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# Asymptotic behaviour of hyperbolic partial differential equations

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Asymptotic behaviour of hyperbolic partial differential equations

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DISSERTATION

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By

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ABSTRACT OF DISSERTATION

Asymptotic behaviour of hyperbolic partial differential equations

We investigate the asymptotic behaviour of solutions to a range of linear and nonlinear hyperbolic equations on asymptotically flat spacetimes. We develop a comprehensive framework for the analysis of pointwise decay of linear and nonlinear wave equations on asymptotically flat manifolds of three space dimensions that are allowed to be time-varying or nonstationary, including quasilinear wave equations. The Minkowski space and time-varying perturbations thereof are included among these spacetimes. A result on scattering for a nonlinear wave equation with finite-energy solutions on nonstationary spacetimes is presented. This work was motivated in part by the investigation of more precise asymptotic behaviour for dispersive equations.

KEYWORDS: Wave equations, Asymptotics, Scattering, Black holes, General relativity, Null condition

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## Chapter 1 Introduction

This dissertation studies certain asymptotic properties of solutions to nonlinear and linear hyperbolic partial differential equations (PDEs), also known as wave equations, in particular their scattering and pointwise decay properties.

Nonlinear hyperbolic equations manifest in the study of various physical systems. The equations describing the motion of elastic materials are hyperbolic equations. The Einstein equations of general relativity are a system of nonlinear hyperbolic equations. The irrotational compressible Euler equations, which describe the dynamics of a compressible gas, provide another example of a nonlinear hyperbolic equation. This is not an exhaustive list of examples of physical situations described by hyperbolic PDEs.

From a more mathematical perspective, hyperbolic equations can be viewed as belonging to a broader family of partial differential equations called dispersive equations. Dispersive equations have origins in geometry, mathematical physics, and other fields. Famous examples of dispersive equations include Schrödinger equations, wave equations, and the Einstein equations. Dispersive equations with variable coefficients (which can depend on position or time) arise both mathematically and in applications, e.g. in optics.

In the subject of general relativity, black holes are modelled as Lorentzian manifolds that solve the vacuum Einstein equations. By studying the asymptotic, or long-time, behaviour of solutions to both nonlinear and linear wave equations, we contribute a step towards the understanding of the behaviour of solutions to the Einstein equations. The Einstein equations belong to a class of hyperbolic equations with a certain structure that allows for global existence with small initial data. In this thesis, we consider a variety of nonlinear hyperbolic equations, such as those satisfying the classical null condition; this condition is a stronger assumption than the one satisfied by the Einstein equations.[1] Nevertheless, improving our understanding of the behaviour of solutions to nonlinear wave equations such as the null condition has contributed to a better understanding of the properties of solutions to the Einstein equations. The Einstein vacuum equations do not satisfy the classical null condition when written in wave coordinates, but they do satisfy a geometric form of the null condition. At present, nonlinear wave equations continue to serve as a model problem for studying the Einstein equations. There are other physical motivations for studying hyperbolic equations, in addition to other mathematical motivations, but we shall not elaborate further on this here.

A traditional direction of research regarding the long-time behavior of dispersive equations is the study of asymptotics, i.e. pointwise values, of solutions. Another direction is scattering, which means that the asymptotic behavior for the nonlinear dispersive equation as $t \to \pm\infty$ is the same as for the linear equation in some norm. We present results in both directions in this thesis.

Compared to the constant-coefficient case, much less is known about the asymptotic properties of linear variable-coefficient problems. Understanding asymptotic properties

[1]A quadratic nonlinearity satisfies the null condition if a derivative transversal to the light cone is always multiplied by a derivative tangent to the light cone.

for linear variable-coefficient problems helps with obtaining asymptotic properties in nonlinear problems, where even less is known about solutions on large timescales. The work in this thesis improves our understanding of long-time behavior of dispersive equations in generic contexts. This includes nonlinear contexts and variable-coefficient contexts.

## 1.1 Outline of thesis

The chapters in this thesis were written to be as self-contained as possible. We summarise the contents of each chapter:

1. The first chapter of this thesis contains background information describing existing work on asymptotic behaviour for linear and nonlinear hyperbolic equations.

2. The second chapter consists of a collection of notation used throughout this thesis.

3. The third chapter of this thesis consists of an introduction to the notion of integrated local energy decay, or ILED for short. ILED estimates can be viewed as core estimates in the study of hyperbolic equations, because many other important estimates in the subject follow once ILED estimates are established. For instance, sharp pointwise decay estimates, such as those obtained in this paper, follow once ILED is established. Strichartz estimates also follow as a consequence of ILED (see [75]).

4. The fourth chapter contains a discussion of sharp pointwise bounds for linear wave equations on a general class of asymptotically flat, nonstationary backgrounds which may be large perturbations of the Minkowski spacetime; all that is assumed, in terms of ILED, is that a weaker form of the standard ILED estimate holds. This weaker form is known to hold for certain black hole spacetimes.

5. The fifth chapter of this thesis contains a proof of the global existence of the wave equation with the classical null condition on nonstationary spacetimes evolving from small initial data, as well as a proof of sharp pointwise decay estimates for these solutions.

6. The sixth chapter also includes a discussion of a scattering result for this nonlinear wave equation. We consider the energy-critical nonlinearity on nonstationary spacetimes evolving from either large or small initial data. Scattering is yet another form of asymptotic behaviour, which means that the asymptotic behavior for the nonlinear hyperbolic equation as $t \to \pm\infty$ is the same as for the linear equation in some norm, which in this case is the energy, or $\dot{H}^1 \times L^2$, norm.

7. The seventh chapter of this thesis contains a proof of the sharp pointwise decay rate of the wave equation with energy-critical nonlinearity on nonstationary spacetimes evolving from either large or small initial data. The largeness of the initial data introduces several new difficulties which we show how to overcome in this chapter. We also prove results for other integer-power nonlinearities.

We briefly remark on some of the difficulties involved in the study of pointwise asymptotic behaviour of linear and nonlinear hyperbolic equations. If we use $\bar{\partial}$ to denote derivatives that are tangent to the level sets of $u = t - r$, then if $\Gamma$ denotes either a Lorentz boost $x_i\partial_t + t\partial_i$ or a translation vector field, a rotation vector field, or the scaling vector field $\partial_t + r\partial_r$, we have for any smooth function $f$

$$|\partial f| \leq \frac{C}{1 + |t - r|} \sum_{|\alpha|=1} |\Gamma^\alpha f|, \qquad |\bar{\partial} f| \leq \frac{C}{1 + |t + r|} \sum_{|\alpha|=1} |\Gamma^\alpha f|.$$

Thus derivatives tangent to the light cones $u = t - r$ are better-behaved than derivatives which transversal to the light cones. However, the Lorentz boosts generate large errors when we work on physically interesting backgrounds, and as we shall not have access to the Lorentz boost in this thesis, we have to work with worse decay estimates than those displayed above. Moreover, with the Lorentz boosts, tools such as the Klainerman-Sobolev inequality give access to stronger decay estimates. Without the Lorentz boosts, and only with integrated local energy decay, we shall show in this thesis how to obtain both sharp pointwise decay bounds and scattering for various linear and nonlinear wave equations.

# Chapter 2 Notation

In this chapter, we collect notation that will be used throughout this thesis. We fix the spatial dimension to be three, but we note that most of the ideas carry over with trivial modifications to other spatial dimensions. We define the d'Alembertian, or wave operator

$$\Box = -\partial_t^2 + \Delta.$$

## Coordinates

In $(1+3)$-dimensional Minkowski space, we shall sometimes use $x^\alpha$ to denote coordinates. In $n$ spatial dimensions, Greek indices in expressions range from 0 through $n$ (that is, both space and time coordinates), and Latin indices range from 1 through $n$ (that is, only space coordinates). We shall occasionally use Einstein summation convention. We shall occasionally raise and lower the indices with the Minkowski metric.

If $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we let $r := |x| = (\sum_{i=1}^3 (x_i)^2)^{1/2}$.

## Vector fields

We shall often commute with collections of vector fields, often denoted by $Z$, and expressions of the form $Z^J$ will denote an ordered application of these vector fields, where $J$ is a multiindex. The collection $Z$ is comprised of the translation vector fields $\partial_\alpha$, the rotation vector fields $\Omega_{ij} = x_i\partial_j - x_j\partial_i$, and the scaling vector field

$S = t\partial_t + r\partial_r$. We note in particular that we shall not be commuting with the Lorentz boosts in this thesis.

**Definition 2.0.1** (Commuting vector fields and function classes $S^Z$)**.** In $\mathbb{R}^{1+3}$, we consider the three (ordered) sets

$$\partial := (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}), \qquad \Omega := (x_i\partial_j - x_j\partial_i), \qquad S := t\partial_t + \sum_{i=1}^{3} x_i\partial_i,$$

which are, respectively, the generators of translations, rotations and scaling.

We introduce the null coordinates $v = t + r$ and $u = t - r$, giving us the coordinate system $(u, v, \omega)$ where $\omega$ is an element of the two-dimensional round sphere $\mathbb{S}^2$. We shall also use the vector fields $L = \partial_t + \partial_r$ and $\underline{L} = \partial_t - \partial_r$. We note that $L$ is a simple rescaling of the coordinate vector field $\partial_v$, and $\underline{L}$ is a simple rescaling of the coordinate vector field $\partial_u$. More generally, the notion of null frames has been used for studying hyperbolic equations: Given a point $p$, and designating polar coordinates denoted by $(t, r, \omega)$ centred at $p$, we can then form the vector fields $L, \underline{L}$ and $e_A$, where $e_A$ denotes a frame tangent to spheres that are the level sets of $r$ intersected with the level sets of $t$.

We shall in particular use $\bar{\partial}$ to represent $L$ and $e_A$, and $\partial$ to represent an arbitrary derivative. We shall often denote the angular derivatives by either $\partial_\omega$ or $\not\partial$.

### Function spaces and norms

We define the function class

$$S^Z(f)$$

to be the collection of functions $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$|Z^J g(t,x)| \lesssim_J |f|$$

whenever $J$ is a multiindex. We let $\langle r \rangle = \langle x \rangle = (1+|x|^2)^{1/2}$. We will frequently use $f = \langle r \rangle^k$ for some real $k$.

We shall also use the alternate notation

$$\phi_J = Z^J \phi$$

and

$$\phi_{\leq m} := (\phi_J)_{J:|J|\leq m}.$$

Given a norm $\|\cdot\|$, and given any nonnegative integer $N \geq 0$, in this thesis we shall write

$$\|g_{\leq N}\|$$

to denote

$$\sum_{|J|\leq N} \|g_J\|.$$

For instance, taking the absolute value as an example of the norm, the notation $|\phi_{\leq m}(t,x)|$ means

$$|\phi_{\leq m}(t,x)| = \sum_{J:|J|\leq m} |\phi_J(t,x)|.$$

**Notation in estimates**

We write either $X \lesssim Y$ or $X = O(Y)$ to indicate that

$$|X| \leq CY$$

(rather than $X \leq CY$) for some absolute constant $C$ which may vary by line. Similarly, $X \sim Y$ means that there are constants $0 < C_1 < C_2$ so that

$$C_1|X| \leq |Y| \leq C_2|X|.$$

**Notation for dyadic numbers and conical subregions**

We work only with dyadic numbers that are at least 1. We denote dyadic numbers by capital letters for that variable; for instance, dyadic numbers that form the ranges for radial (resp. temporal and distance from the cone $\{|x| = t\}$) variables will be denoted by $R$ (resp. $T$ and $U$); thus

$$R, T, U \geq 1.$$

We choose dyadic integers for $T$ and a power $a$ for $R, U$—thus $R = a^k$ for $k \geq 1$—different from 2 but not much larger than 2, for instance in the interval $(2, 5]$, such that for every $j \in \mathbb{N}$, there exists $j' \in \mathbb{N}$ with

$$a^{j'} = \frac{3}{8} 2^j. \tag{2.0.1}$$

**Dyadic decomposition of spacetime**

We decompose the region $\{r \leq t\}$ based on either distance from the cone $\{r = t\}$ or distance from the origin $\{r = 0\}$. We fix a dyadic number $T$.

$$C_T := \begin{cases} \{(t,x) \in [0,\infty) \times \mathbb{R}^3 : T \leq t \leq 2T, \quad r \leq t\} & T > 1 \\ \{(t,x) \in [0,\infty) \times \mathbb{R}^3 : 0 < t < 2, \quad r \leq t\} & T = 1 \end{cases}$$

$$C_T^R := \begin{cases} C_T \cap \{R < r < 2R\} & R > 1 \\ C_T \cap \{0 < r < 2\} & R = 1 \end{cases}$$

$$C_T^U := \begin{cases} \{(t,x) \in [0,\infty) \times \mathbb{R}^3 : T \leq t \leq 2T\} \cap \{U < |t-r| < 2U\} & U > 1 \\ \{(t,x) \in [0,\infty) \times \mathbb{R}^3 : T \leq t \leq 2T\} \cap \{0 < |t-r| < 2\} & U = 1 \end{cases}$$

If a need arises to distinguish between the $R = 1$ and $U = 1$ cases, we shall write $C_T^{R=1}$ and $C_T^{U=1}$ respectively. Note that $|C_T^R| \sim (R^3 T)^{1/2}$ and $|C_T^U| \sim (T^3 U)^{1/2}$. We define

$$C_T^{<3T/4} := \bigcup_{R < 3T/8} C_T^R.$$

Now letting $R > T$, we define

$$C_R^T := \{(t,x) \in [0,\infty) \times \mathbb{R}^3 : r \geq t, T \leq t \leq 2T, R \leq r \leq 2R, R \leq |r - t| \leq 2R\}$$

Note that $|C_R^T| \sim R^2$, as can be seen in the $|r - t|$ and $r$ directions.

$C_T^R, C_T^U$ and $C_R^T$ are where we shall apply Sobolev embedding, which allows us to obtain pointwise bounds from $L^2$ bounds.

**Distinguished subsets of the forward light cone emanating from the space-time origin**

**Definition 2.0.2.** Let

$$\mathcal{R}_1 := \{\text{dyadic numbers } R : R \geq 1, R < \frac{t-r}{8}\}$$

denote the collection of dyadic numbers we shall occasionally call Region 1, and let

$$\mathcal{R}_2 := \{\text{dyadic numbers } R : R \geq 1, \frac{t-r}{8} \leq R < t+r\}$$

denote the collection we shall occasionally call Region 2.

**Definition 2.0.3.** Let $\mathbb{R}_+ := [0, \infty)$.

- Let $D_{tr}$ denote

  $$D_{tr} := \{(\rho, s) \in \mathbb{R}_+^2 : -(t+r) \leq s - \rho \leq t - r,\ |t-r| \leq s + \rho \leq t + r\}.$$

  When we work with $D_{tr}$ we shall use $(\rho, s)$ as variables, and $D_{tr}^R$ is short for $D_{tr}^{\rho \sim R}$.

- For $R > 1$, let

  $$D_{tr}^R := D_{tr} \cap \{(\rho, s) : R < \rho < 2R\}$$

  and let

  $$D_{tr}^{R=1} := D_{tr} \cap \{(\rho, s) : \rho < 2\}.$$

# Chapter 3 Integrated local energy decay

## 3.1 Energy conservation and integrated local energy decay in Minkowski space

In order to motivate the discussion that follows, we devote this section toward proving the integrated local energy decay estimate for the linear problem $\Box u = 0$ in the setting of Minkowski space. Consider the Cauchy problem

$$\begin{cases} \Box u = 0 & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{3.1.1}$$

The energy of the solution $u$ to (3.1.1) is defined to be

$$E^{lin}(t) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla_{t,x} u(t,x)|^2 \, dx$$

and it is conserved: for all $T \geq 0$, $E^{lin}(T) = E^{lin}(0) = E$.

