Geometry of Pipe Dream Complexes

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Digital Object Identifier: https://doi.org/10.13023/etd.2023.367

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Geometry of Pipe Dream Complexes

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

Geometry of Pipe Dream Complexes

In this dissertation we study the geometry of pipe dream complexes with the goal of gaining a deeper understanding of Schubert polynomials. Given a pipe dream complex PD(w) for w a permutation in the symmetric group, we show its boundary is Whitney stratified by the set of all pipe dream complexes PD(v) where $v > w$ in the strong Bruhat order. For permutations w in the symmetric group on n elements, we introduce the pipe dream complex poset $\mathscr{P}(n)$. The dual of this graded poset naturally corresponds to the poset of strata associated to the Whitney stratification of the boundary of the pipe dream complex of the identity element. We examine pipe dream complexes in the case a permutation is a product of commuting adjacent transpositions. Finally, we consider pattern avoidance results. For 132-avoiding permutations, the Rothe diagram forms a Young diagram. In the case a permutation w has exactly one 132-pattern, the associated pipe dream complex is an m-dimensional simplex, where $m = \binom{n}{2} - \ell(w) - 1$ and $\ell(w)$ is the length of w. In the case of exactly two 132 patterns, there are three possible configurations. We include generalizations of these cases.

KEYWORDS: Schubert Polynomials, Pipe Dream Complexes, Pattern Avoidance

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August 9, 2023

Geometry of Pipe Dream Complexes

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Dedicated to my parents who spared no effort affording me the opportunity to write this dissertation.

ACKNOWLEDGMENTS

I would like to thank my advisor Prof. Readdy for working countless hours to help me research and write this dissertation.

I would also like to thank my parents for being supportive and accommodating throughout my years of graduate schooling.

I would also like to thank Prof. Ehrenborg for his comments, showing me his insights, and helping me improve this dissertation.

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Chapter 1 Introduction

The setting of this dissertation lies within the intersection of algebraic geometry, representation theory and combinatorics. Let $\mathcal{F}\ell_n$ be the manifold of complete flags in the vector space $\mathbb{C}^n$. The cohomology ring $H^*(\mathcal{F}\ell_n)$ is equivalent to the polynomial ring $\mathbb{Z}[x_1,\ldots,x_n]$ modulo the ideal generated by all nonconstant homogeneous functions invariant under permutation of the indices [27]. An algebraic basis is given by factors of the monomial $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1$, but does not provide a basis reflecting the geometry of the group. Instead, the Schubert classes, which are cohomology classes of Schubert varieties X_w indexed by permutations $w \in \mathfrak{S}_n$, accommodate the geometry of these structures along with their representative polynomials, known as the Schubert polynomials. Lascoux and Schützenberger developed the Schubert polynomials as a generalization of the Schur polynomials to represent the Schubert classes. It is precisely this object, the Schubert polynomials, that this dissertation will focus upon.

In this chapter we introduce the Schubert polynomial and three methods to compute it: via Rothe diagrams and RC-graphs, **a**-compatible sequences, and orthodontic sequences. We then recall classical pattern avoidance results. We will see that the occurrence of the 132-pattern will play an important role in inequalities for bounding the sum of the coefficients of a given Schubert polynomial, and more generally, in the geometry of the associated pipe dream complex. We end this chapter with a review of the notions of Whitney stratifications and shelling orders.

1.1 The Schubert polynomial

In order to define the Schubert polynomial, we begin by defining the divided difference operator on a multivariate polynomial. For $f = f(x_1,...,x_n) \in \mathbb{Z}[x_1,...,x_n]$ and for $i \in \{1,\ldots,n-1\}$, define the *divided difference operators*

$$\partial_i(f) = \frac{f(x_1,\ldots,x_n) - s_i f(x_1,\ldots,x_n)}{x_i - x_{i+1}},$$

where $s_i = (i, i+1)$ denotes the ith simple reflection in the symmetric group $\mathfrak{S}_n$. The reflection s_i acts on a polynomial by replacing the variable x_i with x_{i+1} and vice versa. For a permutation $w \in \mathfrak{S}_n$, where $w = s_{i_l}\cdots s_{i_1}$ is a reduced word for w read right to left, we denote the operator ∂_w to be the sequence of divided difference operators

$$\partial_w(f) = \partial_{i_l}\cdots\partial_{i_1}(f).$$

The operator ∂_w is well-defined because the divided difference operators satisfy the braid relations

$$\partial_i^2 = 0, \tag{1.1.1}$$
$$\partial_i\partial_j = \partial_j\partial_i, \text{ for } |i-j| \geq 2, \tag{1.1.2}$$
$$\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}. \tag{1.1.3}$$

	1	2	3	4	5	6	7	8	9	10
1	X	X	•							
2	X	X		X	X	X	X	•		
3	X	X		X	•					
4	X	X		X		X	X		X	•
5	X	X		X		X	X		•	
6	X	X		•						
7	X	X				•				
8	•									
9		X					•			
10		•								

Figure 1.1: Rothe's picture of a diagram for the permutation $\pi = (8, 10, 1, 6, 3, 7, 9, 2, 5, 4)$ appearing in [40, page 279] of his 1800 publication, "X. Über Permutationen". This diagram is now known as the Rothe diagram of a permutation, where the X's are replaced with +'s and the dots are suppressed.

For a permutation $w \in \mathfrak{S}_n$, the *Schubert polynomial* $\mathfrak{S}_w$ is

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1,$$

where $w_0 = n \cdots 321$ is the longest word in $\mathfrak{S}_n$.

For example, consider $w = 132 \in \mathfrak{S}_3$. We have $w^{-1}w_0 = s_2s_2s_1s_2 = s_1s_2$. The Schubert polynomial for w is

$$\mathfrak{S}_w = \partial_1\partial_2(x_1^2x_2) = \partial_1(x_1^2) = x_1 + x_2.$$

There are other ways to compute Schubert polyonomials. We will visit these in the next sections.

1.2 Rothe diagrams and RC-graphs

In 1800 Rothe introduced the Rothe diagram for a permutation $w \in \mathfrak{S}_n$ [40]. The *Rothe Diagram* $D(w)$ for w is the set of ordered pairs

$$D(w) = \{(i, j) \in [n]^2 : i < w(j), j < w^{-1}(i)\}.$$

We construct the Rothe Diagram $D(w)$ on an $n \times n$ matrix as follows: Given a permutation $w \in \mathfrak{S}_n$, plot a point at each of the locations $(i, w(i))$ for $i = 1, \ldots, n$. Cross out all of the elements of the hook $(i, w(i))$, that is, the boxes (i, j) for $w(i) \leq j \leq n$ and $(k, w(i))$ for $i \leq k \leq n$. Finally, place a cross in each of the remaining empty boxes. The set of crosses is the Rothe diagram $D(w)$. See Rothe's example from his 1800 publication in Figure 1.1. It is straightforward to check that $D(w^{-1}) = D(w)^T$.

In 1993 Fomin and Kirillov introduced objects to study double Schubert polynomials [15]. These specialize to Schubert polynomials when $y_1 = y_2 = \cdots = y_n = 0$.

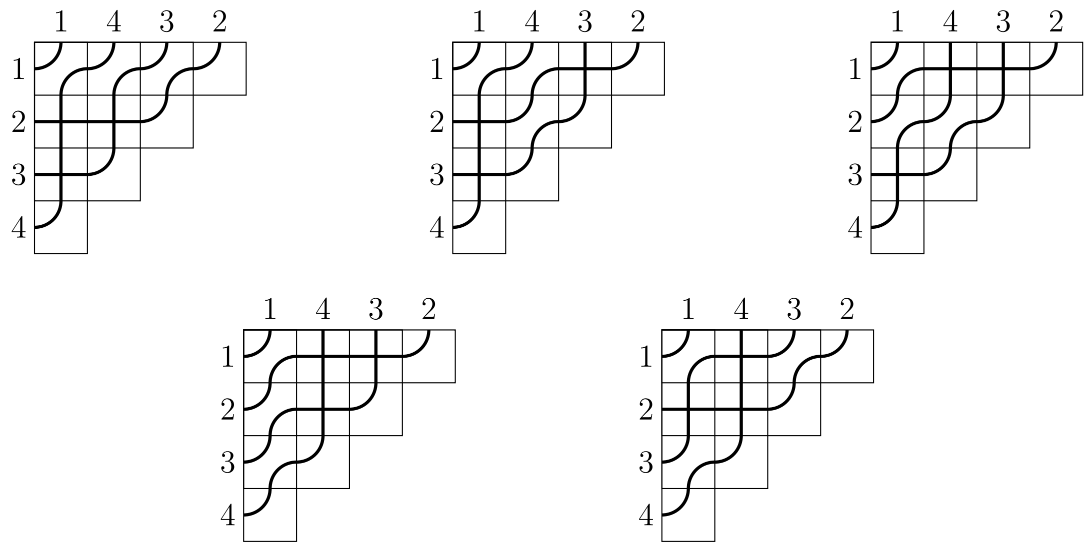


Figure 1.2: The five reduced pipe dreams for the permutation $w = 1432$

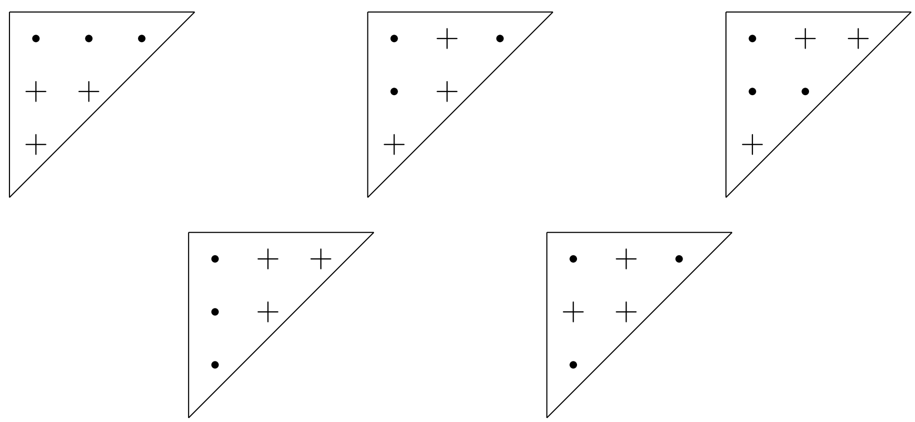
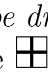


Figure 1.3: Dot-cross diagrams for the reduced pipe dreams of the permutation $w = 1432$

Bergeron and Billey called the resulting diagrams RC-graphs and further studied their properties [3].

For $w \in \mathfrak{S}_n$ a *pipe dream* is a tiling of the order n triangular array with two tiles, namely the cross tile  and the elbow tile , so that if one labels the rows on the left with 1 through n and follow the strands, this gives the one line notation for the permutation w. Furthermore, a pipe dream is *reduced* if no two strands cross more than once. See Figure 1.2 for an example. If some pair of pipes cross more than once, we say the pipe dream is *non-reduced*, and we ignore extra crossings as we follow the strands. See Figure 1.4 for the non-reduced pipe dreams for $w = 1432$.

Bergeron and Billey developed another presentation for pipe dreams where a dot represents a non-crossing tile and a "+" represents a crossing tile [3]. See Figure 1.3 for an example. These are the same five pipe dreams from Figure 1.2 using their notation.

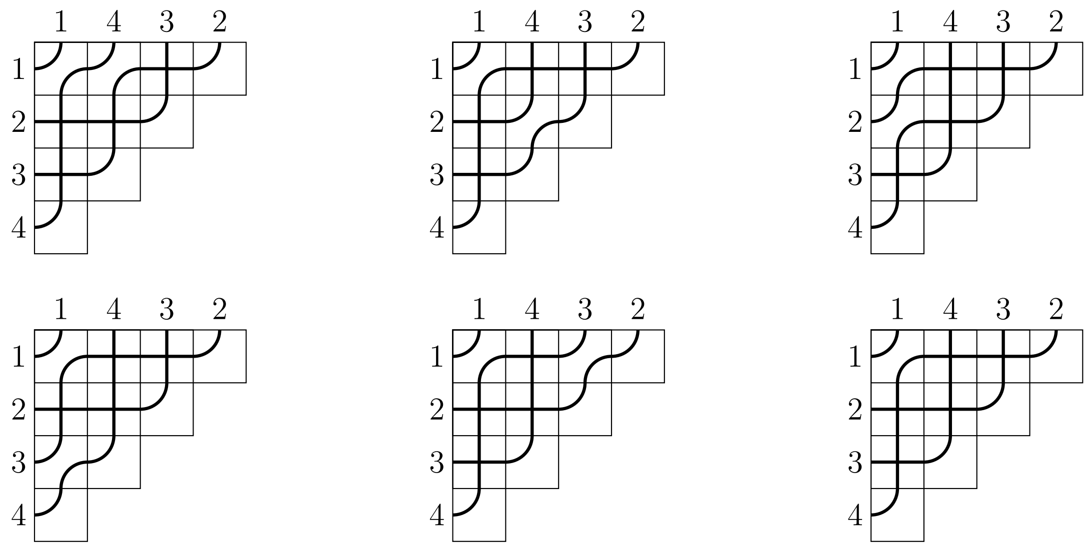


Figure 1.4: The six non-reduced pipe dreams for the permutation $w = 1432$

1.3 Other methods to compute Schubert polynomials

There are other ways to compute Schubert polynomials. We now review three of them.

1.3.1 a-compatible sequences

In 1993 Billey, Jockusch, and Stanley defined Schubert polynomials in terms of **a**-compatible sequences [6]. A *reduced decomposition* of $w \in \mathfrak{S}_n$ is a sequence $\mathbf{a} = (a_1, \ldots, a_p)$ such that $w = s_{a_1} \cdots s_{a_p}$ is a reduced word for w. Let $R(w)$ denote the set of reduced decompositions of w.

Given a reduced decomposition $\mathbf{a} \in R(w)$, define a sequence $\alpha = (i_1, \ldots, i_p)$ of positive integers to be an *a-compatible sequence* if it obeys the following conditions:

$$i_1 \leq i_2 \leq \cdots \leq i_l, \tag{1.3.1}$$

$$i_j \leq a_j, \text{ for } j = 1, \ldots, p, \tag{1.3.2}$$

$$i_j < i_{j+1} \text{ whenever } a_j < a_{j+1}. \tag{1.3.3}$$

Billey–Jockusch–Stanley [6, Theorem 1.1] showed the Schubert polynomial for w can be expressed in the following manner.

Theorem 1.3.1 (Billey–Jockusch–Stanley)**.** *Let $w \in \mathfrak{S}_n$. Then the Schubert polynomial for w is given by*

$$\mathfrak{S}_w = \sum_{\boldsymbol{a} \in R(w)} \sum_{\alpha \in K(\boldsymbol{a})} x_{i_1} x_{i_2} \cdots x_{i_p}, \tag{1.3.4}$$

where $K(\boldsymbol{a})$ is the set of all $\boldsymbol{a}$-compatible sequences generated by $\boldsymbol{a} \in R(w)$.

Billey, Jockusch, and Stanley used this method to show that the Schubert polynomials for permutations avoiding 321 are flag skew Schur functions [6, Theorem 2.1].

1.3.2 Orthodontic sequences.

In 1998 Magyar developed a method to compute the Schubert polynomial for a word using the orthodontic sequence of its Rothe diagram [32].

Let $D = (C_1, \dots C_n)$ be the columns of the Rothe diagram $D(w)$. We view a column C_i as a subset of $\{1, \dots, n\}$, where crosses appear at the matrix coordinates (a, i) for each element $a \in C_i$. We will construct the orthodontic sequence for D by applying the following algorithm and recording our actions.

First we consider all columns of the form $C = [i] = \{1, \dots, i\}$. We define $Mult_D([i])$ to be the number of columns in D with the form $[i]$. We set $k_i = \mathrm{mult}_D([i])$ for $i = 1, \dots, n$ and obtain a new diagram D_- by removing all columns of this form. For a column C not of the form $C = [i]$, we define a *missing tooth* to be a positive integer i such that $i \notin C$ but $i+1 \in C$. We swap the rows i and $i+1$ in the diagram D_- to construct a new diagram $D' = \{s_{i_1}C_1, \dots s_{i_1}C_l\}$. Our objective is to swap rows to tighten the orthodontia so that we obtain columns of the form $C = [i]$.

We repeat the above steps for D', letting $m_1 = \mathrm{mult}_{D'}([i_1])$. We then remove these columns to make a diagram D'_-. We find a missing tooth of D'_-, namely i_2 and perform the row swap s_{i_2} so that $D'' = s_{i_2}D'_-$. Repeat until all columns have been removed. We obtain a reduced word $\mathbf{i} = (i_1, \dots, i_m)$ and a multiplicity list $\mathbf{m} = (k_1, \dots, k_n, m_1, \dots, m_l)$. This sequence $(\mathbf{i}, \mathbf{m})$ is called the *orthodontic sequence* of $D \cong D_{\mathbf{i},\mathbf{m}}$, Magyar then showed that the orthodontic sequence can be used to compute the Schubert polynomials for a word $w \in \mathfrak{S}_n$ [32, Proposition 15]. Magyar defines an operator Λ_i on polynomials where $\Lambda_i(f) = \partial_i(x_i f)$.

Theorem 1.3.2 (Magyar)**.** *If* $(\boldsymbol{i}, \boldsymbol{m})$ *is the orthodontic sequence for the Rothe diagram of* w*, then*

$$\mathfrak{S}_w = \bar{\omega}_1^{k_1} \dots \bar{\omega}_n^{k_n} \Lambda_{i_1}(\bar{\omega}_{i_1}^{m_1} \dots \Lambda_{i_l}(\bar{\omega}_{i_l}^{m_l}) \dots),$$

where $\bar{\omega}_i = x_1 x_2 \dots x_i$.

For example, Figure 1.5 demonstrates the orthodontia algorithm applied to the permutation $w = 1432$. The orthodontic sequence for w is given by

$$\mathbf{i} = (1, 2), \mathbf{m} = (0, 0, 0, 0, 1, 1).$$

Therefore the Schubert polynomial for w is given by

$$\mathfrak{S}_w = \Lambda_1(x_1 \Lambda_2(x_1 x_2)) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3.$$

Note that Magyar's arrangement of a Rothe diagram is the transpose of Rothe's arrangement. Applying this algorithm to a Rothe diagram in Rothe's notation involves performing orthodontia on the rows instead of the columns.

1.3.3 RC-graphs.

As previously mentioned, pipe dreams and Schubert polynomials are intimately related. In 1993 Bergeron and Billey showed that the Schubert polynomials for a

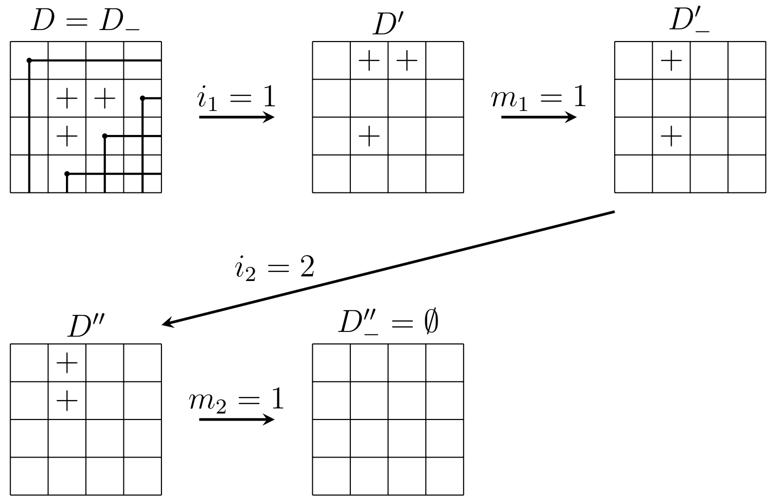


Figure 1.5: Magyar's orthodontic sequence for the permutation 1432. See [32, pp. 27-28] for an example with $w = 24153$.

permutation w is given by the set of crosses in $RP(w)$ according to the following formula [3, Corollary 3.3]. See Fomin and Kirillov for related results [15, Proposition 6.2, Proposition 6.4].

Theorem 1.3.3 (Bergeron–Billey)**.** *The Schubert polynomial for $w \in \mathfrak{S}_n$ is given by*

$$\mathfrak{S}_w = \sum_{P \in RP(w)} x_1^{e_1} x_2^{e_2} \cdots x_{n-1}^{e_{n-1}}, \tag{1.3.5}$$

where e_i is the number of crosses in column i of a reduced RC-graph.

1.4 Pattern occurrence and pattern avoidance

Let $w \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$ be two permutations with $m \leq n$, and let $v = v_1, \ldots, v_m$ be a subsequence of w with m letters. We say v has the *pattern* π if the letters in v have the same relative order as those in π. If w contains a subsequence v that has the pattern π, we say w *contains the pattern* π. If w contains no such subsequence, we say w *avoids the pattern* π, that is w is π*-avoiding*. Moreover, we denote $p_\pi(w)$ to be the number of times the pattern π appears in the permutation w.

An application of pattern avoidance is found in the study of Schubert varieties. Denote B to be the group of upper triangular matrices in $GL_n(\mathbb{C})$. The flag variety FL_n, that is, the set complete flags in $\mathbb{C}^n$, is in bijection with the left cosets of $B \backslash GL_n$. We see this bijection by encoding a complete flag with a set of row vectors $v_i \in V_i \setminus V_{i-1}$ for $i = 1, \ldots, n$ and forming an $n \times n$ matrix. The flag represented by this matrix is invariant under left multiplication by an element of B. Let P_w be the permutation matrix for $w \in \mathfrak{S}_n$. Acting on $B \backslash GL_n$ by the right cosets of B yields the decomposition of double cosets

$$B \backslash GL_n / B = \dot{\bigcup}_{w \in \mathfrak{S}_n} BP_wB.$$

The *Schubert variety* associated to w is the closure of the space BP_wB.

In 1990 Lakshmibai and Sandhya showed a pattern avoidance criteria for the smoothness of the Schubert variety associated to w [29, Theorem 1].

Theorem 1.4.1 (Lakshmibai–Sandhya)**.** *For $w \in \mathfrak{S}_n$, the Schubert variety X_w is smooth if and only if w avoids the patterns* 4231 *and* 3412*.*

The *Poincaré polynomial* of a topological space X is the generating function of its Betti numbers, i.e., the coefficient of x^m is the Betti number $b_m(X)$. Gasharov proved that the same pattern avoidance criteria allows the Poincaré polynomial of the Bruhat order on the symmetric group to factor [18, Theorems 1.1 and 1.2].

Theorem 1.4.2 (Gasharov)**.** *The Poincaré polynomial for $w \in \mathfrak{S}_n$ factors into polynomials of the form $1 + t + t^2 + \cdots + t^r$ if and only if w avoids the patterns* 4231 *and* 3412*. Equivalently, a type A Schubert variety is smooth if and only if the Poincaré polynomial of its cohomology ring factors into polynomials of the form $\sum_{i=0}^{r} t^{2i}$.*

Billey proved a pattern avoidance criteria for rational smoothness for Schubert varieties of type B and C [5, Theorem 4.2]. This involved 26 patterns for signed permutations to avoid.

1.5 132-avoidance and Rothe diagrams

We will see in Chapter 2 that the occurrence of the 132 pattern is integral to understanding the structure of pipe dream complexes. We first consider the case of 132 pattern avoidance.

The following result appears to be folklore.

Theorem 1.5.1. *A permutation $w \in \mathfrak{S}_n$ is* 132*-avoiding if and only if its Rothe diagram $D(w)$ is a Young diagram.*

This follows from Krathenthaler's stated bijection [28, Section 2] between 132-avoiding permutations in $\mathfrak{S}_n$ and Dyck paths from the origin to $(2n, 0)$. Krattenhaler did not realize the connection between the Dyck path he mapped to a 132-avoiding word w and the Young diagram corresponding to the Rothe diagram $D(w)$. Namely, given a Dyck path, look at the partition between it and the path from the origin to $(2n, 0)$ consisting of n northeast steps followed by n southeast steps. See Figure 1.7 for an example. Rotating this region by $\pi/2$ radians counterclockwise and filling it with crosses gives the desired Young diagram. This construction is easily seen to be invertible. We provide a second proof in Section 3.2.

1.6 Coefficients of Schubert polynomials

1.6.1 Bounds on coefficient values

Fink, Mészáros and St. Dizier used Rothe diagrams and Magyar's theory of orthodontic sequences [32] to prove there is a pattern avoidance criteria for Schubert polynomials having coefficients that are only zeros and ones. Namely, they showed if a

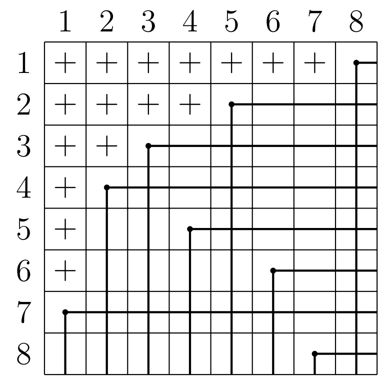


Figure 1.6: Rothe diagram for the 132-avoiding permutation $w = 74352681$. Note that the formation of $+'s$ is a Young diagram.

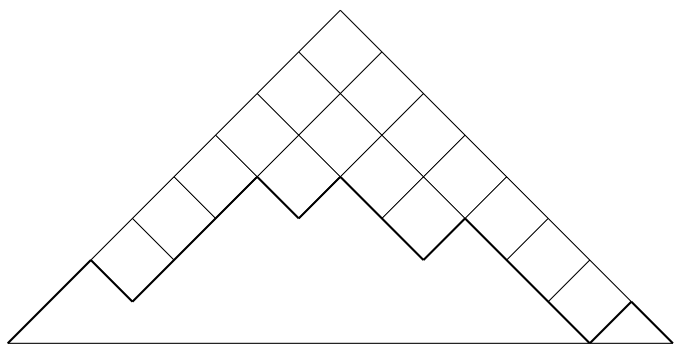


Figure 1.7: Dyck path for the permutation $w = 74352681$ according to Krathenthaler's bijection with 132-avoiding permutations. Note that the grid between the path and the upper triangle is the Young diagram for w rotated clockwise by $\pi/4$.

permutation avoids a set of twelve patterns then the coefficients are only zeros and ones [14, Theorem 1]. They defined a condition known as "multiplicity free" on a Rothe diagram and its orthodontic sequence and then showed that $\mathfrak{S}_w$ has zero-one coefficients if and only if this condition holds [14, Theorem 3.6]. The "multiplicity free" condition is then shown to be equivalent to a permutation w avoiding a set of twelve patterns [14, Theorem 4.8]. Observe each of these patterns contains the pattern 132 at least once.

Theorem 1.6.1 (Fink–Mészáros–St. Dizier)**.** *The Schubert polynomial $\mathfrak{S}_w$ is zero-one if and only if w avoids the patterns 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642.*

Finding a criteria that determines whether the upper bound for Schubert polynomial coefficients is 2, 3, etc. is an open problem. See Conjecture 4.0.1 in Chapter 3. There exist permutations in $\mathfrak{S}_6$ that have coefficients exceeding 2. For example, the permutations 124653, 126543, 134265, 134652, 136254, 136524, 136542, 142653, 143265, 143652, 214653, 216543, and 241653 have coefficients up to 3, and the permutation $w = 132654$ has coefficients up to 4.

Conjecture 1.6.2. *The Schubert polynomial $\mathfrak{S}_w$ has a maximum coefficient of k for $k \geq 2$ if and only if the permutation w avoids a set of patterns.*

1.6.2 The principle specialization

Let $w \in \mathfrak{S}_n$ and $\mathfrak{S}_w$ be its Schubert polynomial. We define the *principle specialization* to be $\mathfrak{S}_w(1, \ldots, 1)$.

In 2017 Weigandt showed a lower bound for the principle specialization based on the number of 132 pattern occurrences [44, Theorem 1.1].

Theorem 1.6.3 (Weigandt). *The following inequality holds:*

$$\mathfrak{S}_w(1, \ldots, 1) \geq p_{132}(w) + 1,$$

where p_π is the number of times the pattern π appears in w.

In 2019 Gao refined Weigandt's lower bound to involve the number of 132 and 1432 occurrences [17, Theorem 2.1].

Theorem 1.6.4 (Gao). *The following inequality holds:*

$$\mathfrak{S}_w(1, \ldots, 1) \geq p_{132}(w) + p_{1432}(w) + 1.$$

Gao also proved that all of the pipe dreams of a permutation w can be connected using simple ladder moves if and only if w avoids the pattern 1432 [17, Theorem 4.1]. See Chapter 2 for a description of ladder moves and simple ladder moves.

In 2020 Fan and Guo showed an upper bound for Schubert polynomials given by a Rothe diagram dominance formula. For a diagram D we denote D_i to be the set of x coordinates in column i of D and $D_{i,k}$ to be the k-th smallest element of D_i. We say for two diagrams D and C that D *dominates* C, or $D \geq C$, if $|D_i| = |C_i|$ for all $i \in [n]$, and $D_{i,k} \geq C_{i,k}$ for all $i, k \in [n]$. Fan and Guo's upper bound is given by the coefficientwise inequality

$$\mathfrak{S}_w \leq \mathrm{Max}_w = \sum_{C \leq D(w)} x^C,$$

where

$$x^C = \prod_{(i,j) \in C} x^i.$$

They then showed a pattern avoidance criteria for when the Schubert polynomial equals this upper bound [13, Theorem 1.1].

Theorem 1.6.5 (Fan–Guo). *The equality $\mathfrak{S}_w = Max_w$ holds if and only if w avoids the patterns 1432 and 1423.*

The *layered permutation* $w(b_k, \ldots, b_1)$ is defined as the permutation

$$w(b_k, \ldots, b_1) = (b_k, b_k - 1, \ldots, 1, b_k + b_{k-1}, \ldots, b_k + 1, \ldots n, \ldots, n - b_1 + 1),$$

where $b_k + \cdots + b_1 = n$. In 2019, Morales, Pak, and Panova showed the maximum value for the principle specialization for $w \in \mathfrak{S}_n$ is found at layered permutations $w(\ldots, b_2, b_1)$, where $b_i \sim \alpha^{i-1}(1-\alpha)n$ for every fixed i and $\alpha \approx 0.43381818312$ is a universal constant [38]. The value of $\mathfrak{S}_w(1, \ldots, 1)$ for these layered permutations approaches $e^{\gamma n^2}$ as $n \to \infty$, where $\gamma \approx 0.2032558981$ is a universal constant [38, Theorem 1.3].

1.7 The Demazure product

Let (W, S) be a Coxeter system, and let $S = \{s_1, \ldots, s_k\}$ be its generators. Let v be a word, not necessarily reduced, from S. We define the *length* of v, denoted $\ell(v)$, to be the length of a minimal word that represents the same group element as v. We define the Demazure product recursively as follows.

$$\delta : S^* \to W, \tag{1.7.1}$$

$$\delta(1) = 1, \tag{1.7.2}$$

$$\delta(s_i v) = \begin{cases} s_i\delta(v), & \text{if } \ell(s_i v) > \ell(v), \\ \delta(v), & \text{if } \ell(s_i v) < \ell(v). \end{cases} \tag{1.7.3}$$

In essence, the Demazure product considers a word one letter at a time. If adding the letter increase the word's length, we include it. If adding the letter decreases the word's length, we omit it.

Note that pipe dream can be read as a word in $\mathfrak{S}_n$ according to the following triangular diagram read right to left and top to bottom, and that the permutation represented by the pipe dream is the Demazure product of the word.

$$\begin{array}{cccccc}
s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} \\
s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} & \\
s_3 & \cdots & s_{n-2} & s_{n-1} & & \\
\vdots & \vdots & \iddots & & & \\
s_{n-2} & s_{n-1} & & & & \\
s_{n-1} & & & & &
\end{array}$$

1.8 The pipe dream complex

Let W be a Coxeter group and $S = \{s_1, \ldots, s_k\}$ be its generators. Let Q be a word using letters from S and $v \in W$. The *subword complex* $\Delta(Q, v)$ is the set of subwords of Q such that their complement with respect to Q contains a reduced expression for v. In particular, the facets of Q are the maximal such subwords.

Pipe dream complexes are a special case of subword complexes, where the Coxeter group W is the symmetric group $\mathfrak{S}_n$, $S = \{s_i : i = 1, \ldots, n-1\}$ is the set of adjacent transpositions $s_i = (i, i+1)$, $Q = Q_{0,n}$ is the triangular word

$$Q_{0,n} = s_{n-1} \cdots s_2 s_1 s_{n-1} \cdots s_3 s_2 \cdots s_{n-1} s_{n-2} s_{n-1},$$

and $v \in \mathfrak{S}_n$.

We denote the pipe dream complex for $w \in \mathfrak{S}_n$ to be $PD(w)$. It is known by the definition of subword complexes that facets for two reduced pipe dreams P_1 and P_2 intersect in a face of codimension 1 if and only if $P_1 - P_2$ is a single cross. More generally, the codimension of their intersection is equal to the number of crosses in $P_1 - P_2$ [26].

1.9 The topological structure of pipe dream complexes

Knutson and Miller proved that the subword complex $\Delta(Q, \pi)$ is homeomorphic to a ball or a sphere [26, Theorem 3.7], where Q is a sequence of reflections in a Coxeter group and π is a reduced word for said Coxeter group. Moreover, if the Demazure product $D(Q) = \pi$, then Δ is homeomorphic to a sphere. Otherwise it is homeomorphic to a ball [26, Theorem 3.8].

Theorem 1.9.1 (Knutson–Miller)**.** *The subword complex $\Delta(Q, \pi)$ is homeomorphic to a ball or sphere of dimension $\ell(w) - 1$. If the Demazure product satisfies $D(Q) = \pi$, then $\Delta(Q, \pi)$ is homeomorphic to a sphere. Otherwise it is homeomorphic to a ball.*

In particular, the word $Q_{0,n}$ generating the pipe dream complex has Demazure product $D(Q_{0,n}) = w_0$, which implies that the pipe dream complex for any word besides the longest word is homeomorphic to a ball. The complex $PD(w_0)$ is the empty set, that is the -1-sphere.

1.10 Whitney stratification

In Chapter 2 we will be giving a Whitney stratification of the boundary of any pipe dream complex. In order to do this, we now review the notion of a Whitney stratification.

Let W be a compact topological space. We say that W has a *Whitney stratification* X if W has a finite decomposition into locally closed subsets $X = \bigcup_{i \in \mathcal{P}} S_i$ where the strata S_i are each smooth manifolds. We further assume this decomposition satisfies the *condition of the frontier*,

$$S_i \cap \overline{S_j} \neq \emptyset \text{ if and only if } S_i \subseteq \overline{S_j},$$

and that this decomposition satisfies two additional conditions:

1. Each $S_i \in \mathcal{P}$ is a locally closed, not necessarily connected, smooth manifold of M,

2. If $S_i <_{\mathcal{P}} S_j$ then Whitney's conditions (A) and (B) hold: Suppose $y_{j,k} \in S_j$ is a sequence of points converging to some $s \in S_i$ and that $s_{i,k}$ converges to s.

Also assume that the secant lines $\ell_k = \overline{s_{i,k}y_{j,k}}$ converge to some limiting line ℓ and the tangent planes $T_{y_{k,j}}S_j$ converge to some limiting plane τ. Then the following hold:

$$(A)\ \ T_{s_{i,k}}S_i \subseteq \tau \quad \text{and} \quad (B)\ \ \ell \subseteq \tau.$$

An example of a Whitney stratified space is to take the boundary of the n-dimensional simplex and decompose it into its open i-dimensional faces for $i = 0, \ldots, n-1$.

1.11 Shelling

For a simplicial complex Δ with face F, the *deletion* and the *link* of F are respectively

$$\begin{aligned} \partial(F,\Delta) &= \{G \in \Delta \ : \ F \cap G = \emptyset\} \\ \mathrm{link}(F,\Delta) &= \{G \in \Delta \ : \ F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}. \end{aligned}$$

A d-dimensional simplicial complex Δ is *vertex decomposable* if it is pure and either it isomorphic to the d-dimensional simplex Δ_d or there exists a vertex $v \in \Delta$ such that both $\partial(v,\Delta)$ and $\mathrm{link}(v,\Delta)$ are each vertex decomposable. The distinguished vertex is called the *shedding vertex*. Provan introduced this notion in his dissertation [39] and with Billera showed it implies shellability [4].

Theorem 1.11.1 (Billera–Provan)**.** *If Δ is a vertex decomposable simplicial complex then Δ is shellable.*

Their argument is simply that if v is a shedding vertex, the shelling order $\sigma_1, \ldots, \sigma_r$ of $\partial(\Delta, v)$ and that of $\tau_1, \ldots, \tau_s$ for $\mathrm{link}(\Delta, v)$ produce a shelling $\sigma_1, \ldots, \sigma_r, \tau_1 \cup v, \ldots, \tau_s \cup v$ of the original complex Δ.

In his dissertation Provan considered the more general setting of *shedding faces* [39].

Informally speaking, a simplicial complex is vertex decomposable if there exists some order in which you can remove the vertices one by one. Strongly vertex decomposability says that you can choose *any* vertex in the first step to remove, and the remaining simplicial complex is itself vertex decomposable.

Knutson and Miller proved that any subword complex $\Delta(Q,w)$ is vertex-decomposable. See [27, Theorem E] and [26, Theorem 2.8]. Furthermore, a subword complex $\Delta(Q,w)$ is homeomorphic to a ball or sphere of dimension $|Q| - \ell(w) - 1$ of the given word w [26, Theorem 3.7].

Theorem 1.11.2 (Knutson–Miller)**.**

(i) The subword complex $\Delta(Q,w)$ is vertex decomposable, and hence, a shellable CW complex.

(ii) The subword complex $\Delta(Q,w)$ is homeomorphic to an either an m-dimensional sphere or an m-dimensional ball, where $m = |Q| - \ell(w) - 1$. Furthermore, a face $Q - P$ of $\Delta(Q,w)$ is on the boundary $\partial\Delta(Q,w)$ if and only if its Demazure product satisfies $\delta(P) \neq w$.

1.12 A known result for the pipe dream complexes of a class of permutations

The pipe dream complexes for permutations of the form $1n \cdots 32$ have a predictable structure, namely the dual of the associahedron of dimension $n-3$. In 2018 Escobar and Mészáros showed that these pipe dreams are the vertex figures of root polytopes of type A_n [12, Theorem 5.3].

Theorem 1.12.1 (Escobar–Mészáros)**.** *The canonical triangulation of the vertex figure of the root polytope of type A_n, that is, $\mathcal{V}(P_n)$, is a geometric realization of the pipe dream complex $PD(1n \cdots 32)$.*

The proved this realization by establishing an equivalence between root polytopes and flow polytopes and by constructing a bijection between the triangulation of $\mathcal{V}(P_n)$ and noncrossing alternating spanning trees of the graph with vertex set $[n]$.

