Uniform Regularity Estimates for the Stokes System in Perforated Domains

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DISSERTATION

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fulfillment of the requirements for
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ABSTRACT OF DISSERTATION

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We consider the Stokes equations in an unbounded domain $\omega_{\varepsilon,\eta}$ perforated by small obstacles, where ε represents the minimal distance between obstacles and η is the ratio between the obstacle size and ε. We are able to obtain uniform $W^{1,q}$ estimates for solutions to the Stokes equations in such domains with bounding constants depending explicitly on ε and η.

KEYWORDS: Stokes Equations, Homogenization, Regularity, Perforated Domains, Analysis, Differential Equations

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Chapter 1 Introduction

1.1 Background

Homogenization theory is a branch of partial differential equations concerned with problems where a differential operator or domain depends on a parameter ε which tends to zero. The homogenization of differential operators has vast applications to physics and materials science, in particular to the study of mixed media consisting of components with different physical properties (electrical/thermal conductivity, elasticity, etc.). As an example, consider a medium consisting of a main material containing small particles of a different material, placed periodically with a distance of ε between them. A differential operator in this medium will have rapidly oscillating coefficients due to the differing properties of the constituent materials. However, by sending both the scale of periodicity ε and the particle size to zero, solutions to equations involving such a differential operator can converge, leading to a homogenized equation.

Typical results of interest in the field of homogenization theory include obtaining qualitative convergence in problems like those described above, obtaining convergence rates once qualitative convergence has been shown, and establishing regularity results for solutions during the homogenization process. To introduce results of historical interest, we consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}(A(x/\varepsilon)\nabla u_\varepsilon) = F & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$ has periodic, bounded, and measurable entries, and satisfies the ellipticity condition

$$A\xi \cdot \xi \geq c_0|\xi|^2$$

for any $\xi \in \mathbb{R}^d$, where $c_0 > 0$. If $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$, it can be shown that u_ε converges weakly in $H^1(\Omega)$ to some u_0 as $\varepsilon \to 0$. Moreover, u_0 satisfies the homogenized equation

$$\begin{cases} -\mathrm{div}(\hat{A}\nabla u_0) = F & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega, \end{cases} \tag{1.1.2}$$

where $\hat{A} \in \mathbb{R}^{d\times d}$ is a constant matrix.

Early work in the field of homogenization, starting with the work of E. de Giorgi and S. Spagnolo in the late 1960's, was concerned with developing methods to prove such qualitative convergence results. The very general viewpoint of Γ and G convergence of operators introduced by de Giorgi gives an abstract framework for obtaining convergence. Later methods include Tartar's method of oscillating test functions, which relies on energy estimates and a careful construction of test functions with similar periodic behavior to u_ε, and the notion of two-scale convergence developed by G. Nguetseng [12] and G. Allaire [3], which is effective for problems with solutions u_ε that don't have limits in a classical sense. Classic monographs ([5], [7], [6], [11]) are available on this topic.

More recent work ([10], [17], [15], [16]) has been concerned with quantitative convergence results, including convergence rates and regularity theorems. In view of the qualitative convergence above, convergence rates allow us to determine how well the homogeneous solution u_0 approximates the inhomogeneous solution u_ε. This is of particular interest since numerically studying solutions to (1.1.1) can prove to be extremely costly, as finite-difference methods require many computations at a scale smaller than ε. By working with the homogenized equation, which we can view as providing an approximation of the macroscopic behavior of solutions to (1.1.1), this problem can be avoided. Regularity results are also of interest, with the main goal being to establish $W^{k,p}$ estimates for solutions to (1.1.1) with explicit dependence on ε. These regularity results can cover multiple cases, making them a powerful tool for understanding behavior of solutions in a wide variety of settings.

1.2 Homogenization of the Stokes Equations

This dissertation focuses on regularity results for solutions to the Stokes equations in a periodically perforated domain. We will begin by introducing periodically perforated domains as well as some known results. For the Stokes equations, we can think of the domain as a porous medium containing particles such as small rocks which obstruct fluid flow. Consider a domain Ω in $\mathbb{R}^d$. Let $Y = [-1/2, 1/2]^d$ be a closed unit cube in $\mathbb{R}^d$ and T the closure of an open subset of Y. We may think of T as a model obstacle. We then define a periodically perforated domain

$$\Omega_\varepsilon = \Omega \backslash \bigcup_{k \in \mathcal{I}} \varepsilon(k + \eta T), \tag{1.2.1}$$

where $\eta = \eta(\varepsilon)$ gives the ratio of obstacle size to periodicity, and the union is taken over all $k \in \mathbb{Z}^d$ such that $\varepsilon(z + Q_1) \subset \Omega$. For $f \in L^2(\Omega; \mathbb{R}^d)$, we aim to study the solutions $(u_\varepsilon, q_\varepsilon) \in H_0^1(\Omega_\varepsilon; \mathbb{R}^d) \times [L^2(\Omega_\varepsilon)/\mathbb{R}]$ to the problem

$$\begin{cases} -\Delta u_\varepsilon + \nabla q_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{1.2.2}$$

with a no-slip boundary condition on the obstacles. When necessary, we extend u_ε by zero into the obstacles, and still call this extension u_ε.

Some qualitative convergence results are known in this setting. The behavior of the solutions to (1.2.2) as $\varepsilon \to 0$ depends on η. In the simplest case, we assume that the ratio of obstacle size to periodicity is constant, namely $\eta = 1$. In this case, it can be shown that there exists an extension of q_ε, which we denote by $\tilde{q}_\varepsilon$, such that

$$\begin{aligned} &\varepsilon^{-2} u_\varepsilon \to u_0 \text{ weakly in } L^2(\Omega; \mathbb{R}^d), \\ &\tilde{q}_\varepsilon \to q_0 \text{ strongly in } L^2(\Omega)/\mathbb{R}, \end{aligned}$$

where (u_0, q_0) satisfies a Darcy law,

$$\begin{cases} u_0 = K(f - \nabla q_0) & \text{in } \Omega, \\ \operatorname{div}(u_0) = 0 & \text{in } \Omega, \\ u_0 \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.3}$$

for a symmetric positive-definite matrix K.

Different assumptions on the scaling of obstacles as $\varepsilon \to 0$ lead to different behavior of the limiting solution. G. Allaire ([1], [2]) studied the case where Ω is a bounded domain and the obstacle size satisfies $\eta \to 0$, known as the vanishing volume fraction case. To explore qualitative convergence results in this setting, we first introduce a parameter σ_ε given by

$$\sigma_\varepsilon = \begin{cases} \varepsilon\eta^{\frac{2-d}{2}} & \text{if } d \geq 3, \\ \varepsilon|\ln(\eta/2)|^{1/2} & \text{if } d = 2. \end{cases} \tag{1.2.4}$$

If the periodicity ε and relative obstacle size η satisfy $\sigma_\varepsilon \to 1$, we consider the obstacle scaling to be "critical." In this case, Allaire showed that the solutions to (1.2.2) satisfy $(u_\varepsilon, \tilde{q}_\varepsilon) \to (u_0, q_0)$ weakly in $H^1(\Omega; \mathbb{R}^d) \times [L^2(\Omega)/\mathbb{R}]$, where (u_0, q_0) satisfies a Brinkman law,

$$\begin{cases} -\Delta u_0 + \nabla q_0 + M u_0 = f & \text{in } \Omega, \\ \operatorname{div}(u_0) = 0 & \text{in } \Omega, \end{cases} \tag{1.2.5}$$

for a symmetric positive definite matrix M. The new term Mu_0 which appears in the equation, called a "strange term" by D. Cioranescu and F. Murat [8], expresses how the obstacles still affect the solution despite having disappeared in the limit.

If the obstacles are small relative to the critical scaling, i.e. $\sigma_\varepsilon \to \infty$, we say the scaling regime is "subcritical." Here, the presence of obstacles will no longer affect the limiting equation. In particular, $(u_\varepsilon, \tilde{q}_\varepsilon) \to (u_0, q_0)$ strongly in $H^1(\Omega; \mathbb{R}^d) \times [L^2(\Omega)/\mathbb{R}]$, where (u_0, q_0) satisfies

$$\begin{cases} -\Delta u_0 + \nabla q_0 = f & \text{in } \Omega, \\ \operatorname{div}(u_0) = 0 & \text{in } \Omega. \end{cases} \tag{1.2.6}$$

Finally, if the obstacles are large relative to the critical scaling, i.e. $\sigma_\varepsilon \to 0$, we are in the "supercritical" scaling regime. The case $\eta = 1$, which led to a Darcy law above, can be viewed as a subcase of the supercritical case. In fact, if $\sigma_\varepsilon \to 0$, we once again obtain a Darcy law for our homogenized equation: $(\sigma_\varepsilon^{-2} u_\varepsilon, \tilde{q}_\varepsilon) \to (u, q)$ strongly in $L^2(\Omega; \mathbb{R}^d) \times [L^2(\Omega)/\mathbb{R}]$, where (u_0, q_0) satisfies (1.2.3) for a symmetric positive definite matrix K.

1.3 Main Results

The main results of this dissertation are regularity estimates for the Stokes equations in an unbounded periodically perforated domain, where the results are independent of the scaling regimes discussed above. As before, let $Y = [-1/2, 1/2]^d$ be a closed unit cube in $\mathbb{R}^d$ and T the closure of an open subset of Y. Throughout this work, we assume $Y \backslash T$ is connected and that

$$B(0, c_0) \subset T \qquad \text{and} \qquad \text{dist}(\partial T, \partial Y) \geq c_0 > 0 \tag{1.3.1}$$

for some $c_0 > 0$. We then define

$$\omega_{\varepsilon,\eta} = \mathbb{R}^d \backslash \bigcup_{k \in \mathbb{Z}^d} \varepsilon(k + \eta T), \tag{1.3.2}$$

where $0 < \varepsilon, \eta < 1$. In this setting, we consider the Stokes equations

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = F + \text{div}(f) & \text{in } \omega_{\varepsilon,\eta}, \\ \text{div}(u_\varepsilon) = 0 & \text{in } \omega_{\varepsilon,\eta}, \\ u_\varepsilon = 0 & \text{on } \partial\omega_{\varepsilon,\eta}. \end{cases} \tag{1.3.3}$$

We wish to establish $W^{1,q}$ estimates of the form

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \leq A_q(\varepsilon, \eta) \|f\|_{L^q(\omega_{\varepsilon,\eta})} + B_q(\varepsilon, \eta) \|F\|_{L^q(\omega_{\varepsilon,\eta})}, \tag{1.3.4}$$

and

$$\|u\|_{L^q(\omega_{\varepsilon,\eta})} \leq C_q(\varepsilon, \eta) \|f\|_{L^q(\omega_{\varepsilon,\eta})} + D_q(\varepsilon, \eta) \|F\|_{L^q(\omega_{\varepsilon,\eta})}, \tag{1.3.5}$$

for $1 < q < \infty$, where the bounding constants $A_q(\varepsilon, \eta), B_q(\varepsilon, \eta), C_q(\varepsilon, \eta), D_q(\varepsilon, \eta)$ depend explicitly on the parameters ε and η.

The following are the main results. The first result addresses the case $d \geq 3$, while the second deals with the case $d = 2$.

Theorem 1.3.1. *Suppose $d \geq 3$ and $1 < q < \infty$. Let $\omega_{\varepsilon,\eta}$ be given by (1.3.2), where T is the closure of an open subset of Y with C^1 boundary. For any $f \in L^q(\omega_{\varepsilon,\eta}; \mathbb{R}^{d\times d})$ and $F \in L^q(\omega_{\varepsilon,\eta}; \mathbb{R}^d)$, the Stokes system (1.3.3) has a unique solution in $W_0^{1,q}(\omega_{\varepsilon,\eta}; \mathbb{R}^d) \times [L^q(\omega_{\varepsilon,\eta})/\mathbb{R}]$. Moreover, the solution satisfies the estimates*

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \leq \begin{cases} C\eta^{-d|\frac{1}{2}-\frac{1}{q}|} \|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon\eta^{1-\frac{d}{2}} \|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 1 < q < 2, \\ C\eta^{-d|\frac{1}{2}-\frac{1}{q}|} \|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon\eta^{1-d+\frac{d}{q}} \|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 2 \leq q < \infty, \end{cases} \tag{1.3.6}$$

and

$$\|u\|_{L^q(\omega_{\varepsilon,\eta})} \leq \begin{cases} C\varepsilon\eta^{1-\frac{d}{q}} \|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon^2\eta^{2-d} \|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 1 < q < 2, \\ C\varepsilon\eta^{1-\frac{d}{2}} \|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon^2\eta^{2-d} \|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 2 \leq q < \infty, \end{cases} \tag{1.3.7}$$

where C depends on $d, q,$ and T.

Theorem 1.3.2. *Suppose $d = 2$ and $1 < q < \infty$. Let $\omega_{\varepsilon,\eta}$ be given by (1.3.2), where T is the closure of an open subset of Y with C^1 boundary. For any $f \in$*

$L^q(\omega_{\varepsilon,\eta};\mathbb{R}^{2\times 2})$ and $F\in L^q(\omega_{\varepsilon,\eta};\mathbb{R}^2)$, the Stokes system (1.3.3) has a unique solution in $W_0^{1,q}(\omega_{\varepsilon,\eta};\mathbb{R}^2)\times[L^q(\omega_{\varepsilon,\eta})/\mathbb{R}]$. Moreover, the solution satisfies the estimates

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})}\le\begin{cases}C\eta^{-2|\frac{1}{2}-\frac{1}{q}|}|\ln(\eta/2)|^{-\frac{1}{2}}\|f\|_{L^q(\omega_{\varepsilon,\eta})}+C\varepsilon|\ln(\eta/2)|^{\frac{1}{2}}\|F\|_{L^q(\omega_{\varepsilon,\eta})}\\ \qquad \text{for } 1<q<2,\\ \|f\|_{L^2(\omega_{\varepsilon,\eta})}+C\varepsilon|\ln(\eta/2)|^{\frac{1}{2}}\|F\|_{L^2(\omega_{\varepsilon,\eta})}\\ \qquad \text{for } q=2,\\ C\eta^{-2|\frac{1}{2}-\frac{1}{q}|}|\ln(\eta/2)|^{-\frac{1}{2}}\|f\|_{L^q(\omega_{\varepsilon,\eta})}+C\varepsilon\eta^{-1+\frac{2}{q}}\|F\|_{L^q(\omega_{\varepsilon,\eta})}\\ \qquad \text{for } 2<q<\infty,\end{cases}\tag{1.3.8}$$

and

$$\|u\|_{L^q(\omega_{\varepsilon,\eta})}\le\begin{cases}C\varepsilon\eta^{1-\frac{2}{q}}\|f\|_{L^q(\omega_{\varepsilon,\eta})}+C\varepsilon^2|\ln(\eta/2)|\|F\|_{L^q(\omega_{\varepsilon,\eta})}\\ \qquad \text{for } 1<q<2,\\ C\varepsilon|\ln(\eta/2)|^{\frac{1}{2}}\|f\|_{L^q(\omega_{\varepsilon,\eta})}+C\varepsilon^2|\ln(\eta/2)|\|F\|_{L^q(\omega_{\varepsilon,\eta})}\\ \qquad \text{for } 2\le q<\infty,\end{cases}\tag{1.3.9}$$

where C depends on q and T.

Existence and uniqueness for this problem are already known [15]. Furthermore, the estimates in Theorems 1.3.1 and 1.3.2 are known for the case of fixed η [15], in which the solutions to (1.3.3) approach the solutions of a Darcy law. The main novelty of this work is that the bounds in (1.3.6)-(1.3.9) feature explicit dependence on η. In particular, Theorems 1.3.1 and 1.3.2 provide information about the behavior of solutions to (1.3.3) for any step of a convergence process in a periodically perforated domain where the obstacles vanish in each periodic cell.

Such $W^{1,q}$ estimates are also known for Laplace's equation in periodically perforated domains. For the problem

$$\begin{cases}-\Delta u=F+\operatorname{div}(f) & \text{in } \omega_{\varepsilon,\eta},\\ \quad u=0 & \text{on } \partial\omega_{\varepsilon,\eta},\end{cases}\tag{1.3.10}$$

where $\omega_{\varepsilon,\eta}$ is given in (1.3.2), the estimates given in Theorems 1.3.1 and 1.3.2 hold with the same bounding constants [17]. Moreover, the estimates are sharp in this case. We expect that the estimates are also sharp in the case of the Stokes equations, but we have not proven this. A recent paper further [13] extended the results to the case of Laplace's equation in bounded perforated domains with non-periodic distribution of obstacles.

The proofs of Theorems 1.3.1 and 1.3.2 rely on large-scale estimates for solutions to the Stokes equations in perforated cubes, which are of interest on their own. We define perforated cubes $Q_R^{\varepsilon,\eta}$ by

$$Q_R^{\varepsilon,\eta}=Q_R\cap\omega_{\varepsilon,\eta},\tag{1.3.11}$$

where $Q_R = (-R/2, R/2)^d$. We then consider the problem in $Q_R^{\varepsilon,\eta}$,

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } Q_R^{\varepsilon,\eta}, \\ \operatorname{div}(u) = 0 & \text{in } Q_R^{\varepsilon,\eta}, \\ u = 0 & \text{on } \partial Q_R^{\varepsilon,\eta} \cap \partial \omega_{\varepsilon,\eta}. \end{cases} \tag{1.3.12}$$

Theorem 1.3.3. *Let $(u,p) \in H^1(Q_R^{\varepsilon,\eta};\mathbb{R}^d) \times L^2(Q_R^{\varepsilon,\eta})$ be a weak solution to (1.3.12) for some $R \geq \varepsilon$. Then if $\varepsilon \leq r \leq R$,*

$$\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} \leq C \left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2}, \tag{1.3.13}$$

where C depends on d.

Theorem 1.3.4. *Let $(u,p) \in H^1(Q_R^{\varepsilon,\eta};\mathbb{R}^d) \times L^2(Q_R^{\varepsilon,\eta})$ be a weak solution to (1.3.12) for some $R \geq \varepsilon$. Then if $\varepsilon \leq r \leq R$,*

$$\left(\fint_{Q_r} |u|^2 \right)^{1/2} \leq C \left(\fint_{Q_R} |u|^2 \right)^{1/2}, \tag{1.3.14}$$

where C depends on d.

Theorem 1.3.3 gives a large-scale Lipschitz estimate, while Theorem 1.3.4 gives a large-scale L^∞ estimate. By large-scale, we mean that averages are taken over cubes whose side lengths are larger than the scale of periodicity. This allows us to exploit the periodic structure to obtain the estimates.

We now describe our approach to proving Theorems 1.3.1 and 1.3.2. Much of the argument, with some notable exceptions, follows the present author's previous work for the case of Laplace's equation in a perforated domain [17]. We first note that the powers of ε in (1.3.6)-(1.3.9) are dictated solely by scaling. This allows us to simplify computations by rescaling so that $\varepsilon = 1$. By localizing and rescaling estimates in a weighted Sobolev space for solutions to the Stokes equations in an exterior domain $\mathbb{R}^d \backslash T$, we are able to reduce the L^q estimates of u and ∇u to the L^q estimates of the average operators

$$T_{\varepsilon,\eta}(F,f)(x) = \left(\fint_{x+\varepsilon Q_2} |u|^2 \right)^{1/2} \tag{1.3.15}$$

and

$$S_{\varepsilon,\eta}(F,f)(x) = \left(\fint_{x+\varepsilon Q_2} |\nabla u|^2 \right)^{1/2} \tag{1.3.16}$$

for $q > 2$. Using a real-variable argument from [14], we establish the L^q boundedness of $T_{\varepsilon,\eta}$ and $S_{\varepsilon,\eta}$ by proving weak reverse Hölder inequalities in a cube Q for solutions to (1.3.3) with $F = 0$ and $f = 0$ in $4Q$. The reverse Hölder inequalities follow from the large-scale estimates in Theorems 1.3.3 and 1.3.4.

The proofs of Theorems 1.3.3 and 1.3.4 are significantly different in the case of the Stokes equations compared to Laplace's equation. This is due to the pressure term causing a Caccioppoli inequality in a perforated cube to only provide useful bounds when applied to cubes whose side length is smaller than σ_ε, where σ_ε is defined in (1.2.4). As such, proofs of the large-scale estimates will be separated into two cases. For small cubes, we use an argument from [17], which relies on a discrete Sobolev inequality for functions defined on $\mathbb{Z}^d$. For large cubes, we will be able to use a modified version of the compactness method from [15], which treats the case of fixed η.

The dissertation is organized as follows: In Chapter 2, we provide some basic results which will be used frequently in the later chapters. In Chapter 3, we establish the large-scale estimates in the case of small cubes, which we call the "subcritical" case. The names of these cases are motivated by [1]. In Chapter 4, we finish the proofs of the large-scale estimates by showing that they also hold in the case of large cubes, which we call the "supercritical" case. In Chapter 5, we establish the bounds for the average operators $T_{\varepsilon,\eta}$ and $S_{\varepsilon,\eta}$ defined in (1.3.15) and (1.3.16). In Chapters 6 and 7, we present the localization argument for solutions in a periodic cell $(1 + c_0)Q_1 \setminus \eta T$. Finally, the proofs of Theorems 1.3.1 and 1.3.2 are given in Chapter 8.

Chapter 2 Preliminaries

This chapter is dedicated to establishing some basic results which will appear frequently in the remainder of the dissertation. We begin by making note of the following Poincaré inequality for functions supported on a perforated domain.

Lemma 2.0.1. *Let Ω be a domain in $\mathbb{R}^d$, and let $\Omega_{\varepsilon,\eta} = \Omega_\varepsilon$ as defined in (1.2.1). Let $u \in H^1(\Omega_{\varepsilon,\eta})$ with $u = 0$ on $\Omega \backslash \Omega_{\varepsilon,\eta}$. If $d \geq 3$, then*

$$\int_{\Omega_{\varepsilon,\eta}} |u|^2 \, dx \leq C\varepsilon^2 \eta^{2-d} \int_{\Omega_{\varepsilon,\eta}} |\nabla u|^2 \, dx, \tag{2.0.1}$$

and if $d = 2$, then

$$\int_{\Omega_{\varepsilon,\eta}} |u|^2 \, dx \leq C\varepsilon^2 |\ln(\eta/2)| \int_{\Omega_{\varepsilon,\eta}} |\nabla u|^2 \, dx, \tag{2.0.2}$$

where C depends only on d and c_0.

Proof. The proof is well known. See, for example, [10]. □

Remark 2.0.2. Let $d \geq 2$. With the definition of σ_ε given in (1.2.4), we may rephrase the conclusion of Lemma 2.0.1 as

$$\int_{\Omega_{\varepsilon,\eta}} |u|^2 \, dx \leq C\sigma_\varepsilon^2 \int_{\Omega_{\varepsilon,\eta}} |\nabla u|^2 \, dx. \tag{2.0.3}$$

2.1 Pressure Estimate

For future use, we establish the following pressure estimate.

Theorem 2.1.1. *Suppose $(u_\varepsilon, p_\varepsilon) \in H^1(Q_1^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_1^{\varepsilon,\eta})$ is a weak solution of*

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in } Q_1^{\varepsilon,\eta} \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } Q_1^{\varepsilon,\eta} \\ u_\varepsilon = 0 & \text{on } Q_1 \cap \partial\omega_{\varepsilon,\eta}. \end{cases} \tag{2.1.1}$$

Then

$$\|p_\varepsilon - \fint_{Q_1^{\varepsilon,\eta}} p_\varepsilon\|_{L^2(Q_1^{\varepsilon,\eta})} \leq C(1 + \sigma_\varepsilon^{-1})(\|\nabla u_\varepsilon\|_{L^2(Q_1^{\varepsilon,\eta})} + \|f\|_{L^2(Q_1^{\varepsilon,\eta})}), \tag{2.1.2}$$

where C depends only on d and T.

The proof of Theorem 2.1.1 will rely on a restriction operator defined in [1] as well as an estimate for a Bogovskiĭ operator in a perforated domain. Define

$$K_\eta = \begin{cases} \eta^{\frac{d-2}{2}} & \text{if } d \geq 3, \\ |\ln(\eta/2)|^{-1/2} & \text{if } d = 2. \end{cases} \tag{2.1.3}$$

We can view K_η as a substitute for σ_ε^{-1} when $\varepsilon = 1$. Since we will frequently rescale so $\varepsilon = 1$ to simplify computations, the term K_η will appear in many results and proofs throughout this work.

Lemma 2.1.2. *There exists a linear continuous operator $L : H^1(Q_1) \to H^1(Q_1 \backslash \eta T)$ such that for any $u \in H^1(Q_1)$, $Lu = u$ on ∂Q_1, $Lu = 0$ on $\partial(\eta T)$ and*

$$\|\nabla(Lu)\|_{L^2(Q_1 \backslash \eta T)} \leq C(\|\nabla u\|_{L^2(Q_1)} + K_\eta \|u\|_{L^2(Q_1)}), \tag{2.1.4}$$

where K_η is given by (2.1.3), and C depends only on d and T.

Proof. See [1]. □

The following classical result provides an estimate for a Bogovskiĭ operator on a domain in $\mathbb{R}^d$.

Lemma 2.1.3. *Let Ω be a bounded, connected, open set in $\mathbb{R}^d$ with Lipschitz boundary. For any $f \in L^2(\Omega)$ with $\int_\Omega f = 0$, there exists $v \in H_0^1(\Omega; \mathbb{R}^d)$ such that* $\text{div}(v) = f$ *in Ω, the map $f \mapsto v$ is linear, and*

$$\|v\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{2.1.5}$$

where C depends only on Ω.

We can extend this result to the case of periodically perforated domains, with explicit dependence on ε and η, as follows.

Lemma 2.1.4. *Let T be the closure of an open subset of Y with C^1 boundary. For any $f \in L^2(Q_1^{\varepsilon,\eta})$ with $\int_{Q_1^{\varepsilon,\eta}} f = 0$, there exists $v \in H_0^1(Q_1^{\varepsilon,\eta}; \mathbb{R}^d)$ such that* $\text{div}(v) = f$ *in $Q_1^{\varepsilon,\eta}$, the map $f \mapsto v$ is linear, and*

$$\|\nabla v\|_{L^2(Q_1^{\varepsilon,\eta})} \leq C(1 + \frac{1}{\sigma_\varepsilon})\|f\|_{L^2(Q_1^{\varepsilon,\eta})}, \tag{2.1.6}$$

where C depends only on d and T.

Proof. The idea of the proof is motivated by [1]. For ease of computations, we will prove the lemma in a rescaled setting. Namely, we will show that for any $f \in L^2(Q_{1/\varepsilon}^{1,\eta})$ with $\int_{Q_{1/\varepsilon}^{1,\eta}} f = 0$, there exists $v \in H_0^1(Q_{1/\varepsilon}^{1,\eta}; \mathbb{R}^d)$ such that $\text{div}(v) = f$ in $Q_{1/\varepsilon}^{1,\eta}$ and

$$\|\nabla v\|_{L^2(Q_{1/\varepsilon}^{1,\eta})} \leq C(1 + \frac{1}{\sigma_\varepsilon})\|f\|_{L^2(Q_{1/\varepsilon}^{1,\eta})}. \tag{2.1.7}$$

By Lemma 2.1.3 and rescaling, we can find $v_* \in H_0^1(Q_{1/\varepsilon};\mathbb{R}^d)$ such that $\mathrm{div}(v_*) = f$ in $Q_{1/\varepsilon}$ and

$$\|\nabla v_*\|_{L^2(Q_{1/\varepsilon})} + \varepsilon \|v_*\|_{L^2(Q_{1/\varepsilon})} \le C\|f\|_{L^2(Q_{1/\varepsilon})}. \tag{2.1.8}$$

Write $\mathbb{Z}^d = \bigcup k_i$, and let $Q^i = k_i + Y$, $T_\eta^i = k_i + \eta T$. On each perforated cube $Q^i \backslash T_\eta^i$, we will find w_i satisfying

$$\begin{cases} \mathrm{div}(w_i) = f - \mathrm{div}(Lv_*) & \text{in } Q^i\backslash T_\eta^i, \\ \quad w_i = 0 & \text{on } \partial T_\eta^i, \\ \quad w_i = 0 & \text{on } \partial Q^i, \end{cases} \tag{2.1.9}$$

and

$$\|\nabla w_i\|_{L^2(Q^i\backslash T_\eta^i)} \le C\|f - \mathrm{div}(Lv_*)\|_{L^2(Q^i\backslash T_\eta^i)}, \tag{2.1.10}$$

where L is the operator in Lemma 2.1.2.

Once we find each w_i, we can obtain the desired function v. Indeed, set $v_i = w_i + Lv_*$ in Q^i. Then let $v = v_i$ in Q^i with $v = v_*$ on ∂Q^i. It follows from (2.1.9) that $\mathrm{div}(v) = f$ in $Q_1^{\varepsilon,\eta}$. To show (2.1.6), we observe

$$\begin{aligned} \|\nabla v_i\|_{L^2(Q^i\backslash T_\eta^i)} &\le C(\|\nabla w_i\|_{L^2(Q^i\backslash T_\eta^i)} + \|\nabla(Lv_*)\|_{L^2(Q^i\backslash T_\eta^i)}) \\ &\le C(\|f\|_{L^2(Q^i\backslash T_\eta^i)} + \|\nabla(Lv_*)\|_{L^2(Q^i\backslash T_\eta^i)}) \\ &\le C(\|f\|_{L^2(Q^i\backslash T_\eta^i)} + \|\nabla v_*\|_{L^2(Q^i\backslash T_\eta^i)} + K_\eta \|v_*\|_{L^2(Q^i\backslash T_\eta^i)}), \end{aligned} \tag{2.1.11}$$

where K_η is defined in Lemma 2.1.2. By adding (2.1.11) over $Q_1^{\varepsilon,\eta}$, we obtain

$$\begin{aligned} \|\nabla v\|_{L^2(Q_{1/\varepsilon}^{1,\eta})} &\le C(\|f\|_{L^2(Q_{1/\varepsilon}^{1,\eta})} + \|\nabla v_*\|_{L^2(Q_{1/\varepsilon}^{1,\eta})} + K_\eta\|v_*\|_{L^2(Q_{1/\varepsilon}^{1,\eta})}) \\ &\le C(1+\frac{1}{\sigma_\varepsilon})\|f\|_{L^2(Q_{1/\varepsilon}^{1,\eta})}, \end{aligned} \tag{2.1.12}$$

where we have used (2.1.8) in the second step.