The solution $u$ to (3.1.1) also satisfies the integrated local energy decay estimate

$$\iint_{[0,T]\times\mathbb{R}^3} \frac{|\nabla_{t,x} u|^2}{\langle r \rangle^{1+\gamma}} + \frac{u^2}{\langle r \rangle^{3+\gamma}} \, dx dt \lesssim E$$

where $\gamma > 0$ is an arbitrarily small fixed constant. Roughly speaking, this estimate is a bound on integrals of the energy density on spacetime cylinders (centred at the origin) for finite-energy solutions.

This estimate is proven by multiplying both sides of the equation by $a(r)u + b(r)\partial_r u + C\partial_t u$ in the region $[0,T] \times \mathbb{R}^3$, with:

$$b(r) = \sum_{j=0}^{\infty} 2^{-j\gamma} \frac{r}{r + 2^j},$$

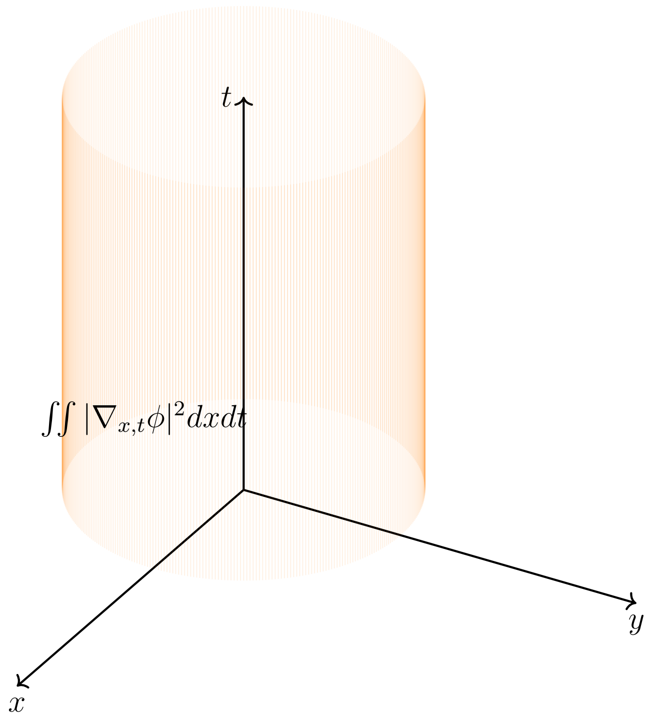


Figure 3.1: Integral of the energy density on a spacetime cylinder centred at the origin.

where for each $j$, this is a function catered to the region $r \approx 2^j$ with the factor $2^{-j\gamma}$, where the small number $\gamma > 0$ is introduced in order to obtain convergence of the series; $a(r) = b(r)/r$; and $C > 0$ is a constant chosen to be sufficiently large.

More precisely, one obtains

$$\iint_{[0,T]\times\mathbb{R}^3} \Box u b(r)\partial_r u \, dxdt = -\iint b'(\partial_r u)^2 + \frac{b}{r}|\partial_\omega u|^2 + \frac{1}{2}(b' + 2\frac{b}{r})(u_t^2 - |\nabla_x u|^2)\, dxdt + \int_{\mathbb{R}^3} -b\partial_r u \partial_t u|_0^T \, dx,$$

where $|\not\partial u|^2 := |\nabla_x u|^2 - |\partial_r u|^2$, and

$$\iint_{[0,T]\times\mathbb{R}^3} \Box u a(r) u \, dxdt = \iint a(u_t^2 - |\nabla_x u|^2) + \frac{1}{2}\Delta a u^2 \, dxdt + \frac{1}{2}\int_{\mathbb{R}^3} -au\partial_t u|_0^T \, dx.$$

Since $a = O(\langle r\rangle^{-1})$ and $b = O(1)$, Hardy's inequality $\int_{\mathbb{R}^3} u^2/r^2\,dx \lesssim \int_{\mathbb{R}^3} |\nabla_x u|^2\,dx$ shows that there exists a sufficiently large constant $C > 0$ such that

$$\int_{\{t\}\times\mathbb{R}^3} b\partial_t u\partial_r u + a\partial_t u u + C|\nabla_{t,x}u|^2\,dx \approx E^{lin}(t)$$

for all $t \geq 0$. We obtain

$$E^{lin}(T) + \iint_{[0,T]\times\mathbb{R}^3} \frac{1}{2}b'(u_r^2 + u_t^2) - (\frac{1}{2}b' - \frac{b}{r})|\not\partial u|^2 - \frac{\Delta a}{2}u^2\,dxdt \lesssim E^{lin}(0)$$

One can check directly that

$$b' \gtrsim \langle r\rangle^{-1-\gamma}, \quad b/r - \frac{1}{2}b' \gtrsim \langle r\rangle^{-1-\gamma}, \quad -\Delta a \gtrsim \langle r\rangle^{-3-\gamma}$$

and thus

$$\frac{1}{2}b'(u_r^2 + u_t^2) + (-\frac{1}{2}b' + \frac{b}{r})|\not\partial u|^2 - \frac{\Delta a}{2}u^2 \gtrsim \frac{|\nabla_{t,x}u|^2}{\langle r\rangle^{1+\gamma}} + \frac{u^2}{\langle r\rangle^{3+\gamma}} \tag{3.1.2}$$

which finishes the proof. □

### 3.2 Prior work on integrated local energy decay (ILED)

The first instance of a local energy estimate was obtained by Morawetz for the Klein-Gordon equation in [79]. Some other work on local energy decay estimates and their applications can be found in, for instance, [1, 45, 47, 68, 75, 103, 107, 110]. For local energy decay estimates for small and time dependent long range perturbations of the Minkowski space-time, see for instance [1], [76], [68] for time dependent perturbations, as well as, e.g., [17], [15], [105] for time independent, nontrapping perturbations. There is a related family of local energy decay estimates for the Schrödinger equation as well.

For Schwarzschild metrics, trapping at the event horizon was shown to be trivial due to an effect guaranteeing energy decay along the trapped rays called the redshift effect. On the other hand, for Kerr metrics, a local energy estimate with derivative loss on the trapped set is often introduced. Weak ILED includes this loss.

For large perturbations of the Minkowski metric, if one assumes the absence of trapping then local energy estimates can still hold; see for instance [15, 77]. For weak enough trapping, Weak ILED has been established; see for instance [16, 20, 85, 122]. If one assumes absence of trapping, then ILED holds; with trapping ILED cannot hold, see [94, 96]. With sufficiently strong trapping, even Weak ILED fails, see [26].

Weak ILED for the Schwarzschild metric was established in [17, 22, 67]. For the Kerr metric with low angular momenta, Weak ILED was proved in [17, 22, 25]. The ILED estimate for Kerr spacetimes with small angular momenta was proven in [119] (see also [6] and [21] for related work), for large angular momentum $|a| < M$ in [25], and for extremal Kerr $|a| = M$ in [7].

We now remark on how ILED relates to two types of asymptotic behaviour, namely pointwise decay rates and scattering. ILED in a compact region on an asymptotically flat region implies pointwise decay rates that are related to how rapidly the metric coefficients decay to the Minkowski metric; see, for example, the works [3, 4, 60, 74, 81, 82, 86, 118]. ILED is also involved in proving scattering (another type of asymptotic behaviour) on variable-coefficient backgrounds. In particular, they imply Strichartz estimates on certain variable-coefficient backgrounds, see [75]. [61] used local energy decay to prove scattering for the version of the defocusing problem considered in this chapter but with only perturbations to the metric (thus $P$ in (7.1.1) included only the $g^{\alpha\beta}$ terms), but the argument extends easily to the version of the problem that includes the lower-order terms and angular terms defined in (7.1.1).

## 3.3 Formal statements of integrated local energy decay, and two weaker forms of integrated local energy decay

We will use the following norms throughout this thesis. In $(1+3)$-dimensions, we define

$$A_R := \{x \in \mathbb{R}^3 : R < |x| < 2R\} \;\; (R \geq 2), \qquad A_{R=1} := \{|x| < 2\}.$$

Given a subinterval $I$ of $[0, \infty)$,

$$\begin{aligned} \|\phi\|_{LE(I)} &:= \sup_R \|\langle r\rangle^{-\frac{1}{2}}\phi\|_{L^2(I\times A_R)}, \\ \|\phi\|_{LE^1(I)} &:= \|\nabla_{t,x}\phi\|_{LE(I)} + \|\langle r\rangle^{-1}\phi\|_{LE(I)}, \\ \|f\|_{LE^*(I)} &:= \sum_R \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(I\times A_R)}. \end{aligned} \tag{3.3.1}$$

We also define

$$\begin{aligned} \|\phi\|_{LE^{1,k}(I)} &= \sum_{|\alpha|\leq k} \|\partial^\alpha \phi\|_{LE^1(I)} \\ \|\phi\|_{LE^{0,k}(I)} &= \sum_{|\alpha|\leq k} \|\partial^\alpha \phi\|_{LE(I)}, \\ \|f\|_{LE^{*,k}(I)} &= \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{LE^*(I)}. \end{aligned}$$

For any norm, an omission of $I$ will denote $I := [0, \infty)$.

**Definition 3.3.1.** We say that the solution to (7.1.2) satisfies the (integrated) local energy decay estimate if the following estimate holds in $[0, \infty)\times\mathbb{R}^3$:

$$\|\phi\|_{LE^{1,k}} \lesssim_k \|\nabla_{t,x}\phi(0)\|_{H^k} + \|f\|_{LE^{*,k}}, \qquad k \geq 0 \tag{3.3.2}$$

Let $\chi(x)$ be a compactly supported and smooth function equalling 1 in a neighbourhood of the trapped set. We define a weaker version each of the $LE^1$ norm that excises the trapped set region when evaluating $\nabla_{t,x}\phi$ in $LE$ norm. We also define the attendant dual weak norm.

$$\|\phi\|_{LE^1_w(I)} := \|(1-\chi)\nabla_{t,x}\phi\|_{LE(I)} + \|\langle r\rangle^{-1}\phi\|_{LE(I)}, \quad \|\phi\|_{LE^{1,k}_w(I)} := \sum_{|\alpha|\le k} \|\partial^\alpha \phi\|_{LE^1_w(I)}$$

$$\|f\|_{LE^*_w(I)} := \|f\|_{LE^*(I)} + \|\chi\nabla_{t,x} f\|_{L^2(I)L^2}, \qquad \|f\|_{LE^{*,k}_w(I)} := \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{LE^*_w(I)}$$

We assume that (7.1.2) satisfies the following weak version of local energy decay Definition 3.3.2, as expressed by the following bounds:

**Definition 3.3.2.** We say (7.1.2) satisfies the weak (integrated) local energy decay estimate if for any real $T_0 \ge 0$ and any integer $k \ge 0$

$$\|\phi\|_{LE^{1,k}_w[T_0,\infty)} \lesssim_k \|\nabla_{t,x}\phi(T_0)\|_{H^k} + \|f\|_{LE^{*,k}_w[T_0,\infty)}. \tag{3.3.3}$$

*Remark* 3.3.3 (Loss of two derivatives in the inhomogeneity)*.* Combining the $k$ and $k+1$ cases of (3.3.3) implies

$$\|\phi\|_{LE^{1,k}[T_0,\infty)} \lesssim_k \|\nabla_{t,x}\phi(T_0)\|_{H^{k+1}} + \|f\|_{LE^{*,k+2}[T_0,\infty)}. \tag{3.3.4}$$

Notice that the right-hand side must have $k+2$ derivatives falling on $f$, since the weak dual norm loses one derivative (at least on $\operatorname{supp}\chi$), and we have applied the $k+1$ case.

*Remark* 3.3.4 (Instances in which weak local energy decay holds)*.* Weak local energy decay is known to hold in the Schwarzschild space-time and the Kerr space-time with small $0 \le |a| \ll M$, where the parameter $M$ denotes the mass of the black hole and the parameter $a$ denotes the angular momentum per unit mass (thus $aM$ denotes the angular momentum of the black hole); more can be found in Section 3.2.

Examples where the assumptions we make on $\phi$ in this chapter are actually satisfied include the following situations:

- the case with small (meaning $O(\epsilon)$ in all compact regions for the function and all its derivatives) and asymptotically flat perturbations $h \in S^Z(\epsilon\langle r\rangle^{-1-\sigma})$ and a small potential $V \in S^Z(\epsilon\langle r\rangle^{-2-\delta})$ for arbitrary real numbers $\delta, \sigma > 0$. See Chapter 7.

- The situation analyzed in [77], which proves local energy decay estimates for solutions to scalar wave equations on nontrapping, asymptotically flat space-times (in particular large perturbations of Minkowski space-time).

**Definition 3.3.5.** The problem (6.1.1) would be said to satisfy stationary (integrated) local energy decay estimates (for derivatives) if for any interval $[T_1, T_2]$ and any integer $k \geq 0$, we have

$$\|\phi\|_{LE^{1,k}[T_1,T_2]} \lesssim_k \sum_{j=1}^{2} \|\nabla_{t,x}\phi(T_j)\|_{H^k} + \|f\|_{LE^{*,k}[T_1,T_2]} + \|\partial_t\phi\|_{LE^{0,k}[T_1,T_2]}. \tag{3.3.5}$$

Both Stationary ILED and Weak ILED are weaker forms of ILED.

# Chapter 4 Pointwise decay for the wave equation on nonstationary spacetimes

## 4.1 Introduction

In this chapter, we examine pointwise decay for linear wave equations on asymptotically flat, nonstationary and stationary backgrounds in $1+3$ dimensions and show how, given certain weak forms of (integrated) local energy decay estimates, the decay rate of the solution depends on the relative rates of the radial decay of the potential, the first-order coefficients and the background geometry. See Theorem 4.1.1 for a simple statement of a special case of the main theorem (Theorem 7.1.1).

We stress that we do not assume integrated local energy decay, but rather weak forms of it; see Definitions 3.3.2 and 3.3.5 for the precise estimates, which accommodate the presence of trapping, and are relevant for black hole spacetimes. The latter can be thought of as an elliptic-type estimate at zero frequency. In particular, the spacetimes we consider here need not be small perturbations of the Minkowski spacetime; with respect to integrated local energy decay, we shall assume only Weak ILED, as defined in Chapter 3.

Let

$$P := \partial_\alpha g^{\alpha\beta}(t,x)\partial_\beta + g^\omega(t,x)\Delta_\omega + \partial_\alpha A^\alpha(t,x) + B^\alpha(t,x)\partial_\alpha + V(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^3 \tag{4.1.1}$$

where the conditions on the potential $V$, the coefficients $A, B, g^\omega$ and the Lorentzian metric $g$ are given Theorem 7.1.1. $\Delta_\omega$ denotes the Laplace operator on the unit sphere.We let $\alpha, \beta$ range across $0, \ldots, 3$ and use the summation convention. Thus $P$ allows for dynamic zeroth- and first-order terms, in addition to dynamic perturbations of the Minkowski metric. We assume asymptotic flatness, thus $\partial_\alpha g^{\alpha\beta}\partial_\beta$ approaches $\Box$ asymptotically as $|x| \to \infty$, while all other terms in $P$ approach zero asymptotically. We assume that the coefficients, such as the potential $V(t,x)$, are sufficiently smooth and obey certain pointwise upper bounds that involve decay as $|x| \to \infty$, but nothing more. For instance, we make no assumptions on the sign of $V$.

While the results in this article were inspired by developments within the subject of general relativity, many of the results in the present article treat compact spatial regions as black boxes, and hence our results apply in more general settings beyond the setting of general relativity.

