Methods of Computing Graph Gonalities

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Methods of Computing Graph Gonalities

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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ABSTRACT OF DISSERTATION

Methods of Computing Graph Gonalities

Chip firing is a category of games played on graphs. The gonality of a graph tells us how many chips are needed to win one variation of the chip firing game. The focus of this dissertation is to provide a variety of new strategies to compute the gonality of various graph families. One family of graphs which this dissertation is particularly interested in is rook graphs. Rook graphs are the Cartesian product of two or more complete graphs and we prove that the gonality of two dimensional rook graphs is the expected value of $(n-1)m$ where n is the size of the smaller complete graph and m is the size of the larger.

KEYWORDS: graph theory, chip firing, combinatorics

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July 19, 2023

Methods of Computing Graph Gonalities

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July 19, 2023
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Dedicated to my loving wife Chamani.
Without your love and support I would not be the mathematician, or the man I am today.

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Chapter 1 Introduction

In [3] Baker and Norine describe the divisor theory of graphs, which is a discrete analogue for the divisor theory of algebraic curves. Divisor theory of graphs is often described as a chip firing game in which a player is given a graph which some number of chips on each vertex, and the player can fire a vertex by transferring one chip away from the vertex along each of its edges. The player wins the game if they can get to a configuration where no vertex has a negative number of chips. The gonality $\text{gon}(G)$ of a graph G is a graph invariant that tells us the fewest number of chips the player would require such that one chip can be stolen and the player could still win.

The primary motivation for computing the gonality of graphs is to better understand the gonality of algebraic curves, which is the minimal degree of a nonconstant rational map from that curve to the projective line. Given some algebraic curve C, we can degenerate it to a union of lines C' and then produce the dual graph of C' which we will call G. Then the minimum gonality over all refinements of G is a lower bound for the gonality of C.

In this paper we introduce multiple new methods that can be used to compute the gonality of many different families of graphs. The first half of this paper is focused on the scramble number of a graph, a newly invented graph invariant denoted $\text{sn}(G)$. The main result from this section is the following.

Theorem 1.0.1. *For any graph G, we have $\text{sn}(G) \leq \text{gon}(G)$.*

Prior to the development of the scramble number, the treewidth of a graph, denoted $\text{tw}(G)$, was shown to be a lower bound on gonality in [9]. The scramble number is in fact a tighter lower bound than treewidth. Because of this improvement, we can use the scramble number to compute the gonality of various graph families which were previously unknown. Many of these graphs arrive naturally from algebraic geometry. We will also discuss interesting properties about the scramble number, including the fact that it is invariant under graph refinement, something that is not true for graph gonality.

The second half of this paper is focused on "Column Equitable Divisor" proof methods. This style of proof can be used to compute the gonality of graphs with a highly regular grid structure. This method is used to compute the gonality of two dimensional rook graphs. One motivation is that rook graphs are the dual graphs of a certain degeneration of complete intersection curves. In [9] van Dobben de Bruyn and Gijswijt raise the question of computing the gonality of $n-$dimensional cubes Q_n, which are examples of rook graphs. In [2] Aidun and Morrison show that $\text{gon}(K_n \square K_m) = (n-1)m$ if $n \leq m$ and $n \leq 5$. One of the main results of this paper is the full generalization of this theorem.

Theorem 1.0.2. *If $n \leq m$, $\text{gon}(K_n \square K_m) = (n-1)m$.*

This result matches the lower bound for the gonality of complete intersection curves given by Lazarsfeld in [6], where Exercise 4.12 shows the complete intersection of hypersurfaces of degrees $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{r-1}$ has gonality $d \geq (a_1 - 1)a_2 \cdots a_{r-1}$. While Theorem 1.0.2 is a significant result, it does not preclude the possibility of some refinement of a rook graph having smaller gonality. However in Chapter 3 we also prove the following.

Theorem 1.0.3. *If* $m \geq (n-2)(n-1)$, $\operatorname{sn}(K_n \square K_m) = (n-1)m$.

This shows that, at least in cases where m is sufficiently larger than n, the gonality of $K_n \square K_m$ does not decrease under any refinement.

In this paper, we also compute the scramble number of some three dimensional rook graphs. In chapter 3, we prove the following result.

Theorem 1.0.4. *Let* $2 \leq n \leq m$, *then* $\operatorname{sn}(K_2 \square K_n \square K_m) = nm$.

The gonality of this family of rook graphs was known to be at most nm, and therefore this result also computes the gonality of all three dimensional rook graphs where the smallest dimension is 2. Further study is required to determine if any other three dimensional rook graphs have a scramble number matching its gonality.

In Chapter 4, we also explore higher gonalities of two dimensional rook graphs. There are not many families of graphs in which higher gonalities are currently known. However, for two dimensional rook graphs, we prove the following.

Theorem 1.0.5. *Let* $n, m \geq 2$, *then* $\operatorname{gon}_2(K_n \square K_m) = nm - 1$ *and* $\operatorname{gon}_3(K_n \square K_m) = nm$.

It is not currently known if the 4-gonality behaves nicely for these graphs as it does with the 2 and 3-gonalities.

Finally, at the end of Chapter 4 we explore queen graphs and use a row equitable divisor argument to prove the following result.

Theorem 1.0.6. *Let* $2 \leq n \leq m$, *then we have*

$$\operatorname{gon}(Q_{n,m}) = \begin{cases} 3 & \text{if } n = m = 2 \\ 7 & \text{if } n = m = 3 \\ n(m-1) & \text{otherwise.} \end{cases}$$

Chapter 2 Preliminaries

We begin by establishing terminology and giving background results.

2.0.1 Graphs

For the entirety of this paper, we will assume that our graphs are connected and without loops. However, multiple edges between vertices are allowed. Given a graph G, we denote the vertex set by $V(G)$ and the edge set by $E(G)$. If $A \subseteq V(G)$ then the complement of A will be denoted as A^c. A *subgraph* of a graph G is a graph that can be obtained from G by deleting edges and deleting isolated vertices. A *minor* of a graph G is a graph that can be obtained from G by contracting edges, deleting edges, and deleting isolated vertices.

Definition 2.0.1. A partition of the vertices into two sets, (A, A^c) is referred to as a *cut*. The *cut-set* $E(A, A^c)$ is the set of edges that have one end in A and the other end in A^c.

Definition 2.0.2. Given a graph G, H is a *minor* of G if H can be obtained by contracting edges of G as well as deleting vertices and edges of G.

Definition 2.0.3. Given two graphs G and H, we can construct a new graph by taking their *Cartesian product* $G\Box H$, with vertex set

$$V(G\Box H) = \{(x,y)|x \in V(G), y \in V(H)\}$$

and edge set

$$E(G\Box H) = \{(x,y_1) \sim (x,y_2)|y_1 \sim y_2 \in E(H)\} \cup \{(x_1,y) \sim (x_2,y)|x_1 \sim x_2 \in E(G)\}.$$

Definition 2.0.4. A *rook graph* is the Cartesian product of 2 or more complete graphs.

The $n \times m$ rook graph $K_n\Box K_m$ can be represented as an $n \times m$ lattice where two lattice points are adjacent if they share either the same row or column. The name rook graph comes from this lattice representation because two vertices are adjacent if they are a rooks move apart.

Similar to the rook graphs we also have the queen graphs in which vertices are adjacent if they are a queen's move apart. we can define queen graphs more formally as follows.

Definition 2.0.5. The *queen graph* $Q_{n,m}$ is the graph with vertex set

$$V(Q_{n,m}) = \{v_{i,j}|i \in [n], j \in [m]\}$$

and edge set

$$E(Q_{n,m}) = \{v_{i,j} \sim v_{i,l}\} \cup \{v_{i,j} \sim v_{k,j}\} \cup \{v_{i,j} \sim v_{k,l}|\ |k-i| = |l-j|\}.$$

2.0.2 Graph Divisors and Chip Firing

In this section we give a brief summary of divisor theory on finite graphs. For a more thorough description, we refer the reader to [4].

Definition 2.0.6. A *divisor* D on a graph G is a $\mathbb{Z}-$linear combination of the vertices in G, or alternatively, an integer vector in $\mathbb{Z}^{V(G)}$.

Divisors on a graph are often described as stacks of poker chips on each vertex, where a negative number on a vertex is thought of as a debt. Because of this, divisors are sometimes referred to as chip configurations.

Definition 2.0.7. The *degree* of a divisor, denoted $\deg(D)$, is the sum of all coordinates of the vector $D \in \mathbb{Z}^{V(G)}$, or simply the sum of all chips and debts.