It remains to find w_i. For simplicity, assume $Q^i = Q_1$. We will find $w \in H^1(Q_1\backslash \eta T;\mathbb{R}^d)$ satisfying

$$\begin{cases} \mathrm{div}(w) = f - \mathrm{div}(Lv_*) & \text{in } Q_1\backslash \eta T, \\ \quad w = 0 & \text{on } \partial(\eta T), \\ \quad w = 0 & \text{on } \partial Q_1 \end{cases} \tag{2.1.13}$$

and

$$\|\nabla w\|_{L^2(Q_1\backslash \eta T)} \le C\|f - \mathrm{div}(Lv_*)\|_{L^2(Q_1\backslash \eta T)}. \tag{2.1.14}$$

Define $\tilde{f}$ by

$$\tilde{f} = f \quad \text{ in } Q_1\backslash \eta T, \qquad \tilde{f} = 0 \quad \text{ in } \eta T. \tag{2.1.15}$$

By Lemma 2.1.3, we can find $u \in H_0^1(Q_1;\mathbb{R}^d)$ such that $\mathrm{div}(u) = \tilde{f} - \mathrm{div}(Lv_*)$ in Q_1 and

$$\|u\|_{H_0^1(Q_1)} \le C\|f - \mathrm{div}(Lv_*)\|_{L^2(Q_1\backslash \eta T)}, \tag{2.1.16}$$

where C depends only on d. We separate into two cases.

Case 1: Assume $d \geq 3$. We will find $\tilde{u} \in H^1(Q_\eta \backslash \eta T)$ satisfying

$$\begin{cases} \operatorname{div}(\tilde{u}) = f - \operatorname{div}(Lv_*) & \text{in } Q_\eta \backslash \eta T, \\ \tilde{u} = 0 & \text{on } \partial(\eta T), \\ \tilde{u} = u & \text{on } \partial Q_\eta. \end{cases} \tag{2.1.17}$$

and

$$\|\nabla \tilde{u}\|_{L^2(Q_\eta \backslash \eta T)} \leq C\|f - \operatorname{div}(Lv_*)\|_{L^2(Q_1 \backslash \eta T)}. \tag{2.1.18}$$

We can then obtain w satisfying (2.1.13) and (2.1.14) by setting $w = u$ in $Q_1 \backslash Q_\eta$ and $w = \tilde{u}$ in $Q_\eta \backslash \eta T$.

It remains to find $\tilde{u}$. We rescale system (2.1.17) as follows: for $x \in Q_1 \backslash T$, let $f_0(x) = (f - \operatorname{div}(Lv_*))(\eta x)$, $u_0(x) = \frac{1}{\eta} u(\eta x)$, and $\tilde{u}_0(x) = \frac{1}{\eta}\tilde{u}(x)$. We obtain the system

$$\begin{cases} \operatorname{div}(\tilde{u}_0) = f_0 & \text{in } Q_1 \backslash T, \\ \tilde{u}_0 = 0 & \text{on } \partial T, \\ \tilde{u}_0 = u_0 & \text{on } \partial Q_1. \end{cases} \tag{2.1.19}$$

By Lemma 2.1.3, there exists a solution $\tilde{u}_0$ to (2.1.19) which satisfies

$$\|\nabla \tilde{u}_0\|_{L^2(Q_1 \backslash T)} \leq C(\|f_0\|_{L^2(Q_1 \backslash T)} + \|u_0\|_{L^2(Q_1 \backslash T)} + \|\nabla u_0\|_{L^2(Q_1 \backslash T)}). \tag{2.1.20}$$

Rescaling, we obtain

$$\|\nabla \tilde{u}\|_{L^2(Q_\eta \backslash \eta T)} \leq C(\|f - \operatorname{div}(Lv_*)\|_{L^2(Q_\eta \backslash \eta T)} + \frac{1}{\eta}\|u\|_{L^2(Q_\eta \backslash \eta T)} + \|\nabla u\|_{L^2(Q_\eta \backslash \eta T)}). \tag{2.1.21}$$

The Hölder inequality in Q_η gives

$$\begin{aligned} \|u\|_{L^2(Q_\eta)} &\leq \|u\|_{L^{\frac{2d}{d-2}}(Q_\eta)} \|1\|_{L^d(Q_\eta)} \\ &\leq C\eta \|u\|_{L^{\frac{2d}{d-2}}(Q_1)}. \end{aligned} \tag{2.1.22}$$

Since $d \geq 3$, we have the Sobolev embedding $H^1 \subset L^{\frac{2d}{d-2}}(Q_1)$. Therefore, (2.1.22) becomes

$$\|u\|_{L^2(Q_\eta)} \leq C\eta \|u\|_{H^1_0(Q_1)}. \tag{2.1.23}$$

As a result, we obtain (2.1.18) from (2.1.21).

Case 2: Assume $d = 2$. In this case, we will divide the interior of Q_1 into annular regions. Let $n \in \mathbb{N}$ such that

$$\frac{1}{2^n} > \eta \geq \frac{1}{2^{n+1}}.$$

Let $A_1 = Q_1 \backslash B_{1/2}$. For $2 \leq i \leq n$, let $A_i = B_{1/2^{i-1}} \backslash B_{1/2^i}$. Finally, let $A_{n+1} = B_{1/2^n} \backslash \eta T$. In A_1, we consider the problem

$$\begin{cases} \operatorname{div}(a_1) = f - \operatorname{div}(Lv_*) & \text{in } A_1, \\ a_1 = 0 & \text{on } \partial Q_1, \\ a_1 = u - \fint_{A_1 \cup A_2} u & \text{on } \partial B_{1/2}. \end{cases} \tag{2.1.24}$$

If $2 \le i \le n$, we consider the problem

$$\begin{cases} \operatorname{div}(a_i) = f - \operatorname{div}(Lv_*) & \text{in } A_i, \\ \quad a_i = u - \fint_{A_{i-1}\cup A_i} u & \text{on } \partial B_{1/2^{i-1}}, \\ \quad a_i = u - \fint_{A_i\cup A_{i+1}} u & \text{on } \partial B_{1/2^{i}}. \end{cases} \tag{2.1.25}$$

Finally, in A_{n+1}, we consider the problem

$$\begin{cases} \operatorname{div}(a_{n+1}) = f - \operatorname{div}(Lv_*) & \text{in } A_{n+1}, \\ \quad a_{n+1} = u - \fint_{A_n\cup A_{n+1}} u & \text{on } \partial B_{1/2^{n}}, \\ \quad a_{n+1} = 0 & \text{on } \partial \eta T. \end{cases} \tag{2.1.26}$$

Let $a_0(x) = 2^i a_i(x/2^i)$, $f_0(x) = (f - \operatorname{div}(Lv_*))(x/2^i)$, and $u_0(x) = 2^i u(x/2^i)$. The problems in (2.1.25) for $3 \le i \le n-1$ become the rescaled problem

$$\begin{cases} \operatorname{div}(a_0) = f_0 & \text{in } B_2, \\ \quad a_0 = u_0 - \fint_{B_4\setminus B_1} u_0 & \text{on } \partial B_2, \\ \quad a_0 = u_0 - \fint_{B_2\setminus B_{1/2}} u_0 & \text{on } \partial B_1. \end{cases} \tag{2.1.27}$$

By Lemma 2.1.3, there exists a solution a_0 to (2.1.27) satisfying

$$\begin{aligned} \|\nabla a_0\|_{L^2(B_2)} &\le C(\|f_0\|_{L^2(B_2)} + \|u_0 - \fint_{B_4\setminus B_1} u_0\|_{L^2(B_2)}) \\ &\quad + C\|u_0 - \fint_{B_2\setminus B_{1/2}} u_0\|_{L^2(B_2)} \\ &\le C(\|f_0\|_{L^2(B_2)} + \|\nabla u_0\|_{L^2(B_4\setminus B_{1/2})}), \end{aligned} \tag{2.1.28}$$

where we have used the Poincaré inequality in the second step. It follows that for $3 \le i \le n-1$,

$$\|\nabla a_i\|_{L^2(A_i)} \le C(\|f - \operatorname{div}(Lv_*)\|_{L^2(A_i)} + \|\nabla u\|_{L^2(A_{i-1}\cup A_i\cup A_{i+1})}). \tag{2.1.29}$$

The same approach can be applied to obtain similar estimates for the problems in A_1, A_2, A_n, and A_{n+1}. By letting $w = a_i$ in A_i, we obtain w satisfying (2.1.13) and (2.1.14), where (2.1.16) has been used.

□

We are now ready to prove the pressure estimate in Theorem 2.1.1.

Proof of Theorem 2.1.1. Note that $\int_{Q_1^{\varepsilon,\eta}}(p_\varepsilon - \fint_{Q_1^{\varepsilon,\eta}} p_\varepsilon) = 0$. Thus we can apply Lemma 2.1.4 to find v_ε satisfying $\mathrm{div}(v_\varepsilon) = p_\varepsilon - \fint_{Q_1^{\varepsilon,\eta}} p_\varepsilon$ in $Q_1^{\varepsilon,\eta}$ and

$$\|\nabla v_\varepsilon\|_{L^2(Q_1^{\varepsilon,\eta})} \leq C(1+\frac{1}{\sigma_\varepsilon})\|f\|_{L^2(Q_1^{\varepsilon,\eta})}. \tag{2.1.30}$$

Then

$$\begin{aligned}
\int_{Q_1^{\varepsilon,\eta}} |p_\varepsilon - \fint_{Q_1^{\varepsilon,\eta}} p_\varepsilon|^2 &= \int_{Q_1^{\varepsilon,\eta}} (p_\varepsilon - \fint_{Q_1^{\varepsilon,\eta}} p_\varepsilon)\mathrm{div}(v_\varepsilon) \\
&\leq \|\nabla p_\varepsilon\|_{H^{-1}(Q_1^{\varepsilon,\eta})}\|\nabla v_\varepsilon\|_{L^2(Q_1^{\varepsilon,\eta})} \\
&\leq C(\|\nabla u_\varepsilon\|_{L^2(Q_1^{\varepsilon,\eta})} + \|f\|_{L^2(Q_1^{\varepsilon,\eta})})(1+\frac{1}{\sigma_\varepsilon})\|p_\varepsilon - \fint_{Q_1^{\varepsilon,\eta}} p_\varepsilon\|_{L^2(Q_1^{\varepsilon,\eta})},
\end{aligned}$$

which yields the desired estimate. □

2.2 Caccioppoli Inequality

The proofs of the large-scale estimates in Theorems 1.3.3 and 1.3.4 will rely on a Caccioppoli inequality for solutions to the Stokes equations in a perforated cube. We give the Caccioppoli inequality in a rescaled setting.

Theorem 2.2.1. *Let $(u,p) \in H^1(Q_R^{1,\eta}) \times L^2(Q_R^{1,\eta})$ be a weak solution of*

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } Q_R^{1,\eta}, \\ \mathrm{div}(u) = 0 & \text{in } Q_R^{1,\eta}, \\ u = 0 & \text{on } Q_R^{1,\eta} \cap \omega_{1,\eta}, \end{cases} \tag{2.2.1}$$

where $1 \leq R \leq \eta^{\frac{2-d}{2}}$ if $d \geq 3$, and $1 \leq R \leq |\ln(\eta/2)|^{1/2}$ if $d = 2$. Then

$$\int_{Q_{R/2}^{1,\eta}} |\nabla u|^2 \leq \frac{C}{R^2}\int_{Q_R^{1,\eta}} |u|^2 + CR^2 \int_{Q_R^{1,\eta}} |f|^2, \tag{2.2.2}$$

where C depends only on d and T.

Proof. Without loss of generality, we may assume $R = 2^k$ for some $k \geq 0$. For otherwise, we can cover Q_R with cubes of side length 2^k and apply the result on each cube. We can also assume $\fint_{Q_R^{1,\eta}} p = 0$. We begin by rescaling as follows: let $\tilde{u}(x) = u(Rx)$, $\tilde{p}(x) = Rp(Rx)$, and $\tilde{f}(x) = R^2 f(Rx)$. We obtain the rescaled system

$$\begin{cases} -\Delta \tilde{u} + \nabla \tilde{p} = \tilde{f} & \text{in } Q_1^{1/R,\eta}, \\ \mathrm{div}(\tilde{u}) = 0 & \text{in } Q_1^{1/R,\eta}, \\ \tilde{u} = 0 & \text{on } Q_1^{1/R,\eta} \cap \omega_{\varepsilon,\eta}. \end{cases} \tag{2.2.3}$$

Note that the assumptions on R imply that $R \le K_\eta^{-1}$. By applying (2.1.2) with $\varepsilon = 1/R$, we obtain

$$
\begin{aligned}
\|p\|_{L^2(Q_R^{1,\eta})} &= R^{d-1}\|\tilde{p}\|_{L^2(Q_1^{1/R,\eta})} \\
&\le CR^{d-1}(1+RK_\eta)(\|\nabla \tilde{u}\|_{L^2(Q_1^{1/R,\eta})} + \|\tilde{f}\|_{L^2(Q_1^{1/R,\eta})}) \\
&\le C(\|\nabla u\|_{L^2(Q_R^{1,\eta})} + R\|f\|_{L^2(Q_R^{1,\eta})}).
\end{aligned}
\tag{2.2.4}
$$

It follows that

$$
\int_{Q_R^{1,\eta}} |p|^2 \le C \int_{Q_R^{1,\eta}} (|\nabla u|^2 + R^2|f|^2). \tag{2.2.5}
$$

Define

$$
\mathcal{I} = \{t \in [1,R] \,:\, \partial Q_t \cap \partial\omega_{1,\eta} = \emptyset\}. \tag{2.2.6}
$$

If $t \in \mathcal{I}$, then (u,p) satisfies

$$
\begin{cases}
-\Delta u + \nabla p = f & \text{in } Q_t^{1,\eta}, \\
\operatorname{div}(u) = 0 & \text{in } Q_t^{1,\eta}, \\
u = 0 & \text{on } Q_t^{1,\eta} \cap \omega_{1,\eta}.
\end{cases}
\tag{2.2.7}
$$

Using u as a test function in (2.2.7) and integrating by parts, we obtain

$$
\int_{Q_t^{1,\eta}} |\nabla u|^2 - \int_{\partial Q_t} u \cdot (\nabla u n) + \int_{\partial Q_t} pu \cdot n = \int_{Q_t^{1,\eta}} f \cdot u. \tag{2.2.8}
$$

Hence

$$
\int_{Q_t^{1,\eta}} |\nabla u|^2 \le \int_{\partial Q_t} (|\nabla u| + |p|)|u| + \int_{Q_t^{1,\eta}} |f||u|. \tag{2.2.9}
$$

Choose $r, s \in \mathcal{I}$ such that $s - r \ge \frac{1}{2}$. Then for $t \in \mathcal{I} \cap [r,s]$,

$$
\int_{Q_r^{1,\eta}} |\nabla u|^2 \le \int_{\partial Q_t} (|\nabla u| + |p|)|u| + \int_{Q_s^{1,\eta}} |f||u|. \tag{2.2.10}
$$

Integrating both sides of (2.2.10) in t over $[r,s] \cap \mathcal{I}$ and using the Cauchy inequality, we find for any $\delta \in (0,1)$,

$$
\begin{aligned}
\int_{Q_r^{1,\eta}} |\nabla u|^2 &\le \frac{C}{s-r}\int_{Q_s^{1,\eta}} (|\nabla u| + |p|)|u| + \int_{Q_s^{1,\eta}} |f||u| \\
&\le \frac{C}{(s-r)^2\delta}\int_{Q_s^{1,\eta}} |u|^2 + C\delta \int_{Q_s^{1,\eta}} (|\nabla u|^2 + |p|^2) + C(s-r)^2 \int_{Q_s^{1,\eta}} |f|^2 \\
&\le \frac{C}{(s-r)^2\delta}\int_{Q_s^{1,\eta}} |u|^2 + C\delta \int_{Q_s^{1,\eta}} |\nabla u|^2 + CR^2 \int_{Q_s^{1,\eta}} |f|^2,
\end{aligned}
\tag{2.2.11}
$$

where (2.2.5) has been used to bound pressure terms. Repeatedly applying (2.2.11) for $r = r_i$ and $s = r_{i+1}$, where $r_i = R(1 - 2^{-i})$, $i = 1, \ldots, k$, we obtain

$$\begin{aligned} \int_{Q^{1,\eta}_{R/2}} |\nabla u|^2 \leq \sum_{i=1}^{k-1} \left(\frac{C(4^{i+1})}{R^2 \delta} (C\delta)^{i-1} \int_{Q^{1,\eta}_{r_{i+1}}} |u|^2 + (C\delta)^{i-1} R^2 \int_{Q^{1,\eta}_{r_{i+1}}} |f|^2 \right) \\ + (C\delta)^{k-1} \int_{Q^{1,\eta}_{R}} |\nabla u|^2. \end{aligned} \tag{2.2.12}$$

Choosing $\delta > 0$ such that $C\delta < 1/4$ and observing that $(1/4)^{k-1} = 4R^{-2}$, we obtain

$$\int_{Q^{1,\eta}_{R/2}} |\nabla u|^2 \leq \frac{C}{R^2} \int_{Q^{1,\eta}_{R}} |u|^2 + \frac{C}{R^2} \int_{Q^{1,\eta}_{R}} |\nabla u|^2 + CR^2 \int_{Q^{1,\eta}_{R}} |f|^2. \tag{2.2.13}$$

To deal with the term involving ∇u in the right side of (2.2.13), we consider the problem

$$\begin{cases} -\Delta v + \nabla q = g & \text{in } Q_{1+\delta} \backslash \eta T, \\ \operatorname{div}(v) = 0 & \text{in } Q_{1+\delta} \backslash \eta T, \\ v = 0 & \text{on } \partial \eta T, \end{cases} \tag{2.2.14}$$

where $\delta > 0$ is small. In this case with only one obstacle, a classical Caccioppoli inequality and classical pressure estimates for the Stokes system yield

$$\int_{Q_1 \backslash \eta T} |\nabla v|^2 \leq C_\delta \int_{Q_{1+\delta} \backslash \eta T} (|v|^2 + |g|^2). \tag{2.2.15}$$

Covering $Q^{1,\eta}_R$ with cubes $Q^{1,\eta}_1$ and applying (2.2.15) in (2.2.13) yields

$$\int_{Q^{1,\eta}_{R/2}} |\nabla u|^2 \leq \frac{C}{R^2} \int_{Q^{1,\eta}_{R+1}} |u|^2 + CR^2 \int_{Q^{1,\eta}_{R+1}} |f|^2. \tag{2.2.16}$$

To obtain the desired result (2.2.2), we cover $Q_{R/2}$ by cubes of side length $\frac{R-1}{2}$ and apply (2.2.16). $\square$

Chapter 3 Large-scale Estimates: Critical and Subcritical Cases

We will now begin to establish the large-scale estimates given in Theorems 1.3.3 and 1.3.4. Due to Theorem 2.2.1 only providing bounds when the cube's side length R is sufficiently small, the proofs of Theorems 1.3.3 and 1.3.4 will be split into multiple cases depending on the size of R. The definitions of the cases are motivated by [1].

Suppose (u, p) satisfies (1.3.12). If $\varepsilon \leq R < \sigma_\varepsilon$, we say that the scaling regime is "subcritical." If $R = \sigma_\varepsilon$, we say that the scaling regime is "critical." Finally, if $R > \sigma_\varepsilon$, we say that the scaling regime is "supercritical." In this chapter, we will prove the large-scale estimates in the case of subcritical or critical scaling regime using an approach from [17].

Theorem 3.0.1. *Let $d \geq 2$, and let $\varepsilon, \eta \in (0,1)$. Suppose $(u,p) \in H^1(Q_R^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_R^{\varepsilon,\eta})$ is a weak solution to (1.3.12) for some $\varepsilon \leq R \leq \sigma_\varepsilon$. Then if $\varepsilon \leq r \leq R$,*

$$\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} \leq C \left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2}, \tag{3.0.1}$$

where C depends on d.

Theorem 3.0.2. *Let $d \geq 2$, and let $\varepsilon, \eta \in (0,1)$. Suppose $(u,p) \in H^1(Q_R^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_R^{\varepsilon,\eta})$ is a weak solution to (1.3.12) for some $\varepsilon \leq R \leq \sigma_\varepsilon$. Then if $\varepsilon \leq r \leq R$,*

$$\left(\fint_{Q_r} |u|^2 \right)^{1/2} \leq C \left(\fint_{Q_R} |u|^2 \right)^{1/2}, \tag{3.0.2}$$

where C depends on d.

Theorem 3.0.1 gives the large-scale Lipschitz estimate in the subcritical and critical cases, while Theorem 3.0.2 gives the large-scale L^∞ estimates in these cases.

3.1 Correctors: The Case $d \geq 3$

For $u \in L^1(\mathbb{R}^d)$ and $z \in \mathbb{Z}^d$, define

$$\widehat{u}(z) = \int_{z+Q_1} u(x)\, dx. \tag{3.1.1}$$

We will see later that Theorem 3.0.1 can be proven under the assumption that $\hat{u}(0) = 0$ by applying the Caccioppoli inequality in Theorem 2.2.1 as well as a discrete Sobolev inequality to u. However, we cannot assume that $\hat{u}(0) = 0$. Furthermore, we cannot apply the argument to $u - \hat{u}(0)$, as this function would not vanish on the obstacles.

To circumvent this issue, we define a matrix of corrector functions M^η which also vanishes on the obstacles. By choosing $\alpha \in \mathbb{R}^d$ such that $w = u - M^\eta \alpha$ satisfies $\hat{w}(0) = 0$ and $w = 0$ in $\mathbb{R}^d \backslash \omega_{\varepsilon,\eta}$, we will then be able to prove Theorem 3.0.1. The corrector functions are defined using solutions to an exterior problem.

Lemma 3.1.1. *Suppose $d \geq 3$. For $k = 1, \dots, d$, let $(w_k, \pi_k) \in H^1(\mathbb{R}^d \backslash T; \mathbb{R}^d) \times L^2(\mathbb{R}^d \backslash T)$ solve the exterior problem*

$$\begin{cases} -\Delta w_k + \nabla \pi_k = 0 & \text{in } \mathbb{R}^d \backslash T \\ \operatorname{div}(w_k) = 0 & \text{in } \mathbb{R}^d \backslash T \\ w_k = 0 & \text{on } \partial T \\ w_k \to e_k & \text{as } |x| \to \infty. \end{cases} \tag{3.1.2}$$

Then (w_k, π_k) satisfies at infinity

$$\begin{aligned} w_k &= e_k - \frac{1}{2S_d r^{d-2}} \left(\frac{F_k}{d-2} + (F_k \cdot e_r) e_r \right) + O\left(\frac{1}{r^{d-1}} \right), \\ \pi_k &= -\frac{1}{S_d r^{d-1}} (F_k \cdot e_r) + O\left(\frac{1}{r^d} \right), \\ \nabla w_k &= O\left(\frac{1}{r^{d-1}} \right), \\ \frac{\partial w_k}{\partial r} - \pi_k e_r &= \frac{1}{2S_d r^{d-1}} (F_k + d(F_k \cdot e_r) e_r) + O\left(\frac{1}{r^d} \right), \end{aligned} \tag{3.1.3}$$

where e_r is the radial unit vector, S_d is the area of the unit sphere in $\mathbb{R}^d$ and

$$F_k = \int_{\partial T} \left(\frac{\partial w_k}{\partial n} - \pi_k n \right). \tag{3.1.4}$$

Moreover,

$$F_k \cdot e_i = \int_{\mathbb{R}^d \backslash T} \nabla w_k \cdot \nabla w_i. \tag{3.1.5}$$

Proof. See [1]. □

For $d \geq 3$, we define correctors (w_k^η, π_k^η) by

$$\begin{cases} w_k^\eta = e_k \\ \pi_k^\eta = 0 \end{cases} \text{in } Y \backslash B_{1/3}, \qquad \begin{cases} -\Delta w_k^\eta + \nabla \pi_k^\eta = 0 \\ \operatorname{div}(w_k^\eta) = 0 \end{cases} \text{in } B_{1/3} \backslash B_{1/4},$$

$$\begin{cases} w_k^\eta = w_k \left(\dfrac{x}{\eta} \right) \\ \pi_k^\eta = \dfrac{1}{\eta} \pi_k \left(\dfrac{x}{\eta} \right) \end{cases} \text{in } B_{1/4} \backslash \eta T, \qquad \begin{cases} w_k^\eta = 0 \\ \pi_k^\eta = 0 \end{cases} \text{in } \eta T. \tag{3.1.6}$$

Since $w_k^\eta = 1$ on ∂Y, we can extend (w_k^η, π_k^η) to $\mathbb{R}^d$ periodically.

Lemma 3.1.2. *Assume $d \geq 3$. Let (w_k^η, π_k^η) be defined by (3.1.6) and extended periodically to $\mathbb{R}^d$. Then*

$$\begin{cases} -\Delta w_k^\eta + \chi \nabla \pi_k^\eta = \eta^{d-2} C_* e_k + \operatorname{div}(f_k^\eta) & \text{in } \omega_{1,\eta}, \\ \operatorname{div}(w_k^\eta) = 0 & \text{in } \omega_{1,\eta}, \\ w_k^\eta = 0 & \text{in } \mathbb{R}^d \backslash \omega_{1,\eta}, \end{cases} \tag{3.1.7}$$

where $C_* \in \mathbb{R}^{d\times d}$ is invertible, χ is the Y-periodic characteristic function such that

$$\chi = \begin{cases} 1 & \text{in } B_{1/4}\backslash \eta T, \\ 0 & \text{elsewhere in } Y, \end{cases} \tag{3.1.8}$$

and f_k^η is Y-periodic and satisfies

$$|f_k^\eta| \le C\eta^{d-2} \qquad \text{in } Y\backslash \eta T, \tag{3.1.9}$$

where C depends only on d and T. Furthermore,

$$\left(\fint_Y |\nabla w_k^\eta|^2 \right)^{1/2} \le C\eta^{\frac{d-2}{2}}, \tag{3.1.10}$$

where C depends only on d and T.

Proof. Let $\varphi \in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ be Y-periodic with $\varphi = 0$ in $\mathbb{R}^d\backslash\omega_{1,\eta}$. We need to show that

$$\int_Y \nabla w_k^\eta \cdot \nabla\varphi \, dx - \int_Y \chi\pi_k^\eta \nabla\cdot\varphi \, dx = \eta^{d-2} C_* e_k \cdot \int_Y \varphi \, dx - \int_Y f_k^\eta \cdot \nabla\varphi \, dx, \tag{3.1.11}$$

where C_* is invertible and f_k^η satisfies (3.1.9). We begin by observing that

$$\begin{aligned} &\int_Y \nabla w_k^\eta \cdot \nabla\varphi \, dx - \int_Y \chi\pi_k^\eta \nabla\cdot\varphi \, dx \\ &\qquad = \int_{B_{1/4}\backslash\eta T} \nabla w_k^\eta \cdot \nabla\varphi \, dx + \int_{B_{1/3}\backslash B_{1/4}} \nabla w_k^\eta \cdot \nabla\varphi \, dx \\ &\qquad\quad - \int_{B_{1/4}\backslash\eta T} \pi_k^\eta \nabla\cdot\varphi \, dx \\ &\qquad = \int_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n - \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) \right) \cdot \varphi \, d\sigma \\ &\qquad\quad + \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) d\sigma \cdot \int_{\partial B_{1/4}} \varphi \, d\sigma \\ &\qquad\quad + \int_{B_{1/3}\backslash B_{1/4}} \nabla w_k^\eta \cdot \nabla\varphi \, dx \\ &\qquad = I_1 + I_2 + I_3. \end{aligned}$$

For I_3, we note that (3.1.3) implies

$$|\nabla w_k^\eta| \le C\eta^{d-2} \qquad \text{in } B_{1/3}\backslash B_{1/4}. \tag{3.1.12}$$

We will now deal with I_1. Observe that (3.1.3) implies

$$\begin{aligned}\left|\int_{\partial B_{1/4}}\left(\frac{\partial w_k^\eta}{\partial n}-\pi_k^\eta n-\fint_{\partial B_{1/4}}\left(\frac{\partial w_k^\eta}{\partial n}-\pi_k^\eta n\right)\right)\right|&\\ \leq 2|\partial B_{1/4}|\max_{x\in\partial B_{1/4}}\left|\left(\frac{\partial w_k^\eta}{\partial n}-\pi_k^\eta n\right)(x)\right|&\\ = 2|\partial B_{1/4}|\max_{x\in\partial B_{1/(4\eta)}}\left|\frac{1}{\eta}\left(\frac{\partial w_k}{\partial n}-\pi_k n\right)(x)\right|&\\ \leq C\eta^{d-2}.&\end{aligned} \tag{3.1.13}$$

Therefore

$$\begin{aligned}|I_1|&\leq C\eta^{d-2}\int_{\partial B_{1/4}}|\varphi-\alpha|\\ &= C\eta^{d-2}\int_{\partial B_{1/4}}(x\cdot n)|\varphi-\alpha|\\ &\leq C\eta^{d-2}\left(\int_{B_{1/4}}|\varphi-\alpha|+\int_{B_{1/4}}|\nabla\varphi|\right)\end{aligned} \tag{3.1.14}$$

for any $\alpha\in\mathbb{R}^d$. Choosing $\alpha=\fint_{B_{1/4}}\varphi$ gives

$$|I_1|\leq C\eta^{d-2}\int_{B_{1/4}}|\nabla\varphi|. \tag{3.1.15}$$

It remains to deal with I_2. Note that

$$\fint_{\partial B_{1/4}}\left(\frac{\partial w_k^\eta}{\partial n}-\pi_k^\eta n\right)\cdot\int_{\partial B_{1/4}}\varphi=\eta^{d-2}\int_{\partial B_{1/(4\eta)}}\left(\frac{\partial w_k}{\partial n}-\pi_k n\right)\cdot\fint_{\partial B_{1/4}}\varphi. \tag{3.1.16}$$

Since (w_k,π_k) satisfies $-\Delta w_k+\nabla\pi_k=0$ in $B_s\backslash B_r$ for any $1/4\leq r<s<\infty$, it follows that

$$\begin{aligned}\int_{\partial B_{1/(4\eta)}}\frac{\partial w_k}{\partial n}-\pi_k n&=\int_{\partial B_1}\frac{\partial w_k}{\partial n}-\pi_k n\\ &=\frac{1}{2S_d}\int_{\partial B_1}F_k+d(F_k\cdot n)n\\ &=C_*e_k\end{aligned} \tag{3.1.17}$$

where C_* is the matrix whose columns are the vectors F_k given by (3.1.4). Next, observe

$$\begin{aligned}\fint_{\partial B_{1/4}}\varphi&=\frac{4}{|\partial B_{1/4}|}\int_{\partial B_{1/4}}(x\cdot n)\varphi\\ &=dC_d\fint_{B_{1/4}}\varphi+C_d\fint_{B_{1/4}}x\cdot\nabla\varphi,\end{aligned} \tag{3.1.18}$$

where $C_d = |B_1|/|\partial B_1|$. Thus we can write

$$\begin{aligned} \left| \fint_{\partial B_{1/4}} \varphi - \int_Y \varphi \right| &\le dC_d \fint_{B_{1/4}} \left| \varphi - \frac{1}{dC_d} \int_Y \varphi \right| + C_d \fint_{B_{1/4}} |x \cdot \nabla \varphi| \\ &\le C \Big(\int_Y |\nabla \varphi| + \int_Y |x \cdot \nabla \varphi| \Big). \end{aligned} \tag{3.1.19}$$

It follows from (3.1.16), (3.1.17), and (3.1.19) that

$$\begin{aligned} \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) \cdot \int_{\partial B_{1/4}} \varphi &= \eta^{d-2} \int_{\partial B_{1/(4\eta)}} \left(\frac{\partial w_k}{\partial n} - \pi_k n \right) \cdot \fint_{\partial B_{1/4}} \varphi \\ &= \eta^{d-2} C_* e_k \cdot \left(\left(\fint_{\partial B_{1/4}} \varphi - \int_Y \varphi \right) + \int_Y \varphi \right) \\ &= \eta^{d-2} C_* e_k \cdot \int_Y \varphi - \int_Y g_k^\eta \cdot \nabla \varphi, \end{aligned} \tag{3.1.20}$$

where $|g_k^\eta| \le C\eta^{d-2}$. We obtain (3.1.11) from (3.1.12), (3.1.15), and (3.1.20).