We consider the linear Cauchy problem

$$P\phi = f, \quad (\phi(0), \bar{N}\phi(0)) = (\phi_0, \phi_1) \tag{4.1.2}$$

where $\bar{N}$ denotes the unit normal derivative to the hypersurface $\{t = 0\}$. We describe our main result in a simple setting:

**Theorem 4.1.1** (Simple and rough statement of main theorem)**.** *Let $\phi$ denote the solution of equation (7.1.2). Assume that the coefficients $B^\alpha, g^{\alpha\beta}$ and $V$ of $P$ obey the following bounds: let $m^{\alpha\beta}$ be the Minkowski metric, let $r = |x|$ and let*

$$|Z^J(g^{\alpha\beta}(t,x) - m^{\alpha\beta})| \leq C(1+r)^{-1-a_1},$$

$$|Z^J V(t,x)| \leq C(1+r)^{-2-a_2},$$

$$|Z^J B^\alpha(t,x)| \leq C(1+r)^{-1-a_3}$$

*for arbitrary real number* $a_1, a_2, a_3 > 0$ *and for all multi-indices* $J$*. Assume that the other coefficients* $A^\alpha$ *and* $g^\omega$ *are equal to zero for simplicity. Assume that the weak and stationary integrated local energy decay bounds (see below) hold. Then*

$$|Z^J \phi(t,x)| \leq C(1+t+r)^{-1}(1+|t-r|)^{-1-a}, \quad a := \min(a_1, a_2, a_3)$$

*for all multi-indices* $J$*. Fix a compact subset* $K \subset \mathbb{R}^3$*; then the above implies that*

$$|Z^J \phi(t,x)| \leq C(1+t)^{-2-a}, \quad x \in K.$$

As Theorem 4.1.1 demonstrates, once the integrated local energy decay bounds are known to hold, then the pointwise decay bounds of the coefficients of the wave operator at spatial infinity are what dictate the pointwise decay bounds of the solution everywhere in the spacetime, even within compact spatial regions. This is stated fully rigorously in Theorem 7.1.1.

**Definition 4.1.2.** We define

$$S^Z_{\text{cone}}(f)$$

to be the collection of $g$ such that $|Z^J g| \lesssim |f|$ in $\{t/2 \leq r \leq 3t/2\}$. Thus $S^Z(f) \subsetneq S^Z_{\text{cone}}(f)$. We define

$$S^Z_{\text{int}}(f)$$

to be the collection of $g$ such that $|Z^J g| \lesssim |f|$ in $\{r < t/2\}$. We define

$$S^Z_{\text{radial}}(f) := \{g \in S^Z(f) : g \text{ is spherically symmetric}\}.$$

Let $\|\cdot\|$ be any norm used in this chapter. Given any nonnegative integer $N \geq 0$, we write $\|g_{\leq N}\|$ to denote $\sum_{|J|\leq N} \|g_J\|$.

Our main result, Theorem 7.1.1, is a pointwise decay estimate for the solution to the following equation:

$$\begin{cases} P\phi(t,x) = 0 \qquad (t,x) \in (0,\infty) \times \mathbb{R}^3,\ P \text{ given in } (7.1.1) \\ (\phi(0,x), \bar{N}\phi(0,x)) = (\phi_0(x), \phi_1(x)) \end{cases} \tag{4.1.3}$$

where $g$ is a Lorentzian metric and $g^{\omega}, A, B, V$ are functions satisfying the following conditions in Theorem 7.1.1. Recall that $h \in S^Z(\langle r\rangle^a)$ means that $|Z^J h(t,x)| \leq C_J \langle r\rangle^a$, where $|J| \geq 0$.

### 4.1.1 Statement of the main theorems

**Theorem 4.1.3.** *Let $m \geq 0$ be an integer and let $N$ be a sufficiently large integer relative to $m$, or $N \gg m$. Let $g^{\alpha\beta}(t,x)$ be a Lorentzian metric such that for all $t_0 \geq 0$ the level sets $\{t = t_0\}$ are space-like, and let $h := g - m$ with $m$ denoting the Minkowski metric. Assume that $\phi$ solving (6.1.1) satisfies the weak and stationary local energy decay (3.3.3) and (3.3.5), and that $\phi_0 \in L^2(\mathbb{R}^3)$.*

1. *Suppose that for some real $0 < \sigma, \delta, \delta' < \infty$,*

$$h \in S^Z(\langle r\rangle^{-1-\sigma})$$

$$A \in S^Z(\langle r\rangle^{-1-\sigma})$$

$$\partial_t A \in S^Z\left(\langle t+r\rangle\langle t-r\rangle^{-1}\langle r\rangle^{-1}\langle r\rangle^{-1-\sigma}\right) \cap S^Z_{cone}(\langle r\rangle^{-1-\sigma})$$

$$\partial A \in S^Z_{int}(\langle r\rangle^{-2-}) \cap S^Z_{cone}(\langle r\rangle^{-2-})$$

$$B \in S^Z(\langle r\rangle^{-1-\sigma})$$

$$\partial_t B \in S^Z(\langle r \rangle^{-2-\sigma})$$

$$V \in S^Z(\langle r \rangle^{-2-\delta})$$

$$g^\omega \in S^Z_{radial}(\langle r \rangle^{-2-\delta'})$$

*Then if* $\kappa := \min(\sigma, \delta, \delta')$, *we have*

$$|\phi(t,x)| \lesssim \langle t+r \rangle^{-1} \langle t-r \rangle^{-1-\kappa} C_0, \qquad r = |x|$$

*where* $C_0$ *is a constant depending on the initial data, which lies in a weighted Sobolev space, with the weight depending on* $\kappa$. *Moreover, if* $m \in \mathbb{N}_0$, *then the vector fields* $\phi_{\leq m}$ *of the solution* $\phi$ *also obey the same decay rates:*

$$|\phi_{\leq m}(t,x)| \lesssim \langle t+r \rangle^{-1} \langle t-r \rangle^{-1-\kappa} \tag{4.1.4}$$

*In addition, we have improved bounds for the derivatives and even better bounds for the time derivative:*

$$|\partial \phi_{\leq m}(t,x)| \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{2+\kappa}} \tag{4.1.5}$$

$$|\partial_t \phi_{\leq m}(t,x)| \lesssim \frac{1}{\langle t+r \rangle \langle t-r \rangle^{2+\kappa}} \tag{4.1.6}$$

*More generally, for every* $k \in \mathbb{N}_0$, $\partial^k \phi_{\leq m}$ *has an upper bound with* $[\langle t+r \rangle / (\langle r \rangle \langle t-r \rangle)]^k$ *better decay than* $\phi_{\leq m}$ *and* $\partial_t^k \phi_{\leq m}$ *has an upper bound with* $\langle t-r \rangle^{-k}$ *better decay than* $\phi_{\leq m}$, *with the implicit constant on the right-hand side depending on* $k$.

**Theorem 4.1.4.** *Suppose that in addition to the assumptions in Theorem 7.1.1, the coefficients* $h, A$ *and* $B$ *satisfy:*

$$\partial_t h \in S^Z\left(\langle r \rangle^{-2-\sigma}\right)$$

$$\partial_t^2 h \in S^Z_{cone}\left(\langle t-r\rangle^{-2}\langle r\rangle^{-1-\sigma}\right)$$

$$A \in S^Z(\langle r\rangle^{-2-\sigma})$$

$$\partial_t A \in S^Z\left(\langle t+r\rangle\langle r\rangle^{-1}\langle t-r\rangle^{-1}\langle r\rangle^{-2-\sigma}\right)$$

$$B \in S^Z(\langle r\rangle^{-2-\sigma})$$

$$\partial_t B \in S^Z(\langle r\rangle^{-3-\sigma})$$

*The assumptions on $h$ here are satisfied if, for example, $h = h(x)$ is time-independent.*

*Then the solution to (6.1.1) satisfies the following improved decay rates in the $\sigma$ category: for $\kappa' := \min(1+\sigma, \delta, \delta')$, we have*

$$|\phi_{\leq m}(t,x)| \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle^{1+\kappa'}} \tag{4.1.7}$$

$$|\partial\phi_{\leq m}(t,x)| \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^{2+\kappa'}} \tag{4.1.8}$$

$$|\partial_t\phi_{\leq m}(t,x)| \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle^{2+\kappa'}}. \tag{4.1.9}$$

*More generally, for every $k \in \mathbb{N}_0$, $\partial^k\phi_{\leq m}$ has an upper bound with $[\langle t+r\rangle/(\langle r\rangle\langle t-r\rangle)]^k$ better decay than $\phi_{\leq m}$ and $\partial_t^k\phi_{\leq m}$ has an upper bound with $\langle t-r\rangle^{-k}$ better decay than $\phi_{\leq m}$, with the implicit constant on the right-hand side depending on $k$.*

*Remark* 4.1.5. All the arguments in this chapter can be adapted to the exterior of a ball and hence the proofs in this chapter can be applied in the case of black hole spacetimes.

*Remark* 4.1.6. We give a novel argument in Proposition 4.9.2 to prove that solutions of equations of the form $\partial_\alpha(g^{\alpha\beta}\partial_\beta\phi) = 0$ decay at the rate $O(\langle v\rangle^{-1}\langle u\rangle^{-2-\sigma})$—thus $t^{-3-\sigma}$ in compact spatial regions—provided the assumptions on the time derivatives in Theorem 4.1.4 are satisfied. This involves taking advantage of the fact that the equation has two derivatives, and intuitively each derivative provides one extra order of $\langle u\rangle^{-1}$ decay, which accounts for the better final decay rate. This is the subject of the third paragraph of the abstract.

*Remark* 4.1.7. We make some remarks supplementing the main theorem.

- Second-order angular operators that have spherically symmetric coefficients of the form $\frac{M}{r}$, thus $\frac{M}{r}\not\partial^2$, can be written as $\frac{M}{r^3}$ potential terms away from the origin. These operators are included by the definition of our operator $P$ in (7.1.1). Indeed, (7.1.1) includes coefficients that have the following form away from the origin: $1/r^a$, $a \in \mathbb{R}_{>0}$. This appears in some equations of physical interest.

- The argument shown in this chapter straightforwardly yields a proof of a more general version of Theorem 7.1.1 which assumes more general decay rates on $A$ and $B$. Namely, given any real $\sigma', \sigma'' > 0$, for part (1) of Theorem 7.1.1 (and similarly for part (2)), if

$$A \in S^Z(\langle r\rangle^{-1-\sigma'})$$

$$\partial_t A \in S^Z\left(\langle v\rangle\langle u\rangle^{-1}\langle r\rangle^{-1}\langle r\rangle^{-1-\sigma'}\right) \cap S^Z_{\text{cone}}(\langle r\rangle^{-1-\sigma'})$$

$$B \in S^Z(\langle r\rangle^{-1-\sigma''})$$

$$\partial_t B \in S^Z(\langle r\rangle^{-2-\sigma''})$$

  (in addition to the assumptions on $h, g^\omega$ and $V$ in part (1), as well as the assumption on the generic derivative $\partial A$) then the same arguments in this chapter automatically give, for instance, with part (1) assumptions,

$$|\phi_{\leq m}(t,x)| \lesssim \frac{1}{\langle v\rangle\langle u\rangle^{1+\min(\sigma,\delta,\delta',\sigma',\sigma'')}}C_0,$$

and the corresponding bounds also hold for $\partial_t\phi_{\leq m}, \partial\phi_{\leq m}$, and so on.

For simplicity of presentation, in this chapter we restrict to the case $\sigma = \sigma' = \sigma''$.

- In item (2) of Theorem 7.1.1, one class of examples of metrics $g^{\alpha\beta}$ satisfying the conditions given are the stationary metrics $g$, that is, those with stationary component

  $$h = h(x).$$

  By substituting the natural number values $\delta \geq 1, \delta \in \mathbb{N}$ and $\sigma \geq 2, \sigma \in \mathbb{N}$, this special case of item (2) of Theorem 7.1.1 recovers a similar result as the main theorem in [80].

- If $h^{\alpha\beta} \in S^Z(\langle r\rangle^{-q})$ for some $q > 0$, then $\sqrt{|g|}h^{\alpha\beta} \in S^Z(\langle r\rangle^{-q})$. This is a consequence of the product rule and the assumption that $-q < 0$. Thus Theorem 7.1.1 also holds if $\partial_\alpha g^{\alpha\beta}\partial_\beta$ is replaced by the geometric wave operator

  $$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha\sqrt{|g|}g^{\alpha\beta}\partial_\beta, \quad |g| := |\det g^{\alpha\beta}|.$$

- An implication of the main theorem is the following: if a local energy decay estimate even with derivative loss is assumed (see (3.3.4)), then one can obtain the pointwise bounds in Theorem 7.1.1.

- For a first reading, since $S^Z\left(\langle v\rangle\langle u\rangle^{-1}\langle r\rangle^{-1}\langle r\rangle^{-1-\sigma}\right)\cap S^Z_{\text{cone}}(\langle r\rangle^{-1-\sigma}) \subset S^Z(\langle r\rangle^{-2-\sigma})$, the reader may wish to keep in mind that $\partial_t A \in S^Z(\langle r\rangle^{-2-\sigma})$ for part 1 of Theorem 7.1.1.

### 4.1.2 Prior literature on pointwise decay

In the 1970s, Price [91,93] conjectured certain decay rates corresponding to a subclass of the $\sigma = \delta = \delta' = 1$ spacetimes above; see also Price–Burko [92]. In [35], a sharp version of Price's law is proved for a class $\mathcal{C}$ of stationary asymptotically flat spacetimes. After performing a change of coordinates (see [118, Section 2.1] for an exposition of this), the class $\mathcal{C}$ can be viewed as a strict subset of the class of (possibly time-dependent) spacetimes we consider in this article, as defined in (7.1.1). The assumptions in [35] correspond to a subclass of the special case $\sigma = \delta = \delta' = 1$ of our chapter, as already mentioned:

$$g^{\alpha\beta} - m^{\alpha\beta} \in S^Z(\langle r\rangle^{-2}),\ V \in S^Z(\langle r\rangle^{-3}), g^\omega \in S^Z_{radial}(\langle r\rangle^{-3}), \quad B^\alpha \equiv 0, A^\alpha \equiv 0.$$

Let $h^{\alpha\beta} := g^{\alpha\beta} - m^{\alpha\beta}$. If $h^{\alpha\beta} = h^{\alpha\beta}(x)$, then the hypotheses on $\partial_t h, \partial_t^2 h$ in Theorem 4.1.4 are satisfied. Taking $\sigma = \delta = \delta' = 1$, we obtain

$$|\phi(t,x)| \lesssim \langle v\rangle^{-1}\langle u\rangle^{-[1+\min(1+1,1,1)]} = \langle v\rangle^{-1}\langle u\rangle^{-2}, \qquad u := t-r,\ v := t+r.$$

Thus, taking Hintz's result into account, Theorem 4.1.4 clarifies that under the hypothesis of stationarity of $h$, the contribution of $h^{\alpha\beta}$ decays at the faster rate $O(\langle v\rangle^{-1}\langle u\rangle^{-3})$, and hence, generically, it does not determine the final Price's law decay rate. The precise meaning of "generic" in this context can be found in [35].

The theorems here can handle all rates of decay that are past a certain threshold of decay for the coefficients, namely $\sigma, \delta, \delta' \in \mathbb{R}_{>0}$, unlike [74] which only treats the aforementioned special case. Due to the small values of $\sigma, \delta, \delta'$ we are not able to use the combination of the fundamental solution of the wave equation and local energy decay directly, and so we instead prove a Sobolev embedding-type lemma giving

$$|\phi_{\leq m}| \lesssim \frac{\langle t-r\rangle^{1/2}}{\langle t+r\rangle}$$

decay. In contrast, due to the fast decay rates of their coefficients, the authors of [74] were able to initiate their iteration with the relatively fast decay rate of

$$|\phi_{\leq m}| \lesssim \frac{\log\langle t-r\rangle}{\langle r\rangle\langle t-r\rangle^{1/2}}.$$

Due to this slower starting decay rate, we use only local energy decay to begin the pointwise decay iteration, and we have to additionally prove an auxiliary result (because the embedding alone gives weak decay near the cone): a Hardy-type inequality in (7.3.6) with a weight that is localised to the cone $\{r = t\}$. Our commutations in Section 4.2 of the more general wave operator in this article and vector fields are provided in more detail than in previous work. The core lemma in our iteration scheme, Lemma 7.4.4, is a new result allowing for inputs with rather general decay rates. This result was not present in previous work, although special cases of it were used in [74], but only implicitly.