We "fire" a vertex v by transferring one chip along each edge connected to v, away from that fired vertex. The number of chips v loses is equal to the degree of v, and every vertex adjacent to v will gain one chip. If two or more vertices are fired, the resulting divisor will be the same regardless of what order the vertices were fired in. Furthermore, if we obtain D' by starting with divisor D and firing some vertex subset A, then we can obtain D from D' by firing A^c. This imposes equivalence classes on the set of divisors of a fixed degree, where two divisors are equivalent if and only if there are some series of chip fires apart from one another.

Definition 2.0.8. A divisor is *effective* if all vertices have a non-negative number of chips, and we say a divisor is *effective away from* v if all vertices, apart from $v \in V(G)$, have a non-negative number of chips.

Definition 2.0.9. Given an effective divisor D, the *support of the divisor* denoted $\mathrm{Supp}(D)$, is the set of vertices with a positive number of chips.

Example 2.0.10. In Figure 2.1, we see two equivalent divisors of degree 2. The divisor D' is effective while D is not.

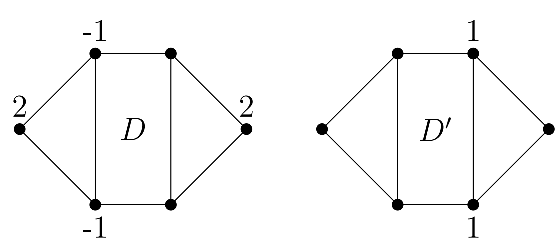

Figure 2.1: Two equivalent divisors D and D'. D' is obtained from D by firing both vertices which had two chips.

Definition 2.0.11. The following terms are needed to understand Dhar's burning algorithm and the proof of Theorem 1.0.2

- A divisor is *v-reduced* if it is effective away from v, and firing any subset of $V(G) \setminus v$ results in a divisor that is not effective away from v.

- A divisor D has *rank* of at least r if, for every effective divisor E of degree r, $D - E$ is equivalent to an effective divisor.

- The *gonality* of a graph G, denoted $\text{gon}(G)$, is the fewest number of chips needed to construct a divisor of rank 1.

Another way to understand the rank of a divisor is in the context of a chip firing game. If a divisor D has rank r, that means if someone were to steal r chips from anywhere on the graph, even from vertices that have 0 or a debt of chips, then there is some series of chip firings one could perform to get back an effective divisor. It also means that there is at least one way to steal $r + 1$ chips that would make getting back to an effective divisor impossible.

Example 2.0.12. In Figure 2.2, we see a divisor of rank 1. If a chip is stolen from the center vertex, the divisor will still be effective. If a chip is stolen from a left vertex, we can fire the center vertex along with the two vertices on the right to obtain an effective divisor. We know the rank of this divisor cannot be greater than 1 because stealing a chip from a vertex on the right and stealing another chip from a vertex on the left results in a divisor that is not equivalent to any effective divisors. There are no ways to construct a rank 1 divisor on this graph with fewer than 2 chips so the gonality of the graph is therefore 2.

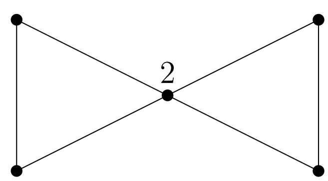

Figure 2.2: A divisor with rank 1.

Lemma 2.0.13. *If a divisor D is v-reduced, and v has 0 or fewer chips, then D does not have positive rank.*

Proof. Let the stolen chip be from the specified vertex v. Then v has a negative number of chips and since D is $v-$reduced, firing any other set of vertices will result in a non-effective divisor. □

Chapter 3 Brambles and Scrambles

We make the following definition.

Definition 3.0.1. A *scramble* in a graph G is a set $\mathscr{S} = \{E_1, \ldots, E_n\}$ of connected subsets of $V(G)$.

We will often refer to the subsets E_i as *eggs*. Scrambles with certain properties have been studied extensively in the graph theory literature.

Definition 3.0.2. A *bramble* is a scramble $\mathscr{S}$ with the property that $E \cup E'$ is connected for every pair $E, E' \in \mathscr{S}$. It is called a *strict bramble* if every pair of elements $E, E' \in \mathscr{S}$ has nonempty intersection.

Definition 3.0.3. A set $C \subseteq V(G)$ is a *hitting set* of a scramble $\mathscr{S}$ if $C \cap E \neq \emptyset$ for all $E \in \mathscr{S}$.

The *order* of a bramble $\mathscr{B}$ is the minimum size of a hitting set. The *bramble number* of a graph G is the maximum order of a bramble in G, and is denoted $\mathrm{bn}(G)$. A result of Seymour and Thomas shows that the bramble number of a graph is closely related to another well-known graph invariant, known as the *treewidth* $\mathrm{tw}(G)$. In particular, $\mathrm{tw}(G) = \mathrm{bn}(G) - 1$ for any graph G [8]. Here, we define some related notions for more general scrambles.

Definition 3.0.4. We say that (A, A^c) is an *egg cut* of our scramble $\mathscr{S}$ if there are two eggs E_1, E_2 such that $E_1 \subseteq A, E_2 \subseteq A^c$. The set of edges $E(A, A^c)$ is referred to as the *egg cut-set* or simply the *cut-set*, when the context is clear that we are talking about an egg cut.

Definition 3.0.5. Given a scramble $\mathscr{S}$ with minimal hitting set C and minimal egg cut-set $E(A, A^c)$, the *order* of $\mathscr{S}$, denoted $||\mathscr{S}||$, is $\min\{|C|, |E(A, A^c)|\}$. In the case where $\mathscr{S}$ is a strict bramble and does not have any egg cuts we say that the size of the minimal egg cut-set is ∞.

We note the following observations about the scramble order of brambles.

Lemma 3.0.6. *The order of a strict bramble is equal to its scramble order.*

Proof. Let $\mathscr{B}$ be a strict bramble of order k. By definition, the scramble order of $\mathscr{B}$ is $\min\{k, \infty\} = k$ □

Lemma 3.0.7. *Let $\mathscr{B}$ be a bramble of order k. Then the scramble order of $\mathscr{B}$ is either k or $k - 1$.*

Proof. By definition, there is a hitting set $C \subset V(G)$ of size k that covers $\mathscr{B}$, and no such set of size less than k. The scramble order of $\mathscr{B}$ is therefore at most k. By [9, Lemma 2.3], if $E, E' \in \mathscr{B}$ and $A \subset V(G)$ is a subset such that $E \subseteq A$ and $E' \subseteq A^c$, then $|E(A, A^c)| \geq k - 1$. It follows that the scramble order of $\mathscr{B}$ is at least $k - 1$. □

Corollary 3.0.8. *For any graph G, we have* $\mathrm{tw}(G) \leq \mathrm{sn}(G)$.

Proof. Let $\mathscr{B}$ be a bramble of maximum order k in G. By [8], we have $\mathrm{tw}(G) = k-1$. By Lemma 3.0.7, the scramble order of $\mathscr{B}$ is at least $k-1$, hence $\mathrm{sn}(G) \geq k-1$. □

3.1 Properties of The Scramble Number

We now prove our main result about the scramble number. Namely, that the scramble number of a graph is a lower bound for the graph's gonality. Our argument follows closely that of [9, Theorem 2.1], which shows that the treewidth of a graph is a lower bound for the graph's gonality. Indeed, we defined the scramble number with the specific goal of stating [9, Theorem 2.1] in its maximum generality.

Theorem 1.0.1. *For any graph G, we have* $\mathrm{sn}(G) \leq \mathrm{gon}(G)$.

Proof. Let $\mathscr{S}$ be a scramble on G, and let D' be an effective divisor of positive rank on G. We will show that $\deg(D') \geq ||\mathscr{S}||$. Among the effective divisors equivalent to D', we choose D such that $\mathrm{Supp}(D)$ intersects a maximum number of eggs in $\mathscr{S}$. If $\mathrm{Supp}(D)$ is a hitting set for $\mathscr{S}$ then, by definition,

$$\deg(D) \geq |\mathrm{Supp}(D)| \geq ||\mathscr{S}||.$$

Conversely, suppose that there is some egg $E \in \mathcal{S}$ that does not intersect $\mathrm{Supp}(D)$, and let $v \in E$. Since D has positive rank and $v \notin \mathrm{Supp}(D)$, it follows that D is not $v-$reduced. Therefore there exists a chain

$$\emptyset \subsetneq U_1 \subseteq \cdots \subseteq U_k \subset V(G) \smallsetminus \{v\}$$

and a sequence of effective divisors $D_0, D_1, \ldots, D_k$ such that:

1. $D_0 = D$,
2. D_k is v-reduced, and
3. D_i is obtained from D_{i-1} by firing the set U_i, for all i.