Chapter 2 A geometric study of pipe dream complexes and Schubert polynomials

Schubert polynomials were introduced by Lascoux and Schützenberger to generalize Schur polynomials [30]. They are representatives of Schubert cycles in flag varieties and provide a basis for $\mathbb{Z}[x_1, \ldots, x_n]/I_n$ where $I_n = \langle e_1, \ldots, e_n \rangle$ is the ideal generated by the elementary symmetric functions e_i for $i = 1, \ldots, n$.

In this chapter, we will study the geometric structures of pipe dream complexes in order to gain a better understanding of Schubert polynomials. In Section 2.1, we introduce the ladder moves of Billey-Bergeron and discuss how to construct the pipe dreams of a permutation using Rothe diagrams and these ladder moves. In Section 2.2, we recall how pipe dream complexes are a subclass of subword complexes and review Knutson and Miller's geometric results for subword complexes. In Section 2.3, we introduce the pipe dream complex poset. We give a natural stratification for a given pipe dream complex (Theorem 2.3.1) and show the dual of the pipe dream complex corresponds to the poset of strata of the boundary of the pipe dream complex of the identity permutation. See Theorem 2.3.3. In Section 2.4, we study the pipe dream complexes of permutations that are a single adjacent transposition, or simple reflection. We extend our results to permutations that are a product of multiple commuting simple reflections in Section 2.5.

2.1 Ladder Moves

Given the Rothe diagram $D(w)$ of a permutation w, we form its set $\mathrm{RC}(w)$ of *RC-graphs*, short for "reduced-word compatible sequence graph", as follows [3, 15]. Let B_w be the triangular diagram formed by left-justifying all of the crosses in $D(w)$ and placing dots in the complementary locations. This is known as the bottom diagram. See Figures 2.1 and 2.2. The top diagram T_w, where there are no more ladder moves possible, is formed by column-justifying all of the crosses in $D(w)$ upwards. Furthermore, the number of crosses in a given RC-graph equals the length $\ell(w)$. There are algebraic moves between these diagrams known as *ladder moves* and *slide moves* due to Bergeron and Billey that generate all of the reduced pipe dreams [3, Section 3]. For an example of the RC-graphs for the permutation $w = 12534$ and the corresponding ladder moves, refer to Figure 2.3.

Theorem 2.1.1. *For a permutation $w \in \mathfrak{S}_n$, its Schubert polynomial is given by*

$$\mathfrak{S}_w = \sum_{G \in RC(w)} x_1^{e_1} x_2^{e_2} \cdots x_{n-1}^{e_{n-1}},$$

where e_i counts the number of crosses in column i of an RC-graph G.

$$
\begin{array}{ccc}
 & j & j+1 \\
i-m & \cdot & \cdot \\
 & + & + \\
 & + & + \\
 & + & + \\
i & + & \cdot
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
 & j & j+1 \\
i-m & \cdot & + \\
 & + & + \\
 & + & + \\
 & + & + \\
i & \cdot & \cdot
\end{array}
$$

Figure 2.1: The ladder move on RC-graphs due to Bergeron and Billey.

2.2 Subword complexes and pipe dreams

Subword complexes were introduced by Knutson and Miller [27]. Within the setting of subword complexes, they show Schubert polynomials occur naturally as recording algebraic data of geometric objects. Although the main focus in this dissertation will be on these geometric manifestations of Schubert polynomials, for completeness we review the notion of subword complexes, and the special case of pipe dream complexes.

Given a Coxeter system (W, S) with Coxeter group W and simple reflections S minimally generating W, let $Q = (q_1, \ldots, q_r)$ be an ordered sequence of elements from S. For $w \in W$, the *subword complex* $\Delta(Q, w)$ is the simplicial complex whose facets consist of subwords P of Q for which the complement $Q - P$ contains a reduced expression of w; see [27, Definition 1.1.8].

Example 2.2.1. *Let $Q = \boldsymbol{q} = (s_3, s_2, s_1, s_3, s_2, s_3)$ and $w = 2143 \in \mathfrak{S}_4$. The permutation w has two reduced expressions, namely, s_1s_3 and s_3s_1. The facets of the complex $\Delta(Q, w)$ are $\{q_1, q_2, q_5, q_6\}$, $\{q_1, q_2, q_4, q_5\}$ and $\{q_2, q_4, q_5, q_6\}$. Geometrically this complex consists of three tetrahedra identified along a common edge. See Figure 2.5 for a geometric realization.*

Given a commutative ring R and Coxeter system (W, S), let $\mathscr{D}$ be a free R-module with generators $\{e_w : w \in W\}$ indexed by the Coxeter groups W. Define a

$$
\begin{array}{cccccc}
 & j-m & & & & j \\
i & \cdot & + & + & + & + \\
i+1 & \cdot & + & + & + & \cdot
\end{array}
$$

$$\big\downarrow$$

$$
\begin{array}{cccccc}
 & j-m & & & & j \\
i & \cdot & + & + & + & \cdot \\
i+1 & + & + & + & + & \cdot
\end{array}
$$

Figure 2.2: The chute move on RC-graphs due to Bergeron and Billey.

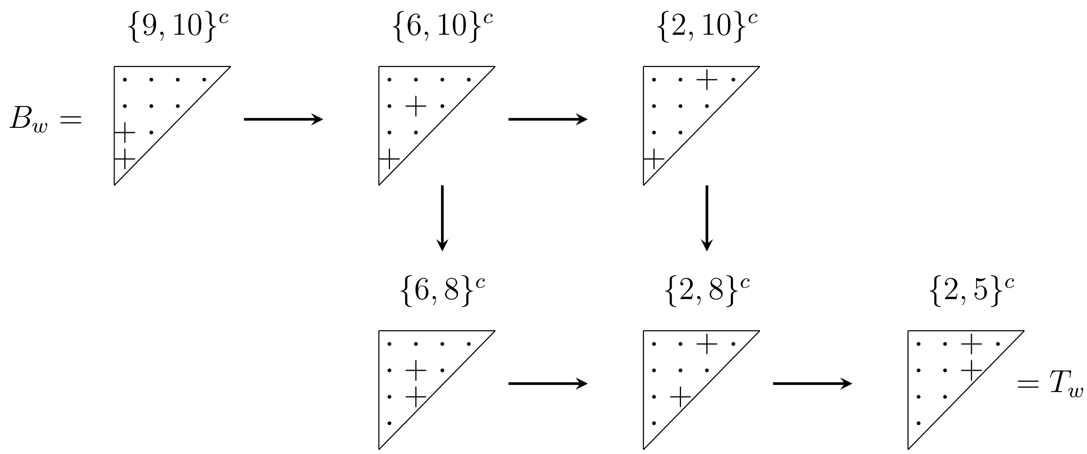


Figure 2.3: RC-graphs for the permutation $w = 12534$ and the corresponding ladder moves. The Schubert polynomial is $\mathfrak{S}_w = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$.

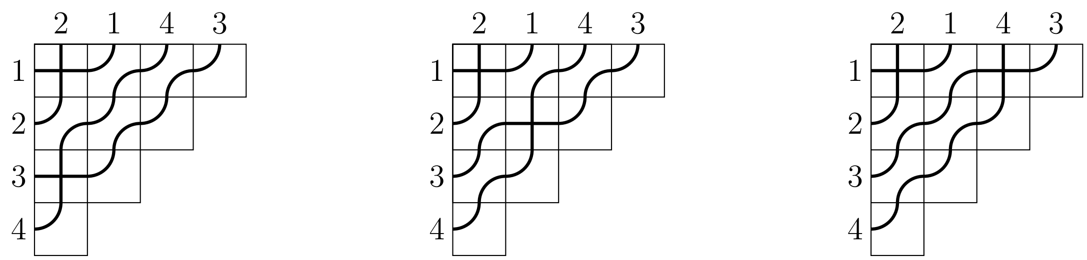


Figure 2.4: The three reduced pipe dreams for the permutation $\pi = 2143$.

multiplication on $\mathscr{D}$ by

$$e_w \cdot e_s = \begin{cases} e_{ws} & \text{if } \ell(ws) > \ell(w), \\ e_w & \text{if } \ell(ws) < \ell(w), \end{cases} \tag{2.2.1}$$

where $s \in S$. This gives the *Demazure algebra* of (W, S) over R. The *Demazure product* of the word $v = s_{i_1} \cdots s_{i_k}$ is

$$\delta(v) = e_{s_{i_1}} \cdots e_{s_{i_k}} = e_{\delta(v)}. \tag{2.2.2}$$

In the case $W = \mathfrak{S}_n$, the *Demazure operators* are defined on $f \in R[x_1, \ldots, x_n]$ by

$$\overline{\partial_i}(f) = \frac{x_{i+1}f - x_i(s_i \cdot f)}{x_{i+1} - x_i},$$

where $i = 1, \ldots, n-1$ and $s_i = (i, i+1)$. See historical comments in [26, Remark 3.3]. As noted in [26] if a word is reduced then the ordered product of the word equals its Demazure product.

In what follows, the symbol $>$ denotes the Bruhat order on the given Coxeter group W.

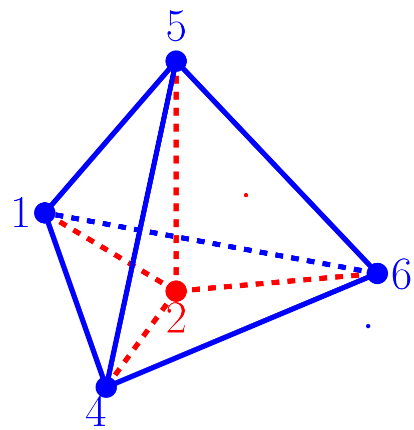


Figure 2.5: The pipe dream complex of the permutation $\pi = 2143$, that is, the subword complex $\Delta(Q, \pi)$ with $Q = \mathbf{q} = (s_3, s_2, s_1, s_3, s_2, s_3)$. Here $\Delta(Q, \pi) \cong \Delta_1 * C_3$.

Lemma 2.2.2. *(Knutson–Miller) For a Coxeter system (W, S) let P be a word in the minimal generating set S of W and let $\nu \in W$.*

(a) The Demazure product satisfies $\delta(P) \geq \nu$ if and only if P contains a subword whose product is ν.

(b) If $\delta(P) = \nu$ then every subword of P containing ν has Demazure product equal to ν.

(c) If $\delta(P) > v$ then P contains a word T representing an element $\tau > \nu$ satisfying $|T| = \ell(\tau) = \ell(\nu) + 1$.

Recall that a d-dimensional simplicial complex Δ is *vertex decomposable* if it is pure and either it isomorphic to the d-dimensional simplex Δ_d or there exists a vertex $v \in \Delta$ such that both the vertex deletion $\partial(v, \Delta) = \{F \in \Delta : F \cap v = \emptyset\}$ and the link of Δ at the vertex $\text{link}(v, \Delta) = \{F \in \Delta : F \cap v = \emptyset \text{ and } F \cup v \in \Delta\}$ are each vertex decomposable. Also recall that the distinguished vertex is called the *shedding vertex*. Provan introduced this notion in his dissertation [39] and with Billera showed it implies shellability [4]. In his dissertation Provan also considered the more general setting of *shedding faces* [39].

Theorem 2.2.3 (Billera–Provan)**.** *If Δ is a vertex decomposable simplicial complex then Δ is shellable.*

Their argument is simply that if v is a shedding vertex, the shelling order $\sigma_1, \ldots, \sigma_r$ of $\partial(\Delta, v)$ and that of $\tau_1, \ldots, \tau_s$ for $\text{link}(\Delta, v)$ produce a shelling $\sigma_1, \ldots, \sigma_r, \tau_1 \cup v, \ldots, \tau_s \cup v$ of the original complex Δ.

Informally speaking, a simplicial complex is vertex decomposable if there exists some order in which you can remove the vertices one by one. *Strongly vertex decomposability* says that you can choose *any* vertex to remove in the first step, and then the remaining simplicial complex is itself vertex decomposable.

Knutson and Miller proved that any subword complex $\Delta(Q, w)$ is vertex decomposable; see [27, Theorem E] and [26, Theorem 2.8]. Furthermore, a subword complex $\Delta(Q, w)$ is homeomorphic to a ball or sphere of dimension $|Q| - \ell(w) - 1$ [26, Theorem 3.7].

Theorem 2.2.4 (Knutson–Miller)**.**

(i) The subword complex $\Delta(Q, w)$ is vertex decomposable, and hence, a shellable CW complex.

(ii) The subword complex $\Delta(Q, w)$ is homeomorphic to an either an m-dimensional sphere or an m-dimensional ball, where $m = |Q| - \ell(w) - 1$. Furthermore, a face $Q - P$ of $\Delta(Q, w)$ is on the boundary $\partial\Delta(Q, w)$ if and only if its Demazure product satisfies $\delta(P) > w$.

Example 2.2.5. *In the case $W = \mathfrak{S}_n$, let*

$$Q_{0,n} = (s_{n-1}, \ldots, s_2, s_1, s_{n-1}, \ldots, s_3, s_2, \cdots, s_{n-1}, s_{n-2}, s_{n-1}).$$

Here the facets of $\Delta(Q_{0,n}, w)$, where $w \in \mathfrak{S}_n$, are the complements of the reduced pipe dreams PD(w)*, that is, RC-graphs for w. See [27, Example 1.8.3](RC-graphs) and [41, Section 3.2].*

Note if one labels a triangular array with

$$\begin{array}{cccccc}
s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} \\
s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} & \\
s_3 & \cdots & s_{n-2} & s_{n-1} & & \\
\vdots & \vdots & \iddots & & & \\
s_{n-2} & s_{n-1} & & & & \\
s_{n-1} & & & & &
\end{array}$$

(read from right to left and top to bottom) then a *reduced* pipe dream corresponds to a reduced word for w where a crossing corresponds to selecting a particular s_i and the number of crossings in a reduced pipe dream for w equals the length $\ell(w)$.

Example 2.2.6. *Let $w = 12534$. The six facets in the pipe dream complex $\mathfrak{S}_w$ are $\{9, 10\}^c$, $\{6, 10\}^c$, $\{2, 10\}^c$, $\{6, 8\}^c$, $\{2, 8\}^c$ and $\{2, 5\}^c$, where the complement is taken with respect to the set $[1, 10]$. See Figure 2.3 for the corresponding ladder moves of Billey–Bergeron.*

For S and T simplicial complexes with disjoint vertex sets $V(S)$ and $V(T)$, the *join* $S * T$ is defined as

$$S * T = \{\sigma \cup \tau \ : \ \sigma \in S \text{ and } \tau \in T\}.$$

For example, we have PD$(2143) \cong \Delta_1 * C_3$ is the simplicial complex in Figure 2.4, where Δ_n denotes the n-dimensional simplex and C_n denotes the n-cycle.

Given d-dimensional simplexes Δ and Δ', the *connected sum* $\Delta \# \Delta'$ is the simplicial complex formed by identifying one facet of Δ with one facet of Δ'.

See Appendix A for an atlas of pipe dream complexes.

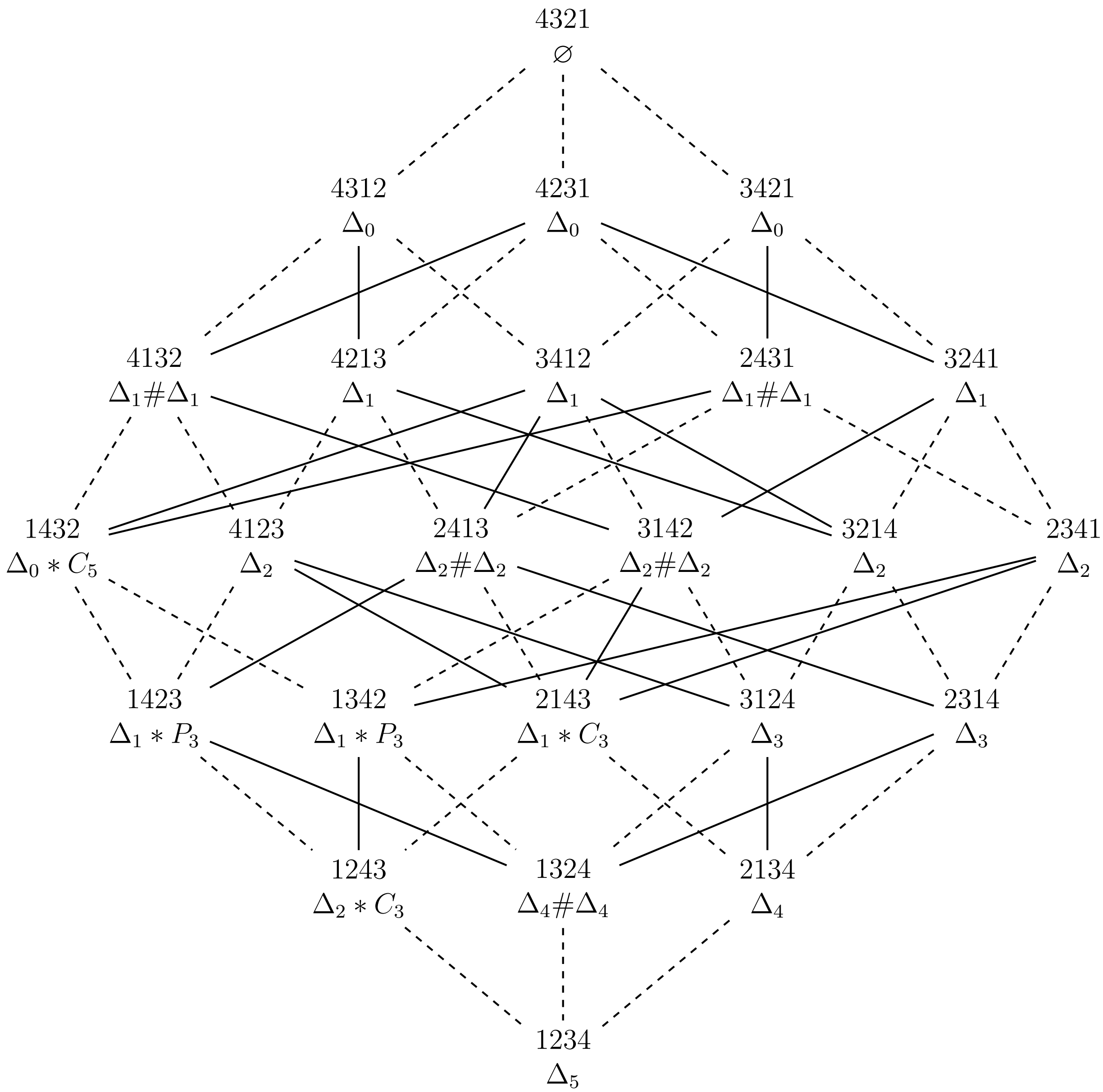


Figure 2.6: The pipe dream complex poset $\mathscr{P}_4$ for the symmetric group $\mathfrak{S}_4$. Dotted lines indicate covers via the weak Bruhat order on the symmetric group, whereas the solid lines indicate covers in the strong Bruhat order that are not in the weak Bruhat order. Here C_i is the cycle on i elements, P_i is the length i path and Δ_i is the i-dimensional simplex, and # is the connected sum along two facets.

2.3 The pipe dream complex poset

Recall the symmetric group $\mathfrak{S}_n$ is generated by simple transpositions $s_i = (i, i+1)$ for $i = 1, \dots, n-1$ with $s_i^2 = \text{id}$, $s_i s_j = s_j s_i$ for $|i-j| \geq 2$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. The length $\ell(x)$ of a permutation w is the minimum number of simple transpositions needed to write w as a product. The symmetric group is partially ordered by the (strong) Bruhat order: $v \leq w$ if $w = s_{j_1} \cdots s_{j_k}$ is a reduced expression for w and v has a reduced expression that is a subword of $s_{j_1} \cdots s_{j_k}$.

We define the *pipe dream complex poset* $\mathscr{P}_n$ to be the poset whose elements consists of the pipe dreams complexes $\text{PD}(w)$ for $w \in \mathfrak{S}_n$ partially ordered with respect to the (strong) Bruhat order on the permutations $w \in \mathfrak{S}_n$, that is, $\text{PD}(v) \leq \text{PD}(w)$ if $v \leq w$ with respect to the strong Bruhat order. See Figure 2.6 for the example when $n = 4$.

There is a natural geometry built into the set of pipe dreams for the symmetric group $\mathfrak{S}_n$.

Theorem 2.3.1. *For the pipe dreams complex poset $\mathscr{P}_n$ the following holds:*

(i) The pipe dreams complex poset $\mathscr{P}_n$ is graded of rank $\binom{n}{2}$. The minimal element $\hat{0}$ is the simplex Δ_m where $m = \binom{n}{2} - 1$, and the maximal element is the empty set.

(ii) The boundary of the pipe dream complex $\text{PD}(v)$ satisfies

$$\partial(\text{PD}(v)) = \dot{\bigcup_{v \prec w}} \text{PD}(w). \tag{2.3.1}$$

(iii) The boundary of the pipe dream complex $\text{PD}(v)$ is stratified by the pipe dream complexes $\text{PD}(w)$ for $w \in (v, w_0]$ where w_0 is the longest word in $\mathfrak{S}_n$.

Proof. For (i) the Rothe diagram of the identity element $w = \text{id}$ is the empty $n \times n$ array. This gives the empty order n triangular RC-graph with $\binom{n}{2}$ dots corresponding to the $(\binom{n}{2} - 1)$-dimensional simplex. The Rothe diagram of the longest element w_0 consists of the left-justified triangular array with $n - i$ crosses in the ith row. The RC-graph is the order $n - 1$ triangular array completely filled with crosses, yielding $\text{PD}(w_0) = \emptyset$.

The pipe dreams complex poset $\mathscr{P}_n$ inherits its rank function and grading from the strong Bruhat order on the symmetric group $\mathfrak{S}_n$, where the rank function ρ is the length function $\ell(w)$, that is, $\rho(\text{PD}(w)) = \ell(w)$.

To show (ii), we first claim that the non-empty intersection of any two facets Δ_1 and Δ_2 of the pipe dream complex $\text{PD}(v)$ gives a non-reduced pipe dream for v. The pipe dream of the intersection $\Delta_1 \cap \Delta_2$ is formed by superimposing the two reduced pipe dreams representing Δ_1 and Δ_2 to form the pipe dream $P(\Delta_1 \cap \Delta_2)$. By reading off the crosses in this pipe dream, one can form the resulting subword π of $Q_{0,n}$. By Theorem 2.2.4 the Demazure product $\delta(\pi) = v$ with the number of crosses in

$P(\Delta_1 \cap \Delta_2) > \ell(v)$. This implies $P(\Delta_1 \cap \Delta_2)$ is a non-reduced pipe dream complex for v, as claimed.

To show (2.3.1) holds, suppose $w \succ v$. Then $w = sv$ for some adjacent transposition s. By definition of the subword complex, each facet of $\mathrm{PD}(w)$ corresponds to a subword w' of $Q_{0,n}$ that is, a reduced subword for w. Since w' is reduced, its Demazure product satisfies $\delta(w') = w$. By Lemma 2.2.2, since $\delta(w') > v$, the word w' contains v as a subword. Since $\ell(w') = \ell(v) + 1$, it is a codimension 1 face of $\mathrm{PD}(v)$ contained in exactly one facet of $\mathrm{PD}(v)$, that is, it lies on the boundary $\partial(\mathrm{PD}(v))$. Hence $\dot{\bigcup}_{v \prec w} \mathrm{PD}(w) \subseteq \partial(\mathrm{PD}(v))$.

Conversely, we claim that every facet of the boundary $\partial(\mathrm{PD}(v)$ appears as a reduced pipe dream in $\mathrm{PD}(w)$ for some $w \succ v$. Let F be a facet of $\partial(\mathrm{PD}(v))$. By Theorem 2.2.4 a face P is on the boundary of $\mathrm{PD}(v)$ if and only if its Demazure product satisfies $\delta(P) > v$. Since F only intersects one facet F' of $\mathrm{PD}(v)$, its pipe dream $P(F)$ has one more cross than the reduced pipe dream $\mathrm{PD}(F')$, implying $P(F)$ is a reduced pipe dream with its subword representation w satisfying $w \succ v$. Hence $\partial(\mathrm{PD}(v)) \subseteq \dot{\bigcup}_{v \prec w} \mathrm{PD}(w)$, giving the desired equality (2.3.1).

Finally, part *(iii)*, follows from iterating *(ii)*. □

Example 2.3.2. *For the permutation* $v = 2143$*, the facets of* $\mathrm{PD}(v)$ *are* $\{1, 2, 5, 6\}$, $\{1, 2, 4, 5\}$ *and* $\{2, 4, 5, 6\}$. *The permutation* $v = 2143$ *is covered by the permutations* 4123*,* 2413*,* 3142 *and* 2341*. The pipe dream complexes are* $\mathrm{PD}(v) = \Delta_1 * P_3$*,* $\mathrm{PD}(4123) = \Delta_2$*,* $\mathrm{PD}(2413) = \Delta_2 \# \Delta_2$*,* $\mathrm{PD}(3142) = \Delta_2 \# \Delta_2$ *and* $\mathrm{PD}(2341) = \Delta_2$*. The facet of* $\mathrm{PD}(4123)$ *is* $\{1, 2, 4\}$*, the facets of* $\mathrm{PD}(2413)$ *are* $\{1, 5, 6\}$ *and* $\{1, 4, 5\}$*, the facets of* $\mathrm{PD}(3142)$ *are* $\{1, 2, 6\}$ *and* $\{2, 4, 6\}$*, and the facet of* $\mathrm{PD}(2341)$ *is* $\{4, 5, 6\}$*. One can check these facets decompose the boundary* $\partial(\mathrm{PD}(v))$*. See Figure 2.5.*

Recall a compact topological space W has a *Whitney stratification* X if W has a decomposition into finitely many smooth manifolds satisfying Whitney's conditions A and B. For details, see [11, Section 6] and the references therein. The strata $X = \bigcup_{i \in \mathcal{P}} S_i$ of a Whitney stratified space form a poset $\mathcal{P}$ where the partial order relation $S \leq_{\mathcal{P}} T$ holds if and only if $S \subset \overline{T}$, and this poset of strata is graded by dimension, that is, $\rho(S) = \dim(S) + 1$. Examples of Whitney stratifications include the (open) faces of a convex polytope, the cells of regular cell complex, real or complex algebraic sets, analytic sets, semi-analytic sets and quotients of smooth manifolds by compact group actions [10, 20, 22, 35, 45].

For a brief history of Whitney stratified spaces, see Goresky's introductory article [21]. The same volume includes a reprint of Mather's proof of Thom's conjecture [35]. For recent work extending the theory of face incidence enumeration to the Whitney stratified and quasi-graded poset settings, see [11].

Theorem 2.3.3. *The dual poset* $\mathscr{P}(n)^*$ *naturally corresponds to the poset of strata associated to a Whitney stratification of the boundary of the pipe dream complex of the identity element, that is,* $\partial(\mathrm{PD}(\mathrm{id}))$*. Furthermore, for* $w \in \mathfrak{S}_n$*, the boundary of its pipe dream complex* $\partial(\mathrm{PD}(w))$ *is Whitney stratified by the set of all pipe dream complexes* $\mathrm{PD}(v)$ *where* $v > w$ *with respect to the strong Bruhat order.*

Proof. Let Δ° denote the interior of a simplicial complex Δ. By Theorem 2.3.1, we have

$$\begin{aligned}\partial(\mathrm{PD}(v)) &= \dot{\bigcup_{v \prec w}} \mathrm{PD}(w) \\ &= \dot{\bigcup_{v \prec w}} \partial(\mathrm{PD}(w)) \dot{\cup} \mathrm{PD}(w)^\circ \\ &= \dot{\bigcup_{v < w}} \mathrm{PD}(w)^\circ.\end{aligned}$$

This gives a stratification of the boundary of $\mathrm{PD}(v)$ via the interiors of all pipe dream complexes $\mathrm{PD}(w)$ with $w > v$. Since $\mathrm{PD}(v) = \partial(\mathrm{PD}(v)) \dot{\cup} \mathrm{PD}(v)^\circ$, we have

$$\mathrm{PD}(v) = \dot{\bigcup_{v \leq w}} \mathrm{PD}(w)^\circ.$$

There is a natural correspondence between a given pipe dream $\mathrm{PD}(w)$ and this stratification, namely mapping $\mathrm{PD}(w)$ to its interior $\mathrm{PD}(w)^\circ$. Under this correspondence, note that $\mathrm{PD}(v) \leq \mathrm{PD}(v')$ in the pipe dream complex poset $\mathscr{P}(n)$ if and only if $v \leq v'$ in the strong Bruhat order if and only if $\mathrm{PD}(v')^\circ \subseteq \mathrm{PD}(v)^\circ$ if and only if $\mathrm{PD}(v')^\circ \subseteq \overline{\mathrm{PD}(v)^\circ}$. Since all of the pipe dream complexes under consideration are homeomorphic to a sphere in the case of $v = \mathrm{id}$ and a ball otherwise, Whitney's conditions A and B hold, so we have a Whitney stratification. □

2.4 The pipe dream complex $\mathrm{PD}(s_k)$

Given a pure simplicial complex Δ, the *cone*, denoted $\mathrm{cone}(\Delta)$, is the set of vertices that appear in all of its facets, while the *core*, denoted $\mathrm{core}(\Delta)$, is the simplicial complex formed by restricting Δ to the set of vertices of Δ not in the cone. By definition the simplicial complex $\mathrm{PD}(w)$ is formed by successively coning over $\mathrm{core}(\mathrm{PD}(w))$ by the vertices in $\mathrm{cone}(\mathrm{PD}(w))$.

Theorem 2.4.1. *For $s_k \in \mathfrak{S}_n$ the pipe dream complex is of the form*

$$\mathrm{PD}(s_k) \cong \Delta_m * \partial\Delta_{k-1},$$

where $m = \binom{n}{2} - k - 1$.

Proof. The Rothe diagram consists of one cross in the entry $(k, k+1)$. The bottom RC-graph B_{s_k} is the triangular array consisting of $\binom{n}{2}$ entries with one cross in the entry $(k, 1)$. Using ladder moves, there are k total facets in $\mathrm{PD}(s_k)$, with the intersection of these facets, that is, the cone being a simplex on $\binom{n}{2} - k$ vertices. The remaining vertices of each facet are a $k-1$ subset of the elements in the triangular array corresponding to the adjacent transposition s_k. □

Corollary 2.4.2. *The Schubert polynomial* $\mathfrak{S}_w$ *for* $w = s_k \in \mathfrak{S}_n$ *is the elementary symmetric function*

$$\mathfrak{S}_w = x_1 + \cdots + x_k = e_k(x_1, \ldots, x_k),$$

with principal specialization

$$\mathfrak{S}_w(1, \ldots, 1) = k.$$

2.5 The pipe dream complex $\mathrm{PD}(s_i s_j)$ with $|i - j| \geq 2$

In this section we consider the case when w is the product of simple transpositions that commute. In section 2.5, we will study when w is a product of adjacent simple transpositions.

Theorem 2.5.1. *Let* $1 \leq i < j \leq n - 1$ *and* $j - i \geq 2$*. Then the pipe dream complex of* $w = s_i s_j$ *is of the form*

$$\mathrm{PD}(s_i s_j) \cong \Delta_m * \partial\Delta_{i-1} * \partial\Delta_{j-1},$$

where $m = \binom{n}{2} - i - j - 1$*. Furthermore, the Schubert polynomial is given by*

$$\mathfrak{S}_w = e_1(x_1, \ldots, x_i) \cdot e_1(x_1, \ldots, x_j),$$

and has principle specialization

$$\mathfrak{S}_w(1, \ldots, 1) = ij.$$

Proof. The bottom RC-graph B_w consists of two crosses, one in the entry $(i + 1, i)$ and the other in the entry $(j + 1, j)$. The $i + 1$st and $j + 1$st antidiagonals are separated by $j - i$ antidiagonals that are empty. Hence the two crosses can perform independent ladder moves, with the cross at entry $(i+1, i)$ having i possible locations on its antidiagonal, and the cross at entry $(j + 1, j)$ having j possible locations on its antidiagonal. The core thus has dimension $m = \binom{n}{2} - (i + j) - 1$ and the pipe dream complex is $\mathrm{PD}(w) = \Delta_m * \partial\Delta_{i-1} * \partial\Delta_{j-1}$, as claimed. □

Recall that a subset $S \subseteq \{1, \ldots, n\}$ is *sparse* if the elements i and $i + 1$ are not both in the set S. By iterating Theorem 2.5.1, we have the immediate corollary.

Corollary 2.5.2. *Let* $w = \prod_{i \in S} s_i$ *where* $S = \{i_1, \ldots, i_k\}$ *is a sparse subset of* $\{1, \ldots, n - 1\}$*. Then*

$$\mathrm{PD}(w) \cong \Delta_m * \partial\Delta_{i_1-1} * \cdots * \partial\Delta_{i_k-1},$$

where $m = \binom{n}{2} - 1 - \sum_{i \in S} i$*. Furthermore, the Schubert polynomial is given by*

$$\mathfrak{S}_w = \prod_{i \in S} e_1(x_1, \ldots, x_i),$$

and has principle specialization

$$\mathfrak{S}_w(1, \ldots, 1) = \prod_{i \in S} i.$$

Chapter 3 The geometry of pattern avoidance

We shift gears in this chapter to return to pattern avoidance. Section 3.1 focuses on the importance of pattern avoidance in the study of Schubert polynomials and pipe dream complexes where the avoidance or occurrence of the pattern 132 plays a particularly important role. In section 3.2 we show that a permutation avoiding the pattern 132 is equivalent to its Rothe diagram is a Young diagram; see Theorem 3.2.2. We furthermore give an explicit description of its Rothe diagram in Corollary 3.2.4 and its pipe dream complex in Corollary 3.2.3. In Section 3.3, we study the case when a permutation contains exactly one occurrence of the 132 pattern. We look at permutations with exactly two occurrences of 132 in Section 3.4, which yields several subcases that generalize to larger but similar patterns. In both sections we classify the homeomorphism type of the pipe dream complex as well as the Schubert polynomial.

3.1 Pattern avoidance and the principle specialization

Recall that, given permutations $w = w_1 \cdots w_n \in \mathfrak{S}_n$ and $\pi = \pi_1 \cdots \pi_k \in \mathfrak{S}_k$, we say that w *contains the pattern* π if there is a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that the elements of the subword $w_{i_1} w_{i_2} \cdots w_{i_k}$ of w are in the same relative order as $\pi_1 \cdots \pi_k$. If this is not the case, we say that w *avoids* the pattern π or that w is π*-avoiding.*

Following [44], for a pattern π and a permutation $w \in \mathfrak{S}_n$, let $p_\pi(w)$ denote the number of occurrences of a pattern π in a permutation w.

Theorem 3.1.1. *Let* $w \in \mathfrak{S}_n$.

(a) If $p_{132}(w) = 1$ *then* $\mathfrak{S}_w(1, \ldots, 1) = 2$.

(b) The following inequality holds:

$$\mathfrak{S}_w(1, \ldots, 1) \geq p_{132}(w) + 1. \tag{3.1.1}$$

(c) The following inequality holds:

$$\mathfrak{S}_w(1, \ldots, 1) \geq p_{132}(w) + p_{1432}(w) + 1. \tag{3.1.2}$$

(d) A permutation $w \in \mathfrak{S}_n$ *is* 1432*-avoiding if and only if any two RC-graphs of* w *are connected with simple ladder moves.*

Parts (a) and (b) of Theorem 3.1.1 are due to Weigandt [44, Corollary 1.3, Theorem 1.2], and parts (c) and (d) are due to Gao [17, Theorem 2.1, Theorem 4.1]. Although the inequality (3.1.2) improves the bound in (3.1.1), it is still quite weak. The following example illustrates this.

Example 3.1.2. *When* $w = 15432$*, we have* $p_{132}(w) = 6$ *and* $p_{1432}(w) = 4$*, giving the lower bounds* 7 *and* 11*, respectively, whereas* $\mathfrak{S}_w(1, \ldots, 1) = 14$*. As another example,* $p_{132}(126543) = 20$*,* $p_{1432}(126543) = 8$*, whereas* $\mathfrak{S}_{126543}(1, \ldots, 1) = 84$*.*

For further work regarding principal specializations of the Schubert polynomial, see Morales–Pak–Panova's asymptotic extremal results for layered permutations [38].

3.2 132-avoiding permutations

Lemma 3.2.1. *There is a bijection between permutations* $w \in \mathfrak{S}_n$ *and the set of Rothe diagrams* $D(\mathfrak{S}_n) = \{D(w) : \ w \in \mathfrak{S}_n\}$*.*

Proof. Given a permutation $w \in \mathfrak{S}_n$, the Rothe diagram $D(w)$ is formed by placing crosses in the complement of the hooks at $(i, w(i))$ for $1 \leq i \leq n$. Conversely, given an $n \times n$ Rothe diagram $D(w)$, we wish to reconstruct the permutation $w \in \mathfrak{S}_n$. Reading from right to left, look at row 1 for the smallest entry j so that the entry $(1, j)$ is not a cross. This implies $w(j) = 1$. Cross off the elements in the hook at $(1, j)$. Repeat this process for each row, looking for the smallest j so that all of the crosses in row i occur to the left of the entry (i, j_i) and this entry has not been crossed off from the hooks in the $i - 1$ rows preceding it. Again, cross out the elements in the hook at (i, j_i), record $w(j_i) = i$, and repeat this process. □

A similar proof of Lemma 3.2.1 follows if one were to instead read column by column in increasing order, or along diagonals, starting from the northwestmost diagonal. In the case of the diagonal argument, the argument requires a slight modification as one may have to select more than one open entry and cross out the elements in each hook.

A Rothe diagram is a *Young diagram* if its crosses form a Ferrers diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$, that is, in the ith row the crosses occur in positions (i, j) for $j = 1, \ldots, \lambda_i$.

The following result is folklore. For completeness we provide a proof.

Theorem 3.2.2. *A permutation* $w \in \mathfrak{S}_n$ *is* 132*-avoiding if and only if the Rothe diagram* $R(w)$ *of* w *is a Young diagram.*

Proof. Let w be a permutation containing a 132-pattern and suppose $w_i w_j w_k$ is a subword of w that has a 132-pattern. In the Rothe diagram $D(w)$, we claim there is an isolated cross in the entry (k, w_j). Observe the entries (k, m) for $m = w_k, \ldots, n$ are crossed off due to the hook at (k, w_k). Similarly, the entries (m, w_j) at and below (k, w_j) for $m = j, \ldots, n$ are crossed off due to the hook at (j, w_j). Finally, all of the entries in the hook at (i, w_i) are crossed off, preventing the cross at (k, w_j) from being adjacent to any crosses above and to the left of this entry, implying the Rothe diagram $R(w)$ is not a Young diagram.