The article of Morgan and Wunsch [81], which appeared at the same time as the article corresponding to the present chapter of this thesis, assumes weak local energy decay as well as certain uniform energy bounds, stationary potential $V$ and metric coefficients, and considers the special case $(\Box_g + V)\phi = 0$ of the equation $P\phi = 0$ we treat here. Moreover, they consider potentials that decay at least as fast as $r^{-4-}$, corresponding to the case $\delta > 1$, rather than the $\delta > 0$ that we handle here; and we consider arbitrary positive $\sigma, \delta, \delta'$ while they assume non-integer values. They assume, roughly speaking, that $|V| \leq Cr^{-3-\eta}$ and $|h| \leq Cr^{-1-\eta}$, as well as similar bounds for derivatives with higher decay, with $\eta \in (1,\infty) \setminus \mathbb{N}$ and prove $|\phi(t,x)| \leq Ct^{-3-\eta}t^{\epsilon}$ for compact sets in $x$, for any $\epsilon > 0$. Their assumptions, in the stationary coefficient case, correspond approximately to our assumptions $h \in S^Z(\langle r\rangle^{-1-\sigma}), V \in S^Z(\langle r\rangle^{-2-\delta})$ with $\delta = 1+\eta, \sigma = \eta$. In contrast to the work [81], which assumed that the operator is self-adjoint and did not analyze first-order terms, the present work includes first-order terms in divergence form ($A^\alpha$) and non-divergence form ($B^\alpha$). The difference between these two with respect to $Z$-vector field decay is clarified and weak decay assumptions on them are made.

We prove results showing how to handle pointwise decay and complete the pointwise iteration in the exterior region $\{(t,x) : r > 3t/2\}$. In contrast, other work such as [74, 81] consider only bounds in the forward light cone, or for $x$ in compact regions. The work [74] considered only compactly supported initial data, but for the more general initial data we consider, we reach the optimal pointwise decay rate stated in the main theorem in this exterior region. As part of this work, we prove new Klainerman-Sobolev embeddings in this exterior region: see Lemma 5.2.2. In a similar vein, we prove in Section 4.4 new $L^2$ estimates for derivatives in the exterior region. The pointwise decay iteration in $\{r > 3t/2\}$ presents certain difficulties for slowly-decaying coefficients, which we explain how to overcome in Section 4.6.

### 4.1.3 Pointwise decay and asymptotic behaviour

It is well-understood that local energy decay in a compact region on an asymptotically flat region implies pointwise decay rates that are related to how rapidly the metric coefficients decay to the Minkowski metric; see, for example, the works [3, 4, 35, 60, 63, 64, 74, 81, 82, 86].

Local energy decay is also involved in proving scattering, another type of asymptotic behaviour, on variable-coefficient backgrounds. In particular, they imply Strichartz estimates on certain variable-coefficient backgrounds, see [75]. The article [61] used local energy decay to prove scattering for the version of the problem (6.1.1) without the potential $V$ and first-order terms $A$ and $B$, although the argument extends straightforwardly to the problem including $V, A$ and $B$ defined above.

In the case of the Schwarzschild metric, Price [93] conjectured that the solution to the wave equation decays at the rate $t^{-3}$ within any compact region; this rate was shown to hold for a variety of spacetimes, including Schwarzschild and Kerr spacetimes with small angular momenta—see [27, 74, 118].

### 4.1.4 The main ideas of the proof

Aside from the standard tools of Sobolev embedding, albeit exploited primarily in dyadic conical subregions, when proving pointwise bounds we take advantage of the reduction to $1+1$ dimensions in spherical symmetry—called the "one-dimensional reduction"—and the positivity of the fundamental solution to the $1+3$ dimensional wave equation. This not only provides a simple setting for the analysis but also allows us to "absorb" pointwise decay from the vector fields of the coefficients $h, V$, and so forth, and transfer them to the decay of the solution $\phi$ or its vector fields. In this way, gradual improvements, starting from an initial decay estimate (7.3.7)—obtained from only Sobolev embedding and local energy decay—are possible, with the improvements arising from the positivity of $\sigma, \delta$ and $\delta'$.

A little more precisely, for components of the wave equation that contain a divergence structure, we analyze them separately in a neighbourhood of the light cone $\{r = t\}$ (see Section 4.9), and in all other regions. However, for components of the wave equation that do not contain a divergence structure, we need not make this distinction.

See Remark 4.5.4 for a simple case of the lemma that is used to convert decay rates of the inhomogeneity (which includes the aforementioned decay rates of the potential, first order terms and background geometry) into decay rates for the solution; this lemma is the core of the iteration scheme that we use to obtain the final pointwise decay rates stated in the main theorem.

### 4.1.5 Summary of sections

In Section 4.2, we commute $P$ with vector fields and prove (weak) local energy estimates for vector fields. In Section 4.3, we prove Sobolev embedding estimates and obtain an initial pointwise decay estimate. We connect pointwise bounds to $L^2$ estimates and norms, thereby connecting local energy decay to pointwise bounds. In Section 4.4, we prove that derivatives of vector fields of the solution decay better at the cost of applying more vector fields.

In Section 4.5, we define more notation that will be used for the pointwise decay iteration, which occupies the remainder of the chapter. We also prove certain lemmas used in the iteration. In Section 4.6, we prove the upper bound in $\{r > t + 1\}$ for components of the solution away from the cone. In Section 4.7, we show how to convert a decay rate of $\langle r\rangle^{-p}$ for the solution $\phi$ and its vector fields to $\langle t + r\rangle^{-p}$ for

$p \leq 1$. In Section 4.8, we prove the upper bound in $\{r < t\}$ for components of the solution away from the cone. In Section 4.9, we prove the upper bound for components of the solution near the cone.

**Enlargements of sets**

Given any subset of these conical regions, a tilde atop the symbol $C$ will denote a slight enlargement of that subset; for example, $\tilde{C}_T^R$ denotes a slightly larger set containing $C_T^R$.

### 4.1.6 More notation for vector fields

We now define more notation for vector fields.

**Subscripts on functions will denote vector fields.**

Given a nonnegative integer $m$ and a triplet $J = (i, j, k)$ of multi-indices $i$, $j$ and $k$ for $(\partial, \Omega, S)$—by this we mean $\partial^i \Omega^j S^k$—we denote $|J| = |i| + 4|j| + 10k$.

**Explaining the counting convention for $|J|$**

In short, we insert the aforementioned counting convention $|i| + 4|j| + 10k$ for $|J|$ because *we shall use extra derivatives in order to control the commutators of $\tilde{Z}$ with the operator $P$ near the trapped set*, where $\tilde{Z} \in \{\Omega, S\}$. More precisely, the coefficient 10 in front of $k$ arises because of the fact $[P, S] - 2P - s_{2+\delta'}\Omega^2 \in \mathcal{C}$, where $\mathcal{C}$ is the class of operators defined in (4.2.2). In particular the presence of $\Omega^2$ as well as loss of derivative considerations (a price of losing two derivatives if one wants to control the full $LE^1$ norm—see (3.3.4)) for the inhomogeneity $P\phi$ in the weak local energy decay Definition 3.3.2 leads to the count $10 = 2 + 2 \cdot 4$. If $g^\omega = 0$, then we would count each $S$ in the same way we would count $\partial^2$, i.e. two derivatives. We put in place these differences in these numerical weights for $i, j$, and $k$ (respectively: 1, 4, and 10) because of the trapped set. See Lemma 4.2.2.

We denote

$$\phi_J := Z^J\phi := \partial^i\Omega^j S^k\phi, \tag{4.1.10}$$

$$\phi_{\le m} := (\phi_J)_{|J|\le m}, \quad \phi_{m_1\le\cdot\le m_2} := (\phi_J)_{m_1\le|J|\le m_2}, \quad \phi_{=m} := (\phi_J)_{|J|=m}$$

$$\partial^{\le m}\phi := (\partial^{i'}\phi)_{|i'|\le m}, \quad \partial^{=m}\phi := (\partial^{i'}\phi)_{|i'|=m}$$

Furthermore, by $Z^{=m}\phi$ we mean $\phi_{=m}$, and so on. We write $J_1 \le J_2$ to mean

$$i_1 \le i_2, \qquad j_1 \le j_2, \qquad k_1 \le k_2,$$

and $J_1 < J_2$ if at least one of the inequalities above is strict. If $I$ is a multiindex of order $\ell$ and $n$ an integer, by $I + n$ we mean

$$\{I + J : |J| = n, J \text{ is an } \ell\text{-multiindex}\}.$$

Throughout the chapter the integer $N$ will denote a fixed and sufficiently large positive numbersignifying the highest total number of vector fields that will ever be applied to the solution $\phi$ to (6.1.1) in the chapter.

We use the convention that the value of $n$ may vary by line.

If $\Sigma$ is a set, we shall use $\tilde{\Sigma}$ to indicate a slight enlargement of $\Sigma$, and we only perform a finite number of slight enlargements in this chapter to dyadic subregions. The symbol $\tilde{\Sigma}$ may vary by line.

If $f$ is a function, we shall typically use $\tilde{f}$ to denote commuting vector fields applied to $f$.

In this chapter, all implicit constants are allowed to depend on the dimension and the initial data $\phi_{\leq N}[0]$, for a fixed $N \in \mathbb{N}$ that is sufficiently large.

We write

$$s_q$$

to denote element of $S^Z(\langle r \rangle^{-q})$. $q$ will denote a nonnegative number.

## 4.2 Commuting with vector fields, and weak local energy decay for vector fields

*Remark* 4.2.1. Let $w$ be a sufficiently smooth function. Then

$$\partial w \in S^Z(\langle r \rangle^{-1})\bar{Z}w + \mu S^Z(1)|\partial_t w| \text{ if } r \geq t/2 \tag{4.2.1}$$

with $\mu = 0, \bar{Z} = \Omega$ for angular derivatives $\partial_\omega w$ on the left-hand side, and $\mu = 1, \bar{Z} = S$ for the radial derivative $\partial_r w$ on the left-hand side.

We define $\mathcal{C}$ to be the collection of real linear combinations of the operators

$$\partial s_{1+q'}\partial,\ s_{1+q'}\partial\partial,\ s_{2+q'},\ \partial s_{1+q'},\ s_{1+q'}\partial \tag{4.2.2}$$

where $q' > 0$ is a number which depends on the assumptions made about the coefficients $h, g^\omega, V, A$, and $B$ in Theorem 7.1.1. That is, schematically, $\mathcal{C} = \{\partial s_{1+q'}\partial + s_{1+q'}\partial\partial + s_{2+q'} + \partial s_{1+q'} + s_{1+q'}\partial\}$.

**Lemma 4.2.2.** *Let $w$ be a sufficiently smooth function. Given $J$ and $k \geq 0$, there are some operators $\dot{C} \in \mathcal{C}$ such that*

$$\Omega^J(S+2)^k P w = P\Omega^J S^k w + \dot{C} w_{\leq 4(|J|-1)+10k} \tag{4.2.3}$$

*where we adopt the following conventions: we interpret $\dot{C} w_{\leq 4(|J|-1)+10k}$ as a sum, and subscripts with negative real value denote the zero multiindex.*

*Proof (sketch).* By the assumptions in the main theorem,

$$[P, \partial] \in \mathcal{C}. \tag{4.2.4}$$

$$[P, \Omega] \in \mathcal{C}. \tag{4.2.5}$$

$$[P, S] - 2P - s_{2+\delta'}\Omega^2 \in \mathcal{C}. \tag{4.2.6}$$

One uses (4.2.4) to (4.2.6) and proves the result by mathematical induction. We omit the details of the proof, except for the following observation. Starting from $\Omega^J(S+2)^k P$ and then commuting the vector fields with $P$, then other than $P\Omega^J S^k$, the terms with the highest vector field count (assuming $g^\omega$ is not the zero function) are those of the form

$$\dot{C}\bar{Z}^{=|J|+k-1} w, \quad \bar{Z} \in \{\Omega, S\}, \quad \dot{C} \in \mathcal{C};$$

more specifically, those of the form $\dot{C}\Omega^{|J|-1}S^k$. This explains the subscript $4(|J| - 1) + 10k$.

**Lemma 4.2.3.** *Given the assumptions in either part 1 or part 2 of Theorem 7.1.1, there exists a positive real number $q' > 0$ such that for any multiindex $J$,*

$$|P\phi_J| \lesssim \frac{|\phi_{\leq |J|-1}|}{\langle r\rangle^{2+q'}} + \frac{|\nabla_{t,x}\phi_{\leq |J|}|}{\langle r\rangle^{1+q'}} + |(P\phi)_{\leq |J|}|.$$

*Proof.* There is a constant $q' > 0$ such that the operator $P$ can be written schematically as $P = \Box + \partial s_{1+q'}\partial + s_{1+q'}\not\partial^2 + s_{2+q'} + s_{1+q'}\partial + \partial s_{1+q'}$. We have $[Z, \partial] = c\partial$ schematically, for some real number $c$ depending on $Z$.

For terms of the form $(\partial\tilde{A})\tilde{\phi}$, where $\tilde{A}, \tilde{\phi}$ denote possible vector fields of $A, \phi$, we apply the assumption

$$\partial A \in S^Z_{int}(\langle r\rangle^{-2-}) \cap S^Z_{cone}(\langle r\rangle^{-2-})$$

on generic derivatives $\partial A$ from part 1 of Theorem 7.1.1 in $\{r < 3t/2\}$, and the assumption on $\partial_t A$ and (7.4.2) in $\{r \geq 3t/2\}$, giving a contribution of the form $\langle r\rangle^{-2-q'}|\phi_{<|J|}|$. For part 2, on the other hand, we in fact need not look at $r < 3t/2$ and $r \geq 3t/2$ separately, because the statement $\partial A \in S^Z(\langle r\rangle^{-2-})$ is already trivially satisfied for any $(t, r)$-pair given the assumption on $A$.

We include the terms arising from $g^\omega\Delta_\omega$ together with the $\langle r\rangle^{-1-}|\nabla_{t,x}\phi_{\leq |J|}|$ term. The rest is clear, and the claim follows. □

We recall the weak local energy decay estimate

$$\|\phi\|_{LE^{1,k}_w} \lesssim_k \|\nabla_{t,x}\phi(T_0)\|_{H^k} + \|f\|_{LE^{*,k}_w},$$

which can be rephrased as

$$\sum_{|\alpha|=k+1} \|(1-\chi)\partial^\alpha \phi\|_{LE[T_0,\infty)} + \|\langle r\rangle^{-1}\phi\|_{LE[T_0,\infty)} + \sum_{1\le|\gamma|\le k} \|\partial^\gamma \phi\|_{LE[T_0,\infty)}$$

$$\lesssim_{k,\chi} \|\nabla_{t,x}\phi(T_0)\|_{H^k} + \|f\|_{LE^{*,k}_w[T_0,\infty)}.$$

**Proposition 4.2.4** (Weak local energy decay for vector fields)**.** *Let $\phi$ be any smooth-enough function solving (6.1.1) and satisfying Definition 3.3.2. Then for any natural number $m \ge 0$,*

$$\|\phi_{\le m}\|_{LE^1} \lesssim \|\nabla_{t,x}\phi_{\le m+1}(0)\|_{L^2} + \|f_{\le m+2}\|_{LE^*}. \tag{4.2.7}$$

*Proof.* We prove (4.2.9) by induction.

