2010

L^p Bounded Point Evaluations for Polynomials and Uniform Rational Approximation

Erin Militzer
University of Kentucky, ermilitzer@gmail.com

ABSTRACT OF DISSERTATION

Erin Militzer

The Graduate School
University of Kentucky
2010

L^p Bounded Point Evaluations for Polynomials and Uniform Rational Approximation

ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Erin Militzer
Lexington, Kentucky

Director: Dr. James E. Brennan, Professor of Mathematics
Lexington, Kentucky 2010

ABSTRACT OF DISSERTATION

L^p Bounded Point Evaluations for Polynomials and Uniform Rational Approximation

A connection is established between uniform rational approximation, and approximation in the mean by polynomials on compact nowhere dense subsets of the complex plane $\mathbb{C}$. Peak points for $R(X)$ and bounded point evaluations for $H^p(X, dA)$, $1 \le p < \infty$, play a fundamental role.

KEYWORDS: polynomial and rational approximation, analytic capacity, peak points, point evaluations.

Author's signature: Erin Militzer

Date: May 4, 2010

L^p Bounded Point Evaluations for Polynomials and Uniform Rational Approximation

By
Erin Militzer

Director of Dissertation: James E. Brennan

Director of Graduate Studies: Qiang Ye

Date: May 4, 2010

RULES FOR THE USE OF DISSERTATIONS

DISSERTATION

Erin Militzer

The Graduate School
University of Kentucky
2010

L^p Bounded Point Evaluations for Polynomials and Uniform Rational Approximation

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Erin Militzer
Lexington, Kentucky

Director: Dr. James E. Brennan, Professor of Mathematics
Lexington, Kentucky 2010

ACKNOWLEDGMENTS

First, I want to acknowledge all the people in my life that have helped me to this point: my family, of course, who have been nothing but supportive and loving throughout my academic career; my mom, who home schooled me early in life and gave me the gifts of mathematics, reading, and writing; my dad, who always listened to me no matter whether I was complaining or celebrating and who was always there whenever I needed him, my brothers and sister, who always make me laugh and who I could not imagine my life without-I hope we are always close and continue to share our lives.

All of my past teachers, to those who believed in me, thanks for that help! To those who didn't, thanks for the motivation! I couldn't have reached to this point without all of the teachers in my life. In particular, Mr.Granlund, one of my believers, thanks for encouraging me to make anything happen, rest in peace. To Dr. Ken Smith, thanks for helping me to find out that I can do math and make money! All those summer research scholarships gave me the tools I needed for graduate school. To my high school math teachers, although you were not my inspiration, your dedication to your students was apparent, and I will always remember what I learned from you.

To my friends, my professors, and my EDGE family who I found in graduate school: Michelle Craddock, Maria Osorio, Emily, Eric, Tricia, Matt, Jessie, Joey, Kelly and Josh, the friendships we have are those of a lifetime. Thank you yesterday, today, and in the future. Maria Clark, thanks for your help with revisions on all my cover letters, which I am sure helped me get my dream job. Nick Kirby, thanks for always being the person I could turn to with any question and for asking those questions that led to a successful research finding. To my EDGE family, whom I love dearly, all of the professors, participants, and mentors, I would not have been convinced I could stay in graduate school without you. To my University of Kentucky professors, for doing your job well and never failing me. Dr. Marion Anton, thank you for your patience and kindness. I would have not continued in graduate school if it were not for your influence.

To my advisor, Dr. James Brennan, who put up with me the most! Dr. Brennan, this thesis exists because you believed in me and never doubted my ability to do mathematics. Our meetings were always insightful and fun. You have a great ability to explain mathematics without giving too much away and always kept me on my toes.

To my future husband, Matthew Benander: I never thought I would meet another math person, but I did, and I am happy to know we will be together. Thanks for letting me cry on your shoulder, for the beautiful sunflowers you sent me the day I passed my first prelim, for always making me laugh, for dancing with me, and for just being you.

Dedicated to those who wanted to go to college but couldn't pay, to those who wanted to move on to graduate school, but didn't think they could do it, and to anyone who is currently struggling with school. At one point I thought all of these were true for me, but in the end you have to remember to believe not only in yourself but also in those who love you. I truly feel that this dissertation proves that no matter the odds, anything is possible.

TABLE OF CONTENTS

Acknowledgments . . . iii

Table of Contents . . . v

Chapter 1 Introduction . . . 1

Chapter 2 Preliminaries . . . 3

2.1 Approximation by Analytic Functions: Background . . . 3

2.2 The Cauchy Transform and Annihilating Measures . . . 5

2.3 The Swiss Cheese . . . 8

Chapter 3 Capacities, Peak Points and Barriers . . . 11

3.1 Analytic Capacity . . . 11

3.2 L^q Capacities . . . 16

3.3 Peak Points and Bishop's Theorem . . . 17

3.4 Barriers . . . 18

Chapter 4 Main Results . . . 23

4.1 Bounded Point Evaluations for $H^p(dA)$ . . . 23

4.2 A Counterexample for $R^p(dA)$ . . . 25

Bibliography . . . 29

Vita . . . 32

Chapter 1 Introduction

Let μ be a finite positive compactly supported regular Borel measure in the complex plane $\mathbb{C}$ having no point masses. For each p, $1 \leq p < \infty$, let $H^p(\mu)$ be the closed subspace of $L^p(\mu)$ that is spanned by the complex analytic polynomials. Over the years considerable attention has been directed to understanding the conditions under which $H^p(\mu) = L^p(\mu)$, due in part to its connection with the invariant subspace problem for subnormal operators on a Hilbert space when $p = 2$. On the other hand, equality evidently fails whenever there exists a point $z_0 \in \mathbb{C}$ such that the map $P \to P(z_0)$ can be extended from the polynomials to a bounded linear functional on $H^p(\mu)$; that is, if

$$|P(z_0)| \leq C\,||P||_{L^p(\mu)} \tag{1.1}$$

for every polynomial P and some absolute constant $C > 0$. Such a point z_0 is said to be a *bounded point evaluation* or *bpe* for $H^p(\mu)$, and the question arises: Is it true that either

(1) $H^p(\mu)$ has a bpe or,

(2) $H^p(\mu) = L^p(\mu)$?

The initial step in dealing with the proposed alternative was taken by Wermer [40] in 1955. At that time he was able to show that if μ is carried on a compact set X having planar measure zero (i.e. if $|X| = 0$) then the alternative is indeed valid. His argument is roughly this; Let $R(X)$ be the class of functions that can be uniformly approximated on X by rational functions whose poles lie outside of X, and let $C(X)$ be the space of all continuous functions on X. If $H^p(\mu)$ has no bpe's then it must contain every rational function analytic on X, and so also $R(X)$ (cf. [7], p. 218). On the other hand, since $|X| = 0$ it follows from a theorem of Hartogs and Rosenthal [17] that $R(X) = C(X)$, and the latter is dense in $L^p(\mu)$. Therefore, if (1) fails then (2) holds .

Although the alternative embodied in (1) and (2) above is now known to be valid for all measures μ (cf. Thomson [33]), we are nevertheless led to consider the relationship between polynomial and rational approximation. Following Wermer's early success the natural question was this: Does the suggested alternative persist if $d\mu << dA$? In particular does it remain in force if $d\mu = dA$ restricted to a compact set X having positive area, but no interior? In an attempt to answer these and similar questions Brennan was led in 1973 to ask: Does there exist a compact set X such that $H^p(X, dA) = L^p(X, dA)$ for all p, $1 \leq p < \infty$, but $R(X) \neq C(X)$ (cf. [4], p. 174). Several years later in a major survey article Mel'nikov and Sinanjan [26] made further reference to the problem just stated, and it has remained open throughout the intervening years. Our goal here in Section 4.1 is to settle the matter in the negative.

Almost a decade prior to the publication of [26] Sinanjan [32] had considered, and answered, the corresponding question for $R^p(X, dA)$, the closed subspace of $L^p(X, dA)$ that is spanned by the rational functions having no poles on X. He showed that there exists a compact set X such that $R(X) \neq C(X)$, but nevertheless $R^p(X, dA) = L^p(X, dA)$ for all p, $1 \leq p < \infty$. In Section 4.2 we present another example of this kind, motivated by an as yet unsolved problem concerning the possible underlying structure of a compact set X where $R(X) \neq C(X)$.

In the sequel $P(X)$ will stand for the closed subspace of $C(X)$ spanned by the polynomials.

Chapter 2 Preliminaries

2.1 Approximation by Analytic Functions: Background

Questions concerning uniform approximation by analytic functions have a long history, dating back at least to 1885 and the papers of Weierstrass [39] and Runge [30]. Of these two works that of Weierstrass is perhaps the more widely known. Here, for the first time, it was shown that every continuous function on a closed bounded interval X on the real line can be approximated arbitrarily closely on X by a sequence of polynomials. Put more succinctly, in this context $P(X) = C(X)$. The earliest generalization of Weierstrass' theorem was obtained by Walsh in 1928 (cf. [38], p. 39). He showed that the same conclusion is valid for polynomial approximation on an arbitrary closed bounded simple Jordan arc lying in $\mathbb{C}$. Evident in these early results are two necessary conditions in order that $P(X) = C(X)$ for any compact set X. They are:

(1) X has no interior,

(2) The complement of X is connected.

If (1) is violated, then clearly the uniform limit on X of any sequence of polynomials must be analytic in the interior of X. Hence $P(X) \neq C(X)$. If on the other hand, (2) is violated let Ω be any bounded component of $\mathbb{C}\backslash X$ and fix a point $a \in \Omega$. Thus, $(z-a)^{-1}$ is continuous on X, and if a sequence of polynomials

$$p_n \to \frac{1}{z-a} \text{ uniformly on } X,$$

then $(z-a)p_n \to 1$ uniformly on X, and by the maximum principle uniformly on Ω. But, this contradicts the fact that $(z-a)p_n = 0$ at a for all $n = 1, 2, 3, ...$ Therefore, we must again conclude that $P(X) \neq C(X)$. Eventually, Lavrentiev [23](cf. also [27] p. 297) showed that conditions (1) and (2) are also sufficient to ensure that $P(X) = C(X)$, thereby establishing a purely topological criterion for uniform polynomial approximation on compact subsets of the plane. It is now known that Lavrentiev's theorem can be obtained by taking advantage of a certain residual continuity enjoyed by the Cauchy transform of a finite compactly supported measure in $\mathbb{C}$; a property that will play an essential role in this investigation (cf. Lemma 3.2).

