involve taking K virtual fixed points and determining if $|K/K \cap H|$ divides n. Since G is abelian, both of these calculations are relatively simple when compared to the general setting.

5.1 Understanding $(G/H)^n$ for G abelian

Any factorization of ψ_n through P_n will require us to provide a value on every basis element of $A(G \times \Sigma_n)$. We'll need to understand the structure of $P_n(G/H) = (G/H)^n$ as a $G \times \Sigma_n$-set to do so. It turns out that we may give a precise description of transitively stabilized orbits when G is abelian.

Definition 5.1.1. For G an abelian group with $H \leq S \leq G$ and $|S/H| = k$ with $k|n$, let $a_{S/H} : S \to \Sigma_k$ be the action map for S acts on S/H. Following this, we denote the graph of this map as $\Gamma(a_{S/H})$.

Using this, we may obtain a subgroup of $G \times \Sigma_n$ as follows: First, assume $n = qk$. Then use the projection $\Gamma(a_{S/H}) \to \Sigma_k$ to have $\Gamma(a_{S/H})$ act on $(\Sigma_q)^k$ by permuting the components of the product. This gives an action that can be used to construct a semidirect product. Consider the semidirect product $(\Sigma_q)^k \rtimes \Gamma(a_{S/H})$, which may be viewed as a subgroup of $G \times ((\Sigma_q)^k \rtimes \Sigma_k)$. Lastly, identify $(\Sigma_q)^k \rtimes \Sigma_k$ as a subgroup of Σ_n using the fact that $qk = n$. This gives us a subgroup of $G \times \Sigma_n$, well-defined up to conjugacy.

Subgroups of this type correspond precisely to transitive stabilizers of tuples which appear in the image of the power operation. The work of Bonventre, Guillou, and Stapleton provides a concise proof for this after identifying the appropriate components.

Lemma 5.1.2. *[2, Lemma 2.24] Let $H \leq S \leq G$ such that $[S : H] = k$ with $k|n$, enumerating the cosets as $\{s_iH\}_{i=1}^{k}$, choosing $s_1H = eH$. Then the following subgroups are conjugate in $G \times \Sigma_n$, where $((s_iH)^{\frac{n}{k}})_{i=1}^{k}$ is concatenated in the order of the indices giving $(s_1H, \cdots, s_1H, s_2H, \cdots, s_2H, \cdots, s_kH, \cdots, s_kH)$.*

$$[\mathrm{Stab}_{G\times\Sigma_n}((s_iH)^{\frac{n}{k}})_{i=1}^{k}] = [(\Sigma_{\frac{n}{k}})^k \rtimes \Gamma(a_{S/H})]$$

Notation 5.1.3. Given $H \leq S \leq G$ with $[S : H] = k_{S,H}$, we denote the associated basis element in $A(G \times \Sigma_n)$ as follows:

$$\Gamma_{H,S} := G \times \Sigma_n((s_iH)^{\frac{n}{k}})_{i=1}^{k} \cong G \times \Sigma_n/((\Sigma_{\frac{n}{k}})^k \rtimes \Gamma(a_{S/H})),$$

where $G \times \Sigma_n$ is the $G \times \Sigma_n$-orbit generated by acting on x.

5.1.1 Decomposition of $(G/H)^n$

Proposition 5.1.4. *Let $H \leq G$, G abelian. The following is a decomposition of $(G/H)^n$ into transitively stabilized $G \times \Sigma_n$ basis elements.*

$$(G/H)^n \cong \left(\coprod_{\substack{H\leq S\leq G \\ k_{H,S}:=[S:H], k_{H,S}|n}} \Gamma_{H,S} \right) \amalg \coprod NTO_H$$

Proof. For the forward direction, suppose $\bar{x} = (g_iH)_{i=1}^{n}$, chosen such that $g_1H = eH$, is a generator for a transitively stabilized orbit of $(G/H)^n$. By the above lemma, it suffices to show its orbit is generated by a tuple of the form $((s_iH)_{i=1}^{k})^{\frac{n}{k}}$ with $s_1H = eH$ and $\{s_iH\}_{i=1}^{k} = S/H$, for some $H \leq S \leq G$ and $[S/H] = k$.

Consider $(g, \sigma) \in \mathrm{Stab}(\bar{x})$. As σ only reorders the tuple, if we consider the set of cosets in $\bar{x}$, we conclude $\{gg_iH\}_{i=1}^{n}$=$\{g_iH\}_{i=1}^{n}$. Since the stabilizer is a group, we also conclude $\langle g \rangle$ acts on $\{g_iH\}_{i=1}^{n}$, the set of cosets in $\bar{x}$. Let π_G be the projection from $G \times \Sigma_n$ to G, then $\pi_G(\mathrm{Stab}(\bar{x}))$ acts on $\{g_iH\}_{i=1}^{n}$ by assembling these actions over the stabilizer, where the permutation portion is not used in the action. Following this, we may define a $\pi_G(\mathrm{Stab}(\bar{x}))$-equivariant function, where $\pi_G(\mathrm{Stab}(\bar{x}))$ acts trivially on $[n]$, the integers from 1 to n.

$$\begin{aligned} f : \{g_iH\}_{i=1}^{n} &\xrightarrow{\hspace{3cm}} [n] \\ g_iH &\longmapsto \text{The number of times } g_iH \text{ appears in } \bar{x}. \end{aligned}$$

This is $\pi_G(\mathrm{Stab}(\bar{x}))$-equivariant, as the fact (g, σ) is invertible means action by $\pi_G(\mathrm{Stab}(\bar{x}))$ may not change the number of cosets of type g_iH present in $\bar{x}$. Furthermore, we observe that $\{g_iH\}_{i=1}^n$ is a transitive $\pi_G(\mathrm{Stab}(\bar{x}))$-set. It suffices to see there is an element of $\pi_G(\mathrm{Stab}(\bar{x}))$ which sends g_jH to eH for each $j \in [n]$. As $\bar{x}$ is transitively stabilized, let $(g_{j,1}, \sigma_{j,1}) \in \pi_G(\mathrm{Stab}(\bar{x}))$ be such that $\sigma_{j,1}^{-1}(1) = j$. Then we may see $g_{j,1}g_{\sigma_{j,1}^{-1}(1)}H = g_{j,1}g_jH = g_1H = eH$, and conclude $(g_{j,1}, \sigma_{j,1})$ is our desired element, and thus the set is a transitive $\pi_G(\mathrm{Stab}(\bar{x}))$-set.

Hence, we may conclude f is a constant function as its image must be a transitive set, and we have a trivial action on the codomain. Let $k \in [n]$ be the image of f, and as each coset must repeat q times, we conclude $q|n$ such that $n = kq$. Since the set is transitive, we can generate $\{g_iH\}_{i=1}^n$ as a $\pi_G(\mathrm{Stab}(\bar{x}))$-set simply by acting on any singular element, so we choose $g_1H = eH$ as the generating element. This allows us to conclude that both $\{g_iH\}_{i=1}^n{=}\pi_G(\mathrm{Stab}(\bar{x}))/H$ and $|\pi_G(\mathrm{Stab}(\bar{x}))/H| = k$.

This is sufficient to see that our orbit is generated by a tuple of the desired type. Let $\{s_iH\}_{i=1}^k = \pi_G(\mathrm{Stab}(\bar{x}))/H = \{g_iH\}_{i=1}^n$. Then $(\{s_iH\}_{i=1}^k)^q$ is a tuple of length n which contains each coset in $\bar{x}$ the same number of times as in $\bar{x}$. Hence, it is the same up to reordering, and thus generates the same orbit. By our lemma, it has the desired stabilizer.

For the return map, we need only show the specified basis elements correspond to transitively stabilized orbits of $(G/H)^n$. Given $H \leq S \leq G$ with $[S : H] = k, k|n$ and $n = kq$, this is sufficient data to construct the tuple $((s_iH)_{i=1}^k)^q$ with $s_iH \in S/H$. This is an element of $(G/H)^n$ and by the previous lemma, it is stabilized precisely by a subgroup conjugate to the desired form.

It is essentially immediate to conclude the two directions compose to identity. If we begin with $\bar{x} = (g_iH)_{i=1}^n$, the construction identifies another tuple in the same $G \times \Sigma_n$-orbit and shows it has the appropriate stabilizer, and the inverse map simply returns this chosen tuple. For the other composite, given $\Gamma_{H,S}$, the associated tuple $((s_iH)^{\frac{n}{k}})_{i=1}^k$ will immediately be of the desired form for the forward direction construction, returning $\Gamma_{H,S}$ immediately. Hence, we obtain the desired bijection, modulo

■

We will provide a formula for α_n, and to check it factors ψ_n as desired, it suffices to check on the image of the marks homomorphism. However, the structure of the G-sets in the image will be convenient, due to the following lemma.