Conversely, suppose $R(w)$ is not a Young diagram. Then there is an entry (b, e) containing a cross and an entry (a, e) not containing a cross, with $a < b$. (See Figure 3.1.) This implies $w(a) = d$ for some $d < e$. Since w is a permutation, we

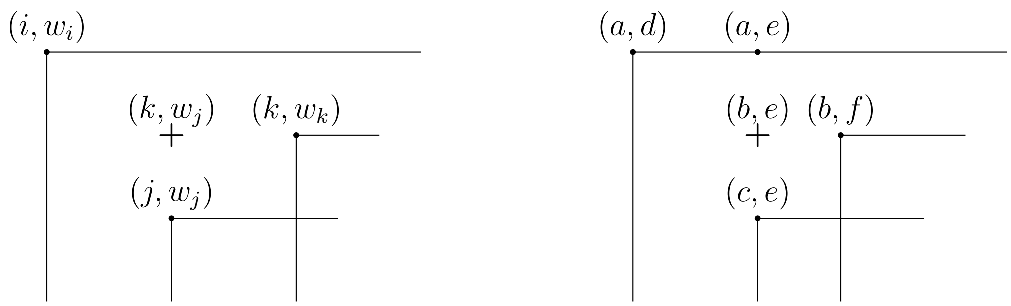


Figure 3.1: Schematic pictures for the proof of Theorem 3.2.2.

must have a hook occurring in the bth row to the right of the entry (b, e), otherwise the entry (b, e) would have been crossed out at a hook to its right in row b, contradicting the fact there is a cross at (b, e). This implies $w(b) = f$ for some $f > e$. Similarly, we must have a hook occurring at an entry (c, e) below the entry (b, e) in the eth column, implying $w(c) = e$ for some $c > a$. Since $a < b < c$ with $d < e < f$, this implies the subword $w_d w_e w_f = acb$ forms a 132-pattern, as claimed. □

We have the immediate corollary.

Corollary 3.2.3. *If $w \in \mathfrak{S}_n$ is 132-avoiding then its pipe dream complex $\mathfrak{S}_w$ consists of exactly one simplex, that is,*

$$\mathrm{PD}(w) \cong \Delta_m,$$

where $m = \binom{n}{2} - \ell(w) - 1$.

Proof. Since the Rothe diagram of w is a Young diagram, there are no further slide moves to perform. The pipe dream complex $\mathrm{PD}(w)$ consists of the simplex generated by vertices labeled by the entries that do not have a plus, that is, the simplex on $\binom{n}{2} - \ell(w)$ vertices, as claimed. □

Recall that (i, j) is an *inversion pair* in the permutation $w \in \mathfrak{S}_n$ if $i < j$ and $w_i > w_j$. Let $\tau_i(w) = (\tau_1 \ldots, \tau_n)$ be the *inversion word* for w, where $\tau_i = \tau_i(w) = |\{(i, j) : (i, j) \text{ is an inversion pair of } w\}|$ for $i = 1, \ldots, n$. When the permutation w is understood, we will suppress the w in the notation.

The following result will be used in the next section.

Corollary 3.2.4. *If $w \in \mathfrak{S}_n$ is 132-avoiding then the ith column of the Rothe diagram $D(w)$ consists of τ_i crosses in locations $(1, i)$ through (τ_i, i). The Rothe diagram $D(w)$ corresponds to the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ where λ is the dual partition to $\tau = (\tau_1, \ldots, \tau_n)$.*

Proof. Since the permutation is 132-avoiding, by Theorem 3.2.2 the Rothe diagram $D(w)$ is a Young diagram, that is, the row lengths $\lambda = (\lambda_1, \ldots, \lambda_n)$ form a weakly decreasing sequence. Since $D(w^{-1}) = D(w)^T$ and w^{-1} is also a 132-avoiding permutation, the column lengths of $D(w)$ also form a weakly decreasing sequence.

We claim the length of the ith column of the Rothe diagram $D(w)$ equals $\tau_i(w)$. This follows since if an element j forms an inversion pair of the form (i, j), the elements in the w_jth row are crossed off at the hook at (w_j, j) occurring to the northeast of the hook at (w_i, i). Furthermore, the column lengths of $D(w)$ satisfy $\tau_1 \geq \cdots \geq \tau_n$, since otherwise if $\tau_i < \tau_{i+1}$ for some i this would imply $w_{i+1} > w_i$ and there is a $j > i + 1$ with $w_{i+1} > w_j > w_i$. This would imply the subword $w_i w_{i+1} w_j$ forms a 132-pattern, contrary to our assumption on the permutation w. □

3.3 The case $p_{132}(w) = 1$

Following [33, Section 2.1.1] the *rank function* of $w \in \mathfrak{S}_n$ is

$$r_w(i, j) = |\{k : \ 1 \leq k \leq i \text{ and } w(k) \leq j\}|.$$

Observe the rank function of a coordinate (i, j) equals the number of hooks between it and the northwest corner $(0, 0)$ of its Rothe diagram $D(w)$. In essence, if there is a cross in location (i, j) of a given Rothe diagram, this statistic gives a rough indication of the number of degrees of freedom this cross can make via ladder moves.

Proposition 3.3.1. *Let $w \in \mathfrak{S}_n$ be a permutation with exactly one occurrence of the* 132 *pattern, that is, $p_{132}(w) = 1$.*

(a) The 132 *pattern in w occurs at the subword $w_i w_{i+1} w_j$ with $1 \leq i < i + 1 < j$ and $w_j = w_i + 1$.*

(b) $w_k > w_j$ for $k = 1, \ldots, i - 1$.

(c) $w_k < w_i$ for $k = i + 2, \ldots, j - 1$.

(d) $w_k < w_i$ or $w_k > w_{i+1}$ for $k = j + 1, \ldots, n$.

(e) The Rothe diagram of w consists of the disjoint union of a northwest justified Young diagram of crosses and exactly one isolated cross in the entry $(w_i+1, i+1)$ of rank $r_w(w_i + 1, i + 1) = 1$.

Proof. For (a) suppose on the contrary that the subword $w_i w_k w_j$ forms the 132-pattern with $i < k < j$ and $k > i + 1$. If the inequality $w_{i+1} > w_j$ holds then the subword $w_i w_{i+1} w_j$ would form a new 132 pattern. Likewise if $w_{i+1} < w_j$ then the subword $w_{i+1} w_k w_j$ would form a new 132 pattern. Finally, suppose on the contrary that $w_j \neq w_i + 1$, that is, suppose $w_k = w_i + 1$. If $k < i$ then $w_k w_{i+1} w_j$ would form a new 132-pattern. Likewise, if $k > i + 1$ with $k \neq j$ then $w_i w_{i+1} w_k$ would form a new would form a new 132-pattern. Hence the assertion holds.

For (b) if there is an index $k \leq i - 1$ with $w_k < w_j$ then the subword $w_k w_{i+1} w_j$ would form another 132 pattern in w, contradicting the fact $p_{132}(w) = 1$.

For (c) if there is an index k with $k \in [i + 2, j - 1]$ and $w_k > w_i$, we have two cases. If we also have $w_k < w_{i+1}$ then the subword $w_i w_{i+1} w_k$ forms a new 132-pattern.

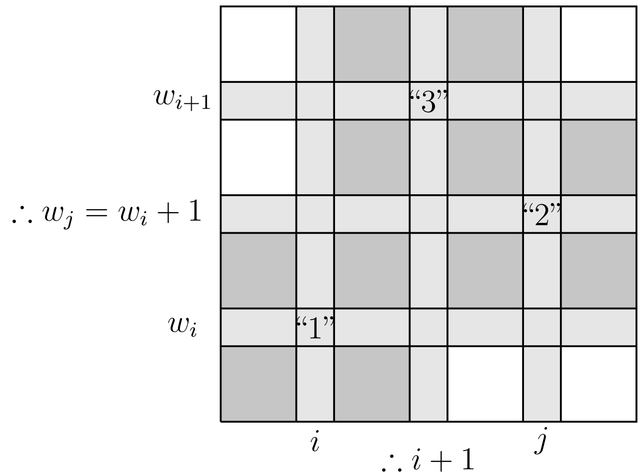


Figure 3.2: Schematic figure for the proof of Proposition 3.3.1 where $p_{132}(w) = 1$.

Similarly, if $w_k > w_{i+1}$ then the subword $w_i w_k w_j$ forms a new 132-pattern. Hence (c) holds.

For (d) if there is an index $k > j$ with $w_i < w_k < w_{i+1}$ then the subword $w_i w_{i+1} w_k$ would form another 132-pattern.

See Figure 3.2 for a schematic diagram for parts (a) through (d).

For part (e), by part (a) assume the 132-pattern occurs at the subword $w_i w_{i+1} w_j$. Consider the subword $w' = w_1 \cdots \widehat{w_j} \cdots w_n$, that is, the word w without the element w_j. This is 132-avoiding, so by Theorem 3.2.2 the associated Rothe diagram $D(w')$ forms a Young diagram. Here we have decreased each of the elements in w' that were greater than w_j by one.

By part (a) we have the vertical condition $w_j = w_i + 1$. Reinserting the element w_j back into the permutation w' and shifting entries back to form w now inserts a hook at the entry (w_i, i) and creates a cross at position $(w_i + 1, i + 1)$. This cross is an isolated cross, since all of the values x with $w_i + 1 < x < w_{i+1}$ occur in the initial subword $w_1 \cdots w_{i-1}$ of w. Thus the hooks have eliminated all possible locations for a cross to occur disjoint from the main Young diagram except for the single cross at $(w_i + 1, i + 1)$. The rank of this isolated cross is one due to the hook at the entry (w_i, i). Furthermore, reinserting the element w_j results in a new w_jth row in $D(w)$ that repeats the previous row w_i since its insertion increases the inversion number of all of the elements in w that satisfy satisfy $w_h > w_j$ for $h < i$. □

Theorem 3.3.2. *Let $w \in \mathfrak{S}_n$ be a permutation with $p_{132}(w) = 1$. Then the pipe dream complex* Pipe(w) *is formed by taking two simplices of dimension $\binom{n}{2} - \ell(w) - 1$ and identifying them along a facet, that is,*

$$\mathfrak{S}_w \cong \Delta_m \# \Delta_m \cong \Delta_m * \partial(\Delta_1),$$

where $m = \binom{n}{2} - \ell(w) - 1$.

Proof. By Proposition 3.3.1, the Rothe diagram $D(w)$ consists of a disjoint union of a northwest justified Young diagram and exactly one isolated cross of rank one in position $(w_i+1, i+1)$. There are two possible pipe dreams, namely B_w, where this cross is left justified to the (w_i+1, i) position, and T_w, where this cross is column justified up to the position $(w_i, i+1)$. These two pipe dreams share a facet, that is, the cone corresponding to the complement of the locations of the crosses appearing in the Young diagram and the two locations (w_i+1, i) and $(w_i, i+1)$. The core is the two vertices corresponding to the locations (w_i+1, i) and $(w_i, i+1)$. $\square$

Corollary 3.3.3. *Let $w \in \mathfrak{S}_n$ be a permutation with $p_{132}(w) = 1$. If the 132-pattern in w occurs at the subword $w_i w_{i+1} w_j$ then the Schubert polynomial of the permutation w is given by*

$$\mathfrak{S}_w = \frac{x_i + x_{i+1}}{x_{i+1}} \prod_{i=1}^{n} \tau_i,$$

where $\tau = (\tau_1, \ldots, \tau_n)$ is the inversion vector of w.

3.4 The case $p_{132}(w) = 2$

In this section we study when a permutation has exactly two occurrences of the 132-pattern. We will show there are two general types of configurations that can occur, namely, the star product of a simplex with a path of length three and the star product of a simplex with the boundary of a 2-dimensional simplex. We include generalizations of these cases yielding the star product of a simplex with a path of length k (Corollary 3.4.3), the boundary of a k-dimensional simplex (Corollary 3.4.6 and Corollary 3.4.9) and the boundary of the kth iterated prism (Corollary 3.4.13).

Proposition 3.4.1. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$. If $p_{1423}(w) = 1$ then the following occurs:*

(a) The 1423-pattern occurs at the subword $w_i w_{i+1} w_k w_l$ where w_i, w_k, w_l are three consecutive integers.

(b) For $h < i$ the entries w_h satisfy $w_h > w_k$ with $w_h \neq w_{i+1}$.

(c) For $i+1 < h < l$ with $h \neq k$ the entries w_h satisfy $w_h < w_i$.

(d) The entries $\{w_h : l < h \leq n\} \subseteq \{1, \ldots, w_i - 1\} \cup \{w_{i+1}+1, \ldots, n\}$.

(e) The Rothe diagram $D(w)$ consists of a disjoint union of a northwest justified Young diagram and two isolated crosses in positions $(w_i+1, i+1)$ and $(w_i+2, i+1)$, each of rank 1. Furthermore, the column lengths of the northwest justified Young diagram are given by $(\tau_1, \ldots, \tau_i, \tau_{i+1}-2, \tau_{i+2}, \ldots, \tau_n)$.

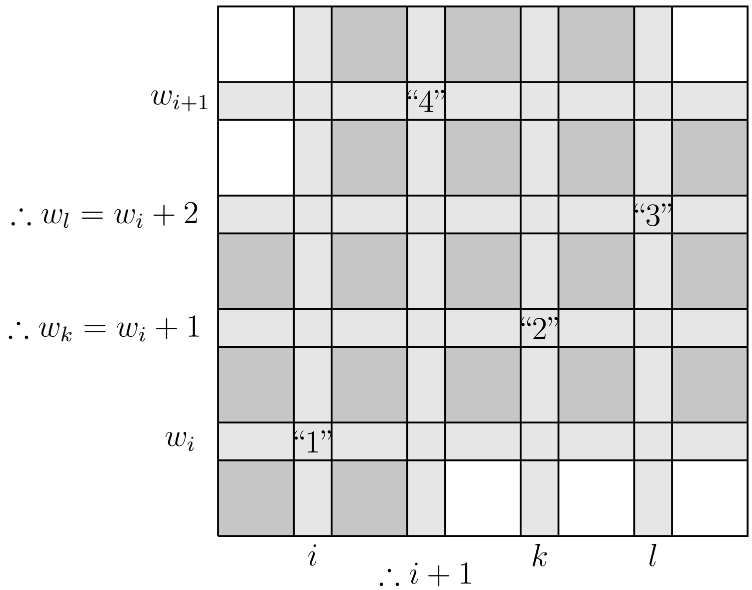


Figure 3.3: Schematic picture for the proof of Proposition 3.4.1 when $p_{132}(w) = 2$ and $p_{1423}(w) = 1$.

Proof. Since $p_{1423}(w) = 1$, we have $p_{132}(w) = 2$. Performing a similar analysis as the proof of Proposition 3.3.1, the 1423-pattern occurs at the subword $w_i w_{i+1} w_k w_l$ with w_i, w_k, w_l three consecutive integers. See Figure 3.3. This gives (a), (b) and (c).

Consider the subword $w' = w_1 \cdots \widehat{w_j} \cdots w_n$, that is, the word w without the element w_j and with all of the elements $w_k > w_j$ shifted down by one. This permutation satisfies $p_{132}(w') = 1$. By the proof of Theorem 3.3.2, the Rothe diagram $D(w')$ consists of a northwest justified Young diagram of crosses and exactly one isolated cross in the entry $(w_i + 1, i + 1)$ of rank $r_{w'}(w_i + 1, i + 1) = 1$. Reinserting the element w_j back into the permutation w' and shifting entries back to form w now inserts a hook at the entry (w_j, j) and an isolated cross at position $(w_j + 1, j + 1)$. The rank of this new isolated cross is one due to the hook at the entry $(w_{i+1}, i + 1)$.

Furthermore, the two isolated crosses in column $i + 1$ in $R(w)$ reduce the length of column $i + 1$ of the northwest justified Young by 2, that is, its length is instead $t_{i+1} - 2$, as claimed. □

Theorem 3.4.2. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$ and $p_{1423}(w) = 1$. Then*

(a) The pipe dream complex $\mathrm{PD}(w)$ *is of the form*

$$\mathrm{PD}(w) \cong \Delta_m * P_3,$$

where $m = \binom{n}{2} - \ell(w) - 3$ and P_3 is the path of length 3.

(b) The Schubert polynomial is given by

$$\mathfrak{S}_w = \frac{x_i^2 + x_i x_{i+1} + x_{i+1}^2}{x_{i+1}^2} \cdot \prod_{i=1}^{n} x_i^{\tau_i(w)},$$

where the 1423*-pattern occurs at the subword* $w_i w_{i+1} w_k w_l$.

(c) *The principal specialization of the Schubert polynomial is given by*

$$\mathfrak{S}_w(1, \dots, 1) = 3.$$

Proof. By Proposition 3.4.1 part (e) the Rothe Diagram $D(w)$ consists of a disjoint union of a northwest justified Young diagram and two isolated crosses in column $i+1$ at positions $(w_i+1, i+1)$ and $(w_i+2, i+1)$. There are three possible pipe dreams, namely B_w, where the two crosses are left justified to positions (w_i+1, i) and (w_i+2, i), T_w, where the two crosses are column justified to positions $(w_i, i+1)$ and $(w_{i+1}-1, i+1) = (w_i+1, i+1)$, and the pipe dream arising from the ladder move with the two crosses at positions $(w_i, i+1)$ and (w_i+2, i). This shows parts (b) and (c).

To show part (a), observe these three pipe dreams share a face, that is, the cone corresponding to the complement of the Young diagram and the four locations $(w_i+1, i), (w_i+2, i), (w_i, i+1)$ and $(w_i+1, i+1)$. This determines a simplex on $\binom{n}{2} - \ell(w) - 2$ vertices. The core is the length 3 path P_3 whose edges correspond to $(a, b), (b, c)$ and (c, d) where $a = (w_i+2, i), b = (w_i+1, i), c = (w_i+1, i+1)$, and $d = (w_i, i+1)$. Hence $PD(w) \cong \Delta_m * P_3$, with $m = \binom{n}{2} - \ell(w) - 3$, verifying part (a).

□

In order to represent longer permutation patterns, let $\mathbf{v} = (v_1, \dots, v_k)$ and let $p_{\mathbf{v}}(w)$ denote the number of occurrences of the pattern $v_1 \cdots v_k$ in the permutation w.

Corollary 3.4.3. *Let* $w \in \mathfrak{S}_n$ *with* $p_{132}(w) = k$ *and* $p_{\boldsymbol{v}}(w) = 1$, *where* $\boldsymbol{v} = (1, k+2, 2, 3, \dots, k+1)$. *Then the pipe dream complex* $\mathrm{PD}(w)$ *is of the form*

$$\mathrm{PD}(w) \cong \Delta_m * P_k,$$

where P_k *is the length* k *path and* $m = \binom{n}{2} - \ell(w) - k - 1$.

Proposition 3.4.4. *Let* $w \in \mathfrak{S}_n$ *with* $p_{132}(w) = 2$. *If* $p_{1243}(w) = 1$ *then the following occurs:*

(a) *The* 1243*-pattern occurs at the subword* $w_i w_{i+1} w_{i+2} w_j$ *where* w_i, w_{i+1} *and* w_{i+2} *are three consecutive integers.*

(b) *For* $h < i$ *the entries* w_h *satisfy* $w_h > w_j$ *with* $w_h \neq w_{i+2}$.

(c) *For* $i+1 < h < j$ *the entries* w_h *satisfy* $w_h < w_i$.

(d) *The entries* $\{w_h : j < h \leq n\} \subseteq \{1, \dots, w_i - 1\} \cup \{w_{i+2}+1, \dots, n\}$.

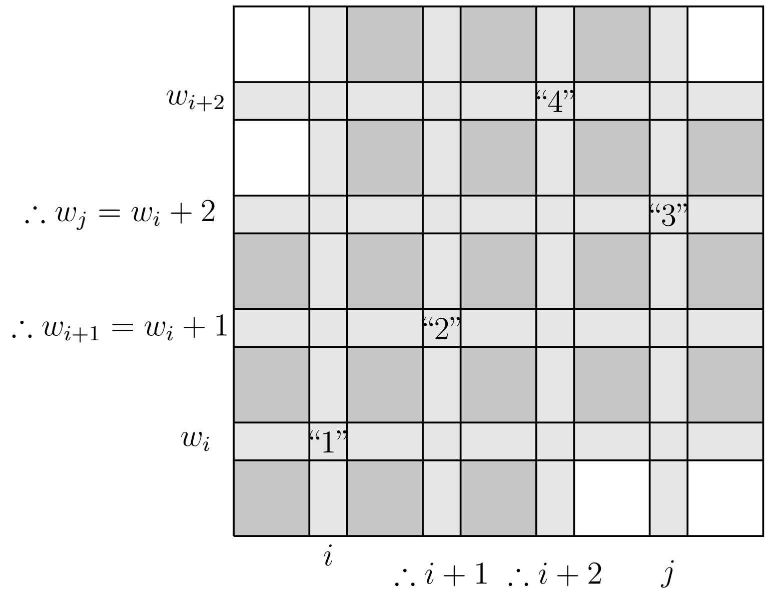


Figure 3.4: Schematic picture for the proof of Theorem 3.4.5 when $p_{132}(w) = 2$ and $p_{1243}(w) = 1$.

(e) *The Rothe diagram $D(w)$ consists of a disjoint union of a northwest justified Young diagram and one isolated cross in position $(w_i + 2, i + 1)$ of rank 2. Furthermore, this isolated cross has two hooks directly above it emanating from (w_i, i) and $w_{i+1}, i + 1)$.*

Proof. We refer to Figure 3.4 for our analysis. Statements (a) through (d) are straightforward. The 1243-pattern occurs at the subword $w_iw_{i+1}w_{i+2}w_j$ with w_i, w_{i+1} and w_j three consecutive integers. Also note that all of the elements w_k for $i+2 < k < j$ satisfy $w_k < w_i$, so the resulting Rothe diagram of $D(w)$ has no crosses occurring in these columns in rows w_i through n.

As in the proof of Proposition 3.4.1, consider the word w without the element w_j, that is, let $w' = w_1 \cdots \widehat{w_j} \cdots w_n$ with the appropriate shift down by one unit of all elements greater than w_j so that $w' \in \mathfrak{S}_{n-1}$. The permutation w' is now 132-avoiding, so its Rothe diagram $D(w')$ is a northwest justified Young diagram. Reinserting w_j into w' to form w, the resulting diagram $D(w)$ the same Young diagram as for $D(w')$ except there is now an inserted row w_j consisting of crosses in positions (w_j, k) for $k = 1, \ldots, i - 1$ plus an isolated cross in the $(w_j, i + 2)$ entry. □

Theorem 3.4.5. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$ and $p_{1243}(w) = 1$. Then the pipe dream complex* PD(w) *is of the form*

$$\mathrm{PD}(w) \cong \Delta_m * \partial(\Delta_2),$$

where $m = \binom{n}{2} - \ell(w) - 2$.

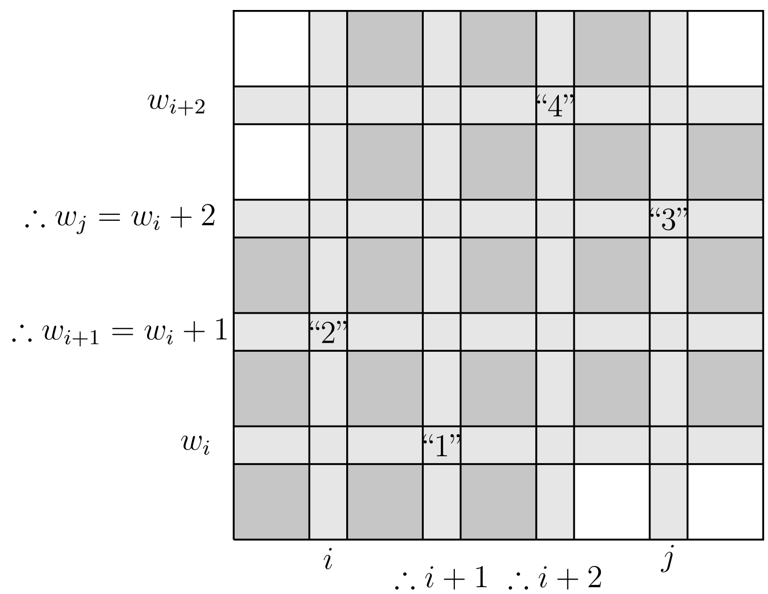


Figure 3.5: Schematic picture for the proof of Theorem 3.4.8 when $p_{132}(w) = 2$ and $p_{2143}(w) = 1$.

Proof. The intersection of all of the facets of $\mathrm{PD}(w)$ consists of a simplex on $\binom{n}{2} - \ell(w) + 1$ vertices. The isolated cross in $D(w)$ has three possible locations in the reduced pipe dreams, namely in positions (w_j, i), $(w_j - 1, i+1)$ and $(w_j - 2, i+2)$. In $\mathrm{PD}(w)$ these correspond to the boundary of the triangle $\partial(\Delta_2)$. $\square$

Iterating the proof of Theorem 3.4.5 gives the following immediate corollary.

Corollary 3.4.6. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = k$ and $p_{\boldsymbol{v}}(w) = 1$, where $\boldsymbol{v} = (1, 2, 3, \ldots, k, k+2, k+1)$. Then the pipe dream complex $\mathrm{PD}(w)$ is of the form*

$$\mathrm{PD}(w) \cong \Delta_m * \partial(\Delta_k),$$

where $m = \binom{n}{2} - \ell(w) - k$.

Proposition 3.4.7. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$. If $p_{2143}(w) = 1$ then the following occurs:*

(a) The 2143-pattern occurs at the subword $w_i w_{i+1} w_{i+2} w_j$ where w_i, w_{i+1}, w_j are three consecutive integers.

(b) For $h < i$ the entries w_h satisfy $w_h > w_j$ with $w_h \neq w_{i+2}$.

(c) For $i + 2 < h < j$ the entries w_h satisfy $w_h < w_i$.

(d) The entries $\{w_h : j < h \leq n\} \subseteq \{1, \ldots, w_i - 1\} \cup \{w_{i+2} + 1, \ldots, n\}$.

(e) *The Rothe diagram $D(w)$ consists of a disjoint union of a northwest justified Young diagram and one isolated cross in position $(w_i + 2, i + 2)$ of rank 2. Furthermore, this isolated cross has two hooks directly above it emanating from positions (w_{i+1}, i) and $(w_i, i + 1)$.*

Proof. We refer to Figure 3.5 for our analysis. Statements (a) through (d) are straightforward to verify.

Similar to Proposition 3.4.4, we consider the word w without the element w_{i+2}. Using a similar analysis, we see that the Rothe diagram $D(w)$ consists of a northwest justified Young diagram with an isolated cross in the entry $(w_j, i + 2)$ of rank 2. □

Theorem 3.4.8. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$ and $p_{2143}(w) = 1$. Then the pipe dream complex $\mathrm{PD}(w)$ is of the form*

$$\mathrm{PD}(w) \cong \Delta_m * \partial(\Delta_2),$$

where $m = \binom{n}{2} - \ell(w) - 3$.

Proof. The intersection of all of the facets of $\mathrm{PD}(w)$ consists of a simplex on $\binom{n}{2} - \ell(w) - 2$ vertices. The isolated cross in $D(w)$ has three possible locations in the reduced pipe dreams, namely in positions (w_j, i), $(w_j - 1, i + 1)$ and $(w_j - 2, i + 2)$. In $PD(w)$ these pairs of positions determine the boundary of a triangle, that is, $\partial(\Delta_2)$. □

We iterate this result to obtain the following corollary.

Corollary 3.4.9. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = k$ and $p_{\boldsymbol{v}}(w) = 1$, where $\boldsymbol{v} = (k, k-1, \ldots, 2, 1, k + 2, k + 1)$. Then the pipe dream complex $\mathrm{PD}(w)$ is of the form*

$$\mathrm{PD}(w) \cong \Delta_m * \partial(\Delta_k),$$

where $m = \binom{n}{2} - \ell(w) - k - 1$.

Proposition 3.4.10. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$. If $p_{1342}(w) = 1$ then the following occurs:*

(a) *The 1342 pattern occurs at the subword $w_i w_{i+1} w_{i+2} w_j$ where w_i and w_j are consecutive integers.*

(b) *For $h < i$ the entries w_h satisfy $w_h > w_j$ with $w_h \notin \{w_{i+1}, w_{i+2}\}$.*

(c) *For $i + 2 < h < j$ the entries w_h satisfy $w_h < w_i$.*

(d) *The entries $\{w_h : j < h \leq n\} \subseteq \{1, \ldots, w_i - 1\} \cup \{w_{i+2} + 1, \ldots, n\}$.*

(e) *The Rothe diagram $D(w)$ consists of a disjoint union of a northwest justified Young diagram and two crosses occurring in positions $(w_j, i+1)$ and $(w_j, i+2)$, each of rank 1. Furthermore, the hook emanating from position (w_i, i) is directly above these two crosses.*

(f) *The column lengths of the northwest justified Young diagram are given by*

$$(\tau_1, \ldots, \tau_i, \tau_{i+1} - 1, \tau_{i+2} - 1, \tau_{i+3}, \ldots, \tau_n).$$

Proof. Using Figure 3.6, statements (a) through (d) are straightforward to check. Notice that there cannot be any entry w_i' between the entries w_i and w_{i+1} satisfying $i < i' < i+1$ with $w_i < w_i' < w_j$, for otherwise the subword $w_i w_i' w_{i+1} w_j$ would form a second 1342-pattern.

To show (e), first observe that since the values w_h satisfying $w_j < w_h < w_{i+2}$ with $w_h \neq w_{i+1}$ (informally, "$2''$ $< w_h <$ "$4''$ with $w_h \neq$ "$3''$) all occur when $h < i$, all of the rows w_j through w_{i+2} ("$1''$ through "$4''$) have been struck out from the hooks occurring in this region of the Rothe diagram of w. By duality, the elements occurring in columns $i+3$ through $j-1$ have struck out all of the elements in the rectangular region of $D(w)$ occurring below the hook emanating from (w_i, i). Next consider the permutation $w' = w_1 \cdots w_{i+1} \widehat{w_{i+2}} \cdots w_n$ with the element w_{i+2} removed and with all of the elements $w_k > w_{i+2}$ shifted down by one. This permutation satisfies $p_{132}(w') = 1$. By the proof of Theorem 3.3.2, the Rothe diagram $D(w')$ consists of a northwest justified Young diagram of crosses and exactly one isolated cross in the entry $(w_j, i+1)$ of rank $r_{w'}(w_j, i+1) = 1$. Reinserting the element w_{i+1} back into the permutation w' after the element w_i and shifting entries back to form w creates an extra cross at the entry $(w_j, i+2)$. The rank of this new cross is one due to the hook at the entry (w_i, i).

Finally, part (f) follows from the fact that the permutation w' has exactly one occurrence of the 132-pattern. Notice the subword $w_{i+3} \cdots w_n$ is 132-avoiding, so its contribution to $D(w)$ occurs as part of the northwest justified Young diagram. Thus the column lengths of the northwest justified Young diagram are given by $(\tau_1, \ldots, \tau_i, \tau_{i+1} - 1, \tau_{i+2} - 1, \tau_{i+3} \cdots, \tau_n)$. □

Theorem 3.4.11. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$ and $p_{1342}(w) = 1$. Then*

(a) *The pipe dream complex* $\mathrm{PD}(w)$ *is of the form*

$$\mathrm{PD}(w) \cong \partial(\Delta_m) * P_3,$$

where $m = \binom{n}{2} - \ell(w) - 5$ and P_3 is the length 3 path.

(b) *The Schubert polynomial is given by*

$$\mathfrak{S}_w = \frac{x_i x_{i+1} + x_i x_{i+2} + x_{i+1} x_{i+2}}{x_{i+1} x_{i+2}} \cdot \prod_{i=1}^{n} x_i^{\tau_i(w)},$$

where the 1342-pattern occurs at the subword $w_i w_{i+1} w_{i+2} w_j$.

(c) *The principal specialization of the Schubert polynomial is given by*

$$\mathfrak{S}_w(1, \ldots, 1) = 3.$$

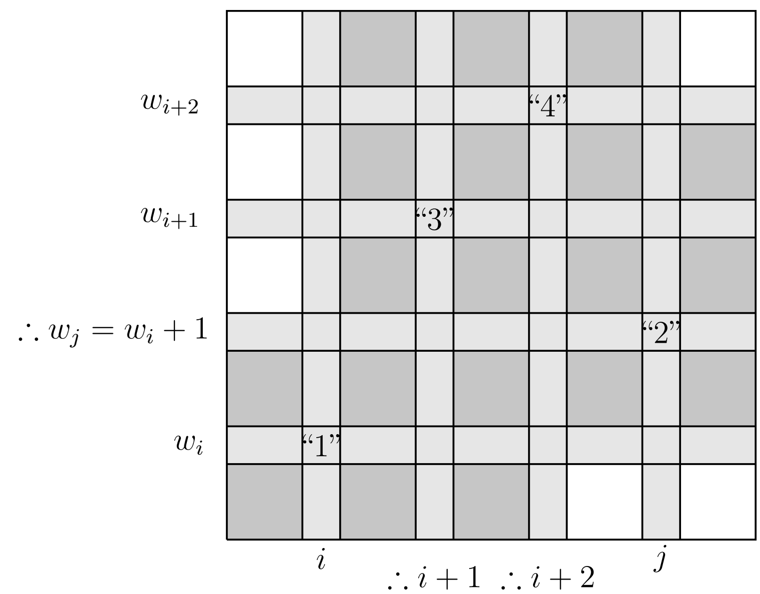


Figure 3.6: Schematic picture for the proof of Theorem 3.4.11 when $p_{132}(w) = 2$ and $p_{1342}(w) = 1$.

Proof. The intersection of all of the facets of $\mathrm{PD}(w)$ consists of a simplex on $\binom{n}{2} - \ell(w) - 4$ vertices. The two isolated crosses in the Rothe diagram $D(w)$ have three possible configurations in the RC-graphs, namely the pairs $\{(w_i+1, i), (w_i+1, i+1)\}$, $\{(w_i+1, i), (w_i, i+2)\}$ and $\{(w_i, i+1), (w_i, i+2)\}$. The complement of each of these vertex pairs is an edge, and together these three edges form a length 3 path. In turn, these are the core of $\mathrm{PD}(w)$, as asserted.

The expression for the Schubert polynomial and its principal specialization follows from the three aforementioned configurations of the vertex pairs. □

The last case of exactly two occurrences of the 132 pattern is when the two occurrences are independent.

Theorem 3.4.12. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = 2$ and $p_{465132}(w) = 1$. Then*

(a) The pipe dream complex $\mathrm{PD}(w)$ is of the form

$$\mathrm{PD}(w) \cong \Delta_m * C_4,$$

where C_4 is the cycle on 4 elements, that is, the boundary of the square, and $m = \binom{n}{2} - \ell(w) - 3$.

(b) The Schubert polynomial is given by

$$\mathfrak{S}_w = \frac{(x_i + x_{i+1})(x_{i'} + x_{i'+1})}{x_{i+1}x_{i'+1}} \prod_{i=1}^{n} x_i^{r_i(w)},$$

where the 465132-pattern occurs at the subword $w_i w_{i+1} w_k w_{i'} w_{i'+1} w_{k'}$.

(c) The principle specialization of the Schubert polynomial is given by

$$\mathfrak{S}_w(1,\ldots,1) = 4.$$

Proof. Let $w_i w_j w_k w_{i'} w_{j'} w_{k'}$ be the subword corresponding to the 465132 pattern. This follows from observing that the Rothe diagram consists of a Young diagram with two isolated crosses, one at the entry $(w_i + 1, j)$ and the other at $(w_{i'} + 1, j')$. Each has rank 1. Furthermore, by Proposition 3.3.1, $j = i + 1$ and $j' = i' + 1$. The intersection of all of the facets in $\mathrm{PD}(w)$ is a simplex on $\binom{n}{2} - \ell(w) - 2$ vertices. The remaining 4 vertices determine the boundary of a square. Since the two isolated crosses act independently, part (b) follows from Corollary 3.3.3 □

Corollary 3.4.13. *Let $w \in \mathfrak{S}_n$ with $p_{132}(w) = k$ and $p_{\boldsymbol{v}} = 1$, where $\boldsymbol{v}\pi = (3k-2)\ 3k\ (3k-1)\ \cdots 4\ 6\ 5\ 1\ 3\ 2$. Then*

$$\mathrm{PD}(w) \cong \partial(\mathrm{Prism}^k(\Delta_0)) * \Delta_m,$$

where $m = \binom{n}{2} - \ell(w) - (k+1)$. Furthermore $\mathfrak{S}_w(1,\ldots,1) = 2^k$

Proof. The result follows from iterating Theorem 3.4.12 and again noting the occurrences of the 132 pattern are independent. □

Chapter 4 Concluding remarks and further research

This dissertation opens a number of avenues for future research.

1. Establishing an upper bound for Schubert polynomial coefficients.

   Fink, Mészáros, and St. Dizier proved a pattern avoidance condition for when the Schubert polynomial coefficients do not exceed one [14].

   While a condition that determines whether the Schubert polynomial coefficients do not exceed 2, 3, or some higher upper bound is unknown, we conjecture that there is a similar pattern avoidance condition equivalent to the Schubert polynomial's coefficients not exceeding k for $k \geq 2$, and that the proof of said condition lies in the observation of Rothe diagrams and orthodontic sequences.

   Conjecture 4.0.1. *Let $k \geq 2$ be a positive integer. The Schubert polynomial $\mathfrak{S}_w$ for $w \in \mathfrak{S}_n$ has no coefficients exceeding k if and only if w avoids a set of patterns.*

   Should this conjecture be true, in the case $k = 2$, the permutation w must avoid 14 patterns in $\mathfrak{S}_6$, namely 124653, 126543, 132654, 134265, 134652, 136254, 136524, 136542, 142653, 143265, 143652, 214653, 216543, 24165. There are likely many patterns of length 7 or more that w would have to avoid also. In the case $k = 3$, there is one pattern of length 6, namely 132654, that w must avoid, with the remainder of patterns having length of at least 7. When $k \geq 4$, all relevant patterns have a minimum length of 7.