In the second of the two articles referenced above Runge [30] initiated the study of rational approximation on compact subsets of $\mathbb{C}$. In particular, he proved that if X

is compact, then any function analytic in a neighborhood of X can be approximated uniformly on X by a sequence of rational functions with poles off X. That, however, left open the question: For which compact sets X is $R(X) = C(X)$? In time it would become clear that there could be no simple geometric criterion as in the case of Lavrentiev's theorem. An early indication of the complexity of the situation was manifested in the theorem of Hartogs and Rosenthal [17] to the effect that $R(X) = C(X)$ whenever $|X| = 0$. Only much later in 1958 did Vitushkin (cf. [37]) establish a necessary and sufficient condition in terms of analytic capacity in order that $R(X) = C(X)$. Nevertheless, there remains considerable debate as to whether such criterion can be considered truly *geometric.*

Beginning with the work of Carleman in the early 1920's attention shifted to questions concerning approximation in the $L^p(dA)$-norm, particularly on sets with interior points (cf. [28] or [6]). But, it would be more than thirty years before the problem of determining necessary and sufficient conditions under which the polynomials are dense in an arbitrary $L^p(\mu)$ would receive added stimulus from a seemingly unrelated problem in operator theory; namely, from the invariant subspace problem for subnormal operators on a Hilbert Space. A bounded linear operator T on an infinite dimensional Hilbert space H is *subnormal* if it has a normal extension to a larger Hilbert space; or equivalently, if T is the restriction of a normal operator to a closed invariant subspace. In general, the invariant subspace problem is to determine whether a bounded operator $T : H \longrightarrow H$ has a *nontrivial closed* invariant subspace. It can therefore be assumed from the outset that T has a *cyclic vector* x; that is, a vector x for which the linear span $x, Tx, T^2x, ...$ is dense in H. Otherwise, invariant subspaces abound and there is nothing to prove. If, in addition, T is subnormal the spectral theorem guarantees that there is a positive measure μ carried on the spectrum of T such that the given operator T is unitarily equivalent to multiplication by the complex identity function z on $H^2(d\mu)$. Thus, the study of subnormal operators leads directly to questions concerning approximation by polynomials in $L^2(d\mu)$ (cf Bram [3], pp. 83-86).

In 1991 Thomson [33] finally established, as a general principle, the alternative described in the first paragraph of the introduction:

Theorem 2.1.1 (Thomson)**.** *For any positive measure μ of compact support, having no point masses, $H^p(d\mu) = L^p(d\mu)$ for $1 \leq p < \infty$ if and only $H^p(d\mu)$ has no bpe.*

Thomson's result, however, leaves unanswered certain questions that had arisen concerning the relation between uniform rational approximation on a compact set X

and the density of the polynomials in $L^p(X, dA)$. These questions are addressed here in Section 4.

2.2 The Cauchy Transform and Annihilating Measures

Let ν be a complex regular Borel measure with compact support X and define

$$\hat{\nu}(z) = \int_X \frac{d\nu(\zeta)}{\zeta - z}$$

to be the *Cauchy transform* of ν. Notice that $\hat{\nu}(z)$ is defined whenever the *Newtonian Potential*, $\tilde{\nu}(z) = \int_X \frac{d|\nu|(\zeta)}{|\zeta - z|}$, converges. Denote by $|\nu|$ the total variation of ν. To gain a clear understanding of how the Cauchy transform will be utilized throughout our discussion, we first indicate a few important properties. We begin by verifying, that the Cauchy transform converges almost everywhere in the plane with respect to area.

It is clear that if z off the set X, that is $z \in \mathbb{C} \setminus X$, then $\tilde{\nu}(z) < \infty$. Choose a square Q in the plane such that $X \subset Q$ and R sufficiently large so that for any $\zeta \in X$, the square Q is contained in the disk $B_r(\zeta) = \{z : |\zeta - z| < R\}$. This implies the following:

$$\int_Q \frac{dA_z}{|\zeta - z|} \leq \int_0^{2\pi} \int_0^R \frac{1}{r} r dr d\theta = 2\pi R < \infty$$

and therefore by Fubini's theorem

$$\int_Q \int_X \frac{d|\nu|(\zeta)}{|\zeta - z|} \, dA_z < \infty = \int_X \int_Q \frac{dA_z}{|\zeta - z|} \, d|\nu|(\zeta) < 2\pi R|\nu|(X)$$

which implies that $\tilde{\nu}(z) < \infty$ a.e.-dA.

The Cauchy transform is also continuous and analytic in $\mathbb{C} \setminus X$. To show continuity, choose z_0 in $\mathbb{C} \setminus X$ and z in a neighborhood U of z_0 such that $\overline{U} \cap X = \emptyset$, then we have that

$$\left| \int_X \frac{d\nu(\zeta)}{\zeta - z_0} - \int_X \frac{d\nu(\zeta)}{\zeta - z} \right| \leq |z_0 - z| \int_X \frac{d|\nu|(\zeta)}{|\zeta - z_0||\zeta - z|} \leq C|z_0 - z|.$$

Where the constant C depends on the distance between z_0 and X.

Choose Γ be a closed curve that lies in $\mathbb{C} \setminus X$ which does not surround X and z_0 as before. Since $\frac{1}{\zeta - z_0}$ is analytic, $\int_\Gamma \frac{1}{\zeta - z_0} = 0$ and therefore

$$\int_\Gamma \left(\int_X \frac{d\nu(\zeta)}{\zeta - z_0} \right) dz = \int_X \left(\int_\Gamma \frac{dz}{\zeta - z_0} \right) d\nu(\zeta) = 0.$$

Since $\hat{\nu}(z)$ is continuous and $\int_\Gamma \hat{\nu}(z) = 0$ over any closed curve Γ off X, we have by Morera's theorem that $\hat{\nu}(z)$ is analytic.

One of the most important properties of the Cauchy transform properties is stated in the following theorem:

Theorem 2.2.1. *Let ν be defined as before. If $\hat{\nu} = 0$ a.e.-dA then $\nu = 0$.*

Proof. We present a proof due to Beurling (cf. [ ], p. and p). First we note that $|\nu| = 0$ is zero on a.e. line parallel to the coordinate axes. If not then $|\nu| > C > 0$ on some infinite set of disjoint lines which would imply that $|\nu|(X) = \infty$. This contradicts the assumption that ν is finite. Now let E be any rectangle in the plane such that $|\nu| = 0$ on ∂E. Assuming that following integral exists and change in integration is permitted we have:

$$\frac{-1}{2\pi i}\int_{\partial E}\int_X \frac{d\nu(\zeta)}{\zeta - z}dz = \frac{1}{2\pi i}\int_X\int_{\partial E}\frac{dz}{\zeta - z}d\nu(\zeta) = 0$$

On the other hand, by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{\partial E}\frac{dz}{\zeta - z} = \chi_E(\zeta) = \begin{cases} 1 & z \in \mathrm{int}(E) \\ 0 & z \notin \mathrm{int}(E) \end{cases}$$

and therefore

$$\frac{1}{2\pi i}\int_X\int_{\partial E}\frac{dz}{\zeta - z}d\mu(\zeta) = \int_X \chi_E(\zeta)d\mu(\zeta) = \mu(int(E)\cap X) = 0.$$

We can carry this out for enough rectangles to conclude that $\mu = 0$.

Suppose A is any compact subset of X and let U be any neighborhood of A. Cover A with rectangles $\{E_j\}$ that have the following properties:

(1.) ν places no mass on ∂E_j.

(2.) $\cup_j E_j \subset U$

It is clear that $|\nu(\cup E_j)| \leq \sum_j |\nu(E_j)| = 0$. Since $\mu(E_j) = 0$ for all j and $\nu(A) = \lim\limits_{U\downarrow A}\nu(U)$ for all $A \subset U$, we have $\nu(A) = 0$. $\square$

Corollary 2.2.2 (Hartogs-Rosenthal)**.** *If $|X| = 0$ then $R(X) = C(X)$.*

To prove this corollary we will argue by duality. We are reminded of some useful results from functional analysis. Take V be any normed vector space and V_0 a subspace of V. Designate V^* to be the dual space of V. The Hahn-Banach theorem states that if L_0 is a bounded linear functional on V_0, that is $||L_0|| < \infty$, then there exists a linear functional L in V^* such that

(i) $L \equiv L_0$ on V_0

(ii) $||L||_{V^*} = ||L_0||_{V_0^*}$

where $||L|| = \sup\{C : |L(f)| \leq C||f||\}$. A consequence is the following:

Theorem 2.2.3 (Closure Theorem)**.** *Let V be a normed vector space and V_0 a subspace and $x \in V$. Then $x \in \overline{V_0}$ if and only if $L \in V^*$ and $L \equiv 0$ on V_0 implies that $L(x) = 0$.*

In our case we are looking at the vector space $C(X)$. The Riesz representation theorem states the following: for every continuous linear functional L on $C(X)$, there exists a uniquely determined measure μ on X such that

$$L(g) = \int g \, d\mu$$

for every $g \in C(X)$ and $||\mu|| = ||L||$. This holds true for linear subspaces of $C(X)$ such as $R(X)$.

Lastly, we define the following: We say μ is an *annihilating measure* for the rational functions analytic on X if for each rational function, $\int_X f \, d\mu = 0$. We will use the notation $\mu \perp R(X)$. Collecting the information up to the point, we are now able to demonstrate a technique to show $R(X) = C(X)$.

Proof of Corollary. Let $\mu \perp R(X)$ with $|X| = 0$, then $\hat{\mu}(z) = \int_X \frac{d\mu(\zeta)}{\zeta - z} = 0$ a.e.-dA and hence $\mu = 0$. We have a measure which annihilates not only the rationals, but everything, therefore $R(X) = C(X)$.

