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Dirichlet Problems in Perforated Domains

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Dirichlet Problems in Perforated Domains

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

Dirichlet Problems in Perforated Domains

We establish $W^{1,p}$ estimates for solutions u_ε to the Laplace equation with Dirichlet boundary conditions in a bounded C^1 domain $\Omega_{\varepsilon,\eta}$ perforated by small holes in $\mathbb{R}^d$. The bounding constants will depend explicitly on ε and η, where ε is the order of the minimal distance between holes, and η denotes the ratio between the size of the holes and ε. The proof relies on a large-scale L^p estimate for ∇u_ε, whose proof is divided into two main parts. First, we show that solutions of an intermediate problem for a Schrödinger operator in Ω can be used to approximate harmonic functions in $\Omega_{\varepsilon,\eta}$ as ε, η approach zero. We then use a real-variable method to establish the large-scale L^p estimate for ∇u_ε. Sharpness is established for these results in all cases except when $d \geq 3$ with $p = d$ or d'.

KEYWORDS: Uniform Estimates; Dirichlet Problem; Perforated Domain; Homogenization

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Dirichlet Problems in Perforated Domains

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Dedicated to the soon to be Jennifer Leigh Righi,
for your love and support.

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1.1 Laplace Equation

The Dirichlet problem for Laplace's equation is given by

$$\begin{cases} -\Delta u_\varepsilon = F + \operatorname{div}(f) & \text{in } \Omega_{\varepsilon,\eta}, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_{\varepsilon,\eta}, \end{cases} \tag{1.1}$$

where $\Omega_{\varepsilon,\eta}$ is a domain perforated with a large number of tiny holes. Given $F \in L^p(\Omega_{\varepsilon,\eta})$ and $f \in L^p(\Omega_{\varepsilon,\eta}; \mathbb{R}^d)$, standard PDE theory emits that the Dirichlet problem (1.1) possesses a unique solution u_ε in $W^{1,p}(\Omega_{\varepsilon,\eta})$, if $1 < p < \infty$ and $\Omega_{\varepsilon,\eta}$ is a bounded C^1 domain in $\mathbb{R}^d$, for $d \geq 2$.

Let $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$ be the smallest constants for which the $W^{1,p}$ estimate,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_{\varepsilon,\eta})} \leq A_p(\Omega_{\varepsilon,\eta})\|f\|_{L^p(\Omega_{\varepsilon,\eta})} + B_p(\Omega_{\varepsilon,\eta})\|F\|_{L^p(\Omega_{\varepsilon,\eta})}, \tag{1.2}$$

holds for solutions u_ε of 1.1. We are interested in the bounds of $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$ that exhibit explicit and sharp dependence on the sizes of the holes as well as on the distances between the holes.

This work is motivated by the study of fluid flow in porous media. A concrete example is given by underground water flow in soil with rocks serving as obstacles, which is analogous to the perforations in our domain. The Laplace equation serves as a toy model for this phenomenon. By studying the Laplace equation we gain insight into more complex equations such as the Stoke's equations that govern this type of fluid flow.

1.2 Perforated Domain

Let $Q(x, r)$ denote the cube centered at x of side length r. To describe the perforated domain $\Omega_{\varepsilon,\eta}$, let $Y = Q(0, 1)$, and $\{Y_z^s : z \in \mathbb{Z}^d\}$ be a sequence of domains with connected and uniform C^1 boundaries, such that

$$B(0, c_0) \subset Y_z^s \subset B(0, 1/8) \tag{1.3}$$

for some $c_0 > 0$. Let $\{x_z : z \in \mathbb{Z}^d\}$ be a sequence of points in $B(0, 1/4)$ and

$$T_z = z + x_z + \eta\overline{Y_z^s}, \tag{1.4}$$

where $\eta \in (0, 1/4)$. For a domain Ω in $\mathbb{R}^d$ and $0 < \varepsilon \leq 1$, define

$$\Omega_{\varepsilon,\eta} = \Omega \setminus \bigcup_z \varepsilon\overline{T_z}, \tag{1.5}$$

where the union is taken over those z's in $\mathbb{Z}^d$ for which $\varepsilon(z + Y) \subset \Omega$. Thus, the perforated domain $\Omega_{\varepsilon,\eta}$ is obtained from Ω by removing a hole $\varepsilon\overline{T_z}$, centered at $\varepsilon(z +$

x_z) and of size $\varepsilon\eta$, from each cube $\varepsilon(z+Y)$ of size ε and contained in Ω. Roughly speaking, the parameter ε represents the scale of the distances between holes, while the parameter η represents the scale of the ratios between the sizes of the holes and ε. We point out that the holes are not identical, nor they are placed periodically, unless the sequences $\{x_z\}$, and $\{Y_z^s\}$ are independent of z.

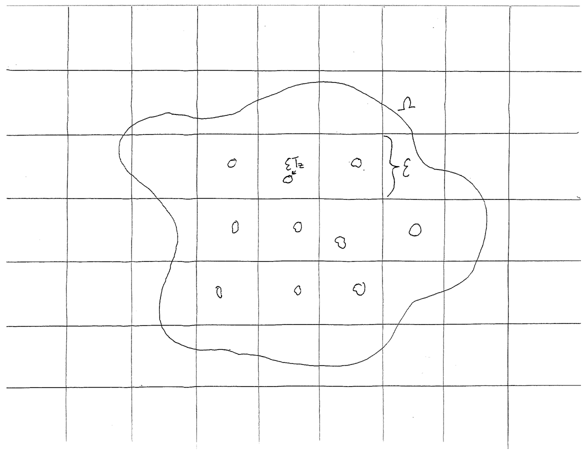

Figure 1.1: Perforated Domain

Figure 1.1 provides a picture of a perforated domain in two dimensions. Notice we remove a hole $\varepsilon\overline{T_z}$ from each epsilon cube $\varepsilon(z+Y)$ entirely contained in Ω.

1.3 Previous Work

Homogenization theory of boundary value problems for elliptic equations in perforated domains has been well studied [8, 5, 1, 2, 7, 4, 9]. This theory takes a multi-scaled problem, with one or more small parameters, and seeks to replace it with a simpler equation. For instance homogenization of the Laplace equation given by (1.1) with $f=0$ has been well studied in the periodic case where $\{x_z\}$ and $\{Y_z^s\}$ are independent of z [7, 4, 9]. That is, as $\varepsilon, \eta \to 0$, solutions u_ε will approach a limiting equation defined in a homogeneous domain. This so called homogeneous equation will depend on the relationship between the size of the holes and the distance between holes. More specifically they are determined by the size of the ratio given by

$$\sigma_\varepsilon = \begin{cases} \varepsilon\eta^{1-\frac{d}{2}} & \text{if } d \geq 3, \\ \varepsilon|\ln\eta|^{\frac{1}{2}} & \text{if } d = 2. \end{cases} \tag{1.6}$$

We point out in the case $d \geq 3$ there is a negative power of η which is large as η is small.

Thus, the case of large holes is given when $\sigma_\varepsilon \to 0$. In this case $\sigma_\varepsilon^{-2}u_\varepsilon \to u$ weakly in $L^2(\Omega)$, where

$$u = \overline{c}F$$

with $\overline{c} \in \mathbb{R}$. Note that we are normalizing by a factor if σ_ε^2. This is due to the fact that in the case of large holes the obstacles are significantly impeding the fluid flow.

The case of small holes is given by $\sigma_\varepsilon \to \infty$. In this setting it is known $u_\varepsilon \to u$ strongly in $H_0^1(\Omega)$, where

$$\begin{cases} -\Delta u = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The homogenized solution satisfies the Laplace equation in Ω. Intuitively, this can be thought of as the holes being so small as to not disrupt the fluid flow.

Finally, the critical case is given by $\sigma_\varepsilon \to 1$. Here $u_\varepsilon \to u$ weakly in $H_0^1(\Omega)$, where

$$\begin{cases} -\Delta u + \mu_* u = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_* is the Newtonian capacity of Y_z^s for $d \geq 3$, and the logarithmic capacity for $d = 2$.

The bounds for both $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$ will depend on which scale setting we are in. Our work provides a complete characterization on each setting.

Homogenization provided the motivation for work into uniform $W^{1,p}$ estimates of the form of (1.2), which were first studied by N. Masmoudi [10] in the case where $\eta = 1$. Furthermore, Z. Shen [14] was first to consider the general case where $0 < \eta < 1$, for both the bounded domain $\Omega_{\varepsilon,\eta}$ and the unbounded domain

$$\omega_{\varepsilon,\eta} = \mathbb{R}^d \setminus \bigcup_{z \in \mathbb{Z}^d} \varepsilon(z + \eta \overline{Y_z^s}),$$

where both domains are not necessarily periodically perforated. In this setting the upper bounds obtained for the constant A_p are not found to be sharp. However, they are only off by an arbitrary small power of η. In the case where $\omega_{\varepsilon,\eta}$ is prescribed the additional requirement of being periodically perforated, J. Wallace and Z. Shen [15] were able to obtain (1.2) with sharp bounding constants A_p and B_p. Their approach makes use of a large-scale Lipschitz estimate for harmonic functions u_ε in perforated domains. The proof of which relies on the fact that the difference $u_\varepsilon(x + \varepsilon e_j) - u_\varepsilon(x)$ is also harmonic. It is unclear how to extend this proof to the non-periodic setting as well as to the setting of bounded domains.

1.4 Poincaré Inequality

The following lemmas will be extremely useful throughout this thesis. The proof of our first lemma when $p = 2$ is well known [1]. A proof for the general case is similar and was shown in [14, Lemma 2.1]. We provide the proof for convenience.

Lemma 1.4.1. *Let $d \geq 2$ and $1 \leq p < \infty$. Suppose that $u \in W^{1,p}(Q(0,\varepsilon))$ and $u = 0$ on $B(x_0, \varepsilon\eta)$ for some $x_0 \in Q(0, \varepsilon/2)$ and $0 < \eta < 1/4$. Then*

$$\int_{Q(0,\varepsilon)} |u|^p dx \leq C \int_{Q(0,\varepsilon)} |\nabla u|^p dx \cdot \begin{cases} \varepsilon^p \eta^{p-d} & \text{if } 1 \leq p < d, \\ \varepsilon^p |\ln \eta|^{d-1} & \text{if } p = d, \\ \varepsilon^p & \text{if } d < p < \infty, \end{cases} \tag{1.7}$$

where C depends on d and p.

Proof. By dilation we may assume $\varepsilon = 1$. It is known

$$\int_{Q(0,1)} |u|^p \leq C \int_{B(x_0,1/2)} |u|^p + C \int_{Q(0,1)} |\nabla u|^p. \tag{1.8}$$

Therefore it suffices to show (1.7) for $B(x_0, 1/2)$ in place of $Q(0,1)$. By translation we consider $B(0,1/2)$ where $u = 0$ on $B(0,\eta)$. This allows us to write

$$u(x) = u(r\omega) - u(\eta\omega) = \int_\eta^r \omega \cdot \nabla u(t\omega) dt,$$

for any $x \in Q(0,1)$, where $r = |x|$ and $\omega = x/|x|$. Applying Hölder's inequality gives

$$|u(x)|^p \leq \int_\eta^r |\nabla u(t\omega)|^p t^{d-1} dt \left(\int_\eta^r t^{-\frac{d-1}{p-1}} dt \right)^{p-1}$$

for $1 < p < \infty$. Then,

$$\int_{S^{d-1}} \int_0^{1/2} |u(x)|^p r^{d-1} dr d\omega \leq \int_{S^{d-1}} \int_0^{1/2} r^{d-1} \int_\eta^r |\nabla u(t\omega)|^p t^{d-1} dt \left(\int_\eta^r t^{-\frac{d-1}{p-1}} dt \right)^{p-1} dr d\omega.$$

Thus,

$$\int_{B(0,1/2)} |u|^p dx \leq C \int_{B(0,1/2)} |\nabla u|^p dx \left(\int_\eta^d t^{-\frac{d-1}{p-1}} dt \right)^{p-1}.$$

This implies

$$\int_{B(x_0,1/2)} |u|^p dx \leq C \int_{Q(0,1)} |\nabla u|^p dx \cdot \begin{cases} \eta^{p-d} & \text{if } 1 < p < d, \\ |\ln \eta|^{d-1} & \text{if } p = d, \\ 1 & \text{if } d < p < \infty. \end{cases} \tag{1.9}$$

Plugging (1.9) into (1.8) and rescaling both sides gives (1.7) for $1 < p < \infty$. Note for $p = 1$ we use

$$\begin{aligned} |u(x)| &\leq \int_\eta^r |\nabla u(t\omega)| \frac{t^{d-1}}{t^{d-1}} dt \\ &\leq \eta^{1-d} \int_\eta^r |\nabla u(t\omega)| t^{d-1} dt \end{aligned}$$

in place of Hölder's inequality and the result follows.

□

Lemma 1.4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be defined in (1.5). Then,*

$$\|u\|_{L^2(\Omega_{\varepsilon,\eta})} \leq C \min(\sigma_\varepsilon, 1) \|\nabla u\|_{L^2(\Omega_{\varepsilon,\eta})} \tag{1.10}$$

for any $u \in H_0^1(\Omega_{\varepsilon,\eta})$, where σ_ε is given in (1.6), and C is independent of ε and η.

Proof. Let $u \in H_0^1(\Omega_{\varepsilon,\eta})$ with $u = 0$ on $\partial\Omega_{\varepsilon,\eta}\backslash\partial\Omega$. Then it follows from Lemma 1.4.1 that

$$\|u\|_{L^2(\Omega_{\varepsilon,\eta})} \leq C\sigma_\varepsilon \|\nabla u\|_{L^2(\Omega_{\varepsilon,\eta})}. \tag{1.11}$$

The standard Poincaré inequality for $u \in H_0^1(\Omega)$ yields

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}. \tag{1.12}$$

Combining (1.11) and (1.12) yields (1.10) for $u \in H_0^1(\Omega_{\varepsilon,\eta})$. □

Lemma 1.4.3. *Let $u_\varepsilon \in H_0^1(\Omega_{\varepsilon,\eta})$ be the solution of the Dirichlet problem* (1.1) *with $F \in L^2(\Omega_{\varepsilon,\eta})$ and $f \in L^2(\Omega_{\varepsilon,\eta};\mathbb{R}^d)$. Then the following estimate holds*

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})} \leq \|f\|_{L^2(\Omega_{\varepsilon,\eta})} + C\min(\sigma_\varepsilon, 1)\|F\|_{L^2(\Omega_{\varepsilon,\eta})}. \tag{1.13}$$

Proof. Assume $\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})} \neq 0$, for otherwise the result is trivial. Then for any $v \in H_0^1(\Omega_{\varepsilon,\eta})$ we have,

$$\int_{\Omega_{\varepsilon,\eta}} \nabla u_\varepsilon \cdot \nabla v = \int_{\Omega_{\varepsilon,\eta}} f\nabla v + \int_{\Omega_{\varepsilon,\eta}} Fv. \tag{1.14}$$

Picking $v = u$,

$$\int_{\Omega_{\varepsilon,\eta}} |\nabla u_\varepsilon|^2 = \int_{\Omega_{\varepsilon,\eta}} f\nabla u_\varepsilon + \int_{\Omega_{\varepsilon,\eta}} Fu_\varepsilon. \tag{1.15}$$

Now we apply Hölder's inequality to obtain

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,\eta})} \leq \|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}\|f\|_{L^2(\Omega_{\varepsilon,\eta})} + \|u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}\|F\|_{L^2(\Omega_{\varepsilon,\eta})} \tag{1.16}$$

Applying Lemma 1.4.2 to the last term on the right-hand side of (1.16) gives

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega_{\varepsilon,\eta})} \leq \|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}\|f\|_{L^2(\Omega_{\varepsilon,\eta})} + C\min(\sigma_\varepsilon, 1)\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}\|F\|_{L^2(\Omega_{\varepsilon,\eta})}. \tag{1.17}$$

Dividing by a factor of $\|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}$ gives the desired estimate. □

1.5 Main Results

We seek to find bounds of $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$, which satisfy (1.2) and exhibit explicit and sharp dependence on ε and η. To state the main results, we note that $A_2(\Omega_{\varepsilon,\eta}) = 1$, and that $A_p(\Omega_{\varepsilon,\eta}) = A_{p'}(\Omega_{\varepsilon,\eta})$, where $p' = \frac{p}{p-1}$, by duality. As a result, it suffices to consider the case $2 < p < \infty$.

The asymptotic behavior of $A_p(\Omega_{\varepsilon,\eta})$, as $\varepsilon, \eta \to 0$, depends on σ_ε. Our first theorem treats the case of relatively large holes, where $\sigma_\varepsilon \leq 1$, while our second theorem handles the case of relatively small holes, where $\sigma_\varepsilon \geq 1$.

Theorem 1.5.1. *Suppose that $0 < \sigma_\varepsilon \leq 1$ and $2 < p < \infty$. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by* (1.5). *Then*

$$A_p(\Omega_{\varepsilon,\eta}) \leq \begin{cases} C\eta^{-d|\frac{1}{2}-\frac{1}{p}|} & \text{if} \quad d \geq 3, \\ C\eta^{-2|\frac{1}{2}-\frac{1}{p}|}|\ln\eta|^{-\frac{1}{2}} & \text{if} \quad d = 2, \end{cases} \tag{1.18}$$

where C depends only on d, p, Ω, and $\{Y_z^s\}$.

Theorem 1.5.2. *Suppose that $\sigma_\varepsilon \geq 1$ and $2 < p < \infty$. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by* (1.5)*. Then*

$$A_p(\Omega_{\varepsilon,\eta}) \leq \begin{cases} C(1+\varepsilon^{-1}\eta^{\frac{d}{p}-1}) & \text{if } d \geq 3 \text{ and } 2 < p < d, \\ C(\varepsilon^{-1} + |\ln \eta|^{1-\frac{1}{d}}) & \text{if } d \geq 3 \text{ and } p = d, \\ C\varepsilon^{-1}\eta^{\frac{d}{p}-1} & \text{if } d \geq 3 \text{ and } d < p < \infty, \\ C\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln \eta|^{-1} & \text{if } d = 2, \end{cases} \tag{1.19}$$

where C depends only on d, p, Ω, and $\{Y_z^s\}$.

The upper bounds for $A_p(\Omega_{\varepsilon,\eta})$ in (1.18) are sharp. Additionally, we remark that the upper bounds of $A_p(\Omega_{\varepsilon,\eta})$ in (1.19) are also sharp for $d = 2$ as well as for $d \geq 3$ and $p \neq d$. Whether the upper bounds are sharp for the remaining case where $d \geq 3$ and $p = d$ is not known. Indeed, if $\Omega_{\varepsilon,\eta}$ is a periodically perforated domain, given by (1.5) with the sequences $\{x_z\}$ and $\{Y_z^s\}$ independent of z, then

$$A_p(\Omega_{\varepsilon,\eta}) \geq \begin{cases} c\,\eta^{-d|\frac{1}{2}-\frac{1}{p}|} & \text{if } d \geq 3, \\ c\,\eta^{-2|\frac{1}{2}-\frac{1}{p}|}|\ln \eta|^{-\frac{1}{2}} & \text{if } d = 2, \end{cases} \tag{1.20}$$

and,

$$A_p(\Omega_{\varepsilon,\eta}) \geq \begin{cases} c(1+\varepsilon^{-1}\eta^{\frac{d}{p}-1}) & \text{if } d \geq 3 \text{ and } 2 < p < d, \\ c\varepsilon^{-1} & \text{if } d \geq 3 \text{ and } p = d, \\ c\varepsilon^{-1}\eta^{\frac{d}{p}-1} & \text{if } d \geq 3 \text{ and } d < p < \infty, \\ c\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln \eta|^{-1} & \text{if } d = 2, \end{cases} \tag{1.21}$$

for the large hole and small hole cases respectively. The constants $c > 0$ depend only on d, p, Ω, and $\{Y_z^s\}$. See Theorem 7.2.2.

The next set of theorems establish the analogous results for $B_p(\Omega_{\varepsilon,\eta})$. The first of which was proved in [14] and handles the case where $1 < p \leq 2$.

Theorem 1.5.3. *Suppose that $1 < p \leq 2$. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by* (1.5)*. Then*

$$B_p(\Omega_{\varepsilon,\eta}) \leq C \begin{cases} \min\{1, \varepsilon\eta^{1-\frac{d}{2}}\} & \text{if } d \geq 3, \\ \min\{1, \varepsilon|\ln \eta|^{1/2}\} & \text{if } d = 2, \end{cases} \tag{1.22}$$

where C depends only on d, p, Ω, and $\{Y_z^s\}$.

Theorem 1.5.4. *Suppose that $0 < \sigma_\varepsilon \leq 1$ and $2 < p < \infty$. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by* (1.5)*. Then*

$$B_p(\Omega_{\varepsilon,\eta}) \leq C\varepsilon\eta^{1-d+\frac{d}{p}}, \tag{1.23}$$

where C depends only on d, p, Ω, and $\{Y_z^s\}$.

Theorem 1.5.5. *Suppose that $\sigma_\varepsilon \geq 1$ and $2 < p < \infty$. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by (1.5). Then*

$$B_p(\Omega_{\varepsilon,\eta}) \leq \begin{cases} C(1+\varepsilon^{-1}\eta^{\frac{d}{p}-1}) & \text{if } \; d\geq 3 \text{ and } 2<p<d, \\ C(\varepsilon^{-1}+|\ln\eta|^{1-\frac{1}{d}}) & \text{if } \; d\geq 3 \text{ and } p=d, \\ C\varepsilon^{-1}\eta^{\frac{d}{p}-1} & \text{if } \; d\geq 3 \text{ and } d<p<\infty, \\ C\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln\eta|^{-1} & \text{if } \; d=2, \end{cases} \tag{1.24}$$

where C depends only on d, p, Ω, and $\{Y_z^s\}$.

The question of sharpness of these estimates is addressed in Chapter 7.

1.6 Thesis Outline

This thesis provides a new approach to establishing $W^{1,p}$ estimates which addresses the non-periodic setting as well as boundary estimates for bounded perforated domains. We construct a unique argument that relies on the key observation that harmonic functions in a perforated domain are well approximated by solutions to a Schrödinger type problem. These so called 'intermediate solutions' exhibit good regularity properties. A more detailed description of our argument is as follows.

In order to prove Theorems 1.5.1 - 1.5.5, we start with estimates on the L^p norm of u_ε. Such estimates were proved extensively in [14]. See Theorem 6.3.1. Using a similar localization argument as in [15] we may reduce the L^p estimates of ∇u_ε to an L^p estimate of the operator T_ε defined by

$$T_\varepsilon(F,f) = \left(\fint_{x+2\varepsilon Y} |\nabla u_\varepsilon|^2 \right)^{1/2}. \tag{1.25}$$

where the solution u_ε to (1.2) has been extended to $\mathbb{R}^d$ by zero. Note is easy to see that

$$\|T_\varepsilon(F,f)\|_{L^2(\mathbb{R}^d)} = \|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}.$$

By energy estimates we obtain,

$$\|T_\varepsilon(F,f)\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\Omega_{\varepsilon,\eta})} + C\min(\sigma_\varepsilon,1)\|F\|_{L^2(\Omega_{\varepsilon,\eta})}.$$

Theorem 1.6.1. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by (1.5). Then for $2 < p < \infty$,*

$$\|T_\varepsilon(F,f)\|_{L^p(\mathbb{R}^d)} \leq C\left\{ \|f\|_{L^p(\Omega_{\varepsilon,\eta})} + \min(\sigma_\varepsilon,1)\|F\|_{L^p(\Omega_{\varepsilon,\eta})} \right\}, \tag{1.26}$$

where C depends only on d, p, Ω, and $\{Y_z^s\}$.

The estimate (1.26) is regarded as a large scale $W^{1,p}$ estimate for u_ε as T_ε is averaging ∇u_ε over a cell of size 2ε. Note that in light of Theorem 1.6.1, this average behaves much better than ∇u_ε in L^p spaces for $p \neq 2$, as when $\sigma_\varepsilon \leq 1$, the operator norm $\|T_\varepsilon(0,\cdot)\|_{L^p \to L^p}$ remains bounded as $\varepsilon, \eta \to 0$, where $A_p(\Omega_{\varepsilon,\eta}) \to \infty$.

Much of our work is henceforth dedicated to proving Theorem 1.6.1. The proof relies on a real-variable argument from [11], which reduces the argument to proving a weak Hölder inequality for harmonic functions u_ε in perforated domains. The proof of the weak Hölder inequality will again rely on the same real-variable argument. This in turn relies on the ability to approximate ∇u_ε, on each subdomain D of size greater than ε, by a function that behaves well in L^p norm. To do this, on each hole we introduce a nonnegative potential supported in a neighborhood around the hole, with V_ε denoting the sum of all such potentials. We then utilize convergence rates of harmonic functions u_ε in a bounded perforated domain to $\chi_{\varepsilon,\eta} v_\varepsilon$, where $\chi_{\varepsilon,\eta}$ is a corrector for $D_{\varepsilon,\eta}$, with $\chi_{\varepsilon,\eta} = 1$ on ∂D and v_ε is the solution to the intermediate equation

$$\begin{cases} (-\Delta + \sigma_\varepsilon^{-2} V_\varepsilon) v_\varepsilon = 0 & \text{in } D, \\ u_\varepsilon = v_\varepsilon & \text{on } \partial D, \end{cases}$$

where D is a non-perforated domain.