It remains to show C_* is invertible. We will show C_* is symmetric and positive-definite. Symmetry follows from (3.1.5). Let W be the matrix whose columns are w_k given by (3.1.2). We must show

$$\xi \cdot C_* \xi = \int_{\mathbb{R}^d \setminus T} |\nabla (W\xi)|^2 \ge c_0 |\xi|^2 \tag{3.1.21}$$

for any $\xi \in \mathbb{R}^d$. It suffices to consider $|\xi| = 1$. Suppose (3.1.21) does not hold. Since $\nabla(W\xi)$ is continuous as a function of ξ, we must have

$$\int_{\mathbb{R}^d \setminus T} |\nabla (W\xi)|^2 = 0 \tag{3.1.22}$$

for some $\xi \in \mathbb{R}^d$ with $|\xi| = 1$. In particular, $|\nabla(W\xi)| = 0$ for all $x \in \mathbb{R}^d \setminus T$. This is a contradiction, because $w_k = 0$ on ∂T and $w_k = e_k$ at infinity. Therefore, we obtain (3.1.21).

Finally, a simple energy estimate shows that (3.1.10) holds. Indeed, using w_k^η as a test function in (3.1.7) yields

$$\begin{aligned} \int_Y |\nabla w_k^\eta|^2 &= \eta^{d-2} \int_Y w_k^\eta \cdot C_* e_k - \int_Y f_k^\eta \cdot \nabla w_k^\eta \\ &\le C\eta^{d-2} \left(\fint_Y |w_k^\eta|^2 \right)^{1/2} + \eta^{d-2} \left(\fint_Y |\nabla w_k^\eta|^2 \right)^{1/2} \\ &\le C\eta^{\frac{d-2}{2}} \left(\fint_Y |\nabla w_k^\eta|^2 \right)^{1/2}. \end{aligned} \tag{3.1.23}$$

□

Given $\eta > 0$, we define $M^\eta \in H^1(\omega_{1,\eta}; \mathbb{R}^{d\times d})$ to be the matrix whose columns are the correctors w_k^η.

Lemma 3.1.3. *Assume $d \geq 3$. The matrix $\widehat{M^\eta}(0)$ is invertible for sufficiently small η, and*

$$|\widehat{M^\eta}(0)^{-1}| \leq C, \tag{3.1.24}$$

where the constant C is independent of η.

Proof. We have

$$\begin{aligned} |\widehat{w}_k^\eta(0) - e_k| &= \left| \int_Y w_k^\eta - e_k \right| \\ &\leq \int_{\eta T} 1 + \int_{B_{1/4}\setminus \eta T} |w_k^\eta - e_k| + \int_{B_{1/3}\setminus B_{1/4}} |w_k^\eta - e_k|. \end{aligned} \tag{3.1.25}$$

Using (3.1.3), we find

$$\int_{B_{1/3}\setminus B_{1/4}} |w_k^\eta - e_k| \leq C\eta^{d-2} \tag{3.1.26}$$

and

$$\begin{aligned} \int_{B_{1/4}\setminus \eta T} |w_k^\eta - e_k| &= \eta^d \int_{B_{1/(4\eta)}\setminus T} |w_k - e_k| \\ &\leq C\eta^d \int_1^{\frac{1}{4\eta}} \frac{1}{r^{d-2}} r^{d-1}\, dr \\ &\leq C\eta^{d-2}. \end{aligned} \tag{3.1.27}$$

It follows that

$$|\widehat{M^\eta}(0) - I| \leq C\eta^{d-2}, \tag{3.1.28}$$

which gives the invertibility of $\widehat{M^\eta}(0)$ for small η as well as (3.1.24). □

3.2 Correctors: The Case $d = 2$

We will now define the corrector matrix in the case $d = 2$. The correctors are again defined using the solutions of an exterior problem.

Lemma 3.2.1. *Suppose $d = 2$. Let $(w_k, \pi_k) \in H^1(\mathbb{R}^2\setminus T; \mathbb{R}^2) \times L^2(\mathbb{R}^2\setminus T)$ solve the exterior problem*

$$\begin{cases} -\Delta w_k + \nabla \pi_k = 0 & \text{in } \mathbb{R}^2\setminus T, \\ \operatorname{div}(w_k) = 0 & \text{in } \mathbb{R}^2\setminus T, \\ w_k = 0 & \text{on } \partial T. \end{cases} \tag{3.2.1}$$

Then (w_k, π_k) *satisfies at infinity*

$$\begin{aligned} w_k &= v_k + \frac{1}{4\pi}\ln(r)e_k + \frac{1}{4\pi}(e_k\cdot e_r)e_r + O\left(\frac{1}{r}\right), \\ \pi_k &= -\frac{1}{2\pi}\frac{e_k\cdot e_r}{r} + O\left(\frac{1}{r^2}\right), \\ \nabla w_k &= O\left(\frac{1}{r}\right), \\ \frac{\partial w_k}{\partial r} - \pi_k e_r &= \frac{1}{4\pi r}(e_k + 2(e_k\cdot e_r)e_r) + O\left(\frac{1}{r^2}\right), \end{aligned} \tag{3.2.2}$$

where $v_k \in \mathbb{R}^2$.

Proof. See [9]. □

For $d = 2$, we define the correctors (w_k^η, π_k^η) by

$$\begin{cases} w_k^\eta = e_k \\ \pi_k^\eta = 0 \end{cases} \text{in } Y\backslash B_{1/3}, \qquad \begin{cases} -\Delta w_k^\eta + \nabla \pi_k^\eta = 0 \\ \operatorname{div}(w_k^\eta) = 0 \end{cases} \text{in } B_{1/3}\backslash B_{1/4},$$
$$\begin{cases} w_k^\eta = \dfrac{4\pi}{|\ln(\eta)|} w_k\left(\dfrac{x}{\eta}\right) \\ \pi_k^\eta = \dfrac{4\pi}{\eta|\ln(\eta)|}\pi_k\left(\dfrac{x}{\eta}\right) \end{cases} \text{in } B_{1/4}\backslash \eta T, \qquad \begin{cases} w_k^\eta = 0 \\ \pi_k^\eta = 0 \end{cases} \text{in } \eta T. \tag{3.2.3}$$

As before, we extend (w_k^η, π_k^η) periodically to $\mathbb{R}^2$.

Lemma 3.2.2. *Assume* $d = 2$. *Let* (w_k^η, π_k^η) *be defined by* (3.2.3) *and extended periodically to* $\mathbb{R}^2$. *Then*

$$\begin{cases} -\Delta w_k^\eta + \chi\nabla\pi_k^\eta = 4\pi|\ln(\eta)|^{-1}e_k + \operatorname{div}(f_k^\eta) & \text{in } \omega_{1,\eta}, \\ \operatorname{div}(w_k^\eta) = 0 & \text{in } \omega_{1,\eta}, \\ w_k^\eta = 0 & \text{in } \mathbb{R}^2\backslash\omega_{1,\eta}, \end{cases} \tag{3.2.4}$$

where χ *is the* Y*-periodic characteristic function such that*

$$\chi = \begin{cases} 1 & \text{in } B_{1/4}\backslash\eta T, \\ 0 & \text{elsewhere in } Y, \end{cases} \tag{3.2.5}$$

and f_k^η *is* Y*-periodic and satisfies*

$$|f_k^\eta| \le C|\ln(\eta)|^{-1} \qquad \text{in } Y\backslash\eta T, \tag{3.2.6}$$

where C *depends only on* d *and* T. *Furthermore,*

$$\left(\fint_Y |\nabla w_k^\eta|^2\right)^{1/2} \le C|\ln(\eta)|^{-1/2}, \tag{3.2.7}$$

where C *depends only on* d *and* T.

Proof. Let $\varphi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be Y-periodic with $\varphi = 0$ in $\mathbb{R}^2 \backslash \omega_{1,\eta}$. We need to show that

$$\int_Y \nabla w_k^\eta \cdot \nabla \varphi \, dx - \int_Y \chi \pi_k^\eta \nabla \cdot \varphi \, dx = 4\pi |\ln(\eta)|^{-1} e_k \cdot \int_Y \varphi \, dx - \int_Y f_k^\eta \cdot \nabla \varphi \, dx, \quad (3.2.8)$$

where f_k^η satisfies (3.2.6). We begin by observing that

$$\begin{aligned}
&\int_Y \nabla w_k^\eta \cdot \nabla \varphi \, dx - \int_Y \chi \pi_k^\eta \nabla \cdot \varphi \, dx \\
&\qquad = \int_{B_{1/4} \backslash \eta T} \nabla w_k^\eta \cdot \nabla \varphi \, dx + \int_{B_{1/3} \backslash B_{1/4}} \nabla w_k^\eta \cdot \nabla \varphi \, dx \\
&\qquad\quad - \int_{B_{1/4} \backslash \eta T} \pi_k^\eta \nabla \cdot \varphi \, dx \\
&\qquad = \int_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n - \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) \right) \cdot \varphi \, d\sigma \\
&\qquad\quad + \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) d\sigma \cdot \int_{\partial B_{1/4}} \varphi \, d\sigma \\
&\qquad\quad + \int_{B_{1/3} \backslash B_{1/4}} \nabla w_k^\eta \cdot \nabla \varphi \, dx \\
&\qquad = I_1 + I_2 + I_3.
\end{aligned}$$

For I_3, we note that (3.2.2) implies

$$|\nabla w_k^\eta| \le C |\ln(\eta)|^{-1} \qquad \text{in } B_{1/3} \backslash B_{1/4}. \quad (3.2.9)$$

We will now deal with I_1. Observe that (3.2.2) implies

$$\begin{aligned}
&\left| \int_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n - \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) \right) \right| \\
&\qquad \le 2 |\partial B_{1/4}| \max_{x \in \partial B_{1/4}} \left| \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right)(x) \right| \\
&\qquad = 2 |\partial B_{1/4}| \max_{x \in \partial B_{1/(4\eta)}} \left| \frac{4\pi}{\eta |\ln(\eta)|} \left(\frac{\partial w_k}{\partial n} - \pi_k n \right)(x) \right| \\
&\qquad \le C |\ln(\eta)|^{-1}.
\end{aligned} \quad (3.2.10)$$

Therefore

$$\begin{aligned}
|I_1| &\le C |\ln(\eta)|^{-1} \int_{\partial B_{1/4}} |\varphi - \alpha| \\
&\le C |\ln(\eta)|^{-1} \int_{\partial B_{1/4}} (x \cdot n) |\varphi - \alpha| \\
&\le C |\ln(\eta)|^{-1} \left(\int_{B_{1/4}} |\varphi - \alpha| + \int_{B_{1/4}} |\nabla \varphi| \right)
\end{aligned} \quad (3.2.11)$$

for any $\alpha \in \mathbb{R}^2$. Choosing $\alpha = \fint_{B_{1/4}} \varphi$ gives

$$|I_1| \le C|\ln(\eta)|^{-1} \int_{B_{1/4}} |\nabla \varphi|. \tag{3.2.12}$$

It remains to deal with I_2. Note that

$$\fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta n \right) \cdot \int_{\partial B_{1/4}} \varphi = 4\pi |\ln(\eta)|^{-1} \int_{\partial B_{1/(4\eta)}} \left(\frac{\partial w_k}{\partial n} - \pi_k n \right) \cdot \fint_{\partial B_{1/4}} \varphi. \tag{3.2.13}$$

Since (w_k, π_k) satisfies $-\Delta w_k + \nabla \pi_k = 0$ in $B_s \backslash B_r$ for any $1/4 \le r < s < \infty$, it follows that

$$\begin{aligned} \int_{\partial B_{1/(4\eta)}} \frac{\partial w_k}{\partial n} - \pi_k n &= \int_{\partial B_1} \frac{\partial w_k}{\partial n} - \pi_k n \\ &= \frac{1}{4\pi} \int_{\partial B_1} e_k + 2x_k n \\ &= e_k. \end{aligned} \tag{3.2.14}$$

Next, observe

$$\begin{aligned} \fint_{\partial B_{1/4}} \varphi &= \frac{4}{|\partial B_{1/4}|} \int_{\partial B_{1/4}} (x \cdot n) \varphi \\ &= 2C_2 \fint_{B_{1/4}} \varphi + C_2 \fint_{B_{1/4}} x \cdot \nabla \varphi, \end{aligned} \tag{3.2.15}$$

where $C_2 = |B_1|/|\partial B_1|$. Thus we can write

$$\begin{aligned} \left| \fint_{\partial B_{1/4}} \varphi - \int_Y \varphi \right| &\le 2C_2 \fint_{B_{1/4}} \left| \varphi - \frac{1}{2C_2} \int_Y \varphi \right| + C_2 \fint_{B_{1/4}} |x \cdot \nabla \varphi| \\ &\le C \left(\int_Y |\nabla \varphi| + C \int_Y |x \cdot \nabla \varphi| \right). \end{aligned} \tag{3.2.16}$$

It follows from (3.2.13), (3.2.14), and (3.2.16) that

$$\begin{aligned} \fint_{\partial B_{1/4}} \left(\frac{\partial w_k^\eta}{\partial n} - \pi_k^\eta \right) \cdot \int_{\partial B_{1/4}} \varphi &= 4\pi |\ln(\eta)|^{-1} \int_{\partial B_{1/(4\eta)}} \left(\frac{\partial w_k}{\partial n} - \pi_k n \right) \cdot \fint_{\partial B_{1/4}} \varphi \\ &= 4\pi |\ln(\eta)|^{-1} e_k \cdot \left(\left(\fint_{\partial B_{1/4}} \varphi - \int_Y \varphi \right) + \int_Y \varphi \right) \\ &= 4\pi |\ln(\eta)|^{-1} e_k \cdot \int_Y \varphi - \int_Y g_k^\eta \cdot \varphi, \end{aligned} \tag{3.2.17}$$

where $|g_k^\eta| \le C|\ln(\eta)|^{-1}$. We obtain (3.2.4) from (3.2.9), (3.2.12), and (3.2.17). To show (3.2.7), we use w_k^η as a test function in (3.2.4) to obtain

$$\begin{aligned}
⨍_Y |\nabla w_k^\eta|^2 &= 4\pi|\ln(\eta)|^{-1}|\int_Y w_k^\eta \cdot e_k + \int_Y \operatorname{div}(f_k^\eta)\cdot w_k^\eta \\
&\le C|\ln(\eta)|^{-1}\left(⨍_Y |w_k^\eta|^2\right)^{1/2} + C|\ln(\eta)|^{-1}\left(⨍_Y |\nabla w_k^\eta|^2\right)^{1/2} \\
&\le C|\ln(\eta)|^{-1/2}\left(⨍_Y |\nabla w_k^\eta|^2\right)^{1/2}.
\end{aligned} \tag{3.2.18}$$

□

As before, we will need invertibility of $\widehat{M}^\eta(0)$, where M^η is the matrix whose columns are the correctors w_k^η.

Lemma 3.2.3. *Assume $d = 2$. The matrix $\widehat{M}^\eta(0)$ is invertible for sufficiently small η, and*

$$|\widehat{M}^\eta(0)^{-1}| \le C, \tag{3.2.19}$$

where the constant C is independent of η.

Proof. We have

$$\begin{aligned}
|\widehat{w}_k^\eta(0) - e_k| &= \left|\int_Y w_k^\eta - e_k\right| \\
&\le \int_{\eta T} 1 + \int_{B_{1/4}\setminus\eta T} |w_k^\eta - e_k| + \int_{B_{1/3}\setminus B_{1/4}} |w_k^\eta - e_k|.
\end{aligned} \tag{3.2.20}$$

Using (3.2.2), we find

$$\begin{aligned}
\int_{B_{1/3}\setminus B_{1/4}} |w_k^\eta - e_k| &\le |B_{1/3}\setminus B_{1/4}| \max_{x\in\partial B_{1/4}} |w_k^\eta - e_k| \\
&\le C|\ln(\eta)|^{-1}
\end{aligned} \tag{3.2.21}$$

and

$$\begin{aligned}
\int_{B_{1/4}\setminus\eta T} |w_k^\eta - e_k| &= \eta^2 \int_{B_{1/(4\eta)}\setminus T} \left|4\pi|\ln(\eta)|^{-1} w_k - e_k\right| \\
&\le C\eta^2|\ln(\eta)|^{-1}\int_1^{\frac{1}{4\eta}} r\,dr \\
&\le C|\ln(\eta)|^{-1}.
\end{aligned} \tag{3.2.22}$$

It follows that

$$|\widehat{M}^\eta(0) - I| \le C|\ln(\eta)|^{-1}, \tag{3.2.23}$$

which gives the invertibility of $\widehat{M}^\eta(0)$ for small η as well as (3.2.19). □

3.3 Proofs of Large-scale Estimates: Critical and Subcritical Cases

We now introduce some results from [17] to be used in the proofs of Theorems 3.0.1 and 3.0.2.

Lemma 3.3.1. *Let $u \in H^1(Q_{r+2})$, where $r \geq 1$. Then*

$$\left(\fint_{Q_r} |u|^2 \right)^{1/2} \leq C \max_{z \in Z^d \cap Q_{r+2}} |\hat{u}(z)| + C \left(\fint_{Q_{r+2}} |\nabla u|^2 \right)^{1/2}, \tag{3.3.1}$$

where C depends only on d.

Proof. See [17]. □

For a function g defined on $\mathbb{R}^d$ or $\mathbb{Z}^d$, we define

$$\Delta_j g(x) = g(x + e_j) - g(x) \tag{3.3.2}$$

for $1 \leq j \leq d$, where $e_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the jth position. For a multi-index $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, we use the notation

$$\Delta^\gamma g = \Delta_1^{\gamma_1} \Delta_2^{\gamma_2} \cdots \Delta_d^{\gamma_d} g \tag{3.3.3}$$

if $|\gamma| \geq 1$, and $\Delta^\gamma g = g$ if $\gamma = 0$. For an integer $k \geq 0$, let $\partial^k g = (\Delta^\gamma) g_{|\gamma|=k}$ and

$$|\partial^k g| = \left(\sum_{|\gamma|=k} |\Delta^\gamma g|^2 \right)^{1/2}. \tag{3.3.4}$$

It follows from the Fundamental Theorem of Calculus that

$$|\partial^{k+1} \widehat{u}(z)| \leq \left(\int_{z+3Q_1} |\nabla \partial^k u|^2 \, dx \right)^{1/2} \tag{3.3.5}$$

for any $z \in \mathbb{Z}^d$.

Lemma 3.3.2. *Let $u \in H^1(Q_R)$ for some $R \geq 100d$. Then, for any $r \in [1, R/100]$,*

$$\left(\fint_{Q_r} |u - \hat{u}(0)|^2 \right)^{1/2} \leq Cr \sum_{k=0}^{N} R^k \left(\fint_{Q_{R/2}} |\nabla \partial^k u|^2 \right)^{1/2} + C \left(\fint_{Q_{3r}} |\nabla u|^2 \right)^{1/2}, \tag{3.3.6}$$

where $N = [d/2] + 1$ and C depends only on d.

Proof. See [17]. □

The following lemma provides a discrete Sobolev inequality for functions defined on $\mathbb{Z}^d$.

Lemma 3.3.3. *Let g be a function defined on $\mathbb{Z}^d$. Then, for $R \geq 3d$,*

$$\max_{z\in\mathbb{Z}^d\cap Q_R} |g(z)| \leq C \sum_{k=0}^{N} R^k \left(\frac{1}{R^d} \sum_{z\in\mathbb{Z}^d\cap Q_{2R}} |\partial^k g(z)|^2 \right)^{1/2}, \tag{3.3.7}$$

where $N = [d/2] + 1$ and C depends only on d.

Proof. See [18]. □

We are now ready to give the proof of Theorem 3.0.1.

Proof of Theorem 3.0.1. By rescaling, we may assume $\varepsilon = 1$. Since both sides of (3.0.1) only feature ∇u, it is clear that this rescaling does not affect the estimate. Note that after rescaling, the condition $\varepsilon \leq R \leq \sigma_\varepsilon$ becomes $1 \leq R \leq K_\eta^{-1}$. We can also assume R is a large integer satisfying $R \geq \delta^{-2}d$ for some small δ. If R is not an integer, we can cover R by cubes with integer side lengths. If $1 \leq r \leq R < \delta^{-2}d$, then the estimate is trivial. Indeed, in this case

$$\begin{aligned}\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} &\leq \left(\frac{|Q_R|}{|Q_r|} \fint_{Q_R} |\nabla u|^2 \right)^{1/2} \\ &\leq C \left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2},\end{aligned} \tag{3.3.8}$$

where C depends only on d and δ.

Define $w = u - M^\eta \alpha$, where $\alpha \in \mathbb{R}^d$ is chosen so that $\widehat{w}(0) = 0$. It follows from (3.1.24) that

$$|\alpha| \leq C|\widehat{u}(0)|. \tag{3.3.9}$$

Furthermore, Lemmas 3.1.2 and 3.2.2 imply that there exists $q \in L^2(\mathbb{R}^d)$ such that

$$\begin{cases} -\Delta w + \nabla q = F^\eta + \operatorname{div}(f^\eta) & \text{in } Q_R^{1,\eta}, \\ \operatorname{div}(w) = 0 & \text{in } Q_R^{1,\eta}, \\ w = 0 & \text{in } Q_R \cap \partial\omega_{1,\eta}, \end{cases} \tag{3.3.10}$$

where F^η and f^η are Y-periodic and satisfy

$$|F^\eta| + |f^\eta| \leq C|\alpha| K_\eta^2. \tag{3.3.11}$$

For $1 \leq \rho \leq R/2$, the Caccioppoli inequality (2.2.2) and (3.3.11) imply

$$\left(\fint_{Q_\rho} |\nabla w|^2 \right)^{1/2} \leq \frac{C}{\rho} \left(\fint_{Q_{2\rho}} |w|^2 \right)^{1/2} + C\rho|\alpha|K_\eta^2. \tag{3.3.12}$$

Let γ be a multi-index with $1 \leq |\gamma| \leq d$. Since F^η and f^η are Y-periodic, we have

$$\begin{cases} -\Delta(\Delta^\gamma w) + \nabla(\Delta^\gamma q) = 0 & \text{in } Q_{R-3|\gamma|} \cap \omega_{1,\eta}, \\ \operatorname{div}(\Delta^\gamma w) = 0 & \text{in } Q_{R-3|\gamma|} \cap \omega_{1,\eta}, \\ \Delta^\gamma w = 0 & \text{on } Q_{R-3|\gamma|} \cap \partial\omega_{1,\eta}. \end{cases} \tag{3.3.13}$$

In particular, if $k \geq 1$, we may apply Theorem 2.2.1 to $\partial^k w$ to find

$$\begin{aligned}\left(\fint_{Q_\rho} |\nabla \partial^k w|^2\right)^{1/2} &\leq \frac{C}{\rho}\left(\fint_{Q_{2\rho}} |\partial^k w|^2\right)^{1/2} \\ &\leq \frac{C}{\rho}\left(\fint_{Q_{2\rho+3}} |\nabla \partial^{k-1} w|^2\right)^{1/2},\end{aligned} \tag{3.3.14}$$

for any $1 \leq \rho \leq (R-3d)/2$, where for the second inequality we have used the observation that

$$\left(\int_{z+Q_1} |\Delta_j w|^2\right)^{1/2} \leq C\left(\int_{z+3Q_1} |\nabla w|^2\right)^{1/2}. \tag{3.3.15}$$

By induction, we obtain

$$\left(\fint_{Q_\rho} |\nabla \partial^k w|^2\right)^{1/2} \leq \frac{C}{\rho^k}\left(\fint_{Q_{C\rho}} |\nabla w|^2\right)^{1/2} \tag{3.3.16}$$

for any $0 \leq k \leq d$, where C depends only on d. Let $r \in [1, \delta R/2]$. Using (3.3.12), (3.3.16), and (3.3.6) applied to w, we see that

$$\begin{aligned}\left(\fint_{Q_r} |\nabla w|^2\right)^{1/2} &\leq \frac{C}{r}\left(\fint_{Q_{2r}} |w|^2\right)^{1/2} + Cr|\alpha|K_\eta^2 \\ &\leq C\sum_{k=0}^{n} R^k \left(\fint_{Q_{100\delta R}} |\nabla \partial^k w|^2\right)^{1/2} + \frac{C}{r}\left(\fint_{Q_{6r}} |\nabla w|^2\right)^{1/2} + Cr|\alpha|K_\eta^2 \\ &\leq C\left(\fint_{Q_R} |\nabla w|^2\right)^{1/2} + \frac{C}{r}\left(\fint_{Q_{6r}} |\nabla w|^2\right)^{1/2} + CR|\alpha|K_\eta^2,\end{aligned} \tag{3.3.17}$$

where we have used the fact that $\widehat{w}(0) = 0$. It follows that

$$\begin{aligned}\left(\fint_{Q_r} |\nabla u|^2\right)^{1/2} &\leq \left(\fint_{Q_r} |\nabla w|^2\right)^{1/2} + \left(\fint_{Q_r} |\nabla (M^\eta \alpha)|^2\right)^{1/2} \\ &\leq C\left(\fint_{Q_R} |\nabla u|^2\right)^{1/2} + \frac{C}{r}\left(\fint_{Q_{6r}} |\nabla u|^2\right)^{1/2} + CR|\alpha|K_\eta^2 \\ &\quad + C\left(\fint_{Y} |\nabla (M^\eta \alpha)|^2\right)^{1/2},\end{aligned} \tag{3.3.18}$$

where the Y-periodicity of M^η has been used. Then by using the corrector bounds (3.1.10) and (3.2.7) as well as (3.3.9) to bound $|\alpha|$, we obtain

$$\left(\fint_{Q_r} |\nabla u|^2\right)^{1/2} \leq C\left(\fint_{Q_R} |\nabla u|^2\right)^{1/2} + \frac{C}{r}\left(\fint_{Q_{6r}} |\nabla u|^2\right)^{1/2} + CK_\eta |\hat{u}(0)|. \tag{3.3.19}$$

For the last term, we note that (3.3.16) holds with w replaced by u. Together with the discrete Sobolev inequality (3.3.7) and (3.3.5), we see that

$$
\begin{aligned}
|\hat{u}(0)| &\le C\sum_{k=0}^{N} R^k \left(\frac{1}{R^d} \sum_{z\in\mathbb{Z}^d\cap Q_{2\delta R}} |\partial^k \hat{u}(z)|^2 \right)^{1/2} \\
&\le C\left(\fint_{Q_{3\delta R}} |u|^2 \right)^{1/2} + C\sum_{k=1}^{N} R^k \left(\fint_{Q_{3\delta R}} |\nabla \partial^{k-1} u|^2 \right)^{1/2} \\
&\le C\left(\fint_{Q_R} |u|^2 \right)^{1/2} + CR\left(\fint_{Q_{R/2}} |\nabla u|^2 \right)^{1/2} \\
&\le C\left(\fint_{Q_R} |u|^2 \right)^{1/2} \\
&\le K_\eta^{-1}\left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2}
\end{aligned}
\tag{3.3.20}
$$

where the Caccioppoli inequality and the Poincaré inequality (2.0.1)-(2.0.2) have been used. From (3.3.19) and (3.3.20), it follows that

$$
\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} \le C\left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2} + \frac{C}{r}\left(\fint_{Q_{6r}} |\nabla u|^2 \right)^{1/2} \tag{3.3.21}
$$

for any $1 \le r \le \delta R/2$. In particular, for any $1 < s \le R$, we have

$$
\sup_{s\le r\le R}\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} \le C\left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2} + \frac{C}{s}\sup_{s\le r\le R}\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2}, \tag{3.3.22}
$$

where C depends only on d. By choosing s sufficiently large, we obtain

$$
\sup_{s\le r\le R}\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} \le C\left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2}. \tag{3.3.23}
$$

Finally, if $1 \le r \le s$, then

$$
\begin{aligned}
\left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} &\le \left(\frac{|Q_s|}{|Q_r|}\fint_{Q_s} |\nabla u|^2 \right)^{1/2} \\
&\le C\left(\fint_{Q_s} |\nabla u|^2 \right)^{1/2} \\
&\le C\left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2},
\end{aligned}
\tag{3.3.24}
$$

where we have used (3.3.23) in the last inequality. □

Proof of Theorem 3.0.2. By rescaling, we may assume $\varepsilon = 1$. We may also assume R is a large integer satisfying $R \geq \delta^{-2} d$ for some small δ. Let $1 \leq r \leq \delta R/2$. Then by the discrete Sobolev inequality (3.3.7),

$$\begin{aligned}
\max_{z \in \mathbb{Z}^d \cap Q_{r+2}} |\hat{u}(z)| &\leq \max_{z \in \mathbb{Z}^d \cap Q_{\delta R}} |\hat{u}(z)| \\
&\leq C \sum_{k=0}^{N} R^k \left(\frac{1}{R^d} \sum_{z \in \mathbb{Z}^d \cap Q_{2\delta R}} |\partial^k \hat{u}(z)|^2 \right)^{1/2} \\
&\leq C \left(\fint_{Q_{3\delta R}} |u|^2 \right)^{1/2} + C \sum_{k=1}^{N} R^k \left(\fint_{Q_{3\delta R}} |\nabla \partial^{k-1} u|^2 \right)^{1/2},
\end{aligned} \tag{3.3.25}$$

where $N = [d/2] + 1$ and (3.3.5) has been used. As before, (3.3.16) holds with w replaced by u. Using this, (3.3.1), and (3.3.25), we obtain

$$\begin{aligned}
\left(\fint_{Q_r} |u|^2 \right)^{1/2} &\leq C \left(\fint_{Q_R} |u|^2 \right)^{1/2} + CR \left(\fint_{Q_{R/2}} |\nabla u|^2 \right)^{1/2} + C \left(\fint_{Q_{3r}} |\nabla u|^2 \right)^{1/2} \\
&\leq C \left(\fint_{Q_R} |u|^2 \right)^{1/2} + \frac{C}{r} \left(\fint_{Q_{6r}} |u|^2 \right)^{1/2}.
\end{aligned} \tag{3.3.26}$$

As in the proof of Theorem 3.0.1, the result follows from (3.3.26). □

Chapter 4 Large-scale Estimates: Supercritical Case

In this chapter, we will complete the proofs of Theorems 1.3.3 and 1.3.4 by showing that the large-scale estimates (1.3.13) and (1.3.14) also hold in the supercritical case, *i.e.* when $R > \sigma_\varepsilon$. We will utilize an approach from [15], which treats the Darcy law case $\eta = 1$. First, we will use a reverse Hölder inequality to establish a boundary layer estimate for solutions to the Stokes equations in a perforated cube.