□

Here we present a more detailed proof of Wermer's result which demonstrates the use of the Cauchy transform.

Suppose $g \in L^2(X, d\mu)$ which is orthogonal to the polynomials in the sense that $\int Pgd\mu =$ for every polynomial P, and form the cauchy transform

$$\widehat{g\mu}(z) = \int \frac{g(\zeta)}{\zeta - z} d\mu(\zeta)$$

then by Cauchy we find that

$$P(z) = \frac{1}{\widehat{g\mu}(z)} \int \frac{P(\zeta)g(\zeta)}{\zeta - z} d\mu(\zeta) \tag{2.3.1}$$

at every point $z \in \mathbb{C}$ where $\hat{g\mu}(z) = \int \frac{g(\zeta)}{\zeta - z} d\mu(\zeta)$ is defined and $\widehat{g\mu}(z) \neq 0$. In particular, if $z \in \mathbb{C} \setminus X$ and $\widehat{g\mu}(z) \neq 0$ then (2.3.1) holds and since the kernel $(\zeta - z)^{-1}$ is bounded on X,

$$|P(z)| \leq C \int |P||g|d\mu$$

for all polynomials P and a suitable constant C. Hence the inequality (1.1) is also satisfied and $H^2(d\mu)$ has a bpe at z. If therefore $H^2(d\mu)$ has no bpe's it follows that $\widehat{g\mu}(z) = 0$ in $\mathbb{C} \setminus X$; that is, $\widehat{g\mu}(z) = 0$ a.e.-dA in $\mathbb{C}$, since X has area zero. Thus, $g\mu = 0$ as a measure and hence $H^2(d\mu) = L^2(d\mu)$.

2.3 The Swiss Cheese

In order to develop a greater appreciation for what might be valid in the most general situation let us consider initially a special class of compact nowhere dense sets, a typical member of which is often referred to as the Swiss cheese. Such sets were first studied in connection with rational approximation by Alice Roth [29] in 1938, rediscovered in a similar context by Mergeljan [27] in 1952, and are constructed as follows: Remove from the closed unit disk $\overline{D}$ countably many disjoint open disks D_j, $j = 1, 2, 3...$, having radii r_j in such a way that

1. $\overline{D_j} \subset \text{int}(\overline{D})$ for each $j = 1, 2, 3...$
2. $\overline{D_j} \cap \overline{D_k} = \emptyset$ whenever $j \neq k$
3. $\overline{D} \setminus \cup_{j=1}^{\infty} D_j$ has no interior
4. $\sum_j r_j < \infty$.

The resulting set $E = \overline{D} \setminus \cup_{j=1}^{\infty} D_j$ is compact and nowhere dense. Letting $d\mu$ be dz on ∂D and $-dz$ on the remaining circles $\Gamma_j = \partial D_j$, $j = 1, 2, 3...$ we obtain a nonzero measure of finite total variation on E such that

$$\int_E f d\mu = \int_{\partial D} f dz - \sum_j \int_{\Gamma_j} f dz = 0$$

for all $f \in R(E)$. Thus, $R(E) \neq C(E)$ and so by the Hartogs-Rosenthal theorem E has positive area. The space $H^p(E, dA)$ is therefore nonempty, and we can ask whether $H^p(E, dA) = L^p(E, dA)$.

Although the Hartogs-Rosenthal theorem allows us to conclude indirectly that $|E| > 0$, it fails to provide any additional information on the specific geometric structure of the Swiss cheese. For this reason it is important here to recall an argument due to W.K. Allard (cf.[9],p.163) which is considerably more informative on that point. For each $x \in [-1,1]$ let $E_x = \{z \in E : Re\, z = x\}$, and for each $n = 1, 2, 3, ...$ let $I_n(x)$ the number of points in $E_x \cap \Gamma_n$. Evidently, $I_n(x) = 0, 1$ or 2. By the monotone convergence theorem we have

$$\int_{-1}^{1} \sum_{n=1}^{\infty} I_n(x)dx = \sum_{n=1}^{\infty} \int_{-1}^{1} I_n(x)dx = 4 \sum_{n=1}^{\infty} r_n < \infty.$$

Hence, $\sum I_n(x) < \infty$ for almost every $x \in [-1,1]$. For any such x all but finitely many $I_n(x)$ must be zero, and the corresponding set E_x consists of a finite number of non-degenerate intervals. Consequently $|E| > 0$ by Fubini's Theorem.

An equally important implication for the question raised in the introduction is the following:

Lemma 2.3.1. *Let E be a Swiss cheese and for each $z \in E$ let $\mathcal{F}_z$ denote the union of all circles centered at z and lying entirely in E. Then, there exists at least one point $z \in E$ where $|\mathcal{F}_z| > 0$.*

Proof. Let E_x be as above, denote by $l(E_x)$ its total length or linear measure, and set $\Gamma = \cup_{j=1}^{\infty} \Gamma_j$. Since there are uncountably many $x \in [-1,1]$ where E_x consists of a finite number of non-degenerate intervals we can choose a sequence of such points x_n in a manner that $l(E_{x_n}) \geq C > 0$ for some constant C and all $n = 1, 2, 3, ...$ Moreover, we can assume with no loss of generality that $x_n < x_{n+1}$. Now select a finite collection of disjoint open disks $\Delta_{j_1} = \Delta(z_{j_1}, r_{j_1})$ with centers $z_{j_1} \in E_{x_1}$ so that $r_{j_1} < \frac{1}{2}(x_2 - x_1)$ and $\sum r_{j_1} > \frac{C}{2}$. Next, choose another finite collection of disjoint open disks $\Delta_{j_2} = \Delta(z_{j_2}, r_{j_2})$ with centers $z_{j_2} \in E_{x_2}$ so that $r_{j_2} < \frac{1}{2}\min(x_3 - x_2, x_2 - x_1)$ and $\sum r_{j_2} > \frac{C}{2}$. The disks Δ_{j_2} are clearly disjoint from any of the Δ_{j_1}. Continuing in this way we obtain a collection of disjoint open disks which we re-designate as Δ_j with centers $z_j \in E$ and radii r_j such that $\sum r_j = \infty$.

If we assume that $|\mathcal{F}_{z_j}| = 0$ for all $j = 1, 2, 3, ...$, then a.e. circle in each Δ_j with center at z_j will meet Γ and we will be forced to conclude that

$$l(\Gamma) \geq \sum_{j=1}^{\infty} l(\Gamma \cap \Delta_j) \geq \sum_{j=1}^{\infty} r_j = \infty,$$

contradicting our construction ensuring that $l(\Gamma) < \infty$. Thus $|\mathcal{F}_{z_j}| > 0$ for, not only one, but many z_j. $\square$

Theorem 2.3.2. *If E is an arbitrary Swiss cheese there exists a point $x_0 \in E$ which is a bpe for $H^p(E, dA)$, for $1 \leq p < \infty$. In particular, $H^p(E, dA) \neq L^p(E, dA)$ for any $p \geq 1$.*

Proof. Choose a point $x_0 \in E$ for which $|\mathcal{F}_{x_0}| > 0$. Assume for convenience that $x_0 = 0$ and let $X = \mathcal{F}_{x_0} \cap [0, 1]$. For any polynomial P,

$$\int_{\mathcal{F}_{x_0}} P dA = \int_0^{2\pi} \int_X P(re^{i\theta}) r dr d\theta = \int_X \left(\int_0^{2\pi} P(re^{i\theta}) d\theta \right) r dr = 2\pi C P(0),$$

where $C = \int_X r dr$. It follows that

$$|P(0)| \leq \frac{1}{2\pi C} ||P||_{L^1(E,dA)},$$

and so by Hölder's inequality, $x_0 = 0$ is a bpe for $H^p(E, dA)$ whenever $1 \leq p < \infty$. $\square$

In addition to what has been established it can be shown that whenever $|\mathcal{F}_{x_0}| > 0$ every function $f \in H^p(E, dA)$ actually admits an analytic continuation to a fixed neighborhood of x_0. To that end, choose $\epsilon > 0$ and small enough to ensure that the portion of $\mathcal{F}_{x_0}$ lying outside of $D_\epsilon = \{z : |z - x_0| < \epsilon\}$ has positive dA-measure. By the argument in the proof of the preceding theorem, there exists a function $h \in L^\infty$ with support in $E \setminus D_\epsilon$ such that

$$P(x_0) = \int P h \, dA$$

for all polynomials P. Thus $kdA = (z - x_0)hdA$ is an annihilating measure and so for any polynomial P and $\zeta \in \mathbb{C}$, it follows as in the proof of Corollary 2.3 that

$$P(\zeta) = \frac{1}{\hat{k}(\zeta)} \int \frac{P(z)}{z - \zeta} k \, dA$$

at any point where $\hat{k}(\zeta) = \int \frac{k(z)}{z-\zeta} dA$ is defined and not equal to zero. Since $\hat{k}$ is analytic in D_ϵ and $\hat{k}(x_0) = 1$ it follows that $|\hat{k}(\zeta)| \geq C > 0$ in a neighborhood U of x_0. Since the kernel $(z - \zeta)^{-1}$ is bounded on the support of the measure dA we conclude that

$$P(\zeta) \leq K ||P||_{L^1(E,dA)}$$

for some suitable constant K, and therefore ζ is a bpe for $H^1(E, dA)$.

Chapter 3 Capacities, Peak Points and Barriers

3.1 Analytic Capacity

The notion of analytic capacity was introduced by Ahlfors in 1947 in connection with the problem of characterizing sets of removable singularities for bounded analytic functions. In subsequent years others, and Vitushkin in particular, further developed the concept and used it to settle a number of questions concerning uniform approximation by rational functions on compact subsets of the plane.