Lemma 5.1.5. *Given $H \leq S \leq G$ and $L \leq G$ with G abelian, the following fixed points are equal.*

$$(G/S)^L = (G/S)^{\langle H,L\rangle}$$

Proof. As G is abelian and $H \leq S$, we may simply commute all elements of H to the coset. Hence, $\ell h \in \langle H, L\rangle$ fixes $gS \in G/S$ if and only if $\ell \in L$ fixes gS. ■

5.2 Factoring ψ_n through P_n for G abelian

Now that we have a decomposition of $P_n(G/H)$ and the necessary framework, we may define an α_n to factor ψ_n when G is abelian.

Theorem 5.2.1. *If G is an abelian group with $H \leq G$, $n \geq 1$, and $T \leq \Sigma_n$ a transitive subgroup, then we may inductively construct an additive map α_n such that $\alpha_n \circ P_n = \psi_n$ satisfying properties* (1) *and* (3)*. The map is given on basis members and linearly extended. If a basis member of $A(G \times \Sigma_n)$ is not given a specific image, the basis member is in the kernel.*

$$\alpha_n(G \times \Sigma_n / H \times T) = |\Sigma_n/T|(G/H)$$
$$\textit{for } H \leq S \leq G \textit{ with } k_{H,S}|n,\ \alpha_n(\Gamma_{H,S}) = k_{H,S}(G/S) - \sum_{\substack{H \leq J < S \\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})$$

Proof. We first note that the two assignments may contradict, in the event the stabilizer of $\Gamma_{H,S}$ took the form $H' \times T'$. However, if $e \times T$ is a subgroup of the transitive stabilizer, we may conclude the tuple is a single repeated coset, and hence its orbit is of the form $\Gamma_{H,H} = (G \times \Sigma_n / H \times \Sigma_n)$.

Now, we may check this formula factors ψ_n correctly. Let $L \leq G$ be a subgroup of G. Then, by Theorem 2.4.4, it suffices to check the value of $\chi(\alpha_n(P_n([G/H])))[L]$, and Lemma 5.0.1 simplifies this further. Additionally, as $\langle L, H\rangle/H \cong L/(L \cap H)$, we can conclude that the value of $\chi(\alpha_n(P_n([G/H])))[L]$ is correct if and only if the value of $\chi(\alpha_n(P_n([G/H])))[\langle H, L\rangle]$ is correct. Hence, we need only consider subgroups containing H. If the index with H divides n, the value should be $|G/H|$, otherwise it should be 0. For simplicity, let $M = \langle H, L\rangle$.

$$\begin{aligned}
\chi(\alpha_n(P_n([G/H])))[M] &= \sum_{\substack{H \leq S \leq G \\ k_{H,S}|n}} \alpha_n(\Gamma_{H,S})^M \\
&= \sum_{\substack{H \leq S < M \\ \text{or } S \not< M, M \not< S \\ k_{H,S}|n}} (\alpha_n(\Gamma_{H,S})^M) + (\alpha_n(\Gamma_{H,M}))^M + \sum_{\substack{M < \hat{S} \leq G \\ k_{H,\hat{S}}|n}} (\alpha_n(\Gamma_{H,\hat{S}})^M)
\end{aligned}$$

The value of the sum is determined by the value of the three components. First, we observe that $\alpha_n(\Gamma_{H,S})^M$ will always contribute 0 to the total, as any G-set in the sum must be a linear combination of G/S for $S < M$ or incomparable to M, all of which are free M-sets, and thus contribute 0 fixed points.

The second term is more complicated, as it has two possible values, dependent on if $k_{H,M}$ divides n. We focus first on the case that $k_{H,M}$ does not divide n. Following this, $\alpha_n(\Gamma_{H,M}) = 0$, so the middle term contributes no virtual fixed points. Furthermore, if $M < \hat{S}$, we may observe $k_{H,\hat{S}} = k_{H,M}|\hat{S}/M|$, which implies $k_{H,\hat{S}}$ also does not divide n. Thus, the right sum is empty, contributing 0. In total, $\chi(\alpha_n(P_n([G/H])))[M] = 0$, as expected by Corollary 5.0.1.

Second, we focus on the case where $k_{H,M}|n$, specifically on the value of the middle term. By applying the given formula, we obtain the following. The sum over $H \leq J < M$ provides zero fixes points as any G-set in the sum a linear combination of G-sets of the form G/J for $J < M$.

$$\begin{aligned}
\alpha_n(\Gamma_{H,M})^M &= k_{H,M}(G/M)^M - \sum_{\substack{H \leq J < M \\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= k_{H,M}|G/M| - \sum_{\substack{H \leq J < M \\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= |G/H| - 0
\end{aligned}$$

Next, we must consider what each of the terms in the final sum indexed by $M < \hat{S} \leq G$ contribute. First, suppose $\hat{S}$ covers M, i.e there are no other subgroups in the sum such that $M < J < \hat{S}$.

$$\begin{aligned}
\alpha_n(\Gamma_{H,\hat{S}})^M &= k_{H,\hat{S}}(G/\hat{S})^M - \sum_{\substack{H \leq J < \hat{S} \\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= k_{H,M}|G/M| - \alpha_n(\Gamma_{H,M})^M - \sum_{\substack{H \leq J < \hat{S} \\ J \neq M, k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= |G/H| - |G/H| - \sum_{\substack{H \leq J < \hat{S} \\ J \neq M, k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= - \sum_{\substack{H \leq J < \hat{S} \\ J \neq M, k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= 0
\end{aligned}$$

In this case, since $\hat{S}$ covers M, the final sum will be entirely over subgroups who are incomparable to M or are contained in M, and as such only give G-sets with such subgroups as their stabilizers. Hence, by our prior determination, it provides 0 virtual fixed points. Alternatively, if $\hat{S}$ does not cover M, the sum splits analogously, but with more parts. As before, we may pull out the M term to cancel, but now it splits into three sums. The first sum will be over subgroups below M and above H, the second for those which are incomparable to M, and the last for those above M and below $\hat{S}$.

$$\begin{aligned}
\alpha_n(\Gamma_{H,\hat{S}})^M &= k_{H,\hat{S}}(G/\hat{S})^M - \sum_{\substack{H\leq J<\hat{S}\\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= -\sum_{\substack{H\leq J<\hat{S}\\ J\neq M, k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M \\
&= -\left(\sum_{\substack{H\leq J<M\\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M + \sum_{\substack{H<J<\hat{S}\\ J\nleq M, M\nleq J\\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M + \sum_{\substack{M<J<\hat{S}\\ k_{H,J}|n}} \alpha_n(\Gamma_{H,J})^M\right) \\
&= -(0+0+0) = 0
\end{aligned}$$

The first two sums provide zero virtual fixed points by our prior arguments. The last requires an extension of our argument when $\hat{S}$ covers M. Among the J, there are those which cover M, and we know for those $\alpha_n(\Gamma_{H,J})^M = 0$. Now, if J does not cover M, we observe we could write a series of sums, those which cover M, then those which are precisely two subgroup inclusions away, then three, and so on. As each rank of subgroups provides 0 virtual fixed points, as in the covering case, we see the next rank also provides 0 virtual fixed points, leading to a total contribution of 0.

Now with each of the terms handled, we may return to the original question in the case that $k_{H,M}|n$. We then apply each of our prior simplifications to obtain the following.

$$\begin{aligned}
\chi(\alpha_n(P_n([G/H])))[M] &= \sum_{\substack{H\leq S\leq G\\ k_{H,S}|n}} \alpha_n(\Gamma_{H,S})^M \\
&= \sum_{\substack{H\leq S<M\\ \text{or } S\nless M, M\nless S\\ k_{H,S}|n}} (\alpha_n(\Gamma_{H,S}))^M + (\alpha_n(\Gamma_{H,M}))^M + \sum_{\substack{M<\hat{S}\leq G\\ k_{H,\hat{S}}|n}} (\alpha_n(\Gamma_{H,\hat{S}}))^M \\
&= 0 + |G/H| + 0 \\
&= |G/H|
\end{aligned}$$

Hence, our assignment agrees after the marks homomorphism, showing that $\chi(\alpha_n \circ P_n) = \chi(\psi_n)$. Since the marks homomorphism is injective, we conclude $\alpha_n \circ P_n = \psi_n$, i.e α_n factors the Adams operation on the Burnside ring.