4.1 Reverse Hölder Inequalities

Given a function $u_\varepsilon \in H^1(Q_R)$ for some $R \geq \varepsilon$, define

$$g_\varepsilon(x) = \left(\fint_{Q(x,\varepsilon)} (\sigma_\varepsilon |\nabla u_\varepsilon| + |u_\varepsilon|)^2 \right)^{1/2}. \tag{4.1.1}$$

We wish to establish the following reverse Hölder inequality.

Theorem 4.1.1. *Let $(u_\varepsilon, p_\varepsilon) \in H^1(Q_{2R}^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_{2R}^{\varepsilon,\eta})$ be a weak solution of (1.3.12) in $Q_{2R}^{\varepsilon,\eta}$ with $u_\varepsilon = 0$ on $Q_{2R} \cap \partial\omega_{\varepsilon,\eta}$, where $0 < \varepsilon \leq 1$, $R \geq \varepsilon$, and $\sigma_\varepsilon < 1$. Let g_ε be defined by (4.1.1). Then there exist $q > 2$ and $C > 0$ such that*

$$\left(\fint_{Q_R} |g_\varepsilon|^q \right)^{1/q} \leq C \left(\fint_{Q_{2R}} (\sigma_\varepsilon |\nabla u_\varepsilon| + |u_\varepsilon|)^2 \right)^{1/2}, \tag{4.1.2}$$

where C depends only on d and T.

We will need a restriction operator defined in [2].

Lemma 4.1.2. *Suppose $0 < \varepsilon, \eta < 1$ satisfy $\sigma_\varepsilon < 1$. Then for any $r \geq \varepsilon$, there exists a linear operator $R_\varepsilon : H_0^1(Q_r; \mathbb{R}^d) \to H_0^1(Q_r^{\varepsilon,\eta}; \mathbb{R}^d)$ such that*

1. *If $u \in H_0^1(Q_r^{\varepsilon,\eta})$, then $R_\varepsilon u = u$ in $Q_r^{\varepsilon,\eta}$, where u has been extended by zero into the obstacles,*
2. *If $\operatorname{div}(u) = 0$ in Q_r, then $\operatorname{div}(R_\varepsilon u) = 0$ in $Q_r^{\varepsilon,\eta}$, and*
3. *$\|\nabla(R_\varepsilon u)\|_{L^2(Q_r^{\varepsilon,\eta})} \leq C\big(\|\nabla u\|_{L^2(Q_r)} + \frac{1}{\sigma_\varepsilon}\|u\|_{L^2(Q_r)}\big)$, where C does not depend on ε.*

Proof. See [2]. □

Lemma 4.1.3. *Let $(v, \tau) \in H^1(Q_t; \mathbb{R}^d) \times L^2(Q_t)$ be a weak solution of*

$$\begin{cases} -\Delta v + \nabla \tau = 0 & \text{in } Q_t, \\ \operatorname{div}(v) = 0 & \text{in } Q_t, \\ v = h & \text{on } \partial Q_t, \end{cases} \tag{4.1.3}$$

for some $t > 0$, where $h \in H^1(\partial Q_t; \mathbb{R}^d)$ satisfies the compatibility condition $\int_{\partial Q_t} h \cdot n \, d\sigma = 0$. Then there exist $q_0 \in (1,2)$ and $C > 0$, depending only on d, such that

$$\left(\fint_{Q_t} |v|^2 \right)^{1/2} \leq C \left(\fint_{\partial Q_t} |h|^{q_0} \right)^{1/q_0}, \tag{4.1.4}$$

and

$$\left(\fint_{Q_t} |\nabla v|^2 \right)^{1/2} \leq C \left(\fint_{\partial Q_t} |\nabla_{\tan} h|^{q_0} \right)^{1/q_0} \tag{4.1.5}$$

Proof. See [15]. □

We will use the following observation: for c_0 given in (1.3.1), we have

$$\operatorname{dist}(\partial Q_t; \mathbb{R}^d \backslash \omega_{\varepsilon,\eta}) \geq c_0 \varepsilon \qquad \text{if} \qquad \operatorname{dist}(t, \varepsilon\mathbb{N}) \leq c_0 \varepsilon. \tag{4.1.6}$$

Lemma 4.1.4. *Let $(u_\varepsilon, p_\varepsilon) \in H^1(Q_{2R}^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_{2R}^{\varepsilon,\eta})$ be a weak solution of (1.3.12) in $Q_{2R}^{\varepsilon,\eta}$ with $u_\varepsilon = 0$ on $Q_{2R} \cap \partial\omega_{\varepsilon,\eta}$, where $0 < \varepsilon \leq 1$, $R \in \varepsilon\mathbb{N}$, and $\sigma_\varepsilon < 1$. Then*

$$\begin{aligned} \sigma_\varepsilon \left(\fint_{Q_R} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{Q_R} |u_\varepsilon|^2 \right)^{1/2} \leq C\sigma_\varepsilon \left(\fint_{Q_{2R}} |\nabla u_\varepsilon|^{q_0} \right)^{1/q_0} \\ + C \left(\fint_{Q_{2R}} |u_\varepsilon|^{q_0} \right)^{1/q_0}, \end{aligned} \tag{4.1.7}$$

where $q_0 \in (1,2)$ is given by Lemma 4.1.3 and C depends only on d and T.

Proof. We will show that there exists $t \in [R, 2R]$ such that $\operatorname{dist}(t, \varepsilon\mathbb{N}) \leq c_0\varepsilon$ and

$$\int_{\partial Q_t} (\sigma_\varepsilon^{q_0} |\nabla u_\varepsilon|^{q_0} + |u_\varepsilon|^{q_0}) \, d\sigma \leq C_1 \int_{Q_{2R}} (\sigma_\varepsilon^{q_0} |\nabla u_\varepsilon|^{q_0} + |u_\varepsilon|^{q_0}) \, dx, \tag{4.1.8}$$

where C_1 depends on d and T. Suppose to the contrary that for any $t \in [R, 2R]$ with $\operatorname{dist}(t, \varepsilon\mathbb{N}) \leq c_0\varepsilon$,

$$\int_{\partial Q_t} (\sigma_\varepsilon^{q_0} |\nabla u_\varepsilon|^{q_0} + |u_\varepsilon|^{q_0}) \, d\sigma > C_1 \int_{Q_{2R}} (\sigma_\varepsilon^{q_0} |\nabla u_\varepsilon|^{q_0} \, dx + |u_\varepsilon|^{q_0}) \, dx.$$

Then integrating the above inequality with respect to t over the set

$$E_{\varepsilon,R} = \{ t \in (R, 2R) : \operatorname{dist}(t, \varepsilon N) \leq c_0\varepsilon \}$$

and using the observation that $|E_{\varepsilon,R}| \geq c > 0$ gives

$$\int_{Q_{2R} \backslash Q_R} (\sigma_\varepsilon^{q_0} |\nabla u_\varepsilon|^{q_0} + |u_\varepsilon|^{q_0}) \, d\sigma > C_2 C_1 \int_{Q_{2R}} (\sigma_\varepsilon^{q_0} |\nabla u_\varepsilon|^{q_0} + |u_\varepsilon|^{q_0}) \, dx, \tag{4.1.9}$$

where C_2 depends only on d and c_0. By choosing $C_1 = \frac{2}{C_2}$, we reach a contradiction.

Next, let (v, τ) be a weak solution of (4.1.3) in Q_t with Dirichlet data $h = u_\varepsilon$ on ∂Q_t. Let $v_\varepsilon = R_\varepsilon(v)$, where R_ε is the restriction operator defined in Lemma 4.1.2. Since $u_\varepsilon - v_\varepsilon$ satisfies

$$\begin{cases} -\Delta(u_\varepsilon - v_\varepsilon) + \nabla p_\varepsilon = \Delta v_\varepsilon & \text{in } Q_t^{\varepsilon,\eta}, \\ \operatorname{div}(u_\varepsilon - v_\varepsilon) = 0 & \text{in } Q_t^{\varepsilon,\eta}, \\ u_\varepsilon - v_\varepsilon = 0 & \text{on } \partial Q_t^{\varepsilon,\eta}, \end{cases} \tag{4.1.10}$$

we can use $u_\varepsilon - v_\varepsilon$ as a test function in (4.1.10) to see that

$$\begin{aligned} \int_{Q_t^{\varepsilon,\eta}} |\nabla(u_\varepsilon - v_\varepsilon)|^2 &= -\int_{Q_t^{\varepsilon,\eta}} \nabla v_\varepsilon \cdot \nabla(u_\varepsilon - v_\varepsilon) \\ &\le \|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^2(Q_t^{\varepsilon,\eta})} \|\nabla v_\varepsilon\|_{L^2(Q_t^{\varepsilon,\eta})}. \end{aligned} \tag{4.1.11}$$

Therefore

$$\begin{aligned} \|\nabla u\|_{L^2(Q_t^{\varepsilon,\eta})} &\le 2\|\nabla v_\varepsilon\|_{L^2(Q_t^{\varepsilon,\eta})} \\ &\le C(\|\nabla v\|_{L^2(Q_t)} + \frac{1}{\sigma_\varepsilon}\|v\|_{L^2(Q_t)}), \end{aligned} \tag{4.1.12}$$

where we have used the bounds for R_ε given in Lemma 4.1.2 in the second step. Together with Lemma 4.1.3 and (4.1.8), we conclude

$$\begin{aligned} \sigma_\varepsilon \|\nabla u_\varepsilon\|_{L^2(Q_R)} + \|u_\varepsilon\|_{L^2(Q_R)} &\le C(\sigma_\varepsilon \|\nabla v\|_{L^2(Q_t)} + \|v\|_{L^2(Q_t)}) \\ &\le C(\sigma_\varepsilon \|\nabla_{\tan} u_\varepsilon\|_{L^{q_0}(\partial Q_t)} + \|u_\varepsilon\|_{L^{q_0}(\partial Q_t)}) \\ &\le C(\sigma_\varepsilon \|\nabla u_\varepsilon\|_{L^{q_0}(Q_{2R})} + \|u_\varepsilon\|_{L^{q_0}(Q_{2R})}). \end{aligned} \tag{4.1.13}$$

□

Remark 4.1.5. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of (1.3.12) in $Q^{\varepsilon,\eta}(x_0, 4R)$ with $u_\varepsilon = 0$ on $Q(x_0, 4R) \cap \partial\omega_{\varepsilon,\eta}$, where $x_0 \in \mathbb{R}^d$, $0 < \varepsilon \le 1$, $R \ge 2\varepsilon$, and $\sigma_\varepsilon < 1$. Then

$$\begin{aligned} \sigma_\varepsilon \left(\fint_{Q(x_0,R)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{Q(x_0,R)} |u_\varepsilon|^2 \right)^{1/2} &\le C\sigma_\varepsilon \left(\fint_{Q(x_0,4R)} |\nabla u_\varepsilon|^{q_0} \right)^{1/q_0} \\ &\quad + C \left(\fint_{Q(x_0,4R)} |u_\varepsilon|^{q_0} \right)^{1/q_0}, \end{aligned} \tag{4.1.14}$$

where $q_0 \in (1, 2)$ is given by Lemma 4.1.3. Indeed, this follows from (4.1.7) by choosing $y_0 \in \varepsilon\mathbb{Z}^d$ and $R_1 \in \varepsilon\mathbb{N}$ such that

$$Q(x_0, R) \subset Q(y_0, R_1) \qquad \text{and} \qquad Q(y_0, 2R_1) \subset Q(x_0, 4R),$$

which is possible since $R \ge 2\varepsilon$.

Proof of Theorem 4.1.1. We may assume $0 < \varepsilon < cR$ where $c > 0$ is sufficiently small. The case $cR \le \varepsilon \le R$ is trivial. Indeed, in this case, for any $q > 2$, we would have

$$\left(\fint_{Q_R} |g_\varepsilon|^q\right)^{1/q} = \left(\fint_{Q_R} \left(\fint_{Q(x,\varepsilon)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{q/2}\right)^{1/q}$$
$$\leq C\left(\fint_{Q_R} \left(\fint_{Q(x,R)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{q/2}\right)^{1/q}$$
$$\leq C\left(\fint_{Q_R} \left(\fint_{Q_{2R}} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{q/2}\right)^{1/q} \tag{4.1.15}$$
$$= C\left(\fint_{Q_{2R}} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2}.$$

Let $q_0 \in (1,2)$ be given by Lemma 4.1.3. Define

$$G_\varepsilon(y) = \sup \left(\fint_{Q(z,r)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^{q_0}\right)^{1/q_0}, \tag{4.1.16}$$

where the supremum is taken over all cubes $Q(z,r)$ satisfying $y \in Q(z,r)$, $r \geq 2\varepsilon$, and $Q(z,r) \subset Q_{2R}$. We will show that

$$\left(\fint_{Q_R} |G_\varepsilon|^q\right)^{1/q} \leq C\left(\fint_{Q_{2R}} |G_\varepsilon|^2\right)^{1/2} \tag{4.1.17}$$

for some $q > 2$ depending only on d and T. For now, assume (4.1.17) holds. Since the Hardy-Littlewood maximal operator M is bounded on L^{2/q_0}, we have

$$\left(\fint_{Q_{2R}} |G_\varepsilon|^2\right)^{1/2} = \left(\fint_{Q_{2R}} \left(\sup_{Q(z_r)} \left(\fint_{Q(z,r)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^{q_0}\right)\right)^{2/q_0}\right)^{1/2}$$
$$\leq \left(\fint_{Q_{2R}} \left|M\big((\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^{q_0}\big)\right|^{2/q_0}\right)^{1/2} \tag{4.1.18}$$
$$\leq C\left(\fint_{Q_{2R}} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2}.$$

Also, by (4.1.14),

$$g_\varepsilon(x) \leq C\left(\fint_{Q(x,2\varepsilon)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2}$$
$$\leq CG_\varepsilon(x) \tag{4.1.19}$$

for any $x \in Q_R$. Together with (4.1.17) and (4.1.18), we see that

$$\left(\fint_{Q_R} |g_\varepsilon|^q\right)^{1/q} \leq C\left(\fint_{Q_R} |G_\varepsilon|^q\right)^{1/q},$$
$$\leq C\left(\fint_{Q_{2R}} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2}, \tag{4.1.20}$$

which yields the desired estimate (4.1.2)

It remains to prove (4.1.17). By the self-improving property of weak reverse Hölder inequalities, it suffices to show that

$$\left(\fint_{Q(x,t)} |G_\varepsilon|^2\right)^{1/2} \le C \left(\fint_{Q(x,8t)} |G_\varepsilon|^{q_0}\right)^{1/q_0} \tag{4.1.21}$$

for any $x \in Q_R$ and $0 < t < cR$. We divide the proof into two cases.

Case 1. Suppose $0 < t < 4\varepsilon$. Suppose $y, z \in Q(x,t)$. Observe that if a cube $Q(z_0, r)$ contains z and satisfies $r \ge 2\varepsilon$, then $Q(z_0,r) \subset Q(x, r+t)$ and $y \in Q(x, r+t)$. Taking supremum over such cubes, we find

$$\begin{aligned} G_\varepsilon(z) &\le C_1\bigg(\fint_{Q(x,r+t)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^{q_0}\bigg)^{1/q_0} \\ &\le C_1 \sup\left(\fint_{Q(y_0,r)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^{q_0}\right)^{1/q_0} \\ &\le C_1 G_\varepsilon(y), \end{aligned} \tag{4.1.22}$$

where the supremum in the second line is taken over cubes $Q(y_0, r)$ containing y and satisfying $r \ge 2\varepsilon$ and $Q(y_0, r) \subset Q_{2R}$. Since the order of y and z in this argument did not matter, we conclude that there exist $C_0 > 0$ and $C_1 > 0$ depending only on d and T such that

$$C_0 G_\varepsilon(y) \le G_\varepsilon(z) \le C_1 G_\varepsilon(y) \quad \text{for } y, z \in Q(x,t). \tag{4.1.23}$$

As a result, we immediately obtain (4.1.21) in this case.

Case 2. Suppose $4\varepsilon \le t < cR$. For $y \in Q(x,t)$, write

$$G_\varepsilon(y) = \max(G_\varepsilon^{(1)}(y), G_\varepsilon^{(2)}(y)), \tag{4.1.24}$$

where $G_\varepsilon^{(1)}$ is defined as in (4.1.16), but with the supremum taken over all cubes $Q(z,r)$ satisfying $y \in Q(z,r)$, $r \ge 2\varepsilon$, and $Q(z,r) \subset Q(x, 2t)$. By (4.1.14),

$$\begin{aligned} \left(\fint_{Q(x,t)} |G_\varepsilon^{(1)}|^2\right)^{1/2} &\le C\left(\fint_{Q(x,2t)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2} \\ &\le C\left(\fint_{Q(x,8t)} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^{q_0}\right)^{1/q_0} \\ &\le C\left(\fint_{Q(x,8t)} |G_\varepsilon|^{q_0}\right)^{1/q_0}. \end{aligned} \tag{4.1.25}$$

As in case 1,

$$G_\varepsilon^{(2)}(y) \sim G_\varepsilon^{(2)}(z) \quad \text{for } y, z \in Q(x,t). \tag{4.1.26}$$

Hence we have

$$\begin{aligned} \left(\fint_{Q(x,t)} |G_\varepsilon^{(2)}|^2\right)^{1/2} &\le C\left(\fint_{Q(x,t)} |G_\varepsilon^{(2)}|^{q_0}\right)^{1/q_0} \\ &\le C\left(\fint_{Q(x,t)} |G_\varepsilon|^{q_0}\right)^{1/q_0}. \end{aligned} \tag{4.1.27}$$

□

The following corollary gives a boundary layer estimate for solutions to the Stokes equations in a perforated cube.

Corollary 4.1.6. *Let $(u_\varepsilon, p_\varepsilon) \in H^1(Q_3^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_3^{\varepsilon,\eta})$ be a weak solution of (1.3.12) in $Q_3^{\varepsilon,\eta}$ with $u_\varepsilon = 0$ on $Q_3 \cap \partial\omega_{\varepsilon,\eta}$, where $0 < \sigma_\varepsilon < 1$. Then*

$$\left(\int_{Q_{1+\delta}\setminus Q_{1-\delta}} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2} \leq C\delta^\gamma \left(\int_{Q_3} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2} \tag{4.1.28}$$

for any $\delta \in (\varepsilon, 1]$, where C and γ depend only on d and T.

Proof. We can assume $\delta \leq 1/4$. The case $\delta > 1/4$ is trivial, as in this case (4.1.28) follows immediately by expanding the domain of integration from $Q_{1+\delta}\setminus Q_{1-\delta}$ to Q_3. By Fubini's Theorem and Hölder's inequality,

$$\begin{aligned}\left(\int_{Q_{1+\delta}\setminus Q_{1-\delta}} (\sigma_\varepsilon|\nabla u_\varepsilon| + |u_\varepsilon|)^2\right)^{1/2} &\leq C\left(\int_{Q_{1+\delta}\setminus Q_{1-\delta}} |g_\varepsilon|^2\right)^{1/2} \\ &\leq C|Q_{1+\delta}\setminus Q_{1-\delta}|^{\frac{1}{2}-\frac{1}{q}}\left(\int_{Q_{3/2}} |g_\varepsilon|^q\right)^{1/q} \\ &\leq C\delta^\gamma\left(\int_{Q_{3/2}} |g_\varepsilon|^q\right)^{1/q}\end{aligned} \tag{4.1.29}$$

where $q > 2$ is given by Theorem (4.1.1) and $\gamma = \frac{1}{2} - \frac{1}{q} > 0$. the desired estimate follows from (4.1.2). □

4.2 Compactness

As in the previous chapter, we will need a matrix of correctors. We define $(W^\eta, \Pi^\eta) \in H^1(\omega_{1,\eta}; \mathbb{R}^{d\times d}) \times L^2(\omega_{1,\eta}; \mathbb{R}^d)$ by

$$\begin{cases} -\Delta W_i^\eta + \nabla \Pi_i^\eta = K_\eta^2 e_i & \text{in } Y\setminus \eta T, \\ \operatorname{div}(W_i^\eta) = 0 & \text{in } Y\setminus \eta T, \\ W_i^\eta = 0 & \text{on } \partial\eta T, \end{cases} \tag{4.2.1}$$

where W_i^η is the ith column of W^η and Π_i^η is the entry in the ith position of Π^η, and we have extended periodically to $\mathbb{R}^d$.

Remark 4.2.1. An energy estimate combined with the Poincaré inequality (2.0.1) - (2.0.2) yields

$$\|W^\eta\|_{L^2(Q_1)} \leq C \qquad \text{and} \qquad \|\nabla W^\eta\|_{L^2(Q_1)} \leq CK_\eta, \tag{4.2.2}$$

where C depends only on d and T.

Remark 4.2.2. It follows from Remark 4.2.1 that

$$\|\Pi^\eta\|_{L^2(Q_1\backslash\eta T)} \le CK_\eta, \tag{4.2.3}$$

where C depends only on d and T. To see this, apply Lemma 2.1.4 to find $v \in H_0^1(Q_1\backslash\eta T;\mathbb{R}^{d\times d})$ such that

$$\operatorname{div}(v) = \Pi^\eta \quad \text{in } Q_1\backslash\eta T, \tag{4.2.4}$$

and

$$\|\nabla v\|_{L^2(Q_1\backslash\eta T)} \le C\|\Pi^\eta\|_{L^2(Q_1\backslash\eta T)}. \tag{4.2.5}$$

Using the equation for the correctors (4.2.1), we find that

$$\begin{aligned}\|\Pi^\eta\|^2_{L^2(Q_1\backslash\eta T)} &= \int_{Q_1\backslash\eta T} \Pi^\eta\cdot\operatorname{div}(v)\\ &\le \|\nabla\Pi^\eta\|_{H^{-1}(Q_1\backslash\eta T)}\|\nabla v\|_{L^2(Q_1\backslash\eta T)}\\ &\le (\|\nabla W^\eta\|_{L^2(Q_1\backslash\eta T)} + K_\eta^2)\|\Pi^\eta\|_{L^2(Q_1\backslash\eta T)}.\end{aligned} \tag{4.2.6}$$

We obtain (4.2.3) from (4.2.2) and (4.2.6).

Lemma 4.2.3. *The matrix of correctors W^η satisfies $W^\eta \to C_*^{-1}$ in $L^2(Y)$ as $\eta\to 0$, where C_* is the matrix in (3.1.7) if $d\ge 3$, and $C_* = 4\pi I$ if $d = 2$.*

Proof. Let (w_k^η, π_k^η) be the correctors defined in (3.1.6) if $d\ge 3$ or (3.2.3) if $d = 2$. Using w_k^η as a test function in (4.2.1) yields

$$\int_Y \nabla W_i^\eta\cdot\nabla w_k^\eta = K_\eta^2\int_Y e_i\cdot w_k^\eta. \tag{4.2.7}$$

Similarly, using W_i^η as a test function in (3.1.7) or (3.2.4) yields

$$\int_Y \nabla W_i^\eta\cdot\nabla w_k^\eta = K_\eta^2\int_Y C_*e_k\cdot W_i^\eta - \int_Y f_k^\eta\cdot\nabla W_i^\eta, \tag{4.2.8}$$

where $|f^\eta|\le K_\eta^2$. By subtracting equations (4.2.7) and (4.2.8), we obtain

$$\begin{aligned}\left|C_*\int_Y W^\eta - \int_Y M^\eta\right| &\le K_\eta^{-2}\int_Y |f^\eta||\nabla W^\eta|\\ &\le C\bigg(\int_Y|\nabla W^\eta|^2\bigg)^{1/2}\\ &\le CK_\eta,\end{aligned} \tag{4.2.9}$$

where we have used (4.2.2) in the last inequality. It is shown in [1] that

$$M^\eta\to I \text{ in } L^2(Y) \text{ as } \eta\to 0 \tag{4.2.10}$$

if $d \geq 3$. The convergence still holds when $d = 2$. This follows from the observation that

$$\begin{aligned}\int_{B_{1/3}\setminus B_{1/4}} |w_k^\eta - e_k|^2 &\leq |B_{1/3}\setminus B_{1/4}| \max_{x\in\partial B_{1/4}} |w_k^\eta - e_k|^2 \\ &\leq C|\ln(\eta)|^{-2}\end{aligned} \tag{4.2.11}$$

and

$$\begin{aligned}\int_{B_{1/4}\setminus \eta T} |w_k^\eta - e_k|^2 &= \eta^2 \int_{B_{1/(4\eta)}\setminus T} \left|4\pi|\ln(\eta)|^{-1} w_k - e_k\right|^2 \\ &\leq C\eta^2|\ln(\eta)|^{-2} \int_1^{\frac{1}{4\eta}} r\,dr \\ &\leq C|\ln(\eta)|^{-2},\end{aligned} \tag{4.2.12}$$

where the estimates for the correctors (3.2.2) have been used. We find that

$$\begin{aligned}\left(\int_Y |W^\eta - C_*^{-1}|^2\right)^{1/2} &\leq \left(\int_Y |W^\eta - \int_Y W^\eta|^2\right)^{1/2} + \left|\int_Y W^\eta - C_*^{-1}\int_Y M^\eta\right| \\ &\quad + |C_*^{-1}| \left(\int_Y |M^\eta - I|^2\right)^{1/2} \\ &\leq CK_\eta + |C_*|^{-1}\left(\int_Y |M^\eta - I|^2\right)^{1/2},\end{aligned} \tag{4.2.13}$$

where we have used (4.2.2) and (4.2.9). In view of 4.2.10, letting $\eta \to 0$ in (4.2.13) yields the result. □

Remark 4.2.4. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution to (1.3.12) in $Q_R^{\varepsilon,\eta}$. We extend u_ε to Q_R by zero and still denote the extension by u_ε. When necessary, we will use P_ε to denote the extension of the pressure p_ε defined by

$$P_\varepsilon(x) = \begin{cases} p_\varepsilon(x) & \text{if } x \in Q_R^{\varepsilon,\eta}, \\ \fint_{\varepsilon(Y\setminus \eta T + z_k)} p_\varepsilon & \text{if } x \in \varepsilon(\eta T + z_k) \text{ and } \varepsilon(Y + z_k) \subset Q_R \text{ for some } z_k \in \mathbb{Z}^d. \end{cases} \tag{4.2.14}$$

Observe that if $\varepsilon(Y + z_k) \subset Q_R$ for some $z_k \in \mathbb{Z}^d$, then

$$\begin{aligned}\fint_{\varepsilon(Y+z_k)} P_\varepsilon &= \frac{1}{\varepsilon^d}\left(\int_{\varepsilon(Y\setminus\eta T+z_k)} p_\varepsilon + \int_{\varepsilon(\eta T+z_k)} \fint_{\varepsilon(Y\setminus\eta T+z_k)} p_\varepsilon\right) \\ &= \frac{1}{\varepsilon^d}\left(|\varepsilon(Y\setminus\eta T + z_k)| \fint_{\varepsilon(Y\setminus\eta T+z_k)} p_\varepsilon + |\varepsilon(\eta T + z_k)| \fint_{\varepsilon(Y\setminus\eta T+z_k)} p_\varepsilon\right) \\ &= \fint_{\varepsilon(Y\setminus\eta T+z_k)} p_\varepsilon\end{aligned}$$

Then if $R \in \varepsilon\mathbb{N}$, it follows that

$$
\begin{aligned}
\fint_{Q_R} P_\varepsilon &= \sum_{z_k} \fint_{\varepsilon(Y+z_k)} P_\varepsilon \\
&= \sum_{z_k} \fint_{\varepsilon(Y\setminus\eta T+z_k)} p_\varepsilon \\
&= \fint_{Q_R^{\varepsilon,\eta}} p_\varepsilon,
\end{aligned} \tag{4.2.15}
$$

where the sum is taken over all $z_k \in \mathbb{Z}^d$ such that $\varepsilon(Y+z_k) \subset Q_R$.