The base case

$$\|\phi\|_{LE^1} \lesssim \|\nabla_{t,x}\phi_{\le 1}(0)\|_{L^2} + \|f_{\le 2}\|_{LE^*}$$

is simply given by combining Definition 3.3.2 at $k=0$ and $k=1$, which yields

$$\|\phi\|_{LE^1} \lesssim \|\nabla_{t,x}\phi(0)\|_{H^1} + \|\partial^{\le 1} f\|_{LE^*} + \|\chi\partial^{=2} f\|_{L^2L^2},$$

which is clearly bounded by

$$\|\nabla_{t,x}\phi(0)\|_{H^1} + \|\partial^{\le 2} f\|_{LE^*} \le \|\nabla_{t,x}\phi_{\le 1}(0)\|_{L^2} + \|f_{\le 2}\|_{LE^*}.$$

Next, we use Lemma 4.2.2. Let $|(I,J,k)| = m$.

$$\begin{aligned}\|\phi_{(I,J,k)}\|_{LE^1} &\lesssim \|\nabla_{t,x}\Omega^J S^k \phi(0)\|_{H^{|I|+1}} + \|\Omega^J S^k f\|_{LE^{*,|I|+2}} + \|[P,\Omega^J S^k]\phi\|_{LE^{*,|I|+2}} \\ &\lesssim \|\nabla_{t,x}\phi_{\le m+1}(0)\|_{L^2} + \|f_{\le m+2}\|_{LE^*} + \|[P,\Omega^J S^k]\phi\|_{LE^{*,|I|+2}} \\ &\lesssim \|\nabla_{t,x}\phi_{\le m+1}(0)\|_{L^2} + \|f_{\le m+2}\|_{LE^*} + \|\langle r\rangle^{-1-}\nabla_{t,x}\phi_{\le m-2}\|_{LE^*} \\ &\quad + \|\langle r\rangle^{-2-}\phi_{\le m-2}\|_{LE^*} \\ &\lesssim \|\nabla_{t,x}\phi_{\le m+1}(0)\|_{L^2} + \|f_{\le m+2}\|_{LE^*} + \|\phi_{\le m-2}\|_{LE^1} \\ &\lesssim \|\nabla_{t,x}\phi_{\le m+1}(0)\|_{L^2} + \|f_{\le m+2}\|_{LE^*}\end{aligned}$$

In transitioning from the second line to the third line, we used (4.2.3). The third line follows by the assumption that $\Omega$ counts for four partial derivatives.The final line follows by the induction hypothesis. □

*Remark* 4.2.5. The above proof extends to time intervals $[T_1,\infty)$, $T_1 \ge 0$. (The proof above assumes $T_1 = 0$.) The estimate is

$$\|\phi_{\le m}\|_{LE^1[T_1,\infty)} \lesssim \|\nabla_{t,x}\phi_{\le m+1}(T_1)\|_{L^2} + \|f_{\le m+2}\|_{LE^*[T_1,\infty)}.$$

**Proposition 4.2.6** (Stationary local energy decay for vector fields)**.** *Assume*

$$\begin{aligned}\|\partial^{\le m}\phi\|_{LE^1([T_0,T_1)\times\mathbb{R}^3)} \lesssim_m \|\partial\phi(T_0)\|_{H^{m+k_0}(\mathbb{R}^3)} + \|\partial^{\le m}(P\phi)\|_{LE^*([T_0,T_1)\times\mathbb{R}^3)} \\ + \|\partial_t\partial^{\le m}\phi\|_{LE[T_0,T_1)}.\end{aligned} \tag{4.2.8}$$

*Then we have*

$$\|\phi_{\le m}\|_{LE^1([T_0,T_1)\times\mathbb{R}^3)} \lesssim \|\partial\phi_{\le m+k_0}(T_0)\|_{L^2} + \|(P\phi)_{\le m}\|_{(LE^*)([T_0,T_1)\times\mathbb{R}^3)} + \|\partial_t\phi_{\le m}\|_{LE[T_0,T_1)}. \tag{4.2.9}$$

*Proof.* We prove (4.2.9) by induction. The base case holds by the base case of (4.2.8). Then

$$\begin{aligned}
\|\phi_{(I,J,k)}\|_{LE^1} &\lesssim \|\partial\Omega^J S^k\phi(T_0)\|_{H^{|I|+k_0}} + \|\Omega^J S^k(P\phi)\|_{LE^{*,|I|}} + \|[P,\Omega^J S^k]\phi\|_{LE^{*,|I|}} \\
&\quad + \|\partial_t\partial^{\le|I|}\Omega^J S^k\phi\|_{LE} \\
&\lesssim \|\partial\phi_{\le m+k_0}(T_0)\|_{L^2} + \|(P\phi)_{\le m}\|_{LE^*} + \|[P,\Omega^J S^k]\phi\|_{LE^{*,|I|}} + \|\partial_t\phi_{\le m}\|_{LE} \\
&\lesssim \|\partial\phi_{\le m+k_0}(T_0)\|_{L^2} + \|(P\phi)_{\le m}\|_{LE^*} + \|\langle r\rangle^{-1-}\partial\phi_{\le m-2}\|_{LE^*} \\
&\quad + \|\langle r\rangle^{-2-}\phi_{\le m-2}\|_{LE^*} + \|\partial_t\phi_{\le m}\|_{LE} \\
&\lesssim \|\partial\phi_{\le m+k_0}(T_0)\|_{L^2} + \|(P\phi)_{\le m}\|_{LE^*} + \|\phi_{\le m-2}\|_{LE^1} + \|\partial_t\phi_{\le m}\|_{LE} \\
&\lesssim \|\partial\phi_{\le m+k_0}(T_0)\|_{L^2} + \|(P\phi)_{\le m}\|_{LE^*} + \|\partial_t\phi_{\le m}\|_{LE}
\end{aligned}$$

The final line follows by the induction hypothesis. □

### 4.3 Initial $L^\infty$ estimates

We now state the Sobolev embedding estimates localised to our selected conical regions.

**Lemma 4.3.1.** *Let $w \in C^4$.*

- *For all $T \ge 1$ and $1 \le U \le 3T/8$, we have*

$$\|w\|_{L^\infty(C_T^U)} \lesssim \sum_{i\le 1, j\le 2} \frac{1}{(T^3U)^{1/2}}\|S^i\Omega^j w\|_{L^2(\tilde{C}_T^U)} + \left(\frac{U}{T^3}\right)^{\frac{1}{2}}\|\partial_r S^i\Omega^j w\|_{L^2(\tilde{C}_T^U)}. \tag{4.3.1}$$

- *For all $T \ge 1$ and $R > T$, we have*

$$\|w\|_{L^\infty(C_R^T)} \lesssim \sum_{i\le 1, j\le 2} \frac{1}{(R^3T)^{1/2}}\|S^i\Omega^j w\|_{L^2(\tilde{C}_R^T)} + \frac{1}{(RT)^{1/2}}\|\partial_t S^i\Omega^j w\|_{L^2(\tilde{C}_R^T)}. \tag{4.3.2}$$

- *For all $T \geq 1$ and $1 \leq R \leq 3T/8$, we have*

$$\|w\|_{L^\infty(C_T^R)} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{(R^3T)^{1/2}} \|S^i\Omega^j w\|_{L^2(\tilde{C}_T^R)} + \frac{1}{(RT)^{1/2}} \|\partial_r S^i \Omega^j w\|_{L^2(\tilde{C}_T^R)}. \tag{4.3.3}$$

*Proof.* In $C_T^U$ we make the change of coordinates $t = e^s$ and $|r - t| = e^{s+\rho}$. With this change of coordinates, we are now dealing with a region of size 1 in spherical coordinates including $s$. We have $\partial_s = t\partial_t + r\partial_r = S$ and $\partial_\rho = (r - t)\partial_r$. Then we apply the fundamental theorem of calculus in $s$ and also in $\rho$. Finally, we rescale to $C_T^U$, obtaining (5.2.4).

For $C_R^T$, we let $r = e^s$ and $r - t = e^{s+\rho}$. Thus $\partial_s = S$ and $\partial_\rho = (t - r)\partial_t$. We get

$$\|w\|_{L^\infty(C_R^T)} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{(R^3T)^{1/2}} \|S^i\Omega^j w\|_{L^2(\tilde{C}_R^T)} + \frac{R-T}{(R^3T)^{1/2}} \|\partial_t S^i \Omega^j w\|_{L^2(\tilde{C}_R^T)}.$$

This implies (5.2.5) since $R - T \leq R$.

For $C_T^R$, we let $t = e^s$ and $r = e^{s+\rho}$. We obtain $\partial_s = S$ and $\partial_\rho = r\partial_r$ and (5.2.6). □

**Corollary 4.3.2.**

$$\|\phi\|_{L^\infty_{t,x}(C_T^{<3T/4})} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{T^{1/2}} \|S^i\Omega^j \phi\|_{LE^1_{t,x}(\tilde{C}_T^{<3T/4})}. \tag{4.3.4}$$

*Proof.* By rewriting (5.2.6) in the local energy norm by shifting the $R$ weights around, we obtain (4.3.4). □

**Lemma 4.3.3.** *Let $A[a, b] := \{x \in \mathbb{R}^3 : a \leq |x| \leq b\}$. If $f \in C^1([0, \infty)_t \times \mathbb{R}^3_x)$, then*

$$\begin{aligned}\int_{A[t/2,3t/2]} \frac{f(t,x)^2}{\langle t - r\rangle^2} dx \lesssim \int_{A[t/4,7t/4]} |\partial_r f(t,x)|^2 dx \\ + t^{-2} \left( \int_{A[t/4,t/2]} f(t,x)^2 dx + \int_{A[3t/2,7t/4]} f(t,x)^2 dx \right)\end{aligned} \tag{4.3.5}$$

*Proof.* Let $\chi : [0,\infty) \to [0,1]$ be a cutoff such that $\chi(s) = 1$ for $1/2 \le s \le 3/2$ and 0 when $s \le 1/4$ and $s \ge 7/4$. We will show that, if $\gamma > -1/2$, and $\gamma \ne 1/2$, then

$$\int \langle t-r\rangle^{-2-2\gamma}\chi(r/t)f(r,\omega)^2r^2dr \lesssim \int \langle t-r\rangle^{-2\gamma}|\partial_r f(r,\omega)\chi(r/t)|^2r^2dr + \frac{1}{t^2}\int \langle t-r\rangle^{-2\gamma}|f(r,\omega)\chi'(r/t)|^2r^2dr.$$

The conclusion then follows if we take $\gamma = 0$ and integrate over the angular variables $\omega$.

We now begin the calculation. We have

$$f(r,\omega)^2\chi(r/t) - f(7t/4,\omega)^2\chi((7t/4)/t) = -2\int_r^{7t/4} f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))d\rho.$$

Hence

$$f(r,\omega)^2\chi(r/t)r^2 \lesssim f(7t/4,\omega)^2\chi(7/4)t^2 + 2\int_r^{7t/4} |f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))|\rho^2d\rho$$

Recall that $\chi(7/4) = 0$. We multiply by $\langle t-r\rangle^{-2-2\gamma}$ and integrate $r$ from $t/4$ to $7t/4$. Thus

$$\begin{aligned}
&\int_{t/4}^{7t/4} \langle t-r\rangle^{-2-2\gamma}f(r,\omega)^2\chi(r/t)r^2dr \\
&\quad\lesssim 2\int_{t/4}^{7t/4}\langle t-r\rangle^{-2-2\gamma}\int_r^{7t/4} |f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))|\rho^2d\rho dr \\
&\quad\lesssim \int_{t/4}^{7t/4}\int_0^{\rho}\langle t-r\rangle^{-2-2\gamma}dr\,|f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))|\rho^2d\rho \\
&\quad\lesssim \int_{t/4}^{7t/4}\langle t-\rho\rangle^{-1-2\gamma}|f(\rho,\omega)\chi(\rho/t)\cdot\partial_\rho(f(\rho,\omega)\chi(\rho/t))|\rho^2d\rho
\end{aligned}$$

By the chain rule, $|\partial_r(\chi(r/t))| \le Ct^{-1}|\chi'(r/t)|$. Thus by Cauchy-Schwarz and the chain rule

$$
\begin{aligned}
(1-\epsilon)\int_{t/2}^{3t/2}&\langle t-r\rangle^{-2-2\gamma}f(r,\omega)^2r^2dr\\
&\lesssim\int_{t/4}^{7t/4}\langle t-r\rangle^{-2\gamma}|\partial_r f(r,\omega)\chi(r/t)|^2r^2dr\\
&\quad+\frac{1}{\epsilon}t^{-2}\int_{t/4}^{7t/4}\langle t-r\rangle^{-2\gamma}|f(r,\omega)^2||\chi(r/t)\chi'(r/t)|r^2dr.
\end{aligned}
$$

This concludes the proof. □

The following result is an analogue of Theorem 5.3 in [**?**].

**Lemma 4.3.4.** *Let $T$ be fixed and $\phi$ solve (6.1.1) for the times $t\in[T,2T]$. There is a fixed positive integer $k$ such that for any multi-index $J$ with $|J|+k\leq N$, we have:*

$$
|\phi_J|\lesssim_{|J|}\|\phi_{|J|\leq\cdot\leq|J|+k}\|_{LE^1[T,2T]}\frac{\langle u\rangle^{1/2}}{\langle v\rangle}. \tag{4.3.6}
$$

*Proof.* We prove this by looking separately at $(t,x)$-pair values in $C_T^R, C_R^T$ and $C_T^U$.

- (The $C_T^U$ regions, with $1\leq U\leq 3T/8$) In contrast to the "near" region $C_T^R$ and the "far" region $C_R^T$, the regions close to the cone will proceed differently: we utilise a Hardy-like inequality adapted to the cone, namely (7.3.6).

  Let $\chi:\mathbb{R}_+\to\mathbb{R}_+$ be a smooth cutoff function with $\chi(s)=1, s\geq 1/2$ and $\chi(s)=0, s\leq 1/4$. For any smooth-enough function $w$,

$$
\begin{aligned}
\|\frac{w}{U}\|_{L^2(C_T^U)}&\lesssim\Big\|\frac{\chi(\frac{r}{t})w}{\langle u\rangle}\Big\|_{L^2[T,2T]L^2}\\
&\lesssim\|\partial_r(\chi(\frac{r}{t})w)\|_{L^2[T,2T]L^2}+T^{-1}\|\chi(\frac{r}{t})w\|_{L^2_{t,x}([T,2T]\times\{T/8\leq r\leq 15/8T\})}\\
&\lesssim T^{1/2}\|w\|_{LE^1[T,2T]}
\end{aligned}
\tag{4.3.7}
$$

where the second line follows by (7.3.6).

Thus

$$\begin{aligned}\|\phi_J\|_{L^\infty(C_T^U)} &\lesssim \sum_{i\le 1, j\le 2} \frac{1}{(T^3U)^{1/2}} \|S^i\Omega^j\phi_J\|_{L^2(\tilde{C}_T^U)} + \left(\frac{U}{T^3}\right)^{\frac{1}{2}} \|\partial_r S^i\Omega^j\phi_J\|_{L^2(\tilde{C}_T^U)} \\ &\lesssim \left(\frac{U}{T^3}\right)^{\frac{1}{2}} T^{1/2} \sum_{i\le 1, j\le 2} \|S^i\Omega^j\phi_J\|_{LE^1[T,2T]} \\ &\lesssim \frac{U^{1/2}}{T} \|\phi_{|J|\le\cdot\le|J|+k}\|_{LE^1[T,2T]}.\end{aligned}$$

- (The $C_T^R$ regions, for $R$ values sufficiently small relative to $T$) This is essentially Corollary 4.3.2: apply the Sobolev embedding estimate (5.2.6) to $\phi_J$

$$\begin{aligned}\|\phi_J\|_{L^\infty(C_T^R)} &\lesssim \sum_{i\le 1, j\le 2} \frac{1}{(R^3T)^{1/2}} \|S^i\Omega^j\phi_J\|_{L^2(\tilde{C}_T^R)} + \frac{1}{R^{1/2}T^{1/2}} \|\partial_r S^i\Omega^j\phi_J\|_{L^2(\tilde{C}_T^R)} \\ &\lesssim \frac{1}{T^{1/2}} \|\phi_{|J|\le\cdot\le|J|+k}\|_{LE^1[T,2T]},\end{aligned}$$

  and take the supremum over, say, $R < 3T/8$. The second inequality comes from commuting $S^i\Omega^j$ with $Z^J$ in a way that will put it in the form (4.1.10). This is where the integer $k$ arises.

