Toric Bundles as Mori Dream Spaces

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Toric Bundles as Mori Dream Spaces

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

Toric Bundles as Mori Dream Spaces

A projective, normal variety is called a Mori dream space when its Cox ring is finitely generated. These spaces are desirable to have, as they behave nicely under the Minimal Model Program, but no complete classification of them yet exists. Some early work identified that all toric varieties are examples of Mori dream spaces, as their Cox rings are polynomial rings. Therefore, a natural next step is to investigate projectivized toric vector bundles. These spaces still carry much of the combinatorial data as toric varieties, but have more variable behavior that means that they aren't as straightforward as Mori dream spaces. Expanding on Gonzalez's 2012 result that all rank 2 projectivized toric vector bundles are Mori dream spaces, we give a combinatorial sufficient condition for when a rank r bundle is Mori dream, using Kaveh and Manon's description of a toric vector bundle by a linear ideal and an integral matrix. We then address the question: if a toric vector bundle projectivizes to a Mori dream space, when is the projectivization of the direct sum of that bundle with itself a Mori dream space? Expanding on the nice families of bundles found, we compute the positivity-related cones for these bundles and provide a description of additional classes of toric vector bundles that uphold the Fujita conjectures. Finally, we conclude with the subduction and KM algorithms, two Macaulay2-implemented algorithms that allow us to produce finite presentations of Cox rings of projectivized toric vector bundles, provided they exist, allowing for future work in the study of these bundles as Mori dream spaces.

KEYWORDS: Mori dream space, toric vector bundle, toric flag bundle, positivity, subduction

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To all those who supported it and will, understandably, never read it.

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Chapter 1 Background

The goal of this chapter is to establish necessary definitions and results for future chapters. Due to time and length constraints, decisions had to be made about where to begin and end the discussions in this chapter, always with the goal for this document to be as self-contained as possible. If necessary, additional information can be found in the following texts, each of which being the primary reference for the section they correspond to:

- **Section 1.1:** *Algebraic Geometry* by R. Hartshorne [10]
- **Section 1.2:** *Toric Geometry* by Cox, Little, and Schenck [2]
- **Section 1.3:** *Tropical Geometry* by Maclagan and Sturmfels [23]
- **Section 1.4:** *Matroid Theory* by J. Oxley [26]

In Section 1.1, we establish the foundational definitions of algebraic geometry, focused around a discussion of varieties and divisors. Section 1.2 introduces framework definitions from polyhedral geometry before establishing fundamental definitions from toric geometry, including this document's key characters: the toric vector bundle. In Section 1.3, we discuss a relatively-new area of algebraic geometry: tropical geometry, an integral tool for connecting geometry to combinatorics. Finally, we conclude with Section 1.4 as a brief overview of the terms and results we'll need from matroid theory, including a theorem that connects many of the previous sections' objects together.

1.1 Algebraic Geometry

1.1.1 Varieties

Classically, algebraic geometry is the study of varieties. For a field k and a subset $I \subset k[x_1, ..., x_n]$, an **algebraic variety** is the collection

$$V(I) = \{\bar{a} := (a_1, ..., a_n) \in k^n \mid f(\bar{a}) = 0 \text{ for all } f \in I\}.$$

Unless otherwise stated, we will exclusively consider the case where $k = \mathbb{C}$. There are many different adjectives that one may put before "variety" to further impose conditions. These will be introduced as they are needed. The classical first example of a variety is given in Example 1.1.1.

Example 1.1.1. Let $I = \langle y - x^2 \rangle \subset \mathbb{C}[x, y]$. Then the variety associated to I, denoted $V(I)$, is all points (x, y) where $y - x^2 = 0$. Rearranging allows for this to be recognized as $y = x^2$. Therefore, $V(I) = \{(x, y) \mid y = x^2\}$.

There are multiple topologies that can be put on affine varieties, but the one we will primarily consider is the **Zariski topology**, in which the closed sets are subvarieties and the Zariski open sets are their complements. There are instances where some Zariski open subsets are themselves affine varieties. For example, given $f \in \mathbb{C}[V] \backslash \{0\}$, let

$$V_f = \{p \in V \mid f(p) \neq 0\} \subseteq V.$$

Then V_f is Zariski open in V and can also be shown to be an affine variety.

There is a set which acts as a pseudo-inverse to varieties (for a discussion on why these sets are not inverses, see [4]). Given a set $X \subset \mathbb{C}^n$, consider

$$I(X) = \{f \in \mathbb{C}[x_1, ..., x_n] \mid f(\bar{a}) = 0 \text{ for all } \bar{a} \in X\}.$$

It can be shown that $I(X)$ is an ideal. Notice that if two functions, f and g, agree on all points $\bar{x} \in X$, we have $0 = f(\bar{x}) - g(\bar{x}) = (f - g)(\bar{x})$. Therefore, $(f - g) \in I(X)$. We can form the ring where all such functions are identified in what Cox, Little, and Schenck [2] call "the most important algebraic object" associated to a variety.

Definition 1.1.2. For a set $X \subset \mathbb{C}^n$, $\mathbb{C}[X] = \mathbb{C}[x_1, ..., x_n]/I(X)$ is the *coordinate ring* of X.

The coordinate ring (and the relationship between $V(I)$ and $I(X)$) allows us to utilize a ring structure on varieties. For example, we can represent the variety $V(y - x^2)$ by $\mathbb{C}[x, y]/\langle y - x^2 \rangle$. The following details key properties of coordinate rings:

- For an affine variety V, $\mathbb{C}[X]$ is an integral domain if and only if $I(V)$ is a prime ideal.
- A point p of an affine variety V, gives the maximal ideal

  $$\mathfrak{m}_p = \{f \in \mathbb{C}(X) \mid f(p) = 0\}$$

  and, by Hilbert's Nullstellensatz, all maximal ideals of $A(V)$ arise in this way.

- Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic $\mathbb{C}$-algebras.

The second correspondence allows us to define a new ring, given a variety. For an variety X, the **local ring of X at a point** p in X is

$$\mathbb{C}[X]_p = \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[X] \quad \text{and} \quad g(p) \neq 0 \right\}.$$

Note that because of the relationship between p and $\mathfrak{m}_p$, the local ring of X at a point p is occasionally denoted $\mathbb{C}[X]_{\mathfrak{m}_p}$.

Crucially, the last point is no longer true when we begin studying projective varieties. A **projective variety**, $V \subset \mathbb{P}^n_{\mathbb{C}}$, is the vanishing set of finitely many homogeneous polynomials from $\mathbb{C}[x_1, ..., x_n]$. Note that, in this case, we take **homogeneous** to mean that each term of the polynomial has the same **degree** (the sum of the exponents in a given term). The fact that two projective varieties can be isomorphic rings and yet not have isomorphic coordinate rings alludes to the subtleties of projective coordinate rings that we'll discuss more in future chapters.

1.1.2 Divisors

Along with studying various varieties, it is also of interest to study divisors. In general, a **Weil divisor** is a formal sum of prime codimension-1 subvarieties of a space. However, it is not uncommon for authors to simply say "divisor" when what they mean is "Weil divisor." Keeping with this tradition, unless otherwise stated, this text will omit the modifier and assume all divisors are Weil.

Definition 1.1.3. A **(Weil) divisor** is a formal linear combination, $D = \sum a_i d_i$ of codimension-1 subvarieties, where $a_i \in \mathbb{Z}$ and d_i are prime divisors.

The **order of vanishing** of f at d_i and is defined to be

$$\operatorname{ord}_{d_i}(f) = \sup\{p \in \mathbb{N} \mid f \in \mathfrak{m}_{d_i}^p\},$$

where $\mathfrak{m}_{d_i}$ is the maximal ideal associated to the *generic point* of the prime divisor d_i (see [2]). The following establishes some common adjective modifiers associated to divisors.

- A divisor is called **effective** if $a_i \geq 0$ for each i.

- A divisor is **principal** if it is defined by a single function and can be written $(f) := \text{div}(f) = \sum \text{ord}_{d_i}(f) \cdot d_i$.

Definition 1.1.4. For an algebraic variety, X, the *class group* of X is $\mathcal{CL}(X) = \text{Div(X)}/\text{Prin}(X)$, where $\text{Div}(X)$ are the divisors of X and $\text{Prin}(X)$ are the principal divisors of X.

Recall that a variety is called **normal** if the local ring at every point is an integrally closed domain.

Definition 1.1.5. For a normal, projective variety X with finitely generated class group, the *Cox ring* of X, denoted $\text{Cox}(X)$, is defined as

$$\text{Cox}(X) = \bigoplus_{[D]\in\mathcal{CL}(X)} H^0\left(X, D\right)$$

where $H^0(X, D) = \{f \in \mathbb{C}(X) \mid (\text{div}(f) + D) \text{ is effective}\}$ is the group of global sections of D on X.

Note that the sum is taken over divisor classes; therefore, we must choose a representative of each divisor class. However, this computation is independent of this choice of representative since, for any two divisors D and D' which differ by a principal generator (therefore are elements of the same class in $\mathcal{CL}(X)$) we have $H^0(X, D) \cong H^0(X, D')$. However, we do need to take some care that these representatives are chosen *uniformly* (see [1]).

While there was a study of Mori dream spaces prior to Hu and Keel's 2008 paper, the following theorem establishes an equivalence that we use as the definition of a Mori dream space.

Theorem 1.1.6 (Hu, Keel [15])**.** *A normal projective variety X is a Mori dream space if and only if Cox(X) is finitely generated.*

We will continue our discussion of Mori dream spaces much more starting in Chapter 2.

1.2 Toric Geometry

In this section, we conclude the background chapter by summarizing the first chapter of [2]. We introduce toric geometry through the lens of polyhedral geometry. While the mathematics is fairly intuitive, a discussion of polyhedral geometry does require many definitions. The section concludes by introducing the main focus of our study.

A **torus** T of dimension n is an algebraic variety isomorphic to $(\mathbb{C}^\times)^n$ where $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$. Under this isomorphism, T inherits a group structure. We then have the following definition for a toric variety.

Definition 1.2.1. An *affine toric variety* is an irreducible affine variety V containing a torus T as a Zariski open subset such that the action of T on itself extends to an action of T on V.

We are interested in building a toric variety from a *fan*, Σ. We will now go through the results and terms necessary to construct a toric variety in this way.

A **character** of T is a group homomorphism $\chi : T \to \mathbb{C}^\times$ utilizing an isomorphism $T \cong (\mathbb{C}^\times)^n$ then choosing $m = (a_1, ..., a_n) \in \mathbb{Z}^n$, mapping

$$\chi^m : (\mathbb{C}^\times)^n \to \mathbb{C}^\times, (t_1, ..., t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}.$$

As all characters of $(\mathbb{C}^\times)^n$ arise in this way [16], the set of all characters of $(\mathbb{C}^\times)^n$, M, forms a group isomorphic to $\mathbb{Z}^n$. The group M is called the **character lattice** of T.

Similarly, a **one-parameter subgroup** of a torus T is a group homomorphism $\lambda : \mathbb{C}^\times \to T$. For $u = (b_1, ..., b_n) \in \mathbb{Z}^n$, we get the mapping

$$\lambda^u : \mathbb{C}^\times \to (\mathbb{C}^\times)^n, (t) \mapsto (t^{b_1}, ..., t^{b_n}).$$

[16] also shows that all one-parameter subgroups take this form. Letting N denote the set of all one-parameter subgroups, the **lattice of one-parameter subgroups**, we get an isomorphism $N \cong \mathbb{Z}^n$.

M and N have a pairing, meaning there is a natural bilinear map

$$\langle\ ,\ \rangle : M \times N \to \mathbb{Z}.$$

After choosing an isomorphism $T \cong (\mathbb{C}^\times)^n$ identifying $M, N \cong \mathbb{Z}^n$, the pairing can be realized as the usual dot product, ie.

$$\langle m, u\rangle = \sum_{i=1}^{n} a_i b_i$$

where $m = (a_1, ..., a_n)$ and $u = (b_1, ..., b_n)$. This pairing realizes the subgroups as duals of each other, ie. $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

We now introduce cones, a key building block of a fan. There are two constructions used to construct/describe a cone: the V-description ("vertex") and the H-description ("hyperplane"). We start with using the V-description to define a convex polyhedral cone and later use the H-description to define its dual.

A **convex polyhedral cone** in N is a set of the form

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\} \subseteq N$$

where $S \subseteq N$ is finite. Since we will not consider the non-convex case, the "convex" modifier will often be suppressed. The **dimension** of a polyhedral cone, denoted $\dim(\sigma)$, is the smallest subspace $W = \mathrm{Span}(\sigma)$ of N containing σ. Given a polyhedral cone $\sigma \subset N$, we can form the **dual cone** by

$$\sigma^\vee = \{m \in M \mid \langle m, u \rangle \geq 0 \quad \text{for all} \quad u \in \sigma\}.$$

Proposition 1.2.2. *Let $\sigma \subseteq N$ be a polyhedral cone. Then $\sigma^\vee$ is a polyhedral cone in M and $(\sigma^\vee)^\vee = \sigma$.*

Given $0 \neq m \in M$, we define

$$H_m^+ = \{u \in N \mid \langle m, u \rangle \geq 0\}$$

to be a **positive half-space**. When the inequality is instead an equality, we have a **hyperplane**, denoted H_m. A **face** of a polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$. The containment of a face in a cone is denoted $\tau \preceq \sigma$.

Example 1.2.3. The below graphic shows an example of how a hyperplane defines a face of the shaded polytope.

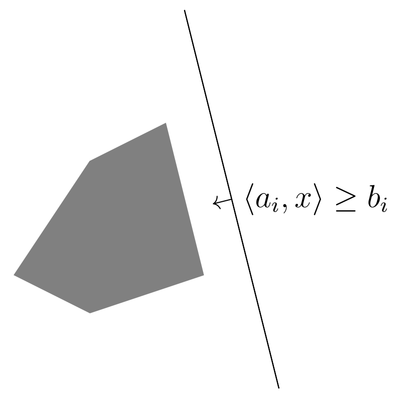

Figure 1.1: Polytope with a face defined by a hyperplane

A **polytope** in N is a set of the form

$$P = \mathrm{Conv}(S) = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0, \sum_{u \in S} \lambda_u = 1 \right\} \subseteq N$$

where $S \subseteq N$ is finite. We say that P is the **convex hull** of S.

Faces of a cone σ have the following properties:

- Every face of σ is a polyhedral cone.
- The intersection of two faces is a face.
- A face of a face is a face.

A **facet** is a face of codimension 1 and an **edge** is a face of dimension 1. When the origin is a face of a cone, that cone is called **strongly convex**. Since the edges of these cones all begin at the origin, they are called **rays**. These rays carry vital information when the cone is rational.

Let N be a lattice where $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Then a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is **rational** if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$.

Example 1.2.4. We have the following example of a rational fan.

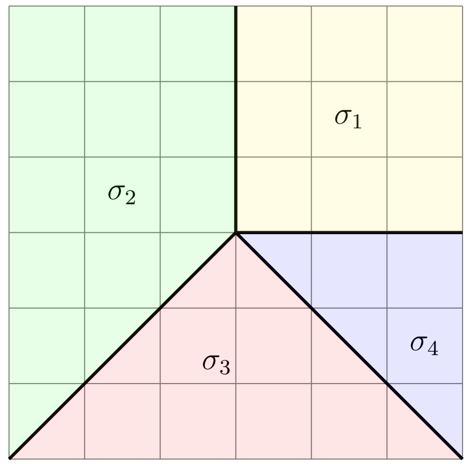

Figure 1.2: Rational fan in $\mathbb{R}^2$.

where the ray generators are $(1,0)$, $(0,1)$, $(-1,-1)$ and $(1,-1)$.

Lemma 1.2.5. *A strongly convex rational cone is generated by its rays.*

Proposition 1.2.6. *(Gordon's Lemma) Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational polyhedral cone. The lattice points*

$$S_\sigma = \sigma^\vee \cap M \subseteq M$$

form a finitely generated semigroup.

This semigroup allows us to build a toric variety from a cone, stated explicitly in the following theorem.

Theorem 1.2.7. *Let $\sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ be a rational polyhedral cone with semigroup S_σ. Then*

$$T_\sigma = Spec(\mathbb{C}[S_\sigma]) = Spec(\sigma^\vee \cap M)$$

is an affine toric variety. Moreover, all affine toric varieties come about in this way.

By Gordon's Lemma, we are able to ensure that $\mathrm{Spec}(\sigma^\vee \cap M)$ is, in fact, a variety. We can now use this information to create other toric varieties by gluing together affine toric varieties. Our first step in this is to consider a collection of cones, called a fan. More formally, a **fan** Σ is a collection of strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that

1. If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.

2. If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

Definition 1.2.8. Given a fan Σ in $N_{\mathbb{R}}$, the abstract toric variety $Y(\Sigma)$ is constructed from affine toric varieties, T_σ for $\sigma \in \Sigma$, by gluing T_σ and $T_{\sigma'}$ along their common open subset $T_{\sigma \cap \sigma'}$ for all $\sigma, \sigma' \in \Sigma$.

Example 1.2.9. Let $r \in \mathbb{N}$ and consider Σ_r in $\mathbb{N}_{\mathbb{R}} = \mathbb{R}^2$ shown below

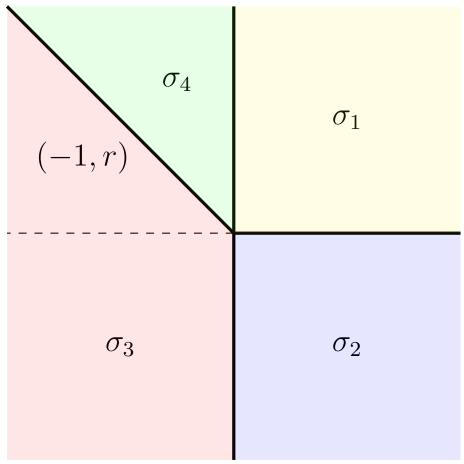

Figure 1.3: The fan structure, Σ_r, for $\mathcal{H}_r$.

The corresponding toric variety $Y(\Sigma_r)$ is covered by open affine subsets

$$\begin{aligned}
U_{\sigma_1} &= \mathrm{Spec}(\mathbb{C}[x, y]) \cong \mathbb{C}^2 \\
U_{\sigma_2} &= \mathrm{Spec}(\mathbb{C}[x, y^{-1}]) \cong \mathbb{C}^2 \\
U_{\sigma_3} &= \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-r}y^{-1}]) \cong \mathbb{C}^2 \\
U_{\sigma_4} &= \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{r}y]) \cong \mathbb{C}^2
\end{aligned}$$

We call $Y(\Sigma_r)$ the **Hirzebruch surface** $\mathcal{H}_r$.

1.2.1 (Projectivized) Toric Vector Bundles

In order to introduce toric vector bundles, we first present vector bundles in general. Conceptually, one can imagine a vector bundle over a space X as the attachment of a vector space V_x to every point $x \in X$ along with a notion of continuity between these vector spaces which reflects the topological structure of X itself. Locally, a vector bundle appears as a product space while its global structure may be more intricate. To formalize this idea, we have the following definition.

Definition 1.2.10. A **vector bundle** consists of:

1. a pair of topological spaces $(\mathcal{E}, X)$
2. a continuous surjection $\pi : \mathcal{E} \to X$ from the total space E to the base space X
3. for every x in X, the structure of a finite-dimensional vector space on the fiber $\pi^{-1}(\{x\})$

where, for every point p in X, there is an open neighborhood $U \subseteq X$ of p, a natural number k, and a homeomorphism

$$\phi : U \times \mathbb{C}^k \to \pi^{-1}(U)$$

such that for all $x \in U$ we have $(\pi \circ \phi)(x, v) = x$ for all $v \in \mathbb{C}^k$ and the map $v \mapsto \phi(x, v)$ is a linear isomorphism of vector spaces.

A **toric vector bundle**, denoted $\mathcal{E}$, is a vector bundle over $Y(\Sigma)$ along with the information of a torus action on $\mathcal{E}$ which is linear on the fibers and for which the projection map π is equivariant, i.e. the following diagram commutes when the vertical maps are the projection π and the horizontal maps are the action of the torus $\mathbb{T}$.

$$\begin{tikzcd}
\mathbb{T} \times \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathbb{T} \times Y(\Sigma) \arrow[r] & Y(\Sigma)
\end{tikzcd}$$

We say that the **rank** of a toric vector bundle is the dimension of its fiber.

Example 1.2.11. Let E_p be the fiber over the point $p \in Y(\Sigma)$. In the case where $\dim(E_p) = 1$, the vector bundle is referred to as a **line bundle**. There are two foundational examples of line bundles we consider. First, the **tautological bundle** over $\mathbb{P}^n$ is formed by making the fiber over a point $p \in \mathbb{P}^n$ the associated line $l_p \in \mathbb{C}^{n+1}$. The sheaf of sections of the dual of the tautological bundle is called the **twisting sheaf**. As a nod to this duality, the twisting sheaf is denoted $\mathcal{O}(1)$ while the tautological bundle is denoted $\mathcal{O}(-1)$. It is well-known that toric line bundles are classified by piecewise linear functions on Σ. When we are not referring specifically to one of these bundles, a line bundle will be denoted $\mathcal{L}$.

Another example of toric vector bundles we consider at tangent bundles. A **tangent space** to a toric variety at a point is the collection of all tangent vectors to that point; then, the **tangent bundle** is all such tangent spaces for the points of the variety. To see that the tangent bundle is a toric vector bundle, consider the torus action $\Phi_t : p \mapsto p \cdot t$ for $t \in \mathbb{T}$. Letting $\mathcal{T}_p$ be the tangent bundle at the point p, we can extend Φ_t to

$$\mathrm{d}\Phi_t|_p : \mathcal{T}_p \mapsto \mathcal{T}_{p\cdot t}.$$

The product rule for derivatives then defines an algebraic action of t on $\mathcal{T}$. We predominately consider the tangent (and cotangent) bundle of projective spaces to serve as examples (and nonexamples) as Mori dream spaces (see Theorem 2.1.3.)

We then obtain a **projectivized toric vector bundle** from a toric vector bundle by replacing each fiber $\mathcal{E}_p$ with $\mathbb{P}\mathcal{E}_p^\vee$. Projectivized toric vector bundles exhibit some nice behavior. For example, projectivized toric vector bundles are smooth and projective over the base field. Also, the Picard group of a projectivized toric vector bundle $\mathbb{P}\mathcal{E}$, denoted $\mathrm{Pic}(\mathbb{P}\mathcal{E})$ can be decomposed as

$$\mathrm{Pic}(\mathbb{P}\mathcal{E}) \cong \mathrm{Pic}(Y(\Sigma)) \times \mathbb{Z} \cong \mathcal{CL}(\mathbb{P}\mathcal{E})$$

where the line bundles on $\mathbb{P}\mathcal{E}$ are obtained from pullbacks from $Y(\Sigma)$ and the isomorphism to the class group is a result of the bundles being smooth [14].