In what follows, we will use rescaled velocity terms. For a solution $(u_\varepsilon, p_\varepsilon)$ to (1.3.3), we write

$$
\tilde{u}_\varepsilon = \frac{u_\varepsilon}{\sigma_\varepsilon^2}. \tag{4.2.16}
$$

We will need the following compactness theorem.

Theorem 4.2.5. *Let $\{(u_{\varepsilon_j}, p_{\varepsilon_j})\}$ be a sequence of weak solutions to*

$$
\begin{cases}
-\Delta u_{\varepsilon_j} + \nabla p_{\varepsilon_j} = 0 & \text{in } Q_4^{\varepsilon_j,\eta_j}, \\
\operatorname{div}(u_{\varepsilon_j}) = 0 & \text{in } Q_4^{\varepsilon_j,\eta_j}, \\
u_{\varepsilon_j} = 0 & \text{on } Q_4 \cap \partial\omega_{\varepsilon_j,\eta_j},
\end{cases} \tag{4.2.17}
$$

where $\varepsilon_j \to 0$ and $\eta_j \to 0$ satisfying $\sigma_{\varepsilon_j} \to 0$. Assume

$$
\|u_{\varepsilon_j}\|_{L^2(Q_4)} \le \sigma_{\varepsilon_j}^2. \tag{4.2.18}
$$

Then there exists a subsequence, still denoted $(u_{\varepsilon_j}, p_{\varepsilon_j})$, and $p_0 \in H^1(Q_1)$ such that

$$
P_{\varepsilon_j} - \fint_{Q_1} P_{\varepsilon_j} \to p_0 \qquad \text{in } L^2(Q_1), \tag{4.2.19}
$$

and

$$
\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0 \to 0 \qquad \text{in } L^2(Q_1;\mathbb{R}^d), \tag{4.2.20}
$$

where P_{ε_j} is the extension of p_{ε_j} defined in (4.2.14).

Proof. The assumption (4.2.18) implies

$$
\|\tilde{u_j}\|_{L^2(Q_4)} \le C. \tag{4.2.21}
$$

Note that Theorem 2.1.1 implies

$$
\|P_{\varepsilon_j} - \fint_{Q_1} P_{\varepsilon_j}\|_{L^2(Q_1)} \le C\sigma_{\varepsilon_j}^{-1}\|\nabla u_{\varepsilon_j}\|_{L^2(Q_1)} \le C, \tag{4.2.22}
$$

where we have summed the Caccioppoli inequality (2.2.2) on cubes of size σ_ε contained in Q_1 to justify the second step. Therefore, by passing to a subsequence, we can assume

$$\tilde{u}_{\varepsilon_j} \to u_0 \quad \text{weakly in } L^2(Q_1;\mathbb{R}^d) \tag{4.2.23}$$

and

$$P_{\varepsilon_j} - \fint_{Q_1} P_{\varepsilon_j} \to p_0 \quad \text{weakly in } L^2(Q_1). \tag{4.2.24}$$

We can also assume that $\fint_{Q_1} p_0 = 0$.

Step 1. We show that

$$P_{\varepsilon_j} - \fint_{Q_1} P_{\varepsilon_j} \to p_0 \qquad \text{in } L^2(Q_1). \tag{4.2.25}$$

Let $\psi \in H_0^1(Q_1;\mathbb{R}^d)$ and let R_{ε_j} be the restriction operator in Lemma 4.1.2. Then

$$\begin{aligned}
|\langle \nabla P_{\varepsilon_j}, \psi\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)}| &= |\langle \nabla p_{\varepsilon_j}, R_{\varepsilon_j}(\psi)\rangle_{H^{-1}(Q_1^{\varepsilon_j,\eta_j})\times H_0^1(Q_1^{\varepsilon_j,\eta_j})}| \\
&= |\langle \Delta \tilde{u}_{\varepsilon_j}, R_{\varepsilon_j}(\psi)\rangle_{H^{-1}(Q_1^{\varepsilon_j,\eta_j})\times H_0^1(Q_1^{\varepsilon_j,\eta_j})}| \\
&\le \|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1^{\varepsilon_j,\eta_j})} \|\nabla R_{\varepsilon_j}(\psi)\|_{L^2(Q_1^{\varepsilon_j,\eta_j})} \\
&\le C\|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1^{\varepsilon_j,\eta_j})} \left(\|\nabla \psi\|_{L^2(Q_1)} + \frac{1}{\sigma_{\varepsilon_j}} \|\psi\|_{L^2(Q_1)} \right).
\end{aligned} \tag{4.2.26}$$

Suppose to the contrary that (4.2.25) does not hold. Since

$$\|P_{\varepsilon_j} - \fint_{Q_1} P_{\varepsilon_j} - p_0\|_{L^2(Q_1)} \le C\|\nabla P_{\varepsilon_j} - \nabla p_0\|_{H^{-1}(Q_1)}, \tag{4.2.27}$$

it follows that ∇P_{ε_j} does not converge to ∇p_0 in $H^{-1}(Q_1)$. In particular, there exists a sequence $\{\psi_j\} \subset H_0^1(Q_1;\mathbb{R}^d)$ satisfying $\|\psi_j\|_{H_0^1(Q_1)} = 1$ and, after passing to a subsequence,

$$|\langle \nabla P_{\varepsilon_j} - \nabla p_0, \psi_j\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)}| \ge C_0 > 0. \tag{4.2.28}$$

By passing to another subsequence we can assume $\psi_n \to \psi_0$ weakly in $H_0^1(Q_1;\mathbb{R}^d)$. We decompose

$$\begin{aligned}
\langle \nabla P_{\varepsilon_j} - \nabla p_0, \psi_j\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)} &= \langle \nabla P_{\varepsilon_j}, \psi_j - \psi_0\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)} \\
&\quad - \langle \nabla p_0, \psi_j - \psi_0\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)} \\
&\quad + \langle \nabla P_{\varepsilon_j} - \nabla p_0, \psi_0\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)}.
\end{aligned} \tag{4.2.29}$$

Since the weak convergence in (4.2.24) implies

$$\langle \nabla P_{\varepsilon_j} - \nabla p_0, \psi_0\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)} \to 0, \tag{4.2.30}$$

it follows from (4.2.28) that

$$|\langle \nabla P_{\varepsilon_j}, \psi_j - \psi_0\rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)}| \ge C_0/2 \tag{4.2.31}$$

whenever j is sufficiently large. However, using $\psi = \psi_j - \psi_0$ in (4.2.26) yields

$$\begin{aligned}
|\langle \nabla P_{\varepsilon_j}, \psi_j - \psi_0 \rangle_{H^{-1}(Q_1)\times H_0^1(Q_1)}| &\leq C\|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)}\|\nabla(\psi_j - \psi_0)\|_{L^2(Q_1)} \\
&\quad + C\sigma_{\varepsilon_j}^{-1}\|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)}\|\psi_j - \psi_0\|_{L^2(Q_1)} \\
&\leq C\sigma_{\varepsilon_j}.
\end{aligned} \tag{4.2.32}$$

Since $\sigma_{\varepsilon_j} \to 0$ as $j \to \infty$, we reach a contradiction.
Step 2. We show that

$$u_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0 \to 0 \qquad \text{in } L^2(Q_1;\mathbb{R}^d). \tag{4.2.33}$$

By the Poincaré inequality, we have

$$\|\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0\|_{L^2(Q_1)} \leq C\sigma_{\varepsilon_j}\|\nabla(\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\|_{L^2(Q_1)}. \tag{4.2.34}$$

Let $\varphi_\delta \in C^\infty(Q_1)$ be a cutoff function satisfying $\varphi_\delta = 1$ on $Q_{1-\delta}$, $\varphi_\delta = 0$ on $Q_1 \backslash Q_{1-\delta/2}$ and $|\nabla \varphi_\delta| \leq \frac{C}{\delta}$. To bound the right side of (4.2.34), we write

$$\begin{aligned}
&\sigma_{\varepsilon_j}^2\|\nabla(\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\|_{L^2(Q_1)}^2 \\
&= \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla(\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot\nabla((\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\varphi_\delta) \\
&\qquad + \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla(\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot\nabla((\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)\nabla p_0)(1-\varphi_\delta)) \\
&= I_{1,j} + I_{2,j}.
\end{aligned} \tag{4.2.35}$$

We can further decompose

$$\begin{aligned}
I_{1,j} &= \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla \tilde{u}_{\varepsilon_j}\cdot\nabla(\tilde{u}_{\varepsilon_j}\varphi_\delta) + \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot\nabla(\tilde{u}_{\varepsilon_j}\varphi_\delta) \\
&\quad + \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla \tilde{u}_{\varepsilon_j}\cdot\nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta) \\
&\quad + \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot\nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta) \\
&= I_{1,j}^1 + I_{1,j}^2 + I_{1,j}^3 + I_{1,j}^4.
\end{aligned} \tag{4.2.36}$$

We will treat each term separately. First, we have

$$\begin{aligned}
I_{1,j}^1 &= -\sigma_{\varepsilon_j}^2 \int_{Q_1} \Delta \tilde{u}_{\varepsilon_j}\cdot\tilde{u}_{\varepsilon_j}\varphi_\delta \\
&= -\int_{Q_1} \nabla p_{\varepsilon_j}\cdot\tilde{u}_{\varepsilon_j}\varphi_\delta \\
&= \int_{Q_1} (P_{\varepsilon_j} - \int_{Q_1} P_{\varepsilon_j})\tilde{u}_{\varepsilon_j}\cdot\nabla\varphi_\delta \\
&\to \int_{Q_1} p_0 u_0\cdot\nabla\varphi_\delta
\end{aligned} \tag{4.2.37}$$

as $j \to \infty$. For $I_{1,j}^2$, we have

$$
\begin{aligned}
I_{1,j}^2 &= -\sigma_{\varepsilon_j}^2 \int_{Q_1} \Delta(W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta \\
&= -\sigma_{\varepsilon_j}^2 \int_{Q_1} \left(\frac{1}{{\varepsilon_j}^2}\Delta W^{\eta_j}(x/\varepsilon_j)\nabla p_0 + \frac{2}{\varepsilon_j}\nabla W^{\eta_j}(x/\varepsilon_j)\nabla^2 p_0\right)\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta \\
&\quad - \sigma_{\varepsilon_j}^2 \int_{Q_1} W^{\eta_j}(x/\varepsilon_j)\Delta(\nabla p_0)\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta.
\end{aligned}
\tag{4.2.38}
$$

Note that regularity results for the Stokes equations imply that $p_0 \in C^\infty(\mathbb{R}^d)$. It follows from the corrector estimates (4.2.2) that

$$
\begin{aligned}
\sigma_{\varepsilon_j}^2 \left|\int_{Q_1} W^{\eta_j}(x/\varepsilon_j)\Delta(\nabla p_0)\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta\right| &\le C\sigma_{\varepsilon_j}^2 \|W^{\eta_j}\|_{L^2(Q_1)}\|\tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)} \\
&\le C\sigma_{\varepsilon_j}^2 \\
&\to 0
\end{aligned}
\tag{4.2.39}
$$

as $j \to \infty$, where we have used the periodicity of W^{η_j}. Similarly,

$$
\begin{aligned}
\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\left|\int_{Q_1} \nabla W^{\eta_j}(x/\varepsilon_j)\nabla^2 p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta\right| &\le C\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\|\nabla W^{\eta_j}\|_{L^2(Q_1)}\|\tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)} \\
&\le C\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}K_{\eta_j} \\
&= C\sigma_{\varepsilon_j} \\
&\to 0
\end{aligned}
\tag{4.2.40}
$$

as $j \to \infty$. It remains to deal with the first term on the right in (4.2.38). Using (4.2.1), we find

$$
\begin{aligned}
-\frac{\sigma_{\varepsilon_j}^2}{{\varepsilon_j}^2}\int_{Q_1} \Delta W^{\eta_j}(x/\varepsilon_j)\nabla p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta &= \frac{\sigma_{\varepsilon_j}^2}{{\varepsilon_j}^2}\int_{Q_1} (K_{\eta_j}^2 I - \varepsilon_j\nabla\Pi^{\eta_j}(x/\varepsilon_j))\nabla p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta \\
&= \int_{Q_1} \nabla p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta - \frac{\sigma_{\varepsilon_j}^2}{{\varepsilon_j}^2}\int_{Q_1} \nabla\Pi^{\eta_j}(x/\varepsilon_j)\nabla p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta.
\end{aligned}
\tag{4.2.41}
$$

Integrating by parts and applying the weak convergence of $\tilde{u}_{\varepsilon_j}$, we see that

$$
\int_{Q_1} \nabla p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta \to -\int_{Q_1} p_0 u_0\cdot\nabla\varphi_\delta \quad \text{as } j\to\infty. \tag{4.2.42}
$$

Finally,

$$
-\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\int_{Q_1} \nabla\Pi^{\eta_j}(x/\varepsilon_j)\nabla p_0\cdot \tilde{u}_{\varepsilon_j}\varphi_\delta = \frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\int_{Q_1} \Pi^{\eta_j}(x/\varepsilon_j)\cdot(\nabla^2 p_0\varphi_\delta + \nabla p_0\nabla\varphi_\delta)\tilde{u}_{\varepsilon_j}. \tag{4.2.43}
$$

By Remark 4.2.2, the right side of (4.2.43) satisfies

$$
\begin{aligned}
\left|\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\int_{Q_1}\Pi^{\eta_j}(x/\varepsilon_j)\cdot(\nabla^2 p_0\varphi_\delta+\nabla p_0\nabla\varphi_\delta)\tilde{u}_{\varepsilon_j}\right| &\leq C\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\|\Pi^{\eta_j}\|_{L^2(Q_1)}\|\tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)}\\
&\leq C\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}K_{\eta_j}\\
&= C\sigma_{\varepsilon_j}\\
&\to 0
\end{aligned}
\tag{4.2.44}
$$

as $j\to\infty$, where we have used the periodicity of Π^{η_j}. Therefore

$$
I_{1,j}^2 \to -\int_{Q_1} p_0 u_0\cdot\nabla\varphi_\delta \quad \text{as } j\to\infty. \tag{4.2.45}
$$

Next,

$$
\begin{aligned}
I_{1,j}^3 &= -\sigma_{\varepsilon_j}^2\int_{Q_1}\Delta\tilde{u}_{\varepsilon_j}\cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta\\
&= -\int_{Q_1}\nabla p_{\varepsilon_j}\cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta\\
&= \int_{Q_1}(P_{\varepsilon_j}-\int_{Q_1}P_{\varepsilon_j})(W^{\eta_j}(x/\varepsilon_j)\nabla p_0\cdot\nabla\varphi_\delta+\varphi_\delta W^{\eta_j}(x/\varepsilon_j)\cdot\nabla^2 p_0)\\
&\to \int_{Q_1}(p_0 C_*^{-1}\nabla p_0\cdot\nabla\varphi_\delta+\varphi_\delta p_0 C_*^{-1}\cdot\nabla^2 p_0),
\end{aligned}
\tag{4.2.46}
$$

where we have used (4.2.24) as well as Lemma 4.2.3. Finally,

$$
\begin{aligned}
I_{1,j}^4 &= -\sigma_{\varepsilon_j}^2\int_{Q_1}\Delta(W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta\\
&= -\sigma_{\varepsilon_j}^2\int_{Q_1}\frac{1}{{\varepsilon_j}^2}\Delta W^{\eta_j}(x/\varepsilon_j)\nabla p_0\cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta\\
&\quad -\sigma_{\varepsilon_j}^2\int_{Q_1}\left(\frac{2}{\varepsilon_j}\nabla W^{\eta_j}(x/\varepsilon_j)\nabla^2 p_0+W^{\eta_j}(x/\varepsilon_j)\Delta(\nabla p_0)\right)\cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta.
\end{aligned}
\tag{4.2.47}
$$

As before, we use the corrector estimates (4.2.2) to see that

$$
\begin{aligned}
&\left|\sigma_{\varepsilon_j}^2\int_{Q_1}\left(\frac{2}{\varepsilon_j}\nabla W^{\eta_j}(x/\varepsilon_j)\nabla^2 p_0+W^{\eta_j}(x/\varepsilon_j)\Delta(\nabla p_0)\right)\cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta\right|\\
&\qquad\leq C\sigma_{\varepsilon_j}^2\left({\varepsilon_j}^{-1}\|\nabla W^{\eta_j}\|_{L^2(Q_1)}+\|W^{\eta_j}\|_{L^2(Q_1)}\right)\|W^{\eta_j}\|_{L^2(Q_1)}\\
&\qquad\leq C\sigma_{\varepsilon_j}\\
&\qquad\to 0
\end{aligned}
\tag{4.2.48}
$$

as $j \to \infty$. We will now treat the first term in (4.2.47). Using the corrector equation (4.2.1), we obtain

$$\begin{aligned}
-\frac{\sigma_{\varepsilon_j}^2}{{\varepsilon_j}^2}\int_{Q_1} &\Delta W^{\eta_j}(x/\varepsilon_j)\nabla p_0 \cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta \\
&= \frac{\sigma_{\varepsilon_j}^2}{{\varepsilon_j}^2}\int_{Q_1}(K_{\eta_j}^2 I - \varepsilon_j\nabla\Pi^{\eta_j}(x/\varepsilon_j))\nabla p_0 \cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta \\
&= \int_{Q_1}\nabla p_0 \cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta \\
&\quad - \frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\int_{Q_1}\nabla\Pi^{\eta_j}(x/\varepsilon_j)\nabla p_0 \cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta.
\end{aligned} \tag{4.2.49}$$

Integrating by parts and using the convergence of correctors in Lemma 4.2.3, we find

$$\begin{aligned}
\int_{Q_1}\nabla p_0 \cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta &= -\int_{Q_1} p_0 W^{\eta_j}(x/\varepsilon_j)\cdot(\nabla^2 p_0\varphi_\delta + \nabla p_0\nabla\varphi_\delta) \\
&\to -\int_{Q_1} p_0 C_*^{-1}\cdot\nabla^2 p_0\varphi_\delta - \int_{Q_1} p_0 C_*^{-1}\nabla p_0\cdot\nabla\varphi_\delta.
\end{aligned} \tag{4.2.50}$$

Since

$$\begin{aligned}
\left|\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\int_{Q_1}\nabla\Pi^{\eta_j}(x/\varepsilon_j)\nabla p_0 \cdot W^{\eta_j}(x/\varepsilon_j)\nabla p_0\varphi_\delta\right| &\le C\frac{\sigma_{\varepsilon_j}^2}{\varepsilon_j}\|\Pi^{\eta_j}\|_{L^2(Q_1)}\|W^{\eta_j}\|_{L^2(Q_1)} \\
&\le C\sigma_{\varepsilon_j} \\
&\to 0
\end{aligned} \tag{4.2.51}$$

as $j \to \infty$, we conclude that

$$I_{1,j}^4 \to -\int_{Q_1} p_0 C_*^{-1}\cdot\nabla^2 p_0\varphi_\delta - \int_{Q_1} p_0 C_*^{-1}\nabla p_0\cdot\nabla\varphi_\delta. \tag{4.2.52}$$

Using (4.2.37), (4.2.45), (4.2.46), and (4.2.52), we obtain

$$I_{1,j} \to 0 \quad \text{as } j \to \infty. \tag{4.2.53}$$

We now wish to deal with $I_{2,j}$. We begin by decomposing $I_{2,j}$ as

$$\begin{aligned}
I_{2,j} = \sigma_{\varepsilon_j}^2&\int_{Q_1}\nabla\tilde{u}_{\varepsilon_j}\cdot\nabla(\tilde{u}_{\varepsilon_j}(1-\varphi_\delta)) + \sigma_{\varepsilon_j}^2\int_{Q_1}\nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot\nabla(\tilde{u}_{\varepsilon_j}(1-\varphi_\delta)) \\
&+ \sigma_{\varepsilon_j}^2\int_{Q_1}\nabla\tilde{u}_{\varepsilon_j}\cdot\nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0(1-\varphi_\delta)) \\
&+ \sigma_{\varepsilon_j}^2\int_{Q_1}\nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0)\cdot\nabla(W^{\eta_j}(x/\varepsilon_j)\nabla p_0(1-\varphi_\delta)) \\
= I_{2,j}^1 &+ I_{2,j}^2 + I_{2,j}^3 + I_{2,j}^4.
\end{aligned} \tag{4.2.54}$$

Using properties of φ_δ, we have

$$\begin{aligned}|I_{2,j}^1| &\le \sigma_{\varepsilon_j}^2 \int_{Q_1} |\nabla \tilde{u}_{\varepsilon_j}|^2 (1-\varphi_\delta) + \sigma_{\varepsilon_j}^2 \int_{Q_1} |\nabla \tilde{u}_{\varepsilon_j} \tilde{u}_{\varepsilon_j} \cdot \nabla (1-\varphi_\delta)| \\ &\le \sigma_{\varepsilon_j}^2 \int_{Q_1 \setminus Q_{1-\delta}} |\nabla \tilde{u}_{\varepsilon_j}|^2 + C \frac{\sigma_{\varepsilon_j}^2}{\delta} \|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)} \|\tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)} \\ &\le \sigma_{\varepsilon_j}^2 \int_{Q_1 \setminus Q_{1-\delta}} |\nabla \tilde{u}_{\varepsilon_j}|^2 + C \frac{\sigma_{\varepsilon_j}}{\delta}.\end{aligned} \tag{4.2.55}$$

By the boundary layer estimate (4.1.28), we have

$$\begin{aligned}\sigma_{\varepsilon_j}^2 \int_{Q_1 \setminus Q_{1-\delta}} |\nabla \tilde{u}_{\varepsilon_j}|^2 &\le C\delta^\gamma \int_{Q_3} (\sigma_{\varepsilon_j}^2 |\nabla \tilde{u}_{\varepsilon_j}|^2 + |\tilde{u}_{\varepsilon_j}|^2) \\ &\le C\delta^\gamma.\end{aligned} \tag{4.2.56}$$

It follows that

$$\limsup_{j\to\infty} |I_{2,j}^1| \le C\delta^\gamma. \tag{4.2.57}$$

Similarly,

$$\begin{aligned}|I_{2,j}^2| &\le \sigma_{\varepsilon_j}^2 \int_{Q_1} |\nabla (W^{\eta_j}(x/\varepsilon_j)\nabla p_0)||\nabla \tilde{u}_{\varepsilon_j}|(1-\varphi_\delta) \\ &\quad + \sigma_{\varepsilon_j}^2 \int_{Q_1} |\nabla (W^{\eta_j}(x/\varepsilon_j)\nabla p_0)||\tilde{u}_{\varepsilon_j}||\nabla(1-\varphi_\delta)| \\ &\le C\sigma_{\varepsilon_j} \|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1\setminus Q_{1-\delta})} + C\frac{\sigma_{\varepsilon_j}}{\delta} \|\tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)} \\ &\le C\delta^\gamma + C\frac{\sigma_{\varepsilon_j}}{\delta}\end{aligned} \tag{4.2.58}$$

and

$$\begin{aligned}|I_{2,j}^3| &\le \sigma_{\varepsilon_j}^2 \int_{Q_1} |\nabla \tilde{u}_{\varepsilon_j}||\nabla (W^{\eta_j}(x/\varepsilon)\nabla p_0)|(1-\varphi_\delta) \\ &\quad + \sigma_{\varepsilon_j}^2 \int_{Q_1} |\nabla \tilde{u}_{\varepsilon_j}||W^{\eta_j}(x/\varepsilon_j)\nabla p_0||\nabla(1-\varphi_\delta)| \\ &\le C\sigma_{\varepsilon_j} \|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1\setminus Q_{1-\delta})} + C\frac{\sigma_{\varepsilon_j}^2}{\delta} \|\nabla \tilde{u}_{\varepsilon_j}\|_{L^2(Q_1)} \\ &\le C\delta^\gamma + C\frac{\sigma_{\varepsilon_j}}{\delta}.\end{aligned} \tag{4.2.59}$$

It follows that

$$\limsup_{j\to\infty} |I_{2,j}^2| \le C\delta^\gamma, \qquad \text{and} \qquad \limsup_{j\to\infty} |I_{2,j}^3| \le C\delta^\gamma. \tag{4.2.60}$$

Finally, we write

$$\begin{aligned}I_{2,j}^4 &= \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla (W^{\eta_j}(x/\varepsilon_j)\nabla p_0) \cdot \nabla (W^{\eta_j}(x/\varepsilon_j)\nabla p_0)(1-\varphi_\delta) \\ &\quad + \sigma_{\varepsilon_j}^2 \int_{Q_1} \nabla (W^{\eta_j}(x/\varepsilon_j)\nabla p_0) W^{\eta_j}(x/\varepsilon_j)\nabla p_0 \cdot \nabla(1-\varphi_\delta).\end{aligned} \tag{4.2.61}$$

Since W^{η_j} is periodic, we have

$$\begin{aligned} |I^4_{2,j}| &\le C\sigma^2_{\varepsilon_j}\|\nabla W^{\eta_j}\|^2_{L^2(Q_1)}|Q_1\backslash Q_{1-\delta}| + C\frac{\sigma^2_{\varepsilon_j}}{\delta}\|\nabla W^{\eta_j}\|_{L^2(Q_1)}\|W^{\eta_j}\|_{L^2(Q_1)} \\ &\le C\delta + C\frac{\sigma_{\varepsilon_j}}{\delta}. \end{aligned} \tag{4.2.62}$$

Therefore

$$\limsup_{j\to\infty}|I^4_{2,j}| \le C\delta. \tag{4.2.63}$$

Letting $\delta \to 0$ gives (4.2.33). □

Corollary 4.2.6. *Under the same assumptions as Theorem 4.2.5, we can conclude*

$$\tilde{u}_{\varepsilon_j} \to -C_*^{-1}\nabla p_0 \quad \text{in } L^2(Q_1;\mathbb{R}^d), \tag{4.2.64}$$

where p_0 is the limit in (4.2.24).

Proof. This follows from (4.2.33) and Lemma 4.2.3. □

4.3 One-step Improvement

We can now prove the following theorem, known as a one-step improvement result.

Theorem 4.3.1. *Let $0 < \beta < 1$. There exist $\theta \in (0, 1/4)$ and $\sigma_0 \in (0, 1/4)$ such that*

$$\inf_{E\in\mathbb{R}^d}\left(\fint_{Q^{\varepsilon,\eta}_\theta}|\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2\right)^{1/2} \le \theta^\beta\left(\fint_{Q^{\varepsilon,\eta}_1}|\tilde{u}_\varepsilon|^2\right)^{1/2} \tag{4.3.1}$$

whenever $0 < \sigma_\varepsilon < \sigma_0$, and $(u_\varepsilon, p_\varepsilon) \in H^1(Q^{\varepsilon,\eta}_1) \times L^2(Q^{\varepsilon,\eta}_1)$ is a weak solution of (1.3.12) in $Q^{\varepsilon,\eta}_1$ with $u_\varepsilon = 0$ in $Q_1 \cap \partial\omega_{\varepsilon,\eta}$.

Proof. The theorem is proved by contradiction. We first choose $\theta \in (0, 1/4)$ such that $C_0\theta \le (1/2)\theta^\beta$, where C_0 is the constant in (4.3.6). This is possible because $\beta < 1$. Suppose no σ_0 with the desired properties exists for this θ. Then there exists a sequence of weak solutions $(u_{\varepsilon_j}, p_{\varepsilon_j})$ of the Stokes equations

$$\begin{cases} -\Delta u_{\varepsilon_j} + \nabla p_{\varepsilon_j} = 0 & \text{in } Q^{\varepsilon_j,\eta_j}_1, \\ \operatorname{div}(u_{\varepsilon_j}) = 0 & \text{in } Q^{\varepsilon_j,\eta_j}_1, \\ u_{\varepsilon_j} = 0 & \text{on } Q_1 \cap \partial\omega_{\varepsilon_j,\eta_j}, \end{cases} \tag{4.3.2}$$

such that $\sigma_{\varepsilon_j} \to 0$,

$$\left(\fint_{Q_1}|\tilde{u}_{\varepsilon_j}|^2\right)^{1/2} \le 1, \tag{4.3.3}$$

and

$$\inf_{E\in\mathbb{R}^d}\left(\fint_{Q_\theta}|\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)E|^2\right)^{1/2} > \theta^\beta. \tag{4.3.4}$$

It follows that

$$
\begin{aligned}
\theta^\beta &< \inf_{E\in\mathbb{R}^d} \left(\fint_{Q_\theta} |\tilde{u}_{\varepsilon_j} + W^{\eta_j}(x/\varepsilon_j)E|^2 \right)^{1/2} \\
&\le \left(\fint_{Q_\theta} |\tilde{u}_{\varepsilon_j} + C_*^{-1}\nabla p_0|^2 \right)^{1/2} + \inf_{E\in\mathbb{R}^d} \left(\fint_{Q_\theta} |C_*^{-1}E - C_*^{-1}\nabla p_0|^2 \right)^{1/2} \\
&\quad + \inf_{E\in\mathbb{R}^d} \left(\fint_{Q_\theta} |W^{\eta_j}(x/\varepsilon_j)E - C_*^{-1}E|^2 \right)^{1/2} \\
&\le \left(\fint_{Q_\theta} |\tilde{u}_{\varepsilon_j} + C_*^{-1}\nabla p_0|^2 \right)^{1/2} + C|C_*^{-1}|\theta \\
&\quad + \left(\fint_{Q_\theta} |W^{\eta_j}(x/\varepsilon_j) - C_*^{-1}|^2 \right)^{1/2} \|\nabla p_0\|_{L^\infty(Q_\theta)},
\end{aligned}
\tag{4.3.5}
$$

where C_*^{-1} is the matrix in Lemma 4.2.3 and we have let $E = \nabla p_0(0)$. By letting $j \to \infty$, we conclude

$$\theta^\beta \le C_0\theta. \tag{4.3.6}$$

This is a contradiction with the choice of θ. □

Remark 4.3.2. Note that if $v_\varepsilon = W_i^\eta(x/\varepsilon)$ and $q_\varepsilon = \frac{1}{\varepsilon}\Pi_i^\eta(x/\varepsilon) - \frac{1}{\sigma_\varepsilon^2}x_i$, then

$$
\begin{cases}
-\Delta v_\varepsilon + \nabla q_\varepsilon = 0 & \text{in } \mathbb{R}^d\backslash\omega_{\varepsilon,\eta}, \\
\operatorname{div}(v_\varepsilon) = 0 & \text{in } \mathbb{R}^d\backslash\omega_{\varepsilon,\eta}, \\
v_\varepsilon = 0 & \text{on } \partial\omega_{\varepsilon,\eta}.
\end{cases}
\tag{4.3.7}
$$

This allows us to replace $\tilde{u}_\varepsilon$ by $\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E_0$ in Theorem 4.3.1 for any $E_0 \in \mathbb{R}^d$. In particular, we can replace (4.3.1) by

$$\inf_{E\in\mathbb{R}^d} \left(\fint_{Q_\theta^{\varepsilon,\eta}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2} \le \theta^\beta \inf_{E\in\mathbb{R}^d} \left(\fint_{Q_1^{\varepsilon,\eta}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2}. \tag{4.3.8}$$

This will allow us to repeatedly apply Theorem (4.3.1) in an induction argument to obtain the following result.