Following this, we must also show it satisfies the desired properties. Regarding property (1), α_n was defined on basis members and linearly extended, hence it is immediately additive. Furthermore, only elements with transitive stabilizer were given a non-zero image, hence it factors through the transfer ideal. Lastly, we may observe that property (3) holds for our earlier choice of $\alpha_{e,n}$, as if we begin with $G/H \otimes \Sigma_n/T$, both directions give $|\Sigma_n/T|G/H$. Hence, α_n also satisfies our desired properties of (1) and (3).

■

5.3 Naturality of $\alpha_{G,n}$

Next we will show that, when p is prime, α_p is natural with respect to restriction. We expect this to be true for all n. We will first prove some lemmas.

Assume that $\varphi\colon G' \to G$ is a homomorphism between finite abelian groups. For $H \subseteq G$, let $H' = \varphi^{-1}(H)$. Since H' is the kernel of the composite $G' \to G \to G/H$, there is an induced injection $G'/H' \hookrightarrow G/H$. Let $\ell_H = |G/H|/|G'/H'|$. This number will give us control over the restriction maps.

Recall $\varphi^*\colon A(G) \to A(G')$ is the restriction map.

Lemma 5.3.1. *With the notation described above, we have*

$$\varphi^*([G/H]) = \ell_H[G'/H'].$$

Proof. We are interested in the G'-set G/H. We have an injection of groups $G'/H' \hookrightarrow G/H$. Since the groups are abelian, we have $G/H \cong \coprod_{\ell_H} G'/H'$ as G'/H'-sets. It follows that we have the same decomposition as G'-sets. ■

Now fix $H \subseteq S \subseteq G$ and let $S' = \varphi^{-1}(S)$. As above, there is an induced injection $S'/H' \hookrightarrow S/H$. We will denote this injection by $\varphi_{H,S}$.

Lemma 5.3.2. *If the injection $\varphi_{H,S}\colon S'/H' \hookrightarrow S/H$ is an isomorphism, then for all $H \subseteq J \subseteq S$, we have $\ell_H = \ell_J$.*

Proof. We have

$$|G/H| = |J/H| \cdot |G/J|$$

and

$$|G'/H'| = |J'/H'| \cdot |G'/J'|.$$

Since $\varphi_{H,S}$ is an isomorphism, $|J/H| = |J'/H'|$. It follows that

$$|G/H|/|G'/H'| = |G/J|/|G'/J'|.$$

■

With this in hand, we turn our attention to the restriction of $G \times \Sigma_p$-sets of the form $\Gamma_{H,S}$. Let $\varphi_p = \varphi \times \Sigma_p\colon G' \times \Sigma_p \to G \times \Sigma_p$. It is important to understand $\varphi_p^*\Gamma_{H,S}$ when $|S/H| = p$. Recall that there is an isomorphism of $G \times \Sigma_p$-sets

$$\Gamma_{H,S} \cong (G \times \Sigma_p)(s_iH),$$

where (s_iH) is an ordered tuple consisting of the cosets in S/H. Note that the subgroup $H \times e \subseteq G \times \Sigma_p$ acts trivially on the tuple (s_iH), so the tuple may be considered as a $G/H \times \Sigma_p$-set. Further, H is a normal subgroup of $\Gamma_{H,S}$ and $\Gamma_{H,S}/H$ is the graph subgroup of $G/H \times \Sigma_p$ for the action of S/H on itself by left multiplication.

5.3.1 Restriction of $\Gamma_{H,S}$ type basis elements

Among the relevant basis elements of $A(G \times \Sigma_p$, sets of the form $\Gamma_{H,S}$ are particularly important to α_n, as as such we will need to understand restrictions of such sets.

Proposition 5.3.3. *Assume $|S/H| = p$. There is an isomorphism of $G' \times \Sigma_p$-sets*

$$\varphi_p^*(\Gamma_{H,S}) \cong \coprod_{(G'/H')\backslash(G/H)/(S/H)} (G' \times \Sigma_p)/\Gamma(a_{S'/H'}).$$

Proof. This is a consequence of the double coset formula (Proposition 2.1.10) for $\mathrm{Res}^{G\times\Sigma_p}_{G'\times\Sigma_p} \mathrm{Tr}^{G\times\Sigma_p}_{\Gamma(a_{S/H})}$. Since $G' \times \Sigma_p \to G'/H' \times \Sigma_p$ is surjective and $G'/H' \times \Sigma_p$ and $\Gamma(a_{S/H})/H$ are both subgroups of $G/H \times \Sigma_p$, it is more convenient to work with the more complicated looking double coset formula for:

$$\mathrm{Res}^{G/H\times\Sigma_p}_{G'/H'\times\Sigma_p} \mathrm{Tr}^{G/H\times\Sigma_p}_{\Gamma(a_{S/H})/H}.$$

First consider the double cosets

$$(G'/H' \times \Sigma_p)\backslash(G/H \times \Sigma_p)/(\Gamma(a_{S/H})/H),$$

Since the quotient of the middle term by the subgroup on the left hand side is just $(G/H)/(G'/H')$, the double coset formula simplifies to

$$((G/H)/(G'/H'))/(\Gamma(a_{S/H})/H),$$

where $\Gamma(a_{S/H})/H \subseteq G/H \times \Sigma_p$ acts through the projection onto G/H. This is just the action of S/H on G/H through right multiplication (we remind the reader that these are abelian groups). This is in bijective correspondence with $(G'/H')\backslash(G/H)/(S/H)$.

Secondly, note that $G'/H' \times \Sigma_p$ is normal in $G/H \times \Sigma_p$, so it is conjugation invariant. Further, we have

$$(G'/H' \times \Sigma_p) \cap (\Gamma(a_{S/H})/H) = \Gamma(a_{S'/H'})/H'$$

as the intersection is just the restriction of the domain of the graph to G'/H'. This gives us an inclusion $\Gamma(a_{S'/H'})/H' \hookrightarrow \Gamma(a_{S/H})/H$.

Thus the double coset formula simplifies to

$$\mathrm{Res}^{G/H\times\Sigma_p}_{G'/H'\times\Sigma_p} \mathrm{Tr}^{G/H\times\Sigma_p}_{\Gamma(a_{S/H})/H} = \sum_{(G'/H')\backslash(G/H)/(S/H)} \mathrm{Tr}^{G'/H'\times\Sigma_p}_{\Gamma(a_{S'/H'})/H'} \mathrm{Res}^{\Gamma(a_{S/H})/H}_{\Gamma(a_{S'/H'})/H'},$$

where the terms that we are summing over have no influence on the transfer or restriction.

Applying this formula to the singleton $(\Gamma(a_{S/H})/H)$-set $(\Gamma(a_{S/H})/H)/(\Gamma(a_{S/H})/H)$, we learn that the restriction of $(G/H \times \Sigma_p)/(\Gamma(a_{S/H})/H)$ to $G'/H' \times \Sigma_p$ is

$$\coprod_{(G'/H')\backslash(G/H)/(S/H)} (G'/H' \times \Sigma_p)/(\Gamma(a_{S'/H'})/H').$$

Finally, restricting this along the surjection $G' \times \Sigma_p \to G'/H' \times \Sigma_p$ gives the $G' \times \Sigma_p$-set

$$\coprod_{(G'/H')\backslash(G/H)/(S/H)} (G' \times \Sigma_p)/(\Gamma(a_{S'/H'})).$$

■

Corollary 5.3.4. *If $\varphi_{H,S} \colon S'/H' \hookrightarrow S/H$ is an isomorphism and $|S/H| = p$, then*

$$\varphi_p^*(\Gamma_{H,S}) = \ell_H \Gamma_{H',S'}.$$

Proof. Apply Proposition 5.3.3. Since $\varphi_{H,S}$ is an isomorphism, $\Gamma(a_{S'/H'})$ is a transitive subgroup of $G' \times \Sigma_p$ and $(G' \times \Sigma_p)/\Gamma(a_{S'/H'}) = \Gamma_{H',S'}$. Since $\varphi_{H,S}$ is an isomorphism, we have $(G'/H')\backslash(G/H)/(S/H) = (G/H)/(G'/H')$ and $|(G/H)/(G'/H')| = \ell_H$ (by definition). ■

Proposition 5.3.5. *If $\varphi_{H,S}$ is an isomorphism and $|S/H| = p$, then*

$$\varphi^*(\alpha_p(\Gamma_{H,S})) = \alpha_p(\varphi_p^*(\Gamma_{H,S})).$$

Proof. By Corollary 5.3.4,

$$\varphi_p^*(\Gamma_{H,S}) = \ell_H \Gamma_{H',S'}.$$

So

$$\alpha_p(\varphi_p^*(\Gamma_{H,S})) = \ell_H(|S'/H'|G'/S' - G'/H').$$

On the other hand,

$$\alpha_p(\Gamma_{H,S}) = |S/H|G/S - G/H,$$

so

$$\varphi^*(\alpha_p(\Gamma_{H,S})) = |S/H|\ell_S G'/S' - \ell_H G'/H'.$$

Lemma 5.3.1 implies that $\ell_H = \ell_S$ and our assumption that $\varphi_{H,S}$ is an isomorphism implies that $|S/H| = |S'/H'|$. The result follows. ■

Now we will consider the case that $\varphi_{H,S}$ is not an isomorphism.