Since D has positive rank, we see that $v \in \mathrm{Supp}(D_k)$ and hence $\mathrm{Supp}(D_k)$ intersects E. By assumption, $\mathrm{Supp}(D_k)$ does not intersect more eggs than $\mathrm{Supp}(D)$, so there is at least one egg E' that intersects $\mathrm{Supp}(D)$ but not $\mathrm{Supp}(D_k)$. Let $i \leq k$ be the smallest index such that there is some $E' \in \mathscr{S}$ that intersects $\mathrm{Supp}(D)$ but not $\mathrm{Supp}(D_i)$. Then $E' \cap \mathrm{Supp}(D_{i-1}) \neq \emptyset$ and $E' \cap \mathrm{Supp}(D_i) = \emptyset$. By [9, Lemma 2.2], it follows that $E' \subseteq U_i$.

Again, by assumption, $\mathrm{Supp}(D_{i-1})$ does not intersect more eggs than $\mathrm{Supp}(D)$, so $\mathrm{Supp}(D_{i-1})$ does not intersect E. Let $j \geq i$ be the smallest index such that $E \cap \mathrm{Supp}(D_{j-1}) = \emptyset$ and $E \cap \mathrm{Supp}(D_j) \neq \emptyset$. Since D_{j-1} can be obtained from D_j by firing U_j^c, we see that $E \subseteq U_j^c \subseteq U_i^c$. Since $E \subseteq U_i^c$ and $E' \subseteq U_i$, it follows by the definition of a scramble that $|E(U_i, U_i^c)| \geq ||\mathscr{S}||$. Since

$$\deg(D_{i-1}) \geq \sum_{u \in U_i} D_{i-1}(u) \geq |E(U_i, U_i^c)|,$$

we have

$$\deg(D_{i-1}) \geq ||\mathscr{S}||.$$

□

One of the major advantages of the treewidth bound from [9] is that the treewidth is minor monotone. In other words, if G' is a graph minor of a graph G, then $\text{tw}(G') \leq \text{tw}(G)$. This is not true of the scramble number, as the following example shows.

Example 3.1.1. Let G be the graph depicted in Figure 3.1. If v is the green vertex, then the divisor $3v$ (the divisor with 3 chips on v) has positive rank. It follows that the gonality of G is at most 3, and thus the scramble number of G is at most 3 by Theorem 1.0.1.

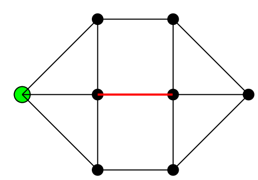

Figure 3.1: A graph G with scramble number 3.

Now, let G' be the graph pictured in Figure 3.2, obtained by contracting the red edge in G. The 4 colored subsets are the eggs of a scramble $\mathscr{S}$, which we now show has scramble order 4. Because the 4 eggs are disjoint, there is no hitting set of size less than 4. Now, let $A \subset V(G')$ be a set with the property that both A and A^c contain an egg. By exchanging the roles of A and A^c, we may assume that A contains the center red vertex. If A consists solely of this vertex, then $|E(A, A^c)| = 6$. Otherwise, A contains some, but not all, of the vertices on the hexagonal outer ring. We then see that $E(A, A^c)$ contains at least two edges in the hexagonal outer ring, and at least two edges that have the center red vertex as an endpoint. Thus, $|E(A, A^c)| \geq 4$.

While the scramble number is not minor monotone, it is subgraph monotone.

Proposition 3.1.2. *If G' is a subgraph of G, then $\text{sn}(G') \leq \text{sn}(G)$.*

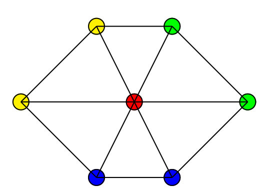

Figure 3.2: The graph G' is a graph minor of G with higher scramble number.

Proof. Let $\mathscr{S}'$ be a scramble on G', and let $\mathscr{S}$ be the scramble on G with the same eggs as $\mathscr{S}'$ on G'. We will show that $||\mathscr{S}|| \geq ||\mathscr{S}'||$. If $C \subset V(G)$ is a hitting set for $\mathscr{S}$, then $C \cap V(G')$ is a hitting set for $\mathscr{S}'$. Now, let A be a subset of $V(G)$ such that A and A^c both contain eggs of $\mathscr{S}$. Then $A \cap V(G')$ is a subset of $V(G')$ with the property that both it and its complement contain eggs of $\mathscr{S}'$, and $|E(A, A^c)| \geq |E(A \cap V(G'), A^c \cap V(G'))|$. It follows that $||\mathscr{S}|| \geq ||\mathscr{S}'||$. □

The scramble number is also invariant under refinement.

Proposition 3.1.3. *If G' is a refinement of G, then* $\mathrm{sn}(G) = \mathrm{sn}(G')$.

Proof. By induction, it suffices to consider the case where G has one fewer vertex than G'. Let v and w be adjacent vertices in G, and let G' be the graph obtained by subdividing an edge between v and w, introducing a vertex u between them.

First, we will show that $\mathrm{sn}(G) \leq \mathrm{sn}(G')$. To see this, let $\mathscr{S}$ be a scramble on G. For each egg $E \in \mathscr{S}$, we define a connected subset $E' \subset V(G')$ as follows. If $v \notin E$, then $E' = E$, and if $v \in E$, then $E' = E \cup \{u\}$. Let

$$\mathscr{S}' = \{E' | E \in \mathscr{S}\}.$$

We will show that $||\mathscr{S}'|| \geq ||\mathscr{S}||$.

Let $C \subset V(G')$ be a hitting set for $\mathscr{S}'$. If $u \notin C$, then C is also a hitting set for $\mathscr{S}$. On the other hand, if $u \in C$, then since every egg in $\mathscr{S}'$ that contains u also contains v, the set $C' = C \cup \{v\} \smallsetminus \{u\}$ is a hitting set for $\mathscr{S}'$ with the property that $u \notin C'$ and $|C'| \leq |C|$. Now, let A be a subset of $V(G')$ such that both A and A^c contain eggs of $\mathscr{S}'$. By exchanging A and A^c, we may assume that $u \notin A$. We may then think of A also as a subset of $V(G)$ with the property that both A and A^c contain eggs of $\mathscr{S}$. If both v and w are contained in A, then the number of edges leaving A in $V(G)$ is 1 fewer than the number of edges leaving A in $V(G')$. Otherwise, these two numbers are equal. It follows that $||\mathscr{S}'|| \geq ||\mathscr{S}||$.

We now show that $\mathrm{sn}(G) \geq \mathrm{sn}(G')$. To see this, let $\mathscr{S}'$ be a scramble on G' of maximal scramble order. If $\mathrm{sn}(G) = 1$, then by Corollary 3.1.6 below, we see that G is a tree. It follows that G' is a tree as well, and $\mathrm{sn}(G') = 1$ by another application of Corollary 3.1.6. We may therefore assume that $\mathrm{sn}(G) \geq 2$, and for contradiction that $||\mathscr{S}'|| \geq 3$.

If every egg in $\mathscr{S}'$ contains u, then $\mathscr{S}'$ has a hitting set of size 1, a contradiction. It follows that if $\{u\} \in \mathscr{S}'$, then the set $A = \{u\}$ has the property that both A and A^c contain eggs of $\mathscr{S}'$. Thus, $||\mathscr{S}'|| \leq |E(A, A^c)| = 2$, another contradiction. We may therefore assume that $\{u\} \notin \mathscr{S}'$. Let

$$\mathscr{S} = \{E' \cap V(G) | E' \in \mathscr{S}'\}.$$

We will show that $||\mathscr{S}|| \geq ||\mathscr{S}'||$.

Let $C \subset V(G)$ be a hitting set for $\mathscr{S}$. Since $\{u\} \notin \mathscr{S}'$, we see that C is also a hitting set for $\mathscr{S}'$. Now, let A be a subset of $V(G)$ with the property that both A and A^c contain eggs of $\mathscr{S}$. As above, define the set A' as follows. If $v \notin A$, then $A' = A$, and if $v \in A$, then $A' = A \cup \{u\}$. We see that $|E(A, A^c)| = |E(A', A'^c)|$. It follows that $||\mathscr{S}|| \geq ||\mathscr{S}'||$. □

Example 3.1.4. The graph on the left in Figure 3.3 has gonality 2. By Theorem 1.0.1, its scramble number is at most 2. Since it is not a tree, by Corollary 3.1.6 below, its scramble number is exactly 2.

On the other hand, the graph on the right has gonality 3. Since it is a refinement of the graph on the left, however, by Proposition 3.1.3 the two graphs have the same scramble number. Thus, the graph on the right is an example where the gonality and scramble number disagree.