As initially conceived, the *analytic capacity* of a compact set X, denoted $\gamma(X)$, is defined as follows:

$$\gamma(X) = \sup |f'(\infty)|,$$

where the supremum is extended over all functions f analytic in $\hat{\mathbb{C}} \backslash X$ and normalized so that

(a) $||f||_\infty = \sup_{\hat{\mathbb{C}} \backslash X} |f| \leq 1$

(b) $f(\infty) = 0$,

where $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$ and $\hat{\mathbb{C}}$ is the extended complex plane or Riemann sphere. In this case f is called an admissible function for $\gamma(X)$. A normal families argument involving Montel's Theorem establishes the existence of an extremal function g, such that $g'(\infty) = \gamma(X)$; this function is in fact unique and is called the *Ahlfors function* for X (cf. [14], p. 197).

For an arbitrary planar set X we let $\gamma(E) = \sup \gamma(X)$, the supremum now being taken over all compact sets $X \subseteq E$. For a more thorough discussion of analytic capacity and its properties see [14] and [42] where it is shown that

(1) If E and F are compact sets in $\mathbb{C}$ with $E \subset F$ then $\gamma(E) \leq \gamma(F)$.

(2) $\gamma(B_r) = r$ for every disk B_r of radius r;

(3) $\gamma(X) \approx \operatorname{diam}(X)$ whenever X is compact and connected; in particular, $\gamma(X) \leq \operatorname{diam}(X) \leq 4\gamma(X)$.

From the outset it was not known and is still not known whether γ is subadditive, and so possibly not a capacity in the usual sense. We now know, however, that γ is at least semiadditive in that

$$\gamma(E \cup F) \leq C\,(\gamma(E) + \gamma(F))$$

for all compact (and even Borel) sets $E, F \subseteq \mathbb{C}$ and some absolute constant C. The key point is that γ is equivalent to a second auxiliary capacity γ^+ defined as follows: For a compact set X and positive measure ν supported on X we form the Cauchy integral

$$\hat{\nu}(z) = \int \frac{d\nu(\zeta)}{\zeta - z} \tag{3.1}$$

and we define

$$\gamma^+(X) = \sup_{\nu} \nu(X)$$

to be the supremum over all positive measures ν such that $\hat{\nu} \in L^\infty(\mathbb{C})$ and $||\hat{\nu}||_\infty \leq 1$. Since $\hat{\nu}$ is analytic in $\hat{\mathbb{C}} \backslash X$ and $|\hat{\nu}'(\infty)| = \nu(X)$, the function $\hat{\nu}$ is admissible for γ and so

$$\gamma^+(X) \leq \gamma(X).$$

As before, if E is an arbitrary planar set we let $\gamma^+(E) = \sup \gamma^+(X)$ where X is compact and $X \subset E$.

The essential equivalence of γ and γ^+ was established by Tolsa [34]. Here is what he proved: There exists an absolute constant $C > 0$ so that

(4) $\gamma^+(E) \leq \gamma(E) \leq C\gamma^+(E)$ for all sets $E \subseteq \mathbb{C}$

(5) If E_n, $n = 1, 2, 3, \ldots$, are Borel sets, then $\gamma(\cup_n E_n) \leq C \sum_n \gamma(E_n)$.

Since Tolsa had previously shown that γ^+ is itself countably semiadditive, (4) implies (5).

Because γ^+ is defined directly in terms of the Cauchy integral, it can be used to establish a certain lower semi-continuity enjoyed by such integrals. Proofs of the following two lemmas can be found in [7]. For the sake of completeness, however, we have included a sketch of the proof of the second lemma since it will be used repeatedly. The argument here can be viewed as a modification of ideas in [7], Lemma 2 and [10], Lemma 1.

Lemma 3.1.1. *Let ν be a finite positive Borel measure of compact support in the complex plane $\mathbb{C}$ with the property that $|\hat{\nu}(z)| \leq C$ a.e.-dA for some constant C. Then $|\hat{\nu}(x_0)| \leq C$ at every point x_0 where $\tilde{\nu}(x_0) = \int \frac{d\nu(\zeta)}{|\zeta - x_0|} < \infty$.*

Lemma 3.1.2. *Let μ be a finite complex, compactly supported, Borel measure in $\mathbb{C}$, and let x_0 be any point where $\tilde{\mu}(x_0) < \infty$. For each $r > 0$ let $B_r = B_r(x_0)$ be the disk with center at x_0 and radius r, and let E be a set with the property that for every $r > 0$ there is a relatively large subset $E_r \subseteq (E \cap B_r)$ on which $\tilde{\mu}$ is bounded; that is,*

(1) $\tilde{\mu} \leq M_r < \infty$ on E_r,

(2) $\gamma(E_r) \geq \epsilon\gamma(E \cap B_r)$ for some absolute constant ϵ.

If, moreover, E is thick at x_0 in the sense that

$$\limsup_{r\to 0} \frac{\gamma(E \cap B_r)}{r} > 0 \tag{3.2}$$

then

$$|\hat{\mu}(x_0)| \leq \limsup_{z\to x_0,\, z\in E} |\hat{\mu}(z)|.$$

An immediate consequence is Lavrentiev's theorem on uniform polynomial approximation.

Theorem 3.1.3 (Lavrentiev)**.** *If X is a compact subset of $\mathbb{C}$, then $P(X) = C(X)$ if and only if X has no interior and the complement $\mathbb{C}\backslash X$ is connected.*

Proof. Suppose that $\mathbb{C}\backslash X$ is connected. By the argument in the proof of the Hartogs-Rosenthal theorem we may also assume that $|X| > 0$.

Let ν be any measure on X such that $\nu \perp P(X)$. Since the Cauchy transform $\hat{\nu}$ is analytic in the connected region $\mathbb{C}\backslash X$ and $\hat{\nu} \equiv 0$ in a neighborhood of ∞, it follows that $\hat{\nu} \equiv 0$ in $\mathbb{C}\backslash X$. Fix a point $x_0 \in X$ where $\tilde{\nu}(x_0) < \infty$ and let $B_r = B_r(x_0)$ be the disk of radius r with center at x_0. Because $\mathbb{C}\backslash X$ is connected we can find an arc $E_r \subset (B_r\backslash X)$ with diam $E_r \geq \frac{r}{2}$, and so that for each $r > 0$, $\hat{\nu}$ is bounded by a constant M_r on E_r. Thus, $\gamma(E_r) \geq \frac{r}{8}$ for all $r > 0$ and

$$\limsup_{r\to 0} \frac{\gamma(B_r\backslash X)}{r} > 0.$$

According to the lemma, then

$$|\hat{\nu}(x_0)| \leq \limsup_{z\to x_0,\, z\in B_r\backslash X} |\hat{\nu}(z)| = 0.$$

Since $\tilde{\nu}(x) < \infty$ a.e-dA on X, it follows that $\hat{\nu} = 0$ a.e-dA, and therefore $P(X) = C(X)$. $\square$

Proof of Lemma 3.2. For convenience assume that $x_0 = 0$. Using the fact that $\gamma \approx \gamma^+$ yields:

$$\limsup_{r \to 0} \frac{\gamma^+(E \cap B_r)}{r} > 0.$$

.

This implies there is a constant $C > 0$ and a sequence of $r \to 0$ such that $\gamma^+(E_r) > C\, r$ for each corresponding r. Consistent with the definition of γ^+, we can select a positive measure σ_r on E_r with

(1) $||\sigma_r|| = \sigma_r(E_r) \geq C\, r$,

(2) $|\hat{\sigma}_r| \leq 1$ a.e.-dA.

Setting $\nu_r = \frac{\sigma_r}{||\sigma_r||}$ we obtain a probability measure on $E_r \subset (E \cap B_r)$, and $|\tilde{\nu_r}| \leq \frac{C}{r}$ a.e.-dA for some absolute constant C. It is easy to check that

(i) $\displaystyle\int \frac{d\nu_r(\zeta)}{z - \zeta} \to \frac{1}{z}$ for every $z \neq 0$ as $r \to 0$,

(ii) $\displaystyle\int \frac{d\nu_r(\zeta)}{|z - \zeta|} \leq \frac{2}{|z|}$ for $|z| \geq 2r$.

Because $\tilde{\mu} \leq M_r$ on E_r, it follows from Fubini's theorem that

$$\int \left(\int \frac{d\nu_r(\zeta)}{|z - \zeta|} \right) d|\mu|(z) = \int \left(\int \frac{d|\mu|(z)}{|z - \zeta|} \right) d\nu_r(\zeta) \leq M_r,$$

and hence $\tilde{\nu}_r < \infty$ a.e. - $d|\mu|$. Therefore, by Lemma 3.1, $|\hat{\nu}_r(z)| \leq \dfrac{C}{r}$ a.e. - $d|\mu|$.

By an interchange in the order of integration,

$$\int \hat{\mu}(\zeta)\, d\nu_r(\zeta) = \int_{|z|<2r} + \int_{|z|\geq 2r} \left\{ \int \frac{d\nu_r(\zeta)}{z - \zeta} \right\} d\mu(z).$$

As a consequence of (i) and (ii) we have

$$\lim_{r \to 0} \int_{|z|\geq 2r} \left\{ \int \frac{d\nu_r(\zeta)}{z - \zeta} \right\} d\mu(z) = \int \frac{d\mu(z)}{z} = \hat{\mu}(0).$$

For the remaining integral over $|z| < 2r$ we have the estimate

$$\left| \int_{|z|<2r} \left\{ \int \frac{d\nu_r(\zeta)}{z - \zeta} \right\} d\mu(z) \right| \leq \int_{|z|<2r} \frac{C}{r}\, d|\mu|(z) \leq 2C \int_{|z|<2r} \frac{d|\mu|(z)}{|z|},$$

and the last integral tends to zero as $r \to 0$ by our assumption that $\tilde{\mu}(0) < \infty$. This establishes property

$$\lim_{r \to 0} \int \hat{\mu}\, d\nu_r = \hat{\mu}(0),$$

and the desired conclusion is immediate; that is,

$$|\hat{\mu}(x_0)| \leq \limsup_{z\to x_0,\, z\in E} |\hat{\mu}(z)|.$$

□

Equally important to us, of course, is Vitushkin's criterion for rational approximation (cf. [37] and [14], p. 207).