Corollary 5.3.6. *If $|S/H| = p$ and $\varphi_{H,S}$ is not an isomorphism, then*

$$\varphi_p^*(\Gamma_{H,S}) = 0$$

is in the transfer ideal $I_{\text{tr}} \subseteq A(G' \times \Sigma_p)$

Proof. Apply Proposition 5.3.3. Since $\varphi_{H,S}$ is not an isomorphism, $S' = H'$ and $\Gamma_{a_{S'/H'}}$ does not have transitive image in Σ_p (the image is just the identity element). Thus the restriction sits in the transfer ideal. ■

Proposition 5.3.7. *Assume that $|S/H| = p$ and $\varphi_{H,S}$ is not an isomorphism, then*

$$\varphi^*(\alpha_p(\Gamma_{H,S})) = 0.$$

Proof. As in the proof of Proposition 5.3.5, we have

$$\varphi^*(\alpha_p(\Gamma_{H,S})) = |S/H|\ell_S G'/S' - \ell_H G'/H'.$$

However, since $\varphi_{H,S}$ is not an isomorphism and $S/H \cong C_p$, we must have that $S'/H' = e$, which implies that $S' = H'$.

Since $|S/H| = p$ and $S' = H'$, we have

$$\begin{aligned}\ell_H &= |G/H|/|G'/H'| \\ &= (|S/H| \cdot |G/S|)/|G'/S'| \\ &= (p|G/S|)/|G'/S'| \\ &= p\ell_S.\end{aligned}$$

■

All together, these results prove that α_p is natural with respect to restriction.

5.3.2 $\alpha_{G,p}$ is natural with respect to restriction

With the above results in hand, we may now show that our $\alpha_{G,p}$ is natural with respect to restriction when G is abelian, i.e it satisfies all of the desired conditions.

Theorem 5.3.8. *Let G be an abelian group and p prime. Then for any group homomorphism $\varphi : G' \to G$, the map α_p constructed in Theorem 5.2.1 is natural with respect to restriction along φ, i.e the following diagram commutes:*

$$\begin{tikzcd} A(G \times \Sigma_p) \arrow[r, "\alpha_{G,p}"] \arrow[d, "\varphi_p^*"] & A(G) \arrow[d, "\varphi^*"] \\ A(G' \times \Sigma_p) \arrow[r, "\alpha_{G',p}"] & A(G') \end{tikzcd}$$

Proof. It suffices to directly check the maps agree on each basis element of $A(G\times\Sigma_p)$. We will classify the basis elements into four collections, then determine the behavior in each case. Suppose M is a subgroup of $G \times \Sigma_p$, then $G \times \Sigma_p$ falls into one of these four collections.

1. $G \times \Sigma_p/M$ is not transitively stabilized
2. $G \times \Sigma_p/M$ is of the form $(G \times \Sigma_p)/(H \times T)$ for a transitive subgroup T of Σ_p and a subgroup H of G.
3. $G \times \Sigma_p/M$ is of the form $\Gamma_{H,S}$ for $H \leq S \leq G$ and $|S/H| = p$ where $\varphi_{H,S}$ is an isomorphism
4. $G \times \Sigma_p/M$ is of the form $\Gamma_{H,S}$ for $H \leq S \leq G$ and $|S/H| = p$ where $\varphi_{H,S}$ is not an isomorphism

First, we note these collections are disjoint and by Proposition 5.1.4 and the definition of α_p in Theorem 5.2.1, any basis element given a non-zero image falls into a unique one of these collections, and basis elements given 0 as their image fall into collection (1).

We first consider $G \times \Sigma_p/M$ in collection (1). We first observe that the $G' \times \Sigma_p$ action factors through the $G \times \Sigma_p$ action. If there is a $G' \times \Sigma_p$-set in the decomposition of $G \times \Sigma_p/M$ which is transitively stabilized, then the stabilizer could be regarded as a subgroup of $G \times \Sigma_p$ and would be the stabilizer of a chosen generating element, but this contradicts our assumption M is non-transitive. Hence, we conclude that the action by $G' \times \Sigma_p$ must decompose the orbit into a collection of sets which are not transitively stabilized. Hence, we see that $\varphi^* \circ \alpha_{G,p}(G \times \Sigma_p/M) = 0 = \alpha_{G',p} \circ \varphi_p^*(G \times \Sigma_p/M)$ for basis elements in collection (1).

Secondly, we consider the case that $G \times \Sigma_p/M$ is in collection (2), i.e of the form $(G \times \Sigma_p)/(H \times T)$ for a transitive subgroup T of Σ_p and a subgroup H of G. As the stabilizer is a product, the effect of the restriction action is limited to the G and G' component respectively. Using this, we then directly calculate the two routes agree:

$$\begin{tikzcd}
(G \times \Sigma_P)/(H \times T) \arrow[r, "\alpha_{G,p}"] \arrow[d, "\varphi_p"] & {|\Sigma_p/T| G/H} \arrow[d, "\varphi^*"] \\
\ell_H G' \times \Sigma_p/H' \times T \arrow[r, "\alpha_{G',p}"] & {|\Sigma_p/T| \ell_H G/H}
\end{tikzcd}$$

Lastly, Proposition 5.3.5 shows α_p is natural for basis elements in collection (3), and together Corollary 5.3.6 with Proposition 5.3.7 show α_p is natural for basis elements in collection (4). As all basis elements have been shown to commute, since $\alpha_{G,p}$ and restriction are additive, we conclude $\varphi^* \alpha_{G,p} = \alpha_{G',p} \varphi_p^*$, as desired. ■

While it remains open to show α_n is natural for an arbitrary n, we expect it to be so and the primary question involves understanding the restriction of $G \times \Sigma_n$ sets of the form $\Gamma_{H,S}$ when $|S/H|$ is not prime. However, when G is not abelian, even understanding the decomposition of $(G/H)^p$ requires significant consideration.

Chapter 6 The $p^{\mathbf{th}}$ Adams operation for p-groups

The setting of abelian groups leads to a number of simplifications regarding the decomposition of $(G/H)^n$ and in computing the image of the Adams operation in Marks(G). If we move to groups which are not necessarily abelian, the complexity of the decomposition changes drastically, but fortunately the computation of the Adams operation remains reasonably simple. If p is prime, its only divisors are 1 and p, meaning the Gay, Morris, and Morris formula simplifies as follows. Additionally, there are notable simplifications to the orbit decomposition if G is a p-group. As such, in this section, we consider G to be a p-group.

Corollary 6.0.1. *For $H, K \leq G$, and r_d equal to the number of orbits of size d when K acts on G/H, we have the following equality.*

$$\chi(\psi_p([G/H]))[K] = r_1 + pr_p$$

Furthermore, we observe that the tuple $(eH)^p \in (G/H)^p$ still generates an orbit of the form $G \times \Sigma_p/H \times \Sigma_p$. To ensure we satisfy property (3) of Definition 3.1.1, any α_p must send this to G/H. After taking K fixed points, this term gives precisely r_1. Thus, our goal is to define an α_p such that the following holds, where orb_p is the set of orbits of size p for the given action:

$$\alpha_p((G/H)^p \setminus G \times \Sigma_p((eH)^p) = p|\text{orb}_p(K \curvearrowright G/H)|.$$

6.1 Goursat's Lemma

As mentioned before, we will require the usage of data from Goursat's lemma to understand the orbit decomposition in the general setting, as $G \times \Sigma_p$ is a product group and we seek to understand its subgroups up to conjugacy. Given a group is of the form of a product, we can create the following diagram, where π_G and π_Σ are the projection maps to the associated subgroup. Consider $M \leq G \times \Sigma_p$, and we then construct an analogous diagram as to 2.1.12 where $G' = \Sigma_p$:

$$\begin{tikzcd}
 & \ker(\pi_G|_M) \arrow[d] \arrow[dr, hook] & \\
\ker(\pi_\Sigma|_M) \arrow[r] \arrow[dr, hook] & M \arrow[r, "\pi_\Sigma"] \arrow[d, "\pi_G"] & \operatorname{im}(\pi_\Sigma|_M) \\
 & \operatorname{im}(\pi_G|_M) &
\end{tikzcd}$$

We may consider $\ker(\pi_G|_M)$ as subgroup of Σ_p, as every element of it is necessarily of the form (e, σ). Similarly, we may take $\ker(\pi_\Sigma|_M)$ to be a subgroup of G. Goursat's lemma provides that $\ker(\pi_\Sigma|_M) \trianglelefteq \operatorname{im}(\pi_G|_M)$ and $\ker(\pi_G|_M) \trianglelefteq \operatorname{im}(\pi_\Sigma|_M)$. In fact, it guarantees the existence of a specific isomorphism encoded by M, notably $\operatorname{im}(\pi_G|_M)/\ker(\pi_\Sigma|_M) \cong \operatorname{im}(\pi_\Sigma|_M)/\ker(\pi_G|_M)$. Finally, given such choices of subgroups and an isomorphism, one obtains a specific subgroup of the product. This is to say, if we want to understand subgroups of $G \times \Sigma_p$, then it suffices to understand the relevant subgroups of G, Σ_p, and chosen isomorphisms.