- (The $C_R^T$ regions) (5.2.5) implies

$$\begin{aligned}\|\phi_J\|_{L^\infty(C_R^T)} &\lesssim \frac{1}{R^{1/2}} \sum_{i\le 1, j\le 2} \|R^{-3/2}S^i\Omega^j\phi_J\|_{L^2(\tilde{C}_R^T)} + \|R^{-1/2}\partial_t S^i\Omega^j\phi_J\|_{L^2(\tilde{C}_R^T)} \\ &\lesssim \frac{1}{R^{1/2}} \sum_{i\le 1, j\le 2} \|S^i\Omega^j\phi_J\|_{LE^1[T,2T]} \\ &\lesssim \frac{1}{R^{1/2}} \|\phi_{|J|\le\cdot\le|J|+k}\|_{LE^1[T,2T]}.\end{aligned}$$

  Then we take the supremum over the relevant $R$ values. In $C_R^T$, we have $v \sim r$ and $u \sim r$.

□

## 4.4 Derivative estimates in $L^2$

**Lemma 4.4.1.** *Suppose that $\sigma$ and $\delta$ from (7.1.1) are nonnegative real numbers, and $\delta' \in [-1, \infty)$. Let $L, L'$ denote dyadic numbers of the form (2.0.1), with $L, L' = 1$ when $h = 0$ and, in general, $L, L' \gg_{h,g^\omega} 1$ are appropriately large relative to 1, depending on $h$ and $g^\omega$.*[1]

- *If $L \leq U, R \leq 3T/8$, then*

  –

$$R\|\nabla_{t,x} w_{\leq m}\|_{L^2\left(C_T^R\right)} \lesssim \|w_{\leq m}\|_{L^2\left(\tilde{C}_T^R\right)} + \|S w_{\leq m}\|_{L^2\left(\tilde{C}_T^R\right)} + R^2 \|(Pw)_{\leq m}\|_{L^2(\tilde{C}_T^R)} \tag{4.4.1}$$

  – *Let $C_{T,1}^U := C_T^U \cap \{r < t\}$ and $C_{T,2}^U := C_T^U \cap \{r > t\}$.*

$$U\|\nabla_{t,x} w_{\leq m}\|_{L^2(C_{T,1}^U)} \lesssim \|w_{\leq m}\|_{L^2(\tilde{C}_{T,1}^U)} + \|S w_{\leq m}\|_{L^2(\tilde{C}_{T,1}^U)} + UT\|(Pw)_{\leq m}\|_{L^2(\tilde{C}_{T,1}^U)} \tag{4.4.2}$$

$$\begin{aligned} U\|\nabla_{t,x} w_{\leq m}\|_{L^2(C_{T,2}^U)} \lesssim \|w_{\leq m}\|_{L^2(\tilde{C}_{T,2}^U)} &+ \sum_{\bar{Z} \in \{\Omega, S\}} \|\bar{Z} w_{\leq m}\|_{L^2(\tilde{C}_{T,2}^U)} \\ &+ UT\|(Pw)_{\leq m}\|_{L^2(\tilde{C}_{T,2}^U)} \end{aligned} \tag{4.4.3}$$

- *If $L' \leq T < R$, i.e. $L' \leq T \leq 3R/8$, then*

$$\begin{aligned} R\|\nabla_{t,x} w_{\leq m}\|_{L^2(C_R^T)} \lesssim \|w_{\leq m}\|_{L^2(\tilde{C}_R^T)} &+ \sum_{\bar{Z} \in \{\Omega, S\}} \|\bar{Z} w_{\leq m}\|_{L^2(\tilde{C}_R^T)} \\ &+ R^2\|(Pw)_{\leq m}\|_{L^2(\tilde{C}_R^T)} \end{aligned} \tag{4.4.4}$$

[1] For example, if $h \in S^Z(\epsilon\langle r\rangle^{-1})$ for a sufficiently small $\epsilon > 0$, then $L = 1$.

*Proof.* We begin by proving (4.4.1). Let $w$ denote a reasonably smooth function. We shall first prove that for $1 \ll R \leq 3T/8$,

$$R\|\nabla_{t,x} w\|_{L^2(C_T^R)} \lesssim \|w\|_{L^2(\tilde{C}_T^R)} + \|Sw\|_{L^2(\tilde{C}_T^R)} + R^2\|Pw\|_{L^2(\tilde{C}_T^R)} \tag{4.4.5}$$

Let $\chi(t,r)$ be a radial cutoff function on $\mathbb{R}^{1+3}$ with $\operatorname{supp}\chi \subset \tilde{C}_T^R$ and $\chi = 1$ on $C_T^R$; a further fixing of $\chi$ will come later in the proof. Two observations are in order:

1. If $r < t$ then for a sufficiently large constant $C'$, we have

$$\chi\left(\frac{u}{t}|\nabla_{t,x} w(t,x)|^2\right) \leq \chi\left(|\nabla_x w|^2 - w_t^2 + \frac{C'}{ut}|Sw|^2\right) \tag{4.4.6}$$

   (which holds without the multiplication by $\chi$ as well) as an expansion of the terms $|Sw|^2, |\nabla_{t,x} w|^2$ reveals; the values $C' \geq 3$ work for every $(r,t)$ such that $0 \leq r < t$.

2. By integration by parts,

$$\int \chi(|\nabla_x w|^2 - w_t^2)\, dxdt = \int \chi w(\partial_t^2 - \Delta) w\, dxdt - \int \frac{1}{2}(\partial_t^2 - \Delta)\chi w^2\, dxdt. \tag{4.4.7}$$

   There are no boundary terms in either time or space because of the compact support of $\chi(t,r)$ in both time and space.

Integrating (4.4.6) in spacetime, we have via (4.4.7)

$$\int \chi\frac{u}{t}|\nabla_{t,x} w|^2\, dxdt \leq \int \chi w(\partial_t^2 - \Delta)w + O(|\Box\chi| w^2) + \frac{C'}{ut}\chi|Sw|^2\, dxdt. \tag{4.4.8}$$

The proof of (4.4.5) will be complete once we incorporate $Pw$ into (4.4.8):

- Let $\Box_h$ denote the second order operator

$$\Box_h := \partial_\alpha h^{\alpha\beta}\partial_\beta.$$

For $\int(\chi w)(\Box_h w)\,dxdt$, we integrate by parts and use Cauchy-Schwarz. A term

$$\int \chi h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\,dxdt = O\left(\int \chi\frac{|\nabla_{t,x}w|^2}{\langle r\rangle}\,dxdt\right)$$

arises, and for this term we use the hypothesis that $L \gg_h 1$ for $h \neq 0$.

Similarly, $\int(\chi w)(g^\omega\Delta_\omega w)\,dxdt$ is treated by integration by parts and Cauchy-Schwarz. We use the smallness of $\langle r\rangle^{-2-\delta'}$ (which is $O(\langle r\rangle^{-1})$ since $\delta' \in [-1,\infty)$) for sufficiently large $R$.

- We use the bound $V \lesssim \langle r\rangle^{-2}$.

- For $\int \chi w B\partial w$ we use Cauchy-Schwarz. For $\int \chi w\partial(Aw)$ we integrate by parts and use Cauchy-Schwarz; it is also possible to bound this using information on $\partial A$ if one does not integrate by parts, but we integrate by parts in order to use fewer assumptions. The bounds we obtain are sufficient to prove the claim (4.4.5) even when $\sigma = 0$, and we only assume $A, B \in S^Z(\langle r\rangle^{-1})$ in this part.

Assuming $\Box\chi \lesssim \langle r\rangle^{-2}$, separating $|\chi w P w| \lesssim \chi[(R^{-1}w)^2 + (RPw)^2]$ in the right-hand side of (4.4.8), and using the reasoning in the bullet points (along with the triangle inequality) to deal with

$$\int(\chi w)((\Box - P)w)\,dxdt,$$

this proves the claim (4.4.5) for $C_T^R$.

The same proof shows the analogue of (4.4.5) for the $C_T^U \cap \{r < t\}$ region,

$$U\|\nabla_{t,x} w\|_{L^2(C_T^U \cap \{r<t\})} \lesssim \|w\|_{L^2(\tilde{C}_T^U \cap \{r<t\})} + \|Sw\|_{L^2(\tilde{C}_T^U \cap \{r<t\})} + UT\|Pw\|_{L^2(\tilde{C}_T^U \cap \{r<t\})} \tag{4.4.9}$$

if we choose a $\chi$ adapted to $C_T^U \cap \{r < t\}$ (rather than $C_T^R$) that satisfies

$$\Box\chi \lesssim \frac{1}{\langle t+r\rangle\langle t-r\rangle}$$

(rather than $\Box\chi \lesssim 1/\langle r\rangle^2$).[2]

Similar arguments show the result for vector fields, (4.4.1) and (4.4.2). The only new thing one has to deal with is $\int \chi w_{\leq m}[P, Z^{\leq m}]w\, dxdt$ and similar arguments involving integration by parts and Cauchy-Schwarz establish the claims (4.4.1) and (4.4.2).

Next, we prove

$$R\|\nabla_{t,x} w\|_{L^2(C_R^T)} \lesssim \|w\|_{L^2(\tilde{C}_R^T)} + \sum_{\bar{Z}\in\{\Omega,S\}} \|\bar{Z}w\|_{L^2(\tilde{C}_R^T)} + R^2\|Pw\|_{L^2(\tilde{C}_R^T)}. \tag{4.4.10}$$

The proof for the region $\{r > t\}$ is essentially a switching of the $r$ and $t$ variables in what has been done for the $C_T^R$ and $C_T^U \cap \{r < t\}$ regions. For any point $(t, x)$ such that $|x| > t$,

$$|\nabla_{t,x} w(t,x)|^2 \leq \frac{r}{r-t}(w_t^2 - w_r^2) + \frac{C'}{(r-t)^2}(Sw)^2 + C\frac{(\Omega w)^2}{r^2} \tag{4.4.11}$$

[2](Note that if $T$ is sufficiently large, then we may even take $L = 1$ for $C_T^U$ and $L' = 1$ for $C_R^T$.)

for some sufficiently large constants $C, C' > 0$. For the angular derivatives, this follows because

$$\not\partial = \sum_j c_j \Omega_j$$

for some coefficients $c_j$ such that

$$|c_j| \lesssim 1/r.$$

We shall only use the weaker estimate

$$|\nabla_{t,x} w(t,x)|^2 \le \frac{r}{r-t}(w_t^2 - |\nabla_x w|^2) + \frac{C'}{(r-t)^2}(Sw)^2 + C\frac{(\Omega w)^2}{r(r-t)}. \tag{4.4.12}$$

We use this because it makes (4.4.13) conceptually cleaner; and because using (4.4.11) would lead to no gain in the final derivative estimates for $C_R^T$, due to the presence of the $(r-t)^{-2}$ coefficient of $(Sw)^2$.

- (Bound in $C_R^T$) Let $\chi(t,r)$ be a radial cutoff function adapted to $C_R^T$. By (4.4.12),

$$\begin{aligned}\int \chi |\nabla_{t,x} w|^2 \, dxdt \le \int \frac{r}{r-t}\chi(w_t^2 - |\nabla_x w|^2) + C\frac{r}{r-t}\chi\left|\frac{(\Omega w)^2}{r(r-t)}\right| \\ + \frac{C'}{(t-r)^2}\chi|Sw|^2 \, dxdt.\end{aligned} \tag{4.4.13}$$

  The analysis henceforth is similar to the three bullet points above. Assuming

$$\Box \chi \lesssim \langle r \rangle^{-2},$$

  we end up with

$$\|\nabla_{t,x} w\|_{L^2(C_R^T)} \lesssim R^{-1}\left(\|w\|_{L^2(\tilde{C}_R^T)} + \sum_{\bar{Z}\in\{\Omega,S\}} \|\bar{Z}w\|_{L^2(\tilde{C}_R^T)}\right) + R\|Pw\|_{L^2(\tilde{C}_R^T)},$$

  i.e., (4.4.10).

- (Bound in $C_T^U \cap \{r > t\}$) We adapt $\chi$ to $C_T^U \cap \{r > t\}$ with

$$\Box \chi \lesssim (\langle t + r \rangle \langle t - r \rangle)^{-1}.$$

  Then by Cauchy-Schwarz,

$$\|\nabla_{t,x} w\|_{L^2(C_T^U \cap \{r>t\})} \lesssim U^{-1} \left( \|w\|_{L^2(\tilde{C}_T^U \cap \{r>t\})} + \sum_{\bar{Z} \in \{\Omega, S\}} \|\bar{Z} w\|_{L^2(\tilde{C}_T^U \cap \{r>t\})} \right) + T \|Pw\|_{L^2(\tilde{C}_T^U \cap \{r>t\})}.$$

The full results for vector fields $w_{\leq m}$ again follow simply by similar integration by parts and Cauchy-Schwarz arguments. □

We will need to bound the second derivative of vector fields in $L^2$ when proving $L^\infty$ estimates for vector fields of a function. Hence we present Corollary 4.4.2 immediately.

**Corollary 4.4.2.** *Assume the hypotheses of Lemma 4.4.1. Then*

-
$$R \|\nabla_{t,x}^2 w_{\leq m}\|_{L^2\left(C_T^R\right)} \lesssim \|\nabla_{t,x} w_{\leq m+n}\|_{L^2\left(\tilde{C}_T^R\right)} + R^2 \|\nabla_{t,x} (Pw)_{\leq m}\|_{L^2(\tilde{C}_T^R)} \quad (4.4.14)$$

-
$$U \|\nabla_{t,x}^2 w_{\leq m}\|_{L^2(C_T^U)} \lesssim \|\nabla_{t,x} w_{\leq m+n}\|_{L^2(\tilde{C}_T^U)} + UT \|\nabla_{t,x} (Pw)_{\leq m}\|_{L^2(\tilde{C}_T^U)} \quad (4.4.15)$$

-
$$R \|\nabla_{t,x}^2 w_{\leq m}\|_{L^2(C_R^T)} \lesssim \|\nabla_{t,x} w_{\leq m+n}\|_{L^2(\tilde{C}_R^T)} + R^2 \|\nabla_{t,x} (Pw)_{\leq m}\|_{L^2(\tilde{C}_R^T)} \quad (4.4.16)$$

*Proof.* Fixing any $\alpha \in \{0,1,2,3\}$ and denoting $\partial_\alpha$ by $\partial$, we substitute $\partial w_{\leq m}$ for the function $w$ in the proof of Lemma 4.4.1.

A new type of term arises, which is

$$\int \chi \partial w_{\leq m} P \partial w_{\leq m} = \int \chi \partial w_{\leq m} (\partial f_{\leq m} + \partial [P, Z^{\leq m}] w + [P, \partial] w_{\leq m}).$$

We can handle the first term on the right-hand side by Cauchy-Schwarz.