The rest of the thesis will be organized in the following way. Chapter 2 will be dedicated to defining our corrector $\chi_{\varepsilon,\eta}$ as well as establishing useful estimates on $\chi_{\varepsilon,\eta}$ which will be used throughout our work. Chapter 3 establishes convergence rates of a non-homogeneous problem to $\chi_{\varepsilon,\eta} v_\varepsilon$. Chapter 4 provides estimates for the intermediate problem. Bounds for the operator $T_\varepsilon(F, f)$ are shown in Chapter 5, which will provide the proof for Theorem 1.6.1. Our main results are shown in Chapter 6, which is dedicated to the proofs of Theorems 1.5.1-1.5.5. Finally, Chapter 7 will provide results pertaining to the sharpness of our estimates.

Chapter 2 Correctors

This chapter is dedicated to constructing and establishing estimates for our corrector $\chi_{\varepsilon,\eta}$. The definition of the correcter will depend on the dimensional setting. Therefore, we must treat the cases where $d \geq 3$ and $d = 2$ separately. Our first section will address the case where $d \geq 3$, while the following section will handle the case for $d = 2$. In both cases we provide a formal definition of the corrector, which is defined piece-wise on cubes of size ε. Additionally, we provide some useful results on the corrector which will be applied in the proofs of the main theorems, which can be found in Chapter 3.

2.1 Definition of the Corrector: Dimension $d \geq 3$

Let $\Omega_{\varepsilon,\eta}$ be defined in (1.5) and T_z be defined in (1.4). For $z \in \mathbb{Z}^d$ let $Q_z = z + Q(0,1)$. Additionally, let $y_z = z + x_z$ where x_z is defined in (1.4). Following the ideas of [1], we define a corrector $\chi_{\varepsilon,\eta}$ on each rescaled cube εQ_z. If εQ_z lies entirely inside of Ω, i.e. $\varepsilon Q_z \subset \Omega$ we define

$$\chi_\varepsilon = \begin{cases} 1 & \text{in } \ \varepsilon Q_z \setminus \varepsilon B(y_z, 1/3), \\ \phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta}) & \text{in } \ \varepsilon B(y_z, 1/4) \setminus \varepsilon T_z, \\ 0 & \text{on } \ \varepsilon T_z, \end{cases} \tag{2.1}$$

where ϕ_*^z are the solutions to the following exterior problem

$$\begin{cases} -\Delta \phi_*^z = 0 & \text{in } \ \mathbb{R}^d \setminus \overline{Y_z^s}, \\ \phi_*^z = 0 & \text{on } \ \partial Y_z^s, \\ \phi_*^z \to 1 & \text{as } \ |x| \to \infty. \end{cases} \tag{2.2}$$

On the region $B(\varepsilon y_z, \varepsilon/3) \setminus B(\varepsilon y_z, \varepsilon/4)$ let $\chi_{\varepsilon,\eta}$ solve the following Dirichlet problem,

$$\begin{cases} -\Delta \chi_{\varepsilon,\eta} = 0 & \text{in } \ B(\varepsilon y_z, \varepsilon/3) \setminus \overline{B(\varepsilon y_z, \varepsilon/4)}, \\ \chi_{\varepsilon,\eta} = \phi_*^z(\dfrac{x - \varepsilon y_z}{\varepsilon\eta}) & \text{on } \ \partial B(\varepsilon y_z, \varepsilon/4), \\ \chi_{\varepsilon,\eta} = 1 & \text{on } \ \partial B(\varepsilon y_z, \varepsilon/3). \end{cases} \tag{2.3}$$

Finally, for cubes that are not entirely contained in Ω we define $\chi_{\varepsilon,\eta} = 1$.

The purpose of the harmonic region serves to bridge the gap between the boundary data of the exterior problem and 1. Thus, by construction $\chi_{\varepsilon,\eta} \in H^1(\Omega)$.

Additionally, as $|x| \to \infty$, it is known that for each ϕ_*^z that satisfies (2.2)

$$\begin{cases} \phi_*^z(x) = 1 - c_*^z |x|^{2-d} + O(|x|^{1-d}), \\ \nabla\phi_*^z(x) = -c_*^z \nabla(|x|^{2-d}) + O(|x|^{-d}), \\ \nabla^2\phi_*^z(x) = -c_*^z \nabla^2(|x|^{2-d}) + O(|x|^{-d-1}). \end{cases} \tag{2.4}$$

where $c_*^z = C_d \mu_*^z$ with

$$\mu_*^z = \int_{\partial Y_z^s} n \cdot \nabla \phi_*^z(y) d\sigma \tag{2.5}$$

and $C_d = \frac{1}{(d-2)|\partial B(0,1)|}$. See [3, 16]. The condition (1.3) on Y_z^s ensures there exists $\mu_0, \mu_1 > 0$ such that $\mu_0 \leq \mu_*^z \leq \mu_1$ for any $z \in \mathbb{Z}^d$.

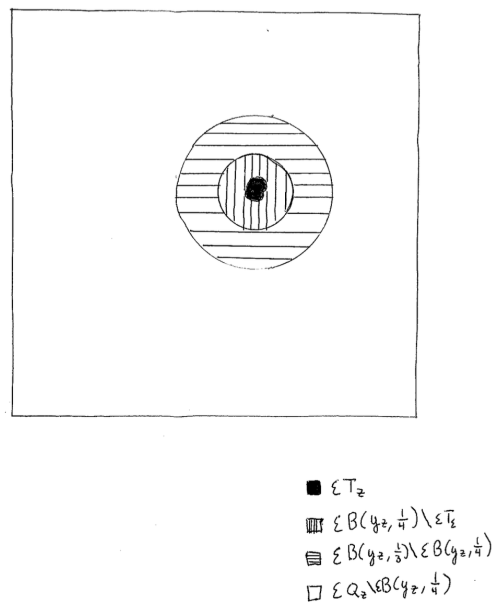

Figure 2.1: Corrector Cell

We have defined our corrector piece-wise in subdomains on each cell entirely contained in Ω. Figure 2.1 provides a description of these subdomains within the cell.

2.2 Estimates on the Corrector: Dimension $d \geq 3$

We now state some useful results on the corrector.

Lemma 2.2.1. *Let* $\chi_{\varepsilon,\eta}$ *be defined in* (2.1)*. Then*

$$\|\chi_{\varepsilon,\eta} - 1\|_{L^p(\Omega_{\varepsilon,\eta})} \leq \begin{cases} C\eta^{d-2} & \text{for} \quad 1 \leq p < \frac{d}{d-2}, \\ C\eta^{d-2}|\ln \eta|^{\frac{d-2}{2}} & \text{for} \quad p = \frac{d}{d-2}, \\ C\eta^{\frac{d}{p}} & \text{for} \quad p > \frac{d}{d-2}, \end{cases} \tag{2.6}$$

where C *does not depend on* ε *or* η*.*

Proof. It suffices to consider the cubes which are entirely contained in Ω, for otherwise $\chi_{\varepsilon,\eta} - 1 = 0$. Suppose $\varepsilon Q_z \subset \Omega$. Recall the corrector $\chi_{\varepsilon,\eta}$ was defined piece-wise in distinct sub-regions of the cube. We consider each region separately. First, from construction $\chi_{\varepsilon,\eta} - 1 = 0$ on the region $\varepsilon Q_z \backslash B(\varepsilon y_z, \varepsilon/3)$.

Now, in $B(\varepsilon y_z, \varepsilon/3)\backslash \overline{B(\varepsilon y_z, \varepsilon/4)}$ we note that $\chi_{\varepsilon,\eta} - 1$ is harmonic. Hence, we may apply the maximum principle in this region to get

$$\begin{aligned}\|\chi_{\varepsilon,\eta} - 1\|_{L^\infty(B(\varepsilon y_z,\varepsilon/3)\backslash B(\varepsilon y_z,\varepsilon/4))} &\leq \|\chi_{\varepsilon,\eta} - 1\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/3)\cup\partial B(\varepsilon y_z,\varepsilon/4))} \\ &= \|\chi_{\varepsilon,\eta} - 1\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/4))}.\end{aligned}$$

By definition, $\chi_{\varepsilon,\eta} = \phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})$ on $\partial B(\varepsilon y_z, \varepsilon/4))$. Furthermore, from (2.4),

$$\begin{aligned}\|\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/4))} &\leq C\left|1 - c_*^z\left|\frac{\varepsilon}{4\varepsilon\eta}\right|^{2-d}\right| \\ &\leq C|1-\eta^{d-2}|.\end{aligned}$$

It directly follows that

$$\|\chi_{\varepsilon,\eta} - 1\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/4))} = \|\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta}) - 1\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/4))} \leq C\eta^{d-2}. \tag{2.7}$$

Moreover,

$$\|\chi_{\varepsilon,\eta} - 1\|^p_{L^p(B(\varepsilon y_z,\varepsilon/3)\backslash B(\varepsilon y_z,\varepsilon/4))} \leq C\varepsilon^d\eta^{p(d-2)}. \tag{2.8}$$

For the remaining region, consider

$$\begin{aligned}\|\chi_{\varepsilon,\eta} - 1\|^p_{L^p(B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z)} &= \int_{B(\varepsilon y_z,\frac{\varepsilon}{4})\backslash\varepsilon T_z} |\chi_{\varepsilon,\eta} - 1|^p dx \\ &= \int_{B(\varepsilon y_z,\frac{\varepsilon}{4})\backslash\varepsilon T_z} |\phi_*(\frac{x-\varepsilon y_z}{\varepsilon\eta}) - 1|^p dx \\ &= \varepsilon^d\eta^d \int_{B(0,\frac{1}{4\eta})\backslash Y_z^s} |\phi_*^z(y) - 1|^p dy.\end{aligned}$$

Now we obtain from (2.4),

$$\begin{aligned}\int_{B(\varepsilon y_z,\frac{\varepsilon}{4})\backslash\varepsilon T_z} |\chi_{\varepsilon,\eta} - 1|^p dx &\leq C\varepsilon^d\eta^d \int_{B(0,\frac{1}{4\eta})\backslash Y_z^s} \frac{1}{|y|^{(d-2)p}} dy \\ &\leq C\varepsilon^d\eta^d \int_1^{\frac{1}{4\eta}} r^{d-1-dp+2p} dr.\end{aligned}$$

Integrating we have,

$$\|\chi_{\varepsilon,\eta} - 1\|^p_{L^p(B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z)} \leq \begin{cases} C\varepsilon^d\eta^{p(d-2)} & \text{for} \quad 1 \leq p < \frac{d}{d-2}, \\ C\varepsilon^d\eta^d|\ln\eta| & \text{for} \quad p = \frac{d}{d-2}, \\ C\varepsilon^d\eta^d & \text{for} \quad p > \frac{d}{d-2}. \end{cases} \tag{2.9}$$

Summing (2.8) and (2.9), we have

$$\|\chi_{\varepsilon,\eta}-1\|^p_{L^p(\varepsilon Q_z\setminus\varepsilon T_z)} \le \begin{cases} C\varepsilon^d\eta^{p(d-2)} & \text{for} \quad 1\le p<\frac{d}{d-2},\\ C\varepsilon^d\eta^d|\ln\eta| & \text{for} \quad p=\frac{d}{d-2},\\ C\varepsilon^d\eta^d & \text{for} \quad p>\frac{d}{d-2}.\end{cases} \tag{2.10}$$

The number of cubes εQ_z entirely contained in Ω is bonded by $C\varepsilon^{-d}$. Hence, summing (2.10) over all such cubes gives,

$$\|\chi_{\varepsilon,\eta}-1\|^p_{L^p(\Omega_{\varepsilon,\eta})} \le \begin{cases} C\eta^{p(d-2)} & \text{for} \quad 1\le p<\frac{d}{d-2},\\ C\eta^d|\ln\eta| & \text{for} \quad p=\frac{d}{d-2},\\ C\eta^d & \text{for} \quad p>\frac{d}{d-2}.\end{cases} \tag{2.11}$$

Taking a $1/p$ to both sides gives the desired estimate. □

Remark 2.2.2. Suppose $\varepsilon Q_z\subset\Omega$. Consider $\xi=\chi_{\varepsilon,\eta}(\varepsilon x+\varepsilon y_z)-1$. This is a harmonic function in $B(0,1/3)\setminus\overline{B(0,1/4)}$ with $\xi=0$ on $\partial B(0,1/3)$ and $\xi=\phi_*^z(\eta^{-1}x)-1$ on $\partial B(0,1/4)$. By (2.4)

$$|\xi|=|\phi_*^z(\eta^{-1}x)-1|\le C\eta^{d-2} \quad \text{on} \quad \partial B(0,1/4).$$

Hence,

$$|\nabla\xi|\le\|\xi\|_{C^{1,1}(\partial B(0,1/4))}\le C\eta^{d-2}.$$

This implies

$$|\nabla\chi_{\varepsilon,\eta}|\le C\varepsilon^{-1}\eta^{d-2}$$

in the region $B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)$.

Lemma 2.2.3. *Let $\chi_{\varepsilon,\eta}$ be defined as in* (2.1)*. Then on each cube $\varepsilon Q_z\subset\Omega$, we have the following estimate,*

$$\left(\fint_{\varepsilon Q_z}|\nabla\chi_{\varepsilon,\eta}|^p\right)^{1/p} \le \begin{cases} C\varepsilon^{-1}\eta^{\frac{d-p}{p}} & \text{if } p>d',\\ C\varepsilon^{-1}\eta^{d-2}|\ln\eta|^{1/p} & \text{if } p=d',\\ C\varepsilon^{-1}\eta^{d-2} & \text{if } p<d',\end{cases} \tag{2.12}$$

where $d'=\frac{d}{d-1}$ and the constant C does not depend on ε or η.

Proof. We start by decomposing along the sub-regions within the cube. This is given by

$$\int_{\varepsilon Q_z}|\nabla\chi_{\varepsilon,\eta}|^p\le\int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)}|\nabla\chi_{\varepsilon,\eta}|^p+\int_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p. \tag{2.13}$$

For the first term on the right-hand side of (2.13), we apply Remark 2.2.2 to obtain

$$\begin{aligned}\int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)}|\nabla\chi_{\varepsilon,\eta}|^p &\le C(\varepsilon^{-1}\eta^{d-2})^p\int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)}|1|\\ &\le C\varepsilon^d(\varepsilon^{-1}\eta^{d-2})^p.\end{aligned} \tag{2.14}$$

For the second term on the right-hand side of (2.13),

$$\begin{aligned}\int_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p &= \int_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}|\nabla(\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta}))|^p dx\\ &= (\varepsilon\eta)^{-p}\int_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}|\nabla\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})|^p dx\\ &= (\varepsilon\eta)^{d-p}\int_{B(0,\frac{1}{4\eta})\setminus Y_z^s}|\nabla\phi_*^z(y)|^p dy\\ &\le C(\varepsilon\eta)^{d-p}\int_{B(0,\frac{1}{4\eta})\setminus Y_z^s}|y|^{(1-d)p}dy\\ &\le C(\varepsilon\eta)^{d-p}\int_1^{\frac{1}{4\eta}} r^{(1-d)p+d-1}dr,\end{aligned}\tag{2.15}$$

where we have used (2.4) and the fact that $\nabla\phi_*^z$ is L^p integrable near ∂Y_z^s under the condition that Y_z^s is uniformly C^1. It follows from integrating (2.15) radially

$$\int_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p \le \begin{cases} C(\varepsilon\eta)^{d-p} & \text{if } p>d',\\ C(\varepsilon\eta)^{d-p}|\ln\eta| & \text{if } p=d',\\ C\varepsilon^{d-p}\eta^{p(d-2)} & \text{if } p<d'.\end{cases}\tag{2.16}$$

This implies,

$$\left(⨍_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p\right)^{1/p} \le \begin{cases} C\varepsilon^{-1}\eta^{\frac{d-p}{p}} & \text{if } p>d',\\ C\varepsilon^{-1}\eta^{\frac{d-p}{p}}|\ln\eta|^{1/p} & \text{if } p=d',\\ C\varepsilon^{-1}\eta^{d-2} & \text{if } p<d'.\end{cases}\tag{2.17}$$

When $p=d'$ note $\frac{d-p}{p}=d-2$. Combining (2.14) and (2.17) gives the result. □

Recall $\sigma_\varepsilon=\varepsilon\eta^{-\frac{d-2}{2}}$ for $d\ge 3$, and that μ_*^z is given by (2.5).

Lemma 2.2.4. *Suppose $\varepsilon Q_z\subset\Omega$. Then*

$$\left|\int_{\varepsilon Q_z}\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi-\int_{\varepsilon Q_z}\sigma_\varepsilon^{-2}\mu_*^z\phi\right|\le C\varepsilon^{-1}\eta^{d-2}\int_{\varepsilon Q_z}|\nabla\phi|\tag{2.18}$$

where $\phi\in H^1(\varepsilon Q_z)$ and $\phi=0$ in εT_z.

Proof.

$$\begin{aligned}&\int_{\varepsilon Q_z}\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi-\int_{\varepsilon Q_z}\sigma_\varepsilon^{-2}\mu_*^z\phi\\ &=\int_{\varepsilon Q_z\setminus B(\varepsilon y_z,\varepsilon/4)}\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi+\int_{B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z}\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi-\int_{\varepsilon Q_z}\sigma_\varepsilon^{-2}\mu_*^z\phi.\end{aligned}\tag{2.19}$$

We start by bounding the first term on the right-hand side of (2.19). Note that

$$\nabla\chi_{\varepsilon,\eta}=0 \qquad \text{on} \qquad \varepsilon Q_z\setminus B(\varepsilon y_z,\varepsilon/3).$$

This fact combined with Remark 2.2.2 yields

$$\begin{aligned}\left|\int_{\varepsilon Q_z\setminus B(\varepsilon y_z,\varepsilon/4)} \nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi\right| &= \left|\int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)} \nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi\right| \\ &\le C\varepsilon^{-1}\eta^{d-2}\int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)} |\nabla\phi| \\ &\le C\varepsilon^{-1}\eta^{d-2}\int_{\varepsilon Q_z} |\nabla\phi|.\end{aligned} \tag{2.20}$$

For the remaining two terms on the right-hand side of (2.19), we use integration by parts to get

$$\begin{aligned}&\int_{\partial B(\varepsilon y_z,\varepsilon/4)} \left(\frac{\partial\chi_{\varepsilon,\eta}}{\partial n} - \fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n}\right)\cdot(\phi-\alpha) \\ &\qquad + \fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n}\int_{\partial B(\varepsilon y_z,\varepsilon/4)}\phi \\ &\qquad - \int_{\varepsilon Q_z} \sigma_\varepsilon^{-2}\mu_*^z\phi,\end{aligned} \tag{2.21}$$

where n denotes the outward unit normal and α is a constant to be determined. We have used the fact that $\chi_{\varepsilon,\eta}$ is harmonic in the region $B(\varepsilon y_z,\varepsilon/4)\setminus\varepsilon T_z$ and that

$$-\alpha\int_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n} + \alpha\int_{\partial B(\varepsilon y_z,\varepsilon/4)}\fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n} = 0.$$

Recall, $\chi_{\varepsilon,\eta} = \phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})$ on $\partial B(\varepsilon y_z,\varepsilon/4)$. Hence by (2.4)

$$\begin{aligned}&\left|\int_{\partial B(\varepsilon y_z,\varepsilon/4)} \left(\frac{\partial\chi_{\varepsilon,\eta}}{\partial n} - \fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n}\right)\cdot(\phi-\alpha)\right| \\ &\le C\varepsilon^{-1}\eta^{d-2}\int_{\partial B(\varepsilon y_z,\varepsilon/4)} |\phi-\alpha| \\ &\le C\varepsilon^{-1}\eta^{d-2}\left(\frac{1}{\varepsilon}\int_{B(\varepsilon y_z,\varepsilon/4)}|\phi-\alpha| + \int_{B(\varepsilon y_z,\varepsilon/4)}|\nabla\phi|\right),\end{aligned} \tag{2.22}$$

where a trace inequality was used in the last step (see Remark 2.2.5). It is in this step in which we prescribe $\alpha = \fint_{B(\varepsilon y_z,\varepsilon/4}\phi$. Now (2.22) becomes

$$\begin{aligned}&C\varepsilon^{-1}\eta^{d-2}\left(\frac{1}{\varepsilon}\int_{B(\varepsilon y_z,\varepsilon/4)}|\phi - \fint_{B(\varepsilon y_z,\varepsilon/4}\phi| + \int_{B(\varepsilon y_z,\varepsilon/4)}|\nabla\phi|\right) \\ &\le C\varepsilon^{-1}\eta^{d-2}\int_{B(\varepsilon y_z,\varepsilon/4)}|\nabla\phi|,\end{aligned} \tag{2.23}$$

where we have used the Poincaré inequality. Note this is the desired bound. Thus, it suffices to bound

$$\left|\fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n}\int_{\partial B(\varepsilon y_z,\varepsilon/4)}\phi - \int_{\varepsilon Q_z}\sigma_\varepsilon^{-2}\mu_*^z\phi\right|. \tag{2.24}$$

To do so we start by moving the average onto the integral of ϕ. Now note

$$\int_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial \chi_{\varepsilon,\eta}}{\partial n} = (\varepsilon\eta)^{-1} \int_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial \phi_*^z}{\partial n}(\frac{x-\varepsilon y_z}{\varepsilon\eta}) d\sigma(x). \tag{2.25}$$

By a change of variables this becomes

$$\begin{aligned}(\varepsilon\eta)^{d-2} \int_{\partial B(0,\frac{1}{4\eta})} \frac{\partial \phi_*^z}{\partial n} d\sigma(y) &= (\varepsilon\eta)^{d-2} \int_{\partial Y_z^s} \frac{\partial \phi_*^z}{\partial n} d\sigma(y) \\ &= \mu_\star^z (\varepsilon\eta)^{d-2},\end{aligned} \tag{2.26}$$

where we have used the fact that ϕ_*^z is harmonic. Thus (2.24) becomes

$$\mu_*^z (\varepsilon\eta)^{d-2} \left| \fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \phi - \fint_{\varepsilon Q_z} \phi \right|, \tag{2.27}$$

which in view of (2.33), is bounded by

$$C\varepsilon^{-1}\eta^{d-2} \int_{\varepsilon Q_z} |\nabla \phi|. \tag{2.28}$$

This completes the proof.

□

Remark 2.2.5. We have used the following trace inequality in (2.22)

$$\int_{\partial B(x_0,r)} |\phi| \leq \frac{d}{r} \int_{B(x_0,r)} |\phi| + \int_{B(x_0,r)} |\nabla \phi|, \tag{2.29}$$

for $\phi \in H^1(B(x_0,r))$. This follows from writing

$$\int_{\partial B(x_0,r)} |\phi| = \frac{1}{r} \int_{\partial B(x_0,r)} |\phi|((x-x_0)\cdot n) d\sigma(x)$$

and applying the divergence theorem. We replace ϕ in (2.29) with $\phi - \alpha$, where $\alpha = \int_{\varepsilon Q_z} \phi$ and $B(x_0,r) = B(\varepsilon y_z, c\varepsilon)$. This gives

$$\int_{\partial B(\varepsilon y_z,c\varepsilon)} |\phi - \fint_{\varepsilon Q_z} \phi| \leq \frac{C}{\varepsilon} \int_{\varepsilon Q_z} |\phi - \fint_{\varepsilon Q_z} \phi| + \int_{\varepsilon Q_z} |\nabla \phi|, \tag{2.30}$$

where we have used the fact $B(\varepsilon y_z, c\varepsilon) \approx \varepsilon Q_z$ and $\nabla \fint_{\varepsilon Q_z} \phi = 0$. We now apply the Poincaré inequality

$$\int_{\varepsilon Q_z} |\phi - \fint_{\varepsilon Q_z} \phi| \leq C\varepsilon \int_{\varepsilon Q_z} |\nabla \phi| \tag{2.31}$$

to the first term in (2.30). This yields

$$\int_{\partial B(\varepsilon y_z,c\varepsilon)} |\phi - \fint_{\varepsilon Q_z} \phi| \leq C \int_{\varepsilon Q_z} |\nabla \phi|. \tag{2.32}$$

It follows from (2.32) that

$$\left| \fint_{\partial B(\varepsilon y_z,c\varepsilon)} \phi - \fint_{\varepsilon Q_z} \phi \right| \leq C\varepsilon \fint_{\varepsilon Q_z} |\nabla \phi|. \tag{2.33}$$

2.3 Definition of the Corrector: Dimension $d = 2$

We now establish a formal definition for $\chi_{\varepsilon,\eta}$ when $d = 2$. First, for $z \in \mathbb{Z}^2$, let ϕ_*^z be the unique solution to the following exterior problem

$$\begin{cases} -\Delta \phi_*^z = 0 & \text{in } \mathbb{R}^2 \backslash \overline{Y_z^s}, \\ \phi_*^z = 0 & \text{on } \partial Y_z^s, \\ \phi_*^z(x) - \ln |x| = O(1) & \text{as } \ |x| \to \infty. \end{cases} \tag{2.34}$$

It is known as $|x| \to \infty$,

$$\begin{cases} \nabla \phi_*^z(x) = \dfrac{x}{|x|^2} + O(|x|^{-2}), \\ \nabla^2 \phi_*^z(x) = O(|x|^{-2}). \end{cases} \tag{2.35}$$

Recall $y_z = z + x_z$ and $Q_z = z + Q(0,1)$. If $\varepsilon Q_z \subset \Omega$, we let

$$\chi_{\varepsilon,\eta} = \begin{cases} 1 & \text{in } \ \varepsilon Q_z \setminus \varepsilon B(y_z, 1/3), \\ \phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})/|\ln \eta| & \text{in } \ \varepsilon B(y_z, 1/4) \setminus \varepsilon T_z, \\ 0 & \text{in } \ \varepsilon T_z. \end{cases} \tag{2.36}$$

For the region $\varepsilon B(y_z, 1/3) \backslash \varepsilon B(y_z, 1/4)$ we let $\chi_{\varepsilon,\eta}$ solve the following Dirichlet problem,

$$\begin{cases} -\Delta \chi_{\varepsilon,\eta} = 0 & \text{in } \ B(\varepsilon y_z, \varepsilon/3) \setminus \overline{B(\varepsilon y_z, \varepsilon/4)}, \\ \chi_{\varepsilon,\eta} = \phi_*^z(\dfrac{x - \varepsilon y_z}{\varepsilon\eta})/|\ln \eta| & \text{on } \ \partial B(\varepsilon y_z, \varepsilon/4), \\ \chi_{\varepsilon,\eta} = 1 & \text{on } \ \partial B(\varepsilon y_z, \varepsilon/3). \end{cases} \tag{2.37}$$

Finally, as in the $d \geq 3$ case, if Q_ε is not entirely contained in Ω, we prescribe $\chi_{\varepsilon,\eta} = 1$ in εQ_z. By construction $\chi_{\varepsilon,\eta} \in H^1(\Omega)$. Additionally, $\chi_{\varepsilon,\eta} = 1$ in Ω^c and $\chi_{\varepsilon,\eta} = 0$ in $\Omega \backslash \Omega_{\varepsilon,\eta}$.