Theorem 3.1.4 (Vitushkin)**.** *$R(X) = C(X)$ if and only if for dA almost all points $x \in X$*

$$\limsup_{r\to 0} \frac{\gamma(B_r(x)\backslash X)}{r} > 0. \tag{3.3}$$

Proof of sufficiency. Let ν be a measure on X with $\nu \perp R(X)$. Then, $\hat{\nu} \equiv 0$ in $\mathbb{C}\backslash X$ and by Lemma 3.2 the capacitary density assumption on the complement of X implies that $\hat{\nu} = 0$ a.e.-dA on X. Hence, $R(X) = C(X)$. For a proof of necessity see [37] or [14], p. 207. □

Vitushkin [37] also shows that for any compact set X, and for almost all $z \in \mathbb{C}$ either

$$\lim_{r\to 0} \frac{\gamma(B_r(z)\backslash X)}{r} = 1$$

or

$$\lim_{r\to 0} \frac{\gamma(B_r(z)\backslash X)}{r^2} = 0,$$

which is often referred to as the instability of capacity. The instability phenomenon will be of critical importance in the construction of the counterexample in Section 4.2. Moreover, the validity of the condition

$$\limsup_{r\to 0} \frac{\gamma(B_r(x)\backslash X)}{r^2} > 0$$

at almost all points $x \in X$ is sufficient to ensure that $R(X) = C(X)$. In a very real sense, the instability of capacity is analogous to a well-known property of Lebesgue measure: In this case, if $E \subset \mathbb{R}^2$ is a Borel set then either

$$\lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1 \quad \text{or} \quad \lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 0$$

at almost every point $x \in E$.

3.2 L^q Capacities

In order to deal with questions concerning rational approximation in the $L^p(X, dA)$ norm we can proceed more or less in the manner outlined above. In this case, however, the relevant dual space is $L^q(X, dA)$ where $\frac{1}{p} + \frac{1}{q} = 1$. And again, it will be essential to take advantage of any underlying continuity associated with Cauchy potentials of the form

$$\hat{k}(z) = \int \frac{k(\zeta)}{\zeta - z} dA_\zeta$$

with $k \in L^q(X, dA)$.

If $q > 2$ then $\hat{k}$ is actually continuous in the entire plane, and no more needs to be said. If, on the other hand, $1 < q < 2$ then in order to describe the exceptional sets for the corresponding Cauchy potential $\hat{k}$ it is necessary to introduce appropriate L^q capacities.

By definition, for any Borel set E and $1 < q \leq 2$,

$$C_q(E)^{1/q} = \sup_\nu \nu(E),$$

the supremum being taken over all positive measures ν concentrated on E for which $||\tilde{\nu}||_{L^p(dA)} \leq 1$. For additional information and background material on these nonlinear capacities the reader is referred to the books [1] , [22], and articles [5], [13], [24] where proofs of the following can be found:

(i) if Φ is a contraction $C_q(\Phi E) \leq k\, C_q(E)$, were k is a constant depending only on q.

(ii) $C_q(B_r) \approx C_q(\operatorname{diam} B_r) \approx r^{2-q}$, $1 < q < 2$, and $C_2(B_r) \approx (\log(\frac{1}{r}))^{-1}$ for any disk B_r of radius r.

Concerning property (i) see, in particular, [1], p. 140 and [5], p. 411.

A property is said to hold *q-quasieverywhere* if the set where it fails has q-capacity zero. As an element of W_1^q the transform $\hat{k}$ is *q-quasicontinuous* in the sense that: Given any $\epsilon > 0$ there exists an open set U such that $C_q(U) < \epsilon$ and $\hat{k}$ is continuous in the complement of U. In addition there is a much subtler pointwise notion of continuity associated with functions in W_1^q, called *fine continuity*. A function $h \in W_1^q$, which we can assume to be defined q-quasieverywhere, is said to be *q-finely continuous* at a point x_0 if there exists a set E that is *thin*, or *sparse*, in a potential theoretic sense at x_0 and

$$\lim_{x \to x_0,\, x \in \mathbb{C} \setminus E} h(x) = h(x_0).$$

If E is not thin at x_0 it is said to be *thick* at that point. In our case it is sufficient to know that E is thick at x_0 if

$$\liminf_{r\to 0} \frac{C_q(E \cap B_r)}{C_q(B_r)} > 0, \tag{3.4}$$

where $B_r = B_r(x_0)$ is the disk with center at x_0 and radius r (cf [7], p. 221).

The following is due to Hedberg [19], p. 161 and [20], p. 316, and can be viewed as an L^p-analogue of Vitushkin's theorem on uniform rational approximation. For $p = 2$ the result goes back to Havin [18].

Theorem 3.2.1 (Hedberg)**.** *Let X be a compact set having no interior, and let $2 \le p < \infty$. Then, $R^p(X, dA) = L^p(X, dA)$ if and only if*

$$\limsup_{r\to 0} \frac{C_q(B_r(x)\backslash X)}{C_q(B_r(x))} > 0$$

for almost every $x \in X$.

Proof of sufficiency. Let $k \in L^q(X, dA)$ with $k \perp R^p(X, dA)$. Then, $\hat{k} \equiv 0$ in $\mathbb{C}\backslash X$ and the argument in Lemma 3.2 can be modified to show that $\hat{k}(z) = 0$ at every point $z \in X$ where $\displaystyle\int \frac{|k(\zeta)|}{|\zeta - z|} dA_\zeta < \infty$. Hence, $\hat{k} = 0$ a.e.-dA. $\square$

The L^q-capacities are also known to be subject to the same sort of instability as analytic capacity; that is, either

$$\lim_{r\to 0} \frac{C_q(B_r(x)\backslash X)}{C_q(B_r(x))} = 1$$

or

$$\lim_{r\to 0} \frac{C_q(B_r(x)\backslash X)}{r^2} > 0$$

(cf. Fernstrom [12], p. 245). Consequently, if the condition

$$\lim_{r\to 0} \frac{C_q(B_r(x)\backslash X)}{r^2} > 0$$

is satisfied at almost all points $x \in X$, then $R^p(X, dA) = L^p(X, dA)$.

3.3 Peak Points and Bishop's Theorem

Peak points play a key role in the theory of uniform rational approximation. By definition, $x \in X$ is a *peak point* for $R(X)$ if there exists a $f \in R(X)$ such that

(1) $||f||_\infty = 1$

(2) $f(x) = 1$

(3) $|f(y)| < 1$ whenever $y \neq x$.

A *representing measure* for a point x is a finite positive Borel measure μ supported on X, of total mass 1, such that

$$f(x) = \int_X f d\mu$$

for every $f \in R(X)$. Any complex measure μ having the same reproducing property is said to be a *complex representing measure*. In general, a given point can have many different representing measures. It can be shown, however, that x is a peak point for $R(X)$ if and only if the unit point mass at x is the only representing measure for x, and this occurs if and only if $\mu(\{x\}) = 1$ for every complex representing measure for x (cf. [14], p. 54). The following establishes an important connection between peak points and annihilating measures:

Theorem 3.3.1. *The point $x_0 \in X$ is a peak point for $R(X)$ if and only if the Cauchy transform, $\hat{\nu}(x) = 0$ whenever $\nu \perp R(X)$ and $\tilde{\nu}(x) < \infty$.*

Proof. If x is not a peak point then there exists a representing measure ρ for x with $\rho(\{x\}) = 0$. Then $\nu = (z - x)\rho \perp R(X)$ and $\hat{\nu}(x) = \tilde{\nu}(x) = 1$.

If there exists a measure ν such that $\nu \perp R(X)$ with $\hat{\nu}(x) \neq 0$ and $\tilde{\nu}(x) < \infty$, then there exists a complex measure μ representing x with $\mu(\{x\}) = 0$ and therefore x is not a peak point. □

Bishop's peak point criterion for rational density is key in our main results (cf [9], p. 172 or [14], p. 54) and follows from theorem 3.4.

Theorem 3.3.2 (Bishop). *$R(X) = C(X)$ if and only if almost every (dA) point of X is a peak point for $R(X)$.*

3.4 Barriers

In order to extend Theorem 2.6 to the most general setting where $R(X) \neq C(X)$ it is essential that we have a way to decide whether a point $x_0 \in X$ is a bpe for $H^p(X, dA)$ or not. With this in mind we shall adopt a scheme due, in broad outline, to Thomson [33] and having its roots in the work of Mel'nikov [25]. Throughout the discussion ν will be a measure on X, not necessarily absolutely continuous with respect to area. For each $\lambda > 0$ the sets $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$ will play a critical role.

Let $x_0 \in X$ be a point where $\tilde{\nu}(x_0) < \infty$ and fix $\lambda > 0$. For each positive integer n form a grid in the plane consisting of lines parallel to the coordinate axes, and intersecting at those points whose coordinates are both integral multiples of 2^{-n}. The resulting collection of squares $\mathcal{G}_n = \{S_j\}_{j=1}^{\infty}$ of side length 2^{-n} is an edge-to-edge tiling of the plane; its members will be referred to as squares of the *n-th generation.*

Beginning with a fixed generation, the n-th say, choose a square $S^* \in \mathcal{G}_n$ with $x_0 \in S^*$. Denote by $\mathcal{G}_n^\lambda$ the collection of all squares in $\mathcal{G}_n$ for which

$$|E_\lambda \cap S| > \frac{1}{100}|S|. \tag{3.5}$$

K_n will denote the union of those squares in $\mathcal{G}_n^\lambda$ that can be joined to S^* by a finite chain of squares lying in $\mathcal{G}_n^\lambda$. If K_n is bounded, or empty, there exists a closed *corridor* or *barrier*, $Q_n = \cup_j S_{nj}$ composed of squares S_{nj} from $\mathcal{G}_n$ abutting $S^* \cup K_n$, separating the latter from ∞, adjacent to one another along their sides, and such that

$$|E_\lambda \cap S_{nj}| \leq \frac{1}{100}|S_{nj}| \tag{3.6}$$

for each j. The polynomial convex hull of Q_n is a polygon Π_n with its boundary Γ_n lying along sides of squares for which (3.6) is satisfied. Thus $|\hat{\nu}| \geq \lambda$ on a large portion of every square from S_{nj} meeting Γ_n. By adjoining to Π_n additional squares from $\mathcal{G}_n$ we obtain another polygon Π_n^* with boundary Γ_n^* in such a way that

(i) $\Gamma_n^* \supseteq \Gamma_n$,

(ii) $n^2 2^{-n} \leq \operatorname{dist}(\Gamma_n^*, \Gamma_n) \leq 3n^2 2^{-n}$.