2. Gao's conjecture: improving the lower bound for the principle specialization.

   In 2019, Gao strengthened Weigandt's lower bound for the principle specialization of the Schubert polynomial with the formula

   $$\mathfrak{S}_w(1, \ldots, 1) \geq 1 + p_{132}(w) + p_{1432}(w).$$

   He then considered how the appearance of more complex patterns in a permutation influences the principle specialization and proposed a conjecture involving patterns up to length 8.

   Conjecture 4.0.2 (Gau)**.** *The lower bound for the principle specialization of the Schubert polynomial $\mathfrak{S}_w$ for $w \in \mathfrak{S}_n$ is given by*

   $$\begin{aligned}\mathfrak{S}_w(1, \ldots, 1) \geq & 1 + p_{132}(w) + p_{1432}(w) + 5p_{12543}(w) + 5p_{21543}(w) + \\ & 37p_{126543}(w) + 37p_{216543}(w) + 342p_{1327654}(w) + 5820p_{13287654}(w).\end{aligned}$$

   Studying the influence of these patterns that are rich in 132-containment, in particular the precise values of the coefficients, as well as other patterns of length 5 or more, is a viable topic for future research.

3. Grünbaum and Sreedharon's polytope and $w = 12543$. We first exhibit a pipe dream complex whose core is a 4-dimensional polytope.

Example 4.0.3. *Consider the pipe dream complex $\mathfrak{S}_w$ for $w = 12543$. Taking complements with respect to the set $[10]$, the facets are:*

$$\{1,2,5\}^c, \{1,2,8\}^c, \{1,2,10\}^c, \{1,6,8\}^c, \{1,6,10\}^c, \{1,9,10\}^c, \{2,5,6\}^c,$$
$$\{2,5,9\}^c, \{2,8,9\}^c, \{5,6,8\}^c, \{5,9,10\}^c, \{5,6,10\}^c, \{6,8,9\}^c, \{8,9,10\}^c.$$

Each facet contains the triangle $\mathrm{cone}(\mathrm{PD}(w)) = \mathrm{conv}\{3,4,7\}$*. Thus, the core of the pipe dream complex consists of the facets*

$$\{6,8,9,10\}, \{5,6,9,10\}, \{5,6,8,9\}, \{2,5,9,10\}, \{2,5,8,9\}, \{2,5,6,8\},$$
$$\{6,8,9,10\}, \{5,6,9,10\}, \{5,6,8,9\}, \{2,5,9,10\}, \{2,5,8,9\}, \{2,5,6,8\},$$
$$\{1,8,9,10\}, \{1,6,8,10\}, \{1,5,6,10\}, \{1,2,9,10\}, \{1,2,6,8\}, \{1,2,8,9\},$$
$$\{1,2,5,10\}, \{1,2,5,6\}.$$

The core in this example corresponds to the 4-dimensional polytope P_5^7 in Grünbaum and Sreedharon [23, Table 1, page 448]. This can be seen by relabeling the vertices of their example via the map $1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 5, 4 \mapsto 6, 5 \mapsto 8, 6 \mapsto 9, 7 \mapsto 10$. *Brücker (1909) was interested in counting simplicial 4-dimensional polytopes with 8 vertices.*

The task of categorizing permutations $w \in \mathfrak{S}_n$ for which the core of the pipe dream complex is a polytope is an open problem.

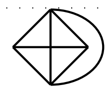
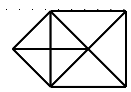
Appendices

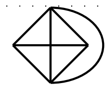
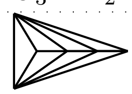
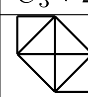
Appendix A: Schubert polynomials, principle specializations, and pipe dream complexes for symmetric groups up to five elements

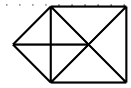
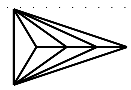
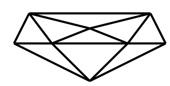
Table 1: Schubert polynomials for the symmetric groups $\mathfrak{S}_1$ through $\mathfrak{S}_5$

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$	$PD(w)$
1	1	1	$\emptyset$
12	1	1	Δ_0
21	x_1	1	$\emptyset$
123	1	1	Δ_2
213	x_1	1	Δ_1
132	$x_1 + x_2$	2	$\Delta_1 \# \Delta_1$
312	x_1^2	1	Δ_0
231	x_1x_2	1	Δ_0
321	$x_1^2x_2$	1	$\emptyset$
1234	1	1	Δ_5
2134	x_1	1	Δ_4
1324	$x_1 + x_2$	2	$\Delta_4 \# \Delta_4$
3124	x_1^2	1	Δ_3
2314	x_1x_2	1	Δ_3
3214	$x_1^2x_2$	1	Δ_2
1243	$x_1 + x_2 + x_3$	3	$C_3 * \Delta_2$
2143	$x_1^2 + x_1x_2 + x_1x_3$	3	$C_3 * \Delta_1$
1423	$x_1^2 + x_1x_2 + x_2^2$	3	$P_3 * \Delta_1$
4123	x_1^3	1	Δ_2
2413	$x_1^2x_2 + x_1x_2^2$	2	$\Delta_2 \# \Delta_2$
4213	$x_1^3x_2$	1	Δ_1
1342	$x_1x_2 + x_1x_3 + x_2x_3$	3	$P_3 * \Delta_1$
3142	$x_1^2x_2 + x_1^2x_3$	2	$\Delta_2 \# \Delta_2$
1432	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$	5	$C_5 * \Delta_0$
4132	$x_1^3x_2 + x_1^3x_3$	2	$\Delta_1 \# \Delta_1$
3412	$x_1^2x_2^2$	1	Δ_1
4312	$x_1^3x_2^2$	1	Δ_0
2341	$x_1x_2x_3$	1	Δ_2
3241	$x_1^2x_2x_3$	1	Δ_1
2431	$x_1^2x_2x_3 + x_1x_2^2x_3$	2	$\Delta_1 \# \Delta_1$
4231	$x_1^3x_2x_3$	1	Δ_0
3421	$x_1^2x_2^2x_3$	1	Δ_0
4321	$x_1^3x_2^2x_3$	1	$\emptyset$

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$	$PD(w)$
12345	1	1	Δ_9
21345	x_1	1	Δ_8
13245	x_1+x_2	2	$\Delta_8 \# \Delta_8$
31245	x_1^2	1	Δ_7
23145	x_1x_2	1	Δ_7
32145	$x_1^2x_2$	1	Δ_6
12435	$x_1+x_2+x_3$	3	$C_3 * \Delta_6$
21435	$x_1^2+x_1x_2+x_1x_3$	3	$C_3 * \Delta_5$
14235	$x_1^2+x_1x_2+x_2^2$	3	$P_3 * \Delta_5$
41235	x_1^3	1	Δ_6
24135	$x_1^2x_2+x_1x_2^2$	2	$\Delta_6 \# \Delta_6$
42135	$x_1^3x_2$	1	Δ_5
13425	$x_1x_2+x_1x_3+x_2x_3$	3	$P_3 * \Delta_5$
31425	$x_1^2x_2+x_1^2x_3$	2	$\Delta_6 \# \Delta_6$
14325	$x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_2^2x_3$	5	$C_5 * \Delta_4$
41325	$x_1^3x_2+x_1^3x_3$	2	$\Delta_5 \# \Delta_5$
34125	$x_1^2x_2^2$	1	Δ_5
43125	$x_1^3x_2^2$	1	Δ_4
23415	$x_1x_2x_3$	1	Δ_6
32415	$x_1^2x_2x_3$	1	Δ_5
24315	$x_1^2x_2x_3+x_1x_2^2x_3$	2	$\Delta_5 \# \Delta_5$
42315	$x_1^3x_2x_3$	1	Δ_4
34215	$x_1^2x_2^2x_3$	1	Δ_4
43215	$x_1^3x_2^2x_3$	1	Δ_3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\dots,1)$	$PD(w)$
12354	$x_1+x_2+x_3+x_4$	4	$\partial\Delta_3 * \Delta_5$
21354	$x_1^2+x_1x_2+x_1x_3+x_1x_4$	4	$\partial\Delta_3 * \Delta_4$
13254	$x_1^2+2x_1x_2+x_2^2+x_1x_3+x_2x_3+x_1x_4+x_2x_4$	8	$\mathrm{Bipyr}(\partial\Delta_3) * \Delta_3$
31254	$x_1^3+x_1^2x_2+x_1^2x_3+x_1^2x_4$	4	$\partial\Delta_3 * \Delta_3$
23154	$x_1^2x_2+x_1x_2^2+x_1x_2x_3+x_1x_2x_4$	4	$\partial\Delta_3 * \Delta_3$
32154	$x_1^3x_2+x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_2x_4$	4	$\partial\Delta_3 * \Delta_2$
12534	$x_1^2+x_1x_2+x_2^2+x_1x_3+x_2x_3+x_3^2$	6	core Δ_3
21534	$x_1^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_1x_3^2$	6	core Δ_2
15234	$x_1^3+x_1^2x_2+x_1x_2^2+x_2^3$	4	$P_4 * \Delta_4$
51234	x_1^4	1	Δ_5
25134	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3$	3	$P_3 * \Delta_3$
52134	$x_1^4x_2$	1	Δ_4
13524	$x_1^2x_2+x_1x_2^2+x_1^2x_3+2x_1x_2x_3+x_2^2x_3+$ $x_1x_3^2+x_2x_3^2$	8	core Δ_2
31524	$x_1^3x_2+x_1^2x_2^2+x_1^3x_3+x_1^2x_2x_3+x_1^2x_3^2$	5	 $* \Delta_2$
15324	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+x_1^2x_2x_3+$ $x_1x_2^2x_3+x_2^3x_3$	7	 $* \Delta_2$
51324	$x_1^4x_2+x_1^4x_3$	2	$\Delta_4 \# \Delta_4$
35124	$x_1^3x_2^2+x_1^2x_2^3$	2	$\Delta_4 \# \Delta_4$
53124	$x_1^4x_2^2$	1	Δ_3
23514	$x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2$	3	$C_3 * \Delta_3$
32514	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2$	3	$C_3 * \Delta_2$
25314	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3$	3	$P_3 * \Delta_2$
52314	$x_1^4x_2x_3$	1	Δ_3
35214	$x_1^3x_2^2x_3+x_1^2x_2^3x_3$	2	$\Delta_3 \# \Delta_3$
53214	$x_1^4x_2^2x_3$	1	Δ_2

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$	$PD(w)$
12453	$x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4$	6	core Δ_3
21453	$x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_1^2x_4+x_1x_2x_4+x_1x_3x_4$	6	core Δ_2
14253	$x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1^2x_4+x_1x_2x_4+x_2^2x_4$	8	core Δ_2
41253	$x_1^3x_2+x_1^3x_3+x_1^3x_4$	3	$C_3 * \Delta_3$
24153	$x_1^2x_2^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_2x_4+x_1x_2^2x_4$	5	 $* \Delta_2$
42153	$x_1^3x_2^2+x_1^3x_2x_3+x_1^3x_2x_4$	3	$C_3 * \Delta_2$
12543	$x_1^2x_2+x_1x_2^2+x_1^2x_3+2x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_1^2x_4+x_1x_2x_4+x_2^2x_4+x_1x_3x_4+x_2x_3x_4+x_3^2x_4$	14	$P_5^7 * \Delta_2$ [23, Table 1, page 448]
21543	$x_1^3x_2+x_1^2x_2^2+x_1^3x_3+2x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+x_1^3x_4+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+x_1x_2x_3x_4+x_1x_3^2x_4$	14	$P_5^7 * \Delta_1$ [23, Table 1, page 448]
15243	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+x_1^2x_2x_3+x_1x_2^2x_3+x_2^3x_3+x_1^3x_4+x_1^2x_2x_4+x_1x_2^2x_4+x_2^3x_4$	11	core Δ_1
51243	$x_1^4x_2+x_1^4x_3+x_1^4x_4$	3	$C_3 * \Delta_2$
25143	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1x_2^3x_4$	8	 $* \Delta_1$
52143	$x_1^4x_2^2+x_1^4x_2x_3+x_1^4x_2x_4$	3	$C_3 * \Delta_1$
14523	$x_1^2x_2^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2$	6	 $* \Delta_2$
41523	$x_1^3x_2^2+x_1^3x_2x_3+x_1^3x_3^2$	3	$P_3 * \Delta_1$
15423	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_3^2+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_2^3x_3^2$	9	 $* \Delta_1$
51423	$x_1^4x_2^2+x_1^4x_2x_3+x_1^4x_3^2$	3	$P_3 * \Delta_1$
45123	$x_1^3x_2^3$	1	Δ_3
54123	$x_1^4x_2^3$	1	Δ_2
24513	$x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2$	3	$P_3 * \Delta_2$
42513	$x_1^3x_2^2x_3+x_1^3x_2x_3^2$	2	$\Delta_3 \# \Delta_3$
25413	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1x_2^3x_3^2$	5	$C_5 * \Delta_2$
52413	$x_1^4x_2^2x_3+x_1^4x_2x_3^2$	2	$\Delta_2 \# \Delta_2$
45213	$x_1^3x_2^3x_3$	1	Δ_2
54213	$x_1^4x_2^3x_3$	1	Δ_1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$	$PD(w)$
13452	$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$	4	$P_4 * \Delta_4$
31452	$x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4$	3	$P_3 * \Delta_3$
14352	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4$	7	 $* \Delta_2$
41352	$x_1^3x_2x_3 + x_1^3x_2x_4 + x_1^3x_3x_4$	3	$P_3 * \Delta_2$
34152	$x_1^2x_2^2x_3 + x_1^2x_2^2x_4$	2	$\Delta_4 \# \Delta_4$
43152	$x_1^3x_2^2x_3 + x_1^3x_2^2x_4$	2	$\Delta_3 \# \Delta_3$
13542	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + 2x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4$	11	core Δ_1
31542	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1^2x_3^2x_4$	8	 $* \Delta_1$
15342	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_2^3x_3x_4$	10	core Δ_0
51342	$x_1^4x_2x_3 + x_1^4x_2x_4 + x_1^4x_3x_4$	3	$P_3 * \Delta_1$
35142	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4$	4	$C_4 * \Delta_1$
53142	$x_1^4x_2^2x_3 + x_1^4x_2^2x_4$	2	$\Delta_2 \# \Delta_2$
14532	$x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2^2x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1^2x_3^2x_4 + x_1x_2x_3^2x_4 + x_2^2x_3^2x_4$	9	 $* \Delta_1$
41532	$x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^3x_2^2x_4 + x_1^3x_2x_3x_4 + x_1^3x_3^2x_4$	5	$C_5 * \Delta_1$
15432	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_2^3x_3^2x_4$	14	$\mathrm{Assoc}_3^* * \Delta_2$
51432	$x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_2^2x_4 + x_1^4x_2x_3x_4 + x_1^4x_3^2x_4$	5	$C_5 * \Delta_0$
45132	$x_1^3x_2^3x_3 + x_1^3x_2^3x_4$	2	$\Delta_2 \# \Delta_2$
54132	$x_1^4x_2^3x_3 + x_1^4x_2^3x_4$	2	$\Delta_1 \# \Delta_1$
34512	$x_1^2x_2^2x_3^2$	1	Δ_3
43512	$x_1^3x_2^2x_3^2$	1	Δ_2
35412	$x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2$	2	$\Delta_2 \# \Delta_2$
53412	$x_1^4x_2^2x_3^2$	1	Δ_1
45312	$x_1^3x_2^3x_3^2$	1	Δ_1
54312	$x_1^4x_2^3x_3^2$	1	Δ_0

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$	$PD(w)$
23451	$x_1x_2x_3x_4$	1	Δ_5
32451	$x_1^2x_2x_3x_4$	1	Δ_4
24351	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4$	2	$\Delta_4\#\Delta_4$
42351	$x_1^3x_2x_3x_4$	1	Δ_3
34251	$x_1^2x_2^2x_3x_4$	1	Δ_3
43251	$x_1^3x_2^2x_3x_4$	1	Δ_2
23541	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4$	3	$C_3 * \Delta_2$
32541	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4$	3	$C_3 * \Delta_1$
25341	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4$	3	$P_3 * \Delta_1$
52341	$x_1^4x_2x_3x_4$	1	Δ_2
35241	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4$	2	$\Delta_2\#\Delta_2$
53241	$x_1^4x_2^2x_3x_4$	1	Δ_1
24531	$x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4$	3	$P_3 * \Delta_1$
42531	$x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4$	2	$\Delta_2\#\Delta_2$
25431	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4$	5	$C_5 * \Delta_0$
52431	$x_1^4x_2^2x_3x_4 + x_1^4x_2x_3^2x_4$	2	$\Delta_1\#\Delta_1$
45231	$x_1^3x_2^3x_3x_4$	1	Δ_1
54231	$x_1^4x_2^3x_3x_4$	1	Δ_0
34521	$x_1^2x_2^2x_3^2x_4$	1	Δ_2
43521	$x_1^3x_2^2x_3^2x_4$	1	Δ_1
35421	$x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4$	2	$\Delta_1\#\Delta_1$
53421	$x_1^4x_2^2x_3^2x_4$	1	Δ_0
45321	$x_1^3x_2^3x_3^2x_4$	1	Δ_0
54321	$x_1^4x_2^3x_3^2x_4$	1	$\emptyset$

Appendix B: Schubert polynomials and principle specializations for the symmetric group on six elements

Table 2: Schubert polynomials for the symmetric group $\mathfrak{S}_6$

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
123456	1	1
213456	x_1	1
132456	$x_1 + x_2$	2
312456	x_1^2	1
231456	x_1x_2	1
321456	$x_1^2x_2$	1
124356	$x_1 + x_2 + x_3$	3
214356	$x_1^2 + x_1x_2 + x_1x_3$	3
142356	$x_1^2 + x_1x_2 + x_2^2$	3
412356	x_1^3	1
241356	$x_1^2x_2 + x_1x_2^2$	2
421356	$x_1^3x_2$	1
134256	$x_1x_2 + x_1x_3 + x_2x_3$	3
314256	$x_1^2x_2 + x_1^2x_3$	2
143256	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$	5
413256	$x_1^3x_2 + x_1^3x_3$	2
341256	$x_1^2x_2^2$	1
431256	$x_1^3x_2^2$	1
234156	$x_1x_2x_3$	1
324156	$x_1^2x_2x_3$	1
243156	$x_1^2x_2x_3 + x_1x_2^2x_3$	2
423156	$x_1^3x_2x_3$	1
342156	$x_1^2x_2^2x_3$	1
432156	$x_1^3x_2^2x_3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
123546	$x_1+x_2+x_3+x_4$	4
213546	$x_1^2+x_1x_2+x_1x_3+x_1x_4$	4
132546	$x_1^2+2x_1x_2+x_2^2+x_1x_3+x_2x_3+x_1x_4+x_2x_4$	8
312546	$x_1^3+x_1^2x_2+x_1^2x_3+x_1^2x_4$	4
231546	$x_1^2x_2+x_1x_2^2+x_1x_2x_3+x_1x_2x_4$	4
321546	$x_1^3x_2+x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_2x_4$	4
125346	$x_1^2+x_1x_2+x_2^2+x_1x_3+x_2x_3+x_3^2$	6
215346	$x_1^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_1x_3^2$	6
152346	$x_1^3+x_1^2x_2+x_1x_2^2+x_2^3$	4
512346	x_1^4	1
251346	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3$	3
521346	$x_1^4x_2$	1
135246	$x_1^2x_2+x_1x_2^2+x_1^2x_3+2x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2$	8
315246	$x_1^3x_2+x_1^2x_2^2+x_1^3x_3+x_1^2x_2x_3+x_1^2x_3^2$	5
153246	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+x_1^2x_2x_3+x_1x_2^2x_3+x_2^3x_3$	7
513246	$x_1^4x_2+x_1^4x_3$	2
351246	$x_1^3x_2^2+x_1^2x_2^3$	2
531246	$x_1^4x_2^2$	1
235146	$x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2$	3
325146	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2$	3
253146	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3$	3
523146	$x_1^4x_2x_3$	1
352146	$x_1^3x_2^2x_3+x_1^2x_2^3x_3$	2
532146	$x_1^4x_2^2x_3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
124536	$x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4$	6
214536	$x_1^2x_2+x_1^2x_3+x_1x_2x_3+x_1^2x_4+x_1x_2x_4+x_1x_3x_4$	6
142536	$x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1^2x_4+x_1x_2x_4+x_2^2x_4$	8
412536	$x_1^3x_2+x_1^3x_3+x_1^3x_4$	3
241536	$x_1^2x_2^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_2x_4+x_1x_2^2x_4$	5
421536	$x_1^3x_2^2+x_1^3x_2x_3+x_1^3x_2x_4$	3
125436	$x_1^2x_2+x_1x_2^2+x_1^2x_3+2x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+$ $x_1^2x_4+x_1x_2x_4+x_2^2x_4+x_1x_3x_4+x_2x_3x_4+x_3^2x_4$	14
215436	$x_1^3x_2+x_1^2x_2^2+x_1^3x_3+2x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+$ $x_1^3x_4+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+x_1x_2x_3x_4+x_1x_3^2x_4$	14
152436	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+x_1^2x_2x_3+x_1x_2^2x_3+x_2^3x_3+$ $x_1^3x_4+x_1^2x_2x_4+x_1x_2^2x_4+x_2^3x_4$	11
512436	$x_1^4x_2+x_1^4x_3+x_1^4x_4$	3
251436	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_2x_4+x_1^2x_2^2x_4+$ $x_1x_2^3x_4$	8
521436	$x_1^4x_2^2+x_1^4x_2x_3+x_1^4x_2x_4$	3
145236	$x_1^2x_2^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2$	6
415236	$x_1^3x_2^2+x_1^3x_2x_3+x_1^3x_3^2$	3
154236	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_3^2+x_1^2x_2x_3^2+$ $x_1x_2^2x_3^2+x_2^3x_3^2$	9
514236	$x_1^4x_2^2+x_1^4x_2x_3+x_1^4x_3^2$	3
451236	$x_1^3x_2^3$	1
541236	$x_1^4x_2^3$	1
245136	$x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2$	3
425136	$x_1^3x_2^2x_3+x_1^3x_2x_3^2$	2
254136	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1x_2^3x_3^2$	5
524136	$x_1^4x_2^2x_3+x_1^4x_2x_3^2$	2
452136	$x_1^3x_2^3x_3$	1
542136	$x_1^4x_2^3x_3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
134526	$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$	4
314526	$x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4$	3
143526	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4$	7
413526	$x_1^3x_2x_3 + x_1^3x_2x_4 + x_1^3x_3x_4$	3
341526	$x_1^2x_2^2x_3 + x_1^2x_2^2x_4$	2
431526	$x_1^3x_2^2x_3 + x_1^3x_2^2x_4$	2
135426	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + 2x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4$	11
315426	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1^2x_3^2x_4$	8
153426	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_2^3x_3x_4$	10
513426	$x_1^4x_2x_3 + x_1^4x_2x_4 + x_1^4x_3x_4$	3
351426	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4$	4
531426	$x_1^4x_2^2x_3 + x_1^4x_2^2x_4$	2
145326	$x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2^2x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1^2x_3^2x_4 + x_1x_2x_3^2x_4 + x_2^2x_3^2x_4$	9
415326	$x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^3x_2^2x_4 + x_1^3x_2x_3x_4 + x_1^3x_3^2x_4$	5
154326	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_2^3x_3^2x_4$	14
514326	$x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_2^2x_4 + x_1^4x_2x_3x_4 + x_1^4x_3^2x_4$	5
451326	$x_1^3x_2^3x_3 + x_1^3x_2^3x_4$	2
541326	$x_1^4x_2^3x_3 + x_1^4x_2^3x_4$	2
345126	$x_1^2x_2^2x_3^2$	1
435126	$x_1^3x_2^2x_3^2$	1
354126	$x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2$	2
534126	$x_1^4x_2^2x_3^2$	1
453126	$x_1^3x_2^3x_3^2$	1
543126	$x_1^4x_2^3x_3^2$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
234516	$x_1x_2x_3x_4$	1
324516	$x_1^2x_2x_3x_4$	1
243516	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4$	2
423516	$x_1^3x_2x_3x_4$	1
342516	$x_1^2x_2^2x_3x_4$	1
432516	$x_1^3x_2^2x_3x_4$	1
235416	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4$	3
325416	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4$	3
253416	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4$	3
523416	$x_1^4x_2x_3x_4$	1
352416	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4$	2
532416	$x_1^4x_2^2x_3x_4$	1
245316	$x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4$	3
425316	$x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4$	2
254316	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4$	5
524316	$x_1^4x_2^2x_3x_4 + x_1^4x_2x_3^2x_4$	2
452316	$x_1^3x_2^3x_3x_4$	1
542316	$x_1^4x_2^3x_3x_4$	1
345216	$x_1^2x_2^2x_3^2x_4$	1
435216	$x_1^3x_2^2x_3^2x_4$	1
354216	$x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4$	2
534216	$x_1^4x_2^2x_3^2x_4$	1
453216	$x_1^3x_2^3x_3^2x_4$	1
543216	$x_1^4x_2^3x_3^2x_4$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
123465	$x_1+x_2+x_3+x_4+x_5$	5
213465	$x_1^2+x_1x_2+x_1x_3+x_1x_4+x_1x_5$	5
132465	$x_1^2+2x_1x_2+x_2^2+x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_1x_5+x_2x_5$	10
312465	$x_1^3+x_1^2x_2+x_1^2x_3+x_1^2x_4+x_1^2x_5$	5
231465	$x_1^2x_2+x_1x_2^2+x_1x_2x_3+x_1x_2x_4+x_1x_2x_5$	5
321465	$x_1^3x_2+x_1^2x_2^2+x_1^2x_2x_3+x_1^2x_2x_4+x_1^2x_2x_5$	5
124365	$x_1^2+2x_1x_2+x_2^2+2x_1x_3+2x_2x_3+x_3^2+x_1x_4+x_2x_4+x_3x_4+x_1x_5+x_2x_5+x_3x_5$	15
214365	$x_1^3+2x_1^2x_2+x_1x_2^2+2x_1^2x_3+2x_1x_2x_3+x_1x_3^2+x_1^2x_4+x_1x_2x_4+x_1x_3x_4+x_1^2x_5+x_1x_2x_5+x_1x_3x_5$	15
142365	$x_1^3+2x_1^2x_2+2x_1x_2^2+x_2^3+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1^2x_4+x_1x_2x_4+x_2^2x_4+x_1^2x_5+x_1x_2x_5+x_2^2x_5$	15
412365	$x_1^4+x_1^3x_2+x_1^3x_3+x_1^3x_4+x_1^3x_5$	5
241365	$x_1^3x_2+2x_1^2x_2^2+x_1x_2^3+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_2x_5+x_1x_2^2x_5$	10
421365	$x_1^4x_2+x_1^3x_2^2+x_1^3x_2x_3+x_1^3x_2x_4+x_1^3x_2x_5$	5
134265	$x_1^2x_2+x_1x_2^2+x_1^2x_3+3x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+x_1x_2x_5+x_1x_3x_5+x_2x_3x_5$	15
314265	$x_1^3x_2+x_1^2x_2^2+x_1^3x_3+2x_1^2x_2x_3+x_1^2x_3^2+x_1^2x_2x_4+x_1^2x_3x_4+x_1^2x_2x_5+x_1^2x_3x_5$	10
143265	$x_1^3x_2+2x_1^2x_2^2+x_1x_2^3+x_1^3x_3+3x_1^2x_2x_3+3x_1x_2^2x_3+x_2^3x_3+x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+x_1x_2x_3x_4+x_2^2x_3x_4+x_1^2x_2x_5+x_1x_2^2x_5+x_1^2x_3x_5+x_1x_2x_3x_5+x_2^2x_3x_5$	25
413265	$x_1^4x_2+x_1^3x_2^2+x_1^4x_3+2x_1^3x_2x_3+x_1^3x_3^2+x_1^3x_2x_4+x_1^3x_3x_4+x_1^3x_2x_5+x_1^3x_3x_5$	10
341265	$x_1^3x_2^2+x_1^2x_2^3+x_1^2x_2^2x_3+x_1^2x_2^2x_4+x_1^2x_2^2x_5$	5
431265	$x_1^4x_2^2+x_1^3x_2^3+x_1^3x_2^2x_3+x_1^3x_2^2x_4+x_1^3x_2^2x_5$	5
234165	$x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2+x_1x_2x_3x_4+x_1x_2x_3x_5$	5
324165	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1^2x_2x_3x_4+x_1^2x_2x_3x_5$	5
243165	$x_1^3x_2x_3+2x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_2x_3x_5+x_1x_2^2x_3x_5$	10
423165	$x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^3x_2x_3^2+x_1^3x_2x_3x_4+x_1^3x_2x_3x_5$	5
342165	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^2x_2^2x_3^2+x_1^2x_2^2x_3x_4+x_1^2x_2^2x_3x_5$	5
432165	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^3x_2^2x_3^2+x_1^3x_2^2x_3x_4+x_1^3x_2^2x_3x_5$	5

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
123645	$x_1^2+x_1x_2+x_2^2+x_1x_3+x_2x_3+x_3^2+x_1x_4+x_2x_4+x_3x_4+x_4^2$	10
213645	$x_1^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_2x_3+x_1x_3^2+x_1^2x_4+x_1x_2x_4+x_1x_3x_4+x_1x_4^2$	10
132645	$x_1^3+2x_1^2x_2+2x_1x_2^2+x_2^3+x_1^2x_3+2x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_1^2x_4+2x_1x_2x_4+x_2^2x_4+x_1x_3x_4+x_2x_3x_4+x_1x_4^2+x_2x_4^2$	20
312645	$x_1^4+x_1^3x_2+x_1^2x_2^2+x_1^3x_3+x_1^2x_2x_3+x_1^2x_3^2+x_1^3x_4+x_1^2x_2x_4+x_1^2x_3x_4+x_1^2x_4^2$	10
231645	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2+x_1^2x_2x_4+x_1x_2^2x_4+x_1x_2x_3x_4+x_1x_2x_4^2$	10
321645	$x_1^4x_2+x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1^2x_2x_3x_4+x_1^2x_2x_4^2$	10
126345	$x_1^3+x_1^2x_2+x_1x_2^2+x_2^3+x_1^2x_3+x_1x_2x_3+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_3^3$	10
216345	$x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+x_1x_3^3$	10
162345	$x_1^4+x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_2^4$	5
612345	x_1^5	1
261345	$x_1^4x_2+x_1^3x_2^2+x_1^2x_2^3+x_1x_2^4$	4
621345	$x_1^5x_2$	1
136245	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+2x_1^2x_2x_3+2x_1x_2^2x_3+x_2^3x_3+x_1^2x_3^2+2x_1x_2x_3^2+x_2^2x_3^2+x_1x_3^3+x_2x_3^3$	15
316245	$x_1^4x_2+x_1^3x_2^2+x_1^2x_2^3+x_1^4x_3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^3x_3^2+x_1^2x_2x_3^2+x_1^2x_3^3$	9
163245	$x_1^4x_2+x_1^3x_2^2+x_1^2x_2^3+x_1x_2^4+x_1^4x_3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_2^4x_3$	9
613245	$x_1^5x_2+x_1^5x_3$	2
361245	$x_1^4x_2^2+x_1^3x_2^3+x_1^2x_2^4$	3
631245	$x_1^5x_2^2$	1
236145	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1x_2x_3^3$	6
326145	$x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1^2x_2x_3^3$	6
263145	$x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1x_2^4x_3$	4
623145	$x_1^5x_2x_3$	1
362145	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^2x_2^4x_3$	3
632145	$x_1^5x_2^2x_3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
124635	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_1^2x_4+2x_1x_2x_4+x_2^2x_4+2x_1x_3x_4+2x_2x_3x_4+x_3^2x_4+x_1x_4^2+ x_2x_4^2 + x_3x_4^2$	20
214635	$x_1^3x_2+x_1^2x_2^2+x_1^3x_3+2x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+ x_1^3x_4+2x_1^2x_2x_4+x_1x_2^2x_4+2x_1^2x_3x_4+2x_1x_2x_3x_4+x_1x_3^2x_4+ x_1^2x_4^2 + x_1x_2x_4^2 + x_1x_3x_4^2$	20
142635	$x_1^3x_2+2x_1^2x_2^2+x_1x_2^3+x_1^3x_3+2x_1^2x_2x_3+2x_1x_2^2x_3+x_2^3x_3+ x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2+x_1^3x_4+2x_1^2x_2x_4+2x_1x_2^2x_4+x_2^3x_4+ x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1^2x_4^2 + x_1x_2x_4^2 + x_2^2x_4^2$	25
412635	$x_1^4x_2 + x_1^3x_2^2 + x_1^4x_3 + x_1^3x_2x_3 + x_1^3x_3^2 + x_1^4x_4 + x_1^3x_2x_4 + x_1^3x_3x_4 + x_1^3x_4^2$	9
241635	$x_1^3x_2^2 + x_1^2x_2^3 + x_1^3x_2x_3 + 2x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^3x_2x_4 + 2x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_4^2$	16
421635	$x_1^4x_2^2+x_1^3x_2^3+x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^3x_2x_3^2+x_1^4x_2x_4+x_1^3x_2^2x_4+ x_1^3x_2x_3x_4 + x_1^3x_2x_4^2$	9
126435	$x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_1^3x_3 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + x_2^3x_3 + x_1^2x_3^2 + 2x_1x_2x_3^2 + x_2^2x_3^2 + x_1x_3^3 + x_2x_3^3 + x_1^3x_4 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_2^3x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_3^3x_4$	25
216435	$x_1^4x_2+x_1^3x_2^2+x_1^2x_2^3+x_1^4x_3+2x_1^3x_2x_3+2x_1^2x_2^2x_3+x_1x_2^3x_3+ x_1^3x_3^2+2x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1^2x_3^3+x_1x_2x_3^3+x_1^4x_4+x_1^3x_2x_4+ x_1^2x_2^2x_4+x_1x_2^3x_4+x_1^3x_3x_4+x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_3^2x_4+ x_1x_2x_3^2x_4 + x_1x_3^3x_4$	25
162435	$x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1x_2^4 + x_1^4x_3 + x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_2^4x_3 + x_1^4x_4 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_2^4x_4$	14
612435	$x_1^5x_2 + x_1^5x_3 + x_1^5x_4$	3
261435	$x_1^4x_2^2+x_1^3x_2^3+x_1^2x_2^4+x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1x_2^4x_3+ x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1x_2^4x_4$	11
621435	$x_1^5x_2^2 + x_1^5x_2x_3 + x_1^5x_2x_4$	3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
146235	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+2x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_3^2+2x_1^2x_2x_3^2+2x_1x_2^2x_3^2+x_2^3x_3^2+x_1^2x_3^3+x_1x_2x_3^3+x_2^2x_3^3$	15
416235	$x_1^4x_2^2+x_1^3x_2^3+x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^4x_3^2+x_1^3x_2x_3^2+x_1^3x_3^3$	7
164235	$x_1^4x_2^2+x_1^3x_2^3+x_1^2x_2^4+x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1x_2^4x_3+x_1^4x_3^2+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_2^4x_3^2$	12
614235	$x_1^5x_2^2+x_1^5x_2x_3+x_1^5x_3^2$	3
461235	$x_1^4x_2^3+x_1^3x_2^4$	2
641235	$x_1^5x_2^3$	1
246135	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+2x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_1^2x_2x_3^3+x_1x_2^2x_3^3$	8
426135	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^3x_2x_3^3$	5
264135	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^2x_2^4x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1x_2^4x_3^2$	7
624135	$x_1^5x_2^2x_3+x_1^5x_2x_3^2$	2
462135	$x_1^4x_2^3x_3+x_1^3x_2^4x_3$	2
642135	$x_1^5x_2^3x_3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
134625	$x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + 3x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2$	15
314625	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + 2x_1^2x_2x_3x_4 + x_1^2x_3^2x_4 + x_1^2x_2x_4^2 + x_1^2x_3x_4^2$	11
143625	$x_1^3x_2x_3 + 2x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^3x_2x_4 + 2x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + 3x_1^2x_2x_3x_4 + 3x_1x_2^2x_3x_4 + x_2^3x_3x_4 + x_1^2x_3^2x_4 + x_1x_2x_3^2x_4 + x_2^2x_3^2x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_4^2 + x_1^2x_3x_4^2 + x_1x_2x_3x_4^2 + x_2^2x_3x_4^2$	26
413625	$x_1^4x_2x_3 + x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^4x_3x_4 + 2x_1^3x_2x_3x_4 + x_1^3x_3^2x_4 + x_1^3x_2x_4^2 + x_1^3x_3x_4^2$	11
341625	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^2x_2^2x_3^2 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^2x_2^2x_3x_4 + x_1^2x_2^2x_4^2$	7
431625	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^3x_2^2x_3x_4 + x_1^3x_2^2x_4^2$	7
136425	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1x_2x_3^3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + 2x_1^2x_2x_3x_4 + 2x_1x_2^2x_3x_4 + x_2^3x_3x_4 + x_1^2x_3^2x_4 + 2x_1x_2x_3^2x_4 + x_2^2x_3^2x_4 + x_1x_3^3x_4 + x_2x_3^3x_4$	21
316425	$x_1^4x_2x_3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^2x_2x_3^3 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^4x_3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1^2x_3^3x_4$	15
163425	$x_1^4x_2x_3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1x_2^4x_3 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1x_2^4x_4 + x_1^4x_3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_2^4x_3x_4$	13
613425	$x_1^5x_2x_3 + x_1^5x_2x_4 + x_1^5x_3x_4$	3
361425	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^2x_2^4x_4$	6
631425	$x_1^5x_2^2x_3 + x_1^5x_2^2x_4$	2