2.4 Estimates on the Corrector: Dimension d = 2

This section will establish estimates for the corrector $\chi_{\varepsilon,\eta}$ analogous to Lemmas 2.2.1, 2.2.3, and 2.2.4 in the $d \geq 3$ case.

Remark 2.4.1. We remark that on the region $\varepsilon B(y_z, 1/3) \backslash \varepsilon B(y_z, 1/4)$ both of the following estimates hold

$$|\chi_{\varepsilon,\eta} - 1| \leq C |\ln \eta|^{-1} \qquad \text{and} \qquad |\nabla \chi_{\varepsilon,\eta}| \leq C \varepsilon^{-1} |\ln \eta|^{-1}. \tag{2.38}$$

For the first estimate in (2.38) we use the fact that $\chi_{\varepsilon,\eta}$ is both harmonic in the region $\varepsilon B(y_z, 1/3) \backslash \varepsilon B(y_z, 1/4)$ and zero on $\partial B(\varepsilon y_z, \varepsilon/3)$. Thus, applying the maximum principle yields

$$\begin{aligned}\|\chi_{\varepsilon,\eta}-1\|_{L^\infty(B(\varepsilon y_z,\varepsilon/3)\backslash B(\varepsilon y_z,\varepsilon/4))} &\le \|\chi_{\varepsilon,\eta}-1\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/4))}\\ &= \|\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})/|\ln\eta|-1\|_{L^\infty(\partial B(\varepsilon y_z,\varepsilon/4))}\\ &= \|\phi_*^z(\eta^{-1}x)/|\ln\eta|-1\|_{L^\infty(\partial B(0,1/4))}\\ &\le |(\ln|\eta^{-1}/4|+C)/|\ln\eta|-1|\\ &\le C|\ln\eta|^{-1},\end{aligned}$$

where we have used $\phi_*^z(x)=O(1)+\ln|x|$ as $|x|\to\infty$. Note in the last inequality we applied a simple log rule and made use of the fact $\ln\eta/|\ln\eta|=-1$. For the second estimate in (2.38), we again consider $\xi=\chi_{\varepsilon,\eta}(\varepsilon x+\varepsilon y_z)-1$. As was the case for $d\ge 3$, ξ is harmonic. We have from a similar calculation as above,

$$|\xi|\le C|\ln\eta|^{-1}\quad\text{on}\quad \partial B(0,1/4),$$

where we have again used (2.34). Hence by the same Lipschitz estimate for harmonic functions as in $d\ge 3$,

$$|\nabla\xi|\le\|\xi\|_{C^{1,1}(\partial B(0,1/4))}\le C|\ln\eta|^{-1}.$$

This implies

$$|\nabla\chi_{\varepsilon,\eta}|\le C\varepsilon^{-1}|\ln\eta|^{-1}$$

in the region $B(\varepsilon y_z,\varepsilon/3)\backslash B(\varepsilon y_z,\varepsilon/4)$.

Lemma 2.4.2. *Let $\chi_{\varepsilon,\eta}$ be defined in* (2.36). *Then for $1<p<\infty$,*

$$\|\chi_{\varepsilon,\eta}-1\|_{L^p(\Omega_{\varepsilon,\eta})}\le C|\ln\eta|^{-1}, \tag{2.39}$$

where C does not depend on ε or η.

Proof. Let $\varepsilon Q_z\subset\Omega$. Consider

$$\begin{aligned}\int_{\varepsilon Q_z\backslash\varepsilon T_z}|\chi_{\varepsilon,\eta}-1|^p &= \int_{B(\varepsilon y_z,\varepsilon/3)\backslash B(\varepsilon y_z,\varepsilon/4)}|\chi_{\varepsilon,\eta}-1|^p\\ &\quad+\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}|\chi_{\varepsilon,\eta}-1|^p.\end{aligned}\tag{2.40}$$

In this step we have decomposed the Left-hand side of (2.40) into subregions of the cube. Note we have used the fact that $\chi_{\varepsilon,\eta}-1=0$ on $\varepsilon Q_z\backslash B(\varepsilon y_z,\varepsilon/3)$, and hence $\int_{\varepsilon Q_z\backslash B(\varepsilon y_z,\varepsilon/3)}|\chi_{\varepsilon,\eta}-1|=0$. By (2.38), the first term on the right-hand side of (2.40) is bounded by $C\varepsilon^2|\ln\eta|^{-p}$. To bound the second term we first use the definition of $\chi_{\varepsilon,\eta}$ to get

$$\begin{aligned}\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}|\chi_{\varepsilon,\eta}-1|^p &= \int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}\left|\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})/|\ln\eta|-1\right|^p\\ &\le\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}\left|(\ln|\frac{x-\varepsilon y_z}{\varepsilon\eta}|+C)/|\ln\eta|-1\right|^p,\end{aligned}\tag{2.41}$$

where we have used (2.34). Rewriting (2.46) gives

$$\begin{aligned}\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}&\left|(\ln|\frac{x-\varepsilon y_z}{\varepsilon}|+C)/|\ln\eta|-\frac{\ln\eta}{|\ln\eta|}-1\right|^p\\ \leq\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}&\left|\ln|\frac{x-\varepsilon y_z}{\varepsilon}|/|\ln\eta|\right|^p dx+\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}|C/|\ln\eta||^p\,dx.\end{aligned} \tag{2.42}$$

Notice the second term in (2.42) is bounded by $C\varepsilon^2|\ln\eta|^{-p}$. By a simple change of variables the first term becomes

$$C\varepsilon^2|\ln\eta|^{-p}\int_{B(0,1)}|\ln|y||^p dy\leq C\varepsilon^2|\ln\eta|^{-p}. \tag{2.43}$$

Summing over all cubes yields

$$\|\chi_{\varepsilon,\eta}-1\|^p_{L^p(\Omega_{\varepsilon,\eta})}\leq C|\ln\eta|^{-p}$$

and hence the result follows. □

Recall that $\sigma_\varepsilon=\varepsilon|\ln\eta|^{1/2}$ for $d=2$.

Lemma 2.4.3. *Suppose $\varepsilon Q_z\subset\Omega$. Then*

$$\left(⨍_{\varepsilon Q_z}|\nabla\chi_{\varepsilon,\eta}|^p\right)^{1/p}\leq\begin{cases}C\sigma_\varepsilon^{-1}|\ln\eta|^{-1/2} & \text{if } 1<p<2,\\ C\sigma_\varepsilon^{-1} & \text{if } p=2,\\ C\sigma_\varepsilon^{-1}\eta^{\frac{2-p}{p}}|\ln\eta|^{-1/2} & \text{if } 2<p<\infty,\end{cases} \tag{2.44}$$

where the constant C does not depend on ε or η.

Proof. Note that

$$\begin{aligned}\int_{\varepsilon Q_z\backslash\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p=&\int_{B(\varepsilon y_z,\varepsilon/3)\backslash B(\varepsilon y_z,\varepsilon/4)}|\nabla\chi_{\varepsilon,\eta}|^p\\ &+\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p,\end{aligned} \tag{2.45}$$

where we have used the fact that $\nabla\chi_{\varepsilon,\eta}=0$ on $\varepsilon Q_z\backslash B(\varepsilon y_z,\varepsilon/3)$. In light of (2.38), the first term on the right-hand side of (2.45) is bounded by $C\varepsilon^{2-p}|\ln\eta|^{-p}$. To bound the second term we first use the definition of $\chi_{\varepsilon,\eta}$ to get

$$\begin{aligned}\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}|\nabla\chi_{\varepsilon,\eta}|^p dx&=|\ln\eta|^{-p}\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}\left|(\varepsilon\eta)^{-1}\nabla\phi_*^z(\frac{x-\varepsilon y_z}{\varepsilon\eta})\right|^p dx\\ &\leq|\ln\eta|^{-p}\int_{B(\varepsilon y_z,\varepsilon/4)\backslash\varepsilon T_z}\left|\frac{C}{|x-\varepsilon y_z|}\right|^p dx\\ &=C\varepsilon^{2-p}|\ln\eta|^{-p}\int_{c\eta}^1 r^{1-p}dr\end{aligned} \tag{2.46}$$

where we have used (2.35). Integrating in r yields

$$\int_{\varepsilon Q_z} |\nabla \chi_{\varepsilon,\eta}|^p \leq \begin{cases} C\varepsilon^{2-p}|\ln\eta|^{-p} & \text{if } 1 < p < 2, \\ C\varepsilon^{2-p}|\ln\eta|^{-p/2} & \text{if } p = 2, \\ C\varepsilon^{2-p}\eta^{\frac{p(2-p)}{p}}|\ln\eta|^{-p/2} & \text{if } 2 < p < \infty. \end{cases} \tag{2.47}$$

Hence the following estimate holds

$$\left(\fint_{\varepsilon Q_z} |\nabla \chi_{\varepsilon,\eta}|^p \right)^{1/p} \leq \begin{cases} C\varepsilon^{-1}|\ln\eta|^{-1} & \text{if } 1 < p < 2, \\ C\varepsilon^{-1}|\ln\eta|^{-1/2} & \text{if } p = 2, \\ C\varepsilon^{-1}\eta^{\frac{(2-p)}{p}}|\ln\eta|^{-1/2} & \text{if } 2 < p < \infty. \end{cases} \tag{2.48}$$

as desired. □

Lemma 2.4.4. *Suppose $\varepsilon Q_z \subset \Omega$. Then*

$$\left| \int_{\varepsilon Q_z} \nabla \chi_{\varepsilon,\eta} \cdot \nabla \phi - 2\pi\sigma_\varepsilon^{-2} \int_{\varepsilon Q_z} \phi \right| \leq \frac{C}{\varepsilon|\ln\eta|} \int_{\varepsilon Q_z} |\nabla \phi| \tag{2.49}$$

where $\phi \in H^1(\varepsilon Q_z)$ and $\phi = 0$ in εT_z.

Proof. The proof is similar to that of Lemma 2.2.4. Note

$$\begin{aligned} &\int_{\varepsilon Q_z} \nabla \chi_{\varepsilon,\eta} \cdot \nabla \phi - 2\pi\sigma_\varepsilon^{-2} \int_{\varepsilon Q_z} \phi \\ &= \int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)} \nabla \chi_{\varepsilon,\eta} \cdot \nabla \phi + \int_{B(\varepsilon y_z,\varepsilon/4)\setminus \varepsilon T_z} \nabla \chi_{\varepsilon,\eta} \cdot \nabla \phi - \frac{2\pi}{|\ln\eta|} \fint_{\varepsilon Q_z} \phi \\ &= \int_{B(\varepsilon y_z,\varepsilon/3)\setminus B(\varepsilon y_z,\varepsilon/4)} \nabla \chi_{\varepsilon,\eta} \cdot \nabla \phi + \left\{ \int_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial \chi_{\varepsilon,\eta}}{\partial n} \phi - \frac{2\pi}{|\ln\eta|} \fint_{\varepsilon Q_z} \phi \right\} \\ &= I_1 + I_2, \end{aligned} \tag{2.50}$$

where we have integrated by parts and used the fact that $\chi_{\varepsilon,\eta}$ is harmonic in $B(\varepsilon y_z, \varepsilon/6)\setminus\varepsilon T_z$. Note that from (2.38), we have

$$|I_1| \leq \frac{C}{\varepsilon|\ln\eta|} \int_{\varepsilon Q_z} |\nabla \phi|. \tag{2.51}$$

To bound I_2 note

$$\begin{aligned} I_2 &= \int_{\partial B(\varepsilon y_z,\varepsilon/4)} \left(\frac{\partial \chi_{\varepsilon,\eta}}{\partial n} - \fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial \chi_{\varepsilon,\eta}}{\partial n} \right) \cdot (\phi - \alpha) \\ &\quad + \left\{ \fint_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial \chi_{\varepsilon,\eta}}{\partial n} \int_{\partial B(\varepsilon y_z,\varepsilon/4)} \phi - \frac{2\pi}{|\ln\eta|} \fint_{\varepsilon Q_z} \phi \right\} \\ &= I_{21} + I_{22}, \end{aligned} \tag{2.52}$$

where α is a constant to be determined. Recall,

$$\chi_{\varepsilon,\eta} = \phi_\star^z(\frac{x - \varepsilon y_z}{\varepsilon\eta})/|\ln\eta| \quad \text{on} \;\; \partial B(\varepsilon y_z, \varepsilon/4).$$

Thus, by (2.38),

$$\begin{aligned} |I_{21}| &\leq \frac{C}{\varepsilon|\ln\eta|}\int_{\partial B(\varepsilon y_z,\varepsilon/4)} |\phi - \alpha| \\ &\leq \frac{C}{\varepsilon|\ln\eta|}\left(\frac{1}{\varepsilon}\int_{B(\varepsilon y_z,\varepsilon/4)} |\phi-\alpha| + \int_{B(\varepsilon y_z,\varepsilon/4)} |\nabla\phi|\right), \end{aligned} \tag{2.53}$$

where we have used the same trace inequality in Remark 2.2.5. As was the case in Lemma 2.2.4, we pick $\alpha = ⨍_{B(\varepsilon y_z,\varepsilon/4)} \phi$. Applying the Poincaré inequality yields

$$|I_{21}| \leq \frac{C}{\varepsilon|\ln\eta|}\int_{B(\varepsilon y_z,\varepsilon/4)} |\nabla\phi|. \tag{2.54}$$

Finally for I_{22}, we first move the average onto the integral of ϕ. Note

$$\int_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n} = \frac{1}{\varepsilon\eta}\int_{\partial B(\varepsilon y_z,\varepsilon/4)} \nabla\frac{\phi_*^z(x - y_z/\varepsilon\eta)}{|\ln\eta|}\cdot n dx.$$

Changing variables yields

$$\begin{aligned} \int_{\partial B(\varepsilon y_z,\varepsilon/4)} \frac{\partial\chi_{\varepsilon,\eta}}{\partial n} &= \frac{1}{|\ln\eta|\eta}\int_{\partial B(0,1)} \nabla\phi_*^z(\eta^{-1}y)\cdot n dy \\ &= \frac{1}{|\ln\eta|\eta}\left\{\int_{\partial B(0,1)} \frac{\eta^{-1}y}{|\eta^{-1}y|^2}\cdot\frac{y}{|y|}dy + O(\eta^2)\right\} \\ &= \frac{1}{|\ln\eta|}\{2\pi + O(\eta)\}, \end{aligned}$$

where we have used (2.35). This combined with (2.33) gives

$$\begin{aligned} |I_{22}| &\leq \frac{C}{\varepsilon|\ln\eta|}\int_{\varepsilon Q_z} |\nabla\phi| + \frac{C\eta}{\varepsilon^2|\ln\eta|}\int_{\varepsilon Q_z} |\phi| \\ &\leq \frac{C}{\varepsilon|\ln\eta|}\int_{\varepsilon Q_z} |\nabla\phi|, \end{aligned}$$

where we have used Lemma 1.4.1 with $p = 1$ for the last inequality. Combining this with (2.54) and 2.51 completes the proof. □

In this chapter we study the Dirichlet problem with nonhomogeneous boundary conditions,

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_{\varepsilon,\eta}, \\ u_\varepsilon = h & \text{on } \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_{\varepsilon,\eta} \backslash \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega_{\varepsilon,\eta}$ is given in (1.5) and $h \in H^{1/2}(\partial\Omega)$. Standard elliptic PDE theory yields that (3.1) emits a unique solution $u_\varepsilon \in H^1(\Omega_{\varepsilon,\eta})$. Let

$$V(y) = \sum_{z \in \mathbb{Z}^d} \mu_*^z \chi_{Q(z,1)}(y), \tag{3.2}$$

where μ_*^z is given in (2.5) for $d \geq 3$, $\mu_*^z = 2\pi$ when $d = 2$, and $\chi_{Q(z,1)}$ denotes the characteristic function of the cell $Q(z,1)$. Note that $c_0 \leq V(y) \leq c_1$ for some $c_0, c_1 > 0$. Let

$$V_\varepsilon(x) = V(x/\varepsilon). \tag{3.3}$$

Finally, let $u_{0,\varepsilon}$ be the solution to the boundary value problem in Ω,

$$\begin{cases} -\Delta u_{0,\varepsilon} + \sigma_\varepsilon^{-2} V_\varepsilon(x) u_{0,\varepsilon} = 0 & \text{in } \Omega, \\ u_{0,\varepsilon} = h & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

which we call the intermediate problem, for a Schrödinger operator $-\Delta + \sigma_\varepsilon^{-2} V_e$. The parameter σ_ε is given in (1.6) and depends on both ε and η.

Our goal is to show that the solution u_ε of (3.1) is well approximated by $\chi_{\varepsilon,\eta} u_{0,\varepsilon}$, where $\chi_{\varepsilon,\eta}$ is the corrector defined extensively in the previous chapter for $\Omega_{\varepsilon,\eta}$. The next two sections will be dedicated to showing the convergence rates for this approximation. The first will handle the case for which $d \geq 3$, while the second treats the case $d = 2$.

3.1 Dimension $d \geq 3$

This section is dedicated to proving the following theorem.

Theorem 3.1.1. *Let u_ε be a solution to* (3.1) *and $u_{0,\varepsilon}$ be a solution to* (3.4). *Let*

$$r_\varepsilon = u_\varepsilon - \chi_{\varepsilon,\eta} u_{0,\varepsilon}, \tag{3.5}$$

where $\chi_{\varepsilon,\eta}$ is defined in Chapter 2. Assume further that $u_{0,\varepsilon} \in W^{1,p}(\Omega)$ for some $2 < p < \infty$. Then for $d \geq 3$,

$$\|\nabla r_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})} \leq \begin{cases} C\eta^{\frac{d-2}{2}} \left(\frac{1}{\sigma_\varepsilon}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} + \|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})}\right) & \text{if } p \geq d, \\ C\eta^{d/q} \left(\frac{1}{\sigma_\varepsilon}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} + \|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})}\right) & \text{if } 2 < p < d, \end{cases} \tag{3.6}$$

where $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ *and* C *does not depend on* ε *or* η.

Note that $\chi_{\varepsilon,\eta} \in W^{1,p}(\Omega)$ for any $p > 2$. Moreover, $\chi_{\varepsilon,\eta} = 1$ on $\partial\Omega$ and $\chi_{\varepsilon,\eta} = 0$ on $\partial\Omega_{\varepsilon,\eta}\backslash\partial\Omega$. It follows that $r_\varepsilon \in H_0^1(\Omega_{\varepsilon,\eta})$. The proof of Theorem 3.1.1 will rely heavily on the estimates for $\chi_{\varepsilon,\eta}$.

Lemma 3.1.2. *Let* $\mathcal{F}$ *be the set of epsilon cubes* εQ_z *with non-empty intersection of* $\partial\Omega$. *Then for each* $\varepsilon Q_z \in \mathcal{F}$

$$\int_{\varepsilon Q_z \cap \Omega} |u| \leq C\varepsilon \int_{2\varepsilon Q_z \cap \Omega} |\nabla u| \tag{3.7}$$

for any $u \in H^1(\varepsilon Q_z)$ *with* $u = 0$ *on* $\overline{\Omega^c \cap \varepsilon Q_z}$.

Proof. By a change of coordinates we may assume that $\Omega = \{(x', x_n) : x_n > 0\}$ and $\varepsilon Q = \{0 \leq x_i \leq \varepsilon\}$. Then by the fundamental theorem of calculus

$$\begin{aligned} |u(x', x_n)| &= |u(x', x_n) - u(x', 0)| \\ &\leq \int_0^{x_n} |\frac{\partial u}{\partial x_n}(x', t)| dt \\ &\leq \int_0^{\varepsilon} |\nabla u(x', t)| dt, \end{aligned} \tag{3.8}$$

where we have used the fact that $dist(x_n, \partial\Omega) \leq \varepsilon$. Integrating both sides in x' gives

$$\begin{aligned} \int_{\varepsilon Q'} |u(x', t)| dx' &\leq C \int_{\varepsilon Q'} \int_0^{\varepsilon} |\nabla u(x', t)| dt dx' \\ &= C \int_{\varepsilon Q} |\nabla u| dx. \end{aligned} \tag{3.9}$$

Now integrating both sides in t we get

$$\begin{aligned} \int_{\varepsilon Q} |u| dx &\leq C \int_0^{\varepsilon} \int_{\varepsilon Q} |\nabla u| dx dt \\ &= C\varepsilon \int_{\varepsilon Q} |\nabla u|. \end{aligned} \tag{3.10}$$

This completes the proof for the upper half space, which implies the result. □

Lemma 3.1.3. *Let* $\phi \in H_0^1(\Omega_{\varepsilon,\eta})$. *Then*

$$\left| \int_{\Omega_{\varepsilon,\eta}} \nabla \chi_{\varepsilon,\eta} \cdot \nabla \phi - \int_{\Omega_{\varepsilon,\eta}} \sigma_\varepsilon^{-2} V_\varepsilon(x) \phi \right| \leq C\varepsilon^{-1}\eta^{d-2} \int_{\Omega_{\varepsilon,\eta}} |\nabla \phi|, \tag{3.11}$$

where $V_\varepsilon(x)$ *is defined in* (3.3).