This can be done by simply adjoining to Π_n additional squares from $\mathcal{G}_n$.

At this point let K_{n+1} denote the union of all squares in $\mathcal{G}_{n+1}^\lambda$ that can be joined to Π_n^* by a chain of squares in $\mathcal{G}_{n+1}^\lambda$. Again, if K_{n+1} is bounded, or empty, there is a second barrier Q_{n+1} abutting $\Pi_n^* \cup K_{n+1}$ and

$$|E_\lambda \cap S| \leq \frac{1}{100}|S|$$

for every square S in Q_{n+1}. The polygon Π_{n+1} is defined to be the polynomial convex hull of Q_{n+1}, and the process continues. In this way we obtain a nested sequence of polygons

$$\Pi_n \subseteq \Pi_{n+1} \subseteq ... \subseteq \Pi_{n+l} \subseteq ...$$

and compact sets $K_j \subseteq \Pi_j \backslash \Pi_{j-1}$, some of which may be empty, such that if $K_j \neq \emptyset$

(a) K_j is the union of squares in $\mathcal{G}_j$ and connects Γ_{j-1}^* to Q_j;

(b) $|E_\lambda \cap S| > \frac{1}{100}|S|$ for each $S \subseteq K_j$;

(c) $\mathrm{dist}(K_j, \Gamma_j^*) \leq \mathrm{dist}(K_j, \Gamma_j) + \mathrm{dist}(\Gamma_j, \Gamma_j^*) < 4j^2 2^{-j}$.

Given an arbitrary disk $B_r = B(x_0, r)$ with center at x_0 there are two mutually exclusive possibilities:

(A) the sets K_j eventually exit B_r;

(B) there exists an infinite sequence of barriers Q_j, $j = n, n+1, n+2, \ldots$ extending outward from x_0 and lying entirely in B_r.

Should the first alternative (A) be the case for all $r > 0$ it can be shown (cf [7], p. 233) that the set E_λ satisfies all the hypothesis of Lemma 3.1, and therefore

$$\limsup_{r \to 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0.$$

The second alternative (B) implies that E_λ surrounds x_0 and contains sufficient mass to ensure that $H^1(X, dA)$ admits a bpe at x_0. This phenomenon replaces the collection of circles lying inside a Swiss cheese as discussed in Chapter 2. It is the subject of our next lemma.

Lemma 3.4.1. *If there exists an infinite sequence of barriers Q_j, $j = n, n+1, n+2, \ldots$ surrounding a point x_0, then $H^1(X, dA)$ admits a bpe at x_0.*

Proof. Because Q_n is a barrier, Γ_n is the union of certain specified sides of n-th generation squares S such that $|E_\lambda \cap S| \leq \frac{1}{100}|S|$; or, setting $F_\lambda = \{z : |\hat{\nu}(z)| \geq \lambda\}$, squares S for which

$$|F_\lambda \cap S| \geq \frac{99}{100}|S|.$$

We can assume for the purpose of argument that $\lambda = 1$, and we set $F = F_1$ and $E = E_1$.

The map $L : P \to P(x_0)$ can be viewed as a bounded linear functional on the space of polynomials when the latter is endowed with the norm $||P||_{L^\infty(\Gamma_n)} = \sup_{\Gamma_n} |P|$. As such, L can be extended in a norm preserving way to $C(\Gamma_n)$, the full space of continuous functions on Γ_n likewise endowed with the uniform norm. Hence, there exists a measure ω of finite total variation on Γ_n such that $||\omega|| = ||L||$ and

$$P(x_0) = \int_{\Gamma_n} P \, d\omega$$

for all polynomials P. The first step in the proof of the Lemma is to replace $\int_{\Gamma_n} P \, d\omega$ by an area integral over $F \cap Q_n$, committing only a small error.

Assume for the moment that P is a *fixed* polynomial. Take $\epsilon > 0$ and let $\Gamma_n = \cup I_j$ be the union of finitely many closed intervals I_j with mutually disjoint interiors chosen so that

$$\left| \int_{\Gamma_n} P\, d\omega - \sum_j P(\xi_j)\omega_j \right| < \epsilon, \tag{3.7}$$

whenever $\xi_j \in I_j$ and $\omega_j = \omega(I_j)$. We can arrange that each I_j is contained entirely in the side of a single square S in the barrier Q_n. Moreover, ω can have no point masses and so there is no ambiguity associated with the approximating sums for $\int_{\Gamma_n} P d\omega$ in (3.5).

For a fixed barrier square $S \subseteq Q_n$ with one or more of its sides in Γ_n, let x_S denote its center, and let ξ_j be one of the points in (3.7) situated on ∂S. Since by construction $\mathrm{dist}(\Gamma_n, \Gamma_{n+1}) \geq n^2 2^{-n}$, a form of Schwarz's lemma implies that

$$|P(\xi_j) - P(x_S)| \leq \frac{2^{n+1}}{n^2}\, |\, \xi_j - x_S|\, ||P||_{L^\infty(\Gamma_{n+1})} \leq \frac{\sqrt{2}}{n^2}\, ||P||_{L^\infty(\Gamma_{n+1})}.$$

If $S_1, S_2, ..., S_k$ represent the totality of squares in Q_n with sides along Γ_n and $B_1, ..., B_k$ are the corresponding inscribed disks, it follows by summing on j and the fact that Γ_n lies entirely inside the region bounded by Γ_{n+1} that

$$\left| P(x_0) - \int_{F\cap Q_n} P\, h_n\, dA \right| \leq \epsilon + \left(\frac{\sqrt{2}}{n^2} + \frac{2}{100} \right) ||\omega||\, ||P||_{L^\infty(\Gamma_{n+1})},$$

where $h_n = \displaystyle\sum_{j=1}^{k} \frac{4}{\pi} 2^{2n} \omega(S_j) \chi_{F\cap B_j}$ and $\chi_{F\cap B_j}$ is the characteristic function of $F \cap B_j$. Since $\epsilon > 0$ is arbitrary it can now be dropped from the inequality, and by choosing n sufficiently large we can arrange that

$$\left| P(x_0) - \int_{F\cap Q_n} P\, h_n\, dA \right| \leq \frac{3}{100} ||\omega||\, ||P||_{L^\infty(\Gamma_{n+1})}$$

for *all* polynomials P, and $||h_n||_\infty \leq \frac{2^{2n+2}}{\pi} ||\omega||$. Now repeat the process noting that the map

$$L_{n+1} : P \longrightarrow P(x_0) - \int_{F\cap Q_n} P\, h_n\, dA$$

can be extended from the space of polynomials and viewed as a bounded linear functional on $C(\Gamma_{n+1})$ with $||L_{n+1}|| \leq \frac{3}{100} ||L||$, where $L = L_n$ and $||L|| = ||\omega||$. Since all squares adjacent to Γ_{n+1} are barrier squares from Q_{n+1}, the argument above gives a function h_{n+1} with support in $F \cap Q_{n+1}$ so that

$$\left| P(x_0) - \int P\, h_n\, dA - \int P\, h_{n+1}\, dA \right| \leq \left(\frac{3}{100} \right)^2 ||L||\, ||P||_{L^\infty(\Gamma_{n+2})},$$

for all polynomials P, and $||h_{n+1}||_\infty \leq \frac{4}{\pi}2^{2(n+1)}\frac{3}{100}||L||$. Continuing in this way we obtain an infinite sequence of functions $h_n, h_{n+1}, ...$ such that for any $k > 0$,

$$\left|P(x_0) - \int P(h_n + ... + h_{n+k})dA\right| \leq \left(\frac{3}{100}\right)^{k+1} ||L|| \, ||P||_{L^\infty(\Gamma_{n+k+1})}.$$

For a given polynomial P the right side tends to zero as $k \to \infty$, since the curves Γ_{n+k} all lie in a bounded portion of the plane. Setting $h = \sum_{k=0}^{\infty} h_{n+k}$ it follows that

$$P(x_0) = \int P\, h\, dA$$

for all polynomials P. Moreover, $h \in L^\infty$ because the individual h_j's have disjoint supports and $||h_{n+k}||_\infty \leq ||h_n||_\infty$ for all $k > 0$; hence, $||h|| = ||h_n||_\infty$.

□

Chapter 4 Main Results

4.1 Bounded Point Evaluations for $H^p(dA)$

We have now reached the point where we can address the question raised in the introduction concerning the relation between uniform rational approximation and the existence of L^p-bounded point evaluations for the polynomials. Even in the most general situation, however, bpe's arise for essentially the same reason as in the case of the Swiss cheese. That is, a point $x_0 \in X$ is a bpe if it is surrounded by a portion of X having sufficient mass to ensure that the inequality (1.1) is satisfied at x_0. From our reasoning it would appear that in some cases the local geometry of X in a neighborhood of a bpe x_0 must be quite complicated, but we have not been able to rule out the possibility that there is a collection of concentric circles about x_0, lying entirely in X and having positive dA measure.

Theorem 4.1.1. *Let X be a compact subset of $\mathbb{C}$ with empty interior. If $R(X) \neq C(X)$, then there exists at least one point x_0 that yields a bpe for every $H^p(X, dA)$, $1 \leq p < \infty$. Moreover, every function $f \in H^p(X, dA)$ admits an analytic extension to a fixed neighborhood of x_0.*

Proof. By assumption there exists a nonzero measure ν such that $\nu \perp R(X)$. Since $\nu \neq 0$ as a measure there is at least one point x_0 such that

(a) $\tilde{\nu}(x_0) < \infty$;

(b) $\hat{\nu}(x_0) \neq 0$.