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
146325	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+2x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_1^2x_2x_3^3+x_1x_2^2x_3^3+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^3x_2x_3x_4+2x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^3x_3^2x_4+2x_1^2x_2x_3^2x_4+2x_1x_2^2x_3^2x_4+x_2^3x_3^2x_4+x_1^2x_3^3x_4+x_1x_2x_3^3x_4+x_2^2x_3^3x_4$	23
416325	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^3x_2x_3^3+x_1^4x_2^2x_4+x_1^3x_2^3x_4+x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^4x_3^2x_4+x_1^3x_2x_3^2x_4+x_1^3x_3^3x_4$	12
164325	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^2x_2^4x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1x_2^4x_3^2+x_1^4x_2^2x_4+x_1^3x_2^3x_4+x_1^2x_2^4x_4+x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1x_2^4x_3x_4+x_1^4x_3^2x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_2^4x_3^2x_4$	19
614325	$x_1^5x_2^2x_3+x_1^5x_2x_3^2+x_1^5x_2^2x_4+x_1^5x_2x_3x_4+x_1^5x_3^2x_4$	5
461325	$x_1^4x_2^3x_3+x_1^3x_2^4x_3+x_1^4x_2^3x_4+x_1^3x_2^4x_4$	4
641325	$x_1^5x_2^3x_3+x_1^5x_2^3x_4$	2
346125	$x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1^2x_2^2x_3^3$	3
436125	$x_1^4x_2^2x_3^2+x_1^3x_2^3x_3^2+x_1^3x_2^2x_3^3$	3
364125	$x_1^4x_2^2x_3^2+x_1^3x_2^3x_3^2+x_1^2x_2^4x_3^2$	3
634125	$x_1^5x_2^2x_3^2$	1
463125	$x_1^4x_2^3x_3^2+x_1^3x_2^4x_3^2$	2
643125	$x_1^5x_2^3x_3^2$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
234615	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2$	4
324615	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1^2x_2x_3x_4^2$	4
243615	$x_1^3x_2x_3x_4+2x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_2x_3x_4^2 + x_1x_2^2x_3x_4^2$	8
423615	$x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^3x_2x_3x_4^2$	4
342615	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^2x_2^2x_3^2x_4 + x_1^2x_2^2x_3x_4^2$	4
432615	$x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^3x_2^2x_3^2x_4 + x_1^3x_2^2x_3x_4^2$	4
236415	$x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1x_2x_3^3x_4$	6
326415	$x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1^2x_2x_3^3x_4$	6
263415	$x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1x_2^4x_3x_4$	4
623415	$x_1^5x_2x_3x_4$	1
362415	$x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^2x_2^4x_3x_4$	3
632415	$x_1^5x_2^2x_3x_4$	1
246315	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+2x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^2x_2x_3^3x_4 + x_1x_2^2x_3^3x_4$	8
426315	$x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^3x_2^2x_3^2x_4 + x_1^3x_2x_3^3x_4$	5
264315	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4 + x_1x_2^4x_3^2x_4$	7
624315	$x_1^5x_2^2x_3x_4 + x_1^5x_2x_3^2x_4$	2
462315	$x_1^4x_2^3x_3x_4 + x_1^3x_2^4x_3x_4$	2
642315	$x_1^5x_2^3x_3x_4$	1
346215	$x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1^2x_2^2x_3^3x_4$	3
436215	$x_1^4x_2^2x_3^2x_4 + x_1^3x_2^3x_3^2x_4 + x_1^3x_2^2x_3^3x_4$	3
364215	$x_1^4x_2^2x_3^2x_4 + x_1^3x_2^3x_3^2x_4 + x_1^2x_2^4x_3^2x_4$	3
634215	$x_1^5x_2^2x_3^2x_4$	1
463215	$x_1^4x_2^3x_3^2x_4 + x_1^3x_2^4x_3^2x_4$	2
643215	$x_1^5x_2^3x_3^2x_4$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
123564	$x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5$	10
213564	$x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_1^2x_4 + x_1x_2x_4 + x_1x_3x_4 + x_1^2x_5 + x_1x_2x_5 + x_1x_3x_5 + x_1x_4x_5$	10
132564	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1^2x_4 + 2x_1x_2x_4 + x_2^2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1^2x_5 + 2x_1x_2x_5 + x_2^2x_5 + x_1x_3x_5 + x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5$	20
312564	$x_1^3x_2 + x_1^3x_3 + x_1^2x_2x_3 + x_1^3x_4 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_1^3x_5 + x_1^2x_2x_5 + x_1^2x_3x_5 + x_1^2x_4x_5$	10
231564	$x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1x_2x_3x_4 + x_1^2x_2x_5 + x_1x_2^2x_5 + x_1x_2x_3x_5 + x_1x_2x_4x_5$	10
321564	$x_1^3x_2^2 + x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1^2x_2x_3x_4 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1^2x_2x_3x_5 + x_1^2x_2x_4x_5$	10
125364	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_1^2x_4 + x_1x_2x_4 + x_2^2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_3^2x_4 + x_1^2x_5 + x_1x_2x_5 + x_2^2x_5 + x_1x_3x_5 + x_2x_3x_5 + x_3^2x_5$	20
215364	$x_1^3x_2 + x_1^2x_2^2 + x_1^3x_3 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_1^3x_4 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_1x_3^2x_4 + x_1^3x_5 + x_1^2x_2x_5 + x_1x_2^2x_5 + x_1^2x_3x_5 + x_1x_2x_3x_5 + x_1x_3^2x_5$	20
152364	$x_1^3x_2+x_1^2x_2^2+x_1x_2^3+x_1^3x_3+x_1^2x_2x_3+x_1x_2^2x_3+x_2^3x_3+x_1^3x_4+x_1^2x_2x_4 + x_1x_2^2x_4 + x_2^3x_4 + x_1^3x_5 + x_1^2x_2x_5 + x_1x_2^2x_5 + x_2^3x_5$	15
512364	$x_1^4x_2 + x_1^4x_3 + x_1^4x_4 + x_1^4x_5$	4
251364	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1x_2^3x_4 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5$	11
521364	$x_1^4x_2^2 + x_1^4x_2x_3 + x_1^4x_2x_4 + x_1^4x_2x_5$	4

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
135264	$x_1^2x_2^2 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + x_1^2x_3^2 + 2x_1x_2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_1^2x_3x_4 + 2x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_1^2x_2x_5 + x_1x_2^2x_5 + x_1^2x_3x_5 + 2x_1x_2x_3x_5 + x_2^2x_3x_5 + x_1x_3^2x_5 + x_2x_3^2x_5$	25
315264	$x_1^3x_2^2 + 2x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^3x_3^2 + x_1^2x_2x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1^2x_3^2x_4 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1^3x_3x_5 + x_1^2x_2x_3x_5 + x_1^2x_3^2x_5$	16
153264	$x_1^3x_2^2 + x_1^2x_2^3 + 2x_1^3x_2x_3 + 2x_1^2x_2^2x_3 + 2x_1x_2^3x_3 + x_1^3x_3^2 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_2^3x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_2^3x_3x_4 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5 + x_1^3x_3x_5 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_2^3x_3x_5$	26
513264	$x_1^4x_2^2 + 2x_1^4x_2x_3 + x_1^4x_3^2 + x_1^4x_2x_4 + x_1^4x_3x_4 + x_1^4x_2x_5 + x_1^4x_3x_5$	8
351264	$x_1^3x_2^3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5$	7
531264	$x_1^4x_2^3 + x_1^4x_2^2x_3 + x_1^4x_2^2x_4 + x_1^4x_2^2x_5$	4
235164	$x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_1x_2x_3^2x_5$	9
325164	$x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1^2x_2x_3^2x_5$	9
253164	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5$	11
523164	$x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_2x_3x_4 + x_1^4x_2x_3x_5$	4
352164	$x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5$	7
532164	$x_1^4x_2^3x_3 + x_1^4x_2^2x_3^2 + x_1^4x_2^2x_3x_4 + x_1^4x_2^2x_3x_5$	4

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
123654	$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_1^2x_4+2x_1x_2x_4+x_2^2x_4+2x_1x_3x_4+2x_2x_3x_4+x_3^2x_4+x_1x_4^2+ x_2x_4^2 + x_3x_4^2 + x_1^2x_5 + x_1x_2x_5 + x_2^2x_5 + x_1x_3x_5 + x_2x_3x_5 + x_3^2x_5 + x_1x_4x_5 + x_2x_4x_5 + x_3x_4x_5 + x_4^2x_5$	30
213654	$x_1^3x_2 + x_1^2x_2^2 + x_1^3x_3 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_3^2 + x_1x_2x_3^2 + x_1^3x_4+2x_1^2x_2x_4+x_1x_2^2x_4+2x_1^2x_3x_4+2x_1x_2x_3x_4+x_1x_3^2x_4+ x_1^2x_4^2+x_1x_2x_4^2+x_1x_3x_4^2+x_1^3x_5+x_1^2x_2x_5+x_1x_2^2x_5+x_1^2x_3x_5+ x_1x_2x_3x_5 + x_1x_3^2x_5 + x_1^2x_4x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_1x_4^2x_5$	30
132654	$x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 + x_1^3x_3 + 3x_1^2x_2x_3 + 3x_1x_2^2x_3 + x_2^3x_3 + x_1^2x_3^2+2x_1x_2x_3^2+x_2^2x_3^2+x_1^3x_4+3x_1^2x_2x_4+3x_1x_2^2x_4+x_2^3x_4+ 2x_1^2x_3x_4+4x_1x_2x_3x_4+2x_2^2x_3x_4+x_1x_3^2x_4+x_2x_3^2x_4+x_1^2x_4^2+ 2x_1x_2x_4^2 + x_2^2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2 + x_1^3x_5 + 2x_1^2x_2x_5 + 2x_1x_2^2x_5+x_2^3x_5+x_1^2x_3x_5+2x_1x_2x_3x_5+x_2^2x_3x_5+x_1x_3^2x_5+ x_2x_3^2x_5 + x_1^2x_4x_5 + 2x_1x_2x_4x_5 + x_2^2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_4^2x_5 + x_2x_4^2x_5$	60
312654	$x_1^4x_2 + x_1^3x_2^2 + x_1^4x_3 + 2x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^3x_3^2 + x_1^2x_2x_3^2 + x_1^4x_4+2x_1^3x_2x_4+x_1^2x_2^2x_4+2x_1^3x_3x_4+2x_1^2x_2x_3x_4+x_1^2x_3^2x_4+ x_1^3x_4^2+x_1^2x_2x_4^2+x_1^2x_3x_4^2+x_1^4x_5+x_1^3x_2x_5+x_1^2x_2^2x_5+x_1^3x_3x_5+ x_1^2x_2x_3x_5 + x_1^2x_3^2x_5 + x_1^3x_4x_5 + x_1^2x_2x_4x_5 + x_1^2x_3x_4x_5 + x_1^2x_4^2x_5$	30
231654	$x_1^3x_2^2 + x_1^2x_2^3 + x_1^3x_2x_3 + 2x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^3x_2x_4 + 2x_1^2x_2^2x_4 + x_1x_2^3x_4 + 2x_1^2x_2x_3x_4 + 2x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_4^2 + x_1x_2x_3x_4^2 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_1x_2x_3^2x_5+x_1^2x_2x_4x_5+x_1x_2^2x_4x_5+x_1x_2x_3x_4x_5+x_1x_2x_4^2x_5$	30
321654	$x_1^4x_2^2 + x_1^3x_2^3 + x_1^4x_2x_3 + 2x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^4x_2x_4 + 2x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + 2x_1^3x_2x_3x_4 + 2x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1^2x_2x_3x_4^2 + x_1^4x_2x_5 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1^2x_2x_3^2x_5+x_1^3x_2x_4x_5+x_1^2x_2^2x_4x_5+x_1^2x_2x_3x_4x_5+x_1^2x_2x_4^2x_5$	30

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
126354	$x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_1^3x_3 + 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + x_2^3x_3 + x_1^2x_3^2 + 2x_1x_2x_3^2 + x_2^2x_3^2 + x_1x_3^3 + x_2x_3^3 + x_1^3x_4 + x_1^2x_2x_4 + x_1x_2^2x_4 + x_2^3x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_2x_3^2x_4 + x_3^3x_4 + x_1^3x_5 + x_1^2x_2x_5 + x_1x_2^2x_5 + x_2^3x_5 + x_1^2x_3x_5 + x_1x_2x_3x_5 + x_2^2x_3x_5 + x_1x_3^2x_5 + x_2x_3^2x_5 + x_3^3x_5$	35
216354	$x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1^4x_3 + 2x_1^3x_2x_3 + 2x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^3x_3^2 + 2x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_3^3 + x_1x_2x_3^3 + x_1^4x_4 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1^2x_3^2x_4 + x_1x_2x_3^2x_4 + x_1x_3^3x_4 + x_1^4x_5 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5 + x_1^3x_3x_5 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_1^2x_3^2x_5 + x_1x_2x_3^2x_5 + x_1x_3^3x_5$	35
162354	$x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1x_2^4 + x_1^4x_3 + x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_2^4x_3 + x_1^4x_4 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_2^4x_4 + x_1^4x_5 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5 + x_2^4x_5$	19
612354	$x_1^5x_2 + x_1^5x_3 + x_1^5x_4 + x_1^5x_5$	4
261354	$x_1^4x_2^2 + x_1^3x_2^3 + x_1^2x_2^4 + x_1^4x_2x_3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1x_2^4x_3 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1x_2^4x_4 + x_1^4x_2x_5 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1x_2^4x_5$	15
621354	$x_1^5x_2^2 + x_1^5x_2x_3 + x_1^5x_2x_4 + x_1^5x_2x_5$	4

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
136254	$x_1^3x_2^2 + x_1^2x_2^3 + 2x_1^3x_2x_3 + 3x_1^2x_2^2x_3 + 2x_1x_2^3x_3 + x_1^3x_3^2 + 3x_1^2x_2x_3^2 + 3x_1x_2^2x_3^2 + x_2^3x_3^2 + x_1^2x_3^3 + 2x_1x_2x_3^3 + x_2^2x_3^3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + 2x_1^2x_2x_3x_4 + 2x_1x_2^2x_3x_4 + x_2^3x_3x_4 + x_1^2x_3^2x_4 + 2x_1x_2x_3^2x_4 + x_2^2x_3^2x_4 + x_1x_3^3x_4 + x_2x_3^3x_4 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5 + x_1^3x_3x_5 + 2x_1^2x_2x_3x_5 + 2x_1x_2^2x_3x_5 + x_2^3x_3x_5 + x_1^2x_3^2x_5 + 2x_1x_2x_3^2x_5 + x_2^2x_3^2x_5 + x_1x_3^3x_5 + x_2x_3^3x_5$	51
316254	$x_1^4x_2^2 + x_1^3x_2^3 + 2x_1^4x_2x_3 + 2x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^4x_3^2 + 2x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^3x_3^3 + x_1^2x_2x_3^3 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^4x_3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1^2x_3^3x_4 + x_1^4x_2x_5 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1^4x_3x_5 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1^3x_3^2x_5 + x_1^2x_2x_3^2x_5 + x_1^2x_3^3x_5$	31
163254	$x_1^4x_2^2 + x_1^3x_2^3 + x_1^2x_2^4 + 2x_1^4x_2x_3 + 2x_1^3x_2^2x_3 + 2x_1^2x_2^3x_3 + 2x_1x_2^4x_3 + x_1^4x_3^2 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_2^4x_3^2 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1x_2^4x_4 + x_1^4x_3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_2^4x_3x_4 + x_1^4x_2x_5 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1x_2^4x_5 + x_1^4x_3x_5 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_2^4x_3x_5$	34
613254	$x_1^5x_2^2 + 2x_1^5x_2x_3 + x_1^5x_3^2 + x_1^5x_2x_4 + x_1^5x_3x_4 + x_1^5x_2x_5 + x_1^5x_3x_5$	8
361254	$x_1^4x_2^3 + x_1^3x_2^4 + x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^2x_2^4x_4 + x_1^4x_2^2x_5 + x_1^3x_2^3x_5 + x_1^2x_2^4x_5$	11
631254	$x_1^5x_2^3 + x_1^5x_2^2x_3 + x_1^5x_2^2x_4 + x_1^5x_2^2x_5$	4
236154	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + 2x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_1x_2x_3^3x_4 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_1^2x_2x_3^2x_5 + x_1x_2^2x_3^2x_5 + x_1x_2x_3^3x_5$	20
326154	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^4x_2x_3^2 + 2x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2x_3^3 + x_1^2x_2^2x_3^3 + x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1^2x_2x_3^3x_4 + x_1^4x_2x_3x_5 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1^3x_2x_3^2x_5 + x_1^2x_2^2x_3^2x_5 + x_1^2x_2x_3^3x_5$	20
263154	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2x_3^2 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1x_2^4x_3^2 + x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1x_2^4x_3x_4 + x_1^4x_2x_3x_5 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1x_2^4x_3x_5$	15
623154	$x_1^5x_2^2x_3 + x_1^5x_2x_3^2 + x_1^5x_2x_3x_4 + x_1^5x_2x_3x_5$	4
362154	$x_1^4x_2^3x_3 + x_1^3x_2^4x_3 + x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^2x_2^4x_3x_4 + x_1^4x_2^2x_3x_5 + x_1^3x_2^3x_3x_5 + x_1^2x_2^4x_3x_5$	11
632154	$x_1^5x_2^3x_3 + x_1^5x_2^2x_3^2 + x_1^5x_2^2x_3x_4 + x_1^5x_2^2x_3x_5$	4

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
125634	$x_1^2x_2^2+x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_3^2+x_1x_2x_3^2+x_2^2x_3^2+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+2x_1x_2x_3x_4+x_2^2x_3x_4+x_1x_3^2x_4+x_2x_3^2x_4+x_1^2x_4^2+x_1x_2x_4^2+x_2^2x_4^2+x_1x_3x_4^2+x_2x_3x_4^2+x_3^2x_4^2$	20
215634	$x_1^3x_2^2+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^3x_3^2+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1^3x_3x_4+2x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_3^2x_4+x_1x_2x_3^2x_4+x_1^3x_4^2+x_1^2x_2x_4^2+x_1x_2^2x_4^2+x_1^2x_3x_4^2+x_1x_2x_3x_4^2+x_1x_3^2x_4^2$	20
152634	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_3^2+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_2^3x_3^2+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1x_2^3x_4+x_1^3x_3x_4+x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_2^3x_3x_4+x_1^3x_4^2+x_1^2x_2x_4^2+x_1x_2^2x_4^2+x_2^3x_4^2$	20
512634	$x_1^4x_2^2+x_1^4x_2x_3+x_1^4x_3^2+x_1^4x_2x_4+x_1^4x_3x_4+x_1^4x_4^2$	6
251634	$x_1^3x_2^3+x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^3x_2x_4^2+x_1^2x_2^2x_4^2+x_1x_2^3x_4^2$	14
521634	$x_1^4x_2^3+x_1^4x_2^2x_3+x_1^4x_2x_3^2+x_1^4x_2^2x_4+x_1^4x_2x_3x_4+x_1^4x_2x_4^2$	6
126534	$x_1^3x_2^2+x_1^2x_2^3+x_1^3x_2x_3+2x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_3^2+2x_1^2x_2x_3^2+2x_1x_2^2x_3^2+x_2^3x_3^2+x_1^2x_3^3+x_1x_2x_3^3+x_2^2x_3^3+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1x_2^3x_4+x_1^3x_3x_4+2x_1^2x_2x_3x_4+2x_1x_2^2x_3x_4+x_2^3x_3x_4+x_1^2x_3^2x_4+2x_1x_2x_3^2x_4+x_2^2x_3^2x_4+x_1x_3^3x_4+x_2x_3^3x_4+x_1^3x_4^2+x_1^2x_2x_4^2+x_1x_2^2x_4^2+x_2^3x_4^2+x_1^2x_3x_4^2+x_1x_2x_3x_4^2+x_2^2x_3x_4^2+x_1x_3^2x_4^2+x_2x_3^2x_4^2+x_3^3x_4^2$	40
216534	$x_1^4x_2^2+x_1^3x_2^3+x_1^4x_2x_3+2x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^4x_3^2+2x_1^3x_2x_3^2+2x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_1^3x_3^3+x_1^2x_2x_3^3+x_1x_2^2x_3^3+x_1^4x_2x_4+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^4x_3x_4+2x_1^3x_2x_3x_4+2x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^3x_3^2x_4+2x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_3^3x_4+x_1x_2x_3^3x_4+x_1^4x_4^2+x_1^3x_2x_4^2+x_1^2x_2^2x_4^2+x_1x_2^3x_4^2+x_1^3x_3x_4^2+x_1^2x_2x_3x_4^2+x_1x_2^2x_3x_4^2+x_1^2x_3^2x_4^2+x_1x_2x_3^2x_4^2+x_1x_3^3x_4^2$	40
162534	$x_1^4x_2^2+x_1^3x_2^3+x_1^2x_2^4+x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1x_2^4x_3+x_1^4x_3^2+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_2^4x_3^2+x_1^4x_2x_4+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1x_2^4x_4+x_1^4x_3x_4+x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_2^4x_3x_4+x_1^4x_4^2+x_1^3x_2x_4^2+x_1^2x_2^2x_4^2+x_1x_2^3x_4^2+x_2^4x_4^2$	26
612534	$x_1^5x_2^2+x_1^5x_2x_3+x_1^5x_3^2+x_1^5x_2x_4+x_1^5x_3x_4+x_1^5x_4^2$	6
261534	$x_1^4x_2^3+x_1^3x_2^4+x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^2x_2^4x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1x_2^4x_3^2+x_1^4x_2^2x_4+x_1^3x_2^3x_4+x_1^2x_2^4x_4+x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1x_2^4x_3x_4+x_1^4x_2x_4^2+x_1^3x_2^2x_4^2+x_1^2x_2^3x_4^2+x_1x_2^4x_4^2$	20
621534	$x_1^5x_2^3+x_1^5x_2^2x_3+x_1^5x_2x_3^2+x_1^5x_2^2x_4+x_1^5x_2x_3x_4+x_1^5x_2x_4^2$	6

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
156234	$x_1^3x_2^3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_3^3 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3 + x_2^3x_3^3$	10
516234	$x_1^4x_2^3 + x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_3^3$	4
165234	$x_1^4x_2^3+x_1^3x_2^4+x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^2x_2^4x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1x_2^4x_3^2+x_1^4x_3^3+x_1^3x_2x_3^3+x_1^2x_2^2x_3^3+x_1x_2^3x_3^3+x_2^4x_3^3$	14
615234	$x_1^5x_2^3 + x_1^5x_2^2x_3 + x_1^5x_2x_3^2 + x_1^5x_3^3$	4
561234	$x_1^4x_2^4$	1
651234	$x_1^5x_2^4$	1
256134	$x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2x_3^3 + x_1^2x_2^2x_3^3 + x_1x_2^3x_3^3$	6
526134	$x_1^4x_2^3x_3 + x_1^4x_2^2x_3^2 + x_1^4x_2x_3^3$	3
265134	$x_1^4x_2^3x_3 + x_1^3x_2^4x_3 + x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2x_3^3 + x_1^3x_2^2x_3^3 + x_1^2x_2^3x_3^3 + x_1x_2^4x_3^3$	9
625134	$x_1^5x_2^3x_3 + x_1^5x_2^2x_3^2 + x_1^5x_2x_3^3$	3
562134	$x_1^4x_2^4x_3$	1
652134	$x_1^5x_2^4x_3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
135624	$x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2^2x_4 + 2x_1^2x_2x_3x_4 + 2x_1x_2^2x_3x_4 + x_1^2x_3^2x_4 + 2x_1x_2x_3^2x_4 + x_2^2x_3^2x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_4^2 + x_1^2x_3x_4^2 + 2x_1x_2x_3x_4^2 + x_2^2x_3x_4^2 + x_1x_3^2x_4^2 + x_2x_3^2x_4^2$	20
315624	$x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^3x_2^2x_4 + 2x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1^3x_3x_4^2 + x_1^2x_2x_3x_4^2 + x_1^2x_3^2x_4^2$	14
153624	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + 2x_1^3x_2x_3x_4 + 2x_1^2x_2^2x_3x_4 + 2x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_2^3x_3^2x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1x_2^3x_4^2 + x_1^3x_3x_4^2 + x_1^2x_2x_3x_4^2 + x_1x_2^2x_3x_4^2 + x_2^3x_3x_4^2$	24
513624	$x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_2^2x_4 + 2x_1^4x_2x_3x_4 + x_1^4x_3^2x_4 + x_1^4x_2x_4^2 + x_1^4x_3x_4^2$	8
351624	$x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2^3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2$	8
531624	$x_1^4x_2^3x_3 + x_1^4x_2^2x_3^2 + x_1^4x_2^3x_4 + x_1^4x_2^2x_3x_4 + x_1^4x_2^2x_4^2$	5
136524	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + 2x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + 2x_1^3x_2x_3x_4 + 3x_1^2x_2^2x_3x_4 + 2x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + 3x_1^2x_2x_3^2x_4 + 3x_1x_2^2x_3^2x_4 + x_2^3x_3^2x_4 + x_1^2x_3^3x_4 + 2x_1x_2x_3^3x_4 + x_2^2x_3^3x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1x_2^3x_4^2 + x_1^3x_3x_4^2 + 2x_1^2x_2x_3x_4^2 + 2x_1x_2^2x_3x_4^2 + x_2^3x_3x_4^2 + x_1^2x_3^2x_4^2 + 2x_1x_2x_3^2x_4^2 + x_2^2x_3^2x_4^2 + x_1x_3^3x_4^2 + x_2x_3^3x_4^2$	44
316524	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^4x_2x_3^2 + 2x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2x_3^3 + x_1^2x_2^2x_3^3 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + 2x_1^4x_2x_3x_4 + 2x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^4x_3^2x_4 + 2x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1^3x_3^3x_4 + x_1^2x_2x_3^3x_4 + x_1^4x_2x_4^2 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1^4x_3x_4^2 + x_1^3x_2x_3x_4^2 + x_1^2x_2^2x_3x_4^2 + x_1^3x_3^2x_4^2 + x_1^2x_2x_3^2x_4^2 + x_1^2x_3^3x_4^2$	30
163524	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2x_3^2 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1x_2^4x_3^2 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^2x_2^4x_4 + 2x_1^4x_2x_3x_4 + 2x_1^3x_2^2x_3x_4 + 2x_1^2x_2^3x_3x_4 + 2x_1x_2^4x_3x_4 + x_1^4x_3^2x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_2^4x_3^2x_4 + x_1^4x_2x_4^2 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1x_2^4x_4^2 + x_1^4x_3x_4^2 + x_1^3x_2x_3x_4^2 + x_1^2x_2^2x_3x_4^2 + x_1x_2^3x_3x_4^2 + x_2^4x_3x_4^2$	32
613524	$x_1^5x_2^2x_3 + x_1^5x_2x_3^2 + x_1^5x_2^2x_4 + 2x_1^5x_2x_3x_4 + x_1^5x_3^2x_4 + x_1^5x_2x_4^2 + x_1^5x_3x_4^2$	8
361524	$x_1^4x_2^3x_3 + x_1^3x_2^4x_3 + x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2^3x_4 + x_1^3x_2^4x_4 + x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^2x_2^4x_3x_4 + x_1^4x_2^2x_4^2 + x_1^3x_2^3x_4^2 + x_1^2x_2^4x_4^2$	13
631524	$x_1^5x_2^3x_3 + x_1^5x_2^2x_3^2 + x_1^5x_2^3x_4 + x_1^5x_2^2x_3x_4 + x_1^5x_2^2x_4^2$	5

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
156324	$x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2x_3^3 + x_1^2x_2^2x_3^3 + x_1x_2^3x_3^3 + x_1^3x_2^3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^3x_3^3x_4 + x_1^2x_2x_3^3x_4 + x_1x_2^2x_3^3x_4 + x_2^3x_3^3x_4$	16
516324	$x_1^4x_2^3x_3+x_1^4x_2^2x_3^2+x_1^4x_2x_3^3+x_1^4x_2^3x_4+x_1^4x_2^2x_3x_4+x_1^4x_2x_3^2x_4+x_1^4x_3^3x_4$	7
165324	$x_1^4x_2^3x_3 + x_1^3x_2^4x_3 + x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2x_3^3 + x_1^3x_2^2x_3^3+x_1^2x_2^3x_3^3+x_1x_2^4x_3^3+x_1^4x_2^3x_4+x_1^3x_2^4x_4+x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1x_2^4x_3^2x_4 + x_1^4x_3^3x_4 + x_1^3x_2x_3^3x_4 + x_1^2x_2^2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_2^4x_3^3x_4$	23
615324	$x_1^5x_2^3x_3+x_1^5x_2^2x_3^2+x_1^5x_2x_3^3+x_1^5x_2^3x_4+x_1^5x_2^2x_3x_4+x_1^5x_2x_3^2x_4+x_1^5x_3^3x_4$	7
561324	$x_1^4x_2^4x_3 + x_1^4x_2^4x_4$	2
651324	$x_1^5x_2^4x_3 + x_1^5x_2^4x_4$	2
356124	$x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 + x_1^2x_2^3x_3^3$	3
536124	$x_1^4x_2^3x_3^2 + x_1^4x_2^2x_3^3$	2
365124	$x_1^4x_2^3x_3^2 + x_1^3x_2^4x_3^2 + x_1^4x_2^2x_3^3 + x_1^3x_2^3x_3^3 + x_1^2x_2^4x_3^3$	5
635124	$x_1^5x_2^3x_3^2 + x_1^5x_2^2x_3^3$	2
563124	$x_1^4x_2^4x_3^2$	1
653124	$x_1^5x_2^4x_3^2$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
235614	$x_1^2x_2^2x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_2x_3x_4^2+x_1x_2^2x_3x_4^2+x_1x_2x_3^2x_4^2$	6
325614	$x_1^3x_2^2x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1^3x_2x_3x_4^2+x_1^2x_2^2x_3x_4^2+x_1^2x_2x_3^2x_4^2$	6
253614	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^3x_2x_3x_4^2+x_1^2x_2^2x_3x_4^2+x_1x_2^3x_3x_4^2$	8
523614	$x_1^4x_2^2x_3x_4+x_1^4x_2x_3^2x_4+x_1^4x_2x_3x_4^2$	3
352614	$x_1^3x_2^3x_3x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2$	5
532614	$x_1^4x_2^3x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^4x_2^2x_3x_4^2$	3
236514	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+2x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^2x_2x_3^3x_4+x_1x_2^2x_3^3x_4+x_1^3x_2x_3x_4^2+x_1^2x_2^2x_3x_4^2+x_1x_2^3x_3x_4^2+x_1^2x_2x_3^2x_4^2+x_1x_2^2x_3^2x_4^2+x_1x_2x_3^3x_4^2$	14
326514	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^4x_2x_3^2x_4+2x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2x_3^3x_4+x_1^2x_2^2x_3^3x_4+x_1^4x_2x_3x_4^2+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1^3x_2x_3^2x_4^2+x_1^2x_2^2x_3^2x_4^2+x_1^2x_2x_3^3x_4^2$	14
263514	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1x_2^4x_3^2x_4+x_1^4x_2x_3x_4^2+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1x_2^4x_3x_4^2$	11
623514	$x_1^5x_2^2x_3x_4+x_1^5x_2x_3^2x_4+x_1^5x_2x_3x_4^2$	3
362514	$x_1^4x_2^3x_3x_4+x_1^3x_2^4x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^2x_2^4x_3^2x_4+x_1^4x_2^2x_3x_4^2+x_1^3x_2^3x_3x_4^2+x_1^2x_2^4x_3x_4^2$	8
632514	$x_1^5x_2^3x_3x_4+x_1^5x_2^2x_3^2x_4+x_1^5x_2^2x_3x_4^2$	3
256314	$x_1^3x_2^3x_3x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2x_3^3x_4+x_1^2x_2^2x_3^3x_4+x_1x_2^3x_3^3x_4$	6
526314	$x_1^4x_2^3x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^4x_2x_3^3x_4$	3
265314	$x_1^4x_2^3x_3x_4+x_1^3x_2^4x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^2x_2^4x_3^2x_4+x_1^4x_2x_3^3x_4+x_1^3x_2^2x_3^3x_4+x_1^2x_2^3x_3^3x_4+x_1x_2^4x_3^3x_4$	9
625314	$x_1^5x_2^3x_3x_4+x_1^5x_2^2x_3^2x_4+x_1^5x_2x_3^3x_4$	3
562314	$x_1^4x_2^4x_3x_4$	1
652314	$x_1^5x_2^4x_3x_4$	1
356214	$x_1^3x_2^3x_3^2x_4+x_1^3x_2^2x_3^3x_4+x_1^2x_2^3x_3^3x_4$	3
536214	$x_1^4x_2^3x_3^2x_4+x_1^4x_2^2x_3^3x_4$	2
365214	$x_1^4x_2^3x_3^2x_4+x_1^3x_2^4x_3^2x_4+x_1^4x_2^2x_3^3x_4+x_1^3x_2^3x_3^3x_4+x_1^2x_2^4x_3^3x_4$	5
635214	$x_1^5x_2^3x_3^2x_4+x_1^5x_2^2x_3^3x_4$	2
563214	$x_1^4x_2^4x_3^2x_4$	1
653214	$x_1^5x_2^4x_3^2x_4$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
124563	$x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+x_1x_2x_5+x_1x_3x_5+x_2x_3x_5+x_1x_4x_5+x_2x_4x_5+x_3x_4x_5$	10
214563	$x_1^2x_2x_3+x_1^2x_2x_4+x_1^2x_3x_4+x_1x_2x_3x_4+x_1^2x_2x_5+x_1^2x_3x_5+x_1x_2x_3x_5+x_1^2x_4x_5+x_1x_2x_4x_5+x_1x_3x_4x_5$	10
142563	$x_1^2x_2x_3+x_1x_2^2x_3+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+x_1x_2x_3x_4+x_2^2x_3x_4+x_1^2x_2x_5+x_1x_2^2x_5+x_1^2x_3x_5+x_1x_2x_3x_5+x_2^2x_3x_5+x_1^2x_4x_5+x_1x_2x_4x_5+x_2^2x_4x_5$	15
412563	$x_1^3x_2x_3+x_1^3x_2x_4+x_1^3x_3x_4+x_1^3x_2x_5+x_1^3x_3x_5+x_1^3x_4x_5$	6
241563	$x_1^2x_2^2x_3+x_1^2x_2^2x_4+x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_2^2x_5+x_1^2x_2x_3x_5+x_1x_2^2x_3x_5+x_1^2x_2x_4x_5+x_1x_2^2x_4x_5$	9
421563	$x_1^3x_2^2x_3+x_1^3x_2^2x_4+x_1^3x_2x_3x_4+x_1^3x_2^2x_5+x_1^3x_2x_3x_5+x_1^3x_2x_4x_5$	6
125463	$x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+2x_1x_2x_3x_4+x_2^2x_3x_4+x_1x_3^2x_4+x_2x_3^2x_4+x_1^2x_2x_5+x_1x_2^2x_5+x_1^2x_3x_5+2x_1x_2x_3x_5+x_2^2x_3x_5+x_1x_3^2x_5+x_2x_3^2x_5+x_1^2x_4x_5+x_1x_2x_4x_5+x_2^2x_4x_5+x_1x_3x_4x_5+x_2x_3x_4x_5+x_3^2x_4x_5$	25
215463	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1^3x_3x_4+2x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_3^2x_4+x_1x_2x_3^2x_4+x_1^3x_2x_5+x_1^2x_2^2x_5+x_1^3x_3x_5+2x_1^2x_2x_3x_5+x_1x_2^2x_3x_5+x_1^2x_3^2x_5+x_1x_2x_3^2x_5+x_1^3x_4x_5+x_1^2x_2x_4x_5+x_1x_2^2x_4x_5+x_1^2x_3x_4x_5+x_1x_2x_3x_4x_5+x_1x_3^2x_4x_5$	25
152463	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1x_2^3x_4+x_1^3x_3x_4+x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_2^3x_3x_4+x_1^3x_2x_5+x_1^2x_2^2x_5+x_1x_2^3x_5+x_1^3x_3x_5+x_1^2x_2x_3x_5+x_1x_2^2x_3x_5+x_2^3x_3x_5+x_1^3x_4x_5+x_1^2x_2x_4x_5+x_1x_2^2x_4x_5+x_2^3x_4x_5$	21
512463	$x_1^4x_2x_3+x_1^4x_2x_4+x_1^4x_3x_4+x_1^4x_2x_5+x_1^4x_3x_5+x_1^4x_4x_5$	6
251463	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^3x_2^2x_5+x_1^2x_2^3x_5+x_1^3x_2x_3x_5+x_1^2x_2^2x_3x_5+x_1x_2^3x_3x_5+x_1^3x_2x_4x_5+x_1^2x_2^2x_4x_5+x_1x_2^3x_4x_5$	15
521463	$x_1^4x_2^2x_3+x_1^4x_2^2x_4+x_1^4x_2x_3x_4+x_1^4x_2^2x_5+x_1^4x_2x_3x_5+x_1^4x_2x_4x_5$	6