Proof. Let $\phi \in H_0^1(\Omega_{\varepsilon,\eta})$. Extend ϕ to $\mathbb{R}^d$ by zero. Note that

$$\begin{aligned}&\left|\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi - \int_{\Omega_{\varepsilon,\eta}} \sigma_\varepsilon^{-2}V_\varepsilon(x)\phi\right| \\ &\leq \sum_{\varepsilon Q_z\in\Omega'_{\varepsilon,\eta}} \left|\int_{\varepsilon Q_z} \nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi - \int_{\varepsilon Q_z} \sigma_\varepsilon^{-2}V_\varepsilon(x)\phi\right| + \left|\int_{\mathcal{F}} \sigma_\varepsilon^{-2}V_\varepsilon(x)\phi\right|,\end{aligned} \tag{3.12}$$

Where $\Omega'_{\varepsilon,\eta}$ is the union of cubes εQ_z lying entirely inside of Ω and $\mathcal{F}$ is the union of cubes with non-empty intersection with the boundary of Ω. By Lemma 2.2.4,

$$\begin{aligned}&\sum_{\varepsilon Q_z\in\Omega'_{\varepsilon,\eta}} \left|\int_{\varepsilon Q_z} \nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi - \int_{\varepsilon Q_z} \sigma_\varepsilon^{-2}V_\varepsilon(x)\phi\right| \\ &\leq C\varepsilon^{-1}\eta^{d-2} \sum_{\varepsilon Q_z\in\Omega'_{\varepsilon,\eta}} \left|\int_{\varepsilon Q_z} |\nabla\phi|\right|,\end{aligned} \tag{3.13}$$

which, after summing cubes is clearly bounded by the right-hand side of (3.11). For the second term on the right-hand side of (3.12),

$$\begin{aligned}\left|\int_{\mathcal{F}} \sigma_\varepsilon^{-2}V_\varepsilon(x)\phi\right| &\leq C\sigma_\varepsilon^{-2} \sum_{\varepsilon Q_z\in\mathcal{F}} \int_{\varepsilon Q_z\cap\Omega} |\phi| \\ &\leq C\varepsilon\sigma_\varepsilon^{-2} \sum_{\varepsilon Q_z\in\mathcal{F}} \int_{2\varepsilon Q_z\cap\Omega} |\nabla\phi| \\ &\leq C\varepsilon^{-1}\eta^{d-2} \int_{\Omega_{\varepsilon,\eta}} |\nabla\phi|,\end{aligned}$$

where we have used Lemma 3.1.2 the second inequality. □

Proof of Theorem 3.1.1. Let u_ε be a solution to (3.1) and $u_{0,\varepsilon}$ be a solution to (3.4). Assume further that $u_{0,\varepsilon} \in W^{1,p}(\Omega)$ for some $p > 2$. Let $r_\varepsilon = u_\varepsilon - \chi_{\varepsilon,\eta}u_{0,\varepsilon}$. Then for any $\phi \in H_0^1(\Omega_{\varepsilon,\eta})$,

$$\begin{aligned}\int_{\Omega_{\varepsilon,\eta}} \nabla r_\varepsilon\cdot\nabla\phi &= -\int_{\Omega_{\varepsilon,\eta}} (\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi)u_{0,\varepsilon} - \int_{\Omega_{\varepsilon,\eta}} \chi_{\varepsilon,\eta}\left(\nabla u_{0,\varepsilon}\cdot\nabla\phi\right) \\ &= -\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla(u_{0,\varepsilon}\phi) + \int_{\Omega_{\varepsilon,\eta}} [\nabla(\chi_{\varepsilon,\eta}-1)\cdot\nabla u_{0,\varepsilon}]\phi \\ &\quad - \int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\left(\nabla u_{0,\varepsilon}\cdot\nabla\phi\right) - \int_{\Omega_{\varepsilon,\eta}} \left(\nabla u_{0,\varepsilon}\cdot\nabla\phi\right).\end{aligned} \tag{3.14}$$

Integrating by parts, the right-hand side of (3.14) becomes

$$\begin{aligned}&-\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla(u_{0,\varepsilon}\phi) - 2\int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\left(\nabla u_{0,\varepsilon}\cdot\nabla\phi\right) \\ &\qquad - \int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\Delta u_{0,\varepsilon}\cdot\phi - \int_{\Omega_{\varepsilon,\eta}} \left(\Delta u_{0,\varepsilon}\cdot\phi\right).\end{aligned}$$

Substituting (3.4) we get

$$-\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla(u_{0,\varepsilon}\phi) - \sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} V_\varepsilon u_{0,\varepsilon}\phi - 2\int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\left(\nabla u_{0,\varepsilon}\cdot\nabla\phi\right) \\ - \sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)V_\varepsilon u_{0,\varepsilon}\cdot\phi.$$

Note that $V_\varepsilon(x) \le c_2$. We now apply Lemma 3.1.3 to get

$$\begin{aligned}\left|\int_{\Omega_{\varepsilon,\eta}} \nabla r_\varepsilon\cdot\nabla\phi\right| &\le C\varepsilon^{-1}\eta^{d-2}\int_{\Omega_{\varepsilon,\eta}} |\nabla(u_{0,\varepsilon}\phi)| + 2\int_{\Omega_{\varepsilon,\eta}} |\chi_{\varepsilon,\eta}-1||\nabla u_{0,\varepsilon}||\nabla\phi| \\ &\quad + C\sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} |\chi_{\varepsilon,\eta}-1||u_{0,\varepsilon}||\phi| \\ &\le C\varepsilon^{-1}\eta^{d-2}\int_{\Omega_{\varepsilon,\eta}} |\nabla u_{0,\varepsilon}||\phi| + C\varepsilon^{-1}\eta^{d-2}\int_{\Omega_{\varepsilon,\eta}} |\nabla\phi||u_{0,\varepsilon}| \\ &\quad + 2\int_{\Omega_{\varepsilon,\eta}} |\chi_{\varepsilon,\eta}-1||\nabla u_{0,\varepsilon}||\nabla\phi| + C\sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} |\chi_{\varepsilon,\eta}-1||u_{0,\varepsilon}||\phi|.\end{aligned}$$

Applying Hölder's inequality yields

$$\begin{aligned}\left|\int_{\Omega_{\varepsilon,\eta}} \nabla r_\varepsilon\cdot\nabla\phi\right| \le & C\sigma_\varepsilon^{-1}\eta^{\frac{d-2}{2}}\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} \\ & + C\sigma_\varepsilon^{-1}\eta^{\frac{d-2}{2}}\|\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} \\ & + C\|\chi_{\varepsilon,\eta}-1\|_{L^q(\Omega_{\varepsilon,\eta})}\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} \\ & + C\sigma_\varepsilon^{-2}\|\chi_{\varepsilon,\eta}-1\|_{L^q(\Omega_{\varepsilon,\eta})}\|\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})},\end{aligned} \tag{3.15}$$

where $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$. We apply the Poincaré inequality $\|\phi\|_{L^2(\Omega_{\varepsilon,\eta})} \le C\sigma_\varepsilon\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}$ in (3.15) to get

$$\begin{aligned}\left|\int_{\Omega_{\varepsilon,\eta}} \nabla r_\varepsilon\cdot\nabla\phi\right| &\le C\eta^{\frac{d-2}{2}}\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\left(\sigma_\varepsilon^{-1}\|u_{0,\varepsilon}\|_{L^p(\Omega)} + \|\nabla u_{0,\varepsilon}\|_{L^p(\Omega)}\right) \\ &\quad + C\|\chi_{\varepsilon,\eta}-1\|_{L^q(\Omega_{\varepsilon,\eta}))}\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\left(\sigma_\varepsilon^{-1}\|u_{0,\varepsilon}\|_{L^p(\Omega)} + \|\nabla u_{0,\varepsilon}\|_{L^p(\Omega)}\right).\end{aligned}$$

Choosing $\phi = r_\varepsilon$, we obtain

$$\begin{aligned}&\|\nabla r_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})} \\ &\quad \le C\left(\eta^{\frac{d-2}{2}} + \|\chi_{\varepsilon,\eta}-1\|_{L^q(\Omega_{\varepsilon,\eta}))}\right)\left(\sigma_\varepsilon^{-1}\|u_{0,\varepsilon}\|_{L^p(\Omega)} + \|\nabla u_{0,\varepsilon}\|_{L^p(\Omega)}\right)\end{aligned} \tag{3.16}$$

Applying Lemma 2.2.1 in (3.16) gives (3.6). □

3.2 Dimension $d = 2$

In this section we consider the case where $d = 2$. We will prove the following analogous result to Theorem 3.1.1.

Theorem 3.2.1. *Let u_ε be a solution of* (3.1) *and $u_{0,\varepsilon}$ be a solution of* (3.4)*. Assume that $u_{0,\varepsilon} \in W^{1,p}(\Omega)$ for some $2 < p < \infty$. Let $r_\varepsilon = u_\varepsilon - \chi_{\varepsilon,\eta} u_{0,\varepsilon}$. Then*

$$\|\nabla r_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})} \leq C|\ln\eta|^{-1/2}\left(\sigma_\varepsilon^{-1}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} + \|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})}\right), \tag{3.17}$$

where C does not depend on ε.

The following lemma will play the role of Lemma 3.1.3.

Lemma 3.2.2. *Let $\chi_{\varepsilon,\eta}$ be the corrector defined in Chapter 2. Then for any $\phi \in H_0^1(\Omega_{\varepsilon,\eta})$,*

$$\left|\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi - 2\pi\sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}}\phi\right| \leq \frac{C}{\varepsilon|\ln\eta|}\int_{\Omega_{\varepsilon,\eta}}|\nabla\phi|. \tag{3.18}$$

Proof. Let $\phi \in H_0^1(\Omega_\varepsilon)$. Extend ϕ by zero to $\mathbb{R}^2$. Note that the left-hand side of (3.18) is bounded by

$$\sum_{\varepsilon Q_z\in\Omega'_{\varepsilon,\eta}}\left|\int_{\varepsilon Q_z}\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi - 2\pi\sigma_\varepsilon^{-2}\int_{\varepsilon Q_z}\phi\right| + \left|\int_{\mathcal{F}} 2\pi\sigma_\varepsilon^{-2}\phi\right|, \tag{3.19}$$

where $\Omega'_{\varepsilon,\eta}$, is the union of cubes εQ_z that lie entirely in Ω, and $\mathcal{F}$ is the union of cubes εQ_z with non-empty intersections with $\partial\Omega$. By Lemma 2.4.4,

$$\begin{aligned}&\sum_{z\in\Omega'_\varepsilon}\left|\int_{\varepsilon Q_z}\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi - 2\pi\sigma_\varepsilon^{-2}\int_{\varepsilon Q_z}\phi\right|\\ &\leq \frac{C}{\varepsilon|\ln\eta|}\sum_{z\in\Omega'_\varepsilon}\left|\int_{\varepsilon Q_z}|\nabla\phi|\right|,\end{aligned} \tag{3.20}$$

which, after summing cubes is clearly bounded by the right-hand side of (3.18). By the Poincaré-Sobolev inequality in Lemma 3.1.2 the second term on the right-hand side of (3.19) is bounded by

$$\begin{aligned}\left|\int_{\mathcal{F}}\sigma_\varepsilon^{-2}V_\varepsilon(x)\phi\right| &\leq C\sigma_\varepsilon^{-2}\sum_{\varepsilon Q_z\in\mathcal{F}}\int_{\varepsilon Q_z\cap\Omega}|\phi|\\ &\leq C\varepsilon\sigma_\varepsilon^{-2}\sum_{\varepsilon Q_z\in\mathcal{F}}\int_{2\varepsilon Q_z\cap\Omega}|\nabla\phi|\\ &\leq \frac{C}{\varepsilon|\ln\eta|}\int_{\Omega_{\varepsilon,\eta}}|\nabla\phi|.\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.2.1. This proof will follow similarly to the proof of Theorem 3.1.1. Let u_ε be a solution of (3.1) and $u_{0,\varepsilon}$ be a solution of (3.4). Assume further that $u_{0,\varepsilon} \in W^{1,p}(\Omega)$ for some $2 < p < \infty$. Let $r_\varepsilon = u_\varepsilon - \chi_{\varepsilon,\eta}u_{0,\varepsilon}$. As was in the dimension $d \geq 3$ case, for $\phi \in H_0^1(\Omega_{\varepsilon,\eta})$ we have

$$\begin{aligned}\int_{\Omega_{\varepsilon,\eta}} \nabla r_\varepsilon \cdot \nabla\phi &= -\int_{\Omega_{\varepsilon,\eta}} (\nabla\chi_{\varepsilon,\eta}\cdot\nabla\phi)u_{0,\varepsilon} - \int_{\Omega_{\varepsilon,\eta}} \chi_{\varepsilon,\eta}\,(\nabla u_{0,\varepsilon}\cdot\nabla\phi)\\ &= -\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla(u_{0,\varepsilon}\phi) + \int_{\Omega_{\varepsilon,\eta}} [\nabla(\chi_{\varepsilon,\eta}-1)\cdot\nabla u_{0,\varepsilon}]\phi \\ &\qquad - \int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\,(\nabla u_{0,\varepsilon}\cdot\nabla\phi) - \int_{\Omega_{\varepsilon,\eta}} (\nabla u_{0,\varepsilon}\cdot\nabla\phi)\,.\end{aligned} \tag{3.21}$$

Integrating by parts, the right-hand side of (3.21) becomes

$$\begin{aligned}&-\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla(u_{0,\varepsilon}\phi) - 2\int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\,(\nabla u_{0,\varepsilon}\cdot\nabla\phi)\\ &\qquad - \int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\Delta u_{0,\varepsilon}\cdot\phi - \int_{\Omega_{\varepsilon,\eta}} (\Delta u_{0,\varepsilon}\cdot\phi)\,.\end{aligned}$$

Substituting (3.4) we get

$$\begin{aligned}&-\int_{\Omega_{\varepsilon,\eta}} \nabla\chi_{\varepsilon,\eta}\cdot\nabla(u_{0,\varepsilon}\phi) + 2\pi\sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} u_{0,\varepsilon}\phi - 2\int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)\,(\nabla u_{0,\varepsilon}\cdot\nabla\phi)\\ &\qquad - 2\pi\sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} (\chi_{\varepsilon,\eta}-1)u_{0,\varepsilon}\phi.\end{aligned}$$

We apply Lemma 3.2.2 to obtain

$$\begin{aligned} I &= \left|\int_{\Omega_{\varepsilon,\eta}} \nabla r_\varepsilon\cdot\nabla\phi\right| \\ &\leq \frac{C}{\varepsilon|\ln\eta|}\int_{\Omega_{\varepsilon,\eta}} |\nabla(u_{0,\varepsilon}\phi)| + 2\int_{\Omega_{\varepsilon,\eta}} |\chi_{\varepsilon,\eta}-1||\nabla u_{0,\varepsilon}||\nabla\phi| \\ &\qquad + 2\pi\sigma_\varepsilon^{-2}\int_{\Omega_{\varepsilon,\eta}} |\chi_{\varepsilon,\eta}-1|\;|u_{0,\varepsilon}||\phi|.\end{aligned}$$

Now applying Hölder's inequality we have

$$\begin{aligned} I \leq C{\sigma_\varepsilon}^{-1}|\ln\eta|^{-\frac{1}{2}}&\left(\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})} + \|\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})}\right)\\ &+ C\|\chi_{\varepsilon,\eta}-1\|_{L^q(\Omega_{\varepsilon,\eta})}(\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|\nabla u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})}\\ &\qquad + \sigma_\varepsilon^{-2}\|\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\|u_{0,\varepsilon}\|_{L^p(\Omega_{\varepsilon,\eta})}),\end{aligned} \tag{3.22}$$

where $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$. By applying the Poincaré inequality $\|\phi\|_{L^2(\Omega_{\varepsilon,\eta})} \leq \sigma_\varepsilon\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}$ and Lemma 2.4.2 to (3.22), we obtain

$$I \leq C|\ln\eta|^{-\frac{1}{2}}\|\nabla\phi\|_{L^2(\Omega_{\varepsilon,\eta})}\left(\|\nabla u_{0,\varepsilon}\|_{L^p(\Omega)} + \sigma_\varepsilon^{-1}\|u_{0,\varepsilon}\|_{L^p(\Omega)}\right).$$

Choosing $\phi = r_\varepsilon$ gives (3.17). □

Chapter 4 An Intermediate Problem

In this Chapter we consider the boundary value problem for the Schrödinger operator,

$$\begin{cases} -\Delta u + \lambda^2 V(x) u = F & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $V = V(x)$ is a potential satisfying the condition $0 < \mu_0 \le V \le \mu_1$.

Lemma 4.0.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $\lambda > 0$. Suppose that $u \in H^1(B(x_0, 2r) \cap \Omega)$ and*

$$\begin{cases} -\Delta u + \lambda^2 V(x) u = 0 & \text{in } \ B(x_0, 2r) \cap \Omega, \\ u = 0 & \text{on } \ B(x_0, 2r) \cap \partial\Omega, \end{cases} \tag{4.2}$$

where $x_0 \in \partial\Omega$ and $0 < r < r_0$. Then

$$\sup_{B(x_0,r)\cap\Omega} |u| \le C \left(\fint_{B(x_0,2r)\cap\Omega} |u|^2 \right)^{1/2}, \tag{4.3}$$

where C depends only on d, Ω, and (μ_0, μ_1). Moreover,

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u|^p \right)^{1/p} \le C \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \right)^{1/2} \tag{4.4}$$

where $2 < p < 3 + \delta$ for $d \ge 3$, $2 < p < 4 + \delta$ for $d = 2$, and $\delta > 0$ depends on d and Ω. The constant C in (4.4) depends on d, p, Ω, and (μ_0, μ_1). If Ω is a bounded C^1 domain, the estimate (4.4) holds for any $2 < p < \infty$.

Proof. Let $\phi \in C_0^\infty(B(x_0, 2r))$. Then

$$\int_{B(x_0,2r)\cap\Omega} \nabla u \cdot \nabla(u\phi^2) + \int_{B(x_0,2r)\cap\Omega} \lambda^2 V u \cdot u\phi^2 = 0.$$

By the product rule

$$\int_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \phi^2 + \int_{B(x_0,2r)\cap\Omega} \nabla u \cdot u \nabla\phi^2 + \int_{B(x_0,2r)\cap\Omega} \lambda^2 V u \cdot u\phi^2 = 0.$$

Hence, by Cauchy's inequality

$$\int_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \phi^2 + \int_{B(x_0,2r)\cap\Omega} \lambda^2 V u^2 \phi^2 \le C \int_{B(x_0,2r)\cap\Omega} |u|^2 |\nabla\phi|^2. \tag{4.5}$$

We choose a cut-off function ϕ such that

$$\phi(x) = \begin{cases} 1 & \text{if } x \in B(x_0, tr), \\ 0 & \text{if } x \in B(x_0, sr)^c, \\ 0 < \phi \le 1 & \text{if } x \in B(x_0, sr) \backslash B(x_0, tr), \end{cases} \tag{4.6}$$

for some $0 < t < s \leq 2$. Then $\nabla\phi \leq \dfrac{C}{r(t-s)}$. We obtain from (4.5)

$$\lambda^2 \int_{B(x_0,tr)\cap\Omega} u^2 \leq \frac{C}{r^2(s-t)^2} \int_{B(x_0,st)\cap\Omega} u^2,$$

where by the Poincaré inequality the gradient on the left-hand side has been absorbed by the right-hand side. Dividing by λ^2 in combination with the fact $\int_{B(x_0,tr)\cap\Omega} u^2 \leq \int_{B(x_0,sr)\cap\Omega} u^2$ gives

$$\int_{B(x_0,tr)\cap\Omega} u^2 \leq \frac{C}{(s-t)^2((1+\lambda)r)^2} \int_{B(x_0,sr)\cap\Omega} u^2. \tag{4.7}$$

We iterate (4.7) in the following manner

$$\begin{aligned}
\left(\fint_{B(x_0,tr)\cap\Omega} u^2\right)^{1/2} &\leq \frac{C_k}{\lambda^2 r^2(s-t)^2+1}\left(\int_{B(x_0,tr+\frac{1}{k}(sr-tr))\cap\Omega} u^2\right)^{1/2} \\
&\leq \left[\frac{C_k}{\lambda^2 r^2(s-t)^2+1}\right]^2\left(\int_{B(x_0,tr+\frac{2}{k}(sr-tr))\cap\Omega} u^2\right)^{1/2},
\end{aligned} \tag{4.8}$$

for any $k \geq 1$, where $1 \leq t < s \leq 2$. By repeating (4.8) k times we get

$$\begin{aligned}
\left(\fint_{B(x_0,tr)\cap\Omega} u^2\right)^{1/2} &\leq \frac{C_{k,t,s}}{(1+(\lambda r)^2)^k}\left(\int_{B(x_0,sr)\cap\Omega} u^2\right)^{1/2} \\
&\leq \frac{C_{k,t,s}}{(1+\lambda r)^{2k}}\left(\int_{B(x_0,sr)\cap\Omega} u^2\right)^{1/2}.
\end{aligned} \tag{4.9}$$

Now since

$$\Delta u = \lambda^2 V u \ \text{ in } B(x_0,2r)\cap\Omega \ \text{ and } \ u = 0 \quad \text{on} \ \ B(x_0,2r)\cap\partial\Omega,$$

the boundary L^∞ estimates for Laplace's equation in Lipschitz domains give

$$\begin{aligned}
\left(\fint_{B(x_0,tr)\cap\Omega} |u|^p\right)^{1/p} &\leq C_{t,s}\left(\fint_{B(x_0,sr)\cap\Omega} |u|^2\right)^{1/2} + C_{t,s} r^2\left(\fint_{B(x_0,sr)\cap\Omega} |\lambda^2 V u|^q\right)^{1/q} \\
&\leq C_{t,s}\left(\fint_{B(x_0,sr)\cap\Omega} |u|^2\right)^{1/2} + C_{t,s} r^2\lambda^2\left(\fint_{B_(x_0,sr)\cap\Omega} |u|^q\right)^{1/q},
\end{aligned}$$

where $0 < \frac{1}{q} - \frac{1}{p} < \frac{2}{d}$, and

$$\sup_{B(x_0,tr)\cap\Omega} |u| \leq C_{t,s}\left(\fint_{B(x_0,sr)\cap\Omega} |u|^2\right)^{1/2} + C_{t,s} r^2\lambda^2\left(\fint_{B(x_0,sr)\cap\Omega} |u|^q\right)^{1/q},$$

where $q > (d/2)$. Hence,

$$\begin{aligned}\sup_{B(x_0,r)\cap\Omega} |u| &\le C(1+r^2\lambda^2)\left(\fint_{B(x_0,\frac{3}{2}r)\cap\Omega} |u|^2\right)^{1/2} \\ &\le C\left(\fint_{B(x_0,2r)\cap\Omega} |u|^2\right)^{1/2},\end{aligned} \tag{4.10}$$

where we have used (4.9). Note that for the $W^{1,p}$ estimate for Laplace's equation in C^1 domains [6] gives

$$\begin{aligned}\left(\fint_{B(x_0,tr)\cap\Omega} |\nabla u|^p\right)^{1/p} \le\; & C\left(\fint_{B(x_0,sr)\cap\Omega} |\nabla u|^2\right)^{1/2} \\ &+ Cr\lambda^2\left(\fint_{B(x_0,sr)\cap\Omega} |u|^q\right)^{1/q}\end{aligned} \tag{4.11}$$

where $1 < t < s < 2$ and $0 < \frac{1}{q} - \frac{1}{p} < \frac{1}{d}$. If Ω is a Lipschitz domain, we need to impose the additional conditions that $2 < p < 3 + \delta$ for $d \ge 3$, and $2 < p < 4 + \delta$ for $d = 2$, where $\delta > 0$ depends on d and Ω. Thus by (4.3) and (4.9),

$$\begin{aligned}\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u|^p\right)^{1/p} &\le \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla u|^2\right)^{1/2} + Cr\lambda^2\left(\fint_{B(x_0,\frac{3}{2}r)\cap\Omega} |u|^2\right)^{1/2} \\ &\le \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla u|^2\right)^{1/2} + Cr^{-1}\left(\fint_{B(x_0,2r)\cap\Omega} |u|^2\right)^{1/2} \\ &\le C\left(\fint_{B(x_0,2r)\cap\Omega} |\nabla u|^2\right)^{1/2},\end{aligned} \tag{4.12}$$

where we have used a Poincaré inequality and the fact that $u = 0$ on $B(x_0, 2r) \cap \partial\Omega$ for the last step. □

Remark 4.0.2. Suppose that $u \in H^1(B(x_0, 2r))$ and $-\Delta u + \lambda^2 V u = 0$ in $B(x_0, 2r)$. Then

$$\sup_{B(x_0,r)} |u| \le C\left(\fint_{B(x_0,2r)} |u|^2\right)^{1/2} \tag{4.13}$$

$$\sup_{B(x_0,r)} |\nabla u| \le C\left(\fint_{B(x_0,2r)} (|\nabla u|^2 + \lambda^2|u|^2)\right)^{1/2}, \tag{4.14}$$

where C depends on d and (μ_0, μ_1).

The proof is similar to that of Lemma 4.0.1. Note that the analogous estimate to (4.11) is given by

$$\sup_{B(x_0,r)} |\nabla u| \le C\left(\fint_{B(x_0,sr)} |\nabla u|^2\right)^{1/2} + Cr\lambda^2\left(\fint_{B(x_0,sr)} |u|^q\right)^{1/q}, \tag{4.15}$$

for $q > d$. It follows that

$$
\begin{aligned}
\sup_{B(x_0,r)} |\nabla u| &\le \left(\fint_{B(x_0,2r)} |\nabla u|^2 \right)^{1/2} + Cr\lambda^2 \left(\fint_{B(x_0,\frac{3}{2}r)} |u|^2 \right)^{1/2} \\
&\le \left(\fint_{B(x_0,2r)} |\nabla u|^2 \right)^{1/2} + C\lambda \left(\fint_{B(x_0,2r)} |u|^2 \right)^{1/2} \\
&\le C \left(\fint_{B(x_0,2r)} (|\nabla u|^2 + \lambda^2 |u|^2) \right)^{1/2}.
\end{aligned} \tag{4.16}
$$

We call an operator sublinear if there exists a constant K such that

$$|T(f+g)| \le K\{|T(f)| + |T(g)|\}. \tag{4.17}$$

The following theorem was proved in [11].

Theorem 4.0.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let T be a bounded sublinear operator on $L^2(\Omega)$ with*

$$\|T\|_{L^2 \to L^2} \le c_0. \tag{4.18}$$

Let $q > 2$. Suppose

$$\left(\fint_{\Omega \cap B(x_0,r)} |T(g)|^q \right)^{1/q} \le N \left(\fint_{\Omega \cap B(x_0,2r)} |T(g)|^2 \right)^{1/2} \tag{4.19}$$

for any ball $B(x_0,r)$ with the property that $0 < r < r_0$ and either $B(x_0,4r) \subset \Omega$ or $x_0 \in \partial\Omega$ and for any $g \in C_0^\infty(\Omega)$ with supp$(g) \subset \Omega \backslash B(x_0,4r)$. Then for any $G \in L^p(\Omega)$,

$$\|T(G)\|_{L^p(\Omega)} \le C\|G\|_{L^p(\Omega)},$$

where $2 < p < q$ and C_p depends on at most $p, q, C_0, N, r_0, \Omega,$ and K.