We can conclude, therefore, that there exists an infinite sequence of barriers relative to the set where $|\hat{\nu}|$ is bounded away from zero, and surrounding the point x_0 as described in Section 3. Such a collection of barriers must, of course, lie entirely in X since $\hat{\nu} \equiv 0$ in $\mathbb{C} \backslash X$.

Suppose for the moment that no such sequence of barriers exists. For an arbitrary, but fixed, $\lambda > 0$ consider the set $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$. By assumption E_λ must in a sense escape from x_0 to ∞. More precisely, we can find a connected set X linking x_0 to ∞ such that X is the union of squares from some generation, the n-th say, and higher, and certain narrow rectangles R_j, $j > n$, where

(1) $|E_\lambda \cap S| > \frac{1}{100}|S|$ for each square $S \subseteq X$,

(2) $\text{diam}(R_j) \approx j^2 2^{-j}$.

Given $r > 0$, let $B_r = B(x_0, r)$. By discarding certain superfluous pieces we can assume that $X \cap B_r$ is connected and joins x_0 to ∂B_r. Thus,

$$\gamma(X \cap B_r) \geq \frac{1}{4} \text{ diam } (X \cap B_r) \geq \frac{r}{8}.$$

On the other hand, it follows from the countable semiadditivity of analytic capacity that

$$\frac{r}{16} \leq \gamma(X \cap B_{r/2}) \leq C \left(\gamma(K) + \sum_{j=n}^{\infty} j^2 2^{-j} \right),$$

where K is the union of squares in X for which (1) is satisfied, and C is an absolute constant. Since we are free to begin with an arbitrary generation, we can let $n \to \infty$ and conclude that

$$\gamma(E_\lambda \cap B_r) \geq Cr$$

(cf. [7] p.233 for details). In particular,

$$\limsup_{r \to 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0,$$

and so Lemma 3.2 implies that

$$|\hat{\nu}(x_0)| \leq \limsup_{z \to x_0,\, z \in E_\lambda} |\hat{\nu}(z)| \leq \lambda.$$

Since this is valid for all $\lambda > 0$, we are led to infer, contrary to assumption, that $\hat{\nu}(x_0) = 0$. Consequently, for some $\lambda > 0$ there exists an infinite sequence of barriers surrounding x_0 that correspond to a set where $|\hat{\nu}| > \lambda$.

Lemma 3.7 now implies that the point x_0 yields a bpe for $H^1(X, dA)$ and so a bpe for all $H^p(X, dA)$, $1 \leq p < \infty$. From the discussion following Theorem 2.6 it follows that there is a fixed neighborhood U of x_0 such that every function $f \in H^p(X, dA)$ extends analytically to U. □

That fact in itself suggests that the conclusion of Theorem 4.1 is linked in a fundamental way to the presence of non-peak points, and that is indeed the case.

Corollary 4.1.2. *If $x_0 \in X$ is not a peak point for $R(X)$, then x_0 yields a bpe for $H^p(X, dA)$, $1 \leq p < \infty$; that is,*

$$|P(x_0)| \leq C_p \, ||P||_{L^p(X,\, dA)}$$

for every polynomial P, and some constant C_p depending only on p.

The proof of this corollary is a direct consequence of Theorem 3.5

The argument in the proof of Theorem 4.1 can therefore proceed exactly as before. It would be remiss, however, if we did not mention that Aleman, Richter and Sundberg [2] have also shown that almost every point where (a) and (b) are both satisfied corresponds to a bpe for $H^p(X, dA)$. That is sufficient for the proof of Theorem 4.1, but not for the corollary. The underlying feature here is that Lemma 3.2 applies at *every point* x_0 where (a) and (b) are satisfied simultaneously.

4.2 A Counterexample for $R^p(dA)$

In 1966 Sinanjan [32] announced the following result which now stands in striking contrast to Theorem 4.1:

Theorem 4.2.1 (Sinanjan). *There exists a compact nowhere dense set X such that $R(X) \neq C(X)$, but nevertheless $R^p(X, dA) = L^p(X, dA)$ for all p, $1 \leq p < \infty$.*

His proof depends on a construction of Mergeljan [27], p. 315 and actually produces a Swiss cheese X with the desired properties. The reader, however, is referred to an earlier paper [31] for many of the computational details. Here we shall describe an entire family of compact nowhere dense sets X having a locally non-rectifiable perimeter such that $R(X) \neq C(X)$, and still $R^p(X, dA) = L^p(X, dA)$ for all $p < \infty$.

Proof of Theorem 4.3. We begin with the construction of a planar Cantor set as follows: Let $Q^0 = [0, 1]\text{x}[0, 1]$ be the closed unit square. Choose 4 closed squares inside Q^0 with side length 1/4, having sides parallel to the coordinate axes, and so that each square contains a vertex of Q^0. Next, apply the same procedure to each of the four squares obtained in the first step. In this way we obtain 16 squares each having side length 1/16. Continuing in this way, at the n-th stage we obtain 4^n closed squares Q^n_j, $j = 1, 2, 3, ..., 4^n$ each having side length $1/4^n$. For each n let

$$E_n = \bigcup_{j=1}^{4^n} Q^n_j$$

and define

$$K = \bigcap_{n=1}^{\infty} E_n.$$

The set K is most commonly referred to as the *corner quarters Cantor set*. It can easily be checked that the orthogonal projection of K onto the line $2y = x$ covers an interval of length $3/\sqrt{5}$; in particular, a line segment of length greater then

$\frac{1}{2}\operatorname{diam}(Q^0)$. Not quite as obvious, however, is the fact that $\gamma(K) = 0$. Cantor sets with these properties were first produced by Vitushkin [36], and his construction was later simplified by Garnett [15] and Ivanov [21], pp 346-348.

Now iterate the procedure outlined above. Decompose Q^0 into 4 congruent squares S_j^1, $j = 1, 2, 3, 4$ by lines through midpoints of the opposite sides. In each square S_j^1 construct a Cantor set K_j^1 similar to K and differing only by a scaling factor of 1/4. Let $K_1 = \cup_j K_j^1$ be the union of the four scaled down Cantor sets, and continue the bisection process in the same manner, thereby obtaining a sequence of Cantor sets $K_1, K_2, K_3, \ldots$ having these properties:

(1) $\gamma(K_n) = 0$ for all $n = 1, 2, 3, \ldots$

(2) $E = \cup_n K_n$ is dense in Q^0

(3) $\Lambda(\operatorname{proj}(K_j^n)) > \frac{1}{2}\operatorname{diam}(S_j^n)$.

Here $K_n = \cup_j K_j^n$, $\operatorname{proj}(K_j^n)$ denotes the orthogonal projection of K_j^n onto the line $2y = x$, and $\Lambda(\operatorname{proj}(K_j^n))$ denotes the 1-dimensional Hausdorff measure or length of this projection. It follows from Tolsa's theorem on the countable semiadditivity of analytic capacity that $\gamma(E) = 0$. In this case, however, where E is the countable union of compact sets of capacity zero the full force of Tolsa's theorem is not needed (cf. [14], p. 237 or [16], p. 12).

Because $\gamma(E) = 0$, and therefore $|E| = 0$, we are able to select a compact set X_0 lying in the interior of $Q = Q^0$, and so that $|X_0| > 0$ and $E \cap X_0 = \emptyset$. Pick $r_1 > 0$, but small enough to ensure that $\{z : \operatorname{dist}(z, X_0) < r_1\}$ lies entirely inside Q. Since K_1 is a compact totally disconnected set with $\gamma(K_1) = 0$, we can cover K_1 by finitely many open rectangles, having mutually disjoint closures with sides parallel to the coordinate axes, and so that the union Ω_1 of the open pieces satisfies $\gamma(\Omega_1) < \frac{1}{2}r_1$. Next, choose $r_2 < r_1$, but small enough that $\{z : \operatorname{dist}(z, X_0) < r_2\}$ does not meet $\overline{\Omega}_1$. Proceed as above to cover $K_2 \backslash \overline{\Omega}_1$ by finitely many open rectangles, with mutually disjoint closures, in such a way that

(i) $\gamma(\Omega_2) < \dfrac{1}{2^2} r_2$

(ii) $\gamma(\Omega_1 \cup \Omega_2) < C\left(\dfrac{r_1}{2} + \dfrac{r_2}{2^2}\right) < Cr_1$,

where C is the absolute constant guaranteed by Tolsa's theorem. Continuing in this way we obtain a decreasing sequence of numbers $r_j \downarrow 0$ and a sequence of open sets $\Omega_1, \Omega_2, \Omega_3, \ldots$ with these properties:

(a) $E \subset \cup_j \Omega_j$

(b) $X_0 \subseteq Q \backslash \cup_j \Omega_j$

(c) $\gamma(\Omega_j) < \dfrac{1}{2^j}\, r_j$

(d) $\gamma(\Omega_j \cup ... \cup \Omega_k) < \dfrac{C}{2^{j-1}} r_j$ for all $j = 1, 2, 3, ...$

Setting $X = Q \backslash \cup_j \Omega_j$ we obtain a compact nowhere dense set with $X_0 \subseteq X$, and we must prove that $R(X) \neq C(X)$, but $R^p(X, dA) = L^p(X, dA)$ for all $p \geq 2$.

By construction, for each point $x \in X_0$ the inequality

$$\frac{\gamma(B(x, r_j) \backslash X)}{r_j} \leq \frac{C}{2^{j-1}}$$

is satisfied for all $j = 1, 2, 3, ...$, where C is an absolute constant throughout. Hence, at each point of X_0 the lower capacitary density of the complement $\mathbb{C} \backslash X$ is zero. It follows from the instability of analytic capacity that

$$\lim_{r \to 0} \frac{\gamma(B(x, r) \backslash X)}{r} = 0$$

at a.e.-dA point $x \in X_0$ (cf. also [14], p. 207). By Vitushkin's Theorem 3.4 it follows that $R(X) \neq C(X)$.