6.2 Understanding $(G/H)^p$ for G a p-group

Just as in the abelian case, we must first understand the structure of the orbits in $(G/H)^p$ which are transitively stabilized. First, we recall a classic result that a transitive subgroup of Σ_p must contain a cycle of length p. Suppose $(\hat{g}_i H)_{i=1}^p$ generates a transitively stabilized orbit. We first act on $(\hat{g}_i H)_{i=1}^p$ by $(\hat{g_1}^{-1}, e)$ to ensure the first entry is eH. Then, as there is some p-cycle $\hat{\sigma}$ in the stabilizer, we may rearrange it by a permutation τ to force $(g, (1, 2, \cdots, p))$ is in the stabilizer. The resulting tuple will be written as $(g_i H)_{i=1}^p$. We then conclude the following:

$$(g_i H)_{i=1}^p = (g, (1, 2 \cdots p))^k (g_i H)_{i=1}^p = (g^k g_{\sigma^{-k}(i)} H)$$

If we focus on the $(k+1)$st component, we observe $g_{k+1}H = g^k H$. Additionally, we note that $g^p H = eH$. Hence, we conclude the orbit generated by $(\hat{g}_i H)_{i=1}^p$ includes an element of the form $\bar{x} = (g^i H)_{i=0}^{p-1}$ for $g \in G$ such that $g^p \in H$.

This will allow us to classify all transitively stabilized orbits, however we must take certain quotients to prevent overcounting.

Definition 6.2.1. For $H \leq G$, we may define the following set of cosets:

$$\operatorname{sub}_p(G/H) := \{gH \in G/H | g^p \in H, g^i \notin H \text{ for } i \in [1, p-1]\}$$

Importantly, note that eH is not in this set.

Alongside each element, we consider the data of a tuple $\bar{x} = (g^i H)_{i=0}^{p-1} \in (G/H)^p$ which generates a $G \times \Sigma_p$-orbit. There are two quotients of this set under which the orbit is invariant. First, there is an action by C_{p-1} on either the left or right, as if $g^p \in H$, then for $i \in [1, p-1]$ we have $(g^i)^p \in H$. As $(g^j H)_{j=0}^{p-1}$ only differs from $(g^{ij} H)_{j=0}^{p-1}$ by a permutation, they generate the same orbit. Additionally, there is a left H-action as follows. Given gH, observe that for any $h \in H$, $hgH = hgh^{-1}H$ and $(hgh^{-1})^p = hg^p h^{-1} \in H$, corresponding to the tuple $(hg^i h^{-1} H)_{i=0}^{p-1}$, which we see is in the same orbit as $(g^i H)_{i=0}^{p-1}$ by acting with the element $(h, e) \in G \times \Sigma_n$.

Definition 6.2.2. Considering all of the above, we define:

$$\operatorname{Sub}_p(G/H) := H \backslash \operatorname{sub}_p(G/H)/C_{p-1}.$$

6.2.1 Transitively Stabilized Orbits of $(G/H)^p$

Just as in the abelian case, since α_p will ideally factor through the transfer ideal, we do not need to understand the exact decomposition of $(G/H)^p$. As such, it suffices to see the decomposition into transitively stabilized orbits, which we show below.

Lemma 6.2.3. *For G a p-group and $H \leq G$, there is a $G \times \Sigma_p$-set map from transitively stabilized orbits of $(G/H)^p$ to $G \times \Sigma_p$-sets corresponding to elements of* $\mathrm{Sub}_p(G/H)$.

$$\textit{Tr. Stabilized Orbits of } (G/H)^p \to G \times \Sigma_p/H \times \Sigma_p \amalg \coprod_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x}=(g^iH)_{i=0}^{p-1}}} G \times \Sigma_p/\operatorname{Stab}(\bar{x}).$$

Proof. We first map the orbit associated to $(eH)_{i=1}^p$ to $G \times \Sigma_p/H \times \Sigma p$. Let $(g_iH)_{i=1}^p \in (G/H)^p$ generate an orbit which is transitively stabilized. First, we act by (g_1^{-1}, e) to obtain the tuple $(g_1^{-1}g_iH)_{i=1}^p$. This is still in the same orbit, and thus is transitively stabilized. As every transitive subgroup of Σ_p contains a p-cycle, this implies there is an element $(g, \sigma) \in \mathrm{Stab}((g_1^{-1}g_iH)_{i=1}^p)$ where σ is a p-cycle. Furthermore, this implies (g^p, e) is in the stabilizer, which gives $g^p \in \mathrm{Sub}_p(G/H)$. Let $k \in [2, p]$, then as σ is a p-cycle, there exists a value $\ell \in [1, p-1]$ so that $\sigma^{-\ell}(k) = 1$. We may use this to see the following:

$$(g_1^{-1}g_iH)_{i=1}^p = (g^\ell, \sigma^\ell)(g_1^{-1}g_iH)_{i=1}^p = (g^\ell g_1^{-1}g_{\sigma^{-\ell}(i)}H)_{i=1}^p \implies g_1^{-1}g_kH = g^\ell H$$

As all cosets of the form $g^jH, j \in [1, p-1]$ are identified under the C_{p-1} quotient, we map this tuple to the equivalence class of the coset $g_1^{-1}g_kH \in \mathrm{Sub}_p(G/H)$. There are two checks to ensure this map is well-defined, the first of which is to ensure it is independent of which element of the orbit we choose, and the second is to see that it is independent of the representatives for the cosets in $(g_iH)_{i=1}^p$. It should be noted that the choice of (g, σ) does not affect our choice of $(g_1^{-1}g_kH)$ as the image, it was only used to show the coset $g_1^{-1}g_kH$ this lies within $\mathrm{Sub}_p(G/H)$.

Suppose $(\hat{g}, \hat{\sigma}) \in G \times \Sigma_p$. Showing the map is independent of our chosen orbit generator is equivalent to showing that acting by $(\hat{g}, \hat{\sigma})$ before mapping does not change the image. We see that $(\hat{g}, \hat{\sigma})(g_iH)_{i=1}^p = (\hat{g}g_{\hat{\sigma}^{-1}(i)}H)$. Continuing with our original construction, we then act by $(g_{\hat{\sigma}^{-1}(1)}^{-1}\hat{g}^{-1}, e)$ to obtain the tuple $(g_{\hat{\sigma}^{-1}(1)}^{-1}g_{\hat{\sigma}^{-1}(i)}H)$. Immediately, we see that the G component of the action will not affect the chosen coset, and additionally, since $\hat{\sigma}$ is a bijection, the tuple is reordered and multiplied by $g_{\hat{\sigma}^{-1}(1)}$ rather than g_1. Hence, it suffices to show that the image is the same regardless of which entry we invert before producing our image. The reordering will not affect our map, as we have already shown all cosets other than eH have the same equivalence class in $\mathrm{Sub}_p(G/H)$.