Remark 4.0.4. In Theorem 4.0.3 we may interchange balls $B(x_0,r)$ with cubes $Q(x_0,r)$.

Theorem 4.0.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u \in W_0^{1,2}(\Omega)$ be the solution of*

$$\begin{cases} -\Delta u + \lambda^2 V(x) u = F & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.20}$$

where $F \in L^2(\Omega)$. Suppose $F \in L^p(\Omega)$, where $2 < p < 3 + \delta$ for $d \ge 3$, and $2 < p < 4 + \delta$ for $d = 2$. Then

$$\lambda \|\nabla u\|_{L^p(\Omega)} + \lambda^2 \|u\|_{L^p(\Omega)} \le C\|F\|_{L^p(\Omega)}, \tag{4.21}$$

where C depends on $d, p, \Omega,$ and (μ_0, μ_1). If Ω is a bounded C^1 domain the estimate (4.21) holds for any $2 < p < \infty$.

Proof. It suffices to apply Theorem 4.0.3 to the operator

$$T_\lambda(F) = \lambda|\nabla u| + \lambda^2|u|.$$

This operator is sublinear by construction. To check condition (4.18) we note when $p = 2$

$$\int_\Omega |\nabla u|^2 + \int_\Omega \lambda^2 V u^2 = \int_\Omega F \cdot u.$$

Since $V \le \mu_1$ we have

$$\begin{aligned}
\int_\Omega |\nabla u|^2 + \lambda^2 \int_\Omega u^2 &\le C \int_\Omega F \cdot u \\
&= C \int_\Omega F \lambda \lambda^{-1} u \\
&\le \frac{c}{\lambda^2} \int_\Omega |F|^2 + \frac{\lambda^2}{2} \int_\Omega |u|^2,
\end{aligned}$$

where we have used Cauchy-Schwartz in the last inequality. This implies

$$\int_\Omega |\nabla u|^2 + \lambda^2 \int_\Omega |u|^2 \le \frac{c}{\lambda^2} \int_\Omega |F|^2$$

and further,

$$\lambda^2 \int_\Omega |\nabla u|^2 + \lambda^4 \int_\Omega |u|^2 \le C \int_\Omega |F|^2.$$

This implies

$$\frac{1}{2}(\|\lambda \nabla u\|_{L^2(\Omega)} + \|\lambda^2 u\|_{L^2(\Omega)})^2 \le \|\lambda \nabla u\|^2_{L^2(\Omega)} + \|\lambda^2 u\|^2_{L^2(\Omega)} \le C\|F\|^2_{L^2(\Omega)}.$$

Thus we see $\|T_\lambda\|_{L^2(\Omega)\to L^2(\Omega)}$. It suffices now to check condition (4.19). Let $f \in C_0^\infty(\Omega)$ such that $f = 0$ in $B(x_0, 2r)$, where $0 < r < r_0$ and either $x_0 \in \partial\Omega$ or $B(x_0, 4r) \subset \Omega$. Let w be a solution of (4.20) with f in the place of F. Then

$$\begin{cases} -\Delta w + \lambda^2 V w = 0 & \text{on } B(x_0, 4r) \cap \Omega, \\ w = 0 & \text{on } B(x_0, 4r) \cap \partial\Omega \text{ (if } x_0 \in \partial\Omega). \end{cases} \tag{4.22}$$

If Ω is a bounded C^1 domain, applying Lemma 4.0.1 and Remark 4.0.2 gives

$$\left(\fint_{B(x_0,r)\cap\Omega} (\lambda|\nabla w| + \lambda^2|w|)^q \right)^{1/q} \le C \left(\fint_{B(x_0,2r)\cap\Omega} (\lambda|\nabla w| + \lambda^2|w|)^2 \right)^{1/2} \tag{4.23}$$

for any $2 < q < \infty$, where C depends on d, q, Ω, and (μ_0, μ_1). Thus we may apply Theorem 4.0.3 to obtain $\|T_\lambda(F)\|_{L^p(\Omega)} \le C\|F\|^p_L(\Omega)$ for any $2 < p < \infty$. If Ω is a Lipschitz domain, the estimate (4.23) holds for $2 < q < 3 + \delta$ if $d \ge 3$, and for $2 < q < 4 + \delta$ if $d = 2$. Hence, we obtain (4.21) for where $2 < p < 3 + \delta$ if $d \ge 3$, and for $2 < p < 4 + \delta$ $\delta > 0$ depends on d and Ω.

□

Theorem 4.0.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with connected boundary. Let $u \in W^{1,2}(\Omega)$ be a solution to*

$$\begin{cases} -\Delta u + \lambda^2 V(x) u = 0 & \text{in } \Omega, \\ \qquad\qquad\quad u = g & \text{on } \partial\Omega, \end{cases} \tag{4.24}$$

where $g \in W^{1,2}(\partial\Omega)$. Then

$$\begin{cases} \quad \|u\|_{L^p(\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}, \\ \|\nabla u\|_{L^p(\Omega)} \leq C\left\{\|\nabla_t g\|_{L^2(\partial\Omega)} + \lambda\|g\|_{L^2(\partial\Omega)}\right\}, \end{cases} \tag{4.25}$$

where $p = \frac{2d}{d-1}$, $\nabla_t g$ denotes the tangential gradient of g on $\partial\Omega$, and C depends on d, Ω, and (μ_0, μ_1).

Proof. We start by solving the Dirichlet problem,

$$\begin{cases} -\Delta G = 0 & \text{in } \Omega, \\ \quad G = g & \text{on } \partial\Omega. \end{cases}$$

By well known nontangential-maximal-function estimates for harmonic functions in Lipschitz domains (see e.g. [16]),

$$\|(\nabla G)^*\|_{L^2(\partial\Omega)} \leq C\|\nabla_t g\|_{L^2(\partial\Omega)} \qquad \text{and} \qquad \|(G)^*\|_{L^2(\Omega)} \leq \|g\|_{L^2(\partial\Omega)},$$

where $\nabla_t g$ denotes the tangential gradient of g on $\partial\Omega$. It follows that

$$\begin{aligned} \|\nabla G\|_{L^p(\Omega)} &\leq \|(\nabla G)^*\|_{L^2(\partial\Omega)} \leq C\|\nabla_t g\|_{L^2(\partial\Omega)}, \\ \|G\|_{L^p(\Omega)} &\leq \|(G)^*\|_{L^2(\partial\Omega)} \leq \|g\|_{L^2(\partial\Omega)}, \end{aligned} \tag{4.26}$$

where $p = \frac{2d}{d-1}$ and we have used the inequality $\|w\|_{L^p(\Omega)} \leq C\|(w)^*\|_{L^2(\partial\Omega)}$ for functions w in Ω.

We first rewrite u in the following way

$$u = u - G + G.$$

By the triangle inequality

$$\begin{aligned} \|u\|_{L^p(\Omega)} &\leq \|u - G\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)} \\ &\leq \|u - G\|_{L^p(\Omega)} + \|g\|_{L^2(\partial\Omega)}, \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \|\nabla u\|_{L^p(\Omega)} &\leq \|\nabla(u - G)\|_{L^p(\Omega)} + \|\nabla G\|_{L^p(\Omega)} \\ &\leq \|\nabla(u - G)\|_{L^p(\Omega)} + \|\nabla_t g\|_{L^2(\partial\Omega)}, \end{aligned} \tag{4.28}$$

where we have made use of (4.26). As a result it suffices to obtain L^p bounds on $u - G$ and $\nabla(u - G)$.

Let $v = u - G$. Then v is a solution to the following Dirichlet problem,

$$\begin{cases} -\Delta v + \lambda^2 V(x) v = F, & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F = -\lambda^2 V(x) G$. Let $p = \frac{2d}{d-1}$. Note that $p = 4$ for $d = 2$, $p = 3$ for $d = 3$, and $p < 3$ for $d \geq 4$. Thus we may apply Theorem 4.0.5 to obtain

$$\begin{aligned} \lambda^2 \|u - G\|_{L^p(\Omega)} &\leq C\|F\|_{L^p(\Omega)} \\ &= C\|\lambda^2 V G\|_{L^p(\Omega)} \\ &\leq C\lambda^2 \|G\|_{L^p(\Omega)}. \end{aligned}$$

Hence,

$$\|u - G\|_{L^p(\Omega)} \leq C\|G\|_{L^p(\Omega)}.$$

This in combination with (4.27) implies

$$\|u\|_{L^p(\Omega)} \leq C\|G\|_{L^p(\Omega)} \leq C\|g\|_{L^2(\partial\Omega)} \tag{4.29}$$

for $p = \frac{2d}{d-1}$. Again from Theorem 4.0.5 we have

$$\begin{aligned} \lambda \|\nabla(u - G)\|_{L^p(\Omega)} &\leq C\|F\|_{L^p(\Omega)} \\ &= C\|\lambda^2 V G\|_{L^p(\Omega)} \\ &\leq C\lambda^2 \|G\|_{L^p(\Omega)}. \end{aligned}$$

Thus,

$$\|\nabla(u - G)\|_{L^p(\Omega)} \leq C\lambda \|G\|_{L^p(\Omega)}.$$

This in combination with (4.28) implies

$$\begin{aligned} \|\nabla u\|_{L^p(\Omega)} &\leq \|\nabla G\|_{L^p(\Omega)} + C\lambda \|G\|_{L^p(\Omega)} \\ &\leq \|\nabla_t g\|_{L^2(\partial\Omega)} + C\lambda \|g\|_{L^2(\partial\Omega)}. \end{aligned} \tag{4.30}$$

for $p = \frac{2d}{d-1}$. Combing (4.29) and (4.30) yields the result. □

Chapter 5 Large Scale $W^{1,p}$ Estimate

Let u_ε be a solution to the Dirichlet problem

$$\begin{cases} -\Delta u_\varepsilon = F + \operatorname{div}(f) & \text{in } \Omega_{\varepsilon,\eta}, \\ \quad u_\varepsilon = 0 & \text{on } \partial\Omega_{\varepsilon,\eta}, \end{cases} \tag{5.1}$$

where $\Omega_{\varepsilon,\eta}$ is given in (1.5). Let

$$T_\varepsilon(F,f) = \left(\fint_{x+2\varepsilon Y} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{5.2}$$

where the solution u_ε has been extended to $\mathbb{R}^d$ by zero. Note is easy to see that

$$\|T_\varepsilon(F,f)\|_{L^2(\mathbb{R}^d)} = \|\nabla u_\varepsilon\|_{L^2(\Omega_{\varepsilon,\eta})}.$$

This fact combined with Lemma 1.4.3 gives

$$\|T_\varepsilon(F,f)\|_{L^2(\mathbb{R}^d)} \le \|f\|_{L^2(\Omega_{\varepsilon,\eta})} + C\min(\sigma_\varepsilon, 1)\|F\|_{L^2(\Omega_{\varepsilon,\eta})}. \tag{5.3}$$

This chapter is henceforth dedicated to the L^p estimates of $T_\varepsilon(F,f)$ which are found in Theorem 1.6.1. We start with proving the estimates for the case $F = 0$. Later we consider the case where $f = 0$. By superimposing the two cases we are able to obtain (1.26).

Theorem 5.0.1. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $\Omega_{\varepsilon,\eta}$ be given by (1.5). Then, for $2 < p < \infty$,*

$$\|T_\varepsilon(0,f)\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{5.4}$$

where C depends only on $d, p, \Omega,$ and $\{Y_z^s\}$.

We will let $0 < \eta < \eta_0$, where η_0 is sufficiently small. The case where $\eta \ge \eta_0$, for any fixed $\eta_0 > 0$, is analogous to the case where $\eta = 1$ which can be found in [10]. In order to prove Theorem 5.0.1, we wish to apply Theorem 4.0.3. Note that the operator T_ε is sublinear by construction. Additionally, the L^2 boundedness condition (4.18) is given by (5.3). Thus it suffices to show the reverse Hölder condition (4.19). That is, if $\Delta u_\varepsilon = 0$ in $Q(x_0, 4R) \cap \Omega_{\varepsilon,\eta}$ and $u_\varepsilon = 0$ on $Q(x_0, 4R) \cap \partial\Omega_{\varepsilon,\eta}$, where $0 < R < c_0$ and either $Q(x_0, 4R) \subset \Omega$ or $x_0 \in \partial\Omega$, then

$$\begin{aligned} &\left(\fint_{Q(x_0,R)\cap\Omega} \left(\fint_{Q(x,2\varepsilon)} |\nabla u_\varepsilon|^2 \right)^{p/2} dx \right)^{1/p} \\ &\qquad \le C \left(\fint_{Q(x_0,2R)\cap\Omega} \left(\fint_{Q(x,2\varepsilon)} |\nabla u_\varepsilon|^2 \right) dx \right)^{1/2} \end{aligned} \tag{5.5}$$

for $p > 2$. The estimate 5.5 holds for $0 < R < C\varepsilon$. The proof of 5.5 for the large-scale case $R \ge C\varepsilon$ uses a real-variable argument and will rely on the convergence rates established in Chapter 3 and the estimates for the intermediate solution obtained in Chapter 4. The rest of this chapter will be divided as followed. First we will consider the interior case where $Q(x_0, 4R) \subset \Omega$. We then handle the boundary case where $x_0 \in \partial\Omega$. Finally we will provide the proofs to Theorem 5.0.1, its analogous result given by Theorem 5.3.1, and the the main result Theorem 1.6.1.

5.1 Interior Case

We begin with the interior case $Q(x_0, 4R) \subset \Omega$. The following theorem was proved in [12, Theorem 3.2].

Theorem 5.1.1. *Let $F \in L^2(Q(x_0, 2R))$ and $2 < p < q$. Suppose that for each cube $Q = Q(y, r)$ with $y \in Q(x_0, R)$ and $0 < r < c_0 R$, there exists two functions R_Q and F_Q such that*

$$|F| \le |R_Q| + |F_Q| \quad \text{in } 2Q, \tag{5.6}$$

$$\left(\fint_{2Q} |R_Q|^q \right)^{1/q} \le N \left(\fint_{8Q} |F|^2 \right)^{1/2}, \tag{5.7}$$

$$\left(\fint_{2Q} |F_Q|^2 \right)^{1/2} \le \delta \left(\fint_{8Q} |F|^2 \right)^{1/2}, \tag{5.8}$$

where $N > 1$ and $0 < c_0 < 1$. Then there exists $\delta_0 > 0$ depending only on d, N, c_0, p, and q, with the property that if $0 \le \delta < \delta_0$ then $F \in L^p(Q(x_0, R))$ and

$$\left(\fint_{Q(x_0,R)} |F|^p \right)^{1/p} \le C \left(\fint_{Q(x_0,2R)} |F|^2 \right)^{1/2}, \tag{5.9}$$

where C depends at most on d, c_0, p, q, and N.

We will use $Q_{\varepsilon,\eta}(x_0, r)$ to denote $\Omega_{\varepsilon,\eta}$, where $\Omega = Q(x_0, r)$. Note that in general,

$$Q_{\varepsilon,\eta}(x_0, r) \neq Q(x_0, r) \cap \Omega,$$

as there is no hole near the boundary of $Q(x_0, r)$ in $Q_{\varepsilon,\eta}(x_0, r)$. Under the assumption $r \ge C\varepsilon$, it is possible to find $Q(y_0, t)$ such that $Q(x_0, r) \subset Q(y_0, t) \subset Q(x_0, 2r)$ with $Q_{\varepsilon,\eta}(y_0, t) = \Omega_{\varepsilon,\eta} \cap Q(y_0, t)$. This observation will be used in the proof of the following lemma.

Lemma 5.1.2. *Let $u \in H^1(Q(x_0, 4r))$, where $r \ge 8\varepsilon$ and $Q(x_0, 8r) \subset \Omega$. Suppose that $\Delta u = 0$ in $Q(x_0, 4r) \cap \Omega_{\varepsilon,\eta}$ and $u = 0$ on $Q(x_0, 4r) \backslash \Omega_{\varepsilon,\eta}$. Then there exists $v \in H^1(Q(x_0, 2r))$ such that*

$$\left(\fint_{Q(x_0,2r)} |\nabla(u - v)|^2 \right)^{1/2} \le C\phi(\eta) \left(\fint_{Q(x_0,3r)} |\nabla u|^2 \right)^{1/2}, \tag{5.10}$$

$$\max_{x \in Q(x_0,r)} \left(\fint_{Q(x,2\varepsilon)} |\nabla v|^2 \right)^{1/2} \le C \left(\fint_{Q(x_0,3r)} |\nabla u|^2 \right)^{1/2}, \tag{5.11}$$

where $\phi(\eta) = \eta^{1/2}$ for $d \ge 3$ and $\phi(\eta) = |\ln \eta|^{-1/2}$ for $d = 2$. The constant C depends only on d and $\{Y_z^s\}$.

Proof. Since $r \geq 8\varepsilon$ without loss of generality, we may assume that $x_0 = 0$ and $r = 2^j\varepsilon$ for some $j \geq 2$. By dilation we may also assume that $r = 1$ where η remains invariant under the dilation. Since $u = 0$ on $Q(0,3)\backslash\Omega_{\varepsilon,\eta}$, it follows from Lemma 1.4.1 that

$$\sigma_\varepsilon^{-2} \int_{Q(0,3)} |u|^2 \leq C \int_{Q(0,3)} |\nabla u|^2. \tag{5.12}$$

We claim that there exists $t \in [2,3]$ such that $Q_{\varepsilon,\eta}(0,t) = Q(0,t) \cap \Omega_{\varepsilon,\eta}$ and

$$\sigma_\varepsilon^{-2} \int_{\partial Q(0,t)} |u|^2 + \int_{\partial Q(0,t)} |\nabla u|^2 \leq C \int_{Q(0,3)} |\nabla u|^2. \tag{5.13}$$

To show this consider the set $E = \{t \in [2,3] : |t - k\varepsilon| \leq c_0\varepsilon \text{ for some } 1 \leq k \leq 2^{j+2}\}$. Note that $|E| \geq c > 0$ and $Q_{\varepsilon,\eta}(0,t) = Q(0,t) \cap \Omega_{\varepsilon,\eta}$ for $t \in E$. Assume (5.13) fails for all $t \in E$. Then

$$\begin{aligned} \int_E \left[\sigma_\varepsilon^{-2} \int_{\partial Q(0,t)} |u|^2 + \int_{\partial Q(0,t)} |\nabla u|^2 \right] dt &> C'|E| \int_{Q(0,3)} |\nabla u|^2 \\ &> C \int_{Q(0,3)} |\nabla u|^2, \end{aligned}$$

for any $C > 0$. Note that

$$\int_E \left[\sigma_\varepsilon^{-2} \int_{\partial Q(0,t)} |u|^2 + \int_{\partial Q(0,t)} |\nabla u|^2 \right] dt \leq C \left[\sigma_\varepsilon^{-2} \int_{Q(0,3)} |u|^2 + \int_{Q(0,3)} |\nabla u|^2 \right].$$

This implies

$$\sigma_\varepsilon^{-2} \int_{Q(0,3)} |u|^2 > C \int_{Q(0,3)} |\nabla u|^2.$$

for any $C > 0$. But this is a contradiction of of (5.12). Thus there must be a $t \in E$ such that (5.13) holds.

Now, let w be a solution of

$$\begin{cases} -\Delta w + \sigma_\varepsilon^{-2} V(x/\varepsilon) w = 0 & \text{in } Q(0,t), \\ \qquad\qquad\qquad\quad w = u & \text{on } \partial Q(0,t), \end{cases} \tag{5.14}$$

where $V(x)$ is given by (3.2). It follows from Theorem 3.1.1 when $d \geq 3$, and Theorem 3.2.1 when $d = 2$, that

$$\|\nabla(u - \chi_{\varepsilon,\eta} w)\|_{L^2(Q(0,t))} \leq C\phi(\eta)\{\sigma_\varepsilon^{-1}\|w\|_{L^p(Q(0,t))} + \|\nabla w\|_{L^p(Q(0,t))}\}, \tag{5.15}$$

where $p = \frac{2d}{d-1}$ and $\chi_{\varepsilon,\eta}$ is the corrector for the domain $Q_{\varepsilon,\eta}(0,t)$. This together with Theorem 4.0.6, yields

$$\begin{aligned} \|\nabla(u - \chi_{\varepsilon,\eta} w)\|_{L^2(Q(0,t))} &\leq C\phi(\eta) \left\{ \sigma_\varepsilon^{-1}\|u\|_{L^2(\partial Q(0,t))} + \|\nabla u\|_{L^2(\partial Q(0,t))} \right\} \\ &\leq C\phi(\eta)\|\nabla u\|_{L^2(Q(0,3))}, \end{aligned} \tag{5.16}$$

where we have used (5.13) in the last step.

Finally, let $v = \chi_{\varepsilon,\eta} w$. Note that estimate (5.10) is given by (5.16). For the estimate (5.11), note that

$$\begin{aligned}\left(\fint_{Q(x,2\varepsilon)} |\nabla v|^2\right)^{1/2} &\leq \max_{Q(x,2\varepsilon)} |w| \left(\fint_{Q(x,2\varepsilon)} |\nabla \chi_{\varepsilon,\eta}|^2\right)^{1/2} + \max_{Q(x,2\varepsilon)} |\chi_{\varepsilon,\eta}||\nabla w| \\ &\leq \max_{Q(x,2\varepsilon)} (\sigma_\varepsilon^{-1}|w| + |\nabla w|),\end{aligned}$$

where we have used Lemma 2.2.3 for $d \geq 3$ and Lemma 2.4.3 for $d = 2$. This combined with Remark 4.0.2 gives

$$\begin{aligned}\left(\fint_{Q(x,2\varepsilon)} |\nabla v|^2\right)^{1/2} &\leq C \max_{Q(0,3/2)} (\sigma_\varepsilon^{-1}|w| + |\nabla w|) \\ &\leq C \left\{\sigma_\varepsilon^{-1}\|w\|_{L^2(Q(0,2))} + \|\nabla w\|_{L^2(Q(0,2))}\right\} \\ &\leq C\|\nabla u\|_{L^2(Q(0,3))}\end{aligned} \tag{5.17}$$

for any $x \in Q(0,1)$. □

Lemma 5.1.3. *Let $u \in H^1(B(x_0, 4R))$, where $R \geq 8\varepsilon$ and $B(x_0, 8R) \subset \Omega$. Suppose that $\Delta u = 0$ in $Q(x_0, 4R) \cap \Omega_{\varepsilon,\eta}$ and $u = 0$ in $Q(x_0, 4R) \backslash \Omega_{\varepsilon,\eta}$. Let*

$$v(x) = \left(\fint_{Q(x,2\varepsilon)} |\nabla u|^2\right)^{1/2}. \tag{5.18}$$

Then for $2 < p < \infty$,

$$\left(\fint_{Q(x_0,R)} |v|^p\right)^{1/p} \leq C \left(\fint_{Q(x_0,2R)} |v|^2\right)^{1/2}, \tag{5.19}$$

where C depends only on d, p, and $\{Y_z^s\}$.

Proof. Without loss of generality assume that $x_0 = 0$. Further, by dilation we may assume $R = 1$. To show (5.19), we will apply Theorem 5.1.1 with $F = v$. Let $Q = Q(y_0, r)$, where $y_0 \in Q(0,1)$ and $0 < r < (1/8)$. If $0 < r < 8\varepsilon$, we let $R_Q = v$ and $F_Q = 0$. Note that

$$\max_{2Q} R_Q \leq C \left(\fint_{Q(y_0,2r+2\varepsilon)} |\nabla u|^2\right)^{1/2} \leq C \left(\fint_{4Q} |v|^2\right)^{1/2},$$

and (5.7) holds pointwise.

Now if $r \geq 8\varepsilon$, we let $R_Q = |\nabla v|$ and $F_Q = \nabla(u - v)$, for v given in Lemma 5.1.2. In view of (5.10) and (5.11) we obtain (5.8) and (5.7) with $\delta = C\phi(\eta)$. Then, it follows from Theorem 5.1.1 that (5.19) holds if $\eta < \eta_0$, where $\eta_0 > 0$ depends on d, p, and $\{Y_z^s\}$. □

5.2 Boundary Case

In this section we treat the boundary case where $x_0 \in \partial\Omega$. Since Ω is Lipschitz, there exists $r_0 > 0$ such that $B(x_0, r_0) \cap \Omega = B(x_0, r_0) \cap D$ and $B(x_0, r_0) \cap \partial\Omega = B(x_0, r_0) \cap \partial D$, where after a rotation of coordinate system, D is given by

$$D = \{(x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \phi(x')\}, \tag{5.20}$$

for some Lipschitz function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$.

The proof of the following theorem can be found in [12] or [13, pp. 79-82].