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
145263	$x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1^2x_2^2x_4+x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_3^2x_4+x_1x_2x_3^2x_4+x_2^2x_3^2x_4+x_1^2x_2^2x_5+x_1^2x_2x_3x_5+x_1x_2^2x_3x_5+x_1^2x_3^2x_5+x_1x_2x_3^2x_5+x_2^2x_3^2x_5$	15
415263	$x_1^3x_2^2x_3+x_1^3x_2x_3^2+x_1^3x_2^2x_4+x_1^3x_2x_3x_4+x_1^3x_3^2x_4+x_1^3x_2^2x_5+x_1^3x_2x_3x_5+x_1^3x_3^2x_5$	8
154263	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^3x_3^2x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_2^3x_3^2x_4+x_1^3x_2^2x_5+x_1^2x_2^3x_5+x_1^3x_2x_3x_5+x_1^2x_2^2x_3x_5+x_1x_2^3x_3x_5+x_1^3x_3^2x_5+x_1^2x_2x_3^2x_5+x_1x_2^2x_3^2x_5+x_2^3x_3^2x_5$	23
514263	$x_1^4x_2^2x_3+x_1^4x_2x_3^2+x_1^4x_2^2x_4+x_1^4x_2x_3x_4+x_1^4x_3^2x_4+x_1^4x_2^2x_5+x_1^4x_2x_3x_5+x_1^4x_3^2x_5$	8
451263	$x_1^3x_2^3x_3+x_1^3x_2^3x_4+x_1^3x_2^3x_5$	3
541263	$x_1^4x_2^3x_3+x_1^4x_2^3x_4+x_1^4x_2^3x_5$	3
245163	$x_1^2x_2^2x_3^2+x_1^2x_2^2x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_2^2x_3x_5+x_1^2x_2x_3^2x_5+x_1x_2^2x_3^2x_5$	7
425163	$x_1^3x_2^2x_3^2+x_1^3x_2^2x_3x_4+x_1^3x_2x_3^2x_4+x_1^3x_2^2x_3x_5+x_1^3x_2x_3^2x_5$	5
254163	$x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^3x_2x_3^2x_5+x_1^2x_2^2x_3^2x_5+x_1x_2^3x_3^2x_5$	12
524163	$x_1^4x_2^2x_3^2+x_1^4x_2^2x_3x_4+x_1^4x_2x_3^2x_4+x_1^4x_2^2x_3x_5+x_1^4x_2x_3^2x_5$	5
452163	$x_1^3x_2^3x_3^2+x_1^3x_2^3x_3x_4+x_1^3x_2^3x_3x_5$	3
542163	$x_1^4x_2^3x_3^2+x_1^4x_2^3x_3x_4+x_1^4x_2^3x_3x_5$	3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
124653	$x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2+x_1^2x_2x_4+x_1x_2^2x_4+x_1^2x_3x_4+3x_1x_2x_3x_4+x_2^2x_3x_4+x_1x_3^2x_4+x_2x_3^2x_4+x_1x_2x_4^2+x_1x_3x_4^2+x_2x_3x_4^2+x_1^2x_2x_5+x_1x_2^2x_5+x_1^2x_3x_5+2x_1x_2x_3x_5+x_2^2x_3x_5+x_1x_3^2x_5+x_2x_3^2x_5+x_1^2x_4x_5+2x_1x_2x_4x_5+x_2^2x_4x_5+2x_1x_3x_4x_5+2x_2x_3x_4x_5+x_3^2x_4x_5+x_1x_4^2x_5+x_2x_4^2x_5+x_3x_4^2x_5$	35
214653	$x_1^3x_2x_3+x_1^2x_2^2x_3+x_1^2x_2x_3^2+x_1^3x_2x_4+x_1^2x_2^2x_4+x_1^3x_3x_4+3x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1^2x_3^2x_4+x_1x_2x_3^2x_4+x_1^2x_2x_4^2+x_1^2x_3x_4^2+x_1x_2x_3x_4^2+x_1^3x_2x_5+x_1^2x_2^2x_5+x_1^3x_3x_5+2x_1^2x_2x_3x_5+x_1x_2^2x_3x_5+x_1^2x_3^2x_5+x_1x_2x_3^2x_5+x_1^3x_4x_5+2x_1^2x_2x_4x_5+x_1x_2^2x_4x_5+2x_1^2x_3x_4x_5+2x_1x_2x_3x_4x_5+x_1x_3^2x_4x_5+x_1^2x_4^2x_5+x_1x_2x_4^2x_5+x_1x_3x_4^2x_5$	35
142653	$x_1^3x_2x_3+2x_1^2x_2^2x_3+x_1x_2^3x_3+x_1^2x_2x_3^2+x_1x_2^2x_3^2+x_1^3x_2x_4+2x_1^2x_2^2x_4+x_1x_2^3x_4+x_1^3x_3x_4+3x_1^2x_2x_3x_4+3x_1x_2^2x_3x_4+x_2^3x_3x_4+x_1^2x_3^2x_4+x_1x_2x_3^2x_4+x_2^2x_3^2x_4+x_1^2x_2x_4^2+x_1x_2^2x_4^2+x_1^2x_3x_4^2+x_1x_2x_3x_4^2+x_2^2x_3x_4^2+x_1^3x_2x_5+2x_1^2x_2^2x_5+x_1x_2^3x_5+x_1^3x_3x_5+2x_1^2x_2x_3x_5+2x_1x_2^2x_3x_5+x_2^3x_3x_5+x_1^2x_3^2x_5+x_1x_2x_3^2x_5+x_2^2x_3^2x_5+x_1^3x_4x_5+2x_1^2x_2x_4x_5+2x_1x_2^2x_4x_5+x_2^3x_4x_5+x_1^2x_3x_4x_5+x_1x_2x_3x_4x_5+x_2^2x_3x_4x_5+x_1^2x_4^2x_5+x_1x_2x_4^2x_5+x_2^2x_4^2x_5$	51
412653	$x_1^4x_2x_3+x_1^3x_2^2x_3+x_1^3x_2x_3^2+x_1^4x_2x_4+x_1^3x_2^2x_4+x_1^4x_3x_4+2x_1^3x_2x_3x_4+x_1^3x_3^2x_4+x_1^3x_2x_4^2+x_1^3x_3x_4^2+x_1^4x_2x_5+x_1^3x_2^2x_5+x_1^4x_3x_5+x_1^3x_2x_3x_5+x_1^3x_3^2x_5+x_1^4x_4x_5+x_1^3x_2x_4x_5+x_1^3x_3x_4x_5+x_1^3x_4^2x_5$	20
241653	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^2x_2^2x_3^2+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^3x_2x_3x_4+3x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_2^2x_4^2+x_1^2x_2x_3x_4^2+x_1x_2^2x_3x_4^2+x_1^3x_2^2x_5+x_1^2x_2^3x_5+x_1^3x_2x_3x_5+2x_1^2x_2^2x_3x_5+x_1x_2^3x_3x_5+x_1^2x_2x_3^2x_5+x_1x_2^2x_3^2x_5+x_1^3x_2x_4x_5+2x_1^2x_2^2x_4x_5+x_1x_2^3x_4x_5+x_1^2x_2x_3x_4x_5+x_1x_2^2x_3x_4x_5+x_1^2x_2x_4^2x_5+x_1x_2^2x_4^2x_5$	31
421653	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^3x_2^2x_3^2+x_1^4x_2^2x_4+x_1^3x_2^3x_4+x_1^4x_2x_3x_4+2x_1^3x_2^2x_3x_4+x_1^3x_2x_3^2x_4+x_1^3x_2^2x_4^2+x_1^3x_2x_3x_4^2+x_1^4x_2^2x_5+x_1^3x_2^3x_5+x_1^4x_2x_3x_5+x_1^3x_2^2x_3x_5+x_1^3x_2x_3^2x_5+x_1^4x_2x_4x_5+x_1^3x_2^2x_4x_5+x_1^3x_2x_3x_4x_5+x_1^3x_2x_4^2x_5$	20

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
126453	$x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1x_2x_3^3 + x_1^3x_2x_4 + x_1^2x_2^2x_4 + x_1x_2^3x_4 + x_1^3x_3x_4 + 2x_1^2x_2x_3x_4 + 2x_1x_2^2x_3x_4 + x_2^3x_3x_4 + x_1^2x_3^2x_4 + 2x_1x_2x_3^2x_4 + x_2^2x_3^2x_4 + x_1x_3^3x_4 + x_2x_3^3x_4 + x_1^3x_2x_5 + x_1^2x_2^2x_5 + x_1x_2^3x_5 + x_1^3x_3x_5 + 2x_1^2x_2x_3x_5 + 2x_1x_2^2x_3x_5 + x_2^3x_3x_5 + x_1^2x_3^2x_5 + 2x_1x_2x_3^2x_5 + x_2^2x_3^2x_5 + x_1x_3^3x_5 + x_2x_3^3x_5 + x_1^3x_4x_5 + x_1^2x_2x_4x_5 + x_1x_2^2x_4x_5 + x_2^3x_4x_5 + x_1^2x_3x_4x_5 + x_1x_2x_3x_4x_5 + x_2^2x_3x_4x_5 + x_1x_3^2x_4x_5 + x_2x_3^2x_4x_5 + x_3^3x_4x_5$	46
216453	$x_1^4x_2x_3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^2x_2x_3^3 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1^4x_3x_4 + 2x_1^3x_2x_3x_4 + 2x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + 2x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_1^2x_3^3x_4 + x_1x_2x_3^3x_4 + x_1^4x_2x_5 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1^4x_3x_5 + 2x_1^3x_2x_3x_5 + 2x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_1^3x_3^2x_5 + 2x_1^2x_2x_3^2x_5 + x_1x_2^2x_3^2x_5 + x_1^2x_3^3x_5 + x_1x_2x_3^3x_5 + x_1^4x_4x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1x_2^3x_4x_5 + x_1^3x_3x_4x_5 + x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_1^2x_3^2x_4x_5 + x_1x_2x_3^2x_4x_5 + x_1x_3^3x_4x_5$	46
162453	$x_1^4x_2x_3 + x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1x_2^4x_3 + x_1^4x_2x_4 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + x_1x_2^4x_4 + x_1^4x_3x_4 + x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_2^4x_3x_4 + x_1^4x_2x_5 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1x_2^4x_5 + x_1^4x_3x_5 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_2^4x_3x_5 + x_1^4x_4x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1x_2^3x_4x_5 + x_2^4x_4x_5$	27
612453	$x_1^5x_2x_3 + x_1^5x_2x_4 + x_1^5x_3x_4 + x_1^5x_2x_5 + x_1^5x_3x_5 + x_1^5x_4x_5$	6
261453	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^2x_2^4x_4 + x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1x_2^4x_3x_4 + x_1^4x_2^2x_5 + x_1^3x_2^3x_5 + x_1^2x_2^4x_5 + x_1^4x_2x_3x_5 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1x_2^4x_3x_5 + x_1^4x_2x_4x_5 + x_1^3x_2^2x_4x_5 + x_1^2x_2^3x_4x_5 + x_1x_2^4x_4x_5$	21
621453	$x_1^5x_2^2x_3 + x_1^5x_2^2x_4 + x_1^5x_2x_3x_4 + x_1^5x_2^2x_5 + x_1^5x_2x_3x_5 + x_1^5x_2x_4x_5$	6

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
146253	$x_1^3x_2^2x_3+x_1^2x_2^3x_3+x_1^3x_2x_3^2+2x_1^2x_2^2x_3^2+x_1x_2^3x_3^2+x_1^2x_2x_3^3+x_1x_2^2x_3^3+x_1^3x_2^2x_4+x_1^2x_2^3x_4+x_1^3x_2x_3x_4+2x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^3x_3^2x_4+2x_1^2x_2x_3^2x_4+2x_1x_2^2x_3^2x_4+x_2^3x_3^2x_4+x_1^2x_3^3x_4+x_1x_2x_3^3x_4+x_2^2x_3^3x_4+x_1^3x_2^2x_5+x_1^2x_2^3x_5+x_1^3x_2x_3x_5+2x_1^2x_2^2x_3x_5+x_1x_2^3x_3x_5+x_1^3x_3^2x_5+2x_1^2x_2x_3^2x_5+2x_1x_2^2x_3^2x_5+x_2^3x_3^2x_5+x_1^2x_3^3x_5+x_1x_2x_3^3x_5+x_2^2x_3^3x_5$	38
416253	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^3x_2x_3^3+x_1^4x_2^2x_4+x_1^3x_2^3x_4+x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^4x_3^2x_4+x_1^3x_2x_3^2x_4+x_1^3x_3^3x_4+x_1^4x_2^2x_5+x_1^3x_2^3x_5+x_1^4x_2x_3x_5+x_1^3x_2^2x_3x_5+x_1^4x_3^2x_5+x_1^3x_2x_3^2x_5+x_1^3x_3^3x_5$	19
164253	$x_1^4x_2^2x_3+x_1^3x_2^3x_3+x_1^2x_2^4x_3+x_1^4x_2x_3^2+x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1x_2^4x_3^2+x_1^4x_2^2x_4+x_1^3x_2^3x_4+x_1^2x_2^4x_4+x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1x_2^4x_3x_4+x_1^4x_3^2x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_2^4x_3^2x_4+x_1^4x_2^2x_5+x_1^3x_2^3x_5+x_1^2x_2^4x_5+x_1^4x_2x_3x_5+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1x_2^4x_3x_5+x_1^4x_3^2x_5+x_1^3x_2x_3^2x_5+x_1^2x_2^2x_3^2x_5+x_1x_2^3x_3^2x_5+x_2^4x_3^2x_5$	31
614253	$x_1^5x_2^2x_3+x_1^5x_2x_3^2+x_1^5x_2^2x_4+x_1^5x_2x_3x_4+x_1^5x_3^2x_4+x_1^5x_2^2x_5+x_1^5x_2x_3x_5+x_1^5x_3^2x_5$	8
461253	$x_1^4x_2^3x_3+x_1^3x_2^4x_3+x_1^4x_2^3x_4+x_1^3x_2^4x_4+x_1^4x_2^3x_5+x_1^3x_2^4x_5$	6
641253	$x_1^5x_2^3x_3+x_1^5x_2^3x_4+x_1^5x_2^3x_5$	3
246153	$x_1^3x_2^2x_3^2+x_1^2x_2^3x_3^2+x_1^2x_2^2x_3^3+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+2x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^2x_2x_3^3x_4+x_1x_2^2x_3^3x_4+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^3x_2x_3^2x_5+2x_1^2x_2^2x_3^2x_5+x_1x_2^3x_3^2x_5+x_1^2x_2x_3^3x_5+x_1x_2^2x_3^3x_5$	19
426153	$x_1^4x_2^2x_3^2+x_1^3x_2^3x_3^2+x_1^3x_2^2x_3^3+x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^3x_2x_3^3x_4+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^4x_2x_3^2x_5+x_1^3x_2^2x_3^2x_5+x_1^3x_2x_3^3x_5$	13
264153	$x_1^4x_2^2x_3^2+x_1^3x_2^3x_3^2+x_1^2x_2^4x_3^2+x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1x_2^4x_3^2x_4+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^2x_2^4x_3x_5+x_1^4x_2x_3^2x_5+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1x_2^4x_3^2x_5$	17
624153	$x_1^5x_2^2x_3^2+x_1^5x_2^2x_3x_4+x_1^5x_2x_3^2x_4+x_1^5x_2^2x_3x_5+x_1^5x_2x_3^2x_5$	5
462153	$x_1^4x_2^3x_3^2+x_1^3x_2^4x_3^2+x_1^4x_2^3x_3x_4+x_1^3x_2^4x_3x_4+x_1^4x_2^3x_3x_5+x_1^3x_2^4x_3x_5$	6
642153	$x_1^5x_2^3x_3^2+x_1^5x_2^3x_3x_4+x_1^5x_2^3x_3x_5$	3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
125643	$x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2^2x_4 + 2x_1^2x_2x_3x_4 + 2x_1x_2^2x_3x_4 + x_1^2x_3^2x_4 + 2x_1x_2x_3^2x_4 + x_2^2x_3^2x_4 + x_1^2x_2x_4^2 + x_1x_2^2x_4^2 + x_1^2x_3x_4^2 + 2x_1x_2x_3x_4^2 + x_2^2x_3x_4^2 + x_1x_3^2x_4^2 + x_2x_3^2x_4^2 + x_1^2x_2^2x_5 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_1^2x_3^2x_5 + x_1x_2x_3^2x_5 + x_2^2x_3^2x_5 + x_1^2x_2x_4x_5 + x_1x_2^2x_4x_5 + x_1^2x_3x_4x_5 + 2x_1x_2x_3x_4x_5 + x_2^2x_3x_4x_5 + x_1x_3^2x_4x_5 + x_2x_3^2x_4x_5 + x_1^2x_4^2x_5 + x_1x_2x_4^2x_5 + x_2^2x_4^2x_5 + x_1x_3x_4^2x_5 + x_2x_3x_4^2x_5 + x_3^2x_4^2x_5$	40
215643	$x_1^3x_2^2x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1^3x_2^2x_4 + 2x_1^3x_2x_3x_4 + 2x_1^2x_2^2x_3x_4 + x_1^3x_3^2x_4 + 2x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1^3x_3x_4^2 + 2x_1^2x_2x_3x_4^2 + x_1x_2^2x_3x_4^2 + x_1^2x_3^2x_4^2 + x_1x_2x_3^2x_4^2 + x_1^3x_2^2x_5 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1^3x_3^2x_5 + x_1^2x_2x_3^2x_5 + x_1x_2^2x_3^2x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1^3x_3x_4x_5 + 2x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_1^2x_3^2x_4x_5 + x_1x_2x_3^2x_4x_5 + x_1^3x_4^2x_5 + x_1^2x_2x_4^2x_5 + x_1x_2^2x_4^2x_5 + x_1^2x_3x_4^2x_5 + x_1x_2x_3x_4^2x_5 + x_1x_3^2x_4^2x_5$	40
152643	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + 2x_1^3x_2x_3x_4 + 2x_1^2x_2^2x_3x_4 + 2x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_2^3x_3^2x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1x_2^3x_4^2 + x_1^3x_3x_4^2 + x_1^2x_2x_3x_4^2 + x_1x_2^2x_3x_4^2 + x_2^3x_3x_4^2 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_1^3x_3^2x_5 + x_1^2x_2x_3^2x_5 + x_1x_2^2x_3^2x_5 + x_2^3x_3^2x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1x_2^3x_4x_5 + x_1^3x_3x_4x_5 + x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_2^3x_3x_4x_5 + x_1^3x_4^2x_5 + x_1^2x_2x_4^2x_5 + x_1x_2^2x_4^2x_5 + x_2^3x_4^2x_5$	44
512643	$x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^4x_2^2x_4 + 2x_1^4x_2x_3x_4 + x_1^4x_3^2x_4 + x_1^4x_2x_4^2 + x_1^4x_3x_4^2 + x_1^4x_2^2x_5 + x_1^4x_2x_3x_5 + x_1^4x_3^2x_5 + x_1^4x_2x_4x_5 + x_1^4x_3x_4x_5 + x_1^4x_4^2x_5$	14
251643	$x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2^3x_4 + 2x_1^3x_2^2x_3x_4 + 2x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1^3x_2x_3x_4^2 + x_1^2x_2^2x_3x_4^2 + x_1x_2^3x_3x_4^2 + x_1^3x_2^3x_5 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1^3x_2x_3^2x_5 + x_1^2x_2^2x_3^2x_5 + x_1x_2^3x_3^2x_5 + x_1^3x_2^2x_4x_5 + x_1^2x_2^3x_4x_5 + x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5 + x_1^3x_2x_4^2x_5 + x_1^2x_2^2x_4^2x_5 + x_1x_2^3x_4^2x_5$	30
521643	$x_1^4x_2^3x_3 + x_1^4x_2^2x_3^2 + x_1^4x_2^3x_4 + 2x_1^4x_2^2x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^4x_2^2x_4^2 + x_1^4x_2x_3x_4^2 + x_1^4x_2^3x_5 + x_1^4x_2^2x_3x_5 + x_1^4x_2x_3^2x_5 + x_1^4x_2^2x_4x_5 + x_1^4x_2x_3x_4x_5 + x_1^4x_2x_4^2x_5$	14

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
126543	$x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + 2x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3 + x_1^3x_2^2x_4 + x_1^2x_2^3x_4 + 2x_1^3x_2x_3x_4 + 3x_1^2x_2^2x_3x_4 + 2x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + 3x_1^2x_2x_3^2x_4 + 3x_1x_2^2x_3^2x_4 + x_2^3x_3^2x_4 + x_1^2x_3^3x_4 + 2x_1x_2x_3^3x_4 + x_2^2x_3^3x_4 + x_1^3x_2x_4^2 + x_1^2x_2^2x_4^2 + x_1x_2^3x_4^2 + x_1^3x_3x_4^2 + 2x_1^2x_2x_3x_4^2 + 2x_1x_2^2x_3x_4^2 + x_2^3x_3x_4^2 + x_1^2x_3^2x_4^2 + 2x_1x_2x_3^2x_4^2 + x_2^2x_3^2x_4^2 + x_1x_3^3x_4^2 + x_2x_3^3x_4^2 + x_1^3x_2^2x_5 + x_1^2x_2^3x_5 + x_1^3x_2x_3x_5 + 2x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_1^3x_3^2x_5 + 2x_1^2x_2x_3^2x_5 + 2x_1x_2^2x_3^2x_5 + x_2^3x_3^2x_5 + x_1^2x_3^3x_5 + x_1x_2x_3^3x_5 + x_2^2x_3^3x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1x_2^3x_4x_5 + x_1^3x_3x_4x_5 + 2x_1^2x_2x_3x_4x_5 + 2x_1x_2^2x_3x_4x_5 + x_2^3x_3x_4x_5 + x_1^2x_3^2x_4x_5 + 2x_1x_2x_3^2x_4x_5 + x_2^2x_3^2x_4x_5 + x_1x_3^3x_4x_5 + x_2x_3^3x_4x_5 + x_1^3x_4^2x_5 + x_1^2x_2x_4^2x_5 + x_1x_2^2x_4^2x_5 + x_2^3x_4^2x_5 + x_1^2x_3x_4^2x_5 + x_1x_2x_3x_4^2x_5 + x_2^2x_3x_4^2x_5 + x_1x_3^2x_4^2x_5 + x_2x_3^2x_4^2x_5 + x_3^3x_4^2x_5$	84
216543	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^4x_2x_3^2 + 2x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2x_3^3 + x_1^2x_2^2x_3^3 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + 2x_1^4x_2x_3x_4 + 3x_1^3x_2^2x_3x_4 + 2x_1^2x_2^3x_3x_4 + x_1^4x_3^2x_4 + 3x_1^3x_2x_3^2x_4 + 3x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^3x_3^3x_4 + 2x_1^2x_2x_3^3x_4 + x_1x_2^2x_3^3x_4 + x_1^4x_2x_4^2 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1^4x_3x_4^2 + 2x_1^3x_2x_3x_4^2 + 2x_1^2x_2^2x_3x_4^2 + x_1x_2^3x_3x_4^2 + x_1^3x_3^2x_4^2 + 2x_1^2x_2x_3^2x_4^2 + x_1x_2^2x_3^2x_4^2 + x_1^2x_3^3x_4^2 + x_1x_2x_3^3x_4^2 + x_1^4x_2^2x_5 + x_1^3x_2^3x_5 + x_1^4x_2x_3x_5 + 2x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1^4x_3^2x_5 + 2x_1^3x_2x_3^2x_5 + 2x_1^2x_2^2x_3^2x_5 + x_1x_2^3x_3^2x_5 + x_1^3x_3^3x_5 + x_1^2x_2x_3^3x_5 + x_1x_2^2x_3^3x_5 + x_1^4x_2x_4x_5 + x_1^3x_2^2x_4x_5 + x_1^2x_2^3x_4x_5 + x_1^4x_3x_4x_5 + 2x_1^3x_2x_3x_4x_5 + 2x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5 + x_1^3x_3^2x_4x_5 + 2x_1^2x_2x_3^2x_4x_5 + x_1x_2^2x_3^2x_4x_5 + x_1^2x_3^3x_4x_5 + x_1x_2x_3^3x_4x_5 + x_1^4x_4^2x_5 + x_1^3x_2x_4^2x_5 + x_1^2x_2^2x_4^2x_5 + x_1x_2^3x_4^2x_5 + x_1^3x_3x_4^2x_5 + x_1^2x_2x_3x_4^2x_5 + x_1x_2^2x_3x_4^2x_5 + x_1^2x_3^2x_4^2x_5 + x_1x_2x_3^2x_4^2x_5 + x_1x_3^3x_4^2x_5$	84
162543	$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2x_3^2 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1x_2^4x_3^2 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^2x_2^4x_4 + 2x_1^4x_2x_3x_4 + 2x_1^3x_2^2x_3x_4 + 2x_1^2x_2^3x_3x_4 + 2x_1x_2^4x_3x_4 + x_1^4x_3^2x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_2^4x_3^2x_4 + x_1^4x_2x_4^2 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1x_2^4x_4^2 + x_1^4x_3x_4^2 + x_1^3x_2x_3x_4^2 + x_1^2x_2^2x_3x_4^2 + x_1x_2^3x_3x_4^2 + x_2^4x_3x_4^2 + x_1^4x_2^2x_5 + x_1^3x_2^3x_5 + x_1^2x_2^4x_5 + x_1^4x_2x_3x_5 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1x_2^4x_3x_5 + x_1^4x_3^2x_5 + x_1^3x_2x_3^2x_5 + x_1^2x_2^2x_3^2x_5 + x_1x_2^3x_3^2x_5 + x_2^4x_3^2x_5 + x_1^4x_2x_4x_5 + x_1^3x_2^2x_4x_5 + x_1^2x_2^3x_4x_5 + x_1x_2^4x_4x_5 + x_1^4x_3x_4x_5 + x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5 + x_2^4x_3x_4x_5 + x_1^4x_4^2x_5 + x_1^3x_2x_4^2x_5 + x_1^2x_2^2x_4^2x_5 + x_1x_2^3x_4^2x_5 + x_2^4x_4^2x_5$	58
612543	$x_1^5x_2^2x_3 + x_1^5x_2x_3^2 + x_1^5x_2^2x_4 + 2x_1^5x_2x_3x_4 + x_1^5x_3^2x_4 + x_1^5x_2x_4^2 + x_1^5x_3x_4^2 + x_1^5x_2^2x_5 + x_1^5x_2x_3x_5 + x_1^5x_3^2x_5 + x_1^5x_2x_4x_5 + x_1^5x_3x_4x_5 + x_1^5x_4^2x_5$	14

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
261543	$x_1^4x_2^3x_3 + x_1^3x_2^4x_3 + x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2^3x_4 + x_1^3x_2^4x_4 + 2x_1^4x_2^2x_3x_4 + 2x_1^3x_2^3x_3x_4 + 2x_1^2x_2^4x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1x_2^4x_3^2x_4 + x_1^4x_2^2x_4^2 + x_1^3x_2^3x_4^2 + x_1^2x_2^4x_4^2 + x_1^4x_2x_3x_4^2 + x_1^3x_2^2x_3x_4^2 + x_1^2x_2^3x_3x_4^2 + x_1x_2^4x_3x_4^2 + x_1^4x_2^3x_5 + x_1^3x_2^4x_5 + x_1^4x_2^2x_3x_5 + x_1^3x_2^3x_3x_5 + x_1^2x_2^4x_3x_5 + x_1^4x_2x_3^2x_5 + x_1^3x_2^2x_3^2x_5 + x_1^2x_2^3x_3^2x_5 + x_1x_2^4x_3^2x_5 + x_1^4x_2^2x_4x_5 + x_1^3x_2^3x_4x_5 + x_1^2x_2^4x_4x_5 + x_1^4x_2x_3x_4x_5 + x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1x_2^4x_3x_4x_5 + x_1^4x_2x_4^2x_5 + x_1^3x_2^2x_4^2x_5 + x_1^2x_2^3x_4^2x_5 + x_1x_2^4x_4^2x_5$	44
621543	$x_1^5x_2^3x_3 + x_1^5x_2^2x_3^2 + x_1^5x_2^3x_4 + 2x_1^5x_2^2x_3x_4 + x_1^5x_2x_3^2x_4 + x_1^5x_2^2x_4^2 + x_1^5x_2x_3x_4^2 + x_1^5x_2^3x_5 + x_1^5x_2^2x_3x_5 + x_1^5x_2x_3^2x_5 + x_1^5x_2^2x_4x_5 + x_1^5x_2x_3x_4x_5 + x_1^5x_2x_4^2x_5$	14
156243	$x_1^3x_2^3x_3 + x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2x_3^3 + x_1^2x_2^2x_3^3 + x_1x_2^3x_3^3 + x_1^3x_2^3x_4 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^3x_3^3x_4 + x_1^2x_2x_3^3x_4 + x_1x_2^2x_3^3x_4 + x_2^3x_3^3x_4 + x_1^3x_2^3x_5 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1^3x_2x_3^2x_5 + x_1^2x_2^2x_3^2x_5 + x_1x_2^3x_3^2x_5 + x_1^3x_3^3x_5 + x_1^2x_2x_3^3x_5 + x_1x_2^2x_3^3x_5 + x_2^3x_3^3x_5$	26
516243	$x_1^4x_2^3x_3 + x_1^4x_2^2x_3^2 + x_1^4x_2x_3^3 + x_1^4x_2^3x_4 + x_1^4x_2^2x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^4x_3^3x_4 + x_1^4x_2^3x_5 + x_1^4x_2^2x_3x_5 + x_1^4x_2x_3^2x_5 + x_1^4x_3^3x_5$	11
165243	$x_1^4x_2^3x_3 + x_1^3x_2^4x_3 + x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2x_3^3 + x_1^3x_2^2x_3^3 + x_1^2x_2^3x_3^3 + x_1x_2^4x_3^3 + x_1^4x_2^3x_4 + x_1^3x_2^4x_4 + x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^2x_2^4x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1x_2^4x_3^2x_4 + x_1^4x_3^3x_4 + x_1^3x_2x_3^3x_4 + x_1^2x_2^2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_2^4x_3^3x_4 + x_1^4x_2^3x_5 + x_1^3x_2^4x_5 + x_1^4x_2^2x_3x_5 + x_1^3x_2^3x_3x_5 + x_1^2x_2^4x_3x_5 + x_1^4x_2x_3^2x_5 + x_1^3x_2^2x_3^2x_5 + x_1^2x_2^3x_3^2x_5 + x_1x_2^4x_3^2x_5 + x_1^4x_3^3x_5 + x_1^3x_2x_3^3x_5 + x_1^2x_2^2x_3^3x_5 + x_1x_2^3x_3^3x_5 + x_2^4x_3^3x_5$	37
615243	$x_1^5x_2^3x_3 + x_1^5x_2^2x_3^2 + x_1^5x_2x_3^3 + x_1^5x_2^3x_4 + x_1^5x_2^2x_3x_4 + x_1^5x_2x_3^2x_4 + x_1^5x_3^3x_4 + x_1^5x_2^3x_5 + x_1^5x_2^2x_3x_5 + x_1^5x_2x_3^2x_5 + x_1^5x_3^3x_5$	11
561243	$x_1^4x_2^4x_3 + x_1^4x_2^4x_4 + x_1^4x_2^4x_5$	3
651243	$x_1^5x_2^4x_3 + x_1^5x_2^4x_4 + x_1^5x_2^4x_5$	3
256143	$x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 + x_1^2x_2^3x_3^3 + x_1^3x_2^3x_3x_4 + x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1^3x_2x_3^3x_4 + x_1^2x_2^2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2^3x_3x_5 + x_1^3x_2^2x_3^2x_5 + x_1^2x_2^3x_3^2x_5 + x_1^3x_2x_3^3x_5 + x_1^2x_2^2x_3^3x_5 + x_1x_2^3x_3^3x_5$	15
526143	$x_1^4x_2^3x_3^2 + x_1^4x_2^2x_3^3 + x_1^4x_2^3x_3x_4 + x_1^4x_2^2x_3^2x_4 + x_1^4x_2x_3^3x_4 + x_1^4x_2^3x_3x_5 + x_1^4x_2^2x_3^2x_5 + x_1^4x_2x_3^3x_5$	8
265143	$x_1^4x_2^3x_3^2 + x_1^3x_2^4x_3^2 + x_1^4x_2^2x_3^3 + x_1^3x_2^3x_3^3 + x_1^2x_2^4x_3^3 + x_1^4x_2^3x_3x_4 + x_1^3x_2^4x_3x_4 + x_1^4x_2^2x_3^2x_4 + x_1^3x_2^3x_3^2x_4 + x_1^2x_2^4x_3^2x_4 + x_1^4x_2x_3^3x_4 + x_1^3x_2^2x_3^3x_4 + x_1^2x_2^3x_3^3x_4 + x_1x_2^4x_3^3x_4 + x_1^4x_2^3x_3x_5 + x_1^3x_2^4x_3x_5 + x_1^4x_2^2x_3^2x_5 + x_1^3x_2^3x_3^2x_5 + x_1^2x_2^4x_3^2x_5 + x_1^4x_2x_3^3x_5 + x_1^3x_2^2x_3^3x_5 + x_1^2x_2^3x_3^3x_5 + x_1x_2^4x_3^3x_5$	23
625143	$x_1^5x_2^3x_3^2 + x_1^5x_2^2x_3^3 + x_1^5x_2^3x_3x_4 + x_1^5x_2^2x_3^2x_4 + x_1^5x_2x_3^3x_4 + x_1^5x_2^3x_3x_5 + x_1^5x_2^2x_3^2x_5 + x_1^5x_2x_3^3x_5$	8
562143	$x_1^4x_2^4x_3^2 + x_1^4x_2^4x_3x_4 + x_1^4x_2^4x_3x_5$	3
652143	$x_1^5x_2^4x_3^2 + x_1^5x_2^4x_3x_4 + x_1^5x_2^4x_3x_5$	3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
145623	$x_1^2x_2^2x_3^2 + x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_1^2x_2^2x_4^2 + x_1^2x_2x_3x_4^2 + x_1x_2^2x_3x_4^2 + x_1^2x_3^2x_4^2 + x_1x_2x_3^2x_4^2 + x_2^2x_3^2x_4^2$	10
415623	$x_1^3x_2^2x_3^2 + x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^3x_2^2x_4^2 + x_1^3x_2x_3x_4^2 + x_1^3x_3^2x_4^2$	6
154623	$x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1^3x_2x_3x_4^2 + x_1^2x_2^2x_3x_4^2 + x_1x_2^3x_3x_4^2 + x_1^3x_3^2x_4^2 + x_1^2x_2x_3^2x_4^2 + x_1x_2^2x_3^2x_4^2 + x_2^3x_3^2x_4^2$	16
514623	$x_1^4x_2^2x_3^2 + x_1^4x_2^2x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^4x_2^2x_4^2 + x_1^4x_2x_3x_4^2 + x_1^4x_3^2x_4^2$	6
451623	$x_1^3x_2^3x_3^2 + x_1^3x_2^3x_3x_4 + x_1^3x_2^3x_4^2$	3
541623	$x_1^4x_2^3x_3^2 + x_1^4x_2^3x_3x_4 + x_1^4x_2^3x_4^2$	3
146523	$x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^2x_2^2x_3^3 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + 2x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^2x_2x_3^3x_4 + x_1x_2^2x_3^3x_4 + x_1^3x_2^2x_4^2 + x_1^2x_2^3x_4^2 + x_1^3x_2x_3x_4^2 + 2x_1^2x_2^2x_3x_4^2 + x_1x_2^3x_3x_4^2 + x_1^3x_3^2x_4^2 + 2x_1^2x_2x_3^2x_4^2 + 2x_1x_2^2x_3^2x_4^2 + x_2^3x_3^2x_4^2 + x_1^2x_3^3x_4^2 + x_1x_2x_3^3x_4^2 + x_2^2x_3^3x_4^2$	26
416523	$x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 + x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^3x_2^2x_3^2x_4 + x_1^3x_2x_3^3x_4 + x_1^4x_2^2x_4^2 + x_1^3x_2^3x_4^2 + x_1^4x_2x_3x_4^2 + x_1^3x_2^2x_3x_4^2 + x_1^4x_3^2x_4^2 + x_1^3x_2x_3^2x_4^2 + x_1^3x_3^3x_4^2$	15
164523	$x_1^4x_2^2x_3^2 + x_1^3x_2^3x_3^2 + x_1^2x_2^4x_3^2 + x_1^4x_2^2x_3x_4 + x_1^3x_2^3x_3x_4 + x_1^2x_2^4x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1x_2^4x_3^2x_4 + x_1^4x_2^2x_4^2 + x_1^3x_2^3x_4^2 + x_1^2x_2^4x_4^2 + x_1^4x_2x_3x_4^2 + x_1^3x_2^2x_3x_4^2 + x_1^2x_2^3x_3x_4^2 + x_1x_2^4x_3x_4^2 + x_1^4x_3^2x_4^2 + x_1^3x_2x_3^2x_4^2 + x_1^2x_2^2x_3^2x_4^2 + x_1x_2^3x_3^2x_4^2 + x_2^4x_3^2x_4^2$	22
614523	$x_1^5x_2^2x_3^2 + x_1^5x_2^2x_3x_4 + x_1^5x_2x_3^2x_4 + x_1^5x_2^2x_4^2 + x_1^5x_2x_3x_4^2 + x_1^5x_3^2x_4^2$	6
461523	$x_1^4x_2^3x_3^2 + x_1^3x_2^4x_3^2 + x_1^4x_2^3x_3x_4 + x_1^3x_2^4x_3x_4 + x_1^4x_2^3x_4^2 + x_1^3x_2^4x_4^2$	6
641523	$x_1^5x_2^3x_3^2 + x_1^5x_2^3x_3x_4 + x_1^5x_2^3x_4^2$	3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
156423	$x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 + x_1^2x_2^3x_3^3 + x_1^3x_2^3x_3x_4 + x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1^3x_2x_3^3x_4 + x_1^2x_2^2x_3^3x_4 + x_1x_2^3x_3^3x_4 + x_1^3x_2^3x_4^2 + x_1^3x_2^2x_3x_4^2 + x_1^2x_2^3x_3x_4^2 + x_1^3x_2x_3^2x_4^2 + x_1^2x_2^2x_3^2x_4^2 + x_1x_2^3x_3^2x_4^2 + x_1^3x_3^3x_4^2 + x_1^2x_2x_3^3x_4^2 + x_1x_2^2x_3^3x_4^2 + x_2^3x_3^3x_4^2$	19
516423	$x_1^4x_2^3x_3^2 + x_1^4x_2^2x_3^3 + x_1^4x_2^3x_3x_4 + x_1^4x_2^2x_3^2x_4 + x_1^4x_2x_3^3x_4 + x_1^4x_2^3x_4^2 + x_1^4x_2^2x_3x_4^2 + x_1^4x_2x_3^2x_4^2 + x_1^4x_3^3x_4^2$	9
165423	$x_1^4x_2^3x_3^2 + x_1^3x_2^4x_3^2 + x_1^4x_2^2x_3^3 + x_1^3x_2^3x_3^3 + x_1^2x_2^4x_3^3 + x_1^4x_2^3x_3x_4 + x_1^3x_2^4x_3x_4 + x_1^4x_2^2x_3^2x_4 + x_1^3x_2^3x_3^2x_4 + x_1^2x_2^4x_3^2x_4 + x_1^4x_2x_3^3x_4 + x_1^3x_2^2x_3^3x_4 + x_1^2x_2^3x_3^3x_4 + x_1x_2^4x_3^3x_4 + x_1^4x_2^3x_4^2 + x_1^3x_2^4x_4^2 + x_1^4x_2^2x_3x_4^2 + x_1^3x_2^3x_3x_4^2 + x_1^2x_2^4x_3x_4^2 + x_1^4x_2x_3^2x_4^2 + x_1^3x_2^2x_3^2x_4^2 + x_1^2x_2^3x_3^2x_4^2 + x_1x_2^4x_3^2x_4^2 + x_1^4x_3^3x_4^2 + x_1^3x_2x_3^3x_4^2 + x_1^2x_2^2x_3^3x_4^2 + x_1x_2^3x_3^3x_4^2 + x_2^4x_3^3x_4^2$	28
615423	$x_1^5x_2^3x_3^2 + x_1^5x_2^2x_3^3 + x_1^5x_2^3x_3x_4 + x_1^5x_2^2x_3^2x_4 + x_1^5x_2x_3^3x_4 + x_1^5x_2^3x_4^2 + x_1^5x_2^2x_3x_4^2 + x_1^5x_2x_3^2x_4^2 + x_1^5x_3^3x_4^2$	9
561423	$x_1^4x_2^4x_3^2 + x_1^4x_2^4x_3x_4 + x_1^4x_2^4x_4^2$	3
651423	$x_1^5x_2^4x_3^2 + x_1^5x_2^4x_3x_4 + x_1^5x_2^4x_4^2$	3
456123	$x_1^3x_2^3x_3^3$	1
546123	$x_1^4x_2^3x_3^3$	1
465123	$x_1^4x_2^3x_3^3 + x_1^3x_2^4x_3^3$	2
645123	$x_1^5x_2^3x_3^3$	1
564123	$x_1^4x_2^4x_3^3$	1
654123	$x_1^5x_2^4x_3^3$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
245613	$x_1^2x_2^2x_3^2x_4 + x_1^2x_2^2x_3x_4^2 + x_1^2x_2x_3^2x_4^2 + x_1x_2^2x_3^2x_4^2$	4
425613	$x_1^3x_2^2x_3^2x_4 + x_1^3x_2^2x_3x_4^2 + x_1^3x_2x_3^2x_4^2$	3
254613	$x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1^3x_2^2x_3x_4^2 + x_1^2x_2^3x_3x_4^2 + x_1^3x_2x_3^2x_4^2 + x_1^2x_2^2x_3^2x_4^2 + x_1x_2^3x_3^2x_4^2$	7
524613	$x_1^4x_2^2x_3^2x_4 + x_1^4x_2^2x_3x_4^2 + x_1^4x_2x_3^2x_4^2$	3
452613	$x_1^3x_2^3x_3^2x_4 + x_1^3x_2^3x_3x_4^2$	2
542613	$x_1^4x_2^3x_3^2x_4 + x_1^4x_2^3x_3x_4^2$	2
246513	$x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1^2x_2^2x_3^3x_4 + x_1^3x_2^2x_3x_4^2 + x_1^2x_2^3x_3x_4^2 + x_1^3x_2x_3^2x_4^2 + 2x_1^2x_2^2x_3^2x_4^2 + x_1x_2^3x_3^2x_4^2 + x_1^2x_2x_3^3x_4^2 + x_1x_2^2x_3^3x_4^2$	11
426513	$x_1^4x_2^2x_3^2x_4 + x_1^3x_2^3x_3^2x_4 + x_1^3x_2^2x_3^3x_4 + x_1^4x_2^2x_3x_4^2 + x_1^3x_2^3x_3x_4^2 + x_1^4x_2x_3^2x_4^2 + x_1^3x_2^2x_3^2x_4^2 + x_1^3x_2x_3^3x_4^2$	8
264513	$x_1^4x_2^2x_3^2x_4 + x_1^3x_2^3x_3^2x_4 + x_1^2x_2^4x_3^2x_4 + x_1^4x_2^2x_3x_4^2 + x_1^3x_2^3x_3x_4^2 + x_1^2x_2^4x_3x_4^2 + x_1^4x_2x_3^2x_4^2 + x_1^3x_2^2x_3^2x_4^2 + x_1^2x_2^3x_3^2x_4^2 + x_1x_2^4x_3^2x_4^2$	10
624513	$x_1^5x_2^2x_3^2x_4 + x_1^5x_2^2x_3x_4^2 + x_1^5x_2x_3^2x_4^2$	3
462513	$x_1^4x_2^3x_3^2x_4 + x_1^3x_2^4x_3^2x_4 + x_1^4x_2^3x_3x_4^2 + x_1^3x_2^4x_3x_4^2$	4
642513	$x_1^5x_2^3x_3^2x_4 + x_1^5x_2^3x_3x_4^2$	2
256413	$x_1^3x_2^3x_3^2x_4 + x_1^3x_2^2x_3^3x_4 + x_1^2x_2^3x_3^3x_4 + x_1^3x_2^3x_3x_4^2 + x_1^3x_2^2x_3^2x_4^2 + x_1^2x_2^3x_3^2x_4^2 + x_1^3x_2x_3^3x_4^2 + x_1^2x_2^2x_3^3x_4^2 + x_1x_2^3x_3^3x_4^2$	9
526413	$x_1^4x_2^3x_3^2x_4 + x_1^4x_2^2x_3^3x_4 + x_1^4x_2^3x_3x_4^2 + x_1^4x_2^2x_3^2x_4^2 + x_1^4x_2x_3^3x_4^2$	5
265413	$x_1^4x_2^3x_3^2x_4 + x_1^3x_2^4x_3^2x_4 + x_1^4x_2^2x_3^3x_4 + x_1^3x_2^3x_3^3x_4 + x_1^2x_2^4x_3^3x_4 + x_1^4x_2^3x_3x_4^2 + x_1^3x_2^4x_3x_4^2 + x_1^4x_2^2x_3^2x_4^2 + x_1^3x_2^3x_3^2x_4^2 + x_1^2x_2^4x_3^2x_4^2 + x_1^4x_2x_3^3x_4^2 + x_1^3x_2^2x_3^3x_4^2 + x_1^2x_2^3x_3^3x_4^2 + x_1x_2^4x_3^3x_4^2$	14
625413	$x_1^5x_2^3x_3^2x_4 + x_1^5x_2^2x_3^3x_4 + x_1^5x_2^3x_3x_4^2 + x_1^5x_2^2x_3^2x_4^2 + x_1^5x_2x_3^3x_4^2$	5
562413	$x_1^4x_2^4x_3^2x_4 + x_1^4x_2^4x_3x_4^2$	2
652413	$x_1^5x_2^4x_3^2x_4 + x_1^5x_2^4x_3x_4^2$	2
456213	$x_1^3x_2^3x_3^3x_4$	1
546213	$x_1^4x_2^3x_3^3x_4$	1
465213	$x_1^4x_2^3x_3^3x_4 + x_1^3x_2^4x_3^3x_4$	2
645213	$x_1^5x_2^3x_3^3x_4$	1
564213	$x_1^4x_2^4x_3^3x_4$	1
654213	$x_1^5x_2^4x_3^3x_4$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
134562	$x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5$	5
314562	$x_1^2x_2x_3x_4 + x_1^2x_2x_3x_5 + x_1^2x_2x_4x_5 + x_1^2x_3x_4x_5$	4
143562	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_1^2x_2x_4x_5 + x_1x_2^2x_4x_5 + x_1^2x_3x_4x_5 + x_1x_2x_3x_4x_5 + x_2^2x_3x_4x_5$	9
413562	$x_1^3x_2x_3x_4 + x_1^3x_2x_3x_5 + x_1^3x_2x_4x_5 + x_1^3x_3x_4x_5$	4
341562	$x_1^2x_2^2x_3x_4 + x_1^2x_2^2x_3x_5 + x_1^2x_2^2x_4x_5$	3
431562	$x_1^3x_2^2x_3x_4 + x_1^3x_2^2x_3x_5 + x_1^3x_2^2x_4x_5$	3
135462	$x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1^2x_2x_3x_5 + x_1x_2^2x_3x_5 + x_1x_2x_3^2x_5 + x_1^2x_2x_4x_5 + x_1x_2^2x_4x_5 + x_1^2x_3x_4x_5 + 2x_1x_2x_3x_4x_5 + x_2^2x_3x_4x_5 + x_1x_3^2x_4x_5 + x_2x_3^2x_4x_5$	14
315462	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1^2x_2x_3^2x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1^3x_3x_4x_5 + x_1^2x_2x_3x_4x_5 + x_1^2x_3^2x_4x_5$	11
153462	$x_1^3x_2x_3x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3x_4 + x_1^3x_2x_3x_5 + x_1^2x_2^2x_3x_5 + x_1x_2^3x_3x_5 + x_1^3x_2x_4x_5 + x_1^2x_2^2x_4x_5 + x_1x_2^3x_4x_5 + x_1^3x_3x_4x_5 + x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_2^3x_3x_4x_5$	13
513462	$x_1^4x_2x_3x_4 + x_1^4x_2x_3x_5 + x_1^4x_2x_4x_5 + x_1^4x_3x_4x_5$	4
351462	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1^3x_2^2x_4x_5 + x_1^2x_2^3x_4x_5$	6
531462	$x_1^4x_2^2x_3x_4 + x_1^4x_2^2x_3x_5 + x_1^4x_2^2x_4x_5$	3
145362	$x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_1^2x_2^2x_3x_5 + x_1^2x_2x_3^2x_5 + x_1x_2^2x_3^2x_5 + x_1^2x_2^2x_4x_5 + x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_1^2x_3^2x_4x_5 + x_1x_2x_3^2x_4x_5 + x_2^2x_3^2x_4x_5$	12
415362	$x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^3x_2^2x_3x_5 + x_1^3x_2x_3^2x_5 + x_1^3x_2^2x_4x_5 + x_1^3x_2x_3x_4x_5 + x_1^3x_3^2x_4x_5$	7
154362	$x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2x_3^2x_4 + x_1^2x_2^2x_3^2x_4 + x_1x_2^3x_3^2x_4 + x_1^3x_2^2x_3x_5 + x_1^2x_2^3x_3x_5 + x_1^3x_2x_3^2x_5 + x_1^2x_2^2x_3^2x_5 + x_1x_2^3x_3^2x_5 + x_1^3x_2^2x_4x_5 + x_1^2x_2^3x_4x_5 + x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5 + x_1^3x_3^2x_4x_5 + x_1^2x_2x_3^2x_4x_5 + x_1x_2^2x_3^2x_4x_5 + x_2^3x_3^2x_4x_5$	19
514362	$x_1^4x_2^2x_3x_4 + x_1^4x_2x_3^2x_4 + x_1^4x_2^2x_3x_5 + x_1^4x_2x_3^2x_5 + x_1^4x_2^2x_4x_5 + x_1^4x_2x_3x_4x_5 + x_1^4x_3^2x_4x_5$	7
451362	$x_1^3x_2^3x_3x_4 + x_1^3x_2^3x_3x_5 + x_1^3x_2^3x_4x_5$	3
541362	$x_1^4x_2^3x_3x_4 + x_1^4x_2^3x_3x_5 + x_1^4x_2^3x_4x_5$	3
345162	$x_1^2x_2^2x_3^2x_4 + x_1^2x_2^2x_3^2x_5$	2
435162	$x_1^3x_2^2x_3^2x_4 + x_1^3x_2^2x_3^2x_5$	2
354162	$x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + x_1^3x_2^2x_3^2x_5 + x_1^2x_2^3x_3^2x_5$	4
534162	$x_1^4x_2^2x_3^2x_4 + x_1^4x_2^2x_3^2x_5$	2
453162	$x_1^3x_2^3x_3^2x_4 + x_1^3x_2^3x_3^2x_5$	2
543162	$x_1^4x_2^3x_3^2x_4 + x_1^4x_2^3x_3^2x_5$	2