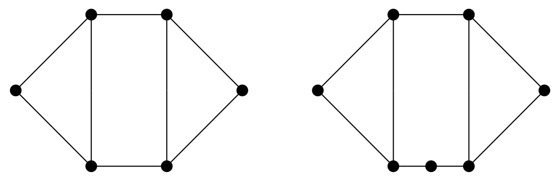

Figure 3.3: Two graphs with the same scramble number, but different gonalities.

We close out this section with some observations about graphs of low scramble number.

Lemma 3.1.5. *If G is a cycle, then* $\mathrm{sn}(G) = 2$.

Proof. For any $v \in V(G)$ consider the scramble $\mathscr{S} = \Big\{\{v\}, V(G) \smallsetminus \{v\}\Big\}$. Because the two eggs are disjoint, any hitting set has size at least two. If A is a subset of the vertices such that both A and A^c contain eggs, then either A or A^c is equal to $\{v\}$. Since $|E(A, A^c)| = 2$, we see that $||\mathscr{S}|| = 2$. There can be no scramble of higher order because, if $A \subsetneq V(G)$ is a connected subset, then $|E(A, A^c)| = 2$. □

Corollary 3.1.6. *The scramble number of a graph G is 1 if and only if G is a tree.*

Proof. If G is a tree, then

$$1 = \mathrm{tw}(G) \leq \mathrm{sn}(G) \leq \mathrm{gon}(G) = 1,$$

so $\mathrm{sn}(G) = 1$. On the other hand, if G is not a tree, then it contains a cycle. By Proposition 3.1.2, the scramble number of G is at least that of the cycle, and by Lemma 3.1.5, the scramble number of the cycle is 2. □

3.2 Using the Scramble Number to Compute the Gonality of Cartesian Product Graphs

In this section, we compute the scramble numbers and gonalities of several well-known families of graphs. The purpose of this section is to illustrate the advantages of the scramble number as a tool for computing gonality, as the constructions are relatively simple in comparison to the preexisting literature.

The following examples all arise as Cartesian products of graphs. Recall that the Cartesian product of two graphs G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and an edge between (u_1, u_2) and (v_1, v_2) if either $u_1 = v_1$ and there is an edge between u_2 and v_2, or $u_2 = v_2$ and there is an edge between u_1 and v_1. For a fixed vertex $v \in G_1$, we refer to the set

$$C_v = \Big\{(v, w) \in V(G_1 \square G_2) | w \in G_2\Big\}$$

as a *column*. Similarly, for $w \in G_2$, we refer to the set

$$R_w = \Big\{(v, w) \in V(G_1 \square G_2) | v \in G_1\Big\}$$

as a *row*. A bound on the gonality of Cartesian products can be found in [2].

Proposition 3.2.1. *[2, Proposition 1.3] For any two graphs G_1 and G_2,*

$$\mathrm{gon}(G_1 \square G_2) \leq \min\Big\{\mathrm{gon}(G_1)|V(G_2)|, \mathrm{gon}(G_2)|V(G_1)|\Big\}.$$

We provide several examples where this bound is achieved. It is a standard result that the $m \times n$ grid graph has treewidth $\min\{m, n\}$, and it is shown in [9] that such graphs have gonality $\min\{m, n\}$ as well. A grid graph is an example of the product of two trees, a family of graphs whose gonality is computed in [2]. We reproduce this result here using the scramble number.

Proposition 3.2.2. *[2, Proposition 3.2] If T_1 and T_2 are trees, then*

$$\mathrm{gon}(T_1 \square T_2) = \mathrm{sn}(T_1 \square T_2) = \min\Big\{|V(T_1)|, |V(T_2)|\Big\}.$$

Proof. By Proposition 3.2.1, the gonality of $T_1 \square T_2$ is at most $\min\{|V(T_1)|, |V(T_2)|\}$. We therefore seek to bound the gonality from below. By Theorem 1.0.1, it suffices to construct a scramble of scramble order $\min\{|V(T_1)|, |V(T_2)|\}$.

Let $\mathscr{S}$ be the set of columns in $T_1 \square T_2$. Any row R_w is a hitting set for $\mathscr{S}$, and $|R_w| = |V(T_1)|$. Moreover, if $v \in T_1$ is a leaf, then $|E(C_v, C_v^c)| = |V(T_2)|$. It follows that

$$||\mathscr{S}|| \leq \min\Big\{|V(T_1)|, |V(T_2)|\Big\}.$$

Since the number of columns is $|V(T_1)|$ and they are disjoint, there is no hitting set of size less than $|V(T_1)|$. Now, let A be a subset of $V(T_1 \square T_2)$ with the property that both A and A^c contain a column. Then every row of $T_1 \square T_2$ contains a vertex in A and a vertex in A^c, so every row contains an edge in $E(A, A^c)$. It follows that $|E(A, A^c)|$ is greater than or equal to the number of rows, which is $|V(T_2)|$. It follows that

$$||\mathscr{S}|| \geq \min\Big\{|V(T_1)|, |V(T_2)|\Big\}.$$

□

In [1], the authors compute the treewidth of the *stacked prism graphs* $Y_{m,n}$, the cartesian product of a cycle with m vertices and a path with n vertices. They show that the gonality of $Y_{m,n}$ is equal to its treewidth, except in the special case where

$m = 2n$. We prove the following generalization, which holds even in this special case. Even in the cases where the gonality has been previously computed, we believe that our constructions, using scrambles rather than brambles, are much simpler. For this reason, we have treated these graphs for all m and n uniformly.

Proposition 3.2.3. *If C is a cycle and T is a tree, then*

$$\text{gon}(C\square T) = \text{sn}(C\square T) = \min\Big\{|V(C)|, 2|V(T)|\Big\}.$$

Proof. By Proposition 3.2.1, we have $\text{gon}(C\square T) \leq \min\{|V(C)|, 2|V(T)|\}$. We now compute a lower bound. By Theorem 1.0.1, it suffices to construct a scramble of scramble order $\min\{|V(C)|, 2|V(T)|\}$.

Again, we let $\mathscr{S}$ be the set of columns in $C\square T$. (See, for example, Figure 3.4.) Any row R_w is a hitting set for $\mathscr{S}$, and $|R_w| = |V(C)|$. Moreover, for any $v \in C$ we have $|E(C_v, C_v^c)| = 2|V(T)|$. It follows that

$$||\mathscr{S}|| \leq \min\Big\{|V(C)|, 2|V(T)|\Big\}.$$

Since the number of columns is $|V(C)|$ and they are disjoint, there is no hitting set of size less than $|V(C)|$. Now, let A be a subset of $V(C\square T)$ with the property that both A and A^c contain a column. Then every row of $C\square T$ contains a vertex in A and a vertex in A^c, so every row contains at least two edges in $E(A, A^c)$. It follows that $|E(A, A^c)|$ is greater than or equal to twice the number of rows, which is $|V(T)|$. It follows that

$$||\mathscr{S}|| \geq \min\Big\{|V(C)|, 2|V(T)|\Big\}.$$

□

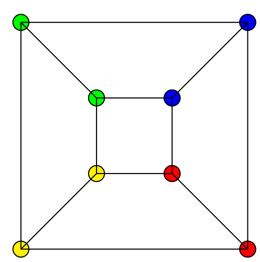

Figure 3.4: The stacked prism graph $Y_{4,2}$ with a scramble of scramble order 4. Note that, by [1, Proposition 3.3], the treewidth of $Y_{4,2}$ is only 3.

Note that in the special case where $m = 2n$, Proposition 3.2.3 shows that the scramble number of the stacked prism graph $Y_{m,n}$ can be strictly greater than the treewidth. In [1], the authors also compute the treewidth of the *toroidal grid graphs* $T_{m,n}$, the product of a cycle with m vertices and a cycle with n vertices. They further show that the gonality of $T_{m,n}$ is equal to its treewidth, except in the special cases where $m = n$ or $m = n \pm 1$. As with the stacked prism graphs, we compute the gonality of these graphs for all m and n uniformly, including the cases not covered in [1].

Proposition 3.2.4. *We have*

$$\text{gon}(T_{m,n}) = \text{sn}(T_{m,n}) = \min\{2m, 2n\}.$$

Proof. By Proposition 3.2.1, $\text{gon}(T_{m,n}) \leq \min\{2m, 2n\}$, so we will compute a lower bound. By Theorem 1.0.1, it suffices to construct a scramble of scramble order $\min\{2m, 2n\}$.