Theorem 4.3.3. *Let $0 < \beta < 1$. Let $\theta, \sigma_0 \in (0, 1/4)$ be given by Theorem 4.3.1. Then*

$$\inf_{E\in\mathbb{R}^d} \left(\fint_{Q_{\theta^k}^{\varepsilon,\eta}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2} \le \theta^{k\beta} \left(\fint_{Q_1^{\varepsilon,\eta}} |\tilde{u}_\varepsilon|^2 \right)^{1/2}, \tag{4.3.9}$$

whenever $0 < \sigma_\varepsilon < \theta^{k-1}\sigma_0$ and $(u_\varepsilon, p_\varepsilon) \in H^1(Q_1^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_1)$ is a weak solution of (1.3.12) in $Q_1^{\varepsilon,\eta}$ with $u_\varepsilon = 0$ in $Q_1 \cap \partial\omega_{\varepsilon,\eta}$.

Proof. We prove the theorem by induction. The case $k = 1$ is given by (4.3.8). Suppose we have the estimate (4.3.9) for some $k \ge 1$. Assume $0 < \sigma_\varepsilon < \theta^k\sigma_0$. Let

$$v = u_\varepsilon(\theta^k x) \quad \text{and} \quad q(x) = \theta^k p_\varepsilon(\theta^k x). \tag{4.3.10}$$

Then

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } Q_1^{\theta^{-k}\varepsilon,\eta}, \\ \operatorname{div}(v) = 0 & \text{in } Q_1^{\theta^{-k}\varepsilon,\eta}, \\ v = 0 & \text{on } Q_1^{\theta^{-k}\varepsilon,\eta} \cap \partial\omega_{\theta^{-k}\varepsilon,\eta}. \end{cases} \tag{4.3.11}$$

Since $\theta^{-k}\sigma_\varepsilon < \sigma_0$, it follows from (4.3.8) that

$$\begin{aligned} \inf_{E\in\mathbb{R}^d} \left(\fint_{Q^{\varepsilon,\eta}_{\theta^{k+1}}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2} &= \inf_{E\in\mathbb{R}^d} \left(\fint_{Q^{\varepsilon,\eta}_{\theta}} |\tilde{v} - W^\eta(x/(\varepsilon\theta^{-k}))E|^2 \right)^{1/2} \\ &\leq \theta \inf_{E\in\mathbb{R}^d} \left(\fint_{Q^{\varepsilon,\eta}_{1}} |\tilde{v} - W^\eta(x/(\varepsilon\theta^{-k}))E|^2 \right)^{1/2} \\ &= \theta \inf_{E\in\mathbb{R}^d} \left(\fint_{Q^{\varepsilon,\eta}_{\theta^k}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2} \\ &\leq \theta^{(k+1)\beta} \inf_{E\in\mathbb{R}^d} \left(\fint_{Q^{\varepsilon,\eta}_{1}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2}, \end{aligned} \tag{4.3.12}$$

where we have used the induction assumption for the last inequality. This completes the induction argument. □

Lemma 4.3.4. *Let $k \geq 1$ be an integer. Suppose $(u_\varepsilon, p_\varepsilon) \in H^1(Q_1^{\varepsilon,\eta};\mathbb{R}^d) \times L^2(Q_1)$ is a weak solution of (1.3.12) in $Q_1^{\varepsilon,\eta}$ with $u_\varepsilon = 0$ in $Q_1 \cap \partial\omega_{\varepsilon,\eta}$, and $0 < \sigma_\varepsilon < \theta^{k-1}\sigma_0$, where $\theta, \sigma_0 \in (0, 1/4)$ are given by Theorem 4.3.1. Define $E(k) \in \mathbb{R}^d$ to be the vector satisfying*

$$\left(\fint_{Q^{\varepsilon,\eta}_{\theta^k}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E(k)|^2 \right)^{1/2} = \inf_{E\in\mathbb{R}^d} \left(\fint_{Q^{\varepsilon,\eta}_{\theta^k}} |\tilde{u}_\varepsilon - W^\eta(x/\varepsilon)E|^2 \right)^{1/2}. \tag{4.3.13}$$

Then

$$|E(k)| \leq C\|\tilde{u}_\varepsilon\|_{L^2(Q_1)}, \tag{4.3.14}$$

where C is independent of k.

Proof. Observe that for any vector $E \in \mathbb{R}^d$ and $r \geq \varepsilon$, it follows from the periodicity of W^η that

$$|E| \leq C \left(\fint_{Q_r} |W^\eta(x/\varepsilon)E|^2 \right)^{1/2}. \tag{4.3.15}$$

Define $E(0) = 0$. For any $l \geq 1$, we use (4.3.15) and (4.3.9) to find

$$
\begin{aligned}
|E(l) - E(l-1)| &\leq C \left(\fint_{Q_{\theta^l}} |W^\eta(x/\varepsilon)(E(l) - E(l-1))|^2 \right)^{1/2} \\
&\leq C \left(\fint_{Q_{\theta^l}} |W^\eta(x/\varepsilon)E(l)|^2 \right)^{1/2} + C \left(\fint_{Q_{\theta^l}} |W^\eta(x/\varepsilon)E(l-1)|^2 \right)^{1/2} \\
&\leq C \left(\fint_{Q_{\theta^l}} |W^\eta(x/\varepsilon)E(l)|^2 \right)^{1/2} + C \left(\fint_{Q_{\theta^{l-1}}} |W^\eta(x/\varepsilon)E(l-1)|^2 \right)^{1/2} \\
&\leq C\theta^{l\beta} \|\tilde{u}_\varepsilon\|_{L^2(Q_1)}.
\end{aligned}
\tag{4.3.16}
$$

Summing, we obtain

$$
\begin{aligned}
|E(k)| &\leq \sum_{l=1}^{k} |E(l) - E(l-1)| \\
&\leq C\|\tilde{u}_\varepsilon\|_{L^2(Q_1)}.
\end{aligned}
\tag{4.3.17}
$$

□

Remark 4.3.5. Suppose $(u_\varepsilon, p_\varepsilon) \in H^1(Q_1^{\varepsilon,\eta}; \mathbb{R}^d) \times L^2(Q_1)$ is a weak solution of (1.3.12) in $Q_1^{\varepsilon,\eta}$ with $u_\varepsilon = 0$ in $Q_1 \cap \partial\omega_{\varepsilon,\eta}$, and $\sigma_\varepsilon < \sigma_0$, where σ_0 is defined in Theorem 4.3.1 It follows from Theorem 4.3.3 and Lemma 4.3.4 that

$$
\left(\fint_{Q_r} |u_\varepsilon|^2 \right)^{1/2} \leq C \left(\fint_{Q_1} |u_\varepsilon|^2 \right)^{1/2} \tag{4.3.18}
$$

whenever $\sigma_\varepsilon \leq r \leq 1$. Indeed, by Theorem 4.3.3 and Lemma 4.3.4, we have

$$
\left(\fint_{Q_{\theta^k}} |\tilde{u}_\varepsilon|^2 \right)^{1/2} \leq C \left(\fint_{Q_1} |\tilde{u}_\varepsilon|^2 \right)^{1/2} \tag{4.3.19}
$$

whenever $\sigma_\varepsilon < \theta^{k-1}\sigma_0$, where θ is defined in Theorem 4.3.1. Therefore we obtain (4.3.18) in the case $\sigma_\varepsilon/\sigma_0 \leq r \leq 1$. We may also obtain (4.3.18) in the case $\sigma_\varepsilon \leq r \leq \sigma_\varepsilon/\sigma_0$ by noting that in this case,

$$
\left(\fint_{Q_r} |u_\varepsilon|^2 \right)^{1/2} \leq C \left(\fint_{Q_{\sigma_\varepsilon/\sigma_0}} |u_\varepsilon|^2 \right)^{1/2}, \tag{4.3.20}
$$

where C depends on σ_0.

4.4 Proofs of Large-scale Estimates

We are now ready to prove the large-scale L^∞ estimate in the general case.

Proof of Theorem 1.3.4. We may assume $R > \sigma_\varepsilon$, as the case $\varepsilon \le R \le \sigma_\varepsilon$ is treated in Theorem 3.0.2. Let $v(x) = u(Rx)$ and $q(x) = Rp(Rx)$. Then (v, q) solves

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } Q_1^{\frac{\varepsilon}{R},\eta}, \\ \operatorname{div}(v) = 0 & \text{in } Q_1^{\frac{\varepsilon}{R},\eta}, \\ v = 0 & \text{on } Q_1^{\frac{\varepsilon}{R},\eta} \cap \partial\omega_{\frac{\varepsilon}{R},\eta}. \end{cases} \tag{4.4.1}$$

Remark 4.3.5 implies

$$\left(\fint_{Q_r} |v|^2 \right)^{1/2} \le C \left(\fint_{Q_1} |v|^2 \right)^{1/2} \tag{4.4.2}$$

whenever $\sigma_\varepsilon / R \le r \le 1$. By rescaling, we obtain the L^∞ estimate

$$\left(\fint_{Q_r} |u|^2 \right)^{1/2} \le C \left(\fint_{Q_R} |u|^2 \right)^{1/2} \tag{4.4.3}$$

whenever $\sigma_\varepsilon \le r \le R$.

It remains to treat the case $\varepsilon \le r \le \sigma_\varepsilon$. In this case, we use Theorem 3.0.2 and (4.4.3) to see that

$$\begin{aligned} \left(\fint_{Q_r} |u|^2 \right)^{1/2} &\le C \left(\fint_{Q_{\sigma_\varepsilon}} |u|^2 \right)^{1/2} \\ &\le C \left(\fint_{Q_R} |u|^2 \right)^{1/2}. \end{aligned} \tag{4.4.4}$$

This concludes the proof. □

Using the large-scale L^∞ estimate, we can prove the large-scale Lipschitz estimate in the general case.

Proof of Theorem 1.3.3. By rescaling, we may assume $\varepsilon = 1$. We may also assume $R > K_\eta^{-1}$, as the case $1 \le R \le K_\eta^{-1}$ is treated in Theorem 3.0.1. If $1 \le r \le K_\eta^{-1}$, it follows from Theorem 3.0.1 and Theorem 2.2.1 that

$$\begin{aligned} \left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} &\le C \Big(\fint_{Q_{K_\eta^{-1}}} |\nabla u|^2 \Big)^{1/2} \\ &\le C K_\eta \Big(\fint_{Q_{K_\eta^{-1}}} |u|^2 \Big)^{1/2}. \end{aligned} \tag{4.4.5}$$

Next, we use the large scale L^∞ estimate (1.3.14) and the Poincaré inequality (2.0.1)-(2.0.2) to see that

$$\begin{aligned} \left(\fint_{Q_{K_\eta^{-1}}} |u|^2 \right)^{1/2} &\le C \Big(\fint_{Q_R} |u|^2 \Big)^{1/2} \\ &\le C K_\eta^{-1} \Big(\fint_{Q_R} |\nabla u|^2 \Big)^{1/2}. \end{aligned} \tag{4.4.6}$$

Together with (4.4.5), we obtain the desired result when $1 \le r \le K_\eta^{-1}$.

If $r > K_\eta^{-1}$, covering Q_r by cubes with side length K_η^{-1} and applying Theorem 2.2.1 yields

$$\begin{aligned} \left(\fint_{Q_r} |\nabla u|^2 \right)^{1/2} &\le C K_\eta \left(\fint_{Q_{2r}} |u|^2 \right)^{1/2} \\ &\le C \left(\fint_{Q_R} |\nabla u|^2 \right)^{1/2}, \end{aligned} \tag{4.4.7}$$

where we have used the large-scale L^∞ estimate and the Poincaré inequality in the second step. □

This chapter is dedicated to establishing L^q estimates for two average operators needed in the proofs of Theorems 1.3.1 and 1.3.2. Let $(u,p) \in H_0^1(\omega_{\varepsilon,\eta}) \times L^2(\omega_{\varepsilon,\eta})$ be a weak solution of

$$\begin{cases} -\Delta u + \nabla p = F + \operatorname{div}(f) & \text{in } \omega_{\varepsilon,\eta}, \\ \operatorname{div}(u) = 0 & \text{in } \omega_{\varepsilon,\eta}, \\ u = 0 & \text{on } \partial\omega_{\varepsilon,\eta}. \end{cases} \tag{5.0.1}$$

Define

$$T_{\varepsilon,\eta}(F,f)(x) = \left(\fint_{x+\varepsilon Q_2} |u|^2 \right)^{1/2}. \tag{5.0.2}$$

It is clear that

$$\|T_{\varepsilon,\eta}(F,f)\|_{L^2(\omega_{\varepsilon,\eta})} = \|u\|_{L^2(\omega_{\varepsilon,\eta})}. \tag{5.0.3}$$

The following theorem gives L^q boundedness of $T_{\varepsilon,\eta}$ for $q \geq 2$.

Theorem 5.0.1. *Let $d \geq 2$. Let $2 \leq q < \infty$ and let $\omega_{\varepsilon,\eta}$ be given by (1.3.2). Then for any $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ and $F \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$,*

$$\|T_{\varepsilon,\eta}(F,f)\|_{L^q(\mathbb{R}^d)} \leq C\varepsilon\eta^{\frac{2-d}{2}} \|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon^2\eta^{2-d}\|F\|_{L^q(\omega_{\varepsilon,\eta})} \tag{5.0.4}$$

if $d \geq 3$, and

$$\|T_{\varepsilon,\eta}(F,f)\|_{L^q(\mathbb{R}^d)} \leq C\varepsilon|\ln(\eta/2)|^{1/2}\|f\|_{L^q(\mathbb{R}^d)} + C\varepsilon^2|\ln(\eta/2)|\|F\|_{L^q(\mathbb{R}^d)} \tag{5.0.5}$$

if $d = 2$, where C depends on d, q, and T.

Remark 5.0.2. The case $q = 2$ follows from the Poincaré inequality (2.0.1)-(2.0.2) and an energy estimate. Indeed, suppose (u,p) solves (5.0.1). Then using u as a test function in (5.0.1) yields

$$\begin{aligned} \int_{\omega_{\varepsilon,\eta}} |\nabla u|^2 &= \int_{\omega_{\varepsilon,\eta}} F \cdot u + \int_{\omega_{\varepsilon,\eta}} \operatorname{div}(f) \cdot u \\ &\leq \|F\|_{L^2(\omega_{\varepsilon,\eta})}\|u\|_{L^2(\omega_{\varepsilon,\eta})} + \|f\|_{L^2(\omega_{\varepsilon,\eta})}\|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})} \\ &\leq C(\|f\|_{L^2(\omega_{\varepsilon,\eta})} + \sigma_\varepsilon\|F\|_{L^2(\omega_{\varepsilon,\eta})})\|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})}. \end{aligned} \tag{5.0.6}$$

Therefore

$$\begin{aligned} \|u\|_{L^2(\omega_{\varepsilon,\eta})} &\leq C\sigma_\varepsilon\|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})} \\ &\leq C(\sigma_\varepsilon\|f\|_{L^2(\omega_{\varepsilon,\eta})} + \sigma_\varepsilon^2\|F\|_{L^2(\omega_{\varepsilon,\eta})}). \end{aligned} \tag{5.0.7}$$

The result follows from (5.0.3) and (5.0.7).

To deal with the case $q > 2$, we apply a real-variable argument and the large-scale L^∞ estimate proven in the previous sections.

An operator T is called sublinear if there is a constant K such that

$$|T(f+g)| \leq K(|T(f)| + |T(g)|). \tag{5.0.8}$$

Theorem 5.0.3. *Let T be a bounded sublinear operator from $L^2(\mathbb{R}^d;\mathbb{R}^m)$ to $L^2(\mathbb{R}^d)$ with $\|T\|_{L^2\to L^2} \leq C_0$. Let $p > 2$. Suppose there exists a constant N such that*

$$\left(\fint_Q |T(g)|^p\right)^{1/p} \leq N\left\{\left(\fint_{2Q} |T(g)|^2\right)^{1/2} + \sup_{Q'\supset Q}\left(\fint_Q |g|^2\right)^{1/2}\right\} \tag{5.0.9}$$

for any cube Q in $\mathbb{R}^d$ and for any $g \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^m)$ with supp$(g) \subset \mathbb{R}^d\backslash 4Q$. Then for any $f \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^m)$,

$$\|T(f)\|_{L^q(\mathbb{R}^d)} \leq C_q\|f\|_{L^q(\mathbb{R}^d)}, \tag{5.0.10}$$

where $2 < q < p$, and C_q depends at most on q, p, C_0, N, and the constant K in (5.0.8).

Proof. See [14]. □

Since, by linearity,

$$T_{\varepsilon,\eta}(F,f)(x) \leq T_{\varepsilon,\eta}(F,0) + T_{\varepsilon,\eta}(0,f), \tag{5.0.11}$$

we can treat the cases $T_{\varepsilon,\eta}(F,0)$ and $T_{\varepsilon,\eta}(0,f)$ separately.

Lemma 5.0.4. *Let $d \geq 2$. Let $2 < q < \infty$ and let $T_{\varepsilon,\eta}$ be defined by (5.0.2). Then*

$$\|T_{\varepsilon,\eta}(F,0)\|_{L^q(\mathbb{R}^d)} \leq \begin{cases} C\varepsilon^2\eta^{2-d}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d \geq 3, \\ C\varepsilon^2|\ln(\eta/2)|\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d = 2, \end{cases} \tag{5.0.12}$$

for any $F \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$, where C depends only on d, q, and T.

Proof. We may rescale so $\varepsilon = 1$. Let $T(f) = K_\eta^2 T_{1,\eta}(F,0)$. Then T satisfies (5.0.8) with $K = 1$, and $\|T\|_{L^2\to L^2} \leq C_0$. Consider a cube $Q \subset \mathbb{R}^d$. We will show that for any $G \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ with supp$(G) \subset \mathbb{R}^d\backslash 4Q$, we have

$$\|T(G)\|_{L^\infty(Q)} \leq C\left(\fint_{2Q} |T(G)|^2\right)^{1/2}. \tag{5.0.13}$$

By Theorem 5.0.3, we then deduce that T is bounded on $L^q(\mathbb{R}^d)$ for any $2 < q < \infty$ which allows us to obtain (5.0.12) for any $F \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$.

Let $Q = Q(x_0,l)$ be a cube centered at x_0 with side length l. Suppose that (u,p) solves

$$\begin{cases} -\Delta u + \nabla p = G & \text{in } \omega_{1,\eta}, \\ \operatorname{div}(u) = 0 & \text{in } \omega_{1,\eta}, \\ u = 0 & \text{on } \partial\omega_{1,\eta}, \end{cases} \tag{5.0.14}$$

where $G \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and supp$(G) \subset \mathbb{R}^d\backslash 4Q$. To show (5.0.13), we will use the geometric observation

$$\left(\fint_{2Q} |T(G)|^2\right)^{1/2} = K_\eta^2\left(\frac{1}{(2l)^d}\int_{Q(x_0,2+2l)} |u(y)|^2|Q(y,2)\cap Q(x_0,2l)|\,dy\right)^{1/2}, \tag{5.0.15}$$

as well as the large-scale L^∞ estimates proven in the previous sections. We consider two cases. In the first case, assume $0 < l \le 2$. Then for any $x \in Q(x_0, l)$, we have

$$T(G)(x) \le K_\eta^2 \left(\int_{Q(x_0, 2+2l)} |u(y)|^2 \, dy \right)^{1/2}. \tag{5.0.16}$$

Since $|Q(y,2) \cap Q(x_0, 2l)| \ge cl^d$ for $y \in Q(x_0, 2+l)$, we obtain (5.0.13) from (5.0.15) and (5.0.16), with C depending only on d.
In the second case, assume $l > 2$. It follows from Theorem 1.3.4 that

$$\left(\fint_{Q(x,2)} |u|^2 \right)^{1/2} \le C \left(\fint_{Q(x,l)} |u|^2 \right)^{1/2} \tag{5.0.17}$$

for any $x \in Q(x_0, l)$. Thus for any $x \in Q(x_0, l)$,

$$\begin{aligned} T(G)(x) &\le C K_\eta^2 \left(\fint_{Q(x,l)} |u|^2 \right)^{1/2} \\ &\le C K_\eta^2 \left(\fint_{Q(x_0,2l)} |u|^2 \right)^{1/2}. \end{aligned}$$

This shows

$$\begin{aligned} \|T(G)\|_{L^\infty(Q)} &\le C K_\eta^2 \left(\fint_{Q(x_0,2l)} |u|^2 \right)^{1/2} \\ &\le C \left(\fint_{2Q} |T(G)|^2 \right)^{1/2}, \end{aligned}$$

where we have used (5.0.15) and the fact that $|Q(y,2) \cap Q(x_0, 2l)| \ge C$ for any $y \in Q(x_0, 2l)$. This gives (5.0.13) for any cube Q. □

Lemma 5.0.5. *Let $d \ge 2$. Let $2 < q < \infty$ and $T_{\varepsilon,\eta}$ be defined by (5.0.2). Then*

$$\|T_{\varepsilon,\eta}(0,f)\|_{L^q(\omega_{\varepsilon,\eta})} \le \begin{cases} C\varepsilon \eta^{\frac{2-d}{2}} \|f\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d \ge 3, \\ C\varepsilon |\ln(\eta/2)|^{1/2} \|f\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d = 2, \end{cases} \tag{5.0.18}$$

for any $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$, where C depends only on d, q, and T.

Proof. Again, we may assume $\varepsilon = 1$ by rescaling. Define the operator T by

$$T(f) = K_\eta T_{1,\eta}(0,f). \tag{5.0.19}$$

Observe that T satisfies (5.0.8) with $K = 1$, and $\|T\|_{L^2\to L^2} \le C_0$. Suppose (u,p) is a weak solution to

$$\begin{cases} -\Delta u + \nabla p = \operatorname{div}(g) & \text{in } \omega_{1,\eta}, \\ \operatorname{div}(u) = 0 & \text{in } \omega_{1,\eta}, \\ u = 0 & \text{on } \partial\omega_{1,\eta}, \end{cases} \tag{5.0.20}$$

with $g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ and $\text{supp}(g) \subset \mathbb{R}^d \backslash 4Q$, where Q is an arbitrary cube. The same argument as in the proof of Lemma 5.0.4 yields

$$\|T(g)\|_{L^\infty(Q)} \le C \left(\fint_{2Q} |T(g)|^2 \right)^{1/2}. \tag{5.0.21}$$

Therefore, by Theorem 5.0.3, we obtain

$$\|T(f)\|_{L^q(\omega_{1,\eta})} \le C\|f\|_{L^q(\omega_{1,\eta})} \tag{5.0.22}$$

for any $2 < q < \infty$. This yields (5.0.18) with $\varepsilon = 1$. □

Proof of Theorem 5.0.1. Utilizing (5.0.11), the estimates in (5.0.4) follow immediately from (5.0.12) and (5.0.18). □

Let $(u, p) \in H_0^1(\omega_{\varepsilon,\eta}) \times L^2(\omega_{\varepsilon,\eta})$ be a weak solution to (5.0.1). Similar to before, we define

$$S_{\varepsilon,\eta}(F, f)(x) = \left(\fint_{x+\varepsilon Q_2} |\nabla u|^2 \right)^{1/2} \tag{5.0.23}$$

and note that

$$\|S_{\varepsilon,\eta}(F, f)\|_{L^2(\omega_{\varepsilon,\eta})} = \|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})}. \tag{5.0.24}$$

Theorem 5.0.6. *Let $d \ge 2$. Let $2 \le q < \infty$ and let $\omega_{\varepsilon,\eta}$ be given by (1.3.2). Then for any $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ and $F \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$,*

$$\|S_{\varepsilon,\eta}(F, f)\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon\eta^{\frac{2-d}{2}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} \tag{5.0.25}$$

if $d \ge 3$, and

$$\|S_{\varepsilon,\eta}(F, f)\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^q(\omega_{\varepsilon,\eta})} + C\varepsilon|\ln(\eta/2)|^{1/2}\|F\|_{L^q(\omega_{\varepsilon,\eta})} \tag{5.0.26}$$

if $d = 2$, where C depends only on d, q, and T.

As before, we will use the linearity of $S_{\varepsilon,\eta}$ to divide into two cases. Namely,

$$S_{\varepsilon,\eta}(F, f)(x) \le S_{\varepsilon,\eta}(F, 0) + S_{\varepsilon,\eta}(0, f), \tag{5.0.27}$$

and we will treat the cases $S_{\varepsilon,\eta}(F, 0)$ and $S_{\varepsilon,\eta}(0, f)$ separately. First, we make the following remark.

Remark 5.0.7. The case $q = 2$ in Theorem (5.0.6) follows from the Poincaré inequality (2.0.1)-(2.0.2) and an energy estimate. As in Remark 5.0.2, using u as a test function in (5.0.1) yields

$$\|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})} \le C(\|f\|_{L^2(\omega_{\varepsilon,\eta})} + \sigma_\varepsilon\|F\|_{L^2(\omega_{\varepsilon,\eta})}). \tag{5.0.28}$$

The result then follows from (5.0.24) and (5.0.28).

Lemma 5.0.8. *Let $d \geq 2$. Let $2 < q < \infty$ and $S_{\varepsilon,\eta}$ be defined by (5.0.23). Then*

$$\|S_{\varepsilon,\eta}(F,0)\|_{L^q(\mathbb{R}^d)} \leq \begin{cases} C\varepsilon\eta^{\frac{2-d}{2}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d \geq 3 \\ C\varepsilon|\ln(\eta/2)|^{1/2}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d = 2, \end{cases} \tag{5.0.29}$$

for any $F \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$, where C depends only on d, q, and T.

Proof. We assume $\varepsilon = 1$ by rescaling. Define the operator S by

$$S(F) = K_\eta S_{1,\eta}(F,0). \tag{5.0.30}$$

Note that S satisfies (5.0.8) with $K = 1$. Furthermore, by Remark 5.0.7 $\|S\|_{L^2\to L^2} \leq C_0$. Suppose (u,p) is a weak solution to (5.0.14) where $G \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $\text{supp}(G) \subset \mathbb{R}^d\backslash 4Q$. As in the proof of Lemma 5.0.4, we make the geometric observation

$$\left(\fint_{2Q}|T(G)|^2\right)^{1/2} = K_\eta\left(\frac{1}{(2l)^d}\int_{Q(x_0,2+2l)}|\nabla u(y)|^2|Q(y,2)\cap Q(x_0,2l)|\,dy\right)^{1/2}. \tag{5.0.31}$$

If $Q(x,l)$ satisfies $0 < l \leq 2$, then we obtain

$$\|S\|_{L^\infty(Q)} \leq C\left(\fint_{2Q}|S(G)|^2\right)^{1/2} \tag{5.0.32}$$

using the same argument as in Lemma 5.0.4. If $l > 2$, we use the large-scale Lipschitz estimates in Theorem 1.3.3 to see that

$$\left(\fint_{Q(x,2)}|\nabla u|^2\right)^{1/2} \leq C\left(\fint_{Q(x,l)}|\nabla u|^2\right)^{1/2} \tag{5.0.33}$$

for any $x \in Q(x_0,l)$. It follows that for any $x \in Q(x_0,l)$,

$$\begin{aligned} S(G)(x) &\leq CK_\eta\left(\fint_{Q(x,l)}|\nabla u|^2\right)^{1/2} \\ &\leq CK_\eta\left(\fint_{Q(x_0,2l)}|\nabla u|^2\right)^{1/2} \\ &\leq C\left(\fint_{2Q}|S(G)|^2\right)^{1/2}, \end{aligned} \tag{5.0.34}$$

where we have used (5.0.31) and the fact that $|Q(y,2)\cap Q(x_0,2l)| \geq C$ for any $y \in Q(x_0,2l)$. This yields (5.0.32) in the case $l > 2$. By Theorem 5.0.3, we obtain (5.0.29). □

Lemma 5.0.9. *Let $d \geq 2$. Let $2 < q < \infty$ and $S_{\varepsilon,\eta}$ be defined by (5.0.23). Then*

$$\|S_{\varepsilon,\eta}\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^q(\omega_{\varepsilon,\eta})}, \tag{5.0.35}$$

where C depends only on d, q, and T.