Theorem 5.2.1. *Let $2 < p < q$ and D be given in (5.20). Let $x_0 \in \partial D$ and $F \in L^2(Q(x_0, 2R) \cap D)$. Suppose that for each cube $Q = Q(y, r)$ with the property that $0 < r < c_0 R$ and either $y \in Q(x_0, R) \cap \partial D$ or $4Q \subset Q(x_0, 2R) \cap D$, there exists two measurable functions R_Q and F_Q in $2Q \cap D$ such that*

$$|F| \le |R_Q| + |F_Q| \quad \text{in } 2Q \cap D, \tag{5.21}$$

$$\left(\fint_{2Q\cap D} |R_Q|^q \right)^{1/q} \le N \left(\fint_{4Q\cap D} |F|^2 \right)^{1/2}, \tag{5.22}$$

$$\left(\fint_{2Q\cap D} |F_Q|^2 \right)^{1/2} \le \gamma \left(\fint_{4Q\cap D} |F|^2 \right)^{1/2}, \tag{5.23}$$

where $N > 1$ and $0 < c_0 < 1$. Then there exists $\gamma_0 > 0$ depending only on d, N, c_0, p, q, and $\|\nabla\phi\|_\infty$, with the property that if $0 \le \gamma < \gamma_0$, then $F \in L^p(Q(x_0, R) \cap D)$ and

$$\left(\fint_{Q(x_0,R)\cap D} |F|^p \right)^{1/p} \le C \left(\fint_{Q(x_0,2R)\cap D} |F|^2 \right)^{1/2}, \tag{5.24}$$

where C depends at most on $d, N, c_0, p, q,$ and $\|\nabla\phi\|_\infty$.

Lemma 5.2.2. *Let Ω be a bounded C^1 domain. Let $u \in H^1(Q(x_0, 4r) \cap \Omega)$, where $r \ge 8\varepsilon$ and $x_0 \in \partial\Omega$. Suppose that $\Delta u = 0$ in $Q(x_0, 4r) \cap \Omega_{\varepsilon,\eta}$ and $u = 0$ on $Q(x_0, 4r) \cap \partial\Omega_{\varepsilon,\eta}$. Then there exists $v \in H^1(Q(x_0, 2r) \cap \Omega)$ such that*

$$\left(\fint_{Q(x_0,r)\cap\Omega} |\nabla(u - v)|^2 \right)^{1/2} \le C\phi(\eta) \left(\fint_{Q(x_0,3r)\cap\Omega} |\nabla u|^2 \right)^{1/2}, \tag{5.25}$$

$$\max_{x\in Q(x_0,r)\cap\Omega} \left(\fint_{Q(x,2\varepsilon)\cap\Omega} |\nabla v|^2 \right)^{1/2} \le C \left(\fint_{Q(x_0,3r)\cap\Omega} |\nabla u|^2 \right)^{1/2}, \tag{5.26}$$

for any $p > 2$, where $\phi(\eta) = \eta^{1/2}$ for $d \ge 3$ and $\phi(\eta) = |\ln\eta|^{-1/2}$ for $d = 2$. The constant C depends only on $d, p, \Omega,$ and $\{Y_z^s\}$.

Proof. The proof will follow similarly to that of Lemma 5.1.2. Without loss of generality assume that $x_0 = 0$, and $r = 1$ by dilation. Once again, it follows from Lemma 1.4.1 that

$$\sigma_\varepsilon^{-2} \int_{Q(0,3)\cap\Omega} |u|^2 \le C \int_{Q(0,3)\cap\Omega} |\nabla u|^2. \tag{5.27}$$

We claim we may choose $t \in [2,3]$ such that $(Q(0,t)\cap\Omega)_{\varepsilon.\eta} = Q(0,t)\cap\Omega_{\varepsilon,\eta}$ with

$$\sigma_\varepsilon^{-2}\int_{\Omega\cap\partial Q(0,t)}|u|^2 + \int_{\Omega\cap\partial Q(0,t)}|\nabla u|^2 \le C\int_{\Omega\cap Q(0,3)}|\nabla u|^2. \tag{5.28}$$

To show (5.28) again consider $E = \{t\in[2,3] : |t-k\varepsilon|\le c_0\varepsilon \text{ for some } 1\le k\le 2^{j+2}\}$. Note that $|E|\ge c>0$ and $(Q(0,t)\cap\Omega)_{\varepsilon,\eta} = Q(0,t)\cap\Omega_{\varepsilon,\eta}$ for $t\in E$. Again if 5.28 fails for all t, integrate the left-hand side of 5.28 with respect to t over E and use 5.27 to obtain a contradiction.

Now, let w be a solution of

$$\begin{cases} -\Delta w + \sigma_\varepsilon^{-2}V(x/\varepsilon)w = 0 & \text{in } Q(0,t)\cap\Omega, \\ w = u & \text{on } \partial(Q(0,t)\cap\Omega), \end{cases} \tag{5.29}$$

where $V(x)$ is given by (3.2). It follows from Theorem 3.1.1 when $d\ge 3$, and Theorem 3.2.1 when $d=2$, that

$$\|\nabla(u-\chi_{\varepsilon,\eta}w)\|_{L^2(Q(0,t)\cap\Omega)} \le C\phi(\eta)\{\sigma_\varepsilon^{-1}\|w\|_{L^p(Q(0,t)\cap\Omega)} + \|\nabla w\|_{L^p(Q(0,t)\cap\Omega)}\}, \tag{5.30}$$

where $p=\frac{2d}{d-1}$ and $\chi_{\varepsilon,\eta}$ is the corrector for the domain $Q_{\varepsilon,\eta}(0,t) = Q(0,t)\cap\Omega_{\varepsilon,\eta}$. This together with Theorem 4.0.6, yields

$$\begin{aligned} \|\nabla(u-\chi_{\varepsilon,\eta}w)\|_{L^2(Q(0,t)\cap\Omega)} &\le C\phi(\eta)\left\{\sigma_\varepsilon^{-1}\|u\|_{L^2(\partial Q(0,t)\cap\Omega)} + \|\nabla u\|_{L^2(\partial Q(0,t)\cap\Omega)}\right\} \\ &\le C\phi(\eta)\|\nabla u\|_{L^2(Q(0,3)\cap\Omega)}, \end{aligned} \tag{5.31}$$

where we have used (5.28) in the last step.

Finally, let $v=\chi_{\varepsilon,\eta}w$. Note that estimate (5.25) is given by (5.31). For the estimate (5.26), note that

$$\begin{aligned} \left(⨍_{Q(x,2\varepsilon)\cap\Omega}|\nabla v|^2\right)^{1/2} &\le \max_{Q(x,2\varepsilon)\cap\Omega}|w|\left(⨍_{Q(x,2\varepsilon)\cap\Omega}|\nabla\chi_{\varepsilon,\eta}|^2\right)^{1/2} \\ &\quad + \max_{Q(x,2\varepsilon)\cap\Omega}|\chi_{\varepsilon,\eta}|\left(⨍_{Q(x,2\varepsilon)\cap\Omega}|\nabla w|^2\right)^{1/2}, \end{aligned}$$

which in light of Lemma 2.2.3 for $d\ge 3$ and Lemma 2.4.3 for $d=2$ gives,

$$\left(⨍_{Q(x,2\varepsilon)\cap\Omega}|\nabla v|^2\right)^{1/2} \le C\sigma_\varepsilon^{-1}\max_{Q(x,2\varepsilon)\cap\Omega}|w| + C\left(⨍_{Q(x,2\varepsilon)\cap\Omega}|\nabla w|^2\right)^{1/2}.$$

Since $w=u=0$ on $Q(0,t)\cap\partial\Omega$, it follows from Lemma 4.0.1 that

$$\begin{aligned} \sigma_\varepsilon^{-1}\max_{Q(0,3/2)\cap\Omega}|w| &\le C\left(⨍_{Q(0,2)\cap\Omega}|\nabla w|^2\right)^{1/2}, \\ \left(⨍_{Q(0,3/2)\cap\Omega}|\nabla w|^p\right)^{1/p} &\le C\left(⨍_{Q(0,2)\cap\Omega}|\nabla w|^2\right)^{1/2}, \end{aligned} \tag{5.32}$$

for $2 < p < \infty$. As a result

$$\int_{Q(0,1)\cap\Omega} \left(\fint_{Q(x,2\varepsilon)\cap\Omega} |\nabla v|^2 \right)^{p/2} \leq C \left(\fint_{Q(0,2)\cap\Omega} |\nabla w|^2 \right)^{p/2}$$
$$\leq C\{\|\nabla u\|_{L^2(\partial Q(0,t)\cap\Omega)} + \sigma_\varepsilon^{-1} \|u\|_{L^2(\partial Q(0,t)\cap\Omega)}\}^p.$$

This combined with (5.28) gives (5.26). □

5.3 Proof of Theorem 1.6.1

In this section we provide the proof of Theorem 1.6.1. We start with the proof of Theorem 5.0.1. We then state and prove the analogous result for when $f \equiv 0$.

Proof of Theorem 5.0.1. As stated before it suffices to show the reverse Hölder condition (4.19), where $0 < R < c_0$ and either $Q(x_0, 4R) \subset \Omega$ or $x_0 \in \partial\Omega$. The interior case is handled by Lemma 5.1.3. For the case in which $x_0 \in \partial\Omega$, we will apply Theorem 5.2.1. As such for each cube $Q = Q(y, r)$ with the property that $0 < r < c_0 R$ and either $4Q \subset Q(x_0, 2R) \cap \Omega$ or $y \in \partial Q(x_0, 2R) \cap \partial\Omega$, we will need to construct two functions F_Q and R_Q for which (5.21)-(5.23) hold. Again the case $4Q \subset Q(x_0, 2R) \cap \Omega$ is given by Lemma 5.1.3. For the case where $y \in \partial\Omega$ and $0 < r < 8\varepsilon$, we let $F_Q = F$ and $R_Q = 0$, which gives the desired bound pointwise. For the final case where $r \geq 8\varepsilon$, apply Lemma 5.2.2. □

Theorem 5.3.1. *Let Ω be a bounded C^1 domain and $\Omega_{\varepsilon,\eta}$ be given by* (1.5)*. Then for any $2 < p < \infty$,*

$$\|T_\varepsilon(F, 0)\|_{L^p(\Omega)} \leq C \min(\sigma_\varepsilon, 1) \|F\|_{L^p(\Omega)}, \tag{5.33}$$

where C depends only on $d, p, \Omega,$ and $\{Y_z^s\}$.

Proof. Consider the operator

$$S_\varepsilon(F) = (\min(\sigma_\varepsilon, 1))^{-1} T_\varepsilon(F, 0). \tag{5.34}$$

This operator is clearly sublinear by construction. We get directly from (5.3), that $\|S_\varepsilon\|_{L^2 \to L^2} \leq C$. Thus it suffices to prove the reverse Hölder condition (4.19), which again we reduce to showing (5.5). As in Theorem 5.0.1, the interior case is handled by Lemma 5.1.3. Additionally, the case in which $x_0 \in \partial\Omega$, we will apply Theorem 5.2.1. We again need to construct two functions F_Q and R_Q for which (5.21)-(5.23) hold. Again the case $4Q \subset Q(x_0, 2R) \cap \Omega$ is given by Lemma 5.1.3. For the case where $y \in \partial\Omega$ and $0 < r < 8\varepsilon$, the same choice of $F_Q = F$ and $R_Q = 0$ gives the desired bound. For the final case where $r \geq 8\varepsilon$ is again handled by Lemma 5.2.2. We apply Theorem 4.0.3 to obtain $\|S_\varepsilon\|_{L^p \to L^p} \leq C$ for any $p > 2$, which from the definition of $S_\varepsilon(F)$, gives (5.33) for $p > 2$. □

Proof of Theorem 1.6.1. Note that

$$T_\varepsilon(F, f) \leq T_\varepsilon(F, 0) + T_\varepsilon(0, f),$$

and as such,

$$\|T_\varepsilon(F, f)\|_{L^p(\mathbb{R}^d)} \leq \|T_\varepsilon(F, 0)\|_{L^p(\Omega)} + \|T_\varepsilon(0, f)\|_{L^p(\Omega)}.$$

Thus, Theorem 1.6.1 follows directly from applying Theorems 5.0.1 and 5.3.1. □

Chapter 6 Main Results

This chapter is dedicated to proving Theorems 1.5.1 - 1.5.5. Following the ideas of [15], our first section will give local $W^{1,p}$ estimates in a cell. The second section will provide the upper bounds for $A_p(\Omega_{\varepsilon,\eta})$. The proof reduces the argument down to large-scale estimates which were proved in Chapter 5. Finally we will provide upper bounds for $B_p(\Omega_{\varepsilon,\eta})$.

6.1 Local Estimates in a Cell

Recall $Q_z = z + Q(0,1)$ and T_z is defined in (1.4). This section establishes $W^{1,p}$ estimates for solutions of

$$\begin{cases} -\Delta u_\varepsilon = F + \operatorname{div}(f) & \text{in } \varepsilon(\widetilde{Q}_z \backslash T_z), \\ u_\varepsilon = 0 & \text{in } \varepsilon T_z, \end{cases} \tag{6.1}$$

where $z \in \mathbb{Z}^d$ and $\widetilde{Q}_z = Q(z, 17/16)$.

We begin with the following estimate on an exterior domain. An extensive proof of this result can be found in [15, Theorem 5.6] which uses a result from [3].

Lemma 6.1.1. *Let $d \geq 2$ and $2 \leq p < \infty$. Let T be the closure of a bounded C^1 domain in $\mathbb{R}^d$ with connected boundary. Let u be a solution of $-\Delta u = F + div(f)$ in $R\widetilde{Y} \backslash T$ with $u = 0$ on ∂T, where $\widetilde{Y} = (1 + c_0)\, Q(0,1)$. Then for $R \geq 3$,*

$$\begin{aligned} &\|\nabla u\|_{L^p(RY\backslash T)} \\ &\quad \leq C\Phi_p(R) \left\{ \|f\|_{L^p(R\tilde{Y}\backslash T)} + R\|F\|_{L^p(R\tilde{Y}\backslash T)} + R^{\frac{d}{p}-\frac{d}{2}-1} \|u\|_{L^2(R\tilde{Y}\backslash B(0,R/3))} \right\} \end{aligned} \tag{6.2}$$

where

$$\Phi_p(R) = \begin{cases} 1 & \text{if } d \geq 3 \text{ and } 2 < p < d, \\ (\ln R)^{1-\frac{1}{d}} & \text{if } d \geq 3 \text{ and } p = d, \\ R^{1-\frac{d}{p}} & \text{if } d \geq 3 \text{ and } d < p < \infty, \\ R^{1-\frac{2}{p}}(\ln R)^{-1} & \text{if } d = 2 \text{ and } 2 < p < \infty, \end{cases} \tag{6.3}$$

and C only depends on d, p and T.

The rest of this section will prove the following Lemma. The case where $\varepsilon = 1$ and $x_z = 0$ was proved in [15, Theorem 6.1]. We provide the details for our case here.

Lemma 6.1.2. *Let $2 < p < \infty$. Let u_ε be a solution of (6.1) with $F \in L^p(\varepsilon\widetilde{Q}_z)$ and $f \in L^p(\varepsilon\widetilde{Q}_z; \mathbb{R}^d)$. Then for $d \geq 3$,*

$$\begin{aligned} \left(\fint_{\varepsilon Q_z} |\nabla u_\varepsilon|^p \right)^{1/p} &\leq C|\alpha|\varepsilon^{-1}\eta^{\frac{d}{p}-1} + C\Phi_p(\eta^{-1}) \left(\fint_{\varepsilon\widetilde{Q}_z} |\varepsilon F|^p + |f|^p dx \right)^{1/p} \\ &\quad + C\varepsilon^{-1}\Phi_p(\eta^{-1}) \left(\fint_{\varepsilon\widetilde{Q}_z \backslash Q(\varepsilon y_z, \frac{\varepsilon}{3})} |u_\varepsilon - \alpha|^2 dx \right)^{1/2}, \end{aligned} \tag{6.4}$$

and for $d=2$

$$\left(\fint_{\varepsilon Q_z} |\nabla u_\varepsilon|^p\right)^{1/p} \leq C|\alpha|\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln\eta|^{-1} + C\Phi_p(\eta^{-1})\left(\fint_{\varepsilon\widetilde{Q}_z}|\varepsilon F|^p + |f|^p dx\right)^{1/p}$$
$$+C\varepsilon^{-1}\Phi_p(\eta^{-1})\left(\fint_{\varepsilon\widetilde{Q}_z\setminus Q(\varepsilon y_z,\frac{\varepsilon}{3})}|u_\varepsilon-\alpha|^2dx\right)^{1/2}, \tag{6.5}$$

where $\alpha\in\mathbb{R}$ *and* C *depends only on* $d, p,$ *and* $\{Y_z^s\}$.

In order to prove Lemma 6.1.2 we must first establish the following lemmas.

Lemma 6.1.3. *Let* $2<p<\infty$. *Let* u_ε *be a solution of* (6.1) *with* $F\in L^p(\varepsilon\widetilde{Q}_z)$ *and* $f\in L^p(\varepsilon\widetilde{Q}_z;\mathbb{R}^d)$. *Then*

$$\|\nabla u_\varepsilon\|_{L^p(\varepsilon Q_z)} \leq C\Phi_p(\eta^{-1})\left\{\varepsilon^{\frac{d}{p}-\frac{d}{2}-1}\|u\|_{L^2(\varepsilon\widetilde{Q}_z\setminus B(\varepsilon y_z,\frac{\varepsilon}{3}))} + \|f\|_{L^p(\varepsilon\widetilde{Q}_z)} + \varepsilon\|F\|_{L^p(\varepsilon\widetilde{Q}_z)}\right\}, \tag{6.6}$$

where Φ_p *is given by* (6.3) *and* C *depends only on* d, p *and* $\{Y_z^s\}$.

Proof. Let u_ε satisfy (6.1). Then let $v(x)=u_\varepsilon(\varepsilon\eta x)$. Then

$$-\Delta v = G + \operatorname{div}(g) \quad \text{in} \quad R(\widetilde{Q}_z\setminus T_z),$$

where $R=\eta^{-1}, G(x)=\varepsilon^2\eta^2F(\varepsilon\eta x)$, and $g(x)=\varepsilon\eta f(\varepsilon\eta x)$. By translation we may apply Lemma 6.1.1. Thus v satisfies

$$\|\nabla v\|_{L^p(\eta^{-1}Y\setminus Y_z^s)} \leq C\Phi_p(\eta^{-1})\left\{\|g\|_{L^p(\eta^{-1}\widetilde{Y}\setminus Y_z^s)} + \eta^{-1}\|G\|_{L^p(\eta^{-1}\widetilde{Y}\setminus Y_z^s)} + \eta^{-\frac{d}{p}+\frac{d}{2}+1}\|v\|_{L^2(\eta^{-1}\widetilde{Y}\setminus B(0,\frac{1}{3\eta}))}\right\}.$$

Substituting we get

$$\varepsilon\eta\|\nabla u_\varepsilon(\varepsilon\eta x)\|_{L^p(\eta^{-1}Y\setminus Y_z^s)} \leq C\Phi_p(\eta^{-1})\left\{\varepsilon\eta\|f(\varepsilon\eta x)\|_{L^p(\eta^{-1}\widetilde{Y}\setminus Y_z^s)} + \eta\varepsilon^2\|F(\varepsilon\eta z)\|_{L^p(\eta^{-1}\widetilde{Y}\setminus Y_z^s)} + \eta^{-\frac{d}{p}+\frac{d}{2}+1}\|u_\varepsilon(\varepsilon\eta x)\|_{L^2(\eta^{-1}\widetilde{Y}\setminus B(0,\frac{1}{3\eta}))}\right\}.$$

Dividing $\varepsilon\eta$ and changing variables gives (6.6). □

Lemma 6.1.4. *Let* $\chi_{\varepsilon,\eta}$ *be defined as in* (2.1) *extended periodically to the whole space. Then*

$$\begin{cases} -\Delta\chi_{\varepsilon,\eta} = F_{\varepsilon,\eta} + \operatorname{div}(f_{\varepsilon,\eta}) & \text{in } \varepsilon Q_z, \\ \chi_{\varepsilon,\eta} = 0 & \text{in } \varepsilon T_z, \end{cases} \tag{6.7}$$

where

$$|\varepsilon F_{\varepsilon,\eta}| + |f_{\varepsilon.\eta}| \leq \varepsilon^{-1}\eta^{d-2} \qquad \text{in } \varepsilon Q_z$$

when $d \geq 3$, *and*

$$|\varepsilon F_{\varepsilon,\eta}| + |f_{\varepsilon.\eta}| \leq \varepsilon^{-1}|\ln\eta|^{-1} \qquad \text{in } \varepsilon Q_z$$

when $d = 2$.

Proof. This follows from rescaling the estimate obtained by [15, Lemma 6.4]. □

We may now state the proof for the main theorem of this section.

Proof of Lemma 6.1.2. Let u_ε solve (6.1) and let $\chi_{\varepsilon,\eta}$ defined in (2.1) be extended periodically to $\mathbb{R}^d$. For any $\alpha \in \mathbb{R}$ we have that $u_\varepsilon - \alpha\chi_{\varepsilon,\eta} = 0$ on $\partial\varepsilon T_z$ and

$$-\Delta(u_\varepsilon - \alpha\chi_{\varepsilon,\eta}) = (F - \alpha F_{\varepsilon,\eta}) + \operatorname{div}(f - \alpha f_{\varepsilon\eta})$$

in $\varepsilon\widetilde{Q}_z$. Then by (6.6) we have

$$\begin{aligned}\left(\int_{\varepsilon Q_z} |\nabla u_\varepsilon|^p dx\right)^{1/p} \leq\ & |\alpha|\left(\int_{\varepsilon Q_z} |\nabla\chi_{\varepsilon,\eta}|^p dx\right)^{1/p} \\ &+ C|\alpha|\varepsilon^{d/p}\Phi_p(\eta^{-1})\left(\varepsilon\|F_{\varepsilon,\eta}\|_\infty + \|f_{\varepsilon,\eta}\|_\infty\right) \\ &+ C\Phi_p(\eta^{-1})\left(\int_{\varepsilon\widetilde{Q}_z} \varepsilon|F|^p + |f|^p dx\right)^{1/p} \\ &+ C\Phi_p(\eta^{-1})\varepsilon^{\frac{d}{p}-\frac{d}{2}-1}\left(\int_{\varepsilon\widetilde{Q}_z\setminus B(y_z,\frac{\varepsilon}{3})} |u_\varepsilon - \alpha|^2 dx\right)^{1/2}.\end{aligned} \tag{6.8}$$

For the first term on the right-hand side of (6.8), when $d \geq 3$ applying Lemma 2.2.3 yields

$$|\alpha|\left(\int_{\varepsilon Q_z} |\nabla\chi_{\varepsilon,\eta}|^p dx\right)^{1/p} \leq C|\alpha|\varepsilon^{\frac{d}{p}-1}\eta^{\frac{2}{p}-1}|\ln\eta|^{-1}.$$

For the second term on the right-hand side of (6.8) applying Lemma 6.1.4 yields

$$\begin{aligned}C|\alpha|\varepsilon^{d/p}\Phi_p(\eta^{-1})\left(\varepsilon\|F_{\varepsilon,\eta}\|_\infty + \|f_{\varepsilon,\eta}\|_\infty\right) &\leq C|\alpha|\varepsilon^{\frac{d}{p}-1}\eta^{d-2}\Phi_p(\eta^{-1}) \\ &\leq C|\alpha|\varepsilon^{\frac{d}{p}-1}\eta^{\frac{d}{p}-1}\end{aligned}$$

when $d \geq 3$.

Similarly when $d = 2$ we apply Lemmas 2.4.3 and 6.1.4 to bound the first two terms by

$$C|\alpha|\varepsilon^{\frac{d}{p}-1}\eta^{\frac{2}{p}-1}|\ln\eta|^{-1}.$$

This gives the result. □

6.2 Upper Bounds for $A_p(\Omega_{\varepsilon,\eta})$

In this section we provide the proofs of Theorems 1.5.1 and 1.5.2.

Theorem 6.2.1. *Let $F \in L^p(\Omega_{\varepsilon,\eta})$ and $f \in L^p(\Omega_{\varepsilon,\eta};\mathbb{R}^d)$ for some $p \geq 2$. Let u_ε be a solution of* (1.1)*, where Ω is a bounded Lipschitz domain. Then*

$$\|u_\varepsilon\|_{L^p(\Omega_{\varepsilon,\eta})} \leq \min(\sigma_\varepsilon, 1)\|f\|_{L^p(\Omega_{\varepsilon,\eta})} + C\min(\sigma_\varepsilon^2, 1)\|F\|_{L^p(\Omega_{\varepsilon,\eta})}, \tag{6.9}$$

where C depends only on $d, p, c_0,$ and Ω.