Let $\mathscr{S}$ be the set of columns in $T_{m,n}$ with one vertex removed. (See, for example, Figure 3.5.) The union of any two rows is a hitting set for $\mathscr{S}$ of size $2m$. Moreover, for any vertex v in the cycle of length m, we see that both C_v and C_v^c contain an egg, and we have $|E(C_v, C_v^c)| = 2n$. It follows that

$$||\mathscr{S}|| \leq \min\{2m, 2n\}.$$

If C is a subset of the vertices of size less than $2m$, then some column contains at most 1 vertex of C, hence C is not a hitting set for $\mathscr{S}$. Now, let A be a subset of $V(T_{m,n})$ with the property that both A and A^c contain eggs. Specifically, suppose that A contains every vertex in column C_v except for possibly (v, w), and that A^c contains every vertex in column $C_{v'}$ except for possibly (v', w'). If $w'' \neq w, w'$ is a vertex in the cycle of length n, then the row $R_{w''}$ contains a vertex in A and a vertex in A^c, so at least two edges in $R_{w''}$ are contained in $E(A, A^c)$. If $(v, w) \notin A$, then the two edges in column C_v with endpoints (v, w) are contained in $E(A, A^c)$, and similarly, if $(v', w') \notin A^c$, then the two edges in column $C_{v'}$ with endpoints (v', w') are contained in $E(A, A^c)$. On the other hand, if $(v, w) \in A$ and $(v', w) \in A^c$, then at least two edges in R_w are contained in $E(A, A^c)$, and similarly, if $(v', w') \in A^c$ and $(v, w') \in A$, then at least two edges in $R_{w'}$ are contained in $E(A, A^c)$. It follows that

$$||\mathscr{S}|| \geq \min\{2m, 2n\}.$$

$\square$

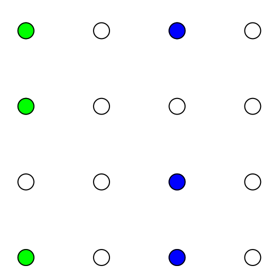

Figure 3.5: Two representative eggs in $T_{4,4}$.

3.3 The Scramble Number of Rook Graphs

Recall that a rook graph is the Cartesian product of two or more complete graphs. The gonality of the complete graph K_n is $n-1$ so by Proposition 3.2.1, $\text{gon}(K_n \square K_m) \leq (n-1)m$ if $n \leq m$. In the next chapter we will show that this bound is actually an equality using other methods. However in the meantime we will use the scramble number to realize this bound for many 2 dimensional rook graphs. We begin this section with the following theorem which will assist us in that computation.

Theorem 3.3.1. *Given a cut* $A \coprod B$ *on* $K_n \square K_m$ *Such that* $|A|, |B| \geq n-1$*, we have* $|E(A,B)| \geq (n-1)m$.

Proof. First, we establish that the proposition holds if either A or B has at most m vertices. Assume without loss of generality that $n-1 \leq |A| \leq m$. Every vertex in $K_n \square K_m$ has degree $n+m-2$, and thus

$$|E(A,B)| = (n+m-2)|A| - 2k,$$

where k is the number of edges with both ends in A. The number k reaches a maximal value of $\binom{|A|}{2}$ when all vertices of A are in a single row or column. Thus we need only to consider the case where A is contained in a single row. Then

$$|E(A,B)| = |A|(m-|A|) + |A|(n-1),$$

where $|A|(m-|A|)$ represents the number of horizontal cut edges between two vertices in the same row, and $|A|(n-1)$ represents the number of vertical cut edges between two vertices contained in the same column.

$$|A|(m-|A|) + |A|(n-1) = |A|(m+n-|A|-1).$$

This product is minimized at the boundary cases when $|A| = n-1$ or $|A| = m$, and the resulting product is $(n-1)m$.

Next, we establish that the proposition holds if both A and B contain an entire row. Without loss of generality, assume the first row is entirely in A and the second row is entirely in B. Then there are m cut edges between those two rows and each of the remaining $(n-2)m$ vertices will have a vertical cut edge between itself and one of the first two rows. Thus $|E(A,B)| \geq (n-1)m$.

Finally, we assume that both $|A|, |B| \geq m+1$, there are $0 \leq i \leq n-2$ rows completely contained in A and no rows completely contained in B. Because $|B| \geq m+1$, there will be at least $i(m+1)$ vertical cut edges between the vertices in B and the i rows in A. The remaining $n-i$ rows contain at least one vertex in A and one vertex in B. Thus each of these rows contains at least $m-1$ horizontal cut edges. Thus

$$|E(A,B)| \geq i(m+1)+(n-i)(m-1) > i(m-1)+(n-i)(m-1) = n(m-1) \geq (n-1)m.$$

□

Lemma 3.3.2. *For all* $m \geq 2$ *we have* $\mathrm{sn}(K_2 \square K_m) = m = \mathrm{gon}(K_2 \square K_m)$

Proof. By Theorem 1.0.1 and Proposition 3.2.1, it suffices to find a scramble of order n. Let $\mathscr{S}$ be the scramble where every vertex is its own egg. Then the minimal hitting set is all of $V(K_2 \square K_m)$. By Theorem 3.3.1, any egg cut of $\mathscr{S}$ will have a cut-set of size greater than or equal to m. Therefore $||\mathscr{S}|| = m$. □

Lemma 3.3.3. *For all* $m \geq 3$*, we have* $\mathrm{sn}(K_3 \square K_m) = 2m = \mathrm{gon}(K_3 \square K_m)$.

Proof. Again by Theorem 1.0.1 and Proposition 3.2.1, it suffices to find a scramble of order $2n$. Let $\mathscr{S}$ be the scramble where the eggs consist of every two adjacent vertices. By Theorem 3.3.1, The minimal egg cut-set size is $2m$. If C is a hitting set of $\mathscr{S}$, then C must contain 2 out of 3 vertices in every column. Then $|C| \geq 2m$ and thus $||\mathscr{S}|| = 2m$. □

Unfortunately, for larger values of n, the scramble number of $K_n \square K_m$ does not always match the gonality, and it becomes increasingly more difficult to compute. A good candidate for a maximal scramble on a rook graph is the scramble where the eggs are all connected subsets of size $n-1$. We will refer to this scramble as $S^*_{n,m}$. By Theorem 3.3.1, the minimal cutset of $S^*_{n,m}$ is always $(n-1)m = \mathrm{gon}(K_n \square K_m)$, so the order of $S^*_{n,m}$ is only less than the gonality if the minimal hitting set is too small. We can find a minimal hitting set indirectly by instead looking for a *maximal avoidance set*, which is the largest subset of vertices that do not contain an entire egg. In the case of $S^*_{n,m}$, we need to avoid all connected subsets of size $n-1$. The complement of a maximal avoidance set is a minimal hitting set. Thus, if a scramble on $K_n \square K_m$ has a maximal avoidance set of size k, then it has a minimal hitting set of size $nm - k$.

Example 3.3.4. We show that $\mathrm{sn}(K_4 \square K_4) = 11$. Note that this is strictly smaller than $\mathrm{gon}(K_4 \square K_4) = 12$.

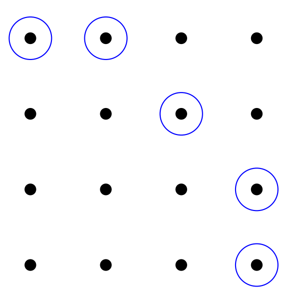

Figure 3.6: A maximal avoidance set of size 5.

In Figure 3.3.4 we see that the 5 circled vertices do not contain a connected subset of 3 or more vertices. However, any collection of 6 vertices must contain a connected subset of 3 vertices. This is because selecting 6 vertices from 4 columns either requires selecting 3 or more vertices from a single column or selecting 2 vertices from 2 separate columns. If the latter of these cases occurs, then either the 4 selected vertices are already connected or every row has one vertex and the next vertex selected will result in a connected subset of 3 vertices.

Therefore the 11 vertices not circled in figure 3.3.4 forms a minimal hitting set for the scramble $S^*_{4,4}$, meaning $||S^*_{4,4}|| = 11$. We then conclude that $\text{sn}(K_n \square K_4) = 11$ because any scramble on this graph with a minimal hitting set larger than 11 would require eggs of size 2 or 1, and any such scramble would have a minimum egg cut set that is size 10 or smaller.