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
134652	$x_1^2x_2x_3x_4+x_1x_2^2x_3x_4+x_1x_2x_3^2x_4+x_1x_2x_3x_4^2+x_1^2x_2x_3x_5+x_1x_2^2x_3x_5+x_1x_2x_3^2x_5+x_1^2x_2x_4x_5+x_1x_2^2x_4x_5+x_1^2x_3x_4x_5+3x_1x_2x_3x_4x_5+x_2^2x_3x_4x_5+x_1x_3^2x_4x_5+x_2x_3^2x_4x_5+x_1x_2x_4^2x_5+x_1x_3x_4^2x_5+x_2x_3x_4^2x_5$	19
314652	$x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1^2x_2x_3^2x_4+x_1^2x_2x_3x_4^2+x_1^3x_2x_3x_5+x_1^2x_2^2x_3x_5+x_1^2x_2x_3^2x_5+x_1^3x_2x_4x_5+x_1^2x_2^2x_4x_5+x_1^3x_3x_4x_5+2x_1^2x_2x_3x_4x_5+x_1^2x_3^2x_4x_5+x_1^2x_2x_4^2x_5+x_1^2x_3x_4^2x_5$	15
143652	$x_1^3x_2x_3x_4+2x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_2x_3x_4^2+x_1x_2^2x_3x_4^2+x_1^3x_2x_3x_5+2x_1^2x_2^2x_3x_5+x_1x_2^3x_3x_5+x_1^2x_2x_3^2x_5+x_1x_2^2x_3^2x_5+x_1^3x_2x_4x_5+2x_1^2x_2^2x_4x_5+x_1x_2^3x_4x_5+x_1^3x_3x_4x_5+3x_1^2x_2x_3x_4x_5+3x_1x_2^2x_3x_4x_5+x_2^3x_3x_4x_5+x_1^2x_3^2x_4x_5+x_1x_2x_3^2x_4x_5+x_2^2x_3^2x_4x_5+x_1^2x_2x_4^2x_5+x_1x_2^2x_4^2x_5+x_1^2x_3x_4^2x_5+x_1x_2x_3x_4^2x_5+x_2^2x_3x_4^2x_5$	34
413652	$x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^3x_2x_3^2x_4+x_1^3x_2x_3x_4^2+x_1^4x_2x_3x_5+x_1^3x_2^2x_3x_5+x_1^3x_2x_3^2x_5+x_1^4x_2x_4x_5+x_1^3x_2^2x_4x_5+x_1^4x_3x_4x_5+2x_1^3x_2x_3x_4x_5+x_1^3x_3^2x_4x_5+x_1^3x_2x_4^2x_5+x_1^3x_3x_4^2x_5$	15
341652	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^2x_2^2x_3^2x_4+x_1^2x_2^2x_3x_4^2+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^2x_2^2x_3^2x_5+x_1^3x_2^2x_4x_5+x_1^2x_2^3x_4x_5+x_1^2x_2^2x_3x_4x_5+x_1^2x_2^2x_4^2x_5$	11
431652	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^3x_2^2x_3^2x_4+x_1^3x_2^2x_3x_4^2+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^3x_2^2x_3^2x_5+x_1^4x_2^2x_4x_5+x_1^3x_2^3x_4x_5+x_1^3x_2^2x_3x_4x_5+x_1^3x_2^2x_4^2x_5$	11
136452	$x_1^3x_2x_3x_4+x_1^2x_2^2x_3x_4+x_1x_2^3x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1x_2x_3^3x_4+x_1^3x_2x_3x_5+x_1^2x_2^2x_3x_5+x_1x_2^3x_3x_5+x_1^2x_2x_3^2x_5+x_1x_2^2x_3^2x_5+x_1x_2x_3^3x_5+x_1^3x_2x_4x_5+x_1^2x_2^2x_4x_5+x_1x_2^3x_4x_5+x_1^3x_3x_4x_5+2x_1^2x_2x_3x_4x_5+2x_1x_2^2x_3x_4x_5+x_2^3x_3x_4x_5+x_1^2x_3^2x_4x_5+2x_1x_2x_3^2x_4x_5+x_2^2x_3^2x_4x_5+x_1x_3^3x_4x_5+x_2x_3^3x_4x_5$	27
316452	$x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1^2x_2x_3^3x_4+x_1^4x_2x_3x_5+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^3x_2x_3^2x_5+x_1^2x_2^2x_3^2x_5+x_1^2x_2x_3^3x_5+x_1^4x_2x_4x_5+x_1^3x_2^2x_4x_5+x_1^2x_2^3x_4x_5+x_1^4x_3x_4x_5+x_1^3x_2x_3x_4x_5+x_1^2x_2^2x_3x_4x_5+x_1^3x_3^2x_4x_5+x_1^2x_2x_3^2x_4x_5+x_1^2x_3^3x_4x_5$	21
163452	$x_1^4x_2x_3x_4+x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1x_2^4x_3x_4+x_1^4x_2x_3x_5+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1x_2^4x_3x_5+x_1^4x_2x_4x_5+x_1^3x_2^2x_4x_5+x_1^2x_2^3x_4x_5+x_1x_2^4x_4x_5+x_1^4x_3x_4x_5+x_1^3x_2x_3x_4x_5+x_1^2x_2^2x_3x_4x_5+x_1x_2^3x_3x_4x_5+x_2^4x_3x_4x_5$	17
613452	$x_1^5x_2x_3x_4+x_1^5x_2x_3x_5+x_1^5x_2x_4x_5+x_1^5x_3x_4x_5$	4
361452	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^2x_2^4x_3x_5+x_1^4x_2^2x_4x_5+x_1^3x_2^3x_4x_5+x_1^2x_2^4x_4x_5$	9
631452	$x_1^5x_2^2x_3x_4+x_1^5x_2^2x_3x_5+x_1^5x_2^2x_4x_5$	3

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
146352	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+2x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^2x_2x_3^3x_4+x_1x_2^2x_3^3x_4+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^3x_2x_3^2x_5+2x_1^2x_2^2x_3^2x_5+x_1x_2^3x_3^2x_5+x_1^2x_2x_3^3x_5+x_1x_2^2x_3^3x_5+x_1^3x_2^2x_4x_5+x_1^2x_2^3x_4x_5+x_1^3x_2x_3x_4x_5+2x_1^2x_2^2x_3x_4x_5+x_1x_2^3x_3x_4x_5+x_1^3x_3^2x_4x_5+2x_1^2x_2x_3^2x_4x_5+2x_1x_2^2x_3^2x_4x_5+x_2^3x_3^2x_4x_5+x_1^2x_3^3x_4x_5+x_1x_2x_3^3x_4x_5+x_2^2x_3^3x_4x_5$	31
416352	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^3x_2x_3^3x_4+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^4x_2x_3^2x_5+x_1^3x_2^2x_3^2x_5+x_1^3x_2x_3^3x_5+x_1^4x_2^2x_4x_5+x_1^3x_2^3x_4x_5+x_1^4x_2x_3x_4x_5+x_1^3x_2^2x_3x_4x_5+x_1^4x_3^2x_4x_5+x_1^3x_2x_3^2x_4x_5+x_1^3x_3^3x_4x_5$	17
164352	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1x_2^4x_3^2x_4+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^2x_2^4x_3x_5+x_1^4x_2x_3^2x_5+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1x_2^4x_3^2x_5+x_1^4x_2^2x_4x_5+x_1^3x_2^3x_4x_5+x_1^2x_2^4x_4x_5+x_1^4x_2x_3x_4x_5+x_1^3x_2^2x_3x_4x_5+x_1^2x_2^3x_3x_4x_5+x_1x_2^4x_3x_4x_5+x_1^4x_3^2x_4x_5+x_1^3x_2x_3^2x_4x_5+x_1^2x_2^2x_3^2x_4x_5+x_1x_2^3x_3^2x_4x_5+x_2^4x_3^2x_4x_5$	26
614352	$x_1^5x_2^2x_3x_4+x_1^5x_2x_3^2x_4+x_1^5x_2^2x_3x_5+x_1^5x_2x_3^2x_5+x_1^5x_2^2x_4x_5+x_1^5x_2x_3x_4x_5+x_1^5x_3^2x_4x_5$	7
461352	$x_1^4x_2^3x_3x_4+x_1^3x_2^4x_3x_4+x_1^4x_2^3x_3x_5+x_1^3x_2^4x_3x_5+x_1^4x_2^3x_4x_5+x_1^3x_2^4x_4x_5$	6
641352	$x_1^5x_2^3x_3x_4+x_1^5x_2^3x_3x_5+x_1^5x_2^3x_4x_5$	3
346152	$x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^2x_2^2x_3^3x_4+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1^2x_2^2x_3^3x_5$	6
436152	$x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^3x_2^2x_3^3x_4+x_1^4x_2^2x_3^2x_5+x_1^3x_2^3x_3^2x_5+x_1^3x_2^2x_3^3x_5$	6
364152	$x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^2x_2^4x_3^2x_4+x_1^4x_2^2x_3^2x_5+x_1^3x_2^3x_3^2x_5+x_1^2x_2^4x_3^2x_5$	6
634152	$x_1^5x_2^2x_3^2x_4+x_1^5x_2^2x_3^2x_5$	2
463152	$x_1^4x_2^3x_3^2x_4+x_1^3x_2^4x_3^2x_4+x_1^4x_2^3x_3^2x_5+x_1^3x_2^4x_3^2x_5$	4
643152	$x_1^5x_2^3x_3^2x_4+x_1^5x_2^3x_3^2x_5$	2

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
135642	$x_1^2x_2^2x_3x_4+x_1^2x_2x_3^2x_4+x_1x_2^2x_3^2x_4+x_1^2x_2x_3x_4^2+x_1x_2^2x_3x_4^2+x_1x_2x_3^2x_4^2+x_1^2x_2^2x_3x_5+x_1^2x_2x_3^2x_5+x_1x_2^2x_3^2x_5+x_1^2x_2^2x_4x_5+2x_1^2x_2x_3x_4x_5+2x_1x_2^2x_3x_4x_5+x_1^2x_3^2x_4x_5+2x_1x_2x_3^2x_4x_5+x_2^2x_3^2x_4x_5+x_1^2x_2x_4^2x_5+x_1x_2^2x_4^2x_5+x_1^2x_3x_4^2x_5+2x_1x_2x_3x_4^2x_5+x_2^2x_3x_4^2x_5+x_1x_3^2x_4^2x_5+x_2x_3^2x_4^2x_5$	26
315642	$x_1^3x_2^2x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1^3x_2x_3x_4^2+x_1^2x_2^2x_3x_4^2+x_1^2x_2x_3^2x_4^2+x_1^3x_2^2x_3x_5+x_1^3x_2x_3^2x_5+x_1^2x_2^2x_3^2x_5+x_1^3x_2^2x_4x_5+2x_1^3x_2x_3x_4x_5+x_1^2x_2^2x_3x_4x_5+x_1^3x_3^2x_4x_5+x_1^2x_2x_3^2x_4x_5+x_1^3x_2x_4^2x_5+x_1^2x_2^2x_4^2x_5+x_1^3x_3x_4^2x_5+x_1^2x_2x_3x_4^2x_5+x_1^2x_3^2x_4^2x_5$	20
153642	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^3x_2x_3x_4^2+x_1^2x_2^2x_3x_4^2+x_1x_2^3x_3x_4^2+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^3x_2x_3^2x_5+x_1^2x_2^2x_3^2x_5+x_1x_2^3x_3^2x_5+x_1^3x_2^2x_4x_5+x_1^2x_2^3x_4x_5+2x_1^3x_2x_3x_4x_5+2x_1^2x_2^2x_3x_4x_5+2x_1x_2^3x_3x_4x_5+x_1^3x_3^2x_4x_5+x_1^2x_2x_3^2x_4x_5+x_1x_2^2x_3^2x_4x_5+x_2^3x_3^2x_4x_5+x_1^3x_2x_4^2x_5+x_1^2x_2^2x_4^2x_5+x_1x_2^3x_4^2x_5+x_1^3x_3x_4^2x_5+x_1^2x_2x_3x_4^2x_5+x_1x_2^2x_3x_4^2x_5+x_2^3x_3x_4^2x_5$	32
513642	$x_1^4x_2^2x_3x_4+x_1^4x_2x_3^2x_4+x_1^4x_2x_3x_4^2+x_1^4x_2^2x_3x_5+x_1^4x_2x_3^2x_5+x_1^4x_2^2x_4x_5+2x_1^4x_2x_3x_4x_5+x_1^4x_3^2x_4x_5+x_1^4x_2x_4^2x_5+x_1^4x_3x_4^2x_5$	11
351642	$x_1^3x_2^3x_3x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1^3x_2^3x_3x_5+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1^3x_2^3x_4x_5+x_1^3x_2^2x_3x_4x_5+x_1^2x_2^3x_3x_4x_5+x_1^3x_2^2x_4^2x_5+x_1^2x_2^3x_4^2x_5$	13
531642	$x_1^4x_2^3x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^4x_2^2x_3x_4^2+x_1^4x_2^3x_3x_5+x_1^4x_2^2x_3^2x_5+x_1^4x_2^3x_4x_5+x_1^4x_2^2x_3x_4x_5+x_1^4x_2^2x_4^2x_5$	8

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
136542	$x_1^3x_2^2x_3x_4+x_1^2x_2^3x_3x_4+x_1^3x_2x_3^2x_4+2x_1^2x_2^2x_3^2x_4+x_1x_2^3x_3^2x_4+x_1^2x_2x_3^3x_4+x_1x_2^2x_3^3x_4+x_1^3x_2x_3x_4^2+x_1^2x_2^2x_3x_4^2+x_1x_2^3x_3x_4^2+x_1^2x_2x_3^2x_4^2+x_1x_2^2x_3^2x_4^2+x_1x_2x_3^3x_4^2+x_1^3x_2^2x_3x_5+x_1^2x_2^3x_3x_5+x_1^3x_2x_3^2x_5+2x_1^2x_2^2x_3^2x_5+x_1x_2^3x_3^2x_5+x_1^2x_2x_3^3x_5+x_1x_2^2x_3^3x_5+x_1^3x_2^2x_4x_5+x_1^2x_2^3x_4x_5+2x_1^3x_2x_3x_4x_5+3x_1^2x_2^2x_3x_4x_5+2x_1x_2^3x_3x_4x_5+x_1^3x_3^2x_4x_5+3x_1^2x_2x_3^2x_4x_5+3x_1x_2^2x_3^2x_4x_5+x_2^3x_3^2x_4x_5+x_1^2x_3^3x_4x_5+2x_1x_2x_3^3x_4x_5+x_2^2x_3^3x_4x_5+x_1^3x_2x_4^2x_5+x_1^2x_2^2x_4^2x_5+x_1x_2^3x_4^2x_5+x_1^3x_3x_4^2x_5+2x_1^2x_2x_3x_4^2x_5+2x_1x_2^2x_3x_4^2x_5+x_2^3x_3x_4^2x_5+x_1^2x_3^2x_4^2x_5+2x_1x_2x_3^2x_4^2x_5+x_2^2x_3^2x_4^2x_5+x_1x_3^3x_4^2x_5+x_2x_3^3x_4^2x_5$	58
316542	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^4x_2x_3^2x_4+2x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2x_3^3x_4+x_1^2x_2^2x_3^3x_4+x_1^4x_2x_3x_4^2+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1^3x_2x_3^2x_4^2+x_1^2x_2^2x_3^2x_4^2+x_1^2x_2x_3^3x_4^2+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^4x_2x_3^2x_5+2x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1^3x_2x_3^3x_5+x_1^2x_2^2x_3^3x_5+x_1^4x_2^2x_4x_5+x_1^3x_2^3x_4x_5+2x_1^4x_2x_3x_4x_5+2x_1^3x_2^2x_3x_4x_5+x_1^2x_2^3x_3x_4x_5+x_1^4x_3^2x_4x_5+2x_1^3x_2x_3^2x_4x_5+x_1^2x_2^2x_3^2x_4x_5+x_1^3x_3^3x_4x_5+x_1^2x_2x_3^3x_4x_5+x_1^4x_2x_4^2x_5+x_1^3x_2^2x_4^2x_5+x_1^2x_2^3x_4^2x_5+x_1^4x_3x_4^2x_5+x_1^3x_2x_3x_4^2x_5+x_1^2x_2^2x_3x_4^2x_5+x_1^3x_3^2x_4^2x_5+x_1^2x_2x_3^2x_4^2x_5+x_1^2x_3^3x_4^2x_5$	44
163542	$x_1^4x_2^2x_3x_4+x_1^3x_2^3x_3x_4+x_1^2x_2^4x_3x_4+x_1^4x_2x_3^2x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1x_2^4x_3^2x_4+x_1^4x_2x_3x_4^2+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1x_2^4x_3x_4^2+x_1^4x_2^2x_3x_5+x_1^3x_2^3x_3x_5+x_1^2x_2^4x_3x_5+x_1^4x_2x_3^2x_5+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1x_2^4x_3^2x_5+x_1^4x_2^2x_4x_5+x_1^3x_2^3x_4x_5+x_1^2x_2^4x_4x_5+2x_1^4x_2x_3x_4x_5+2x_1^3x_2^2x_3x_4x_5+2x_1^2x_2^3x_3x_4x_5+2x_1x_2^4x_3x_4x_5+x_1^4x_3^2x_4x_5+x_1^3x_2x_3^2x_4x_5+x_1^2x_2^2x_3^2x_4x_5+x_1x_2^3x_3^2x_4x_5+x_2^4x_3^2x_4x_5+x_1^4x_2x_4^2x_5+x_1^3x_2^2x_4^2x_5+x_1^2x_2^3x_4^2x_5+x_1x_2^4x_4^2x_5+x_1^4x_3x_4^2x_5+x_1^3x_2x_3x_4^2x_5+x_1^2x_2^2x_3x_4^2x_5+x_1x_2^3x_3x_4^2x_5+x_2^4x_3x_4^2x_5$	43
613542	$x_1^5x_2^2x_3x_4+x_1^5x_2x_3^2x_4+x_1^5x_2x_3x_4^2+x_1^5x_2^2x_3x_5+x_1^5x_2x_3^2x_5+x_1^5x_2^2x_4x_5+2x_1^5x_2x_3x_4x_5+x_1^5x_3^2x_4x_5+x_1^5x_2x_4^2x_5+x_1^5x_3x_4^2x_5$	11
361542	$x_1^4x_2^3x_3x_4+x_1^3x_2^4x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^2x_2^4x_3^2x_4+x_1^4x_2^2x_3x_4^2+x_1^3x_2^3x_3x_4^2+x_1^2x_2^4x_3x_4^2+x_1^4x_2^3x_3x_5+x_1^3x_2^4x_3x_5+x_1^4x_2^2x_3^2x_5+x_1^3x_2^3x_3^2x_5+x_1^2x_2^4x_3^2x_5+x_1^4x_2^3x_4x_5+x_1^3x_2^4x_4x_5+x_1^4x_2^2x_3x_4x_5+x_1^3x_2^3x_3x_4x_5+x_1^2x_2^4x_3x_4x_5+x_1^4x_2^2x_4^2x_5+x_1^3x_2^3x_4^2x_5+x_1^2x_2^4x_4^2x_5$	21
631542	$x_1^5x_2^3x_3x_4+x_1^5x_2^2x_3^2x_4+x_1^5x_2^2x_3x_4^2+x_1^5x_2^3x_3x_5+x_1^5x_2^2x_3^2x_5+x_1^5x_2^3x_4x_5+x_1^5x_2^2x_3x_4x_5+x_1^5x_2^2x_4^2x_5$	8

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
156342	$x_1^3x_2^3x_3x_4+x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2x_3^3x_4+x_1^2x_2^2x_3^3x_4+x_1x_2^3x_3^3x_4+x_1^3x_2^3x_3x_5+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1^3x_2x_3^3x_5+x_1^2x_2^2x_3^3x_5+x_1x_2^3x_3^3x_5+x_1^3x_2^3x_4x_5+x_1^3x_2^2x_3x_4x_5+x_1^2x_2^3x_3x_4x_5+x_1^3x_2x_3^2x_4x_5+x_1^2x_2^2x_3^2x_4x_5+x_1x_2^3x_3^2x_4x_5+x_1^3x_3^3x_4x_5+x_1^2x_2x_3^3x_4x_5+x_1x_2^2x_3^3x_4x_5+x_2^3x_3^3x_4x_5$	22
516342	$x_1^4x_2^3x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^4x_2x_3^3x_4+x_1^4x_2^3x_3x_5+x_1^4x_2^2x_3^2x_5+x_1^4x_2x_3^3x_5+x_1^4x_2^3x_4x_5+x_1^4x_2^2x_3x_4x_5+x_1^4x_2x_3^2x_4x_5+x_1^4x_3^3x_4x_5$	10
165342	$x_1^4x_2^3x_3x_4+x_1^3x_2^4x_3x_4+x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^2x_2^4x_3^2x_4+x_1^4x_2x_3^3x_4+x_1^3x_2^2x_3^3x_4+x_1^2x_2^3x_3^3x_4+x_1x_2^4x_3^3x_4+x_1^4x_2^3x_3x_5+x_1^3x_2^4x_3x_5+x_1^4x_2^2x_3^2x_5+x_1^3x_2^3x_3^2x_5+x_1^2x_2^4x_3^2x_5+x_1^4x_2x_3^3x_5+x_1^3x_2^2x_3^3x_5+x_1^2x_2^3x_3^3x_5+x_1x_2^4x_3^3x_5+x_1^4x_2^3x_4x_5+x_1^3x_2^4x_4x_5+x_1^4x_2^2x_3x_4x_5+x_1^3x_2^3x_3x_4x_5+x_1^2x_2^4x_3x_4x_5+x_1^4x_2x_3^2x_4x_5+x_1^3x_2^2x_3^2x_4x_5+x_1^2x_2^3x_3^2x_4x_5+x_1x_2^4x_3^2x_4x_5+x_1^4x_3^3x_4x_5+x_1^3x_2x_3^3x_4x_5+x_1^2x_2^2x_3^3x_4x_5+x_1x_2^3x_3^3x_4x_5+x_2^4x_3^3x_4x_5$	32
615342	$x_1^5x_2^3x_3x_4+x_1^5x_2^2x_3^2x_4+x_1^5x_2x_3^3x_4+x_1^5x_2^3x_3x_5+x_1^5x_2^2x_3^2x_5+x_1^5x_2x_3^3x_5+x_1^5x_2^3x_4x_5+x_1^5x_2^2x_3x_4x_5+x_1^5x_2x_3^2x_4x_5+x_1^5x_3^3x_4x_5$	10
561342	$x_1^4x_2^4x_3x_4+x_1^4x_2^4x_3x_5+x_1^4x_2^4x_4x_5$	3
651342	$x_1^5x_2^4x_3x_4+x_1^5x_2^4x_3x_5+x_1^5x_2^4x_4x_5$	3
356142	$x_1^3x_2^3x_3^2x_4+x_1^3x_2^2x_3^3x_4+x_1^2x_2^3x_3^3x_4+x_1^3x_2^3x_3^2x_5+x_1^3x_2^2x_3^3x_5+x_1^2x_2^3x_3^3x_5$	6
536142	$x_1^4x_2^3x_3^2x_4+x_1^4x_2^2x_3^3x_4+x_1^4x_2^3x_3^2x_5+x_1^4x_2^2x_3^3x_5$	4
365142	$x_1^4x_2^3x_3^2x_4+x_1^3x_2^4x_3^2x_4+x_1^4x_2^2x_3^3x_4+x_1^3x_2^3x_3^3x_4+x_1^2x_2^4x_3^3x_4+x_1^4x_2^3x_3^2x_5+x_1^3x_2^4x_3^2x_5+x_1^4x_2^2x_3^3x_5+x_1^3x_2^3x_3^3x_5+x_1^2x_2^4x_3^3x_5$	10
635142	$x_1^5x_2^3x_3^2x_4+x_1^5x_2^2x_3^3x_4+x_1^5x_2^3x_3^2x_5+x_1^5x_2^2x_3^3x_5$	4
563142	$x_1^4x_2^4x_3^2x_4+x_1^4x_2^4x_3^2x_5$	2
653142	$x_1^5x_2^4x_3^2x_4+x_1^5x_2^4x_3^2x_5$	2

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
145632	$x_1^2x_2^2x_3^2x_4+x_1^2x_2^2x_3x_4^2+x_1^2x_2x_3^2x_4^2+x_1x_2^2x_3^2x_4^2+x_1^2x_2^2x_3^2x_5+x_1^2x_2^2x_3x_4x_5+x_1^2x_2x_3^2x_4x_5+x_1x_2^2x_3^2x_4x_5+x_1^2x_2^2x_4^2x_5+x_1^2x_2x_3x_4^2x_5+x_1x_2^2x_3x_4^2x_5+x_1^2x_3^2x_4^2x_5+x_1x_2x_3^2x_4^2x_5+x_2^2x_3^2x_4^2x_5$	14
415632	$x_1^3x_2^2x_3^2x_4+x_1^3x_2^2x_3x_4^2+x_1^3x_2x_3^2x_4^2+x_1^3x_2^2x_3^2x_5+x_1^3x_2^2x_3x_4x_5+x_1^3x_2x_3^2x_4x_5+x_1^3x_2^2x_4^2x_5+x_1^3x_2x_3x_4^2x_5+x_1^3x_3^2x_4^2x_5$	9
154632	$x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1^3x_2x_3^2x_4^2+x_1^2x_2^2x_3^2x_4^2+x_1x_2^3x_3^2x_4^2+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1^3x_2^2x_3x_4x_5+x_1^2x_2^3x_3x_4x_5+x_1^3x_2x_3^2x_4x_5+x_1^2x_2^2x_3^2x_4x_5+x_1x_2^3x_3^2x_4x_5+x_1^3x_2^2x_4^2x_5+x_1^2x_2^3x_4^2x_5+x_1^3x_2x_3x_4^2x_5+x_1^2x_2^2x_3x_4^2x_5+x_1x_2^3x_3x_4^2x_5+x_1^3x_3^2x_4^2x_5+x_1^2x_2x_3^2x_4^2x_5+x_1x_2^2x_3^2x_4^2x_5+x_2^3x_3^2x_4^2x_5$	23
514632	$x_1^4x_2^2x_3^2x_4+x_1^4x_2^2x_3x_4^2+x_1^4x_2x_3^2x_4^2+x_1^4x_2^2x_3^2x_5+x_1^4x_2^2x_3x_4x_5+x_1^4x_2x_3^2x_4x_5+x_1^4x_2^2x_4^2x_5+x_1^4x_2x_3x_4^2x_5+x_1^4x_3^2x_4^2x_5$	9
451632	$x_1^3x_2^3x_3^2x_4+x_1^3x_2^3x_3x_4^2+x_1^3x_2^3x_3^2x_5+x_1^3x_2^3x_3x_4x_5+x_1^3x_2^3x_4^2x_5$	5
541632	$x_1^4x_2^3x_3^2x_4+x_1^4x_2^3x_3x_4^2+x_1^4x_2^3x_3^2x_5+x_1^4x_2^3x_3x_4x_5+x_1^4x_2^3x_4^2x_5$	5