Example 1.2.12. Consider $\mathcal{O} \oplus \mathcal{O}(r)$ as a bundle over $\mathbb{P}^1$. We can realize the Hirzebruch surface $\mathcal{H}_r$ as a projectivized toric vector bundle $\mathcal{H}_r \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(r))$.

Klyachko classified toric vector bundles on X in terms of finite dimensional vector spaces E with collections of $\mathbb{Z}$-graded filtrations $\{E_r^\rho\}$, called the **Klyachko data**. This identification is summarized from [20] as follows: Let E be the fiber of $\mathcal{E}$ over the identity point of the base torus $T \subseteq Y(\Sigma)$. For all rays $\rho \in \Sigma(1)$ and $r \in \mathbb{Z}$, we have the subspace $E_r^\rho \subseteq E$ and the filtration

$$E \supseteq \dots \supseteq E_r^\rho \supseteq E_{r+1}^\rho \supseteq \dots \supseteq 0.$$

For all top-dimensional cones in Σ and $m \in M$, we have $\mathcal{L} \subseteq E$ with

$$E \cong \bigoplus_{m \in M} \mathcal{L}_m$$

such that, for all $\rho \in \sigma(1)$ with ray generator u,

$$E_r^\rho = \bigoplus_{\langle u,m \rangle \geq r} \mathcal{L}_m.$$

1.3 Tropical Geometry

Tropical geometry is a relatively new field of mathematics that allows for a beautiful connection between algebra and combinatorics. At the heart of tropical geometry is the **tropical semi-ring**, $\mathbb{R} \cup \{\infty\}$. In this ring, "addition" (denoted $\oplus$) and "multiplication" (denoted $\otimes$) are defined as follows:

$$x \oplus y := \min(x, y) \qquad\qquad x \otimes y := x + y$$

Note that, in some references, tropical addition is defined as $x \oplus y := \max(x, y)$. While these constructions are isomorphic, we will exclusively use the plus-min convention. The following example demonstrated how a computation with these operations may be done.

Example 1.3.1. $2 \otimes (3 \oplus 4) = 2 \otimes (\min(3, 4)) = 2 \otimes 3 = 2 + 3 = 5.$

Many of the expected properties of these operations still hold. For example, commutativity and distributively hold in both operations. However, not everything carries over from the classical operation analogs.

Remark 1.3.2. Tropical multiplication, $\otimes$, has an inverse. For $a \in \mathbb{R}$, we have that $a \otimes -a = a + (-a) = 0$. However, tropical addition, $\oplus$, does not have an inverse. The only way for $\min(a, b) = \infty$ is if $a = b = \infty$. The non-invertibility of $\oplus$ is exactly why $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ is a semi-ring and not a ring.

To see why the identities are what they are, let's consider first tropical addition. We need a number x such that for any $a \in \mathbb{R}$,

$$a \oplus x = \min(a, x) = a.$$

Therefore, the identity for $\oplus$ is $+\infty$. Similarly, considering tropical multiplication, we need a number x such that for any $a \in \mathbb{R}$,

$$a \otimes x = a + x = a.$$

Therefore, the identity for $\otimes$ is 0.
These operations can be extended beyond the real numbers. A **tropical polynomial** is a finite linear combination of tropical monomials:

$$f(x_1, ..., x_n) = a \otimes x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus b \otimes x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots$$

where the coefficients $a, b, ...$ are real numbers are the exponents $i_1, j_1, ...$ are integers. While the tropical multiplication between the x_* have been suppressed, we can apply the tropical operations to rewrite the above as

$$\text{trop}(f(x_1, ..., x_n)) = \min(a + i_1x_1 + ... + i_nx_n, b + j_1x_1 + ... + j_nx_n, ...).$$

This rewrite allows us to make the following observation.

Lemma 1.3.3. *The tropical polynomials in the variables $x_1, ..., x_n$ are precisely the piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.*

Example 1.3.4. Consider $f = (0 \otimes x^3) \oplus (1 \otimes x^2) \oplus (3 \otimes x) \oplus 6$. Applying the tropical operations gives $f = \min(3x, 1 + 2x, 3 + x, 6)$. Each of these functions is plotted below, along with the minimum bolded in black. The piecewise black line is the graph of f.

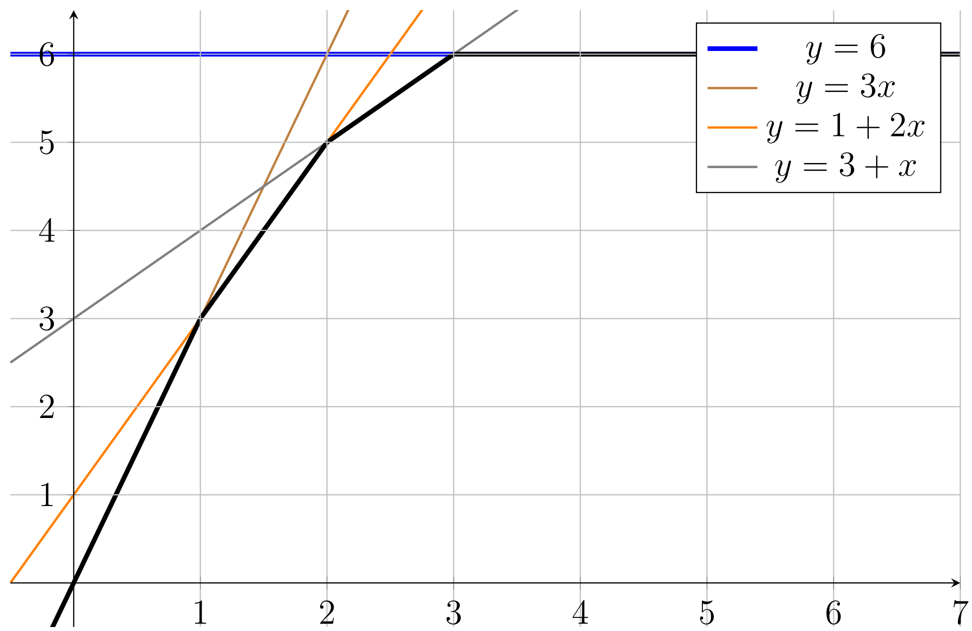

Figure 1.4: The graph of four lines and the corresponding tropical polynomial.

Another similarity between algebraic and tropical geometry is an object that can be formed from polynomials: varieties. Crucially, it is not longer appropriate to define the tropical variety as the zero-locus of a collection of polynomials. Instead, we need to introduce the notion of term ordering. In general, a **term order** is a choice of convention of how to impose an ordering on the terms of a polynomial. For our study, we primarily consider ordering the terms by a weight vector $\mathbf{w}$.

Definition 1.3.5. For a polynomial $f(\mathbf{x}) = \sum_{a \in A} c_a x^a \in K[x_1^{\pm}, ..., x_n^{\pm}]$ and a weight $\mathbf{w} = (w_1, ..., w_n) \in \mathbb{R}^n$, the *initial form of f with respect to* $\mathbf{w}$ is

$$\mathrm{in}_{\mathbf{w}}(f) = \left\{ \sum_{a' \in A} c_{a'} \mathbf{x}^{a'} \quad | \quad \langle a', \mathbf{w} \rangle \leq \langle a, \mathbf{w} \rangle, \text{ for } a \in A \right\}.$$

Example 1.3.6. Consider $f(x, y) = x^2 + xy^2 + y^3$ and $\mathbf{w} = (1, 2)$. Then our exponent vectors, a, are:

$$(2,0) \qquad (1,2) \qquad (0,3)$$

Computing the dot products, we get

$$(1,2) \cdot (2,0) = 2 + 0 = 2 \qquad (1,2) \cdot (1,2) = 5 \qquad (0,3) \cdot (1,2) = 6$$

As 2 is the smallest result, we identify the term of f with exponent vector $(2, 0)$. Therefore, $\mathrm{in}_{(1,2)}(f(x, y)) = x^2$.

However, our initial forms are not limited to being monomial.

Example 1.3.7. Consider $g(x, y) = x^2 + xy^2 + y^3 + y$ and $\mathbf{w} = (1, 2)$. Then:

$$\underbrace{(2,0)}_{x^2} \cdot \underbrace{(1,2)}_{\mathbf{w}} = 2 = \underbrace{(0,1)}_{y} \cdot \underbrace{(1,2)}_{\mathbf{w}}$$

The other dot products can be computed to confirm that 2 is the smallest result. So $\mathrm{in}_{(1,2)}(g(x, y)) = x^2 + y$.

Using the language of initial forms, we can define a tropical variety.

Definition 1.3.8. For a collection of polynomials $F = \{f_1, ..., f_k\} \in K[x_1^{\pm}, ..., x_n^{\pm}]$, the *tropical variety*, $\mathrm{TV}(F)$, is

$$\begin{aligned} \mathrm{TV}(F) &= \{\mathbf{w} \in \mathbb{R}^n \quad | \quad \mathrm{in}_{\mathbf{w}}(f_i) \text{ is not a monomial}\} \\ &= \{\mathbf{w} \in \mathbb{R}^n \quad | \quad \text{the min of } \mathrm{trop}(f_i)(\mathbf{w}) \text{ occurs at least twice}\}. \end{aligned}$$

Note that the condition of the minimum of the tropical polynomial, $\mathrm{trop}(f)$, occurring at least twice is graphically represented by the function changing from one part of the piecewise-linear portion to another.

Example 1.3.9. Referencing the function f and graph shown in Example 1.3.4, we can see that $\mathrm{TV}(f) = \{(1,3),(2,5),(3,6)\}$.

Example 1.3.10. For $f(x,y) = x^2 + xy^2 + y^3 + y$, we've seen in Example 1.3.7 that $\mathrm{in}_{(1,2)}(f) = x^2 + y$. As this is not a monomial, $(1,2) \in \mathrm{TV}(f)$. We have

$$\mathrm{trop}(f)|_{(x,y)=(1,2)} = \min\{2x, x+2y, 3y, y\}|_{(1,2)} = \min\{2,5,3,2\} = 2$$

which occurs when $2x = y \leq x + 2y, 3y$.

Let I be an ideal in $\mathbb{C}[x_1,...,x_n]$. A finite generating set $\mathcal{T}$ of I is a **tropical basis** if, for all vectors $\mathbf{w} \in \mathbb{R}^n$, there is a polynomial $f \in I$ for which the minimum in $\mathrm{trop}(f)(\mathbf{w})$ is achieved only once if and only if there is a $g \in \mathcal{T}$ for which the minimum in $\mathrm{trop}(g)(\mathbf{w})$ is achieved only once.

We can extend the idea of initial forms to all polynomials contained in an ideal. If I is a homogeneous ideal in $\mathbb{C}[x_0,...,x_n]$, then its **initial ideal** is

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle.$$

As this is infinitely generated, it would be preferable if there were a way to finitely describe the initial ideal; however, that would take a very particular generating set. A collection $\mathcal{G} = \{g_1,...,g_k\} \subset I$ is a **Gröbner basis** for I with respect to $\mathbf{w}$ if

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(g_1),...,\mathrm{in}_{\mathbf{w}}(g_k) \rangle.$$

There are a number of ways to confirm that the collection you have is a Gröbner basis. We will use the following:

Proposition 1.3.11. *A collection $\mathcal{G} = \{g_1,...,g_k\}$ is a Gröbner basis if and only if $S(g_i,g_j) = 1$ for all $g_i, g_j \in \mathcal{G}$ where*

$$S(g_i,g_j) = \frac{\mathrm{LCM}(g_i,g_j)}{LT(g_i)} g_i + \frac{\mathrm{LCM}(g_i,g_j)}{LT(g_j)} g_j$$

is called the ***S-pair*** *of g_i and g_j.*

In this proposition, $\mathrm{LT}(g_*)$ is the **leading term**, the first term of g_* after the terms have been ordered under the chosen ordering. We have that $\mathcal{G}$ is a **universal Gröbner basis** if $\mathcal{G}$ is a Gröbner basis with respect to every term order.

Gröbner bases come with an associated fan structure, called the **Gröbner fan**, of polyhedral cones indexing initial ideals. The top-dimensional cones correspond to distinct monomial initial ideals with respect to a term order.

Example 1.3.12. Consider the ideal $L = \langle x_1 - x_2, x_2 - x_3 \rangle \subset \mathbb{C}[x_1, x_2, x_3]$. The Gröbner fan of L is $\mathrm{GFan}(L) \cong \mathbb{R}^3$ with rays corresponding to the term order weights $(1,0,0),(0,1,0)$, and $(0,0,1)$.

Definition 1.3.13. Let L be a linear ideal with *tropicalized linear space*

$$\mathrm{Trop}(L) = \bigcap_{f \in L} TV\left(\mathrm{trop}(f)\right).$$

Then, for τ a maximal face of the Gröbner fan of L, the *apartment associated to* τ, denoted A_τ, is the intersection of τ with the tropicalized linear space of L, i.e.

$$A_\tau = \tau \cap \mathrm{Trop}(L).$$

In Chapter 2, we will use the data of points from $\mathrm{Trop}(L)$ to help define a toric vector bundle. However, we first need to conclude this chapter on background information with one final section.

1.4 Matroid Theory

Matroids are rich combinatorial objects with applications to countless areas of mathematics. While we will only cover information necessary for our study, interested readers should refer to James Oxley's *Matroid Theory* [26].

Definition 1.4.1. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subset 2^E$ is a collection such that:

- $\emptyset \in \mathcal{I}$
- If $I \in \mathcal{J}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
- If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $e \in (I_2 - I_1)$ such that $I_1 \cup e \in \mathcal{I}$.

The elements $I \in \mathcal{I}$ are called **independent sets**. Minimally dependent sets are called **circuits** and the collection of all circuits for a matroid $\mathcal{M}$ is denoted $C(\mathcal{M})$. The **rank** of a set $S \subseteq E$, denoted $\mathrm{rk}(S)$, is given by

$$\mathrm{rk}(S) := \max_{I \subseteq S} |I|.$$

The rank of the matroid $\mathcal{M} = (E, \mathcal{I})$ is then the rank of its ground set, ie. $\mathrm{rk}(\mathcal{M}) := \mathrm{rk}(E)$. Using this notion, we have that a set S is independent if and only if $\mathrm{rk}(S) = |S|$. A **flat** of a matroid is a set $F \subseteq E$ such that

$$\mathrm{rk}(F \cup \{x\}) = \mathrm{rk}(F) + 1$$

for all $x \in E \backslash F$. The collection of all flats of a matroid $\mathcal{M}$ is denoted $\mathcal{L}(\mathcal{M})$. Under inclusion, the pair $(\mathcal{L}(\mathcal{M}), \subseteq)$ forms a geometric lattice called the **lattice of flats**.

Given a linear ideal L, we can define the corresponding matroid $\mathcal{M}(L)$ as the matroid on $y_1, ..., y_m$ where a set I is independent if and only if I/L is independent.

While the combinatorial structure of a matroid can be arbitrarily complex, we will focus our study on two of the nicer, better-understood classes of matroids. First, the **uniform matroid** on n elements, denoted U_n^r, is the matroid whose independent sets are exactly those which contain at most r elements. Note that this means that, for a uniform matroid, the circuits are all those sets which contain exactly $r+1$ elements. Second, a matroid is called **representable** if there exists a matrix A whose columns correspond to the elements of M with linearly independent columns of A corresponding to independent elements of M.

Example 1.4.2. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The corresponding matroid has three elements where any two of them are independent, so A is a representation of U_3^2.

Matroids have an associated fan structure, called the Bergman fan. For a matroid $\mathcal{M}$ on n elements, the **Bergman fan of** $\mathcal{M}$, denoted $B(\mathcal{M})$, is

$$B(\mathcal{M}) = \{\mathbf{w} \in \mathbb{R}^n \mid \text{the minimum } \min_{i \in c} w_i \text{ occurs at least twice } \forall c \in C(\mathcal{M})\}.$$

Example 1.4.3. Consider U_3^2. The Bergman fan is

$$B(U_3^2) = \{\mathbf{w} \in \mathbb{R}^3 \mid x_i = x_j \leq x_k \text{ for } i, j, k \text{ distinct in } \{1, 2, 3\}\}.$$

To see the fan structure of $B(\mathcal{M})$, we need to consider a related object: the matroid polytope. For a matroid $\mathcal{M}$, the **matroid polytope**, $P(\mathcal{M})$ is

$$P(\mathcal{M}) = \text{Conv}\left(\sum_{i \in I} e_i \mid I \in \mathcal{I}\right).$$

That is to say, the matroid polytope is the convex hull of the indicator vectors associated to the independent sets of $\mathcal{M}$.

Example 1.4.4. Consider the uniform matroid U_3^2. Then

$$P(\mathcal{M}) = \text{Conv}(\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\})$$

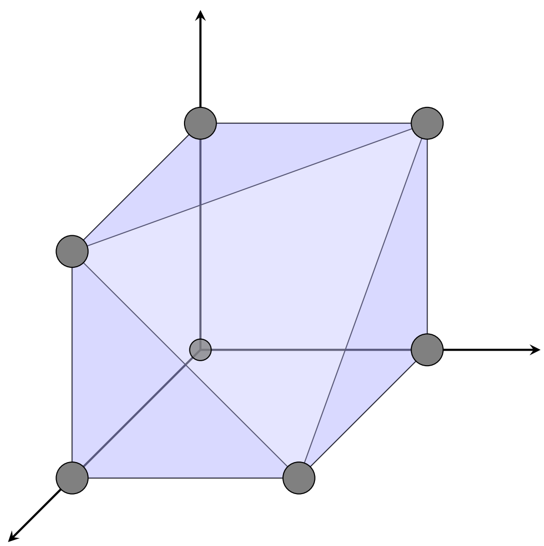

Figure 1.5: The matroid polytope of U_3^2.

Then, taking the inner normal fan to the maximal faces of the matroid polytope, denoted $\mathcal{N}(P(\mathcal{M}))$, we have that $B(\mathcal{M})$ is a subfan of $\mathcal{N}(P(\mathcal{M}))$. The following theorem provides the connection between matroid theory, tropical geometry, and our study of toric vector bundles.

Theorem 1.4.5. *For a linear ideal L, $\mathcal{M}(L)$ is the matroid whose circuits are minimally supported generators of L. Then Trop$(L) = B(\mathcal{M}(L))$ and GFan$(L) = \mathcal{N}(P(\mathcal{M}(L)))$.*

Chapter 2 Toric Vector Bundles

2.1 Past Results

Mori dream spaces are desirable to have, as they are well-behaved in the Minimal Model Program and other areas of birational geometry. However, while there are known examples and counterexamples, no complete classification of them yet exists. Some of the first examples of known Mori dream spaces were given by Cox in [3] where he showed that the Cox ring of a toric variety is a polynomial ring, making them some of the most trivial examples. This led Hering, Payne, and Mustaţă to ask when a projectivized toric vector bundle $\mathbb{P}\mathcal{E}$ is a Mori dream space [13]. Projectivized toric vector bundles still carry much of the combinatorial data of their toric variety (see [20], [18]), but have variable behavior which leads to them not always being Mori dream spaces. The following two theorems describe some of the first results in investigating when a projectivized toric vector bundle is a Mori dream space and Example 2.1.3 demonstrates how things could go wrong.

Theorem 2.1.1 (Hausen, Süß, [12])**.** *Projectivized tangent bundles of toric varieties are Mori dream spaces.*

Notice this result is an example of how a nice bundle on a well-understood example of a Mori dream space can give a Mori dream space. This theorem also gives an infinite family of Mori dream spaces of arbitrarily high rank. The next theorem identifies another natural candidate for nice bundles: bundles of low rank.

Theorem 2.1.2 (González, [6])**.** *For a rank 2 toric vector bundle $\mathcal{E}$, the projectivized toric vector bundle $\mathbb{P}\mathcal{E}$ is a Mori dream space.*

Therefore, this begs the question: *What about rank 3 bundles?* Unfortunately, this case is not as straightforward. In their paper [8], González, Hering, Payne, and Süß, construct the following example of a toric threefold whose projectivized cotangent bundle is not a Mori dream space.

Example 2.1.3 ([8], Example 4.2)**.** Let k be a field of characteristic not equal to two or three. The vectors

$$v_1 = (0,0,1), \quad v_2 = (0,1,0), \quad v_3 = (1,1,1), \quad v_4 = (-1,-2,-2)$$

span the four rays of a unique complete fan $\Sigma_4 \in \mathbb{R}^3$. The corresponding toric variety $Y(\Sigma_4)$ is isomorphic to $\mathbb{P}^3$. We also consider the vectors

$$\begin{aligned}&v_5 = (1,1,2), \quad v_6 = (0,-1,1), \quad v_7 = (1,0,1), \quad v_8 = (1,-1,1),\\&v_9 = (-1,-2,-1), \quad v_{10} = (-1,-1,0), \quad v_{11} = (-1,-1,1),\\&v_{12} = (-1,0,1), \quad v_{13} = (-1,1,1), \quad v_{14} = (0,1,1)\end{aligned}$$

and let Σ_i be the stellar subdivision of Σ_{i-1} along the ray spanned by v_i, for $5 \leq i \leq 14$. Letting $\Sigma = \Sigma_{14}$, the authors then show that the Cox ring of the projectivized cotangent bundle of $Y(\Sigma)$ is isomorphic to the Cox ring of the blowup $\mathrm{Bl}_S(\mathbb{P}^2_k)$, where $S = \{v_1^{\perp}, ..., v_{14}^{\perp}\}$, which is known to not be finitely generated. Therefore, $\mathbb{P}\mathcal{T}^*Y(\Sigma)$ cannot be a Mori dream space.

Therefore, we would like to find a condition that would be sufficient to check that a projectivized toric vector bundle is a Mori dream space. In order to do this, we move from algebraic geometry to combinatorics, associating to each toric vector bundle a matrix D and a linear ideal $L \subseteq \mathbb{C}[y_1, ..., y_n]$. We refer to this matrix D as the diagram associated to the toric vector bundle.