Proof. The case $p = 2$ follows from Lemmas 1.4.1 and 1.4.3. The case for $p > 2$ is proved in [14]. □

Remark 6.2.2. Let u_ε be the solution of (1.1) with $F \in L^p(\Omega)$ and $f \in L^p(\Omega, \mathbb{R}^d)$ for some $p > 2$. Let

$$\Omega'_{\varepsilon,\eta} = \bigcup_z \varepsilon(Q_z \backslash T_z), \tag{6.10}$$

where the union is taken over those $z's$ in $\mathbb{Z}^d$ for which $\varepsilon Q_z \subset \Omega$. Let

$$\mathcal{I} = \left\{x \in \Omega'_{\varepsilon,\eta} : \operatorname{dist}(x, \partial\Omega) \geq c\varepsilon\right\}. \tag{6.11}$$

Picking $\alpha = 0$ in (6.4) and (6.5) along with a covering argument gives

$$\int_{\mathcal{I}} |\nabla u_\varepsilon|^p \leq C[\Phi_p(\eta^{-1})]^p \int_\Omega (\varepsilon|F| + |f|)^p + C\varepsilon^{-p}[\Phi_p(\eta^{-1})]^p \int_\Omega |u_\varepsilon|^p. \tag{6.12}$$

For the region near $\partial\Omega$ and away from the holes we use the boundary $W^{1,p}$ estimates for Laplace's equation in C^1 domains [6] to obtain

$$\int_{B(x,c\varepsilon)} |\nabla u_\varepsilon|^p \leq C\varepsilon^{-p} \int_{B(x,2c\varepsilon)} |u_\varepsilon|^p + C\int_{B(x,2c\varepsilon)} (\varepsilon|F| + |f|)^p, \tag{6.13}$$

where $x \in \Omega$ and $B(x, 2c\varepsilon) \cap \Omega = B(x, 2c\varepsilon) \cap \Omega_{\varepsilon,\eta}$. We may cover $\Omega_{\varepsilon,\eta} \backslash \mathcal{I}$ with such balls to obtain

$$\int_{\Omega_{\varepsilon,\eta}\backslash\mathcal{I}} |\nabla u_\varepsilon|^p \leq C\varepsilon^{-p} \int_{\Omega_{\varepsilon,\eta}} |u_\varepsilon|^p + C\int_\Omega (\varepsilon|F| + |f|)^p. \tag{6.14}$$

Combing (6.12) with (6.14) gives

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^p &\leq C[\Phi_p(\eta^{-1})]^p \int_\Omega (\varepsilon|F| + |f|)^p + C\varepsilon^{-p}[\Phi_p(\eta^{-1})]^p \int_\Omega |u_\varepsilon|^p \\
&\leq C[\Phi_p(\eta^{-1})]^p (1 + \varepsilon^{-1}\min(\sigma_\varepsilon, 1))^p \int_\Omega |f|^p \\
&\qquad + C[\Phi_p(\eta^{-1})]^p (1 + \varepsilon^{-1}\min(\sigma_\varepsilon, 1))^{2p} \int_\Omega |\varepsilon F|^p,
\end{aligned}$$

where we have used (6.9) in the last inequality. As a result, we have shown

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^p(\Omega)} &\leq C\Phi_p(\eta^{-1})(1 + \varepsilon^{-1}\min(\sigma_\varepsilon, 1)\|f\|_{L^p(\Omega)} \\
&\qquad + C\Phi_p(\eta^{-1})(1 + \varepsilon^{-1}\min(\sigma_\varepsilon, 1))^2\|\varepsilon F\|_{L^p(\Omega)}
\end{aligned} \tag{6.15}$$

for $p > 2$ and $d \geq 2$.

Remark 6.2.3. Suppose instead we choose

$$\alpha = \fint_{\varepsilon\widetilde{Q}_z\setminus Q(\varepsilon y_z,\varepsilon/3)} u_\varepsilon$$

in (6.4) and (6.5). Then applying the standard Poincaré inequality gives

$$\begin{aligned}\left(\fint_{\varepsilon Q_z}|\nabla u_\varepsilon|^p\right)^{1/p} \le C\varepsilon^{-1}\eta^{\frac{d}{p}-1}\fint_{\varepsilon\widetilde{Q}_z}|u_\varepsilon| + C\Phi_p(\eta^{-1})\left(\fint_{\varepsilon\widetilde{Q}_z}|\varepsilon F|^p+|f|^p dx\right)^{1/p}\\ +C\varepsilon^{-1}\Phi_p(\eta^{-1})\left(\fint_{\varepsilon\widetilde{Q}_z}|\nabla u_\varepsilon|^2\right)^{1/2},\end{aligned} \tag{6.16}$$

for $d\ge 3$, and

$$\begin{aligned}\left(\fint_{\varepsilon Q_z}|\nabla u_\varepsilon|^p\right)^{1/p} \le C\varepsilon^{-1}\Phi_p(\eta^{-1})\fint_{\varepsilon\widetilde{Q}_z}|u_\varepsilon| + C\Phi_p(\eta^{-1})\left(\fint_{\varepsilon\widetilde{Q}_z}|\varepsilon F|^p+|f|^p dx\right)^{1/p}\\ +C\varepsilon^{-1}\Phi_p(\eta^{-1})\left(\fint_{\varepsilon\widetilde{Q}_z}|\nabla u_\varepsilon|^2 dx\right)^{1/2},\end{aligned} \tag{6.17}$$

for $d=2$. Define

$$T_\varepsilon(F,f)(x) = \left(\fint_{Q(x,2\varepsilon)}|\nabla u_\varepsilon|^2\right)^{1/2}.$$

Then from (6.16) and (6.17) and a covering argument

$$\begin{aligned}\int_{\mathcal{I}}|\nabla u_\varepsilon|^p \le C\varepsilon^{-p}\eta^{d-p}\int_\Omega |u_\varepsilon|^p + C|\Phi_p(\eta^{-1})|^p\int_\Omega |\varepsilon F|^p+|f|^p\\ +C|\Phi_p(\eta^{-1})|^p\int_\Omega |T_\varepsilon(F,f)|^p\end{aligned} \tag{6.18}$$

for $d\ge 3$, and

$$\begin{aligned}\int_{\mathcal{I}}|\nabla u_\varepsilon|^p \le C\varepsilon^{-p}|\Phi_p(\eta^{-1}|^p\int_\Omega |u_\varepsilon|^p + C|\Phi_p(\eta^{-1})|^p\int_\Omega |\varepsilon F|^p+|f|^p\\ +C|\Phi_p(\eta^{-1})|^p\int_\Omega |T_\varepsilon(F,f)|^p\end{aligned} \tag{6.19}$$

for $d=2$, where $\mathcal{I}$ is defined in (6.11). For the region near $\partial\Omega$ and away from the holes, we use the local $W^{1,p}$ estimate in C^1 domains,

$$\left(\fint_{B(x,c\varepsilon)}|\nabla u_\varepsilon|^p\right)^{1/p} \le C\left(\fint_{B(x,2c\varepsilon)}|\nabla u_\varepsilon|^2\right)^{1/2} + C\left(\fint_{B(x,2c\varepsilon)}(\varepsilon|F|+|f|)^p\right)^{1/p}, \tag{6.20}$$

where $x\in\Omega$, $\operatorname{dist}(x,\partial\Omega)\le C\varepsilon$, and $B(x,2c\varepsilon)\cap\Omega = B(x,2c\varepsilon)\cap\Omega_{\varepsilon,\eta}$. By covering we obtain,

$$\int_{\Omega_{\varepsilon,\eta}\setminus\mathcal{I}}|\nabla u_\varepsilon|^p \le C\int_\Omega |T_\varepsilon(F,f)|^p + C\int_\Omega(|\varepsilon F|+|f|)^p. \tag{6.21}$$

Remark 6.2.4. We have used a covering argument in (6.12) and (6.18). Let

$$\Omega''_{\varepsilon,\eta} = \{\varepsilon Q_z : \operatorname{dist}(\varepsilon Q_z, \partial\Omega) < c\varepsilon\}\,. \tag{6.22}$$

For cubes $\varepsilon Q_z \in \Omega'_{\varepsilon,\eta} \backslash \Omega''_{\varepsilon,\eta}$, we apply estimates (6.4) and (6.5) directly. For cubes $\varepsilon Q_z \in \Omega'_{\varepsilon,\eta} \cap \Omega''_{\varepsilon,\eta}$ we remove the set of points

$$\mathcal{R} = \{x \in \varepsilon Q_z : \operatorname{dist}(x, \partial\Omega) < C_0\varepsilon\}\,,$$

where C_0 is chosen so every $\widetilde{Q'_z} = \frac{17}{16}(\varepsilon Q_z \backslash \mathcal{R}) \subset \Omega$ and $\mathcal{I} \cap \varepsilon Q_z \subset \varepsilon Q_z \backslash \mathcal{R}$. We apply the estimates (6.4) and (6.5) to $\varepsilon Q_z \backslash \mathcal{R}$ with $\widetilde{Q'_z}$ taking the place of $\varepsilon \widetilde{Q_z}$. This creates a covering of $\mathcal{I}$. In words, for cubes that lie close to the boundary, we remove a small portion near the boundary to ensure the dilation remains in Ω.

Lemma 6.2.5. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$. Let $u_\varepsilon \in H^1_0(\Omega_{\varepsilon,\eta})$ be the solution of* (1.1) *with $F \in L^p(\Omega)$ and $f \in L^p(\Omega; \mathbb{R}^d)$ for some $p > 2$. Then*

$$\begin{aligned}\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le{}& C\varepsilon^{-1}\eta^{\frac{d}{p}-1}\|u_\varepsilon\|_{L^p(\Omega)} \\ &+ C\Phi_p(\eta^{-1})\left\{\|\varepsilon|F| + |f|\|_{L^p(\Omega)} + \|T_\varepsilon(F,f)\|_{L^p(\Omega)}\right\}\end{aligned} \tag{6.23}$$

for $d \ge 3$, and

$$\begin{aligned}\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le{}& C\varepsilon^{-1}\Phi_p(\eta^{-1})\|u_\varepsilon\|_{L^p(\Omega)} \\ &+ C\Phi_p(\eta^{-1})\left\{\|\varepsilon|F| + |f|\|_{L^p(\Omega)} + \|T_\varepsilon(F,f)\|_{L^p(\Omega)}\right\}\end{aligned} \tag{6.24}$$

for $d = 2$, where C depends on $d, p, \{Y^s_z\}$, and Ω.

Proof. Combining (6.18)-(6.19) and (6.21) along with taking $1/p$ powers yields the result. □

We now give the proofs of Theorems 1.5.1 and 1.5.2.

Proof of Theorem 1.5.1. Suppose that $0 < \sigma_\varepsilon \le 1$ and $2 < p < \infty$. Let u_ε be the solution of (1.1) with $F = 0$. We will now split into cases.

Case 1: $d \ge 3$.

First suppose $p > d$. It follows directly from (6.15) that

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le C\eta^{\frac{d}{p}-\frac{d}{2}}\|f\|_{L^p(\Omega)}. \tag{6.25}$$

Note that from Lemma 1.4.3 when $F = 0$, (6.25) holds with $p = 2$. Thus, by the Riesz Thorin Interpolation Theorem, (6.25) holds for $2 < p < \infty$.

Case 2: $d = 2$.

Recall when $d = 2$ that $\sigma_\varepsilon = \varepsilon|\ln\eta|^{1/2}$ and $\Phi_p(\eta^{-1}) = \eta^{\frac{2}{p}-1}|\ln\eta|^{-1}$. Then apply Theorem 1.6.1 to (6.24) we obtain

$$\begin{aligned}\|\nabla u_\varepsilon\|_{L^p(\Omega)} &\le C\varepsilon^{-1}\Phi_p(\eta^{-1})\|u_\varepsilon\|_{L^p(\Omega)} + C\Phi_p(\eta^{-1})\|f\|_{L^p(\Omega)} \\ &\le C\Phi_p(\eta^{-1})\varepsilon^{-1}\sigma_\varepsilon\|f\|_{L^p(\Omega)} \\ &= C\eta^{\frac{2}{p}-1}|\ln\eta|^{-\frac{1}{2}}\|f\|_{L^p(\Omega)},\end{aligned} \tag{6.26}$$

where we have used Theorem 6.2.1 in the second inequality, along with the fact that $\sigma_\varepsilon \le 1$. □

Proof of Theorem 1.5.2. Suppose $\sigma_\varepsilon \geq 1$ and $2 < p < \infty$. Let u_ε be the solution of (1.1) with $F = 0$. We again split by case.

Case 1: *$d \geq 2$ and $d < p < \infty$.*

It follows from (6.15) when $F = 0$,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{-1}\Phi_p(\eta^{-1})\|f\|_{L^p(\Omega)}. \tag{6.27}$$

Then from (6.3) we have,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq \begin{cases} C\varepsilon^{-1}\eta^{\frac{d}{p}-1}\|f\|_{L^p(\Omega)} & \text{if } d \geq 3 \text{ and } p > d, \\ C\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln\eta|^{-1}\|f\|_{L^p(\Omega)} & \text{if } d = 2 \text{ and } 2 < p < \infty, \end{cases} \tag{6.28}$$

which are the desired estimates.

Case 2: *$d \geq 3$ and $2 < p \leq d$.*

It follows from applying Theorem 1.6.1 to (6.23),

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^p(\Omega)} &\leq C\varepsilon^{-1}\eta^{\frac{d}{p}-1}\|u_\varepsilon\|_{L^p(\Omega)} + C\Phi_p(\eta^{-1})\|f\|_{L^p(\Omega)} \\ &\leq C\left(\varepsilon^{-1}\eta^{\frac{d}{p}-1} + \Phi_p(\eta^{-1})\right)\|f\|_{L^p(\Omega)}, \end{aligned} \tag{6.29}$$

where we have used Theorem 6.2.1 in the last inequality. From (6.3) we have

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq \begin{cases} C\left(\varepsilon^{-1}\eta^{\frac{d}{p}-1} + 1\right)\|f\|_{L^p(\Omega)} & \text{if } d \geq 3 \text{ and } 2 < p < d, \\ C\varepsilon^{-1} + |\ln\eta|^{1-\frac{1}{d}}\|f\|_{L^p(\Omega)} & \text{if } d \geq 3 \text{ and } p = d, \end{cases} \tag{6.30}$$

as desired. □

6.3 Upper Bounds for $B_p(\Omega_{\varepsilon,\eta})$

This section will establish upper bounds for $B_p(\Omega_{\varepsilon,\eta})$.

Remark 6.3.1. Let $1 < p < \infty$. Let $F \in L^p(\Omega_{\varepsilon,\eta})$ and $f \in L^p(\Omega_{\varepsilon,\eta}; \mathbb{R}^d)$. Let $u_\varepsilon \in W_0^{1,p}(\Omega_{\varepsilon,\eta})$ be the unique solution Dirichlet problem (1.1). Let $C_p(\Omega_{\varepsilon,\eta})$ and $D_p(\Omega_{\varepsilon,\eta})$ denote the smallest bounding constants for which

$$\|u_\varepsilon\|_{L^p(\Omega_{\varepsilon,\eta})} \leq C_p(\Omega_{\varepsilon,\eta})\|f\|_{L^p(\Omega_{\varepsilon,\eta})} + D_p(\Omega_{\varepsilon,\eta})\|F\|_{L^p(\Omega_{\varepsilon,\eta})} \tag{6.31}$$

holds. Moreover, by Theorem 6.2.1, if $2 \leq p < \infty$,

$$C_p(\Omega_{\varepsilon,\eta}) \leq C\begin{cases} \min\{1, \varepsilon\eta^{1-\frac{d}{2}}\} & \text{if } d \geq 3, \\ \min\{1, \varepsilon|\ln\eta|^{1/2}\} & \text{if } d = 2, \end{cases} \tag{6.32}$$

and

$$D_p(\Omega_{\varepsilon,\eta}) \leq C\begin{cases} \min\{1, \varepsilon^2\eta^{2-d}\} & \text{if } d \geq 3, \\ \min\{1, \varepsilon^2|\ln\eta|\} & \text{if } d = 2, \end{cases} \tag{6.33}$$

where the constants C depends only on d, p and c_0.

Lemma 6.3.2. *Let $1 < p < \infty$. Then*

$$A_p(\Omega_{\varepsilon,\eta}) = A_{p'}(\Omega_{\varepsilon,\eta}), \quad B_p(\Omega_{\varepsilon,\eta}) = C_{p'}(\Omega_{\varepsilon,\eta}), \quad \text{and} \quad D_p(\Omega_{\varepsilon,\eta}) = D_{p'}(\Omega_{\varepsilon,\eta}), \tag{6.34}$$

where $p' = \frac{p}{p-1}$.

The proof can be found in [14, Lemma 4.2].

Remark 6.3.3. The proof of Theorem 1.5.3 follows directly from (6.32) and Lemma 6.3.2.

Proof of 1.5.4. Let $0 < \sigma_\varepsilon \leq 1$ and $2 < p < \infty$. It follows from (6.15) with $f = 0$ that

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\Phi_p(\eta^{-1})(1 + \varepsilon^{-1}\min(\sigma_\varepsilon, 1))^2\|\varepsilon F\|_{L^p(\Omega)}. \tag{6.35}$$

This implies

$$B_p(\Omega_{\varepsilon,\eta}) \leq C\varepsilon\Phi_p(\eta^{-1})(1 + \varepsilon^{-1}\min(\sigma_\varepsilon, 1))^2. \tag{6.36}$$

In light of fact that $0 < \sigma_\varepsilon \leq 1$,

$$B_p(\Omega_{\varepsilon,\eta}) \leq C\Phi_p(\eta^{-1})\varepsilon^{-1}\sigma_\varepsilon^2. \tag{6.37}$$

We again split into cases.

Case 1: $d = 2$

Note that in this case $\Phi_p(\eta^{-1}) = \eta^{\frac{2}{p}-1}|\ln\eta|^{-1}$ and $\sigma_\varepsilon = \varepsilon|\ln\eta|^{1/2}$. Plugging into (6.37), we obtain

$$B_p(\Omega_{\varepsilon,\eta}) \leq C\varepsilon\eta^{\frac{2}{p}-1}, \tag{6.38}$$

which is the desired estimate (1.23).

Case 2: $d \geq 3$

Assume $p > d$. In this case we have $\sigma_\varepsilon = \varepsilon\eta^{1-\frac{d}{2}}$ and $\Phi_p(\eta^{-1}) = \eta^{\frac{d}{p}-1}$. Thus, plugging into (6.37), we have

$$B_p(\Omega_{\varepsilon,\eta}) \leq C\varepsilon\eta^{1-d+\frac{d}{p}}, \tag{6.39}$$

When $p = 2$, Lemma 1.4.3 yields (1.23). Using this fact combined with (6.39), the Riesz-Thorin Interpolation Theorem gives the desired estimate (1.23) for all $2 < p < \infty$. □

Proof of 1.5.5. Assume $\sigma_\varepsilon \geq 1$ and $2 < p < \infty$. Under the assumption $\sigma_\varepsilon \geq 1$ (6.36) now becomes

$$B_p(\Omega_{\varepsilon,\eta}) \leq C\Phi_p(\eta^{-1})\varepsilon^{-1} \tag{6.40}$$

By (6.3), we obtain

$$B_p(\Omega_{\varepsilon,\eta}) \leq \begin{cases} C\varepsilon^{-1}\eta^{\frac{d}{p}-1} & \text{if} \quad d \geq 3 \text{ and } d < p < \infty, \\ C\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln\eta|^{-1} & \text{if} \quad d = 2, \end{cases} \tag{6.41}$$

as desired.

We now handle the remaining case where $d \geq 3$ and $2 < p \leq d$. To do this let $u_\varepsilon \in W_0^{1,p}(\Omega_{\varepsilon,\eta})$ be a solution of $-\Delta u_\varepsilon = F$ in $\Omega_{\varepsilon,\eta}$ where $F \in L^p(\Omega)$. Extend u_ε to $\mathbb{R}^d$ by zero. Then applying Theorem 1.6.1 to (6.23) we obtain

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\varepsilon^{-1}\eta^{\frac{d}{p}-1}\|u_\varepsilon\|_{L^p(\Omega)} + C\Phi_p(\eta^{-1})\|F\|_{L^p(\Omega)}. \tag{6.42}$$

It follows from Theorem 6.2.1

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\left(\varepsilon^{-1}\eta^{\frac{d}{p}-1} + \Phi_p(\eta^{-1})\right)\|F\|_{L^p(\Omega)}, \tag{6.43}$$

where we have used the assumption $\sigma_\varepsilon \geq 1$. Finally applying (6.3) gives

$$B_p(\Omega_{\varepsilon,\eta}) \leq \begin{cases} C(1 + \varepsilon^{-1}\eta^{\frac{d}{p}-1}) & \text{if} \quad d \geq 3 \text{ and } 2 < p < d, \\ C(\varepsilon^{-1} + |\ln\eta|^{1-\frac{1}{d}}) & \text{if} \quad d \geq 3 \text{ and } p = d. \end{cases} \tag{6.44}$$

Combining (6.41) and (6.44) gives (1.24). □

Chapter 7 Sharpness

In this chapter we will provide results for the sharpness of Theorems 1.5.1 and 1.5.2 for $A_p(\Omega_{\varepsilon,\eta})$, and Theorems 1.5.3-1.5.5 for $B_p(\Omega_{\varepsilon,\eta})$. Our approach is similar to the sharpness estimates in [15] for the periodic, unbounded domain,

$$\omega_{\varepsilon,\eta} = \mathbb{R}^d \setminus \bigcup_{z \in \digamma^d} \varepsilon(z + \eta \overline{Y^s}), \tag{7.1}$$

where Y^s is a bounded domain with connected C^1 boundary such that $B(0, c_0) \subset Y^s \subset B((0, 1/4)$.

Following the idea of [7], we consider the ε-periodic corrector given by

$$-\Delta \psi_{\varepsilon,\eta} = \varepsilon^{-2}\eta^{d-2} \text{ in } \omega_{\varepsilon,\eta} \qquad \text{and} \qquad \psi_{\varepsilon,\eta} = 0 \text{ on } \mathbb{R}^d \setminus \omega_{\varepsilon,\eta}. \tag{7.2}$$

Lemma 7.0.1. *Let $\psi_{\varepsilon,\eta}$ be the ε-periodic function defined in (7.2). Then if $d \geq 3$,*

$$\fint_{Q(0,\varepsilon)} \psi_{\varepsilon,\eta} \approx 1 \qquad \textit{and} \qquad \left(\fint_{Q(0,\varepsilon)} |\nabla \psi_{\varepsilon,\eta}|^2 \right)^{1/2} \approx \varepsilon^{-1}\eta^{\frac{d-2}{2}}. \tag{7.3}$$

If $d = 2$, we have

$$\fint_{Q(0,\varepsilon)} \psi_{\varepsilon,\eta} \approx |\ln \eta| \qquad \textit{and} \qquad \left(\fint_{Q(0,\varepsilon)} |\nabla \psi_{\varepsilon,\eta}|^2 \right)^{1/2} \approx \varepsilon^{-1}|\ln \eta|^{1/2}. \tag{7.4}$$

Proof. The $\varepsilon = 1$ case was proven in [14, Lemma 4.4]. The case $0 < \varepsilon < 1$ follows from rescaling.

Lemma 7.0.2. *Let $\psi_{\varepsilon,\eta}$ be the ε-periodic function defined in (7.2). Then if $d \geq 3$,*

$$\left(\fint_{Q(0,\varepsilon)} |\psi_{\varepsilon,\eta}|^p \right)^{1/p} \leq C \qquad \textit{and} \qquad \left(\fint_{Q(0,\varepsilon)} |\nabla \psi_{\varepsilon,\eta}|^p \right)^{1/p} \geq C\varepsilon^{-1}\eta^{\frac{d}{p}-1}. \tag{7.5}$$

If $d = 2$, we have

$$\left(\fint_{Q(0,\varepsilon)} |\psi_{\varepsilon,\eta}|^p \right)^{1/p} \leq C|\ln \eta| \qquad \textit{and} \qquad \left(\fint_{Q(0,\varepsilon)} |\nabla \psi_{\varepsilon,\eta}|^p \right)^{1/p} \geq C\varepsilon^{-1}\eta^{\frac{2}{p}-1}. \tag{7.6}$$

Proof. The $\varepsilon = 1$ case was proved in [14, Lemma 5.3]. The general case follows by rescaling. □

The upper bounds for which were already shown in Theorem 6.3.1. We will then show some estimates for $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$, whose upper bounds are given in Theorems 1.5.1-1.5.5.

7.1 Sharpness for $C_p(\Omega_{\varepsilon,\eta})$ and $D_p(\Omega_{\varepsilon,\eta})$

This section will provide lower bounds for $C_p(\Omega_{\varepsilon,\eta})$ and $D_p(\Omega_{\varepsilon,\eta})$, which are defined in Remark 6.3.1. We start by showing the estimate (6.33) is sharp for $d \geq 2$ and $1 < p < \infty$.

Theorem 7.1.1. *Let $D_p(\Omega_{\varepsilon,\eta})$ be defined in* (6.31). *Let $1 < p < \infty$. Then*

$$D_p(\Omega_{\varepsilon,\eta}) \geq c \min(\sigma_\varepsilon^2, 1), \tag{7.7}$$

where c depends only on d, p, Y^s and Ω.