Theorem 1.0.3. *If* $m \geq (n-2)(n-1)$*, then* $\text{sn}(K_n \square K_m) = ||S^*_{n,m}|| = (n-1)m$*.*

Proof. Recall that $S^*_{n,m}$ is the scramble where every connected vertex subset of size $n-1$ is an egg. We show that any set A with $m+1$ vertices cannot be an avoidance set. Given such a set, there must be one column with at least 2 vertices in A. Without loss of generality, we say $(1,1)$ and $(2,1)$ are in A. Since $(1,1) \sim (2,1)$ all of the vertices in A that are in the first two rows will form a connected component, and therefore A can have at most $n-2$ vertices in the first two columns. This means that A has at least $m+1-(n-2)$ vertices in the remaining $n-2$ rows. Since $m \geq (n-2)(n-1)$, we have

$$m + 1 - (n-2) \geq (n-2)(n-1) - (n-2) + 1 = (n-2)^2 + 1,$$

and therefore A must have at least $n-1$ vertices in a single column, meaning that A cannot be an avoidance set. We conclude that any hitting set of $S^*_{n,m}$ must be of size $(n-1)m$ or larger. □

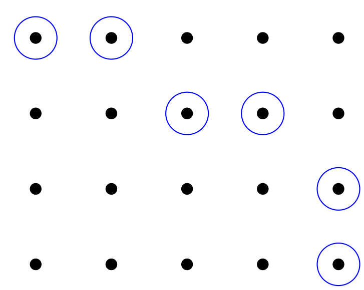

Figure 3.7: An avoidance set of size 6=$m+1$.

Theorem 3.3.5. *If* $m < (n-2)(n-1)$*, then* $||S^*_{n,m}|| < (n-1)m$*.*

Proof. First, we consider the case where $m = k(n-2)+r$ for some nonzero remainder $1 \leq r < n-2$. Then we aim to construct an avoidance set of size $m+1$. In the first k rows, we select $n-2$ vertices per row such that each vertex comes from a unique column. Since $m = k(n-2)+r < (n-2)(n-1)$ we know $k \leq n-2$, and thus there are r columns and at least 2 rows that have no vertices selected. Selecting any set of $r+1$ vertices from these leftover rows and columns will result in an avoidance set of size $m+1$ (see Figure 3.7 for the case $n=4, m=5, k=2, r=1$).

Next, we consider the case where $m = k(n-2)$ for some $k \leq n-2$. We select $n-2$ vertices from each of the first $k-1$ rows, and $n-3$ vertices from row k such that each vertex is from a unique column. From the last remaining column that has not yet had a vertex selected, we select two from rows $k+1, \ldots, n$. The selected vertices then form an avoidance set of size $m+1$.

$\square$

In cases where m and n are relatively close in size, $S^*_{n,m}$ might not be optimal. That is, we can find a scramble of larger order on that graph. This makes computing the scramble number of general rook graphs increasingly difficult as m and n get large.

Example 3.3.6. Given the graph $K_6 \square K_6$, we first consider the scramble $S^*_{6,6}$ consisting of all possible eggs of size 5. This scramble has a hitting set of size 24 since we can construct an avoidance set of size 12 pictured in Figure 3.8. However, if we create a new scramble T^* by augmenting the egg set of $S^*_{6,6}$ to also include all 4-vertex squares, we increase the minimum hitting set to size 27 without decreasing the minimum cut set size. One could also include "S" and "Z" shaped 4-vertex subsets into the egg set without diminishing the size of the minimal cut-set, however, this will not increase the order of the scramble since Figure 3.9 already avoids such eggs. Including any other eggs of size ≤ 4 would decrease the minimal cut set to values less than 27. Thus $\mathrm{sn}(K_6 \square K_6) = 27$.

We note that $K_6 \square K_6$ is the smallest rook graph whose scramble number is strictly greater than the order of $S^*_{n,m}$. Theorem 1.0.3 shows that for $n \leq 5$, $sn(K_n \square K_m) = ||S^*_{n,m}||$ for all but finitely many values of m. These cases can be checked by hand.

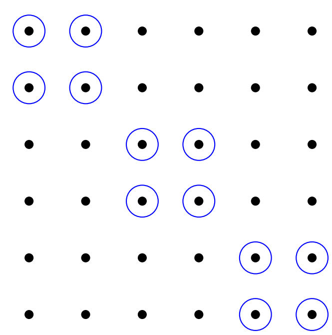

Figure 3.8: A maximal avoidance set of size 12 of the scramble $S^*_{6,6}$.

3.4 Scrambles on 3-Dimensional Rook Graphs

Up until this point, we have only considered 2-dimensional rook graphs. When trying to compute the gonality of graphs which are the product of three or more complete graphs, many of our strategies cannot be immediately generalized. However the

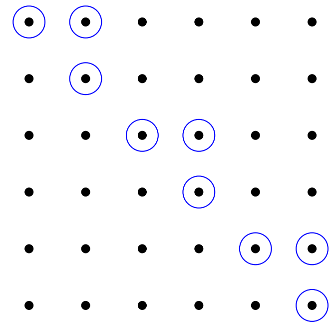

Figure 3.9: A maximal avoidance set of size 9 of the scramble T^*

scramble number can still be used to compute gonality in some circumstances such as in the following theorem.

Theorem 1.0.4. *Let* $2 \leq n \leq m$, *then* $\operatorname{sn}(K_2 \square K_n \square K_m) = \operatorname{gon}(K_2 \square K_n \square K_m) = nm$.

Proof. Consider the scramble $\mathscr{S}$ with an egg set consisting of all connected subsets of n vertices. It suffices to show that $||\mathscr{S}|| = nm$.

We first establish that given any cut $A \coprod B$ such that $|A|, |B| \geq n$ will have $|E(A, B)| \geq nm$. Much of this argument is similar to the proof of Theorem 3.3.1. We begin by considering the case where either A or B has at most m vertices. Assume without loss of generality that $n \leq |A| \leq m$. Every vertex has degree $n + m - 1$, and thus

$$|E(A, B)| = (n + m - 1)|A| - 2k,$$

where k is the number of edges with both ends in A. The value k reaches a maximum value when all vertices of A are in a single copy of K_m. In this case, we would have

$$|E(A, B)| = |A|(m - |A|) + |A|n,$$

where $|A|(m - |A|)$ represents the cut edges contained within the single copy of K_m, and $|A|n$ represents the number of cut edges between two vertices contained in the same copies of K_n or K_2.

$$|A|(m - |A|) + |A|n = |A|(m + n - |A|).$$

This product is minimized at the boundary cases when $|A| = n$ or $|A| = m$, and the resulting product is nm.

Next, we show that the proposition holds if there is at least one copy of $K_2 \square K_n$ with all vertices in A and another copy of $K_2 \square K_n$ with all vertices in B. Without loss of generality, say the first copy of $K_2 \square K_n$ is in A and the second is in B. Between these two copies of $K_2 \square K_n$ there will be $2n$ cut edges. Additionally, for every vertex in the remaining $m - 2$ copies of $K_2 \square K_n$ there will be a cut edge between that vertex and one of the first two copies of $K_2 \square K_n$. Thus we have a minimum of $2n(m - 1) = 2nm - 2n$ cut edges. Since $2 \leq n \leq m$, we have $2n \leq nm$, and thus

$$2nm - 2n \geq 2nm - nm = nm.$$

Finally, we assume there are no copies of $K_2\Box K_n$ that only have vertices in B and there are i copies of $K_2\Box K_n$ that contain only vertices in A, where $0 \le i \le m-1$. because of our first case, we can assume $|B| \ge m+1$ and therefore there are at least $i(m+1)$ cut edges between the vertices in B and the i copies of $K_2\Box K_n$ in A. For each of the remaining $m-i$ copies of $K_2\Box K_n$ that contain at least one vertex in A and B, by Theorem 3.3.1, there are at least n cut edges contained in that copy. Therefore

$$|E(A,B)| \ge n(m-i) + i(m+1) = nm + i(m+1) - in = nm + i(m+1-n).$$

Since $n \le m$, we know $(m+1-n)$ is a positive integer and therefore $|E(A,B)| \ge nm$. This establishes the minimum egg cut set of the scramble $\mathscr{S}$ to be nm.

Next, one needs to show that the minimum hitting set of $\mathscr{S}$ is at least nm. To do this we assume that $A \subset V(K_2\Box K_n\Box K_m)$ only contains $nm-1$ vertices, and then show that it misses some egg in $\mathscr{S}$.

Select the copy of $K_2\Box K_n$ that contains the fewest vertices in A. This copy must contain fewer than n vertices in A so by the pigeon hole principle, there is some copy of K_2 in which neither vertex is contained in A. Therefore all vertices in this copy of $K_2\Box K_n$ which are not in A are connected. Thus, there are at least $n+1$ connected vertices not contained in A, meaning A cannot be a hitting set of $\mathscr{S}$. Therefore $||\mathscr{S}|| = nm$.

□

While it is certainly possible there are other 3-dimensional rook graphs which have scramble number equal to the expected gonality, we also have the following result.