Definition 2.1.4. For a fan Σ and linear ideal L, a *diagram* D of (Σ, L) is a matrix whose rows are indexed by the rays of Σ satisfying:

1. each row of D is a point in $\mathrm{Trop}(L)$,
2. if $p_{i_1}, ..., p_{i_l} \in \Sigma(1)$ are rays contained in the same face, then the corresponding rows live in a common apartment of $\mathrm{Trop}(L)$.

This association of (L, D) to a toric vector bundle is a way of encoding the Klyachko data into combinatorial objects. What's more, this process is reversible; by passing through the Klyachko data, we can associate every toric vector bundle to a pair (L, D) and every (L, D) represents a toric vector bundle. This association is described in [18] and is summarized in the following procedure:

1. Form the matrix D by $D_{(i,j)} = r$ when the last place y_j appears in the i^{th} filtration is r.
2. The ideal L is the ideal of relations that hold amongst the spanning set of each vector space in the filtration.

This collection of matrices, compatible with a linear ideal L and fan Σ, forms a fan, which we'll denote $\Delta(\Sigma, L)$. When no confusion will arise, we will use the notation $\mathcal{E}(L, D)$ to denote the toric vector bundle $\mathcal{E}$ corresponding to the linear ideal L and matrix D.

Remark 2.1.5. For a toric vector bundle $\mathcal{E}$ and a line bundle $\mathcal{L}$, we have that $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$. This action corresponds to adding a fixed integer to every entry in a row of D. Therefore, by tensoring with the appropriate line bundle, we can assume without loss of generality that all entries of D are nonnegative.

Example 2.1.6. Consider $\mathcal{T}\mathbb{P}^2$, the tangent bundle of $\mathbb{P}^2$. The Klychako data for this bundle is

$$E^{\rho_i}(j) = \begin{cases} 0 & j > 1 \\ \langle y_i \rangle & j = 1 \\ E & j < 1 \end{cases} \qquad \forall \rho_i \in \Sigma(1), \quad 0 \leq i \leq 2$$

The rays of the fan of $\mathcal{T}\mathbb{P}^2$ look like:

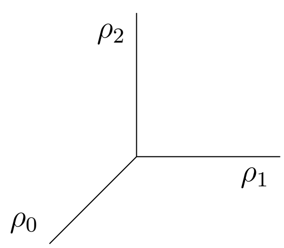

Figure 2.1: The rays of the fan of $\mathcal{T}\mathbb{P}^2$.

with generators ρ_1, ρ_2, and $\rho_0 = -\rho_1 - \rho_2$. Therefore, we get that $L = \langle y_0 + y_1 + y_2 \rangle$ and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.1.1 Subduction and KM Algorithms

The subduction algorithm was introduced in [17] as a way to produce a finite generating set of the associated graded $\mathrm{gr}_\nu(A)$ for a valuation ν. Given an algebra and valuation (A, ν) and elements $f \in A$, the algorithm inductively writes the elements f in terms of a special generating set called a *Khovanskii basis*, $\mathcal{B}$. We will start by introducing the subduction algorithm in general before discussing how this algorithm has been adapted to produce generators of a closely-related Rees algebra, aiding in our study of projectivized toric vector bundles.

Given a totally ordered abelian group Γ, a **(Γ-valued) valuation** of a domain D is a map $\nu : D \to \Gamma \cup \{\infty\}$ such that, for all $a, b \in D$, we have:

- $\nu(a) = \infty$ if and only if $a = 0$
- $\nu(ab) = \nu(a) + \nu(b)$
- $\nu(a+b) \geq \min(\nu(a), \nu(b))$ with equality when $\nu(a) \neq \nu(b)$

A valuation is called **trivial** if $\nu(a) = 0$ for all $a \in D^{\times}$.

Consider a finitely generated $\mathbb{K}-$algebra A of Krull dimension d. A is equipped with a valuation ν,

$$\nu : A \backslash \{0\} \to \mathbb{Q}^r \qquad \text{for some} \quad 0 < r \leq d$$

which restricts to a trivial valuation on $\mathbb{K}$. We then form the **value semigroup** of ν:

$$S(A, \nu) := \mathrm{im}(\nu) = \{\nu(f) \mid 0 \neq f \in A\}$$

For $a \in \mathbb{Q}^r$, the valuation ν gives the filtration $F_{\nu \succeq a}$ on A by

$$F_{\nu \succeq a} = \{f \in A \mid \nu(f) \succeq a\} \cup \{0\}.$$

We can similarly form the space $F_{\nu \succ a}$. Then the **associated graded** is the graded ring

$$\mathrm{gr}_{\nu}(A) = \bigoplus_{a \in \mathbb{Z}^r} F_{\nu \succeq a} / F_{\nu \succ a}$$

Notably, the associated graded is also a domain. For $0 \neq f \in A$, we can consider $\bar{f}$ in $\mathrm{gr}_{\nu}(A)$. These are the images of $f \in F_{\nu \succeq a}/F_{\nu \succ a}$ where $\nu(f) = a$. This notion allows us to define one of the subduction algorithm's crucial elements.

Definition 2.1.7. A set $\mathcal{B} \subseteq A$ is a **Khovanskii basis** for (A, ν) if the image of $\mathcal{B}$ in $\mathrm{gr}_{\nu}(A)$ forms a set of algebra generators.

It should be noted that there is no requirement for our Khovanskii basis to be finite. Rather, it is the goal of the subduction algorithm to write an element $f \in A$ using only finitely many elements of $\mathcal{B}$. Below is a description of the pseudocode provided in [17].

Algorithm 1: Subduction Algorithm

Input : A Khovanskii basis $\mathcal{B} \subset A$ and an element $0 \neq f \in A$

Output: A polynomial expression for f in terms of finitely many elements of $\mathcal{B}$

We find $b_1, ..., b_n \in \mathcal{B}$ and $p(x_1, ..., x_n)$ such that $\bar{f} = p(\bar{b_1}, ..., \bar{b_n})$.

if $f = p(b_1, ...b_n)$ **then**

  return $p(b_1, ..., b_n)$;

else

  set $f_1 = f - p(b_1, ..., b_n)$ and repeat;

end

Kaveh and Manon [17] give conditions to confirm the above algorithm does terminate. However, as is often the case with generating sets, it would be preferable if the Khovanskii basis $\mathcal{B}$ were finite. Also in [17], Kaveh and Manon give an algorithm for obtaining a finite Khovanskii basis, provided one exists. The pseudocode for this algorithm is provided below; however, we first need to establish some concepts.

Let $\mathcal{B} = \{b_1, ..., b_n\}$ be an algebra generating set for A. For $1 \leq i \leq n$, set $\mathcal{A} = \{\nu(b_i)\}$. Let $\mathbf{k}[\mathbf{x}]$ denote the polynomial algebra in indeterminates $\mathbf{x} = (x_1, ..., x_n)$ and consider the surjective homomorphism

$$\varphi : \mathbf{k}[\mathbf{x}] \to A \quad x_i \mapsto b_i$$

and let $I = \ker(\varphi)$. We also consider the homomorphism

$$\psi : \mathbf{k}[\mathbf{x}] \to \mathrm{gr}_\nu(A) \quad x_i \mapsto \bar{b_i}$$

where $\bar{b_i}$ is the image of b_i in $\mathrm{gr}_\nu(A)$. Let $J = \ker(\psi)$.

Let M be the $r \times n$ matrix whose columns are the vectors $\nu(b_1), ..., \nu(b_n)$. Using M, we can define a partial order on the group $\mathbb{Q}^n$ as follows. Given $\alpha, \beta \in \mathbb{Q}^n$, we say that $\alpha \succ_M \beta$ if $M\alpha \succ M\beta$, where $\succ$ is the total order on $\mathbb{Q}^r$ from the definition of ν. Note that it is not necessary that M is square (and therefore not necessarily invertible); therefore, it can happen that $\alpha \neq \beta$ but $M\alpha = M\beta$. In this case, we have that α and β are incompatible under the partial order $\succ_M$. Using this ordering, we can define an initial form of a polynomial $p(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[\mathbf{x}]$ as:

$$\mathrm{in}_M(p)(\mathbf{x}) = \sum_\beta c_\beta \mathbf{x}^\beta$$

where the sum is over all β with $M\beta = \min\{M\alpha \mid c_\alpha \neq 0\}$. Let $\mathrm{in}_M(I)$ be the ideal of $\mathbf{k}[\mathbf{x}]$ generated by $\mathrm{in}_M(p)$ for $p \in I$. We note that, by properties of initial forms, $\mathrm{in}_M(I) \subseteq J$. It will be of interest when $\mathrm{in}_M(I) = J$, justified in the following result.

Theorem 2.1.8. *[[17], Theorem 2.17] Let $\mathcal{B} = \{b_1, ..., b_n\}$ be a set of algebra generators for A. Then the following are equivalent:*

1. *$\mathcal{B}$ is a Khovanskii basis.*
2. *The ideals $in_M(I)$ and J coincide.*
3. *Let $\{p_1, ..., p_s\}$ be a set of homogeneous generators for the ideal J. Then for $1 \leq i \leq s$, the subduction algorithm is applicable to represent $p_i(b_1, ...b_n)$ as a polynomial in the b_i.*

We can now include Kaveh and Manon's algorithm for finding a finite Khovanskii basis starting from a set of algebra generators, provided that such a basis exists.

Algorithm 2: KM Algorithm

Input : A finite set of $\mathbf{k}$-algebra generators $\mathcal{B} = \{b_1, ..., b_n\}$ for A.
Output: An extension of $\{b_1, ..., b_n\}$ to a finite Khovanskii basis $\mathcal{B}$.
1 Let $\varphi : \mathbf{k}[x_1, ..., x_n] \to A, \quad x_i \mapsto b_i$.
2 Let $\psi : \mathbf{k}[x_1, ..., x_n] \to \mathrm{gr}_\nu(A), \quad x_i \mapsto \bar{b_i}$.
3 Set $I = \ker(\varphi)$ and $J = \ker(\psi)$
4 Take an element $g \in J$. Set $h = g(b_1, .., b_n)$ and $\bar{h} = \psi(h)$.
5 Verify if $\bar{h}$ is an element of the subalgebra generated by $\bar{\mathcal{B}}$.
6 If *yes*, find $p(x_1, ..., x_n)$ such that $\bar{h} = p(\bar{b_1}, ..., \bar{b_n})$. Set $h_1 = h - p(b_1, ..., b_n)$. If $h_1 = 0$, go to step (8). Otherwise, replace h with h_1 and go back to step (5).
7 If *no*, append h to $\mathcal{B}$.
8 Repeat until there are no generators left of J.
9 If no elements were added to G then $\mathcal{B}$ is our desired finite Khovanskii basis. Otherwise, return to step (1).

We would like to adapt the KM Algorithm to the case of projectivized toric vector bundles. We use a relationship between the associated graded, Rees algebras (defined below), and the Cox ring of a projectivized toric vector bundles. Recall that, for an ideal $I \subseteq \mathbb{C}[x_1, ..., x_n]$, we can define the initial ideal of I with respect to a weight vector ω. When $\text{in}_\omega(I)$ is prime, the vector ω is called a **prime point**. For prime points, we get an associated **weight valuation**

$$\nu_\omega : A \to \mathbb{Q}^r \cup \{\infty\}, \quad f \ \mapsto \ \max\{\min\{\langle \omega, \alpha \rangle \mid p = \sum c_\alpha x^\alpha\}, \ p \xrightarrow{\phi} f\}$$

Using this valuation, we obtain an additional equivalent condition for certifying our Khovanskii basis:

Theorem 2.1.9. *[17] (continued from Theorem 2.1.8)*

4. *For $\omega = (\nu(f_1), ..., \nu(f_n)) \in Trop(I_{\mathcal{B}})$, the valuations ν and ν_ω are equal.*

This weighted valuation also allows us to define a filtration similar to the one for the associated graded:

$$F_a^\omega = \{f \in A \mid \nu_\omega(f) \succeq a\} \cup \{0\}.$$

Kaveh and Manon [17] then use these filtrations to define the **Rees algebra**

$$\mathcal{R}(\omega, A) = \bigoplus_{a \in \mathbb{Z}} F_a^\omega.$$

We can consider a collection of points $\{\omega_1, ..., \omega_n\}$ and define the Rees algebra for the collection.

Definition 2.1.10. [17] Given the collection $\{\omega_1, ..., \omega_n\}$ of prime points, we can define the multi-Rees algebra as:

$$\mathcal{R}(\{\omega_1, ..., \omega_n\}, A) = \bigoplus_{\bar{a} \in \mathbb{Z}^n} F_{a_1}^{\omega_1} \cap ... \cap F_{a_n}^{\omega_n}.$$

We will use an adaptation of the KM algorithm to compute the Rees algebra for an increasingly large number of prime points. The following proposition from [17] gives a way to iteratively compute these Rees algebras.

Proposition 2.1.11. *The above multi-Rees algebra can be realized as an iterated Rees algebra, where adding a prime point to the collection can be carried through by intersecting by the additional filtration.*

Cox rings of projectivized toric vector bundles appear when the ideal L is linear and the points ω_i are taken from $\text{Trop}(L)$. We think of $\mathbb{C}[\bar{y}]/L$ as $\bigoplus_{\ell \geq 0} \text{Sym}^\ell(E)$, where the direct sum decomposition is by Sym-degree, and E the general fiber of the bundle associated to $D = \left[\omega_1 ... \omega_n\right]^T$ and $\Delta(\Sigma, L)$. In this way, all toric vector bundles are associated to an arrangement of points on a tropicalized linear space. We can then ask: *For which arrangements $S = \{\omega_1, ..., \omega_n\} \subseteq Trop(L)$ is the Rees algebra $\mathcal{R}(S, A)$ finitely generated?* To this end, we use the fact that for a linear ideal L, all points $\omega \in \mathrm{Trop}(L)$ are prime points. Therefore, we can apply Definition 2.1.10 to our projectivized toric vector bundle, allowing us to associate $\mathcal{R}(S, A)$ to $\mathrm{Cox}(\mathbb{P}\mathcal{E})$.

Theorem 2.1.12. *[17] For a projectivized toric vector bundle $\mathbb{P}\mathcal{E}$ corresponding to the diagram $D = \begin{bmatrix}\omega_1 & \dots & \omega_n\end{bmatrix}^T$, the Cox ring is*

$$Cox(\mathbb{P}\mathcal{E}) = \bigoplus_{\bar{a} \in \mathbb{Z}^n} F_{a_1}^{\omega_1} \cap ... \cap F_{a_n}^{\omega_n}$$

with grading by $\mathbb{Z}^n \cong \mathrm{Pic}(Y(\Sigma)) \times M$.

Therefore, if $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ is finitely generated, by Proposition 2.1.11, we can use the KM Algorithm to find a generating set. Moreover, the most (beautifully) convoluted way, we compute a presentation of the Cox ring by computing generators of the Rees algebra by computing generators of the associated graded. We have the following pseudocode that we use to extend our algebra generators to a finite Khovanskii basis. This code is used repeatedly to confirm that a projectivized toric vector bundle is Mori dream, see Example 2.2.6. For a complete version of this code, see Appendix 3.3.

Algorithm 3: KM Algorithm for Toric Vector Bundles

Input : Algebra generators, weight vector
Output: Extension of the given algebra generators to a Khovanskii basis
1 loadPackage **gfanInterface**
2 $R = \mathrm{frac}(\mathbb{Q}[t_1, ..., t_r, y_1, ..., y_n])$
3 $S = \mathbb{Q}[X_1, ..., X_r, Y_1, ..., Y_n]$
4 $\mathrm{in}_{(weight)}(B) = \mathrm{gfanInitialForms}(generators)$
5 $\varphi : S \to R$ with target *generators*
6 $\psi : S \to R$ with target $\mathrm{in}_{(weight)}(B)$
7 Let $I = \ker(\varphi)$ and $J = \ker(\psi)$
8 Set $K = \mathrm{gfanInitialForms}(I)$.
9 Verify whether $K = J$.
10 If *yes*, then return *generators*.
11 If *no*, then choose $g \in J \backslash K$. Add $\varphi(g)$ to *generators* and repeat steps 4-9.

Not only does this code confirm that our projectivized toric vector bundle is a Mori dream space, it also computes a presentation of the Cox ring. However, these presentations could be infinite, so we also need a criterion for this algorithm to stop. The following theorem from [17] gives a necessary and sufficient stopping condition (which has the added bonus of being easily implemented in a programming software.)

Let ν be a valuation and let the pair (A, ν) have Khovanskii basis $\mathcal{B} \subseteq A$. Let $I_{\mathcal{B}}$ be the kernel of

$$\Phi_{\mathbb{B}} : \mathbb{C}[\bar{X}, \bar{Y}] \to \mathbb{C}[t_1^{\pm}, ..., t_n^{\pm}, \bar{y}], \quad X_i \mapsto t_i^{-1},\ Y_i \mapsto b_j t^{D_j}$$

where $D = \begin{bmatrix} \omega_1 & ... & \omega_n \end{bmatrix}^T$ and D_j denotes the j^{th} column of D.

Theorem 2.1.13. *Let $\mathcal{B} \subseteq A$ be a Khovanskii basis and let $I_{\mathcal{B}}$ be as above. Each ω_i lifts to a tropical point $\tilde{\omega_i} \in$ Trop(L). Then $\mathcal{B}$ generates Cox$(\mathbb{P}\mathcal{E})$ if and only if each $\tilde{\omega_i}$ is a prime point if and only if $\langle I_{\mathcal{B}}, X_i \rangle$ is a prime ideal for all $i \in \{1, ..., n\}$.*

Therefore, this algorithm also allows the user to answer a simple question: is $\mathbb{P}\mathcal{E}$ a Mori dream space? While it is not an effective method for answering that question in the negative, the algorithm does serve as part of the experimentation to determine which patterns in (L, D) gave rise to Mori dream space bundles. These patterns identify different classes of bundles and allow us to prove properties about theses classes, which we'll discuss in the next section.

We conclude with a corollary to Theorem 2.1.13 that allows us to extend a toric vector bundle by a line bundle and maintain the Mori dream space property. This result is a direct application to 2.1.13.

Corollary 2.1.14. *Let $\mathcal{E}$ be a toric vector bundle such that $\mathbb{P}\mathcal{E}$ is a Mori dream space. Then $\mathbb{P}(\mathcal{E} \oplus \mathcal{L})$ is a Mori dream space for a line bundle $\mathcal{L}$.*

Proof. Let $\mathcal{B} \subseteq \mathrm{Cox}(\mathbb{P}\mathcal{E})$ be a generating set. Then by Theorem 2.1.13, $\langle X_i, I_{\mathcal{B}} \rangle$ is prime where $I_{\mathcal{B}}$ is the kernel of

$$\Phi_{\mathcal{B}} : \mathbb{C}[\bar{X}, \bar{Y}] \to \mathbb{C}[t_1^{\pm}, ..., t_n^{\pm}, \bar{y}], \quad X_i \mapsto t_i^{-1},\ Y_i \mapsto b_j t^{D_j}.$$

Then for $\mathbb{P}(\mathcal{E} \oplus \mathcal{L})$, we have

$$\Phi_{\mathcal{B}'} : \mathbb{C}[\bar{X}, \bar{Y}, Z] \to \mathbb{C}[t_1^{\pm}, ..., t_n^{\pm}, \bar{y}, z], \quad X_i \mapsto t_i^{-1},\ Y_i \mapsto b_j t^{D_j},\ Z \mapsto z t^{D'}$$

with $I_{\mathcal{B}'} = \ker(\Phi_{\mathcal{B}'})$. The ideal $I_{\mathcal{B}}$ is contained in $\mathbb{C}[\bar{X}, \bar{Y}, Z]$ and $\langle X_i, I_{\mathcal{B}} \rangle = \langle X_i, I_{\mathcal{B}'} \rangle$ since there is no interaction by the variable Z. Then, $\langle X_i, I_{\mathcal{B}'} \rangle$ is prime and $\mathbb{P}(\mathcal{E} \oplus \mathcal{L})$ is a Mori dream space. $\square$

Corollary 2.1.15. *Let $\mathcal{E}$ be a reducible rank 3 toric vector bundle. Then $\mathbb{P}\mathcal{E}$ is a Mori dream space.*

Proof. If $\mathcal{E}$ is reducible, it decomposes as either $\mathcal{L} \oplus \mathcal{L} \oplus \mathcal{L}$ or $\mathcal{E}' \oplus \mathcal{L}$ for $\mathcal{E}'$ a rank 2 bundle. In the first case, $\mathcal{E}$ is a split line bundle and $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ is isomorphic to a polynomial ring. In the second, we have that $\mathbb{P}\mathcal{E}$ is a Mori dream space by Corollary 2.1.14. □

2.2 Rank r Bundles

Expanding on González's result, we are interested in identifying conditions for when a rank r projectivized toric vector bundle is a Mori dream space. We do this by identifying combinatorial conditions on (L, D) which ensure the Mori dream space property holds.

2.2.1 Complete Intersection Bundles

Complete intersection (CI) bundles are introduced in [18] as the class of toric vector bundles $\mathcal{E}$ with linear ideal $L \subset \mathbb{K}[y_1, \ldots, y_s]$ and diagram $D \in \Delta(L, \Sigma)$ whose Cox ring $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ is presented by homogenizations of a minimal generating set of L. In particular, notice that this means that if $\mathcal{E}$ is a CI bundle, then $\mathbb{P}\mathcal{E}$ is a Mori dream space.

Therefore we are interested in more than just confirming that a toric vector bundle is CI. We would also like a procedure to construct these bundles to produce an infinite class of Mori dream spaces. For the following, let M be an $s \times d$ matrix of rank d such that L is generated by the rows of M. For a subset $A \subset [n]$ let M_A be the matrix obtained from M by omitting columns where the rows of the diagram D corresponding to A do not share a common minimal entry. Finally, let m_A be the rank of M_A. The following is [18, Proposition 6.2].

Theorem 2.2.1. *The toric vector bundle $\mathcal{E}(L, D)$ is a complete intersection (CI) bundle if and only if for all $i \in A \subseteq [n]$, $1 + m_{\{i\}} < |A| + m_A$.*

The collection of diagrams D associated to a CI bundle (with respect to some minimal generating set of L) form integral points in a subfan $\mathcal{F}(L, \Sigma) \subseteq \Delta(L, \Sigma)$. The class of complete intersection toric vector bundles contains two distinguished subclasses. First, we say a toric vector bundle $\mathcal{E}$ with diagram D is **sparse** if each row of D has at most one non-zero entry. The class of sparse toric vector bundles contains all vector bundles of rank 2, all tangent bundles of smooth toric varieties, and more generally coincides with those toric vector bundles whose Klyachko filtrations contain at most one step of dimension 1. The projectivizations of sparse toric vector bundles can be shown to belong to a distinguished class of Mori dream spaces called **arrangement varieties** (see [11]).