Hence, fix $j \in [1, p]$, then our goal is to show that $g_j^{-1}g_mH$ for $m \neq j$ is in the same equivalence class as $g_1^{-1}g_mH$ for $m \neq 1$. As before, we choose an element $(g, \sigma) \in \mathrm{Stab}((g_1^{-1}g_iH)_{i=1}^p)$ such that $\sigma^{-\ell}(k) = 1$. We may obtain the tuple $(g_j^{-1}g_iH)_{i=1}^p$ from the tuple $(g_1^{-1}g_iH)_{i=1}^p$ by acting on it with the element $(g_j^{-1}g_1, e)$. This will conjugate the stabilizer appropriately, so we have $(g_j^{-1}g_1gg_1^{-1}g_j, \sigma) \in \operatorname{Stab}((g_j^{-1}g_iH)_{i=1}^p)$. Again, as σ is a p-cycle, for $k \in [1, p], k \neq j$, there exists a power $\hat{\ell} \in [1, p-1]$such that $\sigma^{-\hat{\ell}}(k) = j$, giving us the following:

$$(g_j^{-1}g_iH)_{i=1}^p = (g_j^{-1}g_1g^{\hat{\ell}}g_1^{-1}g_j, \sigma^{\hat{\ell}})(g_j^{-1}g_iH)_{i=1}^p \implies g_j^{-1}g_kH = g_j^{-1}g_1g^{\hat{\ell}}g_1^{-1}g_jH$$

Furthermore, there exists some $\ell' \in [1, p-1]$ such that $g_1^{-1}g_jH = g^{\ell'}H$. Notably, this also implies $g^{-\ell'}g_1^{-1}g_j \in H$. As $\operatorname{Sub}_p(G/H)$ quotients by a left H action, acting by this element will not change the equivalent class of our image, giving us the following:

$$g_j^{-1}g_kH \sim g^{-\ell'}g_1^{-1}g_j \cdot g_j^{-1}g_1g^{\hat{\ell}}g_1^{-1}g_jH = g^{\hat{\ell}-\ell'}g_1^{-1}g_jH = g^{\hat{\ell}-\ell'}g^{\ell'}H = g^{\hat{\ell}}H$$

As $\hat{\ell}$ is guaranteed not to be a multiple of p, this is a coset which is identified with $g^{\ell}H$ from our original image, hence the image of the map is not dependent on the chosen element of the orbit.

Fortunately, seeing the image is independent of the representatives for $(g_iH)_{i=1}^p$ is much simpler. Let $(g_iH)_{i=1}^p = (g_ih_iH)_{i=1}^p$ be another choice of representatives. Our initial procedure first multiplies by $(h_1^{-1}g_1^1, e)$, giving the tuple $(h_1^{-1}g_1^{-1}g_ih_iH)$, which will be mapped to $h_1^{-1}g_1^{-1}g_kh_kH$ for any choice of $k \in [2, p]$. This only differs from the original image by acting with an element of H on the left, and as $\operatorname{Sub}_p(G/H)$ quotients by such an action, our images will be identified under the quotient. Hence, the forward map is well-defined.

■

Lemma 6.2.4. *For G a p-group and $H \leq G$, there is a $G \times \Sigma_p$-set map from sets corresponding to elements of $\operatorname{Sub}_p(G/H)$ to transitively stabilized orbits of $(G/H)^p$.*

$$\textit{Tr. Stabilized Orbits of } (G/H)^p \leftarrow G \times \Sigma_p/H \times \Sigma_p \amalg \coprod_{\substack{gH \in \operatorname{Sub}_p(G/H) \\ \bar{x} = (g^iH)_{i=0}^{p-1}}} G \times \Sigma_p/\operatorname{Stab}(\bar{x}).$$

Proof. We first send $G \times \Sigma_p/H \times \Sigma_p$ to the orbit generated by $(eH)_{i=1}^p$. Our prior work shows that given $gH \in \operatorname{Sub}_p(G/H)$, mapping $G \times \Sigma_p/\operatorname{Stab}(\bar{x})$ to the orbit of $\bar{x}$ is well-defined and respects the $G \times \Sigma_p$-action. ■

Proposition 6.2.5. *The maps in 6.2.3 and 6.2.4 are inverses, and as such, the following is a decomposition of $(G/H)^p$ into transitively stabilized $G \times \Sigma_p$ basis elements.*

Proof. All that remains is to compose these maps in each direction. Given an orbit associated to $(eH)_{i=1}^p$ and the basis element $G \times \Sigma_p/H \times \Sigma_p$, the composition is immediately seen to be identity in each direction.

If we begin with a transitively stabilized orbit $(\hat{g}_iH)_{i=1}^p$ and perform the earlier construction such that we obtain another element of the orbit $(g_iH)_{i=1}^p$ stabilized by an element of the form $(g, (12 \cdots p))$, then the composition returns the orbit of the tuple $(g^iH)_{i=0}^{p-1}$, which is in the same orbit as we have shown $(g^iH)_{i=0}^{p-1} = (g_iH)_{i=1}^{p}$, hence it is identity on the orbit itself.

Composition beginning with $gH \in \mathrm{Sub}_p(G/H)$ is essentially immediate, as it maps to $\bar{x} = (g^iH)_{i=0}^{p-1}$, which in turn maps to the coset $gH \in \mathrm{Sub}_p(G/H)$ by our initial construction.

As the map is well-defined in both directions and each composition is identity, it is a bijection, and we see that alongside a term for a 'diagonal' tuple, $\mathrm{Sub}_p(G/H)$ classifies the transitively stabilized orbits of $(G/H)^p$. ■

Before we can fully classify orbits, we must make an observation regarding the structure of these stabilizers. By our earlier observation, every transitively stabilized orbit is generated by a tuple of the form $\bar{x} := (g^iH)_{i=0}^{p}$ for some $gH \in \mathrm{Sub}_p(G/H)$, with $\mathrm{Stab}\,(\bar{x})$ the stabilizer of this element. As this is a subgroup of $G \times \Sigma_p$, we may apply Goursat's lemma, then observe that $\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))})/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})}) \cong \operatorname{im}(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})/\ker(\pi_G|_{\mathrm{Stab}\,(\bar{x})})$. Following this, an element of $\ker(\pi_G|_{\mathrm{Stab}\,(\bar{x})})$ must be of the form (e, σ). However, as all p elements of $\bar{x}$ are distinct, σ must be trivial. Hence, we conclude this kernel is trivial, showing $\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))})/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})}) \cong \operatorname{im}(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$. As G is a p-group, so is this quotient, and thus $\operatorname{im}(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$ must also be a p-group within Σ_p. Up to conjugacy, the only transitive subgroup of Σ_p which is also a p-group is C_p.

Definition 6.2.6. There is a surjective set map from $G/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$ to $G/\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))})$, which we will denote γ. This follows from the fact that $\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$ is contained in $\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))})$, and thus the following assignment is well-defined:

$$\begin{tikzcd} G/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})}) \arrow[r, "\gamma"] & G/\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))}) \\ \ell\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})}) \arrow[r, mapsto] & \ell\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))}). \end{tikzcd}$$

Furthermore, this also implies that if $\ell\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$ is K-fixed for $K \leq G$, then $\ell\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))})$ must also be K-fixed. Hence, this map is well-defined after taking K-fixed points, though it may no longer be surjective. We will denote the induced map on fixed points by γ^K.

6.2.2 Relating $\mathbf{orb}_p(K \curvearrowright G/H)$ to K-fixed point data

As discussed before, we may check if a proposed α_p factors ψ_p on the image of the marks homomorphism in the ring of marks using 2.4.4. Due to condition (3), the main consideration is how our formula will account for the term which corresponds to orbits of size p for K acting on (G/H). Here, we show this is in bijection with a set of K-fixed points for certain G-sets.

Lemma 6.2.7. *Given $H, K \leq G$, there is a set map from the following collection of K-fixed points to orbits of size p for K acting on G/H.*

$$\coprod_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x}=(g^iH)_{i=0}^{p-1}}} (G/\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})}))^K \setminus \gamma^K(G/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})) \to \mathrm{orb}_p(K \curvearrowright G/H)$$

Proof. For simplicity in notation, let $S_{\bar{x}} := \operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})})$ and $J_{\bar{x}} := \ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$. Following this, given $gH \in \mathrm{Sub}_p(G/H)$ and $\ell S_{\bar{x}}$ such that it is K-fixed but $\ell J_{\bar{x}}$ is not, then we claim the set $\{\ell g^i H\}_{i=0}^{p-1}$ is an orbit of size p for K acts on G/H. As $\mathrm{Sub}_p(G/H)$ involves a quotient by a left H action and a C_{p-1} action, and we made use of a specific representative of $\ell S_{\bar{x}}$, there are three issues to consider before this map is well-defined.

We will first show that $\{\ell g^i H\}_{i=0}^{p-1}$ is a K-set of size p at all, then following this, show the map is well-defined. Let $k \in K$, and consider $k\ell g^i H = \ell\ell^{-1}k\ell g^i H$. As $\ell S_{\bar{x}}$ is K-fixed, we conclude $\ell^{-1}k\ell \in S_{\bar{x}}$ and as such, it will only permute the set $\{g_i H\}_{i=0}^{p-1}$ and the leftmost ℓ will not be affected. Furthermore, $\ell J_{\bar{x}}$ is not K-fixed, there exists a k such that $\ell^{-1}k\ell$ is in $S_{\bar{x}} \setminus J_{\bar{x}}$, making such an element of the form $(\ell^{-1}k\ell, \sigma)$ where σ is of order p. Hence, K acts on this set non-trivially, and as K is a p-group and this set is size p, it is one orbit of size p.