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
146532	$x_1^3x_2^2x_3^2x_4+x_1^2x_2^3x_3^2x_4+x_1^2x_2^2x_3^3x_4+x_1^3x_2^2x_3x_4^2+x_1^2x_2^3x_3x_4^2+x_1^3x_2x_3^2x_4^2+2x_1^2x_2^2x_3^2x_4^2+x_1x_2^3x_3^2x_4^2+x_1^2x_2x_3^3x_4^2+x_1x_2^2x_3^3x_4^2+x_1^3x_2^2x_3^2x_5+x_1^2x_2^3x_3^2x_5+x_1^2x_2^2x_3^3x_5+x_1^3x_2^2x_3x_4x_5+x_1^2x_2^3x_3x_4x_5+x_1^3x_2x_3^2x_4x_5+2x_1^2x_2^2x_3^2x_4x_5+x_1x_2^3x_3^2x_4x_5+x_1^2x_2x_3^3x_4x_5+x_1x_2^2x_3^3x_4x_5+x_1^3x_2^2x_4^2x_5+x_1^2x_2^3x_4^2x_5+x_1^3x_2x_3x_4^2x_5+2x_1^2x_2^2x_3x_4^2x_5+x_1x_2^3x_3x_4^2x_5+x_1^3x_3^2x_4^2x_5+2x_1^2x_2x_3^2x_4^2x_5+2x_1x_2^2x_3^2x_4^2x_5+x_2^3x_3^2x_4^2x_5+x_1^2x_3^3x_4^2x_5+x_1x_2x_3^3x_4^2x_5+x_2^2x_3^3x_4^2x_5$	37
416532	$x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^3x_2^2x_3^3x_4+x_1^4x_2^2x_3x_4^2+x_1^3x_2^3x_3x_4^2+x_1^4x_2x_3^2x_4^2+x_1^3x_2^2x_3^2x_4^2+x_1^3x_2x_3^3x_4^2+x_1^4x_2^2x_3^2x_5+x_1^3x_2^3x_3^2x_5+x_1^3x_2^2x_3^3x_5+x_1^4x_2^2x_3x_4x_5+x_1^3x_2^3x_3x_4x_5+x_1^4x_2x_3^2x_4x_5+x_1^3x_2^2x_3^2x_4x_5+x_1^3x_2x_3^3x_4x_5+x_1^4x_2^2x_4^2x_5+x_1^3x_2^3x_4^2x_5+x_1^4x_2x_3x_4^2x_5+x_1^3x_2^2x_3x_4^2x_5+x_1^4x_3^2x_4^2x_5+x_1^3x_2x_3^2x_4^2x_5+x_1^3x_3^3x_4^2x_5$	23
164532	$x_1^4x_2^2x_3^2x_4+x_1^3x_2^3x_3^2x_4+x_1^2x_2^4x_3^2x_4+x_1^4x_2^2x_3x_4^2+x_1^3x_2^3x_3x_4^2+x_1^2x_2^4x_3x_4^2+x_1^4x_2x_3^2x_4^2+x_1^3x_2^2x_3^2x_4^2+x_1^2x_2^3x_3^2x_4^2+x_1x_2^4x_3^2x_4^2+x_1^4x_2^2x_3^2x_5+x_1^3x_2^3x_3^2x_5+x_1^2x_2^4x_3^2x_5+x_1^4x_2^2x_3x_4x_5+x_1^3x_2^3x_3x_4x_5+x_1^2x_2^4x_3x_4x_5+x_1^4x_2x_3^2x_4x_5+x_1^3x_2^2x_3^2x_4x_5+x_1^2x_2^3x_3^2x_4x_5+x_1x_2^4x_3^2x_4x_5+x_1^4x_2^2x_4^2x_5+x_1^3x_2^3x_4^2x_5+x_1^2x_2^4x_4^2x_5+x_1^4x_2x_3x_4^2x_5+x_1^3x_2^2x_3x_4^2x_5+x_1^2x_2^3x_3x_4^2x_5+x_1x_2^4x_3x_4^2x_5+x_1^4x_3^2x_4^2x_5+x_1^3x_2x_3^2x_4^2x_5+x_1^2x_2^2x_3^2x_4^2x_5+x_1x_2^3x_3^2x_4^2x_5+x_2^4x_3^2x_4^2x_5$	32
614532	$x_1^5x_2^2x_3^2x_4+x_1^5x_2^2x_3x_4^2+x_1^5x_2x_3^2x_4^2+x_1^5x_2^2x_3^2x_5+x_1^5x_2^2x_3x_4x_5+x_1^5x_2x_3^2x_4x_5+x_1^5x_2^2x_4^2x_5+x_1^5x_2x_3x_4^2x_5+x_1^5x_3^2x_4^2x_5$	9
461532	$x_1^4x_2^3x_3^2x_4+x_1^3x_2^4x_3^2x_4+x_1^4x_2^3x_3x_4^2+x_1^3x_2^4x_3x_4^2+x_1^4x_2^3x_3^2x_5+x_1^3x_2^4x_3^2x_5+x_1^4x_2^3x_3x_4x_5+x_1^3x_2^4x_3x_4x_5+x_1^4x_2^3x_4^2x_5+x_1^3x_2^4x_4^2x_5$	10
641532	$x_1^5x_2^3x_3^2x_4+x_1^5x_2^3x_3x_4^2+x_1^5x_2^3x_3^2x_5+x_1^5x_2^3x_3x_4x_5+x_1^5x_2^3x_4^2x_5$	5

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
156432	$x_1^3x_2^3x_3^2x_4+x_1^3x_2^2x_3^3x_4+x_1^2x_2^3x_3^3x_4+x_1^3x_2^3x_3x_4^2+x_1^3x_2^2x_3^2x_4^2+x_1^2x_2^3x_3^2x_4^2+x_1^3x_2x_3^3x_4^2+x_1^2x_2^2x_3^3x_4^2+x_1x_2^3x_3^3x_4^2+x_1^3x_2^3x_3^2x_5+x_1^3x_2^2x_3^3x_5+x_1^2x_2^3x_3^3x_5+x_1^3x_2^3x_3x_4x_5+x_1^3x_2^2x_3^2x_4x_5+x_1^2x_2^3x_3^2x_4x_5+x_1^3x_2x_3^3x_4x_5+x_1^2x_2^2x_3^3x_4x_5+x_1x_2^3x_3^3x_4x_5+x_1^3x_2^3x_4^2x_5+x_1^3x_2^2x_3x_4^2x_5+x_1^2x_2^3x_3x_4^2x_5+x_1^3x_2x_3^2x_4^2x_5+x_1^2x_2^2x_3^2x_4^2x_5+x_1x_2^3x_3^2x_4^2x_5+x_1^3x_3^3x_4^2x_5+x_1^2x_2x_3^3x_4^2x_5+x_1x_2^2x_3^3x_4^2x_5+x_2^3x_3^3x_4^2x_5$	28
516432	$x_1^4x_2^3x_3^2x_4+x_1^4x_2^2x_3^3x_4+x_1^4x_2^3x_3x_4^2+x_1^4x_2^2x_3^2x_4^2+x_1^4x_2x_3^3x_4^2+x_1^4x_2^3x_3^2x_5+x_1^4x_2^2x_3^3x_5+x_1^4x_2^3x_3x_4x_5+x_1^4x_2^2x_3^2x_4x_5+x_1^4x_2x_3^3x_4x_5+x_1^4x_2^3x_4^2x_5+x_1^4x_2^2x_3x_4^2x_5+x_1^4x_2x_3^2x_4^2x_5+x_1^4x_3^3x_4^2x_5$	14
165432	$x_1^4x_2^3x_3^2x_4+x_1^3x_2^4x_3^2x_4+x_1^4x_2^2x_3^3x_4+x_1^3x_2^3x_3^3x_4+x_1^2x_2^4x_3^3x_4+x_1^4x_2^3x_3x_4^2+x_1^3x_2^4x_3x_4^2+x_1^4x_2^2x_3^2x_4^2+x_1^3x_2^3x_3^2x_4^2+x_1^2x_2^4x_3^2x_4^2+x_1^4x_2x_3^3x_4^2+x_1^3x_2^2x_3^3x_4^2+x_1^2x_2^3x_3^3x_4^2+x_1x_2^4x_3^3x_4^2+x_1^4x_2^3x_3^2x_5+x_1^3x_2^4x_3^2x_5+x_1^4x_2^2x_3^3x_5+x_1^3x_2^3x_3^3x_5+x_1^2x_2^4x_3^3x_5+x_1^4x_2^3x_3x_4x_5+x_1^3x_2^4x_3x_4x_5+x_1^4x_2^2x_3^2x_4x_5+x_1^3x_2^3x_3^2x_4x_5+x_1^2x_2^4x_3^2x_4x_5+x_1^4x_2x_3^3x_4x_5+x_1^3x_2^2x_3^3x_4x_5+x_1^2x_2^3x_3^3x_4x_5+x_1x_2^4x_3^3x_4x_5+x_1^4x_2^3x_4^2x_5+x_1^3x_2^4x_4^2x_5+x_1^4x_2^2x_3x_4^2x_5+x_1^3x_2^3x_3x_4^2x_5+x_1^2x_2^4x_3x_4^2x_5+x_1^4x_2x_3^2x_4^2x_5+x_1^3x_2^2x_3^2x_4^2x_5+x_1^2x_2^3x_3^2x_4^2x_5+x_1x_2^4x_3^2x_4^2x_5+x_1^4x_3^3x_4^2x_5+x_1^3x_2x_3^3x_4^2x_5+x_1^2x_2^2x_3^3x_4^2x_5+x_1x_2^3x_3^3x_4^2x_5+x_2^4x_3^3x_4^2x_5$	42
615432	$x_1^5x_2^3x_3^2x_4+x_1^5x_2^2x_3^3x_4+x_1^5x_2^3x_3x_4^2+x_1^5x_2^2x_3^2x_4^2+x_1^5x_2x_3^3x_4^2+x_1^5x_2^3x_3^2x_5+x_1^5x_2^2x_3^3x_5+x_1^5x_2^3x_3x_4x_5+x_1^5x_2^2x_3^2x_4x_5+x_1^5x_2x_3^3x_4x_5+x_1^5x_2^3x_4^2x_5+x_1^5x_2^2x_3x_4^2x_5+x_1^5x_2x_3^2x_4^2x_5+x_1^5x_3^3x_4^2x_5$	14
561432	$x_1^4x_2^4x_3^2x_4+x_1^4x_2^4x_3x_4^2+x_1^4x_2^4x_3^2x_5+x_1^4x_2^4x_3x_4x_5+x_1^4x_2^4x_4^2x_5$	5
651432	$x_1^5x_2^4x_3^2x_4+x_1^5x_2^4x_3x_4^2+x_1^5x_2^4x_3^2x_5+x_1^5x_2^4x_3x_4x_5+x_1^5x_2^4x_4^2x_5$	5
456132	$x_1^3x_2^3x_3^3x_4+x_1^3x_2^3x_3^3x_5$	2
546132	$x_1^4x_2^3x_3^3x_4+x_1^4x_2^3x_3^3x_5$	2
465132	$x_1^4x_2^3x_3^3x_4+x_1^3x_2^4x_3^3x_4+x_1^4x_2^3x_3^3x_5+x_1^3x_2^4x_3^3x_5$	4
645132	$x_1^5x_2^3x_3^3x_4+x_1^5x_2^3x_3^3x_5$	2
564132	$x_1^4x_2^4x_3^3x_4+x_1^4x_2^4x_3^3x_5$	2
654132	$x_1^5x_2^4x_3^3x_4+x_1^5x_2^4x_3^3x_5$	2

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
345612	$x_1^2x_2^2x_3^2x_4^2$	1
435612	$x_1^3x_2^2x_3^2x_4^2$	1
354612	$x_1^3x_2^2x_3^2x_4^2 + x_1^2x_2^3x_3^2x_4^2$	2
534612	$x_1^4x_2^2x_3^2x_4^2$	1
453612	$x_1^3x_2^3x_3^2x_4^2$	1
543612	$x_1^4x_2^3x_3^2x_4^2$	1
346512	$x_1^3x_2^2x_3^2x_4^2 + x_1^2x_2^3x_3^2x_4^2 + x_1^2x_2^2x_3^3x_4^2$	3
436512	$x_1^4x_2^2x_3^2x_4^2 + x_1^3x_2^3x_3^2x_4^2 + x_1^3x_2^2x_3^3x_4^2$	3
364512	$x_1^4x_2^2x_3^2x_4^2 + x_1^3x_2^3x_3^2x_4^2 + x_1^2x_2^4x_3^2x_4^2$	3
634512	$x_1^5x_2^2x_3^2x_4^2$	1
463512	$x_1^4x_2^3x_3^2x_4^2 + x_1^3x_2^4x_3^2x_4^2$	2
643512	$x_1^5x_2^3x_3^2x_4^2$	1
356412	$x_1^3x_2^3x_3^2x_4^2 + x_1^3x_2^2x_3^3x_4^2 + x_1^2x_2^3x_3^3x_4^2$	3
536412	$x_1^4x_2^3x_3^2x_4^2 + x_1^4x_2^2x_3^3x_4^2$	2
365412	$x_1^4x_2^3x_3^2x_4^2 + x_1^3x_2^4x_3^2x_4^2 + x_1^4x_2^2x_3^3x_4^2 + x_1^3x_2^3x_3^3x_4^2 + x_1^2x_2^4x_3^3x_4^2$	5
635412	$x_1^5x_2^3x_3^2x_4^2 + x_1^5x_2^2x_3^3x_4^2$	2
563412	$x_1^4x_2^4x_3^2x_4^2$	1
653412	$x_1^5x_2^4x_3^2x_4^2$	1
456312	$x_1^3x_2^3x_3^3x_4^2$	1
546312	$x_1^4x_2^3x_3^3x_4^2$	1
465312	$x_1^4x_2^3x_3^3x_4^2 + x_1^3x_2^4x_3^3x_4^2$	2
645312	$x_1^5x_2^3x_3^3x_4^2$	1
564312	$x_1^4x_2^4x_3^3x_4^2$	1
654312	$x_1^5x_2^4x_3^3x_4^2$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
234561	$x_1x_2x_3x_4x_5$	1
324561	$x_1^2x_2x_3x_4x_5$	1
243561	$x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5$	2
423561	$x_1^3x_2x_3x_4x_5$	1
342561	$x_1^2x_2^2x_3x_4x_5$	1
432561	$x_1^3x_2^2x_3x_4x_5$	1
235461	$x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_1x_2x_3^2x_4x_5$	3
325461	$x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1^2x_2x_3^2x_4x_5$	3
253461	$x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5$	3
523461	$x_1^4x_2x_3x_4x_5$	1
352461	$x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5$	2
532461	$x_1^4x_2^2x_3x_4x_5$	1
245361	$x_1^2x_2^2x_3x_4x_5 + x_1^2x_2x_3^2x_4x_5 + x_1x_2^2x_3^2x_4x_5$	3
425361	$x_1^3x_2^2x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5$	2
254361	$x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + x_1^2x_2^2x_3^2x_4x_5 +$ $x_1x_2^3x_3^2x_4x_5$	5
524361	$x_1^4x_2^2x_3x_4x_5 + x_1^4x_2x_3^2x_4x_5$	2
452361	$x_1^3x_2^3x_3x_4x_5$	1
542361	$x_1^4x_2^3x_3x_4x_5$	1
345261	$x_1^2x_2^2x_3^2x_4x_5$	1
435261	$x_1^3x_2^2x_3^2x_4x_5$	1
354261	$x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5$	2
534261	$x_1^4x_2^2x_3^2x_4x_5$	1
453261	$x_1^3x_2^3x_3^2x_4x_5$	1
543261	$x_1^4x_2^3x_3^2x_4x_5$	1

w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
234651	$x_1^2x_2x_3x_4x_5 + x_1x_2^2x_3x_4x_5 + x_1x_2x_3^2x_4x_5 + x_1x_2x_3x_4^2x_5$	4
324651	$x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1^2x_2x_3^2x_4x_5 + x_1^2x_2x_3x_4^2x_5$	4
243651	$x_1^3x_2x_3x_4x_5 + 2x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5 + x_1^2x_2x_3^2x_4x_5 + x_1x_2^2x_3^2x_4x_5 + x_1^2x_2x_3x_4^2x_5 + x_1x_2^2x_3x_4^2x_5$	8
423651	$x_1^4x_2x_3x_4x_5 + x_1^3x_2^2x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + x_1^3x_2x_3x_4^2x_5$	4
342651	$x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1^2x_2^2x_3^2x_4x_5 + x_1^2x_2^2x_3x_4^2x_5$	4
432651	$x_1^4x_2^2x_3x_4x_5 + x_1^3x_2^3x_3x_4x_5 + x_1^3x_2^2x_3^2x_4x_5 + x_1^3x_2^2x_3x_4^2x_5$	4
236451	$x_1^3x_2x_3x_4x_5 + x_1^2x_2^2x_3x_4x_5 + x_1x_2^3x_3x_4x_5 + x_1^2x_2x_3^2x_4x_5 + x_1x_2^2x_3^2x_4x_5 + x_1x_2x_3^3x_4x_5$	6
326451	$x_1^4x_2x_3x_4x_5 + x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + x_1^2x_2^2x_3^2x_4x_5 + x_1^2x_2x_3^3x_4x_5$	6
263451	$x_1^4x_2x_3x_4x_5 + x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1x_2^4x_3x_4x_5$	4
623451	$x_1^5x_2x_3x_4x_5$	1
362451	$x_1^4x_2^2x_3x_4x_5 + x_1^3x_2^3x_3x_4x_5 + x_1^2x_2^4x_3x_4x_5$	3
632451	$x_1^5x_2^2x_3x_4x_5$	1
246351	$x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + 2x_1^2x_2^2x_3^2x_4x_5 + x_1x_2^3x_3^2x_4x_5 + x_1^2x_2x_3^3x_4x_5 + x_1x_2^2x_3^3x_4x_5$	8
426351	$x_1^4x_2^2x_3x_4x_5 + x_1^3x_2^3x_3x_4x_5 + x_1^4x_2x_3^2x_4x_5 + x_1^3x_2^2x_3^2x_4x_5 + x_1^3x_2x_3^3x_4x_5$	5
264351	$x_1^4x_2^2x_3x_4x_5 + x_1^3x_2^3x_3x_4x_5 + x_1^2x_2^4x_3x_4x_5 + x_1^4x_2x_3^2x_4x_5 + x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1x_2^4x_3^2x_4x_5$	7
624351	$x_1^5x_2^2x_3x_4x_5 + x_1^5x_2x_3^2x_4x_5$	2
462351	$x_1^4x_2^3x_3x_4x_5 + x_1^3x_2^4x_3x_4x_5$	2
642351	$x_1^5x_2^3x_3x_4x_5$	1
346251	$x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1^2x_2^2x_3^3x_4x_5$	3
436251	$x_1^4x_2^2x_3^2x_4x_5 + x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^2x_3^3x_4x_5$	3
364251	$x_1^4x_2^2x_3^2x_4x_5 + x_1^3x_2^3x_3^2x_4x_5 + x_1^2x_2^4x_3^2x_4x_5$	3
634251	$x_1^5x_2^2x_3^2x_4x_5$	1
463251	$x_1^4x_2^3x_3^2x_4x_5 + x_1^3x_2^4x_3^2x_4x_5$	2
643251	$x_1^5x_2^3x_3^2x_4x_5$	1

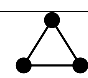
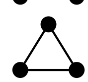
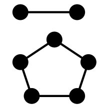
w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
235641	$x_1^2x_2^2x_3x_4x_5 + x_1^2x_2x_3^2x_4x_5 + x_1x_2^2x_3^2x_4x_5 + x_1^2x_2x_3x_4^2x_5 + x_1x_2^2x_3x_4^2x_5 + x_1x_2x_3^2x_4^2x_5$	6
325641	$x_1^3x_2^2x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + x_1^2x_2^2x_3^2x_4x_5 + x_1^3x_2x_3x_4^2x_5 + x_1^2x_2^2x_3x_4^2x_5 + x_1^2x_2x_3^2x_4^2x_5$	6
253641	$x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + x_1^2x_2^2x_3^2x_4x_5 + x_1x_2^3x_3^2x_4x_5 + x_1^3x_2x_3x_4^2x_5 + x_1^2x_2^2x_3x_4^2x_5 + x_1x_2^3x_3x_4^2x_5$	8
523641	$x_1^4x_2^2x_3x_4x_5 + x_1^4x_2x_3^2x_4x_5 + x_1^4x_2x_3x_4^2x_5$	3
352641	$x_1^3x_2^3x_3x_4x_5 + x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1^3x_2^2x_3x_4^2x_5 + x_1^2x_2^3x_3x_4^2x_5$	5
532641	$x_1^4x_2^3x_3x_4x_5 + x_1^4x_2^2x_3^2x_4x_5 + x_1^4x_2^2x_3x_4^2x_5$	3
236541	$x_1^3x_2^2x_3x_4x_5 + x_1^2x_2^3x_3x_4x_5 + x_1^3x_2x_3^2x_4x_5 + 2x_1^2x_2^2x_3^2x_4x_5 + x_1x_2^3x_3^2x_4x_5 + x_1^2x_2x_3^3x_4x_5 + x_1x_2^2x_3^3x_4x_5 + x_1^3x_2x_3x_4^2x_5 + x_1^2x_2^2x_3x_4^2x_5 + x_1x_2^3x_3x_4^2x_5 + x_1^2x_2x_3^2x_4^2x_5 + x_1x_2^2x_3^2x_4^2x_5 + x_1x_2x_3^3x_4^2x_5$	14
326541	$x_1^4x_2^2x_3x_4x_5 + x_1^3x_2^3x_3x_4x_5 + x_1^4x_2x_3^2x_4x_5 + 2x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1^3x_2x_3^3x_4x_5 + x_1^2x_2^2x_3^3x_4x_5 + x_1^4x_2x_3x_4^2x_5 + x_1^3x_2^2x_3x_4^2x_5 + x_1^2x_2^3x_3x_4^2x_5 + x_1^3x_2x_3^2x_4^2x_5 + x_1^2x_2^2x_3^2x_4^2x_5 + x_1^2x_2x_3^3x_4^2x_5$	14
263541	$x_1^4x_2^2x_3x_4x_5 + x_1^3x_2^3x_3x_4x_5 + x_1^2x_2^4x_3x_4x_5 + x_1^4x_2x_3^2x_4x_5 + x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1x_2^4x_3^2x_4x_5 + x_1^4x_2x_3x_4^2x_5 + x_1^3x_2^2x_3x_4^2x_5 + x_1^2x_2^3x_3x_4^2x_5 + x_1x_2^4x_3x_4^2x_5$	11
623541	$x_1^5x_2^2x_3x_4x_5 + x_1^5x_2x_3^2x_4x_5 + x_1^5x_2x_3x_4^2x_5$	3
362541	$x_1^4x_2^3x_3x_4x_5 + x_1^3x_2^4x_3x_4x_5 + x_1^4x_2^2x_3^2x_4x_5 + x_1^3x_2^3x_3^2x_4x_5 + x_1^2x_2^4x_3^2x_4x_5 + x_1^4x_2^2x_3x_4^2x_5 + x_1^3x_2^3x_3x_4^2x_5 + x_1^2x_2^4x_3x_4^2x_5$	8
632541	$x_1^5x_2^3x_3x_4x_5 + x_1^5x_2^2x_3^2x_4x_5 + x_1^5x_2^2x_3x_4^2x_5$	3
256341	$x_1^3x_2^3x_3x_4x_5 + x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1^3x_2x_3^3x_4x_5 + x_1^2x_2^2x_3^3x_4x_5 + x_1x_2^3x_3^3x_4x_5$	6
526341	$x_1^4x_2^3x_3x_4x_5 + x_1^4x_2^2x_3^2x_4x_5 + x_1^4x_2x_3^3x_4x_5$	3
265341	$x_1^4x_2^3x_3x_4x_5 + x_1^3x_2^4x_3x_4x_5 + x_1^4x_2^2x_3^2x_4x_5 + x_1^3x_2^3x_3^2x_4x_5 + x_1^2x_2^4x_3^2x_4x_5 + x_1^4x_2x_3^3x_4x_5 + x_1^3x_2^2x_3^3x_4x_5 + x_1^2x_2^3x_3^3x_4x_5 + x_1x_2^4x_3^3x_4x_5$	9
625341	$x_1^5x_2^3x_3x_4x_5 + x_1^5x_2^2x_3^2x_4x_5 + x_1^5x_2x_3^3x_4x_5$	3
562341	$x_1^4x_2^4x_3x_4x_5$	1
652341	$x_1^5x_2^4x_3x_4x_5$	1
356241	$x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^2x_3^3x_4x_5 + x_1^2x_2^3x_3^3x_4x_5$	3
536241	$x_1^4x_2^3x_3^2x_4x_5 + x_1^4x_2^2x_3^3x_4x_5$	2
365241	$x_1^4x_2^3x_3^2x_4x_5 + x_1^3x_2^4x_3^2x_4x_5 + x_1^4x_2^2x_3^3x_4x_5 + x_1^3x_2^3x_3^3x_4x_5 + x_1^2x_2^4x_3^3x_4x_5$	5
635241	$x_1^5x_2^3x_3^2x_4x_5 + x_1^5x_2^2x_3^3x_4x_5$	2
563241	$x_1^4x_2^4x_3^2x_4x_5$	1
653241	$x_1^5x_2^4x_3^2x_4x_5$	1

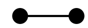
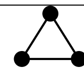
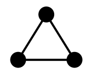
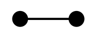
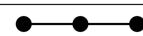
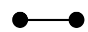
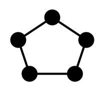
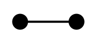
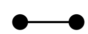
w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
245631	$x_1^2x_2^2x_3^2x_4x_5 + x_1^2x_2^2x_3x_4^2x_5 + x_1^2x_2x_3^2x_4^2x_5 + x_1x_2^2x_3^2x_4^2x_5$	4
425631	$x_1^3x_2^2x_3^2x_4x_5 + x_1^3x_2^2x_3x_4^2x_5 + x_1^3x_2x_3^2x_4^2x_5$	3
254631	$x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1^3x_2^2x_3x_4^2x_5 + x_1^2x_2^3x_3x_4^2x_5 + x_1^3x_2x_3^2x_4^2x_5 + x_1^2x_2^2x_3^2x_4^2x_5 + x_1x_2^3x_3^2x_4^2x_5$	7
524631	$x_1^4x_2^2x_3^2x_4x_5 + x_1^4x_2^2x_3x_4^2x_5 + x_1^4x_2x_3^2x_4^2x_5$	3
452631	$x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^3x_3x_4^2x_5$	2
542631	$x_1^4x_2^3x_3^2x_4x_5 + x_1^4x_2^3x_3x_4^2x_5$	2
246531	$x_1^3x_2^2x_3^2x_4x_5 + x_1^2x_2^3x_3^2x_4x_5 + x_1^2x_2^2x_3^3x_4x_5 + x_1^3x_2^2x_3x_4^2x_5 + x_1^2x_2^3x_3x_4^2x_5 + x_1^3x_2x_3^2x_4^2x_5 + 2x_1^2x_2^2x_3^2x_4^2x_5 + x_1x_2^3x_3^2x_4^2x_5 + x_1^2x_2x_3^3x_4^2x_5 + x_1x_2^2x_3^3x_4^2x_5$	11
426531	$x_1^4x_2^2x_3^2x_4x_5 + x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^2x_3^3x_4x_5 + x_1^4x_2^2x_3x_4^2x_5 + x_1^3x_2^3x_3x_4^2x_5 + x_1^4x_2x_3^2x_4^2x_5 + x_1^3x_2^2x_3^2x_4^2x_5 + x_1^3x_2x_3^3x_4^2x_5$	8
264531	$x_1^4x_2^2x_3^2x_4x_5 + x_1^3x_2^3x_3^2x_4x_5 + x_1^2x_2^4x_3^2x_4x_5 + x_1^4x_2^2x_3x_4^2x_5 + x_1^3x_2^3x_3x_4^2x_5 + x_1^2x_2^4x_3x_4^2x_5 + x_1^4x_2x_3^2x_4^2x_5 + x_1^3x_2^2x_3^2x_4^2x_5 + x_1^2x_2^3x_3^2x_4^2x_5 + x_1x_2^4x_3^2x_4^2x_5$	10
624531	$x_1^5x_2^2x_3^2x_4x_5 + x_1^5x_2^2x_3x_4^2x_5 + x_1^5x_2x_3^2x_4^2x_5$	3
462531	$x_1^4x_2^3x_3^2x_4x_5 + x_1^3x_2^4x_3^2x_4x_5 + x_1^4x_2^3x_3x_4^2x_5 + x_1^3x_2^4x_3x_4^2x_5$	4
642531	$x_1^5x_2^3x_3^2x_4x_5 + x_1^5x_2^3x_3x_4^2x_5$	2
256431	$x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^2x_3^3x_4x_5 + x_1^2x_2^3x_3^3x_4x_5 + x_1^3x_2^3x_3x_4^2x_5 + x_1^3x_2^2x_3^2x_4^2x_5 + x_1^2x_2^3x_3^2x_4^2x_5 + x_1^3x_2x_3^3x_4^2x_5 + x_1^2x_2^2x_3^3x_4^2x_5 + x_1x_2^3x_3^3x_4^2x_5$	9
526431	$x_1^4x_2^3x_3^2x_4x_5 + x_1^4x_2^2x_3^3x_4x_5 + x_1^4x_2^3x_3x_4^2x_5 + x_1^4x_2^2x_3^2x_4^2x_5 + x_1^4x_2x_3^3x_4^2x_5$	5
265431	$x_1^4x_2^3x_3^2x_4x_5 + x_1^3x_2^4x_3^2x_4x_5 + x_1^4x_2^2x_3^3x_4x_5 + x_1^3x_2^3x_3^3x_4x_5 + x_1^2x_2^4x_3^3x_4x_5 + x_1^4x_2^3x_3x_4^2x_5 + x_1^3x_2^4x_3x_4^2x_5 + x_1^4x_2^2x_3^2x_4^2x_5 + x_1^3x_2^3x_3^2x_4^2x_5 + x_1^2x_2^4x_3^2x_4^2x_5 + x_1^4x_2x_3^3x_4^2x_5 + x_1^3x_2^2x_3^3x_4^2x_5 + x_1^2x_2^3x_3^3x_4^2x_5 + x_1x_2^4x_3^3x_4^2x_5$	14
625431	$x_1^5x_2^3x_3^2x_4x_5 + x_1^5x_2^2x_3^3x_4x_5 + x_1^5x_2^3x_3x_4^2x_5 + x_1^5x_2^2x_3^2x_4^2x_5 + x_1^5x_2x_3^3x_4^2x_5$	5
562431	$x_1^4x_2^4x_3^2x_4x_5 + x_1^4x_2^4x_3x_4^2x_5$	2
652431	$x_1^5x_2^4x_3^2x_4x_5 + x_1^5x_2^4x_3x_4^2x_5$	2
456231	$x_1^3x_2^3x_3^3x_4x_5$	1
546231	$x_1^4x_2^3x_3^3x_4x_5$	1
465231	$x_1^4x_2^3x_3^3x_4x_5 + x_1^3x_2^4x_3^3x_4x_5$	2
645231	$x_1^5x_2^3x_3^3x_4x_5$	1
564231	$x_1^4x_2^4x_3^3x_4x_5$	1
654231	$x_1^5x_2^4x_3^3x_4x_5$	1

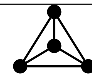
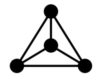
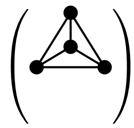
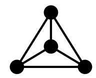
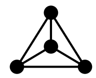
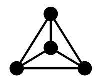
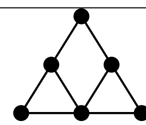
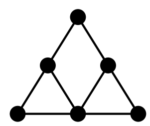
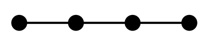
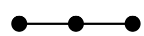
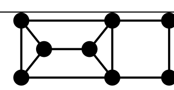
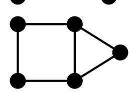
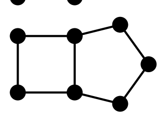
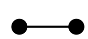
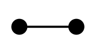
w	$\mathfrak{S}_w(x)$	$\mathfrak{S}_w(1,\ldots,1)$
345621	$x_1^2x_2^2x_3^2x_4^2x_5$	1
435621	$x_1^3x_2^2x_3^2x_4^2x_5$	1
354621	$x_1^3x_2^2x_3^2x_4^2x_5 + x_1^2x_2^3x_3^2x_4^2x_5$	2
534621	$x_1^4x_2^2x_3^2x_4^2x_5$	1
453621	$x_1^3x_2^3x_3^2x_4^2x_5$	1
543621	$x_1^4x_2^3x_3^2x_4^2x_5$	1
346521	$x_1^3x_2^2x_3^2x_4^2x_5 + x_1^2x_2^3x_3^2x_4^2x_5 + x_1^2x_2^2x_3^3x_4^2x_5$	3
436521	$x_1^4x_2^2x_3^2x_4^2x_5 + x_1^3x_2^3x_3^2x_4^2x_5 + x_1^3x_2^2x_3^3x_4^2x_5$	3
364521	$x_1^4x_2^2x_3^2x_4^2x_5 + x_1^3x_2^3x_3^2x_4^2x_5 + x_1^2x_2^4x_3^2x_4^2x_5$	3
634521	$x_1^5x_2^2x_3^2x_4^2x_5$	1
463521	$x_1^4x_2^3x_3^2x_4^2x_5 + x_1^3x_2^4x_3^2x_4^2x_5$	2
643521	$x_1^5x_2^3x_3^2x_4^2x_5$	1
356421	$x_1^3x_2^3x_3^2x_4^2x_5 + x_1^3x_2^2x_3^3x_4^2x_5 + x_1^2x_2^3x_3^3x_4^2x_5$	3
536421	$x_1^4x_2^3x_3^2x_4^2x_5 + x_1^4x_2^2x_3^3x_4^2x_5$	2
365421	$x_1^4x_2^3x_3^2x_4^2x_5 + x_1^3x_2^4x_3^2x_4^2x_5 + x_1^4x_2^2x_3^3x_4^2x_5 + x_1^3x_2^3x_3^3x_4^2x_5 + x_1^2x_2^4x_3^3x_4^2x_5$	5
635421	$x_1^5x_2^3x_3^2x_4^2x_5 + x_1^5x_2^2x_3^3x_4^2x_5$	2
563421	$x_1^4x_2^4x_3^2x_4^2x_5$	1
653421	$x_1^5x_2^4x_3^2x_4^2x_5$	1
456321	$x_1^3x_2^3x_3^3x_4^2x_5$	1
546321	$x_1^4x_2^3x_3^3x_4^2x_5$	1
465321	$x_1^4x_2^3x_3^3x_4^2x_5 + x_1^3x_2^4x_3^3x_4^2x_5$	2
645321	$x_1^5x_2^3x_3^3x_4^2x_5$	1
564321	$x_1^4x_2^4x_3^3x_4^2x_5$	1
654321	$x_1^5x_2^4x_3^3x_4^2x_5$	1

Appendix C: Facet ridge diagrams for the RC-graphs of permutations up to five elements.

Table 3: Facet ridge diagrams for the symmetric groups $\mathfrak{S}_1$ through $\mathfrak{S}_5$

Permutation	Dimension	Facet ridge diagram
1	-1	●
12	0	●
21	-1	●
123	2	●
213	1	●
132	1	●—●
312	0	●
231	0	●
321	-1	●
1234	5	●
2134	4	●
1324	4	●—●
3124	3	●
2314	3	●
3214	2	●
1243	4	
2143	3	
1423	3	●—●—●
4123	2	●
2413	2	●—●
4213	1	●
1342	3	●—●—●
3142	2	●—●
1432	2	
4132	1	●—●
3412	1	●
4312	0	●
2341	2	●
3241	1	●
2431	1	●—●
4231	0	●
3421	0	●
4321	-1	●

Permutation	Dimension	Facet Ridge Diagram
12345	9	
21345	8	
13245	8	
31245	7	
23145	7	
32145	6	
12435	8	
21435	7	
14235	7	
41235	6	
24135	6	
42135	5	
13425	7	
31425	6	
14325	6	
41325	5	
34125	5	
43125	4	
23415	6	
32415	5	
24315	5	
42315	4	
34215	4	
43215	3	

Permutation	Dimension	Facet Ridge Diagram
12354	8	
21354	7	
13254	7	Prism 
31254	6	
23154	6	
32154	5	
12534	7	
21534	6	
15234	6	
51234	5	
25134	5	
52134	4	
13524	6	
31524	5	
15324	5	
51324	4	
35124	4	
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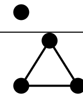
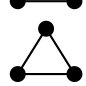
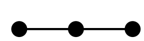
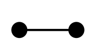
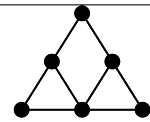
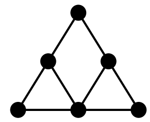
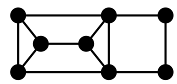
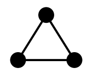
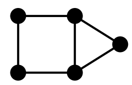
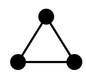
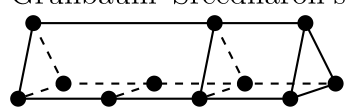
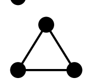
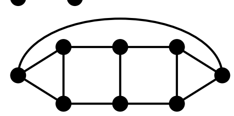
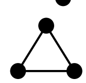
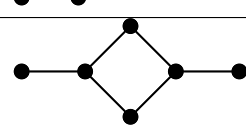
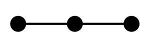
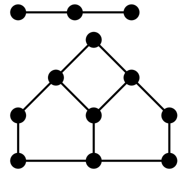
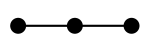
23514	5	
32514	4	
25314	4	
52314	3	
35214	3	
53214	2	

Permutation	Dimension	Facet Ridge Diagram
12453	7	
21453	6	
14253	6	
41253	5	
24153	5	
42153	4	
12543	6	Grunbaum–Sreedharon's P_5^{7*} [23, Table 1]
21543	5	Grunbaum–Sreedharon's P_5^{7*} [23, Table 1]
15243	5	
51243	4	
25143	4	
52143	3	
14523	5	
41523	4	
15423	4	
51423	3	
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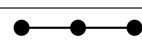
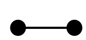
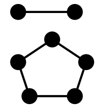
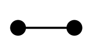
24513	4	
42513	3	
25413	3	
52413	2	
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54213	1	

Permutation	Dimension	Facet Ridge Diagram
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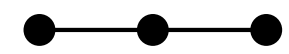
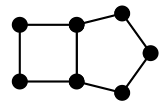
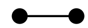
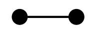
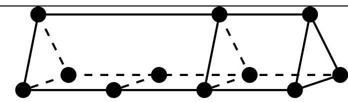
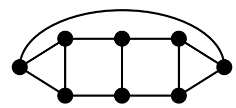
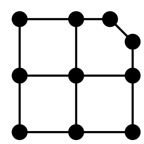
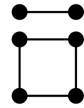
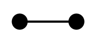
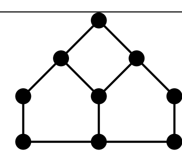
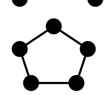
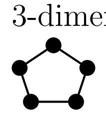
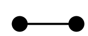
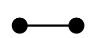
31452	5	
14352	5	
41352	4	
34152	4	
43152	3	
13542	5	
31542	4	
15342	4	
51342	3	
35142	3	
53142	2	
14532	4	
41532	3	
15432	3	3-dimensional associahedron, Mèszaros example
51432	2	
45132	2	
54132	1	
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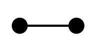
43512	2	
35412	2	
53412	1	
45312	1	
54312	0	

Permutation	Dimension	Facet Ridge Diagram
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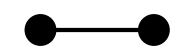
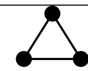
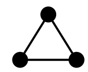
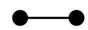
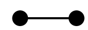
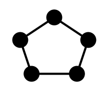
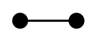
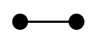
32451	4	
24351	4	
42351	3	
34251	3	
43251	2	
23541	4	
32541	3	
25341	3	
52341	2	
35241	2	
53241	1	
24531	3	
42531	2	
25431	2	
52431	1	
45231	1	
54231	0	
34521	2	
43521	1	
35421	1	
53421	0	
45321	0	
54321	-1	

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