Second, we say a toric vector bundle $\mathcal{E}$ is **uniform** if the matrix M is very general - ie has no vanishing minors. Not all uniform toric vector bundles are complete intersection toric vector bundles, but the condition in Theorem 2.2.1 simplifies considerably for uniform toric vector bundles (see Theorem 2.2.2 below). We say a uniform toric vector bundle is of **type $\mathbf{U^r_m}$** if the matroid corresponding to L is uniform of rank r on m elements. In order to highlight the structure of the uniform matroid, the fan corresponding to uniform toric vector bundles will be denoted $\Delta(U^r_m, \Sigma)$.

The case $s = r+1$ are the sparse hypersurfaces, these toric vector bundles form an extremal family within the uniform sparse toric vector bundles. Any such toric vector bundle has M an all 1's row with $r+1$ entries, and the associated sparse diagram D is always a point in $\Delta(U^r_{r+1}, \Sigma)$, when Σ is any fan with $r+1$ rays.

In the case that our toric vector bundle is uniform, Theorem 2.2.1 is able to be restated as follows.

Theorem 2.2.2. *Let D be the $n \times m$ diagram corresponding to a rank r uniform projectivized toric vector bundle. Then, for every choice of p rows, if all of the nonzero entries appear in $r+p-2$ columns, the corresponding toric vector bundle is a CI bundle.*

Proof. We show this condition is equivalent to that posed in Theorem 2.2.1. By Kaveh and Manon [18], a CI bundle must meet the following inequality:

$$1 + \min\{n-r, s_i\} < |A| + \min\{n-r, s_A\} \quad \forall i \in A$$

where s_A is the number of columns with common nonzero entry from $A \subseteq [r]$. However, $\min\{n-r, s_i\} = n-r$ for all $i \in A$. Therefore, we have:

$$1 + n - r < l + s_A \implies n - r - l + 1 < s_A$$

Therefore, for a bundle to be CI, the nonzero entries in any l rows are trapped in $r + l - 2$ columns. □

When the matrix D has the maximum $r-1$ nonzero entries per row, then we get a more concise, easier to check version of Theorem 2.2.2.

Corollary 2.2.3. *Let D be as in Theorem 2.2.2 and further suppose that D has $r-1$ nonzero entries per row. Then, for every pair of rows in D, if the nonzero entries of those rows are contained in r columns, then the corresponding toric vector bundle is a CI bundle.*

Proof. We show that the containment of nonzero entries of a pair of rows in this setting is equivalent to the conditions stated in Theorem 2.2.2. Suppose every choice of two rows has their nonzero entries contained in r columns. Since each row of D has $r-1$ nonzero entries, this is equivalent to the pair of rows having $r-2$ of their nonzero entries in the same column. Therefore, every row has one entry that is not bound by the constraint and can be anywhere. So p rows have p such "free" entries, meaning that p rows must have nonzero entries in $(r-2)+p$ columns. □

Using this description, we have a quick certification for when our bundle over $\mathbb{P}^n$ is a CI bundle.

Corollary 2.2.4. *Let $\mathcal{E}(U_m^r, D)$ be a rank r bundle over $\mathbb{P}^n$ with D a $(n+1)\times m$ matrix. Then if $m-r<n$, $\mathcal{E}$ is a CI bundle.*

Proof. By Kaveh and Manon [18], any collection of $\leq n$ rows of D lives in a common apartment, so the nonzero entries are contained in $\leq r$ columns. Therefore, we check the case where our collection of rows is all rows.
Suppose that $m-r<n$. Then, adding r and subtracting 1, we get $m \leq n+r-1$. This can be rewritten to be $m \leq (n+1)+r-2$. By Theorem 2.2.2, the bundle is CI. □

Example 2.2.5. Consider $\mathcal{E}$ as a rank 3 bundle over $\mathbb{P}^2$ with $L = \langle \sum_{i=1}^5 y_i, \sum_{i=1}^5 iy_i \rangle$ with

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

One can check that for every choice of subset of the rows of D, the nonzero entries of those rows live in $p+r-2 = p+1$ columns. For example, choosing rows 1 and 2, the nonzero entries are contained in $p+1=3$ columns. Checking all such subsets confirms that D meets the sufficient condition, so $\mathcal{E}$ is a CI bundle.

However, this result is truly only a sufficient condition. We have examples, such as Example 2.2.6 that demonstrate just how fragile this condition is.

Example 2.2.6. Consider $\mathcal{E}$ as a rank 3 bundle over $\mathbb{P}^2$ with $L = \langle \sum_{i=1}^5 y_i, \sum_{i=1}^5 iy_i \rangle$, as in Example 2.2.5, but now suppose

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

where entries $d_{3,2}$ and $d_{3,3}$ have been swapped. Now, any collection containing row 3 does not meet the sufficient condition. However, the KM algorithm confirms that, for this $\mathcal{E}$, we have that $\mathbb{P}\mathcal{E}$ is still a Mori dream space.

2.2.2 New bundles from old

We describe a number of extension theorems, allowing us to form infinitely many new CI bundles of the same type. We describe these new bundles by adding columns to D. We suppose an $m - r \times m$ matrix M has been given such that $L = \ker(M)$. In particular, we take the rows of M to be a minimal generating set of L. In what follows we consider the extension (D', L') of a pair (D, L), where $D' = [D \mid U]$ and $L' = ker(M')$ for $M' = [M \mid X]$. We let $\mathcal{E}$ be the toric vector bundle corresponding to the pair (D, L) and let $\mathcal{E}'$ be the toric vector bundle corresponding to the pair (D', L'). For the sake of simplicity we assume that D and D' are non-negative.

Proposition 2.2.7. *Let $\mathcal{E}$ be a CI toric vector bundle, and suppose that every entry of the i-th row of U is larger than every entry in the i-th row of D, then $\mathcal{E}'$ is CI.*

Proof. Let $p'_1, \ldots, p'_{m-r}$ denote the homogenizations of the rows of M' and $p_1, \ldots, p_{m-r}$ denote the homogenizations of the rows of M. For any row of D', the initial forms of the p'_k agree with those of the p_k. It follows that the initial ideal $\text{in}_{\hat{w}_i}(\mathcal{I})$ is prime and generated by these initial forms, the theorem then follows from Theorem 2.1.13. □

When $\mathcal{E}$ is a uniform bundle it is more straightforward to find extensions.

Proposition 2.2.8. *Let $\mathcal{E}$ be a uniform CI toric vector bundle defined by M a very general matrix, and suppose that M' defined by an X which also makes M' very general, then $\mathcal{E}'$ is a uniform CI bundle for any non-negative D' extending D.*

Proof. The diagram D must satisfy the conditions of Theorem 2.2.2. But it is then immediate that D' also satisfies these conditions. □

Definition 2.2.9. We say that a toric vector bundle $\mathcal{E}(L, D)$ is *monomial* if $\text{in}_\sigma(L)$ is a monomial ideal for all maximal $\sigma \in \Sigma$.

This condition holds precisely when, for each maximal face $\sigma \in \Sigma$, the minimal face of the Gröbner fan of L containing the rows of D corresponding to σ is a maximal face.

Example 2.2.10. When $\mathcal{E}(L, D)$ is uniform, to check if $\mathcal{E}$ is monomial, it suffices to confirm that all choices of $r-1$ rows of D have a unique minimal element. Let $L = \langle y_1 + y_2 + y_3 + y_4 \rangle \subset \mathbb{C}[y_1, ..., y_4]$ and

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\mathcal{E}(L, D)$ is a rank-3 bundle and it suffices to check that every pair of rows has a unique minimal element. This can be quickly confirmed since every pair of rows has a unique zero column, so $\mathcal{E}$ is a monomial bundle.

Remark 2.2.11. We note that there is no implication between a bundle being monomial and a bundle being uniform. However, the fact that the maximal cones of a uniform bundle are spanned by $r-1$ rays does allow for nice combinatorial conditions on when a uniform bundle is monomial. For example, if D is an $m \times n$ matrix then, in order for $\mathcal{E}(L, D)$ to be monomial, the support of each row needs to be disjoint from all others but still have $\lfloor \frac{n}{2} \rfloor - 1$ nonzero entries.

Lemma 2.2.12. *Let $\mathcal{E}$ be a uniform CI toric vector bundle. Further suppose that $D \geq 0$ has no zero columns. Then $\mathcal{E}'$ is a uniform monomial toric vector bundle if and only if every choice of $r-1$ rows of D has exactly one zero column.*

Proof. By Proposition 2.2.8, $\mathcal{E}'$ is CI. Therefore, we need only check that $\mathrm{in}_\sigma(L')$ is monomial for all maximal $\sigma \in \Sigma$. Since $\mathcal{E}$ is uniform, $\Sigma(1) = e_j$ for $1 \leq j \leq m$ and the maximal cones $\sigma \in \Sigma$ are generated by $r-1$ such rays. Since D has no zero columns, every choice of $r-1$ rows of D corresponds to a maximal cone of Σ. $\square$

Theorem 2.2.13. *Let $\mathcal{E}$ be a uniform monomial toric vector bundle. Then $\mathcal{E}'$ is a uniform monomial toric vector bundle if and only if $X \geq 0$ has no more than $r-2$ zero entries per column.*

Proof. By the above lemma, it suffices to confirm that every choice of $r-1$ rows of D' has exactly one zero column. Note that this is only possible if X has no column with more than $r-2$ zero entries since, if there were, choosing these rows would introduce another zero column. $\square$

2.3 Sums of Bundles

The work in this section was obtained in collaboration with Christopher Manon and appears in [5]. We consider, when $\mathbb{P}\mathcal{E}$ and $\mathbb{P}\mathcal{F}$ are Mori dream spaces, whether $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$ is a Mori dream space. We investigate this question in the particular case that $\mathcal{E} = \mathcal{F}$.

Question 2.3.1. *If $\mathbb{P}\mathcal{E}$ is a Mori dream space, when is $\mathbb{P}(\mathcal{E} \oplus \mathcal{E})$ and more generally $\mathbb{P}(\mathcal{E} \otimes V)$ a Mori dream space, for V is a finite-dimensional vector space?*

We find that the answer to this question is closely related to the study of flag varieties. Therefore, before stating the main results, we begin with a review of some key definitions.

Let E be a vector space of dimension r. Given a strictly increasing sequence of integers $I = \{i_1, \ldots, i_d\} \subseteq \{1, ..., r-1\}$, the **flag variety** $\mathcal{FL}_I(E)$ is the space of flags

$$0 \subset V_1 \subset \ldots \subset V_d \subset E$$

where each V_j is a subspace of dimension i_j. Well-known examples of flag varieties include projective spaces and Grassmannians.

Then a flag variety is a **full flag variety** if $I = \{1, ..., r-1\}$. We note that bundles of flag varieties are natural generalizations of projectivized vector bundles. For my work, we focus on the case of flag bundles over toric varieties. A **toric flag bundle** over $Y(\Sigma)$ is a bundle with fibers isomorphic to $\mathcal{FL}_I(E)$, equipped with a compatible torus action. (Notably, any $\mathcal{E}$ gives rise to a $\mathcal{FL}_I(\mathcal{E})$.)

Therefore, we ask: *When is $\mathcal{FL}_I(\mathcal{E})$ a Mori dream space?* In order to address this question, we need to review some background material on representation theory of the general linear group, $\mathrm{GL}(E)$, the group of invertible matrices on E.

2.3.1 Representation Theory of GL(E)

A **linear algebraic group** is a subgroup of $\mathrm{GL}(E)$ defined by polynomial equations. A **representation** of a group G is a homomorphism

$$\Pi : G \to \mathrm{GL}(V)$$

where V is a finite dimensional vector space. A connected linear algebraic group G is a **reductive group** if it has a representation with finite kernel which is a direct sum of irreducible representations (representations with no nontrivial subspaces). For a reductive group G, a G-representation V, and a subspace $N \subset V$, let $\langle N \rangle_G \subseteq V$ denote the **subrepresentation generated by N**. We say that $\langle N \rangle_G$ is the **G-span of N**.

The **dominant weights** of $\mathrm{GL}(E)$ are indexed by integral tuples $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $r = \dim(E)$ and $\lambda_1 \geq \ldots \geq \lambda_r$. Of special importance are the weights with $\lambda_r \geq 0$, which correspond to the **Young tableaux**, a left-justified collection of boxes with (not necessarily strictly) decreasing number of rows. By abuse of notation, we let λ denote the tableau whose i-th row has length λ_i. For a Young tableau λ, define

$$S_\lambda V := c_\lambda(V^{\otimes n}) = a_\lambda(b_\lambda(V^{\otimes n}))$$

where $a_\lambda(V^{\otimes n}) = \bigotimes_i \mathrm{Sym}^{\lambda_i}(V)$ and $b_\lambda(V^{\otimes n}) = \bigotimes_j \bigwedge^{\lambda'_j}(V)$ to be the **Schur function** associated to λ. If $\lambda_r \geq 0$, the corresponding irreducible $\mathrm{GL}(E)$ representation is obtained by evaluating the Schur functor $\mathrm{S}_\lambda : \mathrm{Vect}_{\mathbb{K}} \to \mathrm{Vect}_{\mathbb{K}}$ at E.

For example, consider the **exterior power** algebra, the algebra that uses exterior product (or wedge product) as its multiplication. The exterior power $\bigwedge^\ell E$ corresponds to the weight $\omega_\ell = (1, \ldots, 1, 0, \ldots, 0)$ with ℓ 1's and $r - \ell$ 0's. Any irreducible V_λ of $\mathrm{GL}(E)$ can be realized as a tensor product of a Schur functor and a (possibly negative) power of the determinant: $V_\lambda \cong (\bigwedge^r E)^{\lambda_r} \otimes S_{\bar{\lambda}}(E)$. Here $\bar{\lambda} = \lambda - \lambda_r \omega_r$ corresponds to a tableau with $r-1$ rows. Using this identification, we get that the dual V_{λ^*} is

$$V_{\lambda^*} = (\bigwedge^r E)^{-\lambda_r} \otimes S_{\bar{\lambda}^*}(E),$$

where

$$\bar{\lambda}^* = \sum_{i=1}^{r-1} n_i \omega_{r-1-i} \qquad \text{if} \qquad \bar{\lambda} = \sum_{i=1}^{r-1} n_i \omega_i.$$

Let $|\lambda| = \sum_{i=1}^r \lambda_i$. When $\lambda_r \geq 0$ this is the number of boxes in the tableau corresponding to λ. We let $\mathrm{row}(\lambda)$ be the number of rows in λ.

We need some classical identities from representation theory. The Cauchy identity gives the decomposition of $\mathrm{Sym}^\ell(E \otimes V)$ as a $\mathrm{GL}(E) \times \mathrm{GL}(V)$ representation:

$$\mathrm{Sym}^{\ell}(E \otimes V) = \bigoplus_{|\lambda|=\ell} \mathrm{S}_{\lambda}(E) \otimes \mathrm{S}_{\lambda}(V).$$

If $\mathrm{row}(\lambda) > \mathrm{MIN}\{\dim(E), \dim(V)\}$, then $\mathrm{S}_{\lambda}(E) \otimes \mathrm{S}_{\lambda}(V) = 0$. In this way, the Cauchy identity encodes the $\mathrm{GL}(E) \times \mathrm{GL}(V)$ isotypical decomposition of the polynomial ring generated by $E \otimes V$:

$$\mathrm{Sym}(E \otimes V) = \bigoplus_{\mathrm{row}(\lambda) \leq \mathrm{MIN}\{\dim(E), \dim(V)\}} \mathrm{S}_{\lambda}(E) \otimes \mathrm{S}_{\lambda}(V).$$

A choice of basis $\mathbb{B} = \{e_1, \dots, e_r\} \in E$ determines the maximal torus $T \subset \mathrm{GL}(E)$ of those $g \in \mathrm{GL}(E)$ which are diagonal when expressed in $\mathbb{B}$. Likewise, this choice determines the subgroup B of upper triangular matrices, and its unipotent radical $U \subset B$. (In this setting, U is the set of upper triangular matrices with 1's along the diagonal.) Any $\mathrm{S}_{\lambda}(E)$ has a unique 1-dimensional subspace fixed by the action of U. This subspace is isomorphic to the $1-$dimensional representation of B with weight λ, called the **highest weight**.

2.3.2 Cox rings of flag bundles

Before proving the main theorem of this section, we will need the following results, which appear in [5], courtesy of Christopher Manon.

Theorem 2.3.2. *Let U be the unipotent radical of B. For any toric vector bundle $\mathcal{E}$ we have:*

$$Cox(\mathbb{P}(\mathcal{E} \otimes E))^{U} \cong Cox(\mathcal{FL}(\mathcal{E}))[t], \tag{2.1}$$

where t is a parameter of Sym *degree r.*

Proof. For any Young diagram with $\leq r$ rows we can write $\mathrm{S}_{\lambda}(\mathcal{E}) = (\bigwedge^{r} \mathcal{E})^{\lambda_r} \otimes \mathrm{S}_{\bar{\lambda}}(\mathcal{E})$, where $\bar{\lambda} = \lambda - \lambda_r \omega_r$. Let $\mathcal{D} = \bigoplus_{\mathrm{Pic}(X(\Sigma))} D$, then

$$\mathcal{D} \otimes \mathrm{S}_{\lambda}(\mathcal{E}) \cong \mathcal{D} \otimes (\bigwedge^{r} \mathcal{E})^{\lambda_r} \otimes \mathrm{S}_{\bar{\lambda}}(\mathcal{E}) \cong \mathcal{D} \otimes \mathrm{S}_{\bar{\lambda}}(\mathcal{E}).$$

It follows that

$$\mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E))^U = \bigoplus_{\mathrm{row}(\lambda)\le r} H^0(X(\Sigma),\mathcal{D}\otimes \mathrm{S}_{\bar{\lambda}}(\mathcal{E}))\otimes \mathrm{S}_\lambda(E)^U = \bigoplus_{\bar{\lambda},\lambda_r} H^0(X(\Sigma),\mathcal{D}\otimes \mathrm{S}_{\bar{\lambda}}(\mathcal{E}))t^{\lambda_r}$$

The right hand side is the ring $\mathrm{Cox}(\mathcal{FL}(\mathcal{E}))[t]$. $\square$

Theorem 2.3.2 relates the projectivized sum of bundles to flag bundles. Next, we form another crucial connection by analyzing the representation theory of these sums.

Proposition 2.3.3. *Let $H \subseteq G$ be an inclusion of reductive groups, and suppose that $R \subseteq S$ is an inclusion of H representations such that the H action on S extends to an action by G. Moreover, suppose that every G-highest weight vector in S is in R, then if $M \subseteq S$ is a G representation, we have:*

$$M = \langle M \cap R\rangle_G$$

Proof. The inclusion $\langle M \cap R\rangle_G \subseteq M$ is clear. Let $M = \bigoplus_{\lambda\in\Lambda_+} M_\lambda \otimes V_\lambda$ be the G-isotypical decomposition of M, where M_λ are the U_+ invariants in M of weight λ. For any $\lambda \in \Lambda_+$ we have $\langle M_\lambda\rangle_G = M_\lambda \otimes V_\lambda$. By assumption $M_\lambda \subseteq R$, so $M_\lambda \otimes V_\lambda = \langle M_\lambda\rangle_G \subseteq \langle M \cap R\rangle_G$. As a consequence we conclude that $M \subseteq \langle M \cap R\rangle_G$. $\square$

Lemma 2.3.4. *Let $W \subseteq V$ be finite dimensional vector spaces. Then there is an inclusion Cox$(\mathbb{P}(\mathcal{E}\otimes W)) \subseteq$ Cox$(\mathbb{P}(\mathcal{E}\otimes V))$ of $\mathrm{GL}(W)$ algebras. If dim$(V) \ge$ dim$(W) \ge r$, then every $\mathrm{GL}(V)$ highest weight vector in Cox$(\mathbb{P}(\mathcal{E}\otimes V))$ is in Cox$(\mathbb{P}(\mathcal{E}\otimes W))$.*

Proof. We show inclusion by proving there exists a monomorphism $\mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes W)) \to \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$. The inclusion $W \to V$ induces a map of sheaves $\mathrm{Sym}(\mathcal{E}\otimes W) \to \mathrm{Sym}(\mathcal{E}\otimes V)$. By checking the map on affine neighborhoods, we confirm this map is a monomorphism. For a chosen d we have:

$$\mathrm{Sym}^d(\mathcal{E}\otimes W) = \bigoplus_{|\lambda|=d} \mathrm{S}_\lambda(\mathcal{E})\otimes \mathrm{S}_\lambda(W) \to \bigoplus_{|\lambda|=d} \mathrm{S}_\lambda(\mathcal{E})\otimes \mathrm{S}_\lambda(V) = \mathrm{Sym}^d(\mathcal{E}\otimes V) \quad (2.2)$$

We still have a monomorphism after tensoring (2.2) with any line bundle $\mathcal{L}$, and taking global sections commutes with direct sums, so we obtain a monomorphism $\mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes W)) \to \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$.
After choosing compatible bases, we view $W \to V$ as the inclusion of the first $\dim(W)$ basis members, inducing the upper-left inclusion $\mathrm{GL}(W) \to \mathrm{GL}(V)$. Any compatible ordering on this basis gives a choice of Borel subgroups in $\mathrm{GL}(W)$ and $\mathrm{GL}(V)$. We

have $\mathrm{row}(\lambda) \leq r$, so all highest weight vectors corresponding to these Borel subgroups in the $\mathrm{S}_\lambda(V)$ only involve the first r members of the basis. This implies that any such highest weight vector of $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes V))$ lies in $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes W))$. □

Theorem 2.3.5. *Let* $\dim(V) = \ell$. *The projectivized toric vector bundle* $\mathbb{P}(\mathcal{E} \otimes V)$ *is a Mori dream space if and only if the flag bundle* $\mathcal{FL}_I(\mathcal{E})$ *is a Mori dream space for all* $|I| \leq \ell$. *Moreover, the full flag bundle* $\mathcal{FL}(\mathcal{E})$ *is a Mori dream space if and only if* $\mathbb{P}(\mathcal{E} \otimes V)$ *is a Mori dream space for all finite dimensional vector spaces* V.