We recall that the C_{p-1} action quotients the cosets gH and g^jH where $j \in [1, p-1]$. This gives the image as the set $\{\ell g^{ij} H\}_{i=0}^{p-1}$. As $g^p \in H$, it follows $g^{ij} \notin H$ for all $i \in [1, p-1]$, this will range through the same collection of cosets as gH would produce, since they are all distinct. Hence, we obtain the same image regardless of the representative we choose under the C_{p-1} quotient.

Next, suppose $\ell S_{\bar{x}} = \hat{\ell} S_{\bar{x}}$, hence we must show $\{\ell g^i H\}_{i=0}^{p-1} = \{\hat{\ell} g^i H\}_{i=0}^{p-1}$. As $\ell S_{\bar{x}} = \hat{\ell} S_{\bar{x}}$, we conclude $\hat{\ell}^{-1}\ell \in S_{\bar{x}}$. Since it is in the stabilizer of $\bar{x}$, acting by it will only reorder the tuple, thus the set is unchanged, so $\{\hat{\ell}^{-1}\ell g^i H\}_{i=0}^{p-1} = \{g^i H\}_{i=0}^{p=1}$. This is sufficient to construct a bijection between $\{\ell g^i H\}_{i=0}^{p-1}$ and $\{\hat{\ell} g^i H\}_{i=0}^{p-1}$, and due to the C_{p-1} invariance of the image, the choice of representative does not change the image.

Lastly, we must consider the quotient by the left H action on $\mathrm{sub}_p(G/H)$. Let $h \in H$, and we first note that the stabilizer of $(hg^iH)_{i=0}^{p-1}$ is not $S_{\bar{x}}$, but instead $hS_{\bar{X}}h^{-1}$. However, as this is only conjugate, this still corresponds to the same $G \times \Sigma_p$ orbit and still gives a map from the desired domain. Given that $\ell S_{\bar{x}}$ is K-fixed, then we have $\ell h^{-1} \cdot hS_{\bar{x}}h^{-1}$ is K-fixed, and so our produced orbit will be $\{\ell h^{-1} \cdot (hg)^i H\}_{i=0}^{p-1}$. This intersects with the original set, and as it is a K-orbit, must be the same set as $\{\ell g^i H\}_{i=0}^{p-1}$, so action by H does not change the image.

Hence, the map gives an element of the desired codomain and is well-defined. ■

Lemma 6.2.8. *Given $H, K \leq G$, there is a set map from the orbits of size p for K acting on G/H to the following collection of K-fixed points.*

$$\coprod_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x}=(g^iH)_{i=0}^{p-1}}} (G/\operatorname{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})}))^K \setminus \gamma^K(G/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})) \leftarrow \mathrm{orb}_p(K \curvearrowright G/H)$$

Proof. As before, we will use the following notation for simplicity. Let $S_{\bar{x}} := \mathrm{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})})$ and $J_{\bar{x}} := \ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$. To define this map, suppose $\{g_iH\}_{i=1}^{p}$ is a K-set of size p. Choose any element of this set, say g_1H. As this set is a K-orbit, K acts transitively on it. As K is a p-group, there exists an element $k \in K$ such that $\{k^ig_1H\}_{i=0}^{p-1} = \{g_iH\}_{i=0}^{p-1}$. Providing an image requires an $\bar{x} \in (G/H)^p$, an element of $\mathrm{Sub}_p(G/H)$, and a coset of $G/S_{\bar{x}}$ which is K-fixed but the associated $G/J_{\bar{x}}$ coset is not K-fixed. We define the image to be given by $\bar{x} = (g_1^{-1}k^ig_1H)_{i=0}^{p-1}$, where $g_1^{-1}kg_1H \in \mathrm{Sub}_p(G/H)$ as $k^pg_1H = g_1$, and the coset is $g_1S_{\bar{x}}$. Furthermore, this coset is K-fixed, as let $\hat{k} \in K$. It suffices to show that multiplication on the left by $g_1\hat{k}g_1$ only permutes $\{g_1^{-1}k^ig_1\}_{i=0}^{p-1}$. This multiplication gives $\{g_1^{-1}\hat{k}k^ig_1\}_{i=0}^{p-1}$. As $\hat{k} \in K$ and K acts on the set $\{k^ig_1\}_{i=0}^{p-1}$, then $\hat{k}$ either fixes everything or permutes them, so $g_1S_{\bar{x}}$ is K-fixed. However, $g_1J_{\bar{x}}$ is not K-fixed, as the chosen k must not fix it, since fixing $g_1J_{\bar{x}}$ is equivalent to acting trivially on g_1H. Hence, this image satisfies all of the desired quotients, but we need ensure this map is does not depend on our choices. Notably, we must show it is independent of which element of the orbit we call g_1H, and which element $k \in K$ we use to generate the orbit from g_1H.

Suppose we had chosen another element of $\{g_iH\}_{i=1}^{p}$. Fix the chosen $k \in K$ to be such that $\{k^jg_1H\}_{j=0}^{p-1} = \{g_jH\}_{j=0}^{p-1}$ and $g_iH = k^ig_1H$, which we may do by taking appropriate powers of k and re-indexing the set as necessary. We then would obtain $g_i^{-1}kg_iH \in \mathrm{Sub}_p(G/H)$. Observe that $g_i^{-1}kg_iH = g_i^{-1}k^{i+1}g_1H$, and furthermore, as $g_1^{-1}k^{-i}g_i \in H$ and $\mathrm{Sub}_p(G/H)$ is quotiented by the left H action, our image will be in the same equivalent class as $g_1^{-1}k^{-i}g_ig_i^{-1}k^{i+1}g_1 = g_1^{-1}kg_1H$. As we obtain the same $\mathrm{Sub}_p(G/H)$ equivalence class, $\bar{x}$ is unchanged up to reordering, and hence $S_{\bar{x}}$ is the same up to conjugacy. Furthermore, we may conclude that $g_1S_{\bar{x}} = g_iS_{\bar{x}}$, as this is equivalent to observing $\{g_1^{-1}k^jg_1H\}_{j=0}^{p-1} = \{g_i^{-1}k^{j+i}g_i\}_{j=0}^{p+1}$, which follows as they share an equivalence class in $\mathrm{Sub}_p(G/H)$.

Suppose $\hat{k}$ is another element of k such that $\{\hat{k}^ig_1H\} = \{g_iH\}$. This will reorder the original $\bar{x}$, but not change the elements of, hence the stabilizer will not change up to conjugacy. Hence, as the two tuples are simply permutations of one another, this implies $g_1^{-1}kg_1H = g_1^{-1}\hat{k}^ig_1H$ for some $i \in [1, p-1]$. Since the latter is in the same equivalence class as $g_1^{-1}\hat{k}g_1H$ in $\mathrm{Sub}_p(G/H)$, the image is independent of our choice of $k \in K$ with the desired properties.

Hence, the map gives an element of the desired codomain and is well-defined. ■

Proposition 6.2.9. *The maps in 6.2.7 and 6.2.8 are inverses, and as such, we may classify orbits of size p for K acting on G/H using that data. That is to say, the following bijection holds:*

$$\coprod_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x} = (g^iH)_{i=0}^{p-1}}} (G/\,\mathrm{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})}))^K \setminus \gamma^K(G/\ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})) \cong \mathrm{orb}_p(K \curvearrowright G/H)$$

Proof. We must show each composition is identity. As before, let $S_{\bar{x}} := \mathrm{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})})$ and $J_{\bar{x}} := \ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$. First, we begin with a K-fixed point of the appropriate form. Let $gH \in \mathrm{Sub}_p(G/H)$ and $\ell S_{\bar{x}}$, which then maps to $\{\ell g^iH\}_{i=0}^{p-1}$. For the return map, choose $k \in K$ which acts transitively on this set, then we obtain $\ell^{-1}k\ell H \in \mathrm{Sub}_p(G/H)$ which defines $\bar{x}' = (\ell^{-1}k^i\ell H)_{i=0}^{p-1}$ and gives $\ell S_{\bar{x}'}$. As k acts transitively on the orbit, $k\ell H = \ell g^j H$ for some $j \in [1, p-1]$, which gives $\ell^{-1}k\ell H = g^j H$, hence, they identify under the C_{p-1} quotient and $S_{\bar{x}}$ is conjugate to $S_{\bar{x}'}$.