Theorem 3.4.1. *If $n \ge 3$, then $\text{sn}(K_n\Box K_n\Box K_n)$ is strictly less than $(n-1)n^2$.*

Proof. Consider the following set $A \subset V(K_n\Box K_n\Box K_n)$:

$$A = \{(1,1,k)|2 \le k \le n\} \cup \{(1,j,1)|2 \le j \le n\} \cup \{(i,k,k)|2 \le i \le n, 1 \le k \le n\}$$

The set A contains $n+2$ different connected components, each component having $n-1$ vertices. Since

$$|A| = (n+2)(n-1) = n^2 + n - 2 \ge n^2 + 1,$$

we have $|A^c| \le (n-1)n^2 - 1$. Furthermore, $|A^c|$ will intersect with every connected vertex subset of size at least n. If $\mathscr{S}$ is a scramble on $K_n\Box K_n\Box K_n$, then either, A^c is a hitting set of $\mathscr{S}$, or $\mathscr{S}$ has an egg contained in one of the connected components of A. If the former is true then $||\mathscr{S}|| \le |A^c| \le (n-1)n^2 - 1$. If the latter is true, then $\mathscr{S}$ has an egg E, such that E has $n-1$ or fewer vertices, and every vertex in E is adjacent to one another. If E has i vertices, then the egg cut-set $E(E, E^c)$ will consist of $i(n-i) + i(2n-2)$ edges. This is because $i(n-i)$ counts the number of edges between the i vertices in E and the $n-i$ vertices not in E, but in the same line as E. Each vertex in E is also adjacent to $2n-2$ other vertices which are not in the same line as E. Then we use the fact that $n \ge 3$ and $i \le n-1$ to get

$$i(n-i) + i(2n-2) = i(3n-i-2) \le i(n^2-i-2) \le (n-1)(n^2-i-2) < (n-i)n^2.$$

Thus $\mathscr{S}$ will have either a hitting set or an egg cut set smaller than $(n-1)n^2$. □

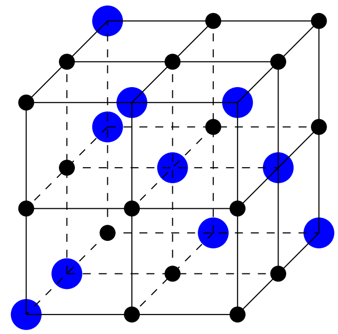

Figure 3.10: The set A for $K_3 \Box K_3 \Box K_3$ as described in the proof of Theorem 6.3

Chapter 4 Column Equitable Divisor Methods

4.1 Dhar's burning algorithm

Here we briefly review Dhar's burning algorithm, which was first introduced in [5]. This algorithm tells us what series of chip fires need to occur in order to get a divisor D to an equivalent divisor D' that is v-reduced for some chosen vertex v. We begin the algorithm by starting a fire at our chosen vertex. It should be noted by the reader that this notion of starting a fire is separate from chip firing. The fire spreads to all edges adjacent to a burning vertex, and a vertex will burn if it has more burning adjacent edges than it does chips. If the entire graph burns, then the divisor is reduced at the vertex v. Otherwise, if there is some set of unburnt vertices left, we fire these vertices and start a new fire at v. The process continues until we have a divisor in which all vertices will burn.

Lemma 4.1.1. *If D is an effective divisor of degree at most $n-2$ on K_n, then a fire started on any vertex with no chips will burn the entire graph.*

Proof. Let k be the number of unburnt vertices left after starting a fire on some vertex with no chips. For these k vertices to not catch fire, they each must have a minimum of $n-k$ chips. Since our divisor is effective and has degree at most $n-2$, we must have $k(n-k) \leq (n-2)$. The only values for which this inequality holds are $k=0$ and $k=n$. We know $k \neq n$ because one vertex was burnt at the beginning. Therefore all vertices must burn. □

4.2 Gonality of Rook Graphs

For the remainder of the paper, we will assume without loss of generality that $n \leq m$.

Definition 4.2.1. Given a divisor D on $K_n \square K_m$, a *poorest column* of D refers to a copy of K_n that contains the fewest number of chips. Similarly, a *poorest row* refers to a copy of K_m that contains the fewest number of chips.

Note that a poorest row/column need not be unique.

Theorem 1.0.2. *If $n \leq m$, $\text{gon}(K_n \square K_m) = (n-1)m$.*

Proof. First, we observe that $\text{gon}(K_n \square K_m) \leq (n-1)m$ since the divisor that has one chip on every vertex except for one row which is left empty, has degree $(n-1)m$ and positive rank. We then need to show that every divisor of degree $(n-1)m-1$ does not have positive rank.

Let D be an effective divisor of degree $(n-1)m-1$, and without loss of generality, we assume that D has a maximal number of chips in its poorest column among all equivalent effective divisors. A poorest column contains at most $n-2$ chips and by Lemma 4.1.1, starting a fire at one of the vertices with no chips will make the entire column burn. Additionally, the poorest row of D will have at most $m-2$ chips since $(n-1)m-1 \leq n(m-1)-1$. Because an entire column has burned, the poorest row must also burn. We then assume for contradiction that there is some nonempty subset $U \subset V(K_n \square K_m)$ which is left unburnt. We can then fire U to obtain a new effective divisor D'.

If column k has $j > 0$ vertices in U, then firing U will transfer $j(n-j)$ chips from the j vertices in U to the other $n-j$ vertices in column k. Since D' is effective, column k will have at least $j(n-j)$ chips in D'. Because the poorest row has burned, $j < n$ and thus, $j(n-j) \geq n-1$. Therefore any column that intersects with U, will not become the poorest column of D'. However, every column that doesn't intersect with U will gain $|U|$ chips once U is fired. This is a contradiction since we assumed D maximized the number of chips in the poorest column. Therefore all vertices of $K_n \square K_m$ must burn, meaning D is v-reduced for some vertex that has 0 chips. Then by Lemma 2.0.13, we conclude that D does not have positive rank. □

Recall that the gonality of a graph G is the fewest number of chips needed to construct a divisor of rank 1. In other words, if $\text{gon}(G) = j$ then there is some divisor D of degree j such that if someone were to take one chip from any vertex, we could get back to an effective divisor through a series of chip fires. We can expand on this idea by asking how many chips we need to make a divisor that can withstand the theft of more than just one chip.

Definition 4.2.2. Given a graph G, the $k-$*gonality* of G, denoted $\text{gon}_k(G)$ is the fewest number of chips needed to construct a divisor of rank k.

Lemma 4.2.3. *Given a graph G, $\text{gon}_k(G) \leq \text{gon}_{k+1}(G) - 1$.*

Proof. Let D be a divisor on G of degree $\text{gon}_{k+1}(G)$, and having rank $k+1$. If we take away a single chip from any vertex which has a positive number of chips, we are left with a divisor of degree $\text{gon}_{k+1}(G) - 1$ and rank k. Therefore $\text{gon}_k(G) \leq \text{gon}_{k+1}(G) - 1$. □

With this lemma, we have enough information to modify the proof of 1.0.2 in order to compute the 2 and 3-gonalities of two dimensional rook graphs.

Theorem 1.0.5. *Let $n, m \geq 2$, then $\text{gon}_2(K_n \square K_m) = nm - 1$ and $\text{gon}_3(K_n \square K_m) = nm$.*

Proof. There are two main components to this proof. First, we will show that there exists a divisor of degree nm of rank at least 3. Second, we will show that no divisor of degree $nm-2$ has rank 2.

Given a graph G, one can show that the divisor D has positive rank if for every vertex $v \in V(G)$, there is some equivalent effective divisor D', where v has a positive number of chips. Similarly, D has rank at least k if we can take away any $k-1$ chips and still be left with a divisor such that for any vertex v, there is some equivalent effective divisor where v has a positive number of chips. Let D be the divisor on $K_n \square K_m$ in which every vertex has exactly 1 chip. We then consider all possible ways to take away 2 chips from D.

- If 2 chips are taken from a single vertex v, we can fire all vertices except v. Then v will have $n+m-3$ chips and all other vertices have either 1 or 0 chips. Assuming $n, m \geq 2$, v will have a positive number of chips.
- If chips are taken from v_1 and v_2 which lie in the same row or column, we can fire all of the other rows or columns to produce an effective divisor where both v_1 and v_2 have a positive number of chips.
- If chips are taken from v_1 and v_2 which lie in different rows and columns, we can fire all vertices except v_1 to produce an effective divisor where v_1 has a positive number of chips. Similarly, we can fire all vertices except v_2 to produce a different effective divisor where v_2 has a positive number of chips.

Therefore D has rank at least 3 and $\text{gon}_3(K_n \square k_m) \leq nm$.