Proof. We consider

$$\mathrm{Cox}(\mathcal{FL}_{[\ell]}(\mathcal{E})) = \bigoplus_{\mathrm{row}(\lambda) \leq \ell} H^0(X(\Sigma), \mathrm{S}_\lambda(\mathcal{E}) \otimes D)$$

for $\ell < r$. For $\dim(V) = \ell$, this is $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes V))^{U_V}$, and coincides with a graded subring of $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes E))^U$ under the inclusion in Lemma 2.3.4. Moreover, $\mathrm{Cox}(\mathcal{FL}_I(\mathcal{E}))$ is a graded subring of $\mathrm{Cox}(\mathcal{FL}_{[\ell]}(\mathcal{E}))$. These identities imply that $\mathrm{Cox}(\mathcal{FL}_{[\ell]}(\mathcal{E}))$ is finitely generated if and only if $\mathrm{Cox}(\mathcal{FL}_I(\mathcal{E}))$ is finitely generated for all $I \subseteq [\ell]$ if and only if $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes V))$ is finitely generated. □

Observe that in the case $\ell = r - 1$, $\mathcal{FL}(\mathcal{E})$ is finitely generated if and only if $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes E))$ is finitely generated if and only if $\mathrm{Cox}(\mathbb{P}(\mathcal{E} \otimes V))$ is finitely generated for all $dim(V) > r$ by Theorem 2.3.2 and Lemma 2.3.4. We can also obtain general information about the generating sets of these rings.

Corollary 2.3.6. *Let* $\Omega \subset \Lambda_+$ *be a set of Young diagrams such that the corresponding summands generate Cox*$(\mathcal{FL}_{[\ell]}(\mathcal{E}))$*, then:*

1. *the* Ω *components generate Cox*$(\mathbb{P}(\mathcal{E} \otimes V))$ *if* $\dim(V) = \ell < r$.
2. *the* $\Omega \cup \{\omega_r\}$ *components generate Cox*$(\mathbb{P}(\mathcal{E} \otimes V))$ *if* $\dim(V) = \ell \geq r$.

Proof. For a G-algebra R, the components which generate R^{U_+} also generate R [9]. Now take $R = \mathrm{Cox}(\mathcal{E} \otimes V)$ and use Theorem 2.3.2. □

We are able to extend this theorem to consider the direct sum of a bundle with itself an arbitrary number of times.

Theorem 2.3.7. *Let* $dim(V) \geq r$, *then Cox*$(\mathbb{P}(\mathcal{E} \otimes V))$ *is generated by the* $\mathrm{GL}(V)$*-span of the generators of Cox*$(\mathbb{P}(\mathcal{E} \otimes E)) \subseteq$ *Cox*$(\mathbb{P}(\mathcal{E} \otimes V))$.

Proof. From Lemma 2.3.4 we get an inclusion $\mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E)) \to \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$ such that $\langle \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E))\rangle_{\mathrm{GL}(V)} = \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$. Let $F \subset \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E)) \subset \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$ be the vector space spanned by a $\mathbb{K}$-generating set of $\mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E))$. The subspace $\langle F\rangle_{\mathrm{GL}(V)}$ then generates the $GL(V)$ subring $\mathbb{K}\langle F\rangle_{GL(V)} \subseteq \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$. By construction, this subring is generated in the same degree as $\mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E))$. By Lemma 2.3.4 and Proposition 2.3.3 we have:

$$\mathbb{K}\langle F\rangle_{\mathrm{GL}(V)} = \langle \mathbb{K}\langle F\rangle_{\mathrm{GL}(V)} \cap \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes E))\rangle_{\mathrm{GL}(V)} = \langle \mathrm{Cox}(\mathcal{E}\otimes E)\rangle_{\mathrm{GL}(V)} = \mathrm{Cox}(\mathbb{P}(\mathcal{E}\otimes V))$$

□

We find this is related to the theory of representation stability. In particular, we see a connection to the following result of Weyl.

Theorem 2.3.8 (Weyl, [28]). *Let E be a finite dimensional representation of a group Γ. For a vector space V, let Γ act on $E\otimes V$ through E. If the invariant ring $\mathbb{K}[E\ otimes E]^\Gamma$ is generated in degree d, then the invariant ring $\mathbb{K}[E\otimes V]^\Gamma$ is generated in degree d for all finite dimensional vector spaces V.*

We can relate these results to families of bundles we've previously discussed, namely CI and sparse hypersurface bundles. If $\mathcal{E}$ is CI bundle and V is any vector space of dimension ℓ, then we can ask if $\mathcal{E}\otimes V$ is also CI bundle. The following is straightforward using Proposition 2.2.1.

Proposition 2.3.9. *If $\mathcal{E}$ is a CI bundle, then $\mathcal{E}\otimes V$ is CI bundle if and only if for all $i \in A \subseteq [n]$, $1+\ell m_{\{i\}} < |A| + \ell m_A$.*

Since sparse hypersurface bundles are the simplest case of CI bundles, we can apply Proposition 2.3.9 to the following corollary.

Corollary 2.3.10. *If $\mathcal{E}$ is a sparse hypersurface toric vector bundle, then $\mathbb{P}(\mathcal{E}\otimes V)$ and $\mathcal{FL}(\mathcal{E})$ are Mori dream spaces for any vector space V.*

We conclude this section with a few computations that demonstrate the utility of the proceeding theorems.

Corollary 2.3.11. *Let $\mathcal{E}$ be a rank 2 toric vector bundle. Then $\mathbb{P}(\mathcal{E}\otimes V)$ is a Mori dream space for all finite dimensional vector spaces V.*

Proof. Because $\mathcal{E}$ is rank 2, its fibers are isomorphic to $\mathbb{C}^2$. Considering the flag bundle, $\mathcal{FL}(\mathbb{C}^2) \cong \mathbb{P}^1$, so $\mathcal{FL}(\mathcal{E})$ is a rank 2 projectivized toric vector bundle, which is always a Mori dream space by González [7]. Therefore, by 2.3.5, $\mathbb{P}(\mathcal{E}\otimes V)$ is a Mori dream space for all finite dimensional vector spaces V. □

Corollary 2.3.12. *For the toric threefold Z referenced in Theorem 2.1.3, $\mathbb{P}(\mathcal{T}Z \oplus \mathcal{T}Z)$ is not a Mori dream space.*

Proof. We can realize the projectivized cotangent bundle $\mathbb{P}\mathcal{T}^*Z$ as a quotient of $\mathcal{FL}(\mathcal{T}Z)$. Therefore, $\mathbb{P}\mathcal{T}^*Z$ not being a Mori dream space implies $\mathcal{FL}(\mathcal{T}Z)$ is not a Mori dream space implies $\mathbb{P}(\mathcal{T}Z \oplus \mathcal{T}Z)$ is not a Mori dream space. □

2.4 The twisted tangent bundle of $\mathbb{P}^n$

The term "twisted commutative algebra" is described in [27] as "a theory for handling commutative algebras with large groups of linear symmetries." More precisely, a **twisted commutative algebra** is a functor from Vect to the category of associative unital commutative $\mathbb{C}$-algebras such that the resulting functor from Vect to Vect by forgetting the algebra structure is polynomial. In this section we describe presentations for the *twisted Cox ring*, $V \to \mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes V))$, of the tangent bundle $\mathcal{T}_n$ of projective space $\mathbb{P}^n$, and the Cox ring of its full flag bundle $\mathcal{FL}\mathcal{T}_n$.

The tangent bundle $\mathcal{T}_n$ is a sparse hypersurface toric vector bundle, where D is the $n{+}1 \times n{+}1$ identity matrix, and M is the $1 \times n{+}1$ all 1's matrix. By Corollary 2.3.10, $\mathbb{P}(\mathcal{T}_n \otimes V)$, and $\mathcal{FL}\mathcal{T}_n$ are Mori dream spaces for any vector space V, along with any Grassmannian bundle $\mathrm{Gr}_\ell(\mathcal{T}_n)$ by implication. We can extend these observations further with the next lemma. It is a straightforward consequence of [18, Theorem 1.5].

Lemma 2.4.1. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles over toric varieties $Y(\Sigma_1)$ and $Y(\Sigma_2)$, respectively, and suppose that $\mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_2)$ are Mori dream spaces, then $\mathbb{P}(\mathcal{E}_1 \times \mathcal{E}_2)$ is a Mori dream space, where $\mathcal{E}_1 \times \mathcal{E}_2$ is the product toric vector bundle over $Y(\Sigma_1) \times Y(\Sigma_2)$.*

Corollary 2.4.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be as above, and suppose that $\mathcal{FL}(\mathcal{E}_1)$ and $\mathcal{FL}(\mathcal{E}_2)$ are Mori dream spaces, then $\mathcal{FL}(\mathcal{E}_1 \times \mathcal{E}_2)$ and $\mathbb{P}((\mathcal{E}_1 \times \mathcal{E}_2) \otimes V)$ are Mori dream spaces for any V.*

Proof. We apply Lemma 2.4.1 to $(\mathcal{E}_1 \otimes V) \times (\mathcal{E}_2 \otimes V) \cong (\mathcal{E}_1 \times \mathcal{E}_2) \otimes V$. □

We now give the result that allows us to justify our future discussions on Cox ring presentations. To do this, we establish the notation that $\mathbf{n} = (n_1, \ldots, n_m)$ with $n_i > 0$, and let $\mathcal{T}_{\mathbf{n}}$ denote the tangent bundle of $\prod_{i=1}^m \mathbb{P}^{n_i}$.

Corollary 2.4.3. *For any V, $\mathbb{P}(\mathcal{T}_{\mathbf{n}} \otimes V)$ and $\mathcal{FL}\mathcal{T}_{\mathbf{n}}$ are a Mori dream spaces.*

The Cox ring $\mathrm{Cox}(\mathbb{P}\mathcal{T}_2)$ has the following presentation:

$$\mathrm{Cox}(\mathbb{P}\mathcal{T}_2) = \mathbb{K}[x_1, x_2, x_3, Y_1, Y_2, Y_3]/\langle x_1Y_1 + x_2Y_2 + x_3Y_3\rangle$$

The rank of $\mathcal{T}_2$ is 2, so by Theorem 2.3.6 we should expect higher degree generators and relations in the presentation of $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_2 \oplus \mathcal{T}_2))$. We can directly compute (using [18, Algorithm 5.6]) the Cox ring of $\mathbb{P}(\mathcal{T}_2 \oplus \mathcal{T}_2)$ to be the quotient of $\mathbb{K}[x_1, x_2, x_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3, W]$ by the ideal:

$$\begin{aligned} I_{2,2} = \langle &Y_3Z_2 - Y_2Z_3 - x_1W,\ Y_3Z_1 - Y_1Z_3 + x_2W,\ Y_2Z_1 - Y_1Z_2 - x_3W, \\ &x_1Z_1 + x_2Z_2 + x_3Z_3,\ x_1Y_1 + x_2Y_2 + x_3Y_3\rangle. \end{aligned}$$

After a change of coordinates, this ideal is recognizable as the Plücker ideal defining the Grassmannian variety $\mathrm{Gr}_2(5) \subset \mathbb{P}^9$ in its Plücker embedding. In the grading by $\mathrm{Pic}(\mathbb{P}(\mathcal{T}_2 \oplus \mathcal{T}_2)) \cong \mathrm{Pic}(\mathbb{P}^2) \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$, $\deg(x_i) = (-1, 0)$, $\deg(Y_i) = \deg(Z_j) = (1, 1)$, and $\deg(W) = (3, 2)$.

The purpose of the rest of this section is to find the appropriate generalization of these observations. We start with the case $\mathcal{T}_n \otimes \mathbb{K}^m$ with $m < n$.

Proposition 2.4.4. *Let $1 \le m < n$, then $\mathrm{Cox}(\mathcal{T}_n \otimes \mathbb{K}^m)$ has the following presentation:*

$$\mathit{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^m)) = \mathbb{K}[x_j, Y_{ij} \mid 1 \le i \le m, 0 \le j \le n]/\langle \sum x_jY_{ij} \mid 1 \le i \le m\rangle$$

.

Proof. The tangent bundles $\mathcal{T}_n, \mathcal{T}_n \otimes \mathbb{K}^2, \ldots, \mathcal{T}_n \otimes \mathbb{K}^{n-1}$ are complete intersections by Corollary 2.3.10. ☐

In the grading by $\mathrm{Pic}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^m)) \cong \mathrm{Pic}(\mathbb{P}^n) \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$, $\deg(x_j) = (-1, 0)$, $\deg(Y_{ij}) = (1, 1)$. By Corollary 2.3.6 we should expect the case $m = n$ to require an additional generator of Sym degree n. We use [18, Theorem 1.5] and a close relationship with a particular *Zelevinsky quiver variety* to compute the presentation of $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^n))$. For an array $\mathbf{r} = (r_{ij})_{0 \le i \le j \le n}$ of nonnegative integers with

$r_{ii} = r_i$, the **quiver ideal** $I_r \subseteq \mathbb{C}[f]$ is generated by the union over $i < j$ of the size $1 + r_{ij}$ minors in the project $\Phi_{i+1}...\Phi_j$ of matrices of variables:

$$I_r = \langle \text{minors of size } 1 + r_{ij} \text{ in } \Phi_{i+1}...\Phi_j \text{ for } i < j \rangle.$$

The **quiver locus** $\omega_r \subseteq \mathrm{Mat}$ is the zero set of the quiver ideal I_r.

The quiver variety we need corresponds to the *rank array* $\mathbf{r}$:

$$\mathbf{r} \quad = \quad \begin{array}{ccc|c} 2 & 1 & 0 & \\ \hline & & n & 0 \\ & n+1 & n-1 & 1 \\ 1 & 1 & 0 & 2 \end{array}$$

This is the rank array for the quiver:

$$\mathbb{K} \xrightarrow{\Phi_1} \mathbb{K}^{n+1} \xrightarrow{\Phi_2} \mathbb{K}^n,$$

where $rank(\Phi_1) \leq 1$, $rank(\Phi_2) \leq n-1$, and $rank(\Phi_2\Phi_1) = 0$. By [24, Theorem 17.23], the ideal:

$$I_{\mathbf{r}} = \langle \sum x_j Y_{ij}, \det Y(j) \rangle \subset \mathbb{K}[x_j, Y_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq n]$$

is prime and Cohen-Macaulay. Here $Y(j)$ denotes the $n \times n$ minor of the matrix $n \times n+1$ $Y = [Y_{ij}]$ obtained by forgetting the j-th column.

Following [18, Theorem 1.5], we start with a potential presentation of $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^n))$, given as a map Φ between polynomial rings. Letting $I = \ker(\Phi)$, in order to show that $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^n)) = \mathrm{Im}(\Phi)$ it is necessary and sufficient to show that $\langle I, x_j \rangle$ is a prime ideal for $0 \leq j \leq n$, where:

$$
\begin{aligned}
\Phi : \mathbb{K}[x_j, Y_{ij}, W] &\to \mathbb{K}[t_j^{\pm}, y_{ij}] \\
x_j &\to t_j^{-1} \\
W &\to \det[y(0)] t_0 \cdots t_n \\
Y_{i0} &\to (-\sum_{j=1}^{n} y_{ij}) t_0 \\
Y_{i1} &\to y_{i1} t_j \\
&\vdots \\
Y_{in} &\to y_{in} t_n
\end{aligned}
$$

First we must identify the kernel I.

Proposition 2.4.5. *The kernel of the map Φ is the ideal:*

$$\langle \sum x_j Y_{ij}, \det Y(j) - x_j W \mid 1 \leq i \leq n, 0 \leq i \leq n \rangle.$$

Proof. For now let $J = \langle \sum x_j Y_{ij}, \det Y(j) - x_j W \mid 1 \leq i \leq n, 0 \leq i \leq n \rangle$. It is straightforward to check that $J \subseteq I$. We define a partial term order δ by weighting the variables $x_j \to 0$, $Y_{ij} \to 0$, $W \to 1$. We have:

$$I_{\mathbf{r}} \subseteq \mathrm{in}_\delta(J) \subseteq \mathrm{in}_\delta(I)$$

The 0-locus of $\mathrm{in}_\delta(I)$ has dimension equal to that of the 0-locus of I, which is n^2+n+1. This is the same as the dimension of the quiver variety defined by $I_{\mathbf{r}}$. It follows that $I_{\mathbf{r}} = \mathrm{in}_\delta(J) = \mathrm{in}_\delta(I)$, and $I = J$. $\square$

Next, for use of Theorem 2.1.13, we must show that $\langle I, x_j \rangle$ is always prime. The fact that $\mathrm{in}(I) = I_{\mathbf{r}}$ implies that I is a Cohen-Macaulay ideal and that $\langle I, x_i \rangle$ is also Cohen-Macaulay. It is straightforward to show that $\langle I, x_j \rangle$ is generically reduced. This and the Cohen-Macaulay property implies that $\langle I, x_j \rangle$ is reduced and unmixed. We show that the corresponding variety is irreducible by arguing that it has one top-dimensional component. We require the following lemma.

Lemma 2.4.6. *Let $\emptyset \neq S \subseteq [n]$, and let $J_S = \langle \sum_{i \in S} Y_{ji}, \det Y(j) \mid 0 \leq j \leq n \rangle$ be the ideal of the variety F_S. The given generating set of J_S is a Gröbner basis, and $\dim(F_S) = n^2 - 1$.*

Proof. Consider the matrix of variables

$$\begin{bmatrix} Y_{10} & Y_{11} & \dots & Y_{1n} \\ Y_{20} & Y_{21} & \dots & Y_{2n} \\ Y_{30} & Y_{31} & \dots & Y_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n0} & Y_{n1} & \dots & Y_{nn} \end{bmatrix}$$

under the term order:

$$Y_{10} \prec Y_{11} \prec \dots \prec Y_{1n} \prec Y_{20} \prec Y_{21} \prec \dots \prec Y_{2n} \prec Y_{30} \prec Y_{31} \prec \dots \prec Y_{nn},$$

where the Y_{j0} element is the smallest in the row, but the ordering completes the row from left to right before moving on to the next row. Let $|S| = k$. Without loss of generality we can have these forms appear together and at the beginning of the matrix (starting in column two since we will consider deleting the first column). Therefore, we wish to verify that the generators $f_j = \sum_{i=1}^{k} Y_{ji}$, $1 \leq j \leq n$ and $\det Y(j)$ form a Gröbner basis.
We verify that the generators are a Gröbner basis by computing the S-pairs. The f_j and the determinants each independently form their own respective Gröbner bases. In particular, the f_j are linear, and the minors are the usual generating set of the ideal of a determinantal variety. It remains to show that the the S-pair of one of the f_j and one $\det Y(j)$ reduces to zero.
Without loss of generality, we consider $\det Y(0)$. It is straightforward to show that the lead term $\mathrm{LT}(\det Y(0)) = Y_{11}Y_{22}...Y_{nn}$. Observe that $\mathrm{LT}(\det Y(0))$ is disjoint from the lead term of every f_j except for f_1. Therefore, we only need to consider $S(f_1, \det Y(0))$:

$$\begin{aligned} S_1 = S(f_1, \det Y(0)) &= Y_{22}Y_{33}...Y_{nn}(Y_{11} + Y_{12} + ... + Y_{1k}) - \det Y(0) \\ &= Y_{22}Y_{33}...Y_{nn}(Y_{12} + ... + Y_{1k}) - (\det Y(0) - Y_{11}Y_{22}...Y_{nn}). \end{aligned}$$

There are $(n-1)!$ terms of the determinant with the coefficient Y_{11}, so the $\mathrm{LT}(S_1)$ will also be one of these terms. In each of these determinant minors, the Y_{jj} term will lead, so $\mathrm{LT}(S_1) = -Y_{11}Y_{22}Y_{33}...Y_{nn-1}Y_{n-1n}$. Notice $\mathrm{LT}(f_1)|\mathrm{LT}(S_1)$, so we have:

$$
\begin{aligned}
S_2 &= S_1 - (-Y_{22}...Y_{nn-1}Y_{n-1n})(f_1) \\
&= Y_{22}Y_{33}...Y_{nn}(Y_{12} + ... + Y_{1k}) - (\det Y(0) + Y_{11}Y_{22}...Y_{nn}) \\
&\quad - (-Y_{22}...Y_{nn-1}Y_{n-1n})(Y_{11} + Y_{12} + ... + Y_{1k}) \\
&= (Y_{22}Y_{33}...Y_{nn} - Y_{22}...Y_{nn-1}Y_{n-1n})(Y_{12} + ... + Y_{1k}) - \det Y(0) \\
&\quad - Y_{11}Y_{22}...Y_{nn} + Y_{11}Y_{22}...Y_{nn-1}Y_{n-1n}).
\end{aligned}
$$

We continue in this way until we have accounted for all of the terms of the determinant that contain Y_{11}, giving:

$$
\begin{aligned}
S_{(n-1)!} &= (Y_{12} + Y_{13} + ... + Y_{1k})(\det Y(0)_{1,1})) - (\det Y(0)) - Y_{11}(\det Y(0)_{1,1}))) \\
&= (f_1 - Y_{11})(\det Y(0)_{1,1}) - (\det Y(0) - Y_{11}(\det Y(0)_{1,1})),
\end{aligned}
$$

where $Y(0)_{1,1}$ is the minor of $Y(0)$ achieved from deleting the first row and first column. Then, $S_{(n-1)!}$ has no remaining terms that contain Y_{11}, so we move on to the next lowest term: Y_{12}. Notice that, per the term order, the next leading terms of the determinant will contain $Y_{12}Y_{21}$ since Y_{21} is the smallest term in the second row. In fact, the lead terms will appear in the same order as when we considered terms containing Y_{11}, just with the opposite sign and Y_{22} term replaced with Y_{21}. That is to stay, $\text{LT}(S_{(n-1)!}) = -Y_{12}Y_{21}Y_{33}...Y_{nn}$, which is divisible by $\text{LT}(f_2) = Y_{21}$. This gives:

$$
\begin{aligned}
S_{((n-1)!+1)} &= S_{(n-1)!} - (-Y_{12}Y_{33}...Y_{nn})(f_2) \\
&= (f_1 - Y_{11})(\det Y(0)_{1,1}) - [\det Y(0) - Y_{11}(\det Y(0)_{1,1})] \\
&\quad - (-Y_{12}Y_{33}...Y_{nn})(Y_{21} + ... + Y_{2k}) \\
&= (f_1 - Y_{11})(\det Y(0)_{1,1}) + (Y_{22} + ... + Y_{2k})(Y_{12}Y_{33}...Y_{nk}) \\
&\quad - (\det Y(0) - Y_{11}(\det Y(0)_{1,1}) - Y_{12}Y_{21}Y_{33}...Y_{nn}).
\end{aligned}
$$

Continuing in this way, we'll get:

$$
\begin{aligned}
S_{2(n-1)!} &= (f_1 - Y_{11})(\det Y(0)_{1,1}) - (f_2 - Y_{21})(\det Y(0)_{1,2}) \\
&\quad - ((\det Y(0) - Y_{11}(\det Y(0)_{1,1}) + Y_{12}(\det Y(0)_{1,2})).
\end{aligned}
$$

At the end of each $k(n-1)!$ steps, we are pulling the cofactor $Y_{1k}(\det Y(0)_{1,k}$ off $\det Y(0)$ and adding $(f_k - \mathrm{LT}(f_k))(\det Y(0)_{1,k}$. It follows that:

$$S_{n!} = (f_1 - Y_{11})(\det Y(0)_{1,1}) - (f_2 - Y_{21})(\det Y(0)_{2,1}) + ... + (-1)^{n+1}(f_n - Y_{nn-1})(\det Y(0)_{nn-1}).$$

In particular:

$$S_{n!} = \det\left(\begin{bmatrix} (f_1 - Y_{11}) & Y_{12} & \dots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ (f_n - Y_{n1}) & Y_{n2} & \dots & Y_{nn} \end{bmatrix}\right) = 0.$$

as the first column is the sum of the $k-1$ columns used to create f_j, creating a linear dependence. We conclude that the collection of generators forms a Gröbner basis.