Again by construction, for a.e.-dA point $x \in X$, and r sufficiently small depending on x, we have

$$\Lambda(\text{proj}(B(x, r) \setminus X) \geq Cr,$$

where C is an absolute constant independent of r. For a fixed $q < 2$ this implies that

$$C_q(B(x, r) \setminus X) \geq Cr^{2-q},$$

since q-capacity decreases modulo a constant under a contraction. Hence, at a.e.-dA point $x \in X$ the complement $\mathbb{C} \setminus X$ is thick in the sense of q-capacity. Suppose then that $k \in L^q(X, dA)$ and that $k \perp R^p(X, dA)$. Then $\hat{k} \equiv 0$ in $\mathbb{C} \setminus X$ and by fine continuity $\hat{k} = 0$ a.e.-dA on X. Therefore, $k = 0$ a.e. and it follows that $R^p(X, dA) = L^p(X, dA)$, and this holds for all $p > 2$. □

As indicated at the beginning of Section 4 we do not know of a single example of a compact nowhere dense set X such that $R(X) \neq C(X)$, and for which no point $x_0 \in X$ admits a collection of concentric circles $\mathcal{F}_{x_0}$ having positive dA measure and lying entirely inside X. The construction in the proof of Theorem 4.3, however, precludes the corresponding phenomenon for rectangles oriented so as to have two of its sides orthogonal to the line $2y = x$. The projection properties of irregular sets such as those that underlie the entire construction here are studied extensively in [11], chapt. 7. If X happens to be a set of finite perimeter *in the sense of DeGiorgi*, it can then be shown that there exist sufficiently many rectangles contained entirely in X to ensure that $H^p(X, dA)$ has a bounded point evaluation in X, thereby extending Theorem 2.2 to this more general situation (cf Trent [35]).

Bibliography

[1] D.R. Adams and L.I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren Math. Wiss. **314** Springer-Verlag, Berlin, 1996

[2] A. Aleman, S. Richter and C. Sundberg, *Nontangential limits in $P^t(\mu)$-spaces and the index of invariant subspaces*, Ann. of Math. **169** (2009), No. 2., 449-490.

[3] J. Bram, *Subnormal Operators*, Duke Math. J. **22** (1955), 75-94.

[4] J.E. Brennan, *Invariant subspaces and weighted polynomial approximation*, Ark. Mat. **11** (1973), 167-189.

[5] J.E. Brennan, *Point Evaluations, invariant subspaces and approximation in the mean by polynomials*, J. Funct. Anal. **34** (1979), 407-420.

[6] J.E. Brennan, *The Cauchy integral and certain of its applications*, Izv. Nats. Akad. Nauk. Armenii. Mat. **39**, (2004) no. 1, 5-48; English transl., J. Contemp. Math. Anal. **39**, no.1, (2004) no. 1, 2-49.

[7] J.E. Brennan, *Thomson's theorem on mean-square polynomial approximation*, Algebra i analiz **17**, No.2, (2005); English transl., St. Petersburg Math. J. **17**, No.2, (2006), 217-238.

[8] J.E. Brennan and E.R. Militzer, *L^p-bounded point evaluations for the polynomials and uniform rational approximation*, Algebra i analiz, **22**, No. 1 (2010), pp. 57-74; English transl., St. Petersburg Math. J.(to appear).

[9] A. Browder, *Introduction to Function Algebras*, W.A. Benjamin, New York, 1969.

[10] L. Carelson, *Mergelyan's theorem on uniform polynomial approximation*, Math. Scand. **15** (1964), 167-175.

[11] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts in Mathematics **85**, Cambridge Univ. Press, Cambridge (1985).

[12] C. Fernström, *On the instability of capacity*, Ark. Mat. **15** (1977), 241-252.

[13] B. Fuglede, *A simple proof that certain capacities decrease under contraction*, Hiroshima Math J. **19** (1989), 567-573.

[14] T.W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs 1969.

[15] J.B. Garnett, *Positive length but zero analytic capacity*, Proc. Amer. Math. Soc. **24** (1970), 696-699.

[16] J.B. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Math. **297**, Springer-Verlag, Berlin 1972.

[17] F. Hartogs and A. Rosenthal, *Über Folgen analytischer Funktionen*, Math. Ann. **104** (1931), 606-610.

[18] V. P. Havin, *Approximation in the mean by analytic functions*, Dokl. Akad. Nauk SSSR, **178** (1968), 1025-1028; English transl. Soviet Math. Dokl. **9** (1968), 245-248.

[19] L.I. Hedberg, *Approximation in the mean by analytic functions*, Trans. Amer. Math. Soc. **163** (1972), 151-171.

[20] L.I. Hedberg, *Non-linear potentials and approximation in the mean by analytic functions*, Math. Z. **129** (1972), 299-319.

[21] L.D. Ivanov, *Variations of Sets and Functions*, Nauka, Moscow 1975 (Russian).

[22] N.S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss., vol **180**, Springer-Verlag, New York - Heidelberg, 1972.

[23] M.A. Lavrentiev, *Sur les fonctions d'une varibles complex representables par des series de polynomes*, Herman, Paris (1936).

[24] V.G. Maz'ja and V.P. Havin, *Non-linear potential theory*, Uspehi Mat. Nauk **27**, no.6, (1972), 67-138; English transl., Russian Math. Surveys **27**, no.6, (1972), 71-148.

[25] M.S. Mel'nikov, *On the Gleason parts of the algebra $R(X)$*, Mat. Sb. **101** (1976), no. 2, 293-300; English transl., Math. USSR-Sb **30** (1976), 261-268.

[26] M.S. Mel'nikov and S.O. Sinanjan, *Questions in the theory of approximation of functions of one complex variable*, Contemporary Problems of Math., vol. **4**, Itogi Nauki i Tekhniki, VINITI, Moscow (1975), 143-245; English transl., J. Soviet Math. **5** (1976), 688-752.

[27] S.N. Mergeljan, *Uniform approximations to functions of a complex variable*, Uspehi Mat. Nauk (N.S.) **7** (1952), 31-122; English transl., Amer. Math. Soc. Transl. Vol **101**, (1954), 294-391.

[28] S.N. Mergeljan, *On the completeness of systems of analytic functions*, Uspehi Mat. Nauk (N.S.) **8** (1953), 3-63; English transl., Amer. Math. Soc. Transl. **19** (1962), 109-166.

[29] Alice Roth, *Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen*, Comment. Math. Helv. **11** (1938), 477-507.,

[30] C. Runge, *Zur Theorie der eindeutigen analytischen Funktionen*, Acta Math., vol. **6**, pp. 229-244, 1885.

[31] S.O. Sinanjan, *The uniqueness property of analytic functions on closed sets without interior points*, Sibirsk. Mat. Z. **6** (1965), 1365-1381 (Russian)

[32] S.O. Sinanjan, *Approximation by polynomials and analytic functions in the areal mean*, Mat. Sb. **69** (1966), 546-578; English transl., Amer. Math. Soc. Translations **74** (1968), 91-124.

[33] J.E. Thomson, *Approximation in the mean by polynomials*, Ann. of Math. **133** (1991), 477-507.

[34] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), 105-149.

[35] T. Trent, *Sufficient conditions for bounded point evaluations*, Integral Equations Operator Theory **13** (1990), 593-606.

[36] A.G. Vitushkin, *An example of a set of positive length, but zero analytic capacity*, Dokl. Akad. Nauk SSSR **127** (1959), no. 2, 246-249 (Russian).

[37] A.G. Vitushkin, *Analytic capacity of sets and problems in approximation theory*, Uspehi Mat. Nauk **22** (1967), 141-199; Englisgh transl., Russian Math. Surveys **22** (1967), 139-200.

[38] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*,(5th Edition), Math. Surveys Monogr., vol 20, Amer. Math. Soc. 1969.

[39] K. Weierstrass, *Uber die analytishce Darstellbarkeit songenannter willkurlicher Funktioneneiner reelen Veranderlichen*, Sitzungsberichte der Akademie zu Berlin, pp. 633-639, and 789-805; [French transl.: J. Math. Pures Appl., vol **2**, pp. 105-138, 1886].

[40] J. Wermer, *Report on subnormal operators*, Report of an International Conference on Operator Theory and Group Representations (Arden House, Harriman, NY, 1955), Publ. **387**, Nat. Acad. Sci.-Nat. Res. Council, Washington, D.C., 1955, pp 1-3.

[41] J. Wermer, *Banach algebras and analytic functions*, Adv. Math. **1** (1961), 51-102.

[42] L. Zalcman, *Analytic Capacity and Rational Approximation*, Lecture Notes in Math. **50**, Springer 1968.

Vita

Erin Militzer
Born: Kalamazoo, MI
February 2, 1982

Education

- M.A. Mathematics, University of Kentucky, May 2006
- B.S. Mathematics, Cum Laude, Central Michigan University, May 2004

Teaching

Primary Instructor - University of Kentucky - Full class responsibility(Fall 2008, Spring 2009)

Teaching Fellowship Algebra Cubed Fellow (August 2006 - August 2007)

Instructor - University of Kentucky - Collaborative exams(Fall 2005, Fall 2009, Spring 2010)

Recitation Leader - University of Kentucky(Fall 2004, Spring 2005, Fall 2007, Spring 2008)

Research Assistant - University of Kentucky

- Directed by James E. Brennan(Summer 2009)
- Directed by Carl Lee with Emphasis in Math Education(Summer 2005)

Awards and Honors

- Arts and Sciences Certificate for Outstanding Teaching Award, 2008-2009
- University of Kentucky - Emeritus Faculty Fellowship Recipient, 2008-2009
- Richtmeyer-Foust, Whitmore Award Finalist, Central Michigan University – 2004
- Academic Honors Scholarship, Central Michigan University – 2000-2004
- Michigan Educational Assessment Program (MEAP) Scholarship – 2000

Publications

- C. Berkesh, J. Ginn, E. Haller and E.R. Militzer,"A Survey of Relative Difference Sets," Rose Hulman Institute of Technology Undergraduate Math Journal, Volume 4, Issue 2, 2003.

- J.E. Brennan and E.R. Militzer,"L^p Bounded Point Evaluations for the Polynomials and Uniform Rational Approximation," Preprint.