Proof. Again, assume $\varepsilon = 1$ by rescaling. Define the operator S by

$$S(f) = S_{1,\eta}(0, f). \tag{5.0.36}$$

Note that S satisfies (5.0.8) with $K = 1$ and by Remark 5.0.7, we have $\|S\|_{L^2 \to L^2} \leq C_0$. Suppose (u, p) is a weak solution to (5.0.20) with $g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ and $\text{supp}(g) \subset \mathbb{R}^d \backslash 4Q$, where Q is an arbitrary cube. The same argument as in the proof of Lemma 5.0.8 yields

$$\|S\|_{L^\infty(Q)} \leq C \left(\fint_{2Q} |S(G)|^2 \right)^{1/2} \tag{5.0.37}$$

We then apply Theorem 5.0.3 to S to obtain (5.0.35). □

Proof of Theorem 5.0.6. In view of (5.0.27), the desired estimates follow readily from (5.0.29) and (5.0.35). □

Chapter 6 Estimates in an Exterior Domain

In this chapter, we establish an L^q estimate for ∇u, where (u,p) is a solution with compact support of the Stokes equations in an exterior domain. Throughout this section, we assume T is the closure of a bounded C^1 domain in $\mathbb{R}^d$. In $\mathbb{R}^d\backslash T$, consider the problem

$$\begin{cases} -\Delta u + \nabla p = F & \text{in } \mathbb{R}^d\backslash T, \\ \operatorname{div}(u) = g & \text{in } \mathbb{R}^d\backslash T, \\ u = 0 & \text{on } \partial T. \end{cases} \tag{6.0.1}$$

The next theorem provides bounds in a weighted Sobolev space for solutions to (6.0.1). We first introduce some notation. For $1 < q < \infty$ and $q \neq d$, let

$$X^{1,q}(\mathbb{R}^d\backslash T) = \big\{u \in W^{1,q}_{loc}(\mathbb{R}^d\backslash T) : (1+|x|)^{-1}u \in L^q(\mathbb{R}^d\backslash T) \text{ and } \nabla u \in L^q(\mathbb{R}^d\backslash T)\big\}, \tag{6.0.2}$$

with the norm

$$\|u\|_{X^{1,q}(\mathbb{R}^d\backslash T)} = \|(1+|x|)^{-1}u\|_{L^q(\mathbb{R}^d\backslash T)} + \|\nabla u\|_{L^q(\mathbb{R}^d\backslash T)}. \tag{6.0.3}$$

If $q = d$, let

$$X^{1,d}(\mathbb{R}^d\backslash T) = \Big\{u \in W^{1,d}_{loc}(\mathbb{R}^d\backslash T) : ((1+|x|)\ln(2+|x|))^{-1}u \in L^q(\mathbb{R}^d\backslash T) \text{ and } \nabla u \in L^q(\mathbb{R}^d\backslash T)\Big\}, \tag{6.0.4}$$

with

$$\|u\|_{X^{1,d}(\mathbb{R}^d\backslash T)} = \|((1+|x|)\ln(2+|x|))^{-1}u\|_{L^d(\mathbb{R}^d\backslash T)} + \|\nabla u\|_{L^d(\mathbb{R}^d\backslash T)}. \tag{6.0.5}$$

It is shown in [4] that for $u \in X^{1,q}(\mathbb{R}^d\backslash T)$,

$$\begin{aligned} \|u\|_{X^{1,q}(\mathbb{R}^d\backslash T)} &\leq C\|\nabla u\|_{L^q(\mathbb{R}^d\backslash T)} && \text{if } 1 < q < d, \\ \inf_{\alpha\in\mathbb{R}^d} \|u-\alpha\|_{X^{1,q}(\mathbb{R}^d\backslash T)} &\leq C\|\nabla u\|_{L^q(\mathbb{R}^d\backslash T)} && \text{if } d \leq q < \infty. \end{aligned} \tag{6.0.6}$$

Let

$$X^{1,q}_0(\mathbb{R}^d\backslash T) = \{u \in X^{1,q}(\mathbb{R}^d\backslash T) : u = 0 \text{ on } \partial T\}, \tag{6.0.7}$$

and $X^{-1,q}(\mathbb{R}^d\backslash T)$ be the dual of $X^{1,q'}(\mathbb{R}^d\backslash T)$, where $q' = \frac{q}{q-1}$.

We then define the null space

$$V^q_0(\mathbb{R}^d\backslash T) = \{(w,\pi) \in X^{1,q}_0(\mathbb{R}^d\backslash T) \times L^q(\mathbb{R}^d\backslash T) : -\Delta w + \nabla\pi = 0 \text{ in } \mathbb{R}^d\backslash T \text{ and } \operatorname{div}(w) = 0 \text{ in } \mathbb{R}^d\backslash T\}. \tag{6.0.8}$$

Theorem 6.0.1. *Let $d \geq 2$ and $2 \leq q < \infty$. Let T be the closure of a bounded C^1 domain in $\mathbb{R}^d$ with connected boundary. Then, for any $F \in X^{-1,q}(\mathbb{R}^d \backslash T)$ and $g \in L^q(\mathbb{R}^d \backslash T)$, the problem (6.0.1) has a unique solution in $(X_0^{1,q}(\mathbb{R}^d \backslash T) \times L^q(\mathbb{R}^d \backslash T))/V_0^q(\mathbb{R}^d \backslash T)$. Moreover, the solution satisfies*

$$\inf_{(w,\pi) \in V_0^q(\mathbb{R}^d \backslash T)} (\|u + w\|_{X^{1,q}(\mathbb{R}^d \backslash T)} + \|p + \pi\|_{L^q(\mathbb{R}^d \backslash T)}) \leq C(\|F\|_{X^{-1,q}(\mathbb{R}^d \backslash T)} + \|g\|_{L^q(\mathbb{R}^d \backslash T)}), \tag{6.0.9}$$

where C depends on d, q, and T.

Proof. See [4]. □

The following remarks, given in [4], provide a characterization of the null space.

Remark 6.0.2. If $d \geq 3$ and $2 \leq q < d$, or $d = q = 2$, then

$$V_0^q(\mathbb{R}^d \backslash T) = \{(0, 0)\}. \tag{6.0.10}$$

In this case, Theorem 6.0.1 implies that the solution of (6.0.1) is unique and satisfies

$$\|u\|_{X^{1,q}(\mathbb{R}^d \backslash T)} \leq C(\|F\|_{X^{-1,q}(\mathbb{R}^d \backslash T)} + \|g\|_{L^q(\mathbb{R}^d \backslash T)}). \tag{6.0.11}$$

Remark 6.0.3. If $d \geq 3$ and $q \geq d$, then

$$V_0^q(\mathbb{R}^d \backslash T) = \mathrm{span}\{(w_k, \pi_k)\}_{k=1}^d, \tag{6.0.12}$$

where for $k = 1, \ldots, d$, (w_k, π_k) is the unique solution of the exterior problem (3.1.2).

Remark 6.0.4. If $d = 2$ and $q > 2$, then

$$V_0^q(\mathbb{R}^d \backslash T) = \mathrm{span}\{(w_k, \pi_k)\}_{k=1}^2, \tag{6.0.13}$$

where for $k = 1, 2$, (w_k, π_k) is the unique solution of the exterior problem (3.2.1).

The following is the main result of this chapter.

Theorem 6.0.5. *Let $d \geq 2$ and $2 < q < \infty$. Let $(u, p) \in W^{1,q}(\mathbb{R}^d \backslash T) \times L^q(\mathbb{R}^d \backslash T)$ be a solution of*

$$\begin{cases} -\Delta u + \nabla p = F + \mathrm{div}(f) & \text{in } \mathbb{R}^d \backslash T, \\ \mathrm{div}(u) = g & \text{in } \mathbb{R}^d \backslash T, \\ u = 0 & \text{on } \partial T. \end{cases} \tag{6.0.14}$$

suppose that $T \subset B(0, R)$ and $supp(u), supp(F), supp(f), supp(g) \subset B(0, R)$ for some $R \geq 2$. Then

$$\|\nabla u\|_{L^q(\mathbb{R}^d \backslash T)} \leq C\Phi_q(R)(\|f\|_{L^q(\mathbb{R}^d \backslash T)} + R\|F\|_{L^q(\mathbb{R}^d \backslash T)} + \|g\|_{L^q(\mathbb{R}^d \backslash T)}), \tag{6.0.15}$$

where

$$\Phi_q(R) = \begin{cases} 1 & \text{if } d \geq 3 \text{ and } 2 < q < d, \\ (\ln R)^{1-\frac{1}{d}} & \text{if } d \geq 3 \text{ and } q = d, \\ R^{1-\frac{d}{q}} & \text{if } d \geq 3 \text{ and } d < q < \infty, \\ R^{1-\frac{2}{q}}(\ln R)^{-1} & \text{if } d = 2 \text{ and } 2 < q < \infty, \end{cases} \tag{6.0.16}$$

and C depends only on d, q, and T.

Proof. For any $\psi \in X_0^{1,q'}(\mathbb{R}^d\backslash T;\mathbb{R}^d)$, we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d\backslash T} F\cdot\psi\right| &\le \|F\|_{L^q(B(0,R)\backslash T)}\|\psi\|_{L^{q'}(B(0,R)\backslash T)}\\
&= \|F\|_{L^q(B(0,R)\backslash T)}\left(\fint_{B(0,R)\backslash T}|\psi|^{q'}\left(\frac{1+|x|}{1+|x|}\right)^{q'}\right)^{1/q'}\\
&\le (1+R)\|F\|_{L^q(B(0,R)\backslash T)}\|(1+|x|)^{-1}\psi\|_{L^{q'}(B(0,R)\backslash T)}\\
&\le 2R\|F\|_{L^q(\mathbb{R}^d\backslash T)}\|\psi\|_{X^{1,q'}(\mathbb{R}^d\backslash T)},
\end{aligned}
\tag{6.0.17}
$$

where we have used $\operatorname{supp}(F)\subset B(0,R)$. Also note that $q'\neq d$ because $q>2$. Similarly,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d\backslash T}\operatorname{div}(f)\cdot\psi\right| &\le \|f\|_{L^q(B(0,R)\backslash T)}\|\nabla\psi\|_{L^{q'}(B(0,R)\backslash T)}\\
&\le \|f\|_{L^q(\mathbb{R}^d\backslash T)}\|\psi\|_{X^{1,q'}(\mathbb{R}^d\backslash T)},
\end{aligned}
\tag{6.0.18}
$$

where we have used the fact that $\psi=0$ on ∂T. Therefore

$$
\|F+\operatorname{div}(f)\|_{X^{-1,q}(\mathbb{R}^d\backslash T)} \le C(\|f\|_{L^q(\mathbb{R}^d\backslash T)}+R\|F\|_{L^q(\mathbb{R}^d\backslash T)}). \tag{6.0.19}
$$

By Theorem 6.0.1, we obtain,

$$
\inf_{(w,\pi)\in V_0^q(\mathbb{R}^d\backslash T)}\|u+w\|_{X^{1,q}(\mathbb{R}^d\backslash T)} \le C(\|f\|_{L^q(\mathbb{R}^d\backslash T)}+R\|F\|_{L^q(\mathbb{R}^d\backslash T)}+\|g\|_{L^q(\mathbb{R}^d\backslash T)}). \tag{6.0.20}
$$

If $d\ge 3$ and $2<q<d$, then by Remark 6.0.2, $V_0^q(\mathbb{R}^d\backslash T)=\{0,0\}$. It follows from (6.0.20) that

$$
\|\nabla u\|_{L^q(\mathbb{R}^d\backslash T)} \le C(\|f\|_{L^q(\mathbb{R}^d\backslash T)}+R\|F\|_{L^q(\mathbb{R}^d\backslash T)}+\|g\|_{L^q(\mathbb{R}^d\backslash T)}). \tag{6.0.21}
$$

Now suppose $d\ge 3$ and $d\le q<\infty$. By Remark 6.0.3,we know that $V_0^q(\mathbb{R}^d\backslash T)=\operatorname{span}\{(w_k,\pi_k)\}_{k=1}^d$, where (w_k,π_k) solves the exterior problem (3.1.2). Let

$$
\inf_{(w,\pi)\in V_0^q(\mathbb{R}^d\backslash T)}\|u+w\|_{X^{1,q}(\mathbb{R}^d\backslash T)} = \|u-\sum_{k=1}^d\alpha_k w_k\|_{X^{1,q}(\mathbb{R}^d\backslash T)} \tag{6.0.22}
$$

for some $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d$. If $d<q$, it follows from (6.0.20) that

$$
\sum_{k=1}^d|\alpha_k|\||x|^{-1}w_k\|_{L^q(\mathbb{R}^d\backslash B(0,R))} \le C(\|f\|_{L^q(\mathbb{R}^d\backslash T)}+R\|F\|_{L^q(\mathbb{R}^d\backslash T)}+\|g\|_{L^q(\mathbb{R}^d\backslash T)}), \tag{6.0.23}
$$

where we have used that $u=0$ in $\mathbb{R}^d\backslash B(0,R)$. Since $w_k\sim e_k$ for $|x|$ large, we have

$$
\begin{aligned}
\||x|^{-1}w_k\|_{L^q(\mathbb{R}^d\backslash B(0,R))} &\ge C\left(\int_{\mathbb{R}^d\backslash B(0,R)}|x|^{-q}\right)^{1/q}\\
&= CR^{\frac{d}{q}-1}.
\end{aligned}
\tag{6.0.24}
$$

Hence

$$\sum_{k=1}^{d} |\alpha_k| \le CR^{1-\frac{d}{q}}(\|f\|_{L^q(\mathbb{R}^d\backslash T)} + R\|F\|_{L^q(\mathbb{R}^d\backslash T)} + \|g\|_{L^q(\mathbb{R}^d\backslash T)}). \tag{6.0.25}$$

Therefore

$$\begin{aligned}\|\nabla u\|_{L^q(\mathbb{R}^d\backslash T)} &\le \|\nabla(u - \sum_{k=1}^{d}\alpha_k w_k)\|_{L^q(\mathbb{R}^d\backslash T)} + \sum_{k=1}^{d}|\alpha_k|\|\nabla w_k\|_{L^q(\mathbb{R}^d\backslash T)} \\ &\le CR^{1-\frac{d}{q}}(\|f\|_{L^q(\mathbb{R}^d\backslash T)} + R\|F\|_{L^q(\mathbb{R}^d\backslash T)} + \|g\|_{L^q(\mathbb{R}^d\backslash T)}).\end{aligned} \tag{6.0.26}$$

If $q = d$, then (6.0.20) implies

$$\sum_{k=1}^{d} |\alpha_k|\|(|x|\ln|x|)^{-1}w_k\|_{L^q(\mathbb{R}^d\backslash B(0,R))} \le C(\|f\|_{L^q(\mathbb{R}^d\backslash T)} + R\|F\|_{L^q(\mathbb{R}^d\backslash T)} + \|g\|_{L^q(\mathbb{R}^d\backslash T)}), \tag{6.0.27}$$

where we have again used that $u = 0$ in $\mathbb{R}^d\backslash B(0,R)$. By noting that $w_k \sim e_k$ for $|x|$ large, we find

$$\begin{aligned}\|(|x|\ln|x|)^{-1}w_k\|_{L^q(\mathbb{R}^d\backslash B(0,R))} &\ge C\left(\int_{\mathbb{R}^d\backslash B(0,R)}(|x|\ln|x|)^{-q}\right)^{1/q} \\ &\ge C\ln(R)^{\frac{1}{q}-1}.\end{aligned} \tag{6.0.28}$$

Therefore

$$\sum_{k=1}^{d} |\alpha_k| \le CR^{1-\frac{1}{d}}(\|f\|_{L^q(\mathbb{R}^d\backslash T)} + R\|F\|_{L^q(\mathbb{R}^d\backslash T)} + \|g\|_{L^q(\mathbb{R}^d\backslash T)}), \tag{6.0.29}$$

which yields

$$\|\nabla u\|_{L^q(\mathbb{R}^d\backslash T)} \le C(\ln R)^{1-\frac{1}{d}}(\|f\|_{L^q(\mathbb{R}^d\backslash T)} + R\|F\|_{L^q(\mathbb{R}^d\backslash T)} + \|g\|_{L^q(\mathbb{R}^d\backslash T)}). \tag{6.0.30}$$

Finally, suppose $d = 2$ and $2 < q < \infty$. By Remark 6.0.4, we know that $V_0^q(\mathbb{R}^2\backslash T) = \text{span}\{(w_k,\pi_k)\}_{k=1}^2$, where (w_k,π_k) solves the exterior problem (3.2.1). Let

$$\inf_{(w,\pi)\in V_0^q(\mathbb{R}^2\backslash T)} \|u + w\|_{X^{1,q}(\mathbb{R}^2\backslash T)} = \|u - \sum_{k=1}^{2}\alpha_k w_k\|_{X^{1,q}(\mathbb{R}^2\backslash T)} \tag{6.0.31}$$

for some $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{R}^d$. In this case, (6.0.20) implies

$$\sum_{k=1}^{2} |\alpha_k|\||x|^{-1}w_k\|_{L^q(\mathbb{R}^2\backslash B(0,R))} \le C(\|f\|_{L^q(\mathbb{R}^2\backslash T)} + R\|F\|_{L^q(\mathbb{R}^2\backslash T)} + \|g\|_{L^q(\mathbb{R}^2\backslash T)}), \tag{6.0.32}$$

where we have again used that $u = 0$ in $\mathbb{R}^d \backslash B(0, R)$. Since $w_k \sim \ln |x| e_k$ for $|x|$ large, we see that

$$\begin{aligned} \||x|^{-1} w_k\|_{L^q(\mathbb{R}^2 \backslash B(0,R))} &\geq C \ln(R) \left(\int_{\mathbb{R}^2 \backslash B(0,R)} |x|^{-q} \right)^{1/q} \\ &= C \ln(R) R^{\frac{2}{q} - 1}. \end{aligned} \tag{6.0.33}$$

We conclude that

$$\|\nabla u\|_{L^q(\mathbb{R}^2 \backslash T)} \leq C R^{1 - \frac{2}{q}} (\ln R)^{-1} (\|f\|_{L^q(\mathbb{R}^2 \backslash T)} + R \|F\|_{L^q(\mathbb{R}^2 \backslash T)} + \|g\|_{L^q(\mathbb{R}^2 \backslash T)}). \tag{6.0.34}$$

This completes the proof. □

To prove a corollary of Theorem 6.0.5, we will need the following interior estimate for the Stokes equations.

Lemma 6.0.6. *Suppose $2 < q < \infty$. Let $(u, p) \in H^1(B_2; \mathbb{R}^d) \times L^2(B_2)$ be a weak solution of*

$$\begin{cases} -\Delta u + \nabla p = F + \mathrm{div}(f) & \text{in } B_2, \\ \mathrm{div}(u) = 0 & \text{in } B_2. \end{cases} \tag{6.0.35}$$

Then

$$\begin{aligned} \left(\int_{B_1} |u|^q \right)^{1/q} + \left(\int_{B_1} |p - \alpha|^q \right)^{1/q} &\leq C \left(\int_{B_2} |u|^2 \right)^{1/2} + C \left(\int_{B_2} |F|^{q^*} \right)^{1/q^*} \\ &\quad + C \left(\int_{B_2} |f|^q \right)^{1/q} \end{aligned} \tag{6.0.36}$$

for any $\alpha \in \mathbb{R}$, where q^* satisfies $\frac{1}{q^*} = \frac{1}{q} + \frac{1}{d}$ and C depends only on d.

Corollary 6.0.7. *Let $d \geq 2$ and $2 < q < \infty$. Let (u, p) be a solution of*

$$\begin{cases} -\Delta u + \nabla p = F + \mathrm{div}(f) & \text{in } R\tilde{Y} \backslash T, \\ \mathrm{div}(u) = 0 & \text{in } R\tilde{Y} \backslash T, \\ u = 0 & \text{on } \partial T, \end{cases} \tag{6.0.37}$$

where $\tilde{Y} = (1 + c_0) Q_1$. Then for $R \geq 3$,

$$\|\nabla u\|_{L^q(Q_R \backslash T)} \leq C \Phi_q(R) (\|f\|_{L^q(R\tilde{Y} \backslash T)} + R \|F\|_{L^q(R\tilde{Y} \backslash T)} + R^{\frac{d}{q} - \frac{d}{2} - 1} \|u\|_{L^2(R\tilde{Y} \backslash B(0, R/3))}), \tag{6.0.38}$$

where $\Phi_q(R)$ is given by (6.0.16) and C depends only on $d, q,$ and T.

Proof. Let $\varphi \in C_0^\infty((1 + c_0/3) Q_R)$ such that $\varphi = 1$ in Q_R and $|\nabla \varphi| \leq C R^{-1}$, $|\nabla^2 \varphi| \leq C R^{-2}$. Note that

$$-\Delta (u \varphi) = F \varphi + \mathrm{div}(f \varphi) - f \nabla \varphi - 2 \mathrm{div}(u \nabla \varphi) + u \Delta \varphi + p \nabla \varphi \tag{6.0.39}$$

in $\mathbb{R}^d\backslash T$, $\operatorname{div}(u\varphi) = u \cdot \nabla\varphi$ in $\mathbb{R}^d\backslash T$, and $u\varphi = 0$ on ∂T. By Theorem 6.0.5, we have

$$\begin{aligned}\|\nabla u\|_{L^q(Q_R\backslash T)} &\leq \|\nabla(u\varphi)\|_{L^q(\mathbb{R}^d\backslash T)} \\ &\leq \Phi_q(R)\Big\{\|f\|_{L^q(R\tilde{Y}\backslash T)} + R\|F\|_{L^q(R\tilde{Y}\backslash T)} + \frac{1}{R}\|u\|_{L^q((1+c_0/3)Q_R\backslash Q_R)} \\ &\qquad + \|p\|_{L^q((1+c_0/3)Q_R\backslash Q_R)}\Big\},\end{aligned} \tag{6.0.40}$$

where $\Phi_q(R)$ is given by (6.0.16). To bound the last two terms on the right side of (6.0.40), we use Lemma 6.0.6 in a rescaled setting. Let $v(x) = u(Rx)$, $\pi(x) = Rp(Rx)$, $G(x) = F(Rx)$, and $g(x) = Rf(Rx)$. Then (v, π) satisfies

$$\begin{cases} -\Delta v + \nabla\pi = G + \operatorname{div}(g) & \text{in } \tilde{Y}\backslash(R^{-1}T), \\ \operatorname{div}(v) = 0 & \text{in } \tilde{T}\backslash(R^{-1}T), \\ v = 0 & \text{on } \partial(R^{-1}T). \end{cases} \tag{6.0.41}$$

By covering $(1+\frac{c_0}{3})Q_1\backslash Q_1$ with balls and using the interior estimate in Lemma 6.0.6, we obtain

$$\left(\int_{(1+\frac{c_0}{3})Q_1\backslash Q_1} (|v|^q + |\pi|^q)\right)^{1/q} \leq C\left(\int_{\tilde{Y}\backslash B_{1/3}} |v|^2\right)^{1/2} + C\left(\int_{\tilde{Y}\backslash B_{1/3}} (|G|^q + |g|^q)\right)^{1/q}, \tag{6.0.42}$$

where we have used the fact that $q > q^*$. By rescaling, we find that

$$\begin{aligned}\|u\|_{L^q((1+c_0/3)Q_R\backslash Q_R)} + R\|p\|_{L^q((1+c_0/3)Q_R\backslash Q_R)} &\leq CR^{\frac{d}{q}-\frac{d}{2}}\|u\|_{L^2(R\tilde{Y}\backslash T)} + CR\|f\|_{L^q(R\tilde{Y}\backslash T)} \\ &\quad + CR^2\|F\|_{L^q(R\tilde{Y}\backslash T)}.\end{aligned} \tag{6.0.43}$$

Then (6.0.38) follows from (6.0.40) and (6.0.43). □

Chapter 7 Local Estimates in a Cell

In this chapter, we wish to establish the following estimate for solutions to the Stokes equations in a cell with a single obstacle.

Theorem 7.0.1. *Let $2 < q < \infty$. Suppose (u, p) is a solution of*

$$\begin{cases} -\Delta u + \nabla p = F + \operatorname{div}(f) & \text{in } \tilde{Y}\setminus\eta T, \\ \operatorname{div}(u) = 0 & \text{in } \tilde{Y}\setminus\eta T, \\ u = 0 & \text{on } \partial\eta T, \end{cases} \tag{7.0.1}$$

with $F \in L^q(\tilde{Y}\setminus\eta T; \mathbb{R}^d)$ and $f \in L^q(\tilde{Y}\setminus\eta T; \mathbb{R}^{d\times d})$, where $\tilde{Y} = (1 + c_0)Q_1$ and $\eta \in (0, (4d)^{-1})$. Let $\alpha \in \mathbb{R}^d$. Then for $d \geq 3$,

$$\begin{aligned} \|\nabla u\|_{L^q(Y\setminus\eta T)} \leq C|\alpha|\eta^{\frac{d}{q}-1} &+ C\Phi_q(\eta^{-1}) \left(\int_{\tilde{Y}\setminus\eta T} (|F|^q + |f|^q)\, dx \right)^{1/q} \\ &+ C\Phi_q(\eta^{-1}) \left(\int_{\tilde{Y}\setminus B(0,1/3)} |u - \alpha|^2\, dx \right)^{1/2}, \end{aligned} \tag{7.0.2}$$

where Φ_q is defined in (6.0.16), and C depends only on d, q, and T. For $d = 2$,

$$\begin{aligned} \|\nabla u\|_{L^q(Y\setminus\eta T)} \leq C|\alpha|\Phi_q(\eta^{-1}) &+ C\Phi_q(\eta^{-1}) \left(\int_{\tilde{Y}\setminus\eta T} (|F|^q + |f|^q)\, dx \right)^{1/q} \\ &+ C\Phi_q(\eta^{-1}) \left(\int_{\tilde{Y}\setminus B(0,1/3)} |u - \alpha|^2\, dx \right)^{1/2}, \end{aligned} \tag{7.0.3}$$

where C depends only on q and T.

Lemma 7.0.2. *Let $2 < q < \infty$. Let (u, p) be the same as in Theorem 7.0.1. Then*

$$\|\nabla u\|_{L^q(Y\setminus\eta T)} \leq C\Phi_q(\eta^{-1})(\|u\|_{L^2(\tilde{Y}\setminus B(0,1/3))} + \|F\|_{L^q(\tilde{Y}\setminus\eta T)} + \|f\|_{L^q(\tilde{Y}\setminus\eta T)}), \tag{7.0.4}$$

where C depends only on d, q, and T.

Proof. This follows from Corollary 6.0.7 and a rescaling argument. Indeed, let $\tilde{u}(x) = u(\eta x)$, $\tilde{p}(x) = \eta p(\eta x)$, $\tilde{F}(x) = \eta^2 F(\eta x)$, and $\tilde{f}(x) = \eta f(\eta x)$. These rescaled functions satisfy the system

$$\begin{cases} -\Delta \tilde{u} + \nabla \tilde{p} = \tilde{F} + \operatorname{div}(\tilde{f}) & \text{in } \eta^{-1}\tilde{Y}\setminus T, \\ \operatorname{div}(\tilde{u}) = 0 & \text{in } \eta^{-1}\tilde{Y}\setminus T, \\ u = 0 & \text{on } \partial T. \end{cases} \tag{7.0.5}$$

Therefore, by Corollary (6.0.7), we have

$$\begin{aligned}\|\nabla u\|_{L^q(Y\setminus\eta T)} &= \eta^{\frac{d}{q}-1}\|\nabla\tilde{u}\|_{L^q(Q_{\eta^{-1}}\setminus T)}\\ &\leq C\eta^{\frac{d}{q}-1}\Phi(\eta^{-1})\Big\{\|\tilde{f}\|_{L^q(\eta^{-1}\tilde{Y}\setminus T)}+\frac{1}{\eta}\|\tilde{F}\|_{L^q(\eta^{-1}\tilde{Y}\setminus T)}\Big\}\\ &\quad + C\eta^{\frac{d}{2}}\Phi(\eta^{-1})\|\tilde{u}\|_{L^2(\eta^{-1}Y\setminus B(0,\eta^{-1}/3))}\\ &= C\Phi_q(\eta^{-1})(\|u\|_{L^2(\tilde{Y}\setminus B(0,1/3))}+\|F\|_{L^q(\tilde{Y}\setminus\eta T)}+\|f\|_{L^q(\tilde{Y}\setminus\eta T)}).\end{aligned}\tag{7.0.6}$$

□

Note that if (u,p) is a solution of (7.0.1), then $(u-\alpha,p)$ is not a solution of (7.0.1) because $u-\alpha\neq 0$ on the boundary of the obstacle. In order to deduce Theorem 7.0.1 from Lemma 7.0.2, we will need to use the corrector matrix M^η defined in Chapter 3. To bound error terms that appear from applying Lemma 7.0.2 to $u-M^\eta\alpha$, we will need $W^{1,q}$ estimates for each corrector w_k^η in the periodic cell Y.

Lemma 7.0.3. *Let w_k^η be defined by (3.1.6) if $d\geq 3$ and (3.2.3) if $d=2$. If $d\geq 3$,*

$$\|\nabla w_k^\eta\|_{L^q(Y)}\approx\begin{cases}\eta^{\frac{d}{q}-1} & \text{if } d'<q<\infty,\\ \eta^{d-2}|\ln\eta|^{\frac{1}{q}} & \text{if } q=d',\\ \eta^{d-2} & \text{if } 1<q<d',\end{cases}\tag{7.0.7}$$

where $d'=\frac{d}{d-1}$. If $d=2$,

$$\|\nabla w_k^\eta\|_{L^q(Y)}\approx\begin{cases}\eta^{\frac{2}{q}-1}|\ln(\eta)|^{-1} & \text{if } 2<q<\infty,\\ |\ln(\eta)|^{-1/2} & \text{if } q=2,\\ |\ln(\eta)|^{-1} & \text{if } 1<q<2.\end{cases}\tag{7.0.8}$$

Proof. Consider the case $d\geq 3$. Since $w_k^\eta(x)=w_k(x/\eta)$ in $B(0,1/4)\setminus\eta T$, we have

$$\begin{aligned}\int_{B(0,1/4)\setminus\eta T}|\nabla w_k^\eta|^q\,dx &= \eta^{d-q}\int_{B(0,(4\eta)^{-1})\setminus T}|\nabla w_k|^q\,dx\\ &\approx\begin{cases}\eta^{d-q} & \text{if } d'<q<\infty,\\ \eta^{d-q}|\ln(\eta)| & \text{if } q=d',\\ \eta^{(d-2)q} & \text{if } 1<q<d',\end{cases}\end{aligned}\tag{7.0.9}$$

where we have used the asymptotic behavior of w_k given in (3.1.3).