We now compute the dimension of F_S. Consider the collection $\{Y_{01}...Y_{nn}\}$ of $n(n+1)$ variables. We wish to determine the dimension of the initial ideal, $\mathrm{in}(J_S)$, by determining the degree of the largest monomial, M, not divisible by any generator of $\mathrm{in}(J_S)$. Notice that, from the f_j, $\mathrm{in}(J_S)$ contains n degree 1 lead terms, $\{Y_{11}, ..., Y_{1n}\}$, none of which can appear in M. However, the product of the remaining n^2 variables is still divisible by the lead terms of the determinants. In order to remove these, we consider how many variables could possible end up in the top left corner of the of the $n \times n$ minor, Y. These terms come from the first two columns of our general matrix. However, the entire second column has been removed from consideration by the lead terms of the linear forms. Therefore, we consider only the first column. With n rows, there is 1 entry from the first column that could appear in the top left position, all of which appear in the subsequent lead term of the corresponding determinant. Therefore, these terms also cannot appear in M. This is all the terms that need to be removed since the lead term of all the determinants is the product of its diagonal entries, all of which contain an entry from the first two columns of the general $n \times (n+1)$ matrix. Therefore,

$$\dim(F_S) = \deg(M) = n(n+1) - (n) - 1 = n^2 - 1.$$

$\square$

Proposition 2.4.7. *The ideal $\langle I, x_j \rangle$ is prime.*

Proof. It suffices to treat the case $j = 0$. The ideal $\langle I, x_0 \rangle$ is generated by $\sum_{j=1}^{n} x_j Y_{ij}$ for $1 \leq i \leq n$, the minor $\det Y(0)$, and $\det Y(j) - x_j W$ for $1 \leq j \leq n$. The initial ideal $\mathrm{in}_\delta \langle I, x_j \rangle$ contains $\langle I_{\mathbf{r}}, x_j \rangle$, and the dimensions of the varieties of these ideals coincide, so it suffices to check if $\langle I_{\mathbf{r}}, x_j \rangle$ is prime. Let V be the variety of this ideal, then there is a T^n equivariant extension $f^* : \mathbb{K}[x_1, \dots, x_n] \to \mathbb{K}[V]$. Let $\mathcal{O}(S) \subseteq \mathbb{A}^n_{\mathbb{K}}$ be the orbit of T^n where $x_j \neq 0$ for $j \in S$, then the fibers of the map $f : V \to \mathbb{A}^n_{\mathbb{K}}$ over any $\mathcal{O}(S)$ are all isomorphic. We use this to compute the dimension of $f^{-1}(\mathcal{O}(S)) = V_S$.
Starting with the smallest case $S = \emptyset$, we set all $x_j = 0$. The fiber is then the determinantal variety cut out by $\langle \det Y(j) \mid 0 \leq j \neq n \rangle$. This variety has dimension $(n-1)((n+1) + n - (n-1)) = n^2 + n - 2$. If $S \neq \emptyset$, we can consider the fiber F_S over the point p_S, where $X_j(p_S) = 1$ $j \in S$, $X_j(p_S) = 0$ $j \notin S$. We have $\dim(V_S) = \dim(F_S) + |S|$. The variety F_S is cut out by the ideal $\langle \sum_{j \in S} Y_{ij}, \det(Y(j) \rangle$. By Lemma 2.4.6, $\dim(F_S)$ is $n^2 - 1$. As a consequence, $\dim(V_S) = n^2 - 1 + |S| < n^2 + n - 1 = \dim(V_{[n]})$.
We conclude that $V_{[n]}$ has strictly higher dimension than all other V_S, so the closure $\overline{V}_{[n]} \subseteq V$ is a top dimensional component. But the complement of this closure must be composed of constructible sets of strictly smaller dimension. It follows that $V = \overline{V}_{[n]}$, and that V is reduced and irreducible. $\square$

Now by [18, Theorem 1.5] we have:

$$\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^n)) = \mathbb{K}[x_j, Y_{ij}, W] / \langle \sum x_j Y_{ij}, \det Y(j) - x_j W \rangle.$$

Remark 2.4.8. In principle the multigraded Hilbert series of $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^n))$ should be expressible in terms of the K-polynomial of quiver variety determined by the rank array $\mathbf{r}$. This involves the so-called Grothendieck polynomials, see [21].

By Theorem 2.3.7, $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^m))$ for $m \geq n$ is the image of the map:

$$
\begin{aligned}
\Phi_m : \mathbb{K}[x_j, Y_{ij}, W_\tau] \to \mathbb{K}[t_j^{\pm}, y_{ij}]& \\
x_j &\to t_j^{-1} \\
W_\tau &\to \det[y(0,\tau)]t_0 \cdots t_n \\
Y_{i0} &\to (-\sum_{j=1}^{n} y_{ij})t_0 \\
Y_{i1} &\to y_{i1}t_j \\
&\vdots \\
Y_{in} &\to y_{in}t_n
\end{aligned}
$$

where $1 \leq i \leq m$, $0 \leq j \leq n$, and $\tau \in \bigwedge^m[n] = \{S \subset [n] \mid |S| = n\}$. In the grading by $\mathrm{Pic}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^m)) \cong \mathbb{Z} \times \mathbb{Z}$, the addition generators W_τ all have degree $(n+1, n)$. We can rephrase this by saying that there is a surjection of twisted commutative algebras:

$$
\Phi_V : \mathrm{Sym}\left(\mathbb{K}^{n+1} \oplus (\mathbb{K}^{n+1} \otimes V) \oplus \bigwedge^n V\right) \to \mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes V)).
$$

Remark 2.4.9. In future work, it may be of interest to find a description of the functor $V \to \ker(\Phi_V)$.

2.4.1 The full flag bundle $\mathcal{FLT}_n$

By Theorem 2.3.2, the Cox ring $\mathrm{Cox}(\mathcal{FLT}_n)$ is the algebra of invariants $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^{n-1}))^{U_{n-1}}$, where U_{n-1} is the group of $n-1 \times n-1$ lower-triangular matrices. The action of U_{n-1} on $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^{n-1}))$ extends to an action on the presenting polynomial ring $\mathbb{K}[x_j, Y_{ij} \mid 0 \leq j \leq n, 1 \leq i \leq n-1]$, so we obtain a presentation by invariants:

$$
\mathbb{K}[x_j, Y_{ij} \mid 0 \leq j \leq n, 1 \leq i \leq n-1]^{U_{n-1}} \to \mathrm{Cox}(\mathcal{FLT}_n) \to 0.
$$

The algebra $\mathbb{K}[x_j, Y_{ij} \mid 0 \leq j \leq n, 1 \leq i \leq n-1]^{U_{n-1}} \subset \mathbb{K}[x_j, Y_{ij} \mid 0 \leq j \leq n, 1 \leq i \leq n-1]$ is a polynomial ring in $n+1$ variables over the Plücker algebra of minors of the matrix $[Y_{ij}]$. We present $\mathrm{Cox}(\mathcal{FLT}_n)$ as a quotient of the polynomial ring $\mathbb{K}[x_j, P_\tau, P_{0,\tau} \mid 0 \leq j \leq n, \tau \subset [n]]$. We make use of the realization of $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^{n-1}))$ as a subalgebra of $\mathbb{K}[t_j, y_{ij}]$, to get that $\mathrm{Cox}(\mathcal{FLT}_n)$ is the image of the polynomial map $\Psi : \mathbb{K}[x_j, P_\tau, P_{0,\tau} \mid 0 \leq j \leq n, \tau \subset [n]] \to \mathbb{K}[t_j, y_{ij}]$, where:

$$\Psi(x_j) = t_j^{-1},$$

$$\Psi(P_\tau) = \det[y(\tau)]t^\tau,$$

$$\Psi(P_{0,\tau}) = \sum_{j=1}^{n} \det[y(j,\tau)]t_0 t^\tau.$$

Here $y(\tau)$ denotes the minor on the first $|\tau|$ rows and the τ columns of $[y_{ij}]$. The map Ψ factors through the quotient map from $\mathbb{K}[x_j, Y_{ij} \mid 0 \leq j \leq n, 1 \leq i \leq n-1]^{U_{n-1}}$, so the usual quadratic Plücker relations hold among the P_τ and $P_{0,\tau}$. Additionally, we have the relations $\sum_{j \notin \tau} x_j P_{j\tau} = 0$, which are consequences of the defining relations of $\mathrm{Cox}(\mathbb{P}(\mathcal{T}_n \otimes \mathbb{K}^{n-1}))$.

Theorem 2.4.10. *The ideal* $\ker(\Psi)$ *presenting* $(\mathcal{FLT}_n)$ *is generated by the Plücker relations among the* P_τ *and* $P_{0,\tau}$*, along with the quadratics* $\sum_{j \notin \tau} x_j P_{j\tau} = 0$.

For the proof of Theorem 2.4.10 we use a subduction argument [17, Algorithm 1.4] involving a modification of the semigroup GZ_n of Gel'fand-Zetlin patterns with n rows. A **Gel'fand-Zetlin pattern** $g \in GZ_n$ is an array of integers arranged in n rows, where the i-th row has $n+1-i$ entries g_{ij}. These integers satisfy additional *interlacing inequalities*: $g_{ij} \geq g_{i+1,j} \geq g_{i,j+1}$. Let GZ_n^+ be the set of patterns with $g_{1n} = 0$. It is well-known that the Cox ring $\mathrm{Cox}(\mathcal{FL}(\mathbb{K}^n))$ has a discrete valuation $\mathfrak{v}_{GT}$ with value semigroup is GZ_n^+. The generators of GZ_n^+ are in bijection with strict subsets $\tau \subset [n]$, and in turn, with the Plücker generators of $\mathrm{Cox}(\mathcal{FL}(\mathbb{K}^n))$. The pattern $g(\tau)$ corresponding to τ is the unique pattern with $|\tau \cap [n-i+1]|$ 1's and $|[n-i+1] \setminus \tau|$ 0's in row i.

Proof of Theorem 2.4.10. We select a monomial ordering on $\mathbb{K}[t_j, y_{ij}]$ which satisfies $y_{ij} \prec t_\ell$ for all i, j, ℓ, and is diagonal on the y_{ij}. In particular, the initial form $\text{in}_\prec \det[y(\tau)]$ is the product of the diagonal terms. The following construction should be compared to the Gel'fand-Zetlin degeneration of the usual flag variety.

We identify the initial forms $\text{in}_\prec t_j^{-1}$, $\text{in}_\prec \det[y(\tau)]t^\tau$, $\text{in}_\prec \det[y(0,\tau)]t_0 t^\tau$ with certain extended Gel'fand-Zetlin patterns. The form $\text{in}_\prec x_j$ is sent to $(0, -\mathbf{e}_j) \in GZ_n \times \mathbb{Z}^{n+1}$, and $\text{in}_\prec \det[y(\tau)]t^\tau$ is sent to $(g(\tau), \sum_{j\in\tau} \mathbf{e}_j) \in GZ_n \times \mathbb{Z}^{n+1}$. The initial form $\text{in}_\prec \sum_{j=1}^n \det[y(j,\tau)]t_0 t^\tau$ requires some discussion. Observe that this sum can be rewritten as $\sum_{j\notin\tau} \det[y(j,\tau)]t_0 t^\tau$. The initial monomial from these minors will then be the diagonal term of the minor (ℓ, τ) where ℓ is the *first* element of $[n]$ not in τ. We let τ^* denote $\tau \cup \{\ell\}$. Accordingly, we send $\text{in}_\prec \sum_{j=1}^n \det[y(j,\tau)]t_0 t^\tau$ to $(g(\tau^*), \mathbf{e}_0 + \sum_{j\in\tau} \mathbf{e}_j) \in GZ_n \times \mathbb{Z}^{n+1}$.

We must compute a generating set of binomial relations on these extended Gel'fand-Zetlin patterns, then show that each relation can be lifted to an element in the ideal generated by the Plücker relations and the $\sum_{j\notin\tau} x_j P_{j\tau} = 0$. Following [17, Theorem 1.4], we have then shown that these relations generate $\ker(\Psi)$, and that $(\mathcal{FLT}_n)$ has a full rank valuation with Khovanskii basis given by the x_j and P_τ.

To simplify notation, we let $(0, -\mathbf{e}_\ell)$ be denoted by $[-\ell]$, $(g(\tau), \sum_{j\in\tau} \mathbf{e}_j)$ be denoted by $[\tau, 0]$, and $(g(\tau^*), \mathbf{e}_0 + \sum_{j\in\tau} \mathbf{e}_j)$ be denoted by $[\tau^*, \ell]$, where ℓ is the element "replaced" by 0. Observe that $[\tau, a]$ makes sense if and only if $[a] \subset \tau$.

Now we have several natural classes of binomial relations. For any $a \in [n]$ with $[a] \subset \tau$ we have:

$$[-a][\tau, 0] = [0][\tau, a].$$

Next, for any $\tau, \eta \subset [n]$,

$$[\tau, 0][\eta, 0] = [\tau \cup \eta, 0][\tau \cap \eta, 0]$$

For any marked $[\tau, a][\eta, b]$ we can perform relations like this and the there is always a compatible assignment of the markings a, b. We call relations of this type "union/intersection" relations. The first type of relation above lifts to $\sum_{j\notin\tau} x_j P_\tau = 0$, and the union/intersection lifts to a Plücker relation. Therefore, if we check that these relations suffice to generate the binomial ideal which vanishes on the initial forms, we have shown that the required relations generate $\ker(\Psi)$.

Let us suppose we have two words $A_1 \cdots A_n$, $B_1 \cdots B_n$ whose product maps to the same extended Gel'fand-Zetlin pattern. We must show that after applications of the above binomial relations, these words can be taken to a common word. If any element $[-a]$ corresponds to an $[a]$ not supported by a pattern elsewhere in the word, this can be read off the $\mathbb{Z}^{n+1}$ component of the corresponding extended pattern. Moreover, any $[-a]$ for a which appear in some $(\tau, 0)$ can be turned into $[0]$ using the first relation above. The number of these elements can also be read off the extended pattern, so we may assume without loss of generality that both words do not contain any elements $[-a]$ for $a \in \{0\} \cup [n]$. Next, using the union/intersection relations, we can assume that the underlying Gel'fand-Zetlin patterns of the A_i and B_i are the same, with possibly different markings. Select a pattern on both sides, $A_1 = [\tau, a_1]$, $B_1 = [\tau, b_1]$. If both markings are equal (including the case that they are 0), we may factor off this top element and appeal to induction. If not, say $a_1 < b_1$. We must conclude that $[a_1] \subset \tau$ and $[b_1] \subset \tau$. Moreover, there must be some other pattern $A_j = [\eta, b_1]$. Indeed, the set of markings can be deduced by comparing the total Gel'fand-Zetlin pattern of the word to its $\mathbb{Z}^n$ component. Now we can form $[\tau, a_1][\eta, b_1] = [\tau, b_1][\eta, a_1]$ as $[a_1] \subset [b_1] \subset \eta$. We factor off the first element of both words, and once again appeal to induction. This completes the proof. □

Remark 2.4.11. These descriptions are presented for the tangent bundle $\mathcal{T}_n$ since that's the framework for which the results were originally obtained. However, after further consideration, each argument in this section holds up for $\mathcal{E}$ a sparse, hypersurface bundle. We intend to address this in future work.

Chapter 3 Positivity Properties of Toric Vector Bundles

This chapter chronologically followed our study of projectivized toric vector bundles as Mori dream spaces discussed in the previous chapter. In particular, the class of Mori dream space bundles is so well-behaved that it is natural to wonder about the geometry of these bundles. We find that these nicely described (monomial, CI, sparse, etc.) toric vector bundles have desirable implications between the commonly-studied positivity properties. To begin our discussion of these results, we first establish some fundamental ideas. These definitions and foundational results are summarized from [10] and [22].

3.1 Background

Let X be an n-dimensional nonsingular variety and let D be a divisor on X. Then, for a curve C on X, **intersection number** of D with C, written $D \cdot C$, generalizes the number of times two curves intersect in higher dimensions (see [22]). We will use the intersection number as a positivity metric for a divisor.

Recall that a Weil divisor is a formal linear combination, $D = \sum a_i d_i$, where $a_i \in \mathbb{Z}$ and d_i are prime codimension-1 subvarieties. Additionally, we have another classical family of divisors: Cartier divisors. A **Cartier divisor** is a collection $\{U_i, f_i\}$ where $\{U_i\}$ are an open cover of X, $f_i \in K^{\times}(X)$, and $f_i g_{ij} = f_j g_{ij}$ on $U_i \cap U_j$ for g_{ij} a regular function on $U_i \cap U_j$. Over a nonsingular variety, we can realize Cartier divisors as Weil divisors by defining $D = \sum_v \mathrm{ord}_v(f_i)[v]$ for the f_i from the $\{U_i, f_i\}$ pairs that define our Cartier divisors [10]. For a variety X, we will denote the Weil divisors on X by $\mathrm{WDiv}(X)$ and the Cartier divisors by $\mathrm{CDiv}(X)$. We recall that the class group of X is $\mathcal{CL}(X) = \mathrm{WDiv}(X)/\mathrm{Prin}(X)$. The Cartier divisor analogue to the class group is the Picard group, $\mathrm{Pic}(X)$.

To establish the environment we will work in, consider a variety X and Cartier divisors $D, D' \in \mathrm{Div}(X)$. We then say that D is **numerically equivalent** to D', written $D \sim_{\mathrm{num}} D'$, if $D \cdot C = D' \cdot C$ for all irreducible curves C. We then define the **Néron-Severi group** to be $N(X) := \mathrm{CDiv}(X)/\sim_{\mathrm{num}}$. We will consider the Néron-Severi space to be $N_{\mathbb{Q}}(X) := N(X) \otimes \mathbb{Q}$. We then have the following diagram:

$$\begin{tikzcd}
\mathrm{CDiv}(X) \arrow[r] \arrow[d] & \mathrm{WDiv}(X) \arrow[d] \\
\mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathcal{CL}(X) \\
N(X) &
\end{tikzcd}$$

However, we are considering the case where X is a smooth, projective variety, we get that $\mathrm{CDiv}(X) \cong \mathrm{WDiv}(X)$ and $\mathrm{Pic}(X) \cong \mathcal{CL}(X)$ [10]. We then get a natural map $\mathcal{CL}(X) \to N(X)$. For our study, we make the assumption that the class group $\mathcal{CL}(X)$ is finitely generated, meaning $\mathcal{CL}(X) \to N(X)$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Therefore, we work with the definition that $N_{\mathbb{Q}}(X) := \mathcal{CL}(X) \otimes \mathbb{Q}$.

3.1.1 Positivity of Divisors on Varieties

Effective divisor classes give us a first notion of positivity. Recall that a divisor $D = \sum a_i d_i$ is called effective if $a_i \geq 0$ for all i. The **complete linear system** of a divisor D on a variety X, denoted $|D|$, is the set of all effective divisors linearly equivalent D. Recall that two divisors D and E are linearly equivalent if there is a rational function f on X such that $E = D + \mathrm{div}(f)$. A divisor class is called an **effective divisor class** if there exists an $E \in |D|$ where E is effective. It is called a **pseudoeffective divisor class** if there exists an $n \in \mathbb{N}$ such that $n|D|$ is effective. Pseudoeffective classes generate a cone, called the **pseudoeffective cone**, $\mathrm{PsEff}(X) \subseteq N_{\mathbb{Q}}(X)$. Contained in the pseudoeffective cone is the **effective monoid**, $\mathrm{Eff}(X) \subseteq \mathrm{PsEff}(X)$.

Basepoint-free-ness provides the second, more sharpened notion of positivity. The **base locus** of a complete linear system $|D|$, denoted $\mathrm{Bl}(|D|)$, is the common intersection of the supports of effective divisors in $|D|$. More specifically,

$$\mathrm{Bl}(|D|) := \bigcap_{E \in |D|} \mathrm{Supp}(E).$$

Notice that $\mathrm{Bl}(|D|) = X$ if $|D|$ is not effective. D is called **basepoint free (bpf)** if $\mathrm{Bl}(|D|) = \emptyset$. Like in the effective case, we get that the basepoint free divisors form a monoid $\mathrm{BPF}(X) \subseteq N(X)$. A divisor D is called **semi-ample** if there exists an $n \in \mathbb{N}$ such that nD is basepoint free. The collection of semi-ample divisors of D

generate a cone, called the **semi-ample cone**, which we'll denote $\mathrm{SAmp}(X)$.

We recall that the space of global sections $H^0(X, D) = \{f \in K(X) \mid \mathrm{div}(f) + D \geq 0\}$ is a vector space in the space of rational functions on X. When X is complete, we have that $H^0(X, D)$ is finite-dimensional with dimension depending only on $|D|$. Then, choosing a basis $\{f_1, ..., f_d\}$ of $H^0(X, D)$, we have the map:

$$\varphi : X \backslash \mathrm{Bl}(|D|) \to \mathbb{P}H^0(X, D)^\vee, \quad x \mapsto [f_0(x) : ... : f_d(x)].$$

A change of basis amounts to an action by the projective general linear group. We also have that a different choice of $D \in |D|$ defines a natural isomorphism between section spaces. Notice that if X is basepoint free, we have

$$\varphi : X \to \mathbb{P}H^0(X, D)^\vee.$$

Then a divisor D is **very ample** if φ is an embedding. D is called **ample** if there exists $n \in \mathbb{N}$ such that nD is very ample.