For the reverse composition, it suffices to see that the orbit is not entirely lost, as any singular element will generate the orbit. Since the full composition can be chosen to preserve g_1H, it gives identity in this direction as well, hence, the map is a bijection. ■

The proposition allows us to understand r_p as in Gay, Morris, and Morris' theorem. When combined with the fact that orbit associated to $(eH)_{i=1}^{p}$ is $G \times \Sigma_p / H \times \Sigma p$ and must be mapped under α_p to G/H, all that remains is to understand the above fixed points as a G-set.

6.2.3 Translating K-fixed point data into G-sets

Lastly, while this set of K-fixed points is in bijection with orbits of size p, it is not strictly a sum of basis elements in $A(G)$. One final lemma will allow is to translate this to the appropriate data for the Burnside ring.

Lemma 6.2.10. *Given $H, K \leq G$ and $gH \in \mathrm{Sub}_p(G/H)$ with $\bar{x} = (g^iH)_{i=0}^{p-1}$, let $S_{\bar{x}} = \mathrm{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x}))})$ and $J_{\bar{x}} = \ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$. The following equality holds:*

$$p|(G/S_{\bar{x}})^K \setminus \gamma^K(G/J_{\bar{x}})| = p|(G/S_{\bar{x}})^K| - |(G/J_{\bar{x}})^K|.$$

Proof. As $S_{\bar{x}}/J_{\bar{x}} \cong C_p$, the size of the fibers must be 0 or p. In the case the fibers are size p, they can be given explicitly. Suppose $\ell J_{\bar{x}}$ is K-fixed, i.e $\ell^{-1}k\ell \in J_x$ for all $k \in K$. Then we see $\ell g^i J_{\bar{x}}$ is also K-fixed for $i \in [0, p-1]$ as $g^{-i}\ell^{-1}k\ell g^i \cdot g^jH = g^jH$, hence $g^{-i}\ell^{-1}k\ell g^i \in J_{\bar{x}}$. These p cosets are distinct, as otherwise, this implies $g^j \in J_{\bar{x}}$ for some $j \neq p$, which contradicts the fact the entries of $\bar{x}$ are distinct, giving all p possible cosets. However, as $g^j \in S_{\bar{x}}$ for any j, these cosets collapse to $\ell S_{\bar{x}}$ under γ. Hence, as we range over elements of $(G/J_{\bar{x}})^K$ and consider them as elements of $(G/S_{\bar{X_x}})^K$ by applying γ^K, they contribute 0 to the left and cancel to 0 on the right. All other cosets present are those of $(G/S_{\bar{x}})^K \setminus \gamma^K(G/J_{\bar{x}})$, which contribute p to each side. Hence, the equality holds. ■

6.3 Factoring ψ^p through P^p for G a p-group

As before, now that we have a decomposition and the necessary framework, it is possible to define an α_p factoring ψ_p for G a p-group. However, due to the method of this decomposition, it is unclear if this is natural with respect to restriction.

Theorem 6.3.1. *If G is a p-group with $H \leq G$ and p prime, $T \leq \Sigma_p$ a transitive subgroup, then we may define an additive map α_p such that $\alpha_p \circ P_p = \psi_p$, satisfying properties* (1) *and* (3)*.The map is given on basis members and linearly extended. If a basis member of $A(G \times \Sigma_p)$ is not given a specific image, the basis member is in* *the kernel. As above, if* $gH \in \mathrm{Sub}_p(G/H)$ *and* $\bar{x} = (g^iH)_{i=0}^{p-1}$*, let* $S_{\bar{x}} = \mathrm{im}(\pi_G|_{\mathrm{Stab}\,(\bar{x})})$ *and* $J_{\bar{x}} = \ker(\pi_\Sigma|_{\mathrm{Stab}\,(\bar{x})})$*. The map* α_p *is given by*

$$\alpha_p((G \times \Sigma_P)/(H \times T)) = |\Sigma_p/T|(G/H)$$

and, for $gH \in \mathrm{Sub}_p(G/H)$*,*

$$\alpha_p(G \times \Sigma_p/\,\mathrm{Stab}(\bar{x})) = p(G/S_{\bar{x}}) - (G/J_{\bar{x}}).$$

Proof. First, we observe that as $\mathrm{Sub}_p(G/H)$ does not include eH, the two assignments do not interact in any way. Additionally, we see the definition of α_p satisfies (1) as α_p is defined on basis members and linearly extended, hence additive. Furthermore, only basis members with a transitive stabilizer were given a non-zero image, and as such it factors through the transfer ideal. Additionally, property (3) holds for our earlier choice of $\alpha_{e,p}$, as if we begin with $(G/H) \otimes \Sigma_p/T$, both directions give $|\Sigma_p/T|(G/H)$.

By Corollary 6.0.1, it suffices to check that $\chi(\alpha \circ P_p([G/H]))[K] = r_1 + pr_p$ for each conjugacy class of subgroups $K \leq G$. We have

$$\begin{aligned}
\chi(\alpha P_p([G/H]))[K] &= (\alpha_p(P_p(G/H))^K \\
&= \alpha_p(G \times \Sigma_p/H \times \Sigma_p)^K \amalg \coprod_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x} = (g^iH)_{i=0}^{p-1}}} \alpha_p(G \times \Sigma_p/\,\mathrm{Stab}\,(\bar{x}))^K && (6.2.5) \\
&= (G/H)^K + \sum_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x} = (g^iH)_{i=0}^{p-1}}} (p(G/S_{\bar{x}})^K - (G/J_{\bar{x}})^K) \\
&= r_1 + \sum_{\substack{gH \in \mathrm{Sub}_p(G/H) \\ \bar{x} = (g^iH)_{i=0}^{p-1}}} p|(G/S_{\bar{x}})^K \setminus \gamma^K(G/J_{\bar{x}})| && (6.2.10) \\
&= r_1 + p|\mathrm{orb}_p(K \curvearrowright G/H)| && (6.2.9) \\
&= r_1 + pr_p
\end{aligned}$$

As α_p is defined on basis members and linearly extended, it is additive, and furthermore, the transfer ideal is in the kernel by construction. Given $G/H \in A(G)$ and Σ_p/T in $A(\Sigma_p)$, property (3) requires the image under α_p to be $|\Sigma_p/T|(G/H)$ for it to commute with our chosen $\alpha_{e,n}$, which agrees with our assignment. Hence, α_p satisfies conditions (1) and (3) and by Corollary 6.0.1, we see $\psi_p = \alpha_p \circ P_p$.

■

Chapter 7 Future Work

In this chapter, we discuss a number of routes to consider for extending this work, of which there are two of particular interest. First, we ideally would find a classification of groups such that a factorization is known for more values of n, and secondly, if $\alpha_{G,n}$ does not exist with our desired conditions for all groups and n, we would like to find an infinite class of such counter examples.

7.1 Factoring ψ^{p^k} through P^{p^k}

There are extensions to much of the work in Chapter 5 to factoring ψ^{p^k} through P^{p^k}, but they require determining the decomposition of $(G/H)^{p^k}$ and a classification of orbits of size p^j when $j \leq k$ for K acting on G/H. Furthermore, Lemma 6.2.10 relies on the fact that certain quotients are size p, but we believe we can extend this to whenever certain quotients are abelian. Hence, we propose the following conjecture:

Conjecture 7.1.1. *Given p prime and $n \in \mathbb{N}$, then for G a p-group such that all proper sub-quotients of size p^j are abelian for $j \leq k$, there exists an α_{G,p^k} factoring ψ^{p^k} satisfying conditions* (1) *and* (3).

This conjecture would require propositions analogous to 6.2.5, 6.2.9, and 6.2.10 respectively, of which some proposed formulae exist for at this time, but we have omitted for simplicity.

7.2 Determining G, n where $\alpha_{G,n}$ fails to exist

It's likely that the failure of $\alpha_{\Sigma_4,3}$ is not unique, given that collisions in the $A(G \times \Sigma_n)$ decomposition of $(G/H)^n$ can be constructed for symmetric groups with ease. Computational complexity increases sharply, so at this time no other exact results are known. However, based on this work, we propose the following conjecture:

Conjecture 7.2.1. *For $n \geq 4$, there does not exists an $\alpha_{\Sigma_n,3}$ factoring ψ^3 satisfying conditions* (1) *and* (3).

In the case of $G = \Sigma_4$, it was possible to show this failure by only computing the decomposition of three basis elements, but it is likely more will need to be computed for higher symmetric groups, leading to increased computational difficulty.

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