For the $\Box_h, g^\omega \Delta_\omega$ and $V$ contributions to $P$, similar arguments as before using Cauchy-Schwarz and integration by parts work. For the contributions of the $\partial_\alpha A^\alpha$ and $B^\alpha \partial_\alpha$ components to $P$ in both

$$\partial [P, Z^{\leq m}] w$$

and

$$[P, \partial] w_{\leq m},$$

we also use integration by parts and Cauchy-Schwarz, and the fact that $\partial A \in S^Z(\langle r \rangle^{-2})$, i.e. this bound holds for all $(r,t)$ (and hence all three dyadic regions), which follows from the assumptions

$$\partial A \in S^Z_{int}(\langle r \rangle^{-2}) \cap S^Z_{cone}(\langle r \rangle^{-2})$$

and

$$\partial_t A \in S^Z(\langle v \rangle \langle u \rangle^{-1} \langle r \rangle^{-1} \langle r \rangle^{-1-\sigma})$$

because of (7.4.2). More concretely, we have the schematic equalities

$$\int \chi \partial w_{\leq m} \partial [B^\alpha \partial_\alpha, Z^{\leq m}] w = \int (\chi' \partial w_{\leq m} + \chi \partial^2 w_{\leq m}) \tilde{B} \partial w_{\leq m}$$

$$\int \chi \partial w_{\leq m} \partial [\partial_\alpha A^\alpha, Z^{\leq m}] w = \int (\chi' \partial w_{\leq m} + \chi \partial^2 w_{\leq m}) \partial \tilde{A} \cdot w_{\leq m}$$

where tildes denote vector fields. We apply the aforementioned assumptions

$$\partial A \in S^Z(\langle r\rangle^{-2})$$

and

$$B \in S^Z(\langle r\rangle^{-1}).$$

□

**Corollary 4.4.3** ($L^\infty$ estimates for derivatives)**.** *Assume the hypotheses of Corollary 4.4.2. Hence $\sigma$ and $\delta$ from (7.1.1) are* __*nonnegative*__ *real numbers.*

1. *If $1 \ll U \leq 3T/8$, we have*

$$\|\partial w_{\leq m}\|_{L^\infty(C_T^U)} \lesssim \frac{1}{\sqrt{UT^3}}\Big(U^{-1}\|w_{\leq m+n}\|_{L^2(\tilde{C}_T^U)} + T(\|(Pw)_{\leq m+n}\|_{L^2(\tilde{C}_T^U)} + \|U\partial(Pw)_{\leq m}\|_{L^2(\tilde{C}_T^U)})\Big). \tag{4.4.17}$$

2. *Let $1 \ll R \leq 3T/8$. Then we have:*

$$\|\partial w_{\leq m}\|_{L^\infty(C_T^R)} \lesssim \frac{1}{\sqrt{TR^3}}\Big(R^{-1}\|w_{\leq m+n}\|_{L^2(\tilde{C}_T^R)} + R(\|(Pw)_{\leq m+n}\|_{L^2(\tilde{C}_T^R)} + \|R\partial(Pw)_{\leq m}\|_{L^2(\tilde{C}_T^R)})\Big).$$

3. *Let $1 \ll T \leq 3R/8$. Then we have:*

$$\|\partial w_{\leq m}\|_{L^\infty(C_R^T)} \lesssim \frac{1}{\sqrt{TR^3}}\Big(R^{-1}\|w_{\leq m+n}\|_{L^2(\tilde{C}_R^T)} + R(\|(Pw)_{\leq m+n}\|_{L^2(\tilde{C}_R^T)} + \|R\partial(Pw)_{\leq m}\|_{L^2(\tilde{C}_R^T)})\Big).$$

*Proof.* Let $v = w_{\leq m}$. The main idea in this proof is to

- first use the initial $L^\infty$ estimates proved in Section 4.3 on derivatives $\partial v$, and to commute this $\partial$ with the vector fields $S^i\Omega^j$ in both terms of the majorizer in the estimates (5.2.4) to (5.2.6). This results in

$$\begin{aligned}\|\partial v\|_\infty &\lesssim \sum_{i\leq 1, j\leq 2} (W^3W')^{-1/2}\|S^i\Omega^j\partial v\|_2 + (\tilde{W})((W^3W')^{-1/2})\|\nabla_{t,x}S^i\Omega^j\partial v\|_2 \\ &\lesssim (W^3W')^{-1/2}\|\partial v_{\leq n}\|_2 + (\tilde{W})((W^3W')^{-1/2})\|\partial^2 v_{\leq n}\|_2 \\ &= (W^3W')^{-1/2}\Big(\|\partial v_{\leq n}\|_2 + \tilde{W}\|\partial^2 v_{\leq n}\|_2\Big)\end{aligned}$$

  for dyadic weights $W, W'$ and $\tilde{W} \in \{W, W'\}$, where the choices of $W, W'$ and $\tilde{W}$ all depend on the region in question.

- And secondly to use the derivative estimates just proved in Lemma 4.4.1 and Corollary 4.4.2, in order to control $\|\nabla_{t,x}v_{\leq n}\|_2$ and $\tilde{W}\|\nabla^2_{t,x}v_{\leq n}\|_2$ respectively.

In $C_T^U$, one has $W = T$ and $W' = \tilde{W} = U$. Let $k \geq 0$ be any integer. Then

$$\begin{aligned}\|\partial v_{\leq k}\|_2 + \tilde{W}\|\partial^2 v_{\leq k}\|_2 &= \|\partial v_{\leq k}\|_2 + U\|\partial^2 v_{\leq k}\|_2 \\ &\lesssim \|\partial v_{\leq k+1}\|_2 + UT\|\partial(Pv)_{\leq k}\|_2 \\ &\lesssim U^{-1}\|v_{\leq k+2}\|_2 + T\|(Pv)_{\leq k+1}\|_2 + UT\|\partial(Pv)_{\leq k}\|_2\end{aligned}$$

This proves (4.4.17). For the other two regions, the proof is similar. □

**Corollary 4.4.4.** *Let $\phi$ solve (7.1.2), let $\mathcal{R} \in \{C_T^R, C_T^U, C_R^T\}$ and assume the hypotheses of Lemma 4.4.1 (that is, assume the hypotheses on the dyadic parameters for $\mathcal{R}$ and on the exponent parameters of the coefficients of $P$). Then*

$$\|\partial\phi_{\leq m}\|_{L^\infty(\mathcal{R})} \lesssim \|\mu^{-1}\phi_{\leq m+n}\|_{L^\infty(\tilde{\mathcal{R}})} \quad \mu := \langle \min(r, |t-r|)\rangle. \tag{4.4.18}$$

*Proof.* This is an immediate consequence of Corollary 4.4.3 because $P\phi = 0$. □

### 4.5 Setup for pointwise decay iteration

**Lemma 4.5.1** (Maximal vertical length within $D_{tr} \cap \{(\rho, s) \in \mathbb{R}^2_+ : \rho \le s\}$). *Uniformly in the set of $r, t$ values lying in $\{(r,t) : 0 \le r \le t\}$, we have that for any point $(\rho', s') \in D_{tr} \subset \mathbb{R}^+_{\rho'} \times \mathbb{R}^+_{s'}$,*

1. *If $r \le t/3$, then*

$$|D_{tr} \cap \{(\rho', s') : \rho = \rho'\}| \le \min\{2\rho, 2r\}$$

2. *If $t \ge r \ge t/3$, then*

$$|\{s' \ge \rho' \ge 0\} \cap D_{tr} \cap \{(\rho', s') : \rho = \rho'\}| \le t - r$$

*where $|\cdot|$ denotes the length.*

*Proof.* We split the proof into two cases.

1. Let $r \le t/3$; then for each $\rho$, the maximal vertical length within $D_{tr}$ is $2r$ and occurs when $r \le \rho \le \frac{t-r}{2}$; by symmetry, this length, $2r$, is maximal. When $0 \le \rho \le r$, the maximal vertical length of $D_{tr}$ is $2\rho$, which implies that this value of this length is sharp if and only if $0 \le \rho \le r$.

2. Let $r \ge t/3$; then for each $\rho$, the maximal vertical length within $D_{tr} \cap \{s \ge \rho\}$ is $t - r$ and occurs when $\frac{t-r}{2} \le \rho \le r$ and by symmetry once more, this length, $t - r$, is maximal. Furthermore, in a manner precisely analogous to the $r \le t/3$

case, we once more have that when $0 \le \rho \le \frac{t-r}{2}$, the bound $2\rho$ is sharp if and only if $\rho$ lies in this small region.

$\square$

**Definition 4.5.2.** Given $\lambda \in \mathbb{R}$,

$$\kappa(\eta, t-r) := \begin{cases} 1 & \eta > 1 \\ \log\langle t-r\rangle & \eta = 1 \\ \langle t-r\rangle^{1-\eta} & \eta < 1 \end{cases}.$$

In this chapter, this function arises either as

$$\sum_{1 \le R \lesssim \langle t-r\rangle} \frac{1}{R^{\eta-1}} \quad \text{or} \quad \int_0^{t-r} \frac{1}{\langle \upsilon\rangle^{\eta}} d\upsilon.$$

The following lemma allows us to convert pointwise decay rates of inhomogeneities $g$ in $\Box\psi = g$ to pointwise decay rates for $\psi$. A simple case of Lemma 7.4.4 is explained in Remark 4.5.4.

**Lemma 4.5.3.** *Let $m \ge 0$ be an integer and suppose that $\psi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ solves*

$$\Box\psi(t, x) = g(t, x), \qquad (t > 0,\ x \in \mathbb{R}^3)$$

*with vanishing initial data, with*

$$g \lesssim \frac{\log^m\langle t-r\rangle}{\langle r\rangle^{\alpha}\langle t\rangle^{\beta}\langle t-r\rangle^{\eta}},$$

*where the values of $\alpha, \beta, \eta$ will be specified below.*

- *(The case $r \le t$) Assume that $\beta \ge 0$ and $\eta \in \mathbb{R}$. Assume also that $|x| \le t$.*

*If* $1 < \alpha < 3$, *then*

$$\frac{\langle r\rangle\psi(t,x)}{\log^m\langle t-r\rangle} \lesssim \min\left(\frac{\kappa(\alpha-1,t-r)}{\langle t-r\rangle^{\beta+\eta-1}}, \frac{1}{\langle t-r\rangle^{\beta+\eta+\alpha-3}}\right) + \frac{\kappa(\eta,t-r)}{\langle t-r\rangle^{\alpha+\beta-2}} \quad (4.5.1)$$

*If* $\alpha > 3$ *(we will not be needing the cases* $\alpha = 3$ *or* $\alpha \leq 1$*),*

$$\frac{\langle r\rangle\psi(t,x)}{\log^m\langle t-r\rangle} \lesssim \frac{\kappa(\alpha-1,t-r)}{\langle t-r\rangle^{\beta+\eta-1}} + \frac{\kappa(\eta,t-r)}{\langle t-r\rangle^{\alpha+\beta-2}} \quad (4.5.2)$$

- *(The case* $r > t+1$*: in this chapter we will only need* $\alpha > 1, \eta \neq 1$*; a full explanation is given in the proof of Proposition 4.6.2) Let* $\alpha > 1, \eta \in \mathbb{R}$*. Suppose that* $r > t+1$*, and*

$$g \lesssim \frac{1}{\langle r\rangle^\alpha\langle t-r\rangle^\eta}.$$

*Then*

$$\langle r\rangle\psi \lesssim \frac{1}{\langle t-r\rangle^{\alpha-2}}\begin{cases} 1/\langle t-r\rangle^{\eta-1} & \eta > 1 \\ 1/\langle r+t\rangle^{\eta-1} & \eta < 1 \end{cases} \quad (4.5.3)$$

*Proof.* 1. (The case $r \leq t$) We write

$$\int_{D_{tr}} \rho \sup_{S^2} |\Box\psi| ds d\rho = \sum_{\mathcal{R}_1} \int_{D_{tr}^R} \rho \sup_{S^2} |\Box\psi| ds d\rho + \sum_{\mathcal{R}_2} \int_{D_{tr}^R} \rho \sup_{S^2} |\Box\psi| ds d\rho$$

and bound $\Box\psi$ pointwise by the bound in the hypotheses. Throughout $D_{tr}$, we have

$$\frac{1}{s} \lesssim \frac{1}{t-r}$$

and we will use this repeatedly below.

We begin with the first bound in (4.5.1), namely,

$$\sum_{\mathcal{R}_1} \int_{D_{tr}^R} \rho \sup_{S^2} |\Box\psi| ds d\rho \lesssim \frac{\kappa(\alpha-1,t-r)}{\langle t-r\rangle^{\beta+\eta-1}}.$$

In the region $\mathcal{R}_1 := \{1 \leq R < \frac{t-r}{8}\}$ defined at the beginning of this section, we have

$$s - \rho \sim t - r.$$

Therefore, for $R \in \mathcal{R}_1$ and any $\beta \geq 0, \eta \in \mathbb{R}$,

$$\begin{aligned}
\int_{D_{tr}^R} \rho \sup_{S^2} |\Box\psi| ds d\rho &\lesssim \int_{D_{tr}^R} |\frac{\log^m \langle v \rangle}{\langle \rho \rangle^{\alpha-1} \langle s \rangle^\beta \langle v \rangle^\eta}| ds d\rho \\
&\lesssim \frac{\log^m \langle t-r \rangle}{R^{\alpha-1} \langle t-r \rangle^\eta} \int \frac{ds d\rho}{\langle s \rangle^\beta} \\
&\sim \frac{\log^m \langle t-r \rangle}{R^{\alpha-2} \langle t-r \rangle^\eta} \int \frac{ds}{\langle s \rangle^\beta} \\
&\lesssim \frac{\log^m \langle t-r \rangle}{R^{\alpha-2} \langle t-r \rangle^\eta} \frac{1}{\langle t-r \rangle^{\beta-1}} \\
&= \frac{\log^m \langle t-r \rangle}{R^{\alpha-2} \langle t-r \rangle^{\beta+\eta-1}}
\end{aligned}$$

where $v := s - \rho$. Thus

$$\sum_{1 \leq R < \frac{t-r}{8}} \frac{\log^m \langle t-r \rangle}{R^{\alpha-2} \langle t-r \rangle^{\beta+\eta-1}} = \frac{\log^m \langle t-r \rangle}{\langle t-r \rangle^{\beta+\eta-1}} \kappa(\alpha-1, t-r).$$

Next, we prove that when $\alpha < 3$, and $\beta \geq 0, \eta \in \mathbb{R}$, we have

$$\sum_{\mathcal{R}_1} \int_{D_{tr}^R} \rho |\Box\psi| \, ds d\rho \lesssim \frac{\log^m \langle t-r \rangle}{\langle t-r \rangle^{\beta+\eta+\alpha-3}}.$$

This is shown as follows: since $\beta \geq 0$, we have $\langle s \rangle^{-\beta} \lesssim \langle t-r \rangle^{-\beta}$, and

$$\begin{aligned}
\log^{-m} \langle t-r \rangle \sum_{\mathcal{R}_1} \int_{D_{tr}^R} \rho |\Box\psi| ds d\rho &\lesssim \langle t-r \rangle^{-\beta-\eta} \sum_{\mathcal{R}_1} R^{1-\alpha} \int\int ds d\rho \\
&\lesssim \langle t-r \rangle^{-\beta-\eta} \sum_{\mathcal{R}_1} R^{3-\alpha} \\
&\lesssim \langle t-r \rangle^{-\beta-\eta+3-\alpha}
\end{aligned}$$

where the last line follows by the hypothesis $\alpha < 3$.