Recalling intersection numbers, we then say that a divisor D is **nef** if the intersection number of D with a curve C, written $D \cdot C$, is nonnegative for all curves C in X. We let $\mathrm{NEF}(X), \mathrm{Amp}(X) \subseteq N(X)_{\mathbb{Q}}$ be the cones generated by nef and ample divisors on X. Crucially, due to Kleiman, we have the following relationship.

Theorem 3.1.1. *[19] For a projective variety X, the nef cone of X is the closure of the ample cone of X.*

For a variety X, the **canonical bundle**, ω_X, is a line bundle that can be realized as the top exterior power of the tangent bundle of X, $\omega_X = \bigwedge^d \mathcal{T}_X$. A canonical divisor, K_X, is then a divisor such that $\omega_X = \mathcal{L}(K_X)$. An **anticanonical divisor** is any divisor $-K$ such that K is canonical. In terms of the bundles, the anticanonical bundle is the inverse bundle ω^{-1} such that ω is canonical. When $-K_X$ is ample, the space X is called **Fano**.

The study of positivity properties and canonical bundles led Fujita to the following conjectures.

Conjecture 3.1.2. *Let X be smooth of dimension n and let $A \in \mathcal{CL}(X)$ be ample, then*

1. *for $m \geq n + 1$, $K_X + mA$ is basepoint free,*

2. for $m \geq n+2$, $K_X + mA$ is very ample.

We will refer to the first part of the conjecture as *Fujita freeness* and the second part as *Fujita ampleness*. The Fujita conjectures have been confirmed for particular cases, like smooth projective toric varieties. However, since their declaration in 1988, they has not been proven. We approach these conjectures for special bundles over $\mathbb{P}^{\bar{n}}$.

Mori's cone theorem ([22, Theorem 1.5.33]) shows that if X and A are as in 3.1.2, then $K_X + mA$ is nef if $m \geq n+1$ and ample if $m \geq n+2$ (see [22, Section 10.4]). Therefore, to prove Fujita's freeness and ampleness conjectures it is sufficient to show that nef implies basepoint free and ample implies very ample on $\mathbb{P}\mathcal{E}(L, D)$. Note that these implications do not hold in general, and we conclude this chapter with an example of a case where nef does not imply basepoint free.

Remark 3.1.3. I cannot help but share the rant that many around me have heard as I was writing this chapter. The names for these positivity properties are inconsistent and frustrating. You mean to tell me that "semi-ample" is not a property related to ampleness? Maddening. I propose we (as a community) adopt the following naming convention:

$$\begin{aligned} \text{very ample} &\to \text{ample} \\ \text{ample} &\to \text{pseudo-ample} \\ \text{semi-ample} &\to \text{pseudo-basepoint free} \end{aligned}$$

However, no one has given me the power to change these namings. Therefore, for consistency's sake, this document will (unfortunately) continue to use the names as they are classically defined.

3.1.2 Positivity of Mori Dream Spaces

When our space X is a Mori dream space, we have a much more computational description of these positivity monoids and cones. In particular, we consider the case where X is a smooth variety with finitely generated class group.

Let $f_1, ..., f_n$ be a homogeneous (with respect to $\mathcal{CL}(X)$) generating set of $\text{Cox}(X)$. Let $\deg(f_i)$ be the class-grading of $f_i \in \text{Cox}(X)$. We then have the maps

$$\varphi : \mathbb{Q}^n_{\geq 0} \to N(X), \quad (a_1, ..., a_n) \mapsto \sum a_i \deg(f_i)$$

and

$$\psi : \mathbb{Z}^n_{\geq 0} \to N(X), \quad (a_1, ..., a_n) \mapsto \sum a_i \deg(f_i).$$

The image of φ is then the **pseudoeffective cone** of X, PEff(X). The image of ψ is the **effective monoid** of X, Eff(X).

We can define cones of the divisors that have nonvanishing sections at given points. In particular, we can define the monoid at a point p as

$$S_p := \{D \mid \exists s \in H^0(X, D), \quad s(p) \neq 0\}.$$

Note that, with the nonvanishing section, these monoids see the positivity at the point.

Proposition 3.1.4. *Let X be a Mori dream space. Then S_p is generated by* $\deg(f_i)$ *for f_i generators of Cox(X) such that $f_i(p) \neq 0$.*

Proof. Let $D \in S_p$. Then there exists $s \in H^0(X, D)$ such that $s(p) \neq 0$. Write $s = \sum c_\alpha f_1^{\alpha_1} ... f_n^{\alpha_n}$. Since this is homogeneous, we have $D = \sum \alpha_i \deg(f_i)$. Since at least one of the terms of s must be nonzero at p, we have $f_i(p) \neq 0$ for all $\alpha_i > 0$. □

Define $C_p := \mathbb{Q}_{\geq 0} S_p$. We then have the following definition for two of the cones we will consider.

Proposition 3.1.5. *The basepoint free cone of a Mori dream space X is*

$$BPF(X) = \bigcap_{p \in X} S_p$$

and the semi-ample cone is

$$SAmp(X) = \bigcap_{p \in X} C_p.$$

In the above proposition, we denote $\bigcap C_p$ as the semi-ample cone. For a general variety X, this intersection does define only the semi-ample cone. However, in the case that X is a Mori dream space, these we have that the semi-ample cone and the nef cone coincide [1]. As the nef cone is of more interest to our study, we will refer to this intersection as NEF(X).

In summary, for a divisor X, we have the following cone containments:

$$\text{Amp}(X) \subseteq \text{NEF}(X) \subseteq \text{SAmp}(X) \subseteq \text{PsEff}(X).$$

We have a similar inclusion for the associated monoids:

$$\text{BPF}(X) \subseteq \text{Eff}(X).$$

We can see from Propositions 3.1.4 and 3.1.5 that it suffices to consider only finitely many points $p \in X$. In particular, these cones are polyhedral and the monoids are finitely generated.

3.2 Positivity for Mori Dream Space Bundles

Throughout this section, we consider the case where $\mathbb{P}\mathcal{E}$ is a Mori dream space. Presentations of $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ for various projectivized toric vector bundles are constructed in [18]. Let $X_i \in \mathrm{Cox}(\mathbb{P}\mathcal{E})$ for $1 \leq i \leq n$ denote the section of the pullback of the class of the i-th toric divisor on $Y(\Sigma)$ corresponding to the ith ray of Σ. For each generator $y_j \in \mathbb{C}[y_1, \ldots, y_m]$ there is a corresponding section $Y_j \in \mathrm{Cox}(\mathcal{E})$. The j-th column D_j of the diagram D defines an element $\mathbf{d}_j = \sum D_{ij}\mathbf{e}_i \in \mathcal{CL}(Y(\Sigma))$. The divisor classes of these sections are as follows:

$$\deg(X_i) = (-\mathbf{e}_i, 0),$$
$$\deg(Y_j) = (\mathbf{d}_j, 1).$$

We compute the various positivity cones for $\mathbb{P}\mathcal{E}$. Expanding on Proposition 3.1.5, we have the following proposition that details exactly which finite collection of points we need to consider.

Proposition 3.2.1. *For BPF($\mathbb{P}\mathcal{E}$) and NEF($\mathbb{P}\mathcal{E}$) it suffices to consider the points in the T_N-fixed-point fibers $(\mathbb{P}\mathcal{E})_\sigma \subset \mathbb{P}\mathcal{E}$.*

Proof. Let $p \in \mathbb{P}\mathcal{E}$, and let $n \in N$ be such that $p_0 = \lim_{t\to 0} \chi_n(t) \circ p \in \mathbb{P}\mathcal{E}_\sigma$ for a maximal face $\sigma \in \Sigma$. Now if $s \in \mathrm{Cox}(\mathbb{P}\mathcal{E})$ is a T_N-quasi-invariant section, we must have $s(p) = 0$ implies $s(p_0) = 0$. As a consequence, any section which does not vanish at p_0 must not vanish at p, so $S_p \supseteq S_{p_0}$ and $C_p \supseteq C_{p_0}$. □

3.2.1 Bundles over a general toric variety, $Y(\Sigma)$

Now, we want to reduce the number of points that we need to consider. Let $L \subset \mathbb{C}[y_1, \ldots, y_m]$ be a linear ideal. Then consider the lattice of flats of the matroid $\mathcal{M}(\mathrm{in}_\sigma L)$. To each maximal flat in $\mathcal{M}(\mathrm{in}_\sigma L)$, we can associate a point, formalized in the following definition.

Definition 3.2.2. For $\sigma \in \Sigma$ a maximal face, and F a maximal flat of the matroid $\mathcal{M}(\mathrm{in}_\sigma L)$, let $p_{\sigma,F} \in \mathbb{P}\mathcal{E}_\sigma$ be such that an entry is nonzero if and only if that entry corresponds to F. (There is a unique point in the zero-locus of $\text{in}_\sigma L$, up to scaling, since F is maximal.) Let $S_{\sigma,F}$ and $C_{\sigma,F}$ be the monoid and rational cone of nonzero sections associated to $p_{\sigma,F}$.

Theorem 3.2.3. *Let $\mathcal{E}$ be a toric vector bundle over $Y(\Sigma)$ with Cox($\mathbb{P}\mathcal{E}$) generated in Sym-degree 1. Then BPF($\mathbb{P}\mathcal{E}$) $= \bigcap_{\sigma,F} S_{\sigma,F}$, and NEF($\mathbb{P}\mathcal{E}$) $= \bigcap_{\sigma,F} C_{\sigma,F}$.*

Proof. By Proposition 3.2.1, we only need to consider points $p \in V(\text{in}_\sigma L) \subseteq \mathbb{P}^{m+1}$. We have that the $\{y_i \mid y_i(p) = 0\}$ form a flat, F_p. To see this, suppose that $\langle F_p \rangle \neq F_p$. Then let $y \in \langle F_p \rangle \backslash F_p$. This allows us to write $y = \sum a_i y_i$ for $y_i \in F_p$. However, if $y_i(p) = 0$ for all $y_i \in F_p$, then $y(p) = 0$. So F_p is a flat. As a consequence, if F is a maximal flat containing F_p. By Proposition 3.1.4, $C_{p_{\sigma,F}} \subseteq C_p$. □

Recall that a monomial toric vector bundle $\mathcal{E}(L, D)$ is a toric vector bundle where $\text{in}_\sigma(L)$ is a monomial ideal for all maximal $\sigma \in \Sigma$, where Σ. Monomial toric vector bundles are a natural family to consider for positivity properties since the property of being monomial has a nice description in terms of the cones of the Gröbner fan. Additionally, the bundle having a unique minimal generator serves as a sort of "generic case" for a collection of points in $\text{Trop}(L)$.

Lemma 3.2.4. *Let $\mathcal{E}$ be a monomial bundle over $Y(\Sigma)$, and let $\sigma \in \Sigma$ be a maximal face, then any maximal flat F of $\mathcal{M}(\text{in}_\sigma L)$ is the complement of a single element.*

Proof. Each initial ideal $\text{in}_\sigma L$ is a monomial ideal. As a consequence, $\mathcal{M}(\text{in}_\sigma L)$ has a single basis of r elements and $m - r$ loops. □

Lemma 3.2.5. *Let $\mathcal{E}$ be a monomial toric vector bundle on $Y(\Sigma)$ with Cox($\mathbb{P}\mathcal{E}$) generated in Sym-degree 1. Then the monoids $S_{\sigma,F}$ are freely generated and the cones $C_{\sigma,F}$ are smooth.*

Proof. By Lemma 3.2.4, for some $i \in [m]$ we must have $S_{\sigma,F} = -S_\sigma \times \{0\} + \mathbb{Z}_{\geq 0}(\mathbf{d}_i, 1)$. This is always a freely generated monoid. □

Proposition 3.2.6. *Let $\mathcal{E}$ be a monomial toric vector bundle on $Y(\Sigma)$ with Cox($\mathbb{P}\mathcal{E}$) generated in Sym-degree 1, then any nef divisor on $\mathbb{P}\mathcal{E}$ is basepoint free. In particular, $\mathbb{P}\mathcal{E}$ satisfies Fujita's freeness conjecture.*

Proof. By Theorem 3.2.3 we have BPF($\mathbb{P}\mathcal{E}$) $= \bigcap_{\sigma,F} S_{\sigma,F}$. The intersection of a set of saturated monoids is a saturated monoid, so any integral nef divisor is already a member of BPF($\mathbb{P}\mathcal{E}$). □

Corollary 3.2.7. *Let $\mathcal{E}(L, D)$ be a CI bundle with $D = \left[\omega_1 ... \omega_n\right]^T$. If for all σ, $\sum_{i \in \sigma(1)} \omega_i$ lives in the interior of a maximal face of GFan(L), then $\mathbb{P}\mathcal{E}$ satisfies Fujita freeness.*

Proof. The Gröbner fan condition implies that the bundle in monomial. $\square$

3.2.2 Bundles over projective space

Theorem 3.2.8. *Let $\mathcal{E}$ be a bundle over $\mathbb{P}^n$ with Cox$(\mathbb{P}\mathcal{E})$ generated in Sym-degree 1. Then any pseudoeffective class on $\mathbb{P}\mathcal{E}$ is effective and any nef class on $\mathbb{P}\mathcal{E}$ is basepoint free. In particular, $\mathcal{E}$ satisfies Fujita freeness.*

Proof. Similarly to Proposition 3.2.6, Cox$(\mathbb{P}\mathcal{E})$ begin generated in Sym-degree 1 means that any monoid $S_{\sigma,F}$ is of the form $\mathbb{Z}_{\geq 0}\{(-1, 0), (d_j, 1)\}$ for $d_j \in \mathbb{Z}_{\geq 0}$. Letting $d = \max\{d_j\}$, we have that the effective monoid of any such bundle is generated by $(-1, 0)$ and $(d, 1)$. Similarly, by intersecting, the basepoint free monoid of $\mathbb{P}\mathcal{E}$ is generated by $(-1, 0)$ and $(\min\{d_j\}, 1)$. $\square$

Corollary 3.2.9. *Let $\mathcal{E}$ be a uniform bundle over $\mathbb{P}^n$ with $m - r < n$ (as in Corollary 2.2.4). Then $\mathcal{E}$ satisfies Fujita freeness.*

The following example shows that, even in the simplest possible case, the above theorem does not hold over products of projective spaces.

Example 3.2.10. Consider a monomial bundle $\mathcal{E}(L, D)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with $L = \langle y_1 + y_2 + y_3 + y_4 \rangle$ and

$$D = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}.$$

We then map

$$\begin{aligned} \deg(Y_1) &= (a_1, 0, 1) \\ \deg(Y_2) &= (0, a_2, 1) \\ \deg(Y_3) &= (a_3, 0, 1) \\ \deg(Y_4) &= (0, a_4, 1) \end{aligned}$$

and have the cones shown in Figure 3.1.

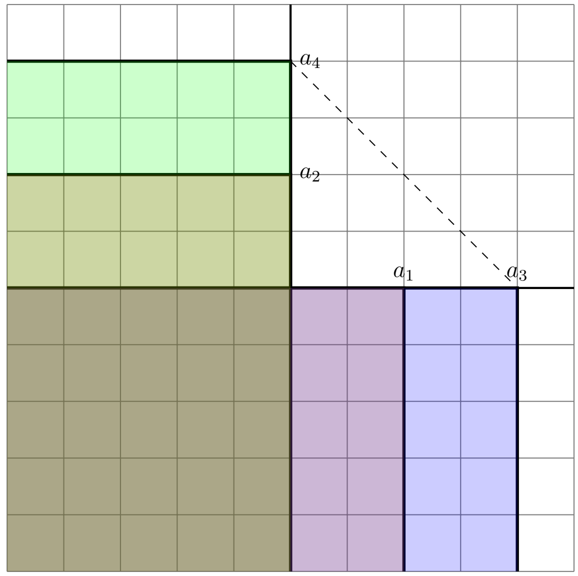

Figure 3.1: The class group, $\mathcal{CL}(\mathbb{P}\mathcal{E})$.

Here, the negative orthant is the (saturated!) nef cone. However, the integral points on the line connecting a_3 and a_4 represent classes that are pseudoeffective but not effective.

We now give an example of a bundle over $\mathbb{P}^n$ that is not generated in Sym-degree 1. As a result, we are able to produce a pseudoeffective class that is not effective.

Example 3.2.11. Let $\mathcal{T}_2$ denote the tangent bundle of $\mathbb{P}^2$. It is known ([12, 8, 18, 5] that $\mathbb{P}\mathcal{T}_2$ is a Mori dream space with Cox ring

$$\mathrm{Cox}(\mathbb{P}\mathcal{T}_2) = \mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3]/\langle X_1Y_1 + X_2Y_2 + X_3Y_3\rangle.$$

In particular, $\mathcal{T}_2$ is a CI toric vector bundle. See [5] for an account of the Cox rings of projectivized toric vector bundles of the form $\mathbb{P}(\mathcal{T}_2 \otimes V)$, where V is a finite dimensional vector space. Here we consider a different operation; let $\mathrm{Sym}^2\mathcal{T}_2$ denote the 2-nd symmetric power of $\mathcal{T}_2$. It can be shown that $\mathbb{P}\mathrm{Sym}^2\mathcal{T}_2$ is isomorphic to the Hilbert space $\mathrm{Hilb}^2(\mathbb{P}^2)$. Below we give the diagram and linear ideal defining $\mathrm{Sym}^2\mathcal{T}_2$.

$$L = \langle y_{11} + y_{12} + y_{13}, y_{12} + y_{22} + y_{23}, y_{13} + y_{23} + y_{33} \rangle \subset \mathbb{C}[y_{12}, y_{13}, y_{23}, y_{11}, y_{22}, y_{33}]$$

$$D = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{bmatrix}$$

The corresponding elements $X_1, X_2, X_3, Y_{12}, Y_{13}, Y_{23}, Y_{11}, Y_{22}, Y_{33} \subset \mathrm{Cox}(\mathbb{P}\mathrm{Sym}^2\mathcal{T}_2)$ do not suffice to generate. However, after an application of the KM algorithm, we find a single new generator $Z \in \mathrm{Cox}(\mathbb{P}\mathrm{Sym}^2\mathcal{T}_2)$. The Cox ring of $\mathbb{P}\mathrm{Sym}^2\mathcal{T}_2 \cong \mathrm{Hilb}^2\mathbb{P}^2$ is then generated by X_1, X_2, X_3 $Y_{12}, Y_{13}, Y_{23}, Y_{11}, Y_{22}, Y_{33}, Z$ subject to the ideal:

$$X_1Y_{13} + X_2Y_{23} + X_3Y_{33}, \quad X_2Y_{12} + X_1Y_{11} + X_3Y_{13}, \quad X_1Y_{12} + X_2Y_{22} + X_3Y_{23},$$

$$X_1X_2Z + Y_{13}Y_{23} - Y_{12}Y_{33}, \quad X_2X_3Z + Y_{12}Y_{13} - Y_{23}Y_{11}, \quad X_1X_3Z + Y_{12}Y_{23} - Y_{13}Y_{22},$$

$$X_2^2Z - Y_{13}^2 + Y_{11}Y_{33}, \quad X_2^2Z - Y_{12}^2 + Y_{11}Y_{22}, \quad X_1^2Z - Y_{23}^2 + Y_{22}Y_{33},$$

$$2Y_{12}Y_{13}Y_{23} - Y_{23}^2Y_{11} - Y_{13}^2Y_{22} - Y_{12}^2Y_{33} + Y_{11}Y_{22}Y_{33}.$$

The degrees of these generators in $\mathcal{CL}(\mathbb{P}\mathrm{Sym}^2\mathcal{T}_2) \cong \mathbb{Z} \times \mathbb{Z}$ are:

$$\deg(X_i) = (-1, 0), \quad \deg(Y_{ij}) = (2, 1), \quad \deg(Z) = (6, 2).$$

In particular, the class $(3, 1) = \frac{1}{2}\deg(Z)$ lies in the pseudo-effective cone but is not effective. This class corresponds to the exceptional divisor of $\mathrm{Hilb}^2\mathbb{P}^2$ when it is viewed as a blow-up of the symmetric power $\mathrm{Sym}^2\mathbb{P}^2$.

3.2.3 Bundles over products of projective space

We now extend to consider the positivity properties of bundles over a product of projective spaces. Similar to in Chapter 2, we let $\bar{n} = (n_1, ..., n_k)$ so $\mathbb{P}^{\bar{n}} = \mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_k}$. We begin by presenting a combinatorial condition for a bundle over $\mathbb{P}^{\bar{n}}$ to satisfy the Fujita conjectures before showing that many of our favorite bundles meet that condition.

Theorem 3.2.12. *Let $\mathcal{E}$ be a toric vector bundle over $\mathbb{P}^{\underline{n}}$ with $\mathcal{R}(\mathbb{P}\mathcal{E})$ generated in* Sym-*degree* 1*, and suppose further that for any $1 \leq j \leq m$ there is a maximal face σ such that y_j is a coloop of $\mathcal{M}(\mathrm{in}_\sigma L)$, then:*

1. *Any Nef class on $\mathbb{P}\mathcal{E}$ is basepoint-free,*
2. *Any ample class on $\mathbb{P}\mathcal{E}$ is very ample,*
3. *$\mathbb{P}\mathcal{E}$ satisfies Fujita's freeness and ampleness conjectures.*

Proof. It suffices to prove 1 and 2. We consider the split toric vector bundle $\mathcal{V}_D = \bigoplus_{j=1}^m \mathcal{O}(D_j)$ determined by the columns of the diagram D. We have a surjection $\mathcal{F}_D \to \mathcal{E}$ and an embedding $\mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{F}_D$, and the Cox ring $\mathcal{R}(\mathbb{P}\mathcal{F}_D)$ presents $\mathcal{R}(\mathbb{P}\mathcal{E})$ via the generators X_i, Y_j. As a consequence, we get inclusions $\mathrm{BPF}(\mathbb{P}\mathcal{V}_D) \subseteq \mathrm{BPF}(\mathbb{P}\mathcal{E})$ as subsets of $\mathcal{CL}(\mathbb{P}\mathcal{V}_D) \cong \mathcal{CL}(\mathbb{P}\mathcal{E})$. Our assumption implies that $\mathrm{BPF}(\mathbb{P}\mathcal{E}) = \bigcap_{j=1}^m (-\mathrm{NEF}(\mathbb{P}^{\underline{n}}), 0) + \mathbb{Z}_{\geq 0}(\mathbf{d}_j, 1) = \mathrm{BPF}(\mathbb{P}\mathcal{V}_D)$. This directly implies 1. Also, it follows that any Nef class or basepoint-free class on $\mathbb{P}\mathcal{E}$ extends to a Nef or basepoint-free class on $\mathbb{P}\mathcal{V}_D$. The space $\mathbb{P}\mathcal{V}_D$ is a smooth toric variety, so any ample class is very ample (in fact projectively normal in this case), so the same must be true for $\mathbb{P}\mathcal{E}$. This proves 2 and 3. $\square$

The conditions of the previous theorem are made easier to establish by the following lemma.