Proof. By Lemma 6.3.2 it suffices to consider when $1 < p \leq 2$. That is, once we have shown this case the result will follow by duality. By translation we may assume the origin is contained within Ω. Let r_0 be the radius of the ball such that

$$B(0, 4r_0) \subset \Omega.$$

Let ε be sufficiently small and let $\varepsilon R \leq r_0$ for some $R \in \mathbb{N}$. Let $\phi \in C_0^\infty(B(0, \varepsilon R))$ be a cutoff function such that $0 \leq \phi \leq 1$ with

$$\phi(x) = 1 \text{ in } B(0, \frac{1}{2}) \qquad \text{and} \qquad \phi(x) = 0 \;\text{ outside }\; B(0,1). \tag{7.8}$$

By rescaling,

$$\phi(\frac{x}{\varepsilon R}) = 1 \text{ in } B(0, \frac{\varepsilon R}{2}) \qquad \text{and} \qquad \phi(\frac{x}{\varepsilon R}) = 0 \;\text{ outside }\; B(0, \varepsilon R). \tag{7.9}$$

It is important to note that $B(0, \varepsilon R) \subset \Omega'_{\varepsilon,\eta}$ where we recall that $\Omega'_{\varepsilon,\eta}$ is defined as the union of all $\varepsilon-$cubes lying entirely inside of Ω. That is, our cutoff function stays sufficiently away from the boundary to avoid cubes that intersect $\partial\Omega$. By construction $|\nabla \phi| \leq C\varepsilon^{-1}R^{-1}$ and $|\nabla^2 \phi| \leq C\varepsilon^{-2}R^{-2}$.
Now,

$$-\Delta(\psi_{\varepsilon,\eta}\phi) = \varepsilon^{-2}\eta^{d-2}\phi - 2\nabla\psi_{\varepsilon,\eta} \cdot \nabla\phi - \psi_{\varepsilon,\eta}\Delta\phi \tag{7.10}$$

in $\Omega_{\varepsilon,\eta}$ with $\psi_{\varepsilon,\eta} = 0$ on $\partial\Omega_{\varepsilon,\eta}$. Thus, from (6.31) we have

$$\begin{aligned} &cR^{d/p}\|\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} \leq \|\psi_{\varepsilon,\eta}\phi\|_{L^p(\Omega_{\varepsilon,\eta})} \\ &\leq CD_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2+\frac{d}{p}}\eta^{d-2}R^{\frac{d}{p}} + \varepsilon^{-1}R^{\frac{d}{p}-1}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} + \varepsilon^{-2}R^{\frac{d}{p}-2}\|\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} \right\}, \end{aligned} \tag{7.11}$$

where we have used the definition of ϕ, the periodicity of $\psi_{\varepsilon,\eta}$, and the fact that the number of cubes constructing $\Omega'_{\varepsilon,\eta}$ is of order $R^{\frac{d}{p}}$. It follows that

$$\begin{aligned} &\varepsilon^{\frac{d}{p}} \fint_{Q(0,\varepsilon)} |\psi_{\varepsilon,\eta}| \leq \\ &CD_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{\frac{d}{p}-2}\eta^{d-2} + R^{-1}\varepsilon^{\frac{d}{p}-1} \left(\fint_{Q(0,\varepsilon)} |\nabla\psi_{\varepsilon,\eta}|^2 \right)^{1/2} + (\varepsilon R)^{-2}\|\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} \right\}. \end{aligned} \tag{7.12}$$

We may now separate by cases.

Case 1: *$d \geq 3$ and $0 < \sigma_\varepsilon \leq 1$.*

We apply Lemmas 7.0.1 and 7.0.2 to (7.12) to obtain

$$c \leq D_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2}\eta^{d-2} + R^{-1}\varepsilon^{-2}\eta^{\frac{d-2}{2}} + (\varepsilon R)^{-2} \right\}. \tag{7.13}$$

Since $\sigma_\varepsilon = \varepsilon\eta^{1-\frac{d}{2}} \leq 1$, we may choose $R \in \mathbb{N}$ such that $R \approx \eta^{-\frac{d-2}{2}}$ in (7.13), which gives

$$c \leq D_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2}\eta^{d-2} \right\}. \tag{7.14}$$

Hence

$$D_p(\Omega_{\varepsilon,\eta}) \geq c\varepsilon^2\eta^{2-d} = c\sigma_\varepsilon^2. \tag{7.15}$$

Note $\sigma_\varepsilon^2 = \min(\sigma_\varepsilon^2, 1)$.

Case 2: *$d \geq 3$ and $\sigma_\varepsilon \geq 1$.*

In this case we choose $R \in \mathbb{N}$ such that $R \approx \varepsilon^{-1}$ in (7.13). This gives

$$c \leq D_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2}\eta^{d-2} + \varepsilon^{-1}\eta^{\frac{d-2}{2}} + 1 \right\}.$$

Since $\sigma_\varepsilon \geq 1$, this implies

$$D_p(\Omega_{\varepsilon,\eta}) \geq c = c\min(\sigma_\varepsilon^2, 1). \tag{7.16}$$

Case 3: *$d = 2$ and $0 < \sigma_\varepsilon \leq 1$.*

Applying Lemmas 7.0.1 and 7.0.2 to (7.12) now yields

$$c|\ln\eta| \leq D_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2} + R^{-1}\varepsilon^{-2}|\ln\eta|^{\frac{1}{2}} + (\varepsilon R)^{-2}|\ln\eta| \right\}. \tag{7.17}$$

Picking $R \in \mathbb{N}$ such that $R \approx |\ln\eta|^{\frac{1}{2}}$ in (7.17) gives

$$D_p(\Omega_{\varepsilon,\eta}) \geq c\varepsilon^2|\ln\eta| = c\min(\sigma_\varepsilon^2, 1). \tag{7.18}$$

Case 4: *$d = 2$ and $\sigma_\varepsilon \geq 1$.*

Choosing $R \in \mathbb{N}$ such that $R \approx \varepsilon^{-1}$ in (7.17) yields

$$\begin{aligned} c|\ln\eta| &\leq D_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2} + \varepsilon^{-1}|\ln\eta|^{\frac{1}{2}} + |\ln\eta| \right\} \\ &\leq D_p(\Omega_{\varepsilon,\eta})|\ln\eta|. \end{aligned} \tag{7.19}$$

Thus,

$$D_p(\Omega_{\varepsilon,\eta}) \geq c = c\min(\sigma_\varepsilon^2, 1). \tag{7.20}$$

Combining (7.15), (7.16), (7.18), and (7.20), along with the fact $D_p(\Omega_{\varepsilon,\eta}) = D_{p'}(\Omega_{\varepsilon,\eta})$, yields the result. □

The estimate (6.32) for $C_p(\Omega_{\varepsilon,\eta})$ is also sharp for $d \geq 2$ and $2 \leq p < \infty$.

Theorem 7.1.2. *Let $C_p(\Omega_{\varepsilon,\eta})$ be defined in* (6.31). *Let $2 \leq p < \infty$.*

$$C_p(\Omega_{\varepsilon,\eta}) \geq c \min(\sigma_\varepsilon, 1), \tag{7.21}$$

where c depends only on d, p, Y^s and Ω.

Proof. By Lemma 6.3.2 it suffices to consider the estimate for $B_p(\Omega_{\varepsilon,\eta})$ when $1 < p \leq 2$. Let ϕ be defined as in (7.9). Then by (7.10) and (1.2) we have the following estimate

$$\begin{aligned}&\|\nabla(\psi_{\varepsilon,\eta}\phi)\|_{L^p(\Omega_{\varepsilon,\eta})} \leq \\ &CB_p(\Omega_{\varepsilon,\eta}) \left\{\varepsilon^{\frac{d}{p}-2}\eta^{d-2}R^{\frac{d}{p}} + \varepsilon^{-1}R^{\frac{d}{p}-1}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} + \varepsilon^{-2}R^{\frac{d}{p}-2}\|\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))}\right\}.\end{aligned} \tag{7.22}$$

Note here by a similar argument to (7.11)

$$\begin{aligned}\varepsilon^{\frac{d}{q}}R^{\frac{d}{q}} \fint_{Q(0,\varepsilon)} |\psi_{\varepsilon,\eta}| &\leq R^{\frac{d}{q}}\|\psi_{\varepsilon,\eta}\|_{L^q(Q(0,\varepsilon))} \\ &\leq \|\psi_{\varepsilon,\eta}\phi\|_{L^q(\Omega_{\varepsilon,\eta})} \\ &\leq \|\nabla(\psi_{\varepsilon,\eta}\phi)\|_{L^p(\Omega_{\varepsilon,\eta})},\end{aligned}$$

where the last inequality is by Sobolev embedding with $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$. Thus, (7.22) becomes

$$\begin{aligned}&\fint_{Q(0,\varepsilon)} |\psi_{\varepsilon,\eta}| \\ &\leq CB_p(\Omega_e)\left\{\varepsilon^{-1}\eta^{d-2}R + \left(\fint_{Q(0,\varepsilon)} |\nabla\psi_{\varepsilon\eta}|^2\right)^{1/2} + \varepsilon^{-1-\frac{d}{p}}R^{-1}\|\psi_{\varepsilon,\eta}\|_{L^p(\Omega_{\varepsilon,\eta})}\right\}.\end{aligned} \tag{7.23}$$

We again separate by cases here,

Case 1: *$d \geq 3$ and $\sigma_\varepsilon \leq 1$.*

We apply Lemmas 7.0.1 and 7.0.2 to (7.23) which yields

$$c \leq B_p(\Omega_{\varepsilon,\eta})\left\{\varepsilon^{-1}\eta^{d-2}R + \varepsilon^{-1}\eta^{\frac{d-2}{2}} + \varepsilon^{-1}R^{-1}\right\}. \tag{7.24}$$

Choose $R \in \mathbb{N}$ such that $R \approx \eta^{-\frac{d-2}{2}}$ in (7.24). Then

$$c \leq B_p(\Omega_{\varepsilon,\eta})\{\varepsilon^{-1}\eta^{\frac{d-2}{2}}\}.$$

Thus,

$$B_p(\Omega_{\varepsilon,\eta}) \geq c\sigma_\varepsilon = c\min(\sigma_\varepsilon, 1). \tag{7.25}$$

Case 2: *$d \geq 3$ and $\sigma_\varepsilon \geq 1$.*

Choose $R \in \mathbb{N}$ such that $R \approx \varepsilon^{-1}$ in (7.24). Then

$$c \le B_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2} + \varepsilon^{-1} \eta^{\frac{d-2}{2}} + 1 \right\}.$$

Thus using the fact $\sigma_\varepsilon \ge 1$,

$$B_p(\Omega_{\varepsilon,\eta}) \ge c = c \min(\sigma_\varepsilon, 1). \tag{7.26}$$

Case 3: *$d = 2$ and $\sigma_\varepsilon \le 1$.*

Applying Lemmas 7.0.1 and 7.0.2 to (7.23) now yields

$$c|\ln \eta| \le B_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-1} R + \varepsilon^{-1} |\ln \eta|^{\frac{1}{2}} + (\varepsilon R)^{-1} |\ln \eta| \right\}. \tag{7.27}$$

Picking $R \in \mathbb{N}$ such that $R \approx |\ln \eta|^{\frac{1}{2}}$ in (7.27) gives

$$c|\ln \eta| \le B_p(\Omega_{\varepsilon,\eta}) \{ \varepsilon^{-1} |\ln \eta|^{\frac{1}{2}} \}.$$

Thus,

$$B_p(\Omega_{\varepsilon,\eta}) \ge c\varepsilon |\ln \eta|^{\frac{1}{2}} = c \min(\sigma_\varepsilon, 1). \tag{7.28}$$

Case 4: *$d = 2$ and $\sigma_\varepsilon \ge 1$*

Choose $R \in \mathbb{N}$ so $R \approx \varepsilon^{-1}$ in (7.27). This gives

$$\begin{aligned} C|\ln \eta| &\le B_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-2} + \varepsilon^{-1} |\ln \eta|^{\frac{1}{2}} + \varepsilon^{-2} |\ln \eta| \right\} \\ &\le B_p(\Omega_{\varepsilon,\eta}) |\ln \eta|. \end{aligned}$$

This implies

$$B_p(\Omega_{\varepsilon,\eta}) \ge c = c \min(\sigma_\varepsilon, 1). \tag{7.29}$$

Combining (7.25),(7.26),(7.28), and (7.29) with the fact that $C_{p'}(\Omega_{\varepsilon,\eta}) = B_p(\Omega_{\varepsilon,\eta})$ yields the result. □

7.2 Sharpness for $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$

This section will provide lower bounds for both $A_p(\Omega_{\varepsilon,\eta})$ and $B_p(\Omega_{\varepsilon,\eta})$. The upper bounds are provided in Theorems 1.5.1-1.5.5.

We now consider $B_p(\Omega_{\varepsilon,\eta})$ for $2 < p < \infty$. The case where $1 < p \le 2$ was already shown in the proof of Theorem 7.1.2. We state the result for the remaining case here.

Theorem 7.2.1. *Let $B_p(\Omega_{\varepsilon,\eta})$ be defined in (1.2). Assume $d \ge 3$. Then if $0 < \sigma_\varepsilon \le 1$ we have*

$$B_p(\Omega_{\varepsilon,\eta}) \ge c\varepsilon \eta^{1-d+\frac{d}{p}}.$$

If $\sigma_\varepsilon \geq 1$ *with* $2 < p < d$*, we have*

$$B_p(\Omega_{\varepsilon,\eta}) \geq c(1 + \varepsilon^{-1}\eta^{\frac{d}{p}-1}),$$

where the constants c *only depend on* d, p*, and* Y^s*, and* Ω*.*

Proof. We separate by cases.

Case 1: $2 < p < d$ *and* $0 < \sigma_\varepsilon \leq 1$.

It follows from Lemma 1.4.1, along with the standard Poincaaré inequality, that

$$\|u\|_{L^p(\Omega_{\varepsilon,\eta})} \leq C \min{(\varepsilon\eta^{1-\frac{d}{p}}, 1)} \|\nabla u\|_{L^p(\Omega_{\varepsilon,\eta})} \tag{7.30}$$

for $u \in W_0^{1,p}(\Omega_{\varepsilon,\eta})$. As a result, we have

$$D_p(\Omega_{\varepsilon,\eta}) \leq C \min{(\varepsilon\eta^{1-\frac{d}{p}}, 1)} B_p(\Omega_{\varepsilon,\eta}). \tag{7.31}$$

By Theorem 7.1.1 we have

$$D_p(\Omega_{\varepsilon,\eta}) \geq c\sigma_\varepsilon^2 = c\varepsilon^2\eta^{2-d}.$$

This combined with (7.31) gives

$$B_p(\Omega_{\varepsilon,\eta}) \geq c\varepsilon\eta^{1-d+\frac{d}{p}}. \tag{7.32}$$

Case 2: $d \leq p < \infty$ *and* $0 < \sigma_\varepsilon \leq 1$.

We proceed with a convexity argument. Choose $2 < q < d$ and $t \in (0, 1)$ so that

$$\frac{1}{q} = \frac{1-t}{2} + \frac{t}{p}.$$

By the Riesz-Thorin Theorem we have

$$B_q(\Omega_{\varepsilon,\eta}) \leq [B_2(\Omega_{\varepsilon,\eta})]^{1-t}[B_p(\Omega_{\varepsilon,\eta})]^t.$$

It follows that

$$B_p(\Omega_{\varepsilon,\eta}) \geq [B_2(\Omega_{\varepsilon,\eta})]^{1-\frac{1}{t}}[B_q(\Omega_{\varepsilon,\eta})]^{\frac{1}{t}}.$$

The desired estimate follows from using the fact $B_2(\Omega_{\varepsilon,\eta}) \leq C\sigma_\varepsilon$ and $B_q(\Omega_{\varepsilon,\eta}) \geq c\varepsilon\eta^{1-d+\frac{d}{q}}$.

Case 3: $2 < p < d$ *and* $\sigma_\varepsilon \geq 1$.

Now from (7.31), we have

$$C\varepsilon\eta^{1-\frac{d}{p}} B_p(\Omega_{\varepsilon,\eta}) \geq 1,$$

which gives

$$B_p(\Omega_{\varepsilon,\eta}) \geq c\varepsilon^{-1}\eta^{\frac{d}{p}-1}. \tag{7.33}$$

From (7.31) we have

$$B_p(\Omega_{\varepsilon,\eta}) \geq cD_p(\Omega_{\varepsilon,\eta}) \geq c. \tag{7.34}$$

Combining (7.32), (7.33), and (7.34) yields the result. □

Finally, we establish the lower bounds for $A_p(\Omega_{\varepsilon,\eta})$.

Theorem 7.2.2. *Let $A_p(\Omega_{\varepsilon,\eta})$ be defined in* (1.2)*. Then for $2 < p < \infty$ and $\sigma_\varepsilon \leq 1$ we have*

$$A_p(\Omega_{\varepsilon,\eta}) \geq \begin{cases} c\eta^{-d|\frac{1}{2}-\frac{1}{p}|} & \text{if } d \geq 3, \\ c\eta^{-2|\frac{1}{2}-\frac{1}{p}|}|\ln \eta|^{-\frac{1}{2}} & \text{if } d = 2, \end{cases}$$

where c only depends on d, p, and c_0. Furthermore, if $\sigma_\varepsilon \geq 1$ we have

$$A_p(\Omega_{\varepsilon,\eta}) \geq \begin{cases} c(1+\varepsilon^{-1}\eta^{\frac{d}{p}-1}) & \text{if } 2 < p < d, \\ c\varepsilon^{-1} & \text{if } p = d, \\ c\varepsilon^{-1}\eta^{\frac{d}{p}-1} & \text{if } d < p < \infty, \\ c\varepsilon^{-1}\eta^{\frac{2}{p}-1}|\ln \eta|^{-1} & \text{if } d = 2, \end{cases}$$

where c only depends on d, p, Y^s, and Ω.

Proof. Let u_ε be a weak solution to $-\Delta u_\varepsilon = \text{div}(f)$ in $\Omega_{\varepsilon,\eta}$ with $u_\varepsilon = 0$ on $\partial\Omega_{\varepsilon,\eta}$. By a Sobolev embedding, we have

$$\|u_\varepsilon\|_{L^{q'}(\Omega_{\varepsilon,\eta})} \leq C\|\nabla u_\varepsilon\|_{L^{p'}(\Omega_{\varepsilon,\eta})} \leq A_p(\Omega_{\varepsilon,\eta})\|f\|_{L^{p'}(\Omega_{\varepsilon,\eta})},$$

where $\frac{1}{q'} = \frac{1}{p'} + \frac{1}{d}$ and $1 < p' < d$. By duality this implies that if $-\Delta v_\varepsilon = G$ in $\Omega_{\varepsilon,\eta}$ and $v_\varepsilon = 0$ on $\partial\Omega_{\varepsilon,\eta}$, then

$$\|\nabla v_\varepsilon\|_{L^p(\Omega_{\varepsilon,\eta})} \leq CA_p(\Omega_{\varepsilon,\eta})\|G\|_{L^q(\Omega_{\varepsilon,\eta})}.$$

Thus if $-\Delta u_\varepsilon = F + \text{div}(f)$ in $\Omega_{\varepsilon,\eta}$ with $u_\varepsilon = 0$ on $\partial\Omega_{\varepsilon,\eta}$, then

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_{\varepsilon,\eta})} \leq CA_p(\Omega_{\varepsilon,\eta})\left\{\|F\|_{L^q(\Omega_{\varepsilon,\eta})} + \|f\|_{L^p(\Omega_{\varepsilon,\eta})}\right\}, \tag{7.35}$$

where $d' < p < \infty$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$.
Let ϕ and $\psi_{\varepsilon,\eta}$ be as defined in (7.9) and (7.2) respectively. Then

$$-\Delta(\psi_{\varepsilon,\eta}\phi) = \varepsilon^{-2}\eta^{d-2}\phi - 2div(\psi_{\varepsilon,\eta}\nabla\phi) + \psi_{\varepsilon,\eta}\Delta\phi$$

in $\Omega_{\varepsilon,\eta}$ with $\psi_{\varepsilon,\eta}\phi = 0$ on $\partial\Omega_{\varepsilon,\eta}$. Hence from (7.35) we have

$$\|\nabla(\psi_{\varepsilon,\eta}\phi)\|_{L^p(\Omega_{\varepsilon,\eta})} \leq$$
$$CA_p(\Omega_{\varepsilon,\eta})\left\{\varepsilon^{\frac{d}{q}-2}\eta^{d-2}R^{\frac{d}{q}} + \varepsilon^{-1}R^{\frac{d}{p}-1}\|\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} + \varepsilon^{-2}R^{\frac{d}{q}-2}\|\psi_{\varepsilon,\eta}\|_{L^q(Q(0,\varepsilon))}\right\}.$$

Note from the periodicity of $\psi_{\varepsilon,\eta}$

$$\|\nabla(\psi_{\varepsilon,\eta}\phi)\|_{L^p(\Omega_{\varepsilon,\eta})} \geq \|\nabla\psi_{\varepsilon,\eta}\|_{L^p(B(0,\varepsilon R))} \approx R^{\frac{d}{p}}\|\nabla\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))}.$$

This combined with the fact that $\frac{d}{q} = \frac{d}{p} + 1$ gives

$$\begin{aligned} &\|\nabla\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} \\ &\leq A_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{\frac{d}{p}-1}\eta^{d-2}R + \varepsilon^{-1}R^{-1}\|\psi_{\varepsilon,\eta}\|_{L^p(Q(0,\varepsilon))} + \varepsilon^{-2}R^{-1}\|\psi_{\varepsilon,\eta}\|_{L^q(Q(0,\varepsilon))} \right\}. \end{aligned} \tag{7.36}$$

We will now separate by cases.

Case 1: *$d \geq 3$ and $0 < \sigma_\varepsilon \leq 1$.*

Applying Lemma 7.0.2 to (7.36) yields

$$\varepsilon^{\frac{d}{p}-1}\eta^{\frac{d}{p}-1} \leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{\frac{d}{p}-1}\eta^{d-2}R + \varepsilon^{\frac{d}{p}-1}R^{-1} \right\}.$$

Dividing over gives

$$\eta^{\frac{d}{p}-1} \leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ \eta^{d-2}R + R^{-1} \right\}. \tag{7.37}$$

Picking $R \in \mathbb{N}$ such that $R \approx \eta^{-\frac{d-2}{2}}$ in (7.37) gives

$$\eta^{\frac{d}{p}-1} \leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ \eta^{\frac{d-2}{2}} \right\}.$$

This implies

$$A_p(\Omega_{\varepsilon,\eta}) \geq C\eta^{\frac{d}{p}-\frac{d}{2}} = C\eta^{-d|-\frac{1}{p}+\frac{1}{2}|}. \tag{7.38}$$

Case 2: *$d \geq 3$ and $\sigma_\varepsilon \geq 1$*

Choosing $R \approx \varepsilon^{-1}$ in (7.37) gives

$$\begin{aligned} \eta^{\frac{d}{p}-1} &\leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ \eta^{d-2}\varepsilon^{-1} + \varepsilon \right\} \\ &\leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ \eta^{\frac{d-2}{2}} + \varepsilon \right\} \\ &\leq C\varepsilon A_p(\Omega_{\varepsilon,\eta}). \end{aligned}$$

Thus we have

$$A_p(\Omega_{\varepsilon,\eta}) \geq c\varepsilon^{-1}\eta^{\frac{d}{p}-1}. \tag{7.39}$$

Additionally, we have in the case where $2 < p < d$,

$$A_p(\Omega_{\varepsilon,\eta}) \geq C_p(\Omega_{\varepsilon,\eta}) = B_{p'}(\Omega_{\varepsilon,\eta}) \geq c, \tag{7.40}$$

where the first inequality stems from a standard Poincaré inequality and the second inequality was shown in Theorem 7.1.2.

Case 3: *$d = 2$ and $0 < \sigma_\varepsilon \leq 1$.*

Applying Lemma 7.0.2 to (7.36) now yields

$$\eta^{\frac{2}{p}-1} \leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ R + |\ln \eta| R^{-1} \right\}. \tag{7.41}$$

Picking $R \approx |\ln \eta|^{\frac{1}{2}}$ in (7.41) gives

$$\eta^{\frac{2}{p}-1} \leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ |\ln \eta|^{\frac{1}{2}} \right\}.$$

This implies

$$A_p(\Omega_{\varepsilon,\eta}) \geq c\eta^{-2|\frac{1}{2}-\frac{1}{p}|} |\ln \eta|^{-\frac{1}{2}}. \tag{7.42}$$

Case 4: *$d = 2$ and $\sigma_\varepsilon \geq 1$*

Now take $R \approx \varepsilon^{-1}$ in (7.41). This gives

$$\begin{aligned} \eta^{\frac{2}{p}-1} &\leq CA_p(\Omega_{\varepsilon,\eta}) \left\{ \varepsilon^{-1} + \varepsilon |\ln \eta| \right\} \\ &\leq C\varepsilon A_p(\Omega_{\varepsilon,\eta}) |\ln \eta|. \end{aligned}$$

This implies

$$A_p(\Omega_{\varepsilon,\eta}) \geq c\varepsilon^{-1} \eta^{\frac{2}{p}-1} |\ln \eta|^{-1}. \tag{7.43}$$

Combining (7.38),(7.39),(7.40), (7.42), and (7.43) completes the proof.

□

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