Next, we let E be an effective divisor of degree $nm - 2$. We assume E has a maximal number of chips in its poorest column. First, we consider the case where the poorest columns have $n - 1$ chips. We begin by selecting one of the poorest columns, and removing a chip from a vertex that has a positive number of chips. By Lemma 4.1.1, starting a fire on any vertex in this column with no chips will result in the entire column burning. Now that an entire column is burning, every other column with n-1 chips must also burn due to the same argument as in the proof of Lemma 4.1.1. We can understand this by thinking of the original burning column as a singular burning vertex that is adjacent to every vertex in a given column. This then reduces to the case where we have the graph K_{n+1} with only $n - 1$ chips.

Since E is degree $nm-2$, and the poorest column has $n-1$ chips, if a column has $n+i$ chips for $i \geq 0$, then there must be at least $i+2$ other columns with $n-1$ chips. All of these $i + 2$ columns with $n - 1$ chips will burn, which will lead to the column with $n + i$ chips burning as well. Therefore the entire graph burns and E does not have rank 2.

Next, we assume that E has a poorest column with $\leq n - 2$ chips. We remove a chip from a poorest row, which would have at most $m - 1$ chips. We then start a fire on some vertex with zero chips in that row. This row now has at most $m - 2$ chips, and thus it must burn entirely. Any column with $\leq n-2$ chips will also burn. We then assume for contradiction that some subset $U \subset V(K_n \square K_m)$ will be left unburnt. If we fire the vertices in U to produce the new divisor E', we know from the proof of Theorem 1.0.2 that any column which intersects with U will have at least $n - 1$ chips in E'. Every other column will increase the number of chips it has by $|U|$. This contradicts the fact that E maximized the chips in the poorest column, and the poorest column had $\leq n - 2$ chips. Therefore the entire graph must burn and E does not have rank 2. Thus we conclude that $\text{gon}_2(K_n \square K_m) = nm - 1$ and $\text{gon}_3(K_n \square K_m) = nm$. □

4.3 Gonality of Queen Graphs

The $n \times m$ queen graph can be depicted as an $n \times m$ grid where two vertices are adjacent if they are a queen's move apart (i.e they lie in the same row or column, or they are diagonal from one another). Because the rook graph $K_n \square K_m$ is a spanning subgraph of $Q_{n,m}$, we can modify the column equitable divisor method to help us compute the gonality of $Q_{n,m}$.

Lemma 4.3.1. *Let $n, m \geq 4$, then* $\text{gon}(Q_{n,m}) \leq n(m-1)$.

Proof. It is sufficient to show that for $n, m \geq 4$, there always exists a divisor of degree $n(m-1)$ having positive rank. A result from [7] states that for $n \geq 4$ you can place n queens on an $n \times n$ chessboard such that no two queens can attack one another. This means that for $Q_{n,n}$ there exists a subset of n vertices which are all independent of one another (pairwise non-adjacent). This also means there will exist n independent vertices within the first n columns of $Q_{n,m}$.

Let I be an independent set of n vertices on $Q_{n,m}$ and consider the divisor which places one chip on all vertices except the n vertices in I. If a chip is stolen from one of the vertices not in I then the resulting divisor is already effective. If a chip is stolen from some vertex $v \in I$ then we can fire all vertices except v to obtain an effective divisor. Note that since I is an independent set, all vertices adjacent to v are not in I and therefore have one chip which will transfer to v. □

Theorem 4.3.2. *Let $n \leq m$, then* $\text{gon}(Q_{n,m}) \geq n(m-1)$.

Proof. Let D be an effective divisor on $Q_{n,m}$ of degree $n(m-1)-1$. Without loss of generality, assume that D maximizes the number of chips in its poorest row among all effective divisors equivalent to D. We then run Dhar's burning algorithm starting on a vertex that lies in the poorest row and has no chips. The poorest row will have at most $m-2$ chips and by Lemma 4.1.1, the entire row will burn. Next we consider the following two cases.

In Case 1 we assume that every other row has at least one vertex burn. We then assume for contradiction that there is some nonempty subset $U \subset V(Q_{n,m})$ which is left unburnt. We can then fire U to obtain a new effective divisor D'.

If row k has $j > 0$ vertices in U, then firing U will transfer $j(m-j)$ chips from the j vertices in U to the other $m-j$ vertices in row k. Since D' is effective, row k will have at least $j(m-j)$ chips in D'. Since we assumed that at least one vertex in row k has burned, $j < m$ and thus, $j(m-j) \geq m-1$. Therefore any row that intersects with U, will not become the poorest row of D'. However, every row that doesn't intersect with U will gain more than $|U|$ chips once U is fired. This is a contradiction since we assumed D maximized the number of chips in the poorest column.

In Case 2 we assume that at least one row is entirely contained in the set of unburnt vertices U. We then consider the cut set $E(U, U^c)$. Since both U and U^c contain an entire row, they both contain at least $m \geq n-1$ vertices and by Theorem 3.3.1, $E(U, U^c)$ has at least $(n-1)m$ non-diagonal cut edges. The first and last row of a queen graph shares $2(m-n+1)$ diagonal edges between them and every other pair of rows will have more diagonal edges. thus

$$\begin{aligned}|E(U,U^c)| &\geq (n-1)m + 2(m-n+1)\\ &= nm - m + 2m - 2n + 2\\ &= nm + m - 2n + 2\\ &\geq nm - n + 2\\ &= (m-1)n + 2.\end{aligned}$$

This is a contradiction because firing U should produce another effective divisor, which is impossible if $|E(U,U^c)| > deg(D)$. Therefore the entire graph must burn and D does not have positive rank. □

By combining Lemma 4.3.1 and Theorem 4.3.2, we can now classify the gonality of all queen graphs.

Theorem 1.0.6. *Let $2 \leq n \leq m$, then we have*

$$\text{gon}(Q_{n,m}) = \begin{cases} 3 & \text{if } n = m = 2\\ 7 & \text{if } n = m = 3\\ n(m-1) & \text{otherwise.}\end{cases}$$

Proof. The only cases cases which are not yet proven by Lemma 4.3.1 and Theorem 4.3.2 are when $n = 2$ or $n = 3$. By a similar argument as in Lemma 4.3.1 the gonality of these graphs is bounded above by $nm - \alpha(Q_{n,m})$ where $\alpha(Q_{n,m})$ is the independence number, or in this context, the largest number of non-attacking queens that be placed on an $n \times m$ chess board. Fortunately the only two chess boards where the number of non-attacking queens is less than n, are the 2×2 case and the 3×3 case. $Q_{2,2} \cong K_4$ so the gonality is 3.

The largest number of non-attacking queens which can be placed on a 3×3 chess board is 2, so $\text{gon}(Q_{3,3}) \leq 7$. We can then use the scramble number to show equality. Let $\mathscr{S}$ be the scramble in which every pair of adjacent edges is an egg. because the largest independent set on $Q_{3,3}$ has size 2, a minimal hitting set of $\mathscr{S}$ has size 7. By Theorem 3.3.1, any egg cut set has at least 6 non-diagonal edges. The only way to partition the vertices of $Q_{3,3}$ into a cut that has no diagonal cut edges would be to partition them in the same manner a chess board is separated into black and white squares. However this cut would have has 12 edges in the cut set. Therefore $\text{sn}(Q_{3,3}) = 7 = \text{gon}(Q_{3,3})$. □

Bibliography

[1] I. Aidun, F. Dean, R. Morrison, T. Yu, and J. Yuan. Treewidth and gonality of glued grid graphs. *Discrete Appl. Math.*, 279:1–11, 2020.

[2] I. Aidun and R. Morrison. On the gonality of Cartesian products of graphs. *Electron. J. Combin.*, 27(4):Paper No. 4.52, 35, 2020.

[3] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[4] S. Corry and D. Perkinson. *Divisors and Sandpiles: An Introduction to Chip-firing.* American Mathematical Society, 2018.

[5] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64(14):1613–1616, 1990.

[6] R. Lazarsfeld. Lectures on linear series. In *Complex algebraic geometry (Park City, UT, 1993)*, volume 3 of *IAS/Park City Math. Ser.*, pages 161–219. Amer. Math. Soc., Providence, RI, 1997. With the assistance of Guillermo Fernández del Busto.

[7] E. Pauls. Das maximalproblem der damen auf dem schachbrete. *Deutsche Schachzeitung. Organ für das Gesammte Schachleben*, 29(5):129–134, 1874.

[8] P. D. Seymour and R. Thomas. Graph searching and a min-max theorem for tree-width. *J. Combin. Theory Ser. B*, 58(1):22–33, 1993.

[9] J. van Dobben de Bruyn and D. Gijswijt. Treewidth is a lower bound on graph gonality. *Algebr. Comb.*, 3(4):941–953, 2020.