Lemma 3.2.13. *Let $\mathcal{E}$ be toric vector bundle over $X(\Sigma)$ associated to (L, D), and suppose that every circuit of the ideal L has at least one 0 entry in each row of D. Then if $D_j \neq 0$ there is some maximal $\sigma \in \Sigma$ such that y_j is a coloop of $\mathcal{M}(\mathrm{in}_\sigma L)$.*

Proof. We will show that there must be some $\sigma \in \Sigma$ for which y_j is not supported on any linear polynomial in $\mathrm{in}_\sigma(L)$. If this is the case, we can find a standard basis for $\mathrm{in}_\sigma(L)$ containing y_j, and define p by setting $y_j(p) = 1, y_k(p) = 0$ $k \neq j$.

Observe that if y_j is supported on a linear form in $\mathrm{in}_\sigma(L)$ then y_j must be supported on some linear form in every row of D corresponding to the elements of $\sigma(1)$. It follows that if y_j is supported on a linear form in each $\mathrm{in}_\sigma(L)$, then y_j is supported on at least one linear form in the initial ideal of each row of D. Moreover, the circuits of L form a universal Gröbner basis, so we conclude that for each row β of D there is a circuit $C_\beta \in L$ such that y_j is supported on the initial form $\mathrm{in}_\beta(C_\beta)$. But the minimum entry in a row supported on a circuit must be 0. We conclude that $D_j = 0$, which is a contradiction. $\square$

The diagram condition in Lemma 3.2.13 is satisfied for both sparse bundles and uniform bundles, so we obtain the following corollary.

Corollary 3.2.14. *Let $\mathcal{E}$ be a sparse toric vector bundle or a uniform toric vector bundle over $\mathbb{P}^{\underline{n}}$ with $\mathcal{R}(\mathbb{P}\mathcal{E})$ generated in* Sym*-degree* 1*, then:*

1. *Any Nef class on $\mathbb{P}\mathcal{E}$ is basepoint-free,*
2. *Any ample class on $\mathbb{P}\mathcal{E}$ is very ample,*
3. *$\mathbb{P}\mathcal{E}$ satisfies Fujita's freeness and ampleness conjectures.*

Notice that this gives us a way to produce an infinite family of bundles over $\mathbb{P}^{\bar{n}}$ that satisfy the Fujita conjectures, since CI bundles are uniform and generated in Sym-degree 1.

3.3 Example where NEF does not imply basepoint free

We conclude this chapter by examining an example of a toric vector bundle $\mathcal{E}$ whose projectivization $\mathbb{P}\mathcal{E}$ has classes that is nef, but not basepoint free. We consider $Y(\Sigma) = \mathbb{P}^1 \times \mathbb{P}^1$ blown up at 2 points, resulting in the following fan structure.

We let $\mathcal{E}(L, D)$ be the toric vector bundle corresponding to

$$L = \langle y_1 + y_2 + y_3 \rangle \subset \mathbb{C}[y_1, y_2, y_3],$$

$$D = \begin{bmatrix} c_1 & 0 & 0 \\ c_2 & 0 & 0 \\ c_3 & 0 & 0 \\ c_4 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & c_6 \end{bmatrix}$$

Computation of the nef cones in Macaulay2 allow us to test multiple values of $c_1, ..., c_6$. For this example, we consider the case where $c_1 = 3, c_2 = 6, c_3 = 9, c_4 = 2, c_5 = 9, c_6 = 6$; however, these values can be replaced with any nonnegative values and maintain properties of $\mathbb{P}\mathcal{E}$. Toric vector bundles of this form have previously found use as counterexamples to global generation in work of Nødland [25]. The Cox ring $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ is presented as follows:

$$\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6, Y_1, Y_2, Y_3]/\langle x_1^3 x_2^6 x_3^9 x_4^2 Y_1 + x_5^9 Y_2 + x_6^6 Y_3 \rangle.$$

We let the classes of the toric divisors corresponding to the rays of Σ be denoted $e_i \in \mathcal{CL}(Y(\Sigma))$ $1 \leq i \leq 6$. The class group $\mathcal{CL}(Y(\Sigma))$ is freely generated by e_1, e_2, e_3, e_4 with $e_5 = e_1 + e_2 + e_3$ and $e_6 = e_2 + 2e_3 + e_4$. As a consequence, the generators of $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ have the following classes in $\mathcal{CL}(\mathbb{P}\mathcal{E}) \cong \mathcal{CL}(Y(\Sigma)) \times \mathbb{Z}$:

$$\begin{aligned} [x_i] &= (-e_i, 0) \\ [Y_1] &= (3e_1 + 6e_2 + 9e_3 + 2e_4, 1) \\ [Y_2] &= (9e_1 + 9e_2 + 9e_3, 1) \\ [Y_6] &= (6e_2 + 12e_3 + 6e_4, 1) \end{aligned}$$

We label the cones of Σ counter clockwise: $\sigma_1 = \mathbb{Q}_{\geq 0}\{(1,0),(1,1)\}$, $\sigma_2 = \mathbb{Q}_{\geq 0}\{(1,1),(1,2)\}$, $\sigma_3 = \mathbb{Q}_{\geq 0}\{(1,2),(0,1)\}$, $\sigma_4 = \mathbb{Q}_{\geq 0}\{(0,1),(-1,0)\}$, $\sigma_5 = \mathbb{Q}_{\geq 0}\{(-1,0),(0,-1)\}$, and $\sigma_6 = \mathbb{Q}_{\geq 0}\{(0,-1),(1,0)\}$ (See Figure 3.2).

Each initial ideal $\mathrm{in}_{\sigma_i} L$ has two minimal elements which we denote with a $0, 1$ vector indicating the support of the element. For example, over σ_1 the initial ideal is

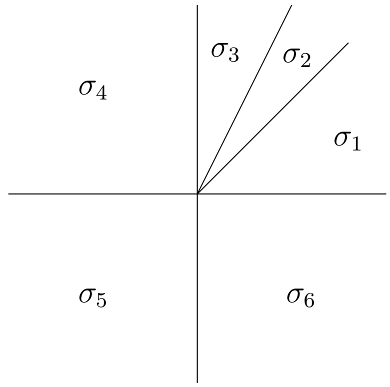

Figure 3.2: The fan of $\mathbb{P}^1 \times \mathbb{P}^1$ with labeled cones

$\mathrm{in}_{\sigma_1} L = \langle y_2 + y_3 \rangle$, which has minimally supported solution types 100 (e.g. $(3,0,0)$ and 011 (e.g. $(0,-5,5)$). There are 12 corresponding monoids:

$$S_1^{100} = \langle (-e_3, 0), (-e_4, 0), (-e_5, 0), (-e_6, 0), (3e_1 + 6e_2 + 9e_3 + 2e_4, 1) \rangle$$

$$S_1^{011} = \langle (-e_3, 0), (-e_4, 0), (-e_5, 0), (-e_6, 0), (9e_1 + 9e_2 + 9e_3, 1), (6e_2 + 12e_3 + 6e_4, 1) \rangle$$

$$S_2^{100} = \langle (-e_1, 0), (-e_4, 0), (-e_5, 0), (-e_6, 0), (3e_1 + 6e_2 + 9e_3 + 2e_4, 1) \rangle$$

$$S_2^{011} = \langle (-e_1, 0), (-e_4, 0), (-e_5, 0), (-e_6, 0), (9e_1 + 9e_2 + 9e_3, 1), (6e_2 + 12e_3 + 6e_4, 1) \rangle$$

$$S_3^{100} = \langle (-e_1, 0), (-e_2, 0), (-e_5, 0), (-e_6, 0), (3e_1 + 6e_2 + 9e_3 + 2e_4, 1) \rangle$$

$$S_3^{011} = \langle (-e_1, 0), (-e_2, 0), (-e_5, 0), (-e_6, 0), (9e_1 + 9e_2 + 9e_3, 1), (6e_2 + 12e_3 + 6e_4, 1) \rangle$$

$$S_4^{100} = \langle (-e_1, 0), (-e_2, 0), (-e_3, 0), (-e_6, 0), (3e_1 + 6e_2 + 9e_3 + 2e_4, 1) \rangle$$

$$S_4^{010} = \langle (-e_1, 0), (-e_2, 0), (-e_3, 0), (-e_6, 0), (9e_1 + 9e_2 + 9e_3, 1) \rangle$$

$$S_5^{010} = \langle (-e_1, 0), (-e_2, 0), (-e_3, 0), (-e_4, 0), (9e_1 + 9e_2 + 9e_3, 1), \rangle$$

$$S_5^{001} = \langle (-e_1, 0), (-e_2, 0), (-e_3, 0), (-e_4, 0), (6e_2 + 12e_3 + 6e_4, 1) \rangle$$

$$S_6^{100} = \langle (-e_2, 0), (-e_3, 0), (-e_4, 0), (-e_5, 0), (3e_1 + 6e_2 + 9e_3 + 2e_4, 1), \rangle$$

$$S_6^{001} = \langle (-e_2, 0), (-e_3, 0), (-e_4, 0), (-e_5, 0), (6e_2 + 12e_3 + 6e_4, 1) \rangle$$

Every monoid except $S_1^{011}, S_2^{011}, S_3^{011}$ is smooth. Intuitively, it is the distance between the rays generated by $[Y_2]$ and $[Y_3]$ which is allowing for integral points in the convex hulls of these monoids to be missed.

We let C_i^{***} be the cone generated by S_i^{***} so that $\mathrm{NEF}(\mathbb{P}\mathcal{E}) = \bigcap C_i^{***}$ and $\mathrm{BPF}(\mathbb{P}\mathcal{E}) = \bigcap S_i^{***}$.

We get the following vectors as the generators of our Hilbert basis:

$$\begin{bmatrix} -2 \\ -2 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 10 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

We the check which, if any, of the basis elements is not able to be written as an integral combination of the ray generators that define each of our unsaturated cones, $C_1^{011}, C_2^{011}, C_3^{011}$. We get that the pairs

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ -1 \\ 0 \end{bmatrix}, C_1^{011} \right\} \quad \left\{ \begin{bmatrix} 0 \\ 5 \\ 10 \\ 0 \\ 2 \end{bmatrix}, C_2^{011} \right\} \quad \left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ -1 \\ 0 \end{bmatrix}, C_3^{011} \right\}$$

have exactly this issue. Therefore, these classes are not basepoint free.

Appendices

Appendix A: KM Algorithm Code

Disclaimer: This code, as it appears, is workable for rank 3 bundles with $L = \langle \sum y_i, \sum iy_i \rangle$. The two places where these pieces of data are noted in comments. Changing the rank is easy. Changing the linear ideal is more complicated, as Macaulay2 is particular about which ring elements live in and it is possible for the user to input their ideal in a way incompatible to the *adaptedBasis* code. It is very much the intention to continue improving this code so that it works as generally as possible.

Crucially, we need to load the gfan package:

```
    loadPackage "gfanInterface"
```

Given two ideals, in(I) and J, *findMissingGen* finds an element of J that is not in in(I) and returns that element.

```
findMissingGen = (K,J) -> (
    newGens0 = {};
    L1 = flatten entries mingens J;
    for i from 0 to #L1-1 do(newGens0 = append(newGens0, L1_i % K));
    newGens = delete(0_S,newGens0);
    return newGens#0;
    )
```

Despite not being the main called function, *khovanskiiBasis* is what runs the KM Algorithm given the row of the diagram D as the weight w.

```
khovanskiiBasis = (w, polyRing, algebraGens, limit, D) -> (
    R = polyRing;
    n = #algebraGens;
    S = QQ[X_1..X_#D, Y_1..Y_(n-#D)];

    B = new MutableList from toList(limit:0);
```

```
for i from 0 to n - 1 do (
     B#i = algebraGens#i;
 );

R' = QQ[y_1..y_(#transpose(D)),t_1..t_#D];

subB = apply((toList B)_(toList((#D)..n-1)), i -> sub(i,R'));

inB = apply(flatten prepend((toList B)_(toList(0..(#D-1))),
    gfanInitialForms(subB, -w)), i -> sub(i, R));

phi = map(R,S, (toList B)_(toList(0..n-1)));
psi = map(R,S, inB);

I = ker phi;
J = ker psi;

u = for b in inB list (
E = matrix{(exponents(b))#0};
    matrix{w}*transpose(E))_(0,0)
    ;

if I == 0 then (
    return toList B;
 ) else (
    K = ideal gfanInitialForms(flatten entries mingens I, -u,
        "ideal" => true);

    while (J != K) do (
    if n > limit then (
    break
    ) else (
    g = sub(findMissingGen(K,J), S);
    B#n = phi(g);
        n = n + 1;
    subB = apply((toList B)_(toList((#D)..n-1)), i -> sub(i,R'));
    inB = apply(flatten prepend((toList B)_(toList(0..(#D-1))),
```

```
            gfanInitialForms(subB, -w)), i -> sub(i, R));
        S = QQ[X_1..X_#D, Y_1..Y_(n-#D)];

        phi = map(R,S, (toList B)_(toList(0..n-1)));
        psi = map(R,S, inB);

        I = ker phi;
        J = ker psi;

        u = for b in inB list (
            E = matrix{(exponents(b))#0};
            (matrix{w}*transpose(E))_(0,0)
            );

     K = ideal gfanInitialForms(flatten entries mingens I, -u,
        "ideal" => true);
         )
    );

    if n > limit then (
    print "limit reached";
    return toList B;
    ) else (
    return toList B;
        )
      )
    )
```

pickFirstL identifies the locations of the nonzero entries of the input vector v, then selects other entries (as many needed for the rank r). This function is called by adaptedBasis to choose the appropriate variables to serve as the adapted basis.

```
pickFirstL = (v) -> (
     ones = delete(null, for i from 0 to #transpose(D) - 1 list if
                v_i != 0 then i + 1);
     howMany = 3 - #ones; --here, 3 = rank. Needs to be changed if rank changes.
     comp = toList ((set toList {1..#transpose(D)}_0) - set ones);
     comp = comp_(toList(0..howMany - 1));
```

```
    return ones|comp
   )
```

pickL serves a similar function to *pickFirstL*. However, it takes into account the adapted basis chosen from the previous iteration and tries to pick as many of the same variables as possible.

```
pickL = (v,w) -> (
onesV = delete(null, for i from 0 to #transpose(D) - 1
            list if v_i != 0 then i + 1);
onesW = ones = delete(null, for i from 0 to #transpose(D) - 1
            list if w_i != 0 then i + 1);
ones = onesV | onesW;
if commonest ones == sort ones then return pickFirstL(v) else
            return unique ones;
)
```

The function *adaptedBasis* identifies the variables that form the adapted basis, then writes the complementary variables in terms of the basis with respect to the relations identified in I.

```
adaptedBasis = (posL,gensL,originalRing) -> (
    agens = new MutableList from toList(y_1..y_(#transpose(D)));
    n = #gensL - #transpose(D);
    wts = new MutableList from toList(#D+#transpose(D):1);
    M = toList ((set toList {1..#transpose(D)}_0) - set posL);
    for i in M do wts#(i-1) = 10^3;
    wts = toList wts;
    S = QQ[y_1..y_(#transpose(D)), t_1..t_#D, Weights => wts];

    A = sum (for i from 1 to #transpose(D) list y_i);
    B = sum (for i from 1 to #transpose(D) list i*y_i);
    I = ideal(A,B); --This says that the ideal L=<A,B>.
                This needs to be changed if the linear ideal changes.
    adB = M / (i -> (y_i % I));
    adB = adB / (f -> sub(f,originalRing));
```

```
    p1 = M_0 -1;
    p2 = M_1 -1;

    agens#p1 = adB_0;
    agens#p2 = adB_1;


    if n > 0 then (for i from #transpose(D) to (#gensL-1) do
        (agens = append(agens,sub(gensL_i, S) % I)));

    agens = toList agens;
    agens = agens / (f -> sub(f,originalRing));

    return agens;
    )
```

QRing is a quick function that forms the appropriate quotient ring based on the number of columns of the diagram D.

```
QRing = (D) -> (
R'=QQ[y_1..y_(#transpose(D)),t_1..t_#D, s_1..s_#D];
Ilist = {};
for i from 1 to #D do Ilist = append(Ilist, t_i*s_i-1);
R = R'/ideal(Ilist);

return R;
)
```

Finally, the star of the show, the lead function: *allRows*. Given your diagram D and a positive integer, this functions runs the KM algorithm on an algebra generating set inductively using the rows of D as the weight vectors. The provided integer is the number of iterations the function will go through *per row* before returning an message that it has exceeded its limit.

```
allRows = (D, limit) -> (
    R = QRing(D);
    zeroList = toList(2*#D: 0);
    oneList= toList(#D: 1);
```

```
    expT_0 = toList(#D+#transpose(D):1);
    kBasis_0 = toList(s_1..s_#D)|toList {y_1..y_(#transpose(D))}_0;

    for k from 0 to (#D-1) do(
    if k == 0 then  (
        almostAlgGens_k = flatten prepend(toList(s_1..s_#D),
            adaptedBasis(pickFirstL(D_k,r),(kBasis_k)_{#D..(#kBasis_k)-1},R));)
    else (
        almostAlgGens_k = flatten prepend(toList(s_1..s_#D),
            adaptedBasis(pickL(D_k, D_(k-1)),(kBasis_k)_{#D..(#kBasis_k)-1},R));););

    algGens = new MutableList from almostAlgGens_k;
    for j from 0 to k do(for l from 0 to (#expT_j)-1 do(
        algGens#l = algGens#l*(expT_j)_l));
    alggens_k = toList algGens;

    w_k=flatten append(D_k, zeroList);
    newBasis_k = delete(0, khovanskiiBasis(w_k,R,alggens_k,limit,D));

    expT_(k+1) = flatten prepend(oneList,apply(D_k, j->t_(k+1)^j));
    extraTs = {};
    for i from 1 to (#newBasis_k - #(expT_(k+1))) do
    (extraTs = append(extraTs,t_(k+1)));
    expT_(k+1) = flatten append((expT_(k+1)), extraTs);
    expT_(k+1) = apply((expT_(k+1)), j -> sub(j, R));

    kBasis_(k+1) = {};

    for j from 0 to #(expT_(k+1))-1 do(
        kBasis_(k+1) = append(kBasis_(k+1), (newBasis_k)_j*((expT_(k+1))_j));
);
return kBasis_(#D);
)
```

Appendix B: Positivity

```
needsPackage "Polyhedra"
```

The function *doesNEF* was written to allow us to run many test cases at once. The input variable n is the number of rays the user wants the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ to have and L is the max integer that an entry of the C-touple may obtain.

```
doesNEF = (n,L) -> (
    cValues = (n:1)..(n:L);
    goodVal = {};
    for i in cValues do if abs(i_-1 - i_-2) > 1 then goodVal = append(goodVal, i);
    resultValues = {};
    for i in goodVal do resultValues = append(resultValues, difN(n,i));
    return resultValues
);
```

difN takes the n value input in *doesNEF* and, one at a time, the C-touple produced. It then computes all the C_p cones and intersects to compute the NEF cone. It then computes a Hilbert Basis and tests if each Hilbert Basis element can be written as an integral combination of the rays generating each of the C_p. If yes, move on. Otherwise, return this C-touple.

```
difN = (n, C) -> (

for i from 1 to #C do c_i = C_(i-1);

rList= {};
for i from 0 to n-3 do rList = append(rList, replace(i, -1, toList(n-1:0)));
rList = append(rList, -append(append(toList(n-3:1),0),0));
rList = append(rList, -append(append(prepend(0,toList(1..n-4)),1),0));


for i from 1 to n-1 do Rays_(i,i+1) = transpose matrix drop(rList, {i-1,i});
Rays_(n,1) = transpose matrix drop(drop(rList, -1), {0,0});

for i from 1 to n-1 do Cones_(i,i+1) = coneFromVData Rays_(i,i+1);
Cones_(n,1) = coneFromVData Rays_(n,1);

type1 = append(for i from 1 to n-2 list c_i,1);
```

```
type2 = append(append(toList(n-3 : c_(n-1)),0),1);
type3 = append(append(prepend(0,for i from 1 to n-4 list i*c_n),c_n),1);

allMat = {};
for i from 1 to n-3 do allMat = append(allMat,
    M_(i,i+1,100) = (transpose matrix {type1}) | rays Cones_(i, i+1));
for i from 1 to n-3 do allMat = append(allMat,
    M_(i,i+1,011) = (transpose matrix {type2, type3}) | rays Cones_(i,i+1));


allMat = append(allMat,
    M_(n-2, n-1, 100) = (transpose matrix {type1}) | rays Cones_(n-2, n-1));
allMat = append(allMat,
    M_(n-2, n-1, 010) = (transpose matrix {type2}) | rays Cones_(n-2, n-1));
allMat = append(allMat,
    M_(n-1, n, 010) = (transpose matrix {type2}) | rays Cones_(n-1, n));
allMat = append(allMat,
    M_(n-1, n, 001) = (transpose matrix {type3}) | rays Cones_(n-1, n));
allMat = append(allMat,
    M_(n,1, 100) = (transpose matrix {type1}) | rays Cones_(n, 1));
allMat = append(allMat,
    M_(n,1, 001) = (transpose matrix {type3}) | rays Cones_(n, 1));

allCones = {};
for i in allMat do allCones = append(allCones, coneFromVData i);

intCone = intersection(allCones_0, allCones_1);
for i from 2 to (#allCones-1) do intCone = intersection(intCone, allCones_i);

HB = hilbertBasis intCone;

nonsatCones = {};
for i from 1 to n-3 do nonsatCones = append(nonsatCones, M_(i,i+1,011));

latPts = {};

for i in HB do (
    for j in nonsatCones do (
```

```
        latPts = append(latPts,
        latticePoints(polyhedronFromHData(-id_(ZZ^(#entries transpose j)),
            transpose matrix{toList(#entries transpose j : 0)}, j, i)))
));

if number(latPts, i -> i=={}) == 0 then "{}" else C

);
```

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