To bound ∇w_k^η on $B(0,1/3)\setminus B(0,1/4)$, we note that $\psi_k=w_k^\eta-e_k$ satisfies

$$\begin{cases}-\Delta\psi_k+\nabla\pi_k^\eta=0 & \text{in } B(0,1/3)\setminus B(0,1/4),\\ \operatorname{div}(\psi_k)=0 & \text{in } B(0,1/3)\setminus B(0,1/4),\end{cases}\tag{7.0.10}$$

with $\psi_k = 0$ on $\partial B(0, 1/3)$ and $\psi_k = w_k(x/\eta) - e_k$ on $\partial B(0, 1/4)$. Using (3.1.3) and regularity estimates for solutions to (7.0.10), we obtain

$$\begin{aligned} |\nabla w_k^\eta| &= |\nabla \psi_k| \\ &\leq \max_{x \in \partial B(0,(4\eta)^{-1})} |w_k(x) - e_k| \\ &\leq C\eta^{d-2} \end{aligned} \tag{7.0.11}$$

in $B(0, 1/3)\backslash B(0, 1/4)$. Together with (7.0.9), we obtain (7.0.7).

Now consider the case $d = 2$. Since $w_k^\eta(x) = \frac{4\pi}{|\ln(\eta)|} w_k(x/\eta)$ in $B(0, 1/4)\backslash \eta T$, we have

$$\begin{aligned} \int_{B(0,1/4)\backslash \eta T} |\nabla w_k^\eta|^q \, dx &= \left(\frac{4\pi}{|\ln(\eta)|} \right)^q \eta^{2-q} \int_{B(0,(4\eta)^{-1})\backslash T} |\nabla w_k|^q \, dx \\ &\approx (4\pi)^q \begin{cases} \eta^{2-q} |\ln(\eta)|^{-q} & \text{if } 2 < q < \infty, \\ |\ln(\eta)|^{-1} & \text{if } q = 2, \\ |\ln(\eta)|^{-q} & \text{if } 1 < q < 2, \end{cases} \end{aligned} \tag{7.0.12}$$

where we have used the asymptotic behavior of w_k given in (3.2.2). As before, defining $\psi_k = w_k^\eta - e_k$ and applying regularity estimates for the Stokes equations gives

$$|\nabla w_k^\eta| \leq C |\ln(\eta)|^{-1} \qquad \text{in } B(0, 1/3)\backslash B(0, 1/4). \tag{7.0.13}$$

Together with (7.0.12), we obtain (7.0.8). □

Proof of Theorem 7.0.1. Let (u, p) be a solution of (7.0.1). Let M^η be the matrix whose columns are the correctors w_k^η, and let P^η be the vector whose entries are π_k^η, where (w_k^η, π_k^η) are defined in (3.1.6) if $d \geq 3$ and (3.2.3) if $d = 2$. For any $\alpha \in \mathbb{R}^d$, we have $u - M^\eta \alpha = 0$ on $\partial(\eta T)$, and Lemmas 3.1.2 and 3.2.2 imply that there exist F_α^η and f_α^η such that

$$-\Delta(u - M^\eta \alpha) + \nabla(p - P^\eta \cdot \alpha) = (F - F_\alpha^\eta) + \operatorname{div}(f - f_\alpha^\eta) \tag{7.0.14}$$

in $Y\backslash \eta T$, where $|F_\alpha^\eta| \leq C|\alpha| K_\eta^2$ and $|f_\alpha^\eta| \leq C|\alpha| K_\eta^2$. It follows from Lemma 7.0.2 that

$$\begin{aligned} \left(\int_{Y\backslash \eta T} |\nabla u|^q \, dx \right)^{1/q} \leq |\alpha| &\left(\int_{Y\backslash \eta T} |\nabla M^\eta|^q \, dx \right)^{1/q} + C\Phi_q(\eta^{-1})(\|f_\alpha^\eta\|_\infty + \|F_\alpha^\eta\|_\infty) \\ &+ C\Phi_q(\eta^{-1}) \left(\int_{\tilde{Y}\backslash \eta T} (|F|^q + |f|^q) \, dx \right)^{1/q} \\ &+ C\Phi_q(\eta^{-1}) \left(\int_{\tilde{Y}\backslash B(0,1/3)} |u - \alpha|^2 \, dx \right)^{1/2}. \end{aligned} \tag{7.0.15}$$

Suppose $d \geq 3$. By Lemma 7.0.3 the first two terms on the right side are bounded by

$$C|\alpha| \eta^{\frac{d}{q} - 1} + C|\alpha| \Phi_q(\eta^{-1}) \eta^{d-2} \leq C|\alpha| \eta^{\frac{d}{q} - 1}. \tag{7.0.16}$$

This yields (7.0.2). Similarly, if $d = 2$, the first two terms on the right side of (7.0.15) are bounded by $C|\alpha| \Phi_q(\eta^{-1})$. In this case, we obtain (7.0.3). □

Corollary 7.0.4. *Suppose $d \geq 2$, and let $2 < q < \infty$. Let (u, p) be the same as in Theorem 7.0.1. Then*

$$\|u\|_{L^q(Y\setminus\eta T)} \leq C\left(\int_{\tilde{Y}\setminus\eta T}(|F|^q + |f|^q)\,dx\right)^{1/q} + C\left(\int_{\tilde{Y}}|T_{1,\eta}(F,f)(x)|^q\,dx\right)^{1/q}, \tag{7.0.17}$$

where $T_{1,\eta}(F, f)$ is given by (5.0.2), and C depends only on d, q, and T.

Proof. By a Sobolev inequality,

$$\left(\int_{Y\setminus\eta T}|u|^q\,dx\right)^{1/q} \leq C\left(\int_{Y\setminus\eta T}|u|^2\,dx\right)^{1/2} + C\left(\int_{Y\setminus\eta T}|\nabla u|^{q^*}\,dx\right)^{1/q^*}, \tag{7.0.18}$$

where $\frac{1}{q^*} = \frac{1}{q} + \frac{1}{d}$. We split the proof into two cases. First, suppose $d \geq 3$. Noting that $\Phi_{q^*} = 1$ since $q^* < d$, we can apply Theorem 7.0.1 to obtain

$$\begin{aligned}\left(\int_{Y\setminus\eta T}|u|^q\,dx\right)^{1/q} &\leq C\left(\int_{Y\setminus\eta T}|u|^2\,dx\right)^{1/2} + C\left(\int_{\tilde{Y}\setminus\eta T}(|F|^{q*} + |f|^{q*})\,dx\right)^{1/q^*}\\ &\quad + C\left(\int_{\tilde{Y}\setminus B_{1/3}}|u|^2\,dx\right)^{1/2}\\ &\leq C\left(\int_{\tilde{Y}}|u|^2\,dx\right)^{1/2} + C\left(\int_{\tilde{Y}\setminus\eta T}(|F|^q + |f|^q)\,dx\right)^{1/q}.\end{aligned} \tag{7.0.19}$$

Using the observation that

$$\left(\int_{\tilde{Y}}|u|^2\,dx\right)^{1/2} \leq C\left(\int_{\tilde{Y}}|T_{1,\eta}(F,f)(x)|^q\,dx\right)^{1/q}, \tag{7.0.20}$$

we obtain (7.0.17) from (7.0.19).

Next, suppose $d = 2$. Since $q^* < 2$, we can write

$$\|\nabla u\|_{L^{q^*}(Y\setminus\eta T)} \leq C\|\nabla u\|_{L^2(Y\setminus\eta T)}. \tag{7.0.21}$$

Therefore

$$\begin{aligned}\left(\int_{Y\setminus\eta T}|u|^q\,dx\right)^{1/q} &\leq C\left(\int_{Y\setminus\eta T}|u|^2\,dx\right)^{1/2} + C\left(\int_{Y\setminus\eta T}|\nabla u|^2\,dx\right)^{1/2}\\ &\leq C\left(\int_{Y\setminus\eta T}|u|^2\,dx\right)^{1/2} + C\left(\int_{Y\setminus\eta T}(|F|^2 + |f|^2)\,dx\right)^{1/2},\end{aligned} \tag{7.0.22}$$

where the second inequality follows from an energy estimate. As before, (7.0.17) follows from (7.0.21). □

We are now ready to prove the $W^{1,q}$ estimates in Theorems 1.3.1 and 1.3.2. To simplify the proofs, we first make note of a relationship between the bounding constants in (1.3.4) and (1.3.5). Let $1 < q < \infty$, and let $A_q(\varepsilon,\eta), B_q(\varepsilon,\eta), C_q(\varepsilon,\eta)$, and $D_q(\varepsilon,\eta)$ be the smallest bounding constants such that (1.3.4) and (1.3.5) hold. It follows from a duality argument that $B_{q'}(\varepsilon,\eta) = C_q(\varepsilon,\eta)$, $A_{q'}(\varepsilon,\eta) = A_q(\varepsilon,\eta)$, and $D_{q'}(\varepsilon,\eta) = D_q(\varepsilon,\eta)$, where $q' = \frac{q}{q-1}$. Indeed, suppose (u_1, p_1) solves

$$\begin{cases} -\Delta u_1 + \nabla p_1 = F + \operatorname{div}(f) & \text{in } \omega_{\varepsilon,\eta}, \\ \operatorname{div}(u_1) = 0 & \text{in } \omega_{\varepsilon,\eta}, \\ u_1 = 0 & \text{on } \partial\omega_{\varepsilon,\eta}, \end{cases} \tag{8.0.1}$$

and (u_2, p_2) solves

$$\begin{cases} -\Delta u_2 + \nabla p_2 = G + \operatorname{div}(g) & \text{in } \omega_{\varepsilon,\eta}, \\ \operatorname{div}(u_2) = 0 & \text{in } \omega_{\varepsilon,\eta}, \\ u_2 = 0 & \text{on } \partial\omega_{\varepsilon,\eta}. \end{cases} \tag{8.0.2}$$

Then using u_2 as a test function in (8.0.1) and u_1 as a test function in (8.0.2) yields

$$\begin{aligned} \int_{\omega_{\varepsilon,\eta}} G \cdot u_1 - \int_{\omega_{\varepsilon,\eta}} \nabla u_1 \cdot g &= \int_{\omega_{\varepsilon,\eta}} \nabla u_1 \cdot \nabla u_2 \\ &= \int_{\omega_{\varepsilon,\eta}} F \cdot u_2 - \int_{\omega_{\varepsilon,\eta}} \nabla u_2 \cdot f. \end{aligned} \tag{8.0.3}$$

By choosing $g = 0$ and $F = 0$, we find that

$$\begin{aligned} \left| \int_{\omega_{\varepsilon,\eta}} G \cdot u_1 \right| &= \left| \int_{\omega_{\varepsilon,\eta}} \nabla u_2 \cdot f \right| \\ &\leq \|\nabla u_2\|_{L^{q'}(\omega_{\varepsilon,\eta})} \|f\|_{L^q(\omega_{\varepsilon,\eta})} \\ &\leq B_{q'}(\varepsilon,\eta) \|G\|_{L^{q'}(\omega_{\varepsilon,\eta})} \|f\|_{L^q(\omega_{\varepsilon,\eta})}. \end{aligned} \tag{8.0.4}$$

It follows that

$$\begin{aligned} \|u_1\|_{L^q(\omega_{\varepsilon,\eta})} &= \sup_{\substack{G \in L^{q'}(\omega_{\varepsilon,\eta};\mathbb{R}^d) \\ \|G\|_{L^{q'}(\omega_{\varepsilon,\eta})} = 1}} \left| \int_{\omega_{\varepsilon,\eta}} G \cdot u_1 \right| \\ &\leq B_{q'}(\varepsilon,\eta) \|f\|_{L^q(\omega_{\varepsilon,\eta})}. \end{aligned} \tag{8.0.5}$$

Therefore $C_q(\varepsilon,\eta) = B_{q'}(\varepsilon,\eta)$. Similarly, if we choose $G = 0$ and $F = 0$, we can use (8.0.3) to see that

$$\begin{aligned} \left| \int_{\omega_{\varepsilon,\eta}} \nabla u_1 \cdot g \right| &= \left| \int_{\omega_{\varepsilon,\eta}} \nabla u_2 \cdot f \right| \\ &\leq \|\nabla u_2\|_{L^{q'}(\omega_{\varepsilon,\eta})} \|f\|_{L^q(\omega_{\varepsilon,\eta})} \\ &\leq A_{q'}(\varepsilon,\eta) \|g\|_{L^{q'}(\omega_{\varepsilon,\eta})} \|f\|_{L^q(\omega_{\varepsilon,\eta})}, \end{aligned} \tag{8.0.6}$$

which yields

$$\|\nabla u_1\|_{L^q(\omega_{\varepsilon,\eta})} \leq A_{q'}(\varepsilon,\eta)\|f\|_{L^q(\omega_{\varepsilon,\eta})} \tag{8.0.7}$$

Therefore $A_{q'}(\varepsilon,\eta) = A_q(\varepsilon,\eta)$. To see that $D_{q'}(\varepsilon,\eta) = D_q(\varepsilon,\eta)$, we choose $f = 0$ and $g = 0$ in (8.0.3) and follow the same argument as above.

In view of the duality discussed above, we will be able to only consider the case $q > 2$ in many of the proofs in this chapter. We begin with an estimate for $\|u\|_{L^q(\omega_{\varepsilon,\eta})}$.

Theorem 8.0.1. *Let $1 < q < \infty$. For any $F \in L^q(\omega_{\varepsilon,\eta};\mathbb{R}^d)$ and $f \in L^q(\omega_{\varepsilon,\eta};\mathbb{R}^{d\times d})$, the Stokes system (1.3.3) has a unique solution in $W_0^{1,q}(\omega_{\varepsilon,\eta};\mathbb{R}^d)\times[L^q(\omega_{\varepsilon,\eta})/\mathbb{R}]$. Moreover, if $2 \leq q < \infty$, the solution satisfies*

$$\|u\|_{L^q(\omega_{\varepsilon,\eta})} \leq C(\varepsilon^2\eta^{2-d}\|F\|_{L^q(\omega_{\varepsilon,\eta})} + \varepsilon\eta^{1-\frac{d}{2}}\|f\|_{L^q(\omega_{\varepsilon,\eta})}) \tag{8.0.8}$$

if $d \geq 3$, and

$$\|u\|_{L^q(\omega_{\varepsilon,\eta})} \leq C(\varepsilon^2|\ln(\eta/2)|\|F\|_{L^q(\omega_{\varepsilon,\eta})} + \varepsilon|\ln(\eta/2)|^{1/2}\|f\|_{L^q(\omega_{\varepsilon,\eta})}) \tag{8.0.9}$$

for $d = 2$. The constant C depends only on d, q, and T.

Proof. By rescaling, we may assume $\varepsilon = 1$. If $q \geq 2$, summing (7.0.17) over the whole space and using Theorem 5.0.1 to bound terms involving $T_{1,\eta}$ yields

$$\|u\|_{L^q(\omega_{1,\eta})} \leq C(\eta^{2-d}\|F\|_{L^q(\omega_{1,\eta})} + \eta^{1-\frac{d}{2}}\|f\|_{L^q(\omega_{1,\eta})}) \tag{8.0.10}$$

if $d \geq 3$ and

$$\|u\|_{L^q(\omega_{1,\eta})} \leq C(|\ln(\eta/2)\|F\|_{L^q(\omega_{1,\eta})} + |\ln(\eta/2)|^{1/2}\|f\|_{L^q(\omega_{1,\eta})}) \tag{8.0.11}$$

if $d = 2$. This gives the desired estimate when $\varepsilon = 1$. □

We will now give estimates for $\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})}$. We begin with the case $F = 0$.

Theorem 8.0.2. *Let $1 < q < \infty$. For any $f \in L^q(\omega_{\varepsilon,\eta};\mathbb{R}^{d\times d})$, the solution of the Stokes system*

$$\begin{cases} -\Delta u + \nabla p = \operatorname{div}(f) & \text{in } \omega_{\varepsilon,\eta}, \\ \operatorname{div}(u) = 0 & \text{in } \omega_{\varepsilon,\eta}, \\ u = 0 & \text{on } \partial\omega_{\varepsilon,\eta}, \end{cases} \tag{8.0.12}$$

satisfies the estimate

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \leq C\eta^{-d|\frac{1}{2}-\frac{1}{q}|}\|f\|_{L^q(\omega_{\varepsilon,\eta})} \tag{8.0.13}$$

for $d \geq 3$ and $q \neq 2$, and

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \leq C\eta^{-2|\frac{1}{2}-\frac{1}{q}|}|\ln(\eta/2)|^{-1/2}\|f\|_{L^q(\omega_{\varepsilon,\eta})} \tag{8.0.14}$$

for $d = 2$ and $q \neq 2$, where C depends only on d, q, and T.

Remark 8.0.3. If (u,p) solves (8.0.12), we can obtain estimates for $q=2$ by a simple energy estimate. Indeed, using u as a test function in (8.0.12) yields

$$\int_{\omega_{\varepsilon,\eta}} |\nabla u|^2 \leq C\|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})}\|f\|_{L^2(\omega_{\varepsilon,\eta})}.$$

Therefore

$$\|\nabla u\|_{L^2(\omega_{\varepsilon,\eta})} \leq C\|f\|_{L^2(\omega_{\varepsilon,\eta})}. \tag{8.0.15}$$

Proof of Theorem 8.0.2. By rescaling, we may assume $\varepsilon = 1$. Note that we are looking for the bounding constant $A_q(1,\eta)$ in (1.3.4). By duality, we know that $A_q(1,\eta) = A_{q'}(1,\eta)$. Thus we may assume $q>2$. Consider the case $d\geq 3$. Suppose (u,p) is a solution of (8.0.12) with $\varepsilon = 1$. For any $k\in\mathbb{Z}^d$ and $\alpha\in\mathbb{R}^d$, it follows from Theorem 7.0.1 that

$$\begin{aligned}\int_{k+(Y\setminus\eta T)} |\nabla u|^q\,dx \leq C|\alpha|^q\eta^{d-q} + C[\Phi_q(\eta^{-1})]^q \int_{k+(\tilde{Y}\setminus\eta T)} |f|^q\,dx \\ + C[\Phi_q(\eta^{-1})]^q \left(\int_{k+(\tilde{Y}\setminus B(0,1/3))} |u-\alpha|^2\,dx\right)^{q/2}.\end{aligned} \tag{8.0.16}$$

By choosing

$$\alpha = \fint_{k+(\tilde{Y}\setminus B(0,1/3))} u\,dx$$

and applying the Poincaré inequality, we obtain

$$\begin{aligned}\int_{k+(Y\setminus\eta T)} |\nabla u|^q\,dx &\leq C\eta^{d-q}\int_{k+(\tilde{Y}\setminus\eta T)} |u|^q\,dx + C[\Phi_q(\eta^{-1})]^q\int_{k+(\tilde{Y}\setminus\eta T)} |f|^q\,dx \\ &\quad + C[\Phi_q(\eta^{-1})]^q\left(\int_{k+(\tilde{Y}\setminus\eta T)} |\nabla u|^2\,dx\right)^{q/2} \\ &\leq C\eta^{d-q}\int_{k+(\tilde{Y}\setminus\eta T)} |u|^q\,dx + C[\Phi_q(\eta^{-1})]^q\int_{k+(\tilde{Y}\setminus\eta T)} |f|^q\,dx \\ &\quad + C[\Phi_q(\eta^{-1})]^q\int_{k+Y} |S_{1,\eta}(0,f)|^q\,dx,\end{aligned} \tag{8.0.17}$$

where $S_{1,\eta}$ is defined in (5.0.23). Then summing over $k\in\mathbb{Z}^d$ yields

$$\begin{aligned}\|\nabla u\|_{L^q(\omega_{1,\eta})} &\leq C\eta^{\frac{d}{q}-1}\|u\|_{L^q(\omega_{1,\eta})} + C\Phi_q(\eta^{-1})(\|f\|_{L^q(\omega_{1,\eta})} + \|S_{1,\eta}(0,f)\|_{L^q(\mathbb{R}^d)}) \\ &\leq C\eta^{\frac{d}{q}-\frac{d}{2}}\|f\|_{L^q(\omega_{1,\eta})} + C\Phi_q(\eta^{-1})\|f\|_{L^q(\omega_{1,\eta})} \\ &\leq C\eta^{\frac{d}{q}-\frac{d}{2}}\|f\|_{L^q(\omega_{1,\eta})},\end{aligned} \tag{8.0.18}$$

where we have used the estimates (8.0.10) and (5.0.25) as well as the fact $\Phi_q(\eta^{-1}) \leq C\eta^{\frac{d}{q}-\frac{d}{2}}$ when $d\geq 3$. We deduce (8.0.13) when $\varepsilon = 1$ and $q>2$ in the case $d\geq 3$.

Similarly, if $d=2$ we use (7.0.3) with $\alpha = \fint_{k+(\tilde{Y}\setminus B(0,1/3))} u\,dx$ to obtain

$$\int_{k+(Y\setminus\eta T)} |\nabla u|^q\,dx \le C[\Phi_q(\eta^{-1})]^q \int_{k+(\tilde{Y}\setminus\eta T)} (|u|^q + |f|^q)\,dx + C[\Phi_q(\eta^{-1})]^q \left(\int_{k+(\tilde{Y}\setminus\eta T)} |\nabla u|^2\,dx\right)^{q/2}, \tag{8.0.19}$$

where the Poincaré inequality has been used. Summing over $k \in \mathbb{Z}^d$, we see that

$$\|\nabla u\|_{L^q(\omega_{1,\eta})} \le C\eta^{\frac{2}{q}-1}|\ln(\eta)|^{-1}(\|u\|_{L^q(\omega_{1,\eta})}+\|f\|_{L^q(\omega_{1,\eta})}+\|S_{1,\eta}(0,f)\|_{L^q(\omega_{1,\eta})}). \tag{8.0.20}$$

The desired result then follows from (8.0.11) and (5.0.26). □

We now consider the case $f=0$.

Theorem 8.0.4. *Let $1<q<\infty$. For any $F\in L^q(\omega_{\varepsilon,\eta};\mathbb{R}^d)$, the solution (u,p) of the Stokes system*

$$\begin{cases} -\Delta u + \nabla p = F & \text{in } \omega_{\varepsilon,\eta}, \\ \operatorname{div}(u) = 0 & \text{in } \omega_{\varepsilon,\eta}, \\ u = 0 & \text{on } \partial\omega_{\varepsilon,\eta}, \end{cases} \tag{8.0.21}$$

satisfies the estimate

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \le \begin{cases} C\varepsilon\eta^{1-\frac{d}{2}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 1<q\le 2, \\ C\varepsilon\eta^{1-d+\frac{d}{q}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 2<q<\infty \end{cases} \tag{8.0.22}$$

for $d\ge 3$ and

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \le \begin{cases} C\varepsilon|\ln(\eta/2)|^{1/2}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 1<q\le 2, \\ C\varepsilon\eta^{-1+\frac{2}{q}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{for } 2<q<\infty \end{cases} \tag{8.0.23}$$

for $d=2$, where C depends only on d, q, and T.

Proof. We first treat the case $1<q\le 2$ using a duality argument. We have

$$\|\nabla u\|_{L^q(\omega_{\varepsilon,\eta})} \le C_{q'}(\varepsilon,\eta)\|F\|_{L^q(\omega_{\varepsilon,\eta})}. \tag{8.0.24}$$

Noting that $q'>2$, we may use Theorem 8.0.1 to obtain (8.0.22) and (8.0.23) in the case $1<q\le 2$.

Now consider the case $2<q<\infty$. We may assume $\varepsilon=1$ by rescaling. Let (u,p) be a solution of (8.0.21). Suppose $d\ge 3$. As in the proof of Theorem 8.0.2, it follows from Theorem 7.0.1 and summation that

$$\begin{aligned} \|\nabla u\|_{L^q(\omega_{1,\eta})} &\le C\eta^{\frac{d}{q}-1}\|u\|_{L^q(\omega_{1,\eta})} + C\Phi_q(\eta^{-1})(\|F\|_{L^q(\omega_{1,\eta})} + \|S_{1,\eta}(F,0)\|_{L^q(\mathbb{R}^d)}) \\ &\le C\eta^{1-d+\frac{d}{q}}\|F\|_{L^q(\omega_{1,\eta})} + C\Phi_q(\eta^{-1})(\|F\|_{L^q(\omega_{1,\eta})} + \eta^{1-\frac{d}{2}}\|F\|_{L^q(\omega_{1,\eta})}) \\ &\le C\eta^{1-d+\frac{d}{q}}\|F\|_{L^q(\omega_{1,\eta})}, \end{aligned} \tag{8.0.25}$$

where we have used (8.0.10) and (5.0.25) in the second inequality. This gives (8.0.22) when $\varepsilon = 1$. If $d = 2$, Theorem 7.0.1 gives

$$\begin{aligned} \|\nabla u\|_{L^q(\omega_{1,\eta})} &\leq C\eta^{\frac{2}{q}-1}|\ln(\eta)|^{-1}(\|u\|_{L^q(\omega_{1,\eta})} + \|F\|_{L^q(\omega_{1,\eta})} + \|S_{1,\eta}(F,0)\|_{L^q(\omega_{1,\eta})}) \\ &\leq C\eta^{-1+\frac{2}{q}}\|F\|_{L^q(\omega_{1,\eta})}, \end{aligned} \tag{8.0.26}$$

where we have used (8.0.11) and (5.0.26) in the second inequality. We obtain (8.0.23) in the case $\varepsilon = 1$. □

Given $1 < q < \infty$ and $\varepsilon, \eta \in (0, 1]$, let $A_q(\varepsilon, \eta), B_q(\varepsilon, \eta), C_q(\varepsilon, \eta)$, and $D_q(\varepsilon, \eta)$ be the smallest constants such that (1.3.4) and (1.3.5) hold. It follows from Theorem 8.0.2 and Remark 8.0.3 that

$$A_q(\varepsilon, \eta) \leq \begin{cases} C\eta^{-d|\frac{1}{2}-\frac{1}{q}|} & \text{if } d \geq 3, \\ C\eta^{-2|\frac{1}{2}-\frac{1}{q}|}|\ln(\eta/2)|^{-1/2} & \text{if } d = 2 \text{ and } q \neq 2, \\ 1 & \text{if } d \geq 2 \text{ and } q = 2. \end{cases} \tag{8.0.27}$$

By Theorem 8.0.4, we have

$$B_q(\varepsilon, \eta) = C_{q'}(\varepsilon, \eta) = \begin{cases} C\varepsilon\eta^{1-\frac{d}{2}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d \geq 3 \text{ and } 1 < q \leq 2, \\ C\varepsilon\eta^{1-d+\frac{d}{q}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d \geq 3 \text{ and } 2 < q < \infty, \\ C\varepsilon|\ln(\eta/2)|^{1/2}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d = 2 \text{ and } 1 < q \leq 2, \\ C\varepsilon\eta^{-1+\frac{2}{q}}\|F\|_{L^q(\omega_{\varepsilon,\eta})} & \text{if } d = 2 \text{ and } 2 < q < \infty. \end{cases} \tag{8.0.28}$$

Finally, it follows from Theorem 8.0.1 and duality that

$$D_q(\varepsilon, \eta) \leq \begin{cases} C\varepsilon^2\eta^{2-d} & \text{if } d \geq 3 \\ C\varepsilon^2|\ln(\eta/2)| & \text{if } d = 2. \end{cases} \tag{8.0.29}$$

Proof of Theorems 1.3.1 and 1.3.2. The estimates (1.3.6) and (1.3.8) follow from (8.0.27) and (8.0.28), while the estimates (1.3.7) and (1.3.9) follow from (8.0.28) and (8.0.29). □

Bibliography

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113:209–259, 1991.

[2] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. Noncritical sizes of the holes for a volume distribution and a surface distribution. *Arch. Rational Mech. Anal.*, 113:261–298, 1991.

[3] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[4] F. Alliot and C. Amrouche. Weak solutions for the exterior Stokes problem in weighted Sobolev spaces. *Math. Methods Appl. Sci.*, 23(6):575–600, 2000.

[5] A. Bensoussan, J. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North Holland, 1978.

[6] G. A. Chechkin, A. L. Pianitski, and A. S. Shamaev. *Homogenization: Methods and Applications*. American Mathematical Society, Translations of Mathematical Monographs vol. 234, 2007.

[7] D. Cioranescu and P. Donato. *An Introduction to Homogenization*. Oxford University Press, 1999.

[8] D. Cioranescu and F. Murat. Un terme étrange venu d'ailleurs. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, volume 60 of *Res. Notes in Math.*, pages 98–138, 389–390. Pitman, Boston, Mass.-London, 1982.

[9] G. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Volume I: Linearised Steady Problems*. Springer Tracts in Natural Philosophy. Springer New York, 1998.

[10] W. Jing. A unified homogenization approach for the Dirichlet problem in perforated domains. *SIAM J. Math. Anal.*, 52(2):1192–1220, 2020.

[11] S. M. Koslov, O. A. Oleinik, and V. V. Zhikov. *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, 1994.

[12] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[13] R. Righi and Z. Shen. Dirichlet problems in perforated domains, 2024. Preprint, available at `https://arxiv.org/abs/2402.13021`.

[14] Z. Shen. Bounds of Riesz transforms on L^p spaces for second order elliptic operators. *Ann. Inst. Fourier (Grenoble)*, 55(1):173–197, 2005.

[15] Z. Shen. Compactness and large-scale regularity for Darcy's law. *J. Math Pures Appl. (9)*, 163:673–701, 2022.

[16] Z. Shen. Uniform estimates for Dirichlet problems in perforated domains. *Vietnam J. Math.*, 51(4):845–867, 2023.

[17] Z. Shen and J. Wallace. Uniform $W^{1,p}$ estimates and large-scale regularity for Dirichlet problems in perforated domains. *J. Funct. Anal.*, 285(10, Paper No. 110118), 2023.

[18] R. Stevenson. Discrete Sobolev spaces and regularity of elliptic difference schemes. *RAIRO Modél. Math. Anal. Numér.*, 25(5):607–640, 1991.

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- C. Frayer and J. Wallace, *Geometry of a Family of Complex Polynomials*, The PUMP J. Undergrad. Res. **2** (2019), 95-106.

- K. Clinton, C. Frayer, and J. Wallace, *More Geometry of Polynomials with Three Roots*, Pi Mu Epsilon J. **14** (2019), no. 10, 613-620.