Finally, we show that when $\alpha > 1$ and $\beta \geq 0$, then

$$\int_{\bigcup_{R\in\mathcal{R}_2} D_{tr}^R} \rho|\Box\psi| ds d\rho \lesssim \log^m\langle t-r\rangle \frac{\kappa(\eta, t-r)}{\langle t-r\rangle^{\alpha+\beta-2}}$$

which will complete the proof. For $R \in \mathcal{R}_2$, we employ the fact that when $\beta \geq 0$ we have

$$\langle\rho\rangle^{-\beta} \lesssim \langle t-r\rangle^{-\beta}$$

to find that

$$\begin{aligned}\log^{-m}\langle t-r\rangle \int_{D_{tr}^R} \rho|\Box\psi| ds d\rho &\lesssim \langle t-r\rangle^{1-\alpha} \int_{D_{tr}^R} \langle s\rangle^{-\beta}\langle \upsilon\rangle^{-\eta} ds d\rho \\ &\lesssim \langle t-r\rangle^{1-\alpha}\langle t-r\rangle^{-\beta} \int ds \int_0^{t-r} \langle \upsilon\rangle^{-\eta} d\upsilon \\ &\lesssim \frac{1}{\langle t-r\rangle^{\beta+\alpha-2}} \kappa(\eta, t-r)\end{aligned}$$

with the last line following by Lemma 4.5.1.

2. (The case $r > t$) We now prove (4.5.3). Assume that $\alpha > 1$. A straightforward integration shows that

$$\int_{D_{tr}} \rho \frac{ds d\rho}{\langle\rho\rangle^{\alpha}\langle s-\rho\rangle^{\eta}} \lesssim \frac{1}{\langle t-r\rangle^{\alpha-2}} \begin{cases} \frac{1}{\langle t-r\rangle^{\eta-1}} & \eta > 1 \\ \ln \frac{\langle r+t\rangle}{\langle t-r\rangle} & \eta = 1 \\ \frac{1}{\langle r+t\rangle^{\eta-1}} & \eta < 1 \end{cases}$$

which shows (4.5.3).

□

In Remark 4.5.4 we state a simple version of Lemma 7.4.4. A particularly relevant case for the present article will be the case $2 < \alpha < 3$, which we cover now.

*Remark* 4.5.4 (Simple special case of Lemma 7.4.4). Let us set $m = 0$, which is the case that there are no logarithms involved. The $r \leq t$ claim (4.5.1) states in the special case $2 < \alpha < 3$ (that is, the inhomogeneity has $\langle r \rangle^{-\alpha}$ decay where $2 < \alpha < 3$) that

$$r\psi \lesssim \langle u \rangle^{-(\alpha+\beta+\eta-3)} + \langle u \rangle^{-(\alpha+\beta-1+\tilde{\eta})}$$

where

$$\tilde{\eta} := \begin{cases} -1 & \eta > 1 \\ \eta - 2 & \eta < 1. \end{cases}$$

This simplifies to

$$r\psi \lesssim \langle u \rangle^{-(\alpha+\beta-1+\tilde{\eta})}.$$

Consider the problem

$$(\Box + V(t,x))\psi = 0$$

with

$$V \in S^Z(\langle r \rangle^{-2-\delta}).$$

This is (7.1.2) but with only a potential.We have $\alpha = 2 + \delta$, so we are in the special case stated in this remark. Suppose also that we use (7.3.7) as an initial bound on vector fields of $\phi$. Thus $\eta = -1/2, \beta = 1$. Accordingly,

$$\begin{aligned} r\psi &\lesssim \langle u \rangle^{-(\alpha+\beta-1+\tilde{\eta})} \\ &= \langle u \rangle^{-(\alpha+\beta-1+(\eta-2))} \\ &\lesssim \langle u \rangle^{-((2+\delta)+(1)+(-1/2)-3)} \\ &= \langle u \rangle^{-(\delta-1/2)} = \langle u \rangle^{1/2-\delta}. \end{aligned}$$

Thus at least in the region $\{r \geq t/2\}$, our pointwise bound on $\psi$ has gained a factor of $\langle u\rangle^{-\delta}$.

The strategy of the proof of this article is then to propagate this improved bound into the region $\{r < t/2\}$ using Theorem 4.7.4, which we explain in Section 4.7. In this way, we obtain the bound $\langle t+r\rangle\psi \lesssim \langle t-r\rangle^{1/2-\delta}$. We then apply Lemma 7.4.4 again and iterate in this fashion. Eventually, we reach the final pointwise bound for the solution $\phi$ (and for $\phi_{\leq m}$) stated in the main theorem.

**Definition 4.5.5** (Cutoff functions)**.** Let

$$\chi_{\text{exte}}(t,x)$$

denote a smooth radial cutoff function adapted to $\{r \geq t, r-t \sim r\}$. Let

$$\chi_{\text{inte}}(t,x)$$

denote a smooth radial cutoff function adapted to $\{r \leq t, t-r \sim t\}$. Let

$$\chi^{\text{cone}}(t,x)$$

be a smooth radial cutoff function equalling $1-(\chi_{\text{inte}}+\chi_{\text{exte}})$. We also assume $\operatorname{supp}\chi^{\text{cone}} \subset \{r/2 \leq t \leq 3t/2\}$. Thus: in $C_T$, for instance, $\chi_{\text{inte}}$ and $\chi^{\text{cone}}$ sum to 1, while in $([T,2T]\times\mathbb{R}^3)\setminus C_T$, $\chi_{\text{exte}}$ and $\chi^{\text{cone}}$ sum to 1.

In the following sections, we shall finish the proof of Theorem 7.1.1; by the product rule, and also (7.4.2), it will suffice to prove pointwise decay for

$$\Box\phi_{\leq m} = (\tilde{V} + \partial\tilde{B})\phi_{\leq m} + \partial([\tilde{A} + \tilde{B}]\phi_{\leq m}) + \partial(\tilde{h}\partial\phi_{\leq m}) + \tilde{g^{\omega}}\partial^2\phi_{\leq m},$$
$$(\phi_{\leq m}(0), \bar{N}\phi_{\leq m}(0)) = ((\phi_0)_{\leq m}, (\phi_1)_{\leq m-1}) \tag{4.5.4}$$

where ˜ denotes vector fields.

Before commencing the pointwise decay iteration in the next section, we note that:

- By (7.3.7), the desired decay rate in Theorem 7.1.1 already holds in the region $\{|u| \leq 1\}$. Henceforth in this article, we shall assume that $|u| > 1$, i.e., $|t-r| > 1$. Thus we work away from the light cone $\{r = t\}$.
- Due to the domain of dependence properties of the wave equation, we shall first complete the iteration in $\{r > t+1\}$, which is the content of Section 4.6. For the iteration in $\{r < t-1\}$, the decay rates obtained from the fundamental solution are insufficient in the region $\{r < t/2\}$. To remedy this, we prove Theorem 4.7.4. With the new decay rates obtained from Theorem 4.7.4, we are then able to obtain new decay rates for the solution and its vector fields. At every step of the iteration, Lemma 7.4.4 is used to turn the decay gained at previous steps into new decay rates.

## 4.6 The upper bound in $\{r > t + 1\}$

Before embarking on the pointwise decay iteration for the equation in (7.1.2), we explain how we deal with the initial data in (7.1.2)—see the following remark.

*Remark* 4.6.1 (The initial data). Let

$$w := S(t,0)(\phi_0,\phi_1)$$

denote the solution to the free wave equation at time $t$ with initial data $(\phi_0,\phi_1)$ at time 0. Thus

$$w_J(t,x) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} (\phi_0)_J(y) + \nabla_y(\phi_0)_J(y)\cdot(y-x) + t(\phi_1)_J(y)\,dS(y).$$

Let

$$\alpha - 1 \in \{1+\min(\sigma,\delta,\delta'), 1+\min(\sigma+1,\delta,\delta')\}$$

with the first (resp. second) number in the set as the value of $\alpha-1$ assuming hypotheses from part 1 (resp. part 2) of Theorem 7.1.1. For any multiindex $J$, we now show that

$$w_J \lesssim \langle v\rangle^{-1}\langle u\rangle^{-\alpha+1}$$

by the Kirchhoff formula and the weighted $L^2$ decay assumption on the initial data. We use Cauchy-Schwarz and Sobolev embedding to control the free wave pointwise by the weighted $L^2$ bound assumed on the initial data. When $r \gg t$ and $y \in \partial B(x,t)$,

$$|(\phi_0)_J(y)| + |\nabla(\phi_0)_J(y)\cdot(y-x)| + |t(\phi_1)_J(y)| \lesssim \langle r\rangle^{-\alpha}$$

so that

$$w_J \lesssim \langle r\rangle^{-\alpha} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-(\alpha-1)}.$$

Similarly, when $r \ll t$ and $y \in \partial B(x,t)$,

$$|(\phi_0)_J(y)| + |\nabla(\phi_0)_J(y)\cdot(y-x)| + |t(\phi_1)_J(y)| \lesssim \langle t\rangle^{-\alpha}$$

so that

$$w_J \lesssim \langle t\rangle^{-\alpha} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-(\alpha-1)}.$$

When $r \sim t$, we have

$$w_J \lesssim \langle v\rangle^{-1}.$$

Recalling (4.5.4), in this section we prove that the solution to

$$\Box w_{(m)} = O(\langle r\rangle^{-2-\min(\delta',\delta,\sigma+1)})\phi_{\leq m+n} \tag{4.6.1}$$

obeys the maximal decay rate

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^{1+\min(1+\sigma,\delta,\delta')}}$$

in $\{r > t+1\}$ assuming that

$$B \in S^Z(\langle r\rangle^{-2-\sigma}), \partial_t B \in S^Z(\langle r\rangle^{-3-\sigma}).$$

We used the results from Section 4.4 in transitioning from (4.5.4) to (4.6.1).

If

$$B \in S^Z(\langle r\rangle^{-1-\sigma}), \partial_t B \in S^Z(\langle r\rangle^{-2-\sigma})$$

then we instead have

$$\Box w_{(m)} = O(\langle r\rangle^{-2-\min(\delta',\delta,\sigma)})\phi_{\leq m+n}$$

and the final bound

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^{1+\min(\sigma,\delta,\delta')}};$$

the argument shown below proving Proposition 4.6.2 covers this case equally well. For the sake of simplicity and concreteness, we pick and fix the assumption (4.6.1).

(4.6.1) includes all the terms in (4.5.4) except for the parts of the right-hand side of (4.5.4) that are supported near the cone; we prove estimates for those parts in Section 4.9.

**Proposition 4.6.2.** *Assume that $r > t+1$. Assuming the hypotheses of part 2 of Theorem 7.1.1,*

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle \langle u\rangle^{1+\min(1+\sigma,\delta,\delta')}}.$$

*Assuming the hypotheses of part 1 of Theorem 7.1.1,*

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle \langle u\rangle^{1+\min(\sigma,\delta,\delta')}}.$$

*Proof.* We only prove the case assuming the hypotheses of part 2; thus

$$\Box w_{(m)} = O(\langle r\rangle^{-2-\min(\delta',\delta,\sigma+1)})\phi_{\leq m+n}$$

since the other case is similar.

Given

$$\Box w_{(m)} = \bar{G}\phi_{\leq m+n}$$

with

$$\bar{G} = O(1/\langle r\rangle^{1+\beta})$$

(here, $\beta = 1+\min(\sigma+1,\delta,\delta')$), the first step is to use (7.3.7) and Lemma 7.4.4, which yields

$$w_{(m)} \lesssim \frac{1}{\langle v\rangle^{-1/2}\langle u\rangle^{\beta}}. \tag{4.6.2}$$

Then, a second application of Lemma 7.4.4 yields

$$w_{(m)} \lesssim \begin{cases} \langle v\rangle^{-1}\langle u\rangle^{-(2\beta-5/2)} & \beta-\frac{1}{2}>1 \text{ i.e. } \min(\delta',\delta,\sigma+1)>\frac{1}{2} \\ \langle v\rangle^{1/2-\beta}\langle u\rangle^{1-\beta} & \beta-\frac{1}{2}<1 \text{ i.e. } \min(\delta',\delta,\sigma+1)<\frac{1}{2} \end{cases}.$$

Note that the sum of exponents in the denominator, call it $i_n$ if we are at step $n$, has increased by $\min(\sigma+1,\delta,\delta')$.

The case $\eta=1$ in Lemma 7.4.4, whenever $r>t$, arises if $n\min(\delta,\delta')=1$ for some integer $n\geq 1$, but in this case we incur an arbitrarily small polynomial loss in $\langle t-r\rangle$; so we avoid having to apply Lemma 7.4.4 for the case when $\eta$ takes the value 1. For $r>t+1$, given a certain fixed value of $i_n$, we always have $i_{n+1}-i_n=\min(\sigma+1,\delta,\delta')$ or $i_{n+1}-i_n=\min(\sigma+1,\delta,\delta')-$, with the latter occurring if there is a borderline case which leads to an arbitrarily small loss.

Let $a:=\min(1+\sigma,\delta,\delta')$. The general pattern after the first iterate (4.6.2) is as follows. Suppose that $a=\min(\delta,\delta')<1/2$ (the case $a=1/2$ is similar because we just incur an arbitrarily small polynomial loss). For some integer $N\geq 1$, one has either

$$\text{(A) } w_{(m)} \lesssim \langle r\rangle^{-\frac{1}{2}}\langle r\rangle^{-N\tilde{a}}\langle u\rangle^{-N\tilde{a}} \qquad \text{or} \qquad \text{(B) } w_{(m)} \lesssim \langle u\rangle^{-\frac{1}{2}}\langle r\rangle^{-N\tilde{a}}\langle u\rangle^{-(N+1)\tilde{a}} \tag{4.6.3}$$

for some $\tilde{a}\in(0,a]$ which we may (and do) choose to be arbitrarily close to $a$ in the event of a borderline case, whereas $\tilde{a}=a$ if and only if we are in a non-borderline case.

The pattern will cycle between these two, starting at (A) for an integer value $N$, going to (B) for that same integer $N$, and then going to (A) for the integer value

$N+1$, then to (B) for the integer $N+1$, and so on.

There are correspondingly two kinds of integrals, as follows: writing $N$ now for the previous value of $N+1$, so that we work with $a$ instead of $\tilde{a}$, by Lemma 7.4.4 the function $rw_{(m)}$ is bounded by one of

$$\min\left(\frac{1}{\langle u\rangle^{\frac{1}{2}+(N+1)a}}, \frac{1}{\langle u\rangle^{1+a}}\right)\cdot\begin{cases}\langle u\rangle^{-(Na-1)} & \text{if } Na>1\\ \langle r\rangle^{-(Na-1)} & \text{if } Na<1\end{cases}$$

$$\frac{1}{\langle u\rangle^{(N+1)a}}\begin{cases}\langle u\rangle^{-((N+1)a-\frac{1}{2})} & \text{if } \frac{1}{2}+(N+1)a>1\\ \langle r\rangle^{-((N+1)a-\frac{1}{2})} & \text{if } \frac{1}{2}+(N+1)a<1\end{cases}$$

This iteration continues until $Na>1$. Then respectively

$$w_{(m)}\lesssim\frac{1}{\langle r\rangle\langle t-r\rangle^{-\frac{1}{2}+(2N+1)a}},\qquad w_{(m)}\lesssim\frac{1}{\langle r\rangle\langle t-r\rangle^{-\frac{1}{2}+2(N+1)a}}.$$

For the minimal integer $N$ satisfying $Na>1$, by Lemma 7.4.4 we have

$$\langle r\rangle w_{(m)}\lesssim 1/\langle t-r\rangle^{1+a-\frac{3}{2}+(2N+1)a}\le 1/\langle t-r\rangle^{1+a},$$

$$\langle r\rangle w_{(m)}\lesssim 1/\langle t-r\rangle^{1+a-\frac{3}{2}+2(N+1)a}\le 1/\langle t-r\rangle^{1+a}.$$

Suppose that $a>1/2$. After one iteration, by Lemma 7.4.4,

$$w_{(m)}\lesssim\langle v\rangle^{3/2}/\langle u\rangle^{1+\min(1+\sigma,\delta,\delta')}.$$

After the second iteration,

$$w_{(m)}\lesssim\frac{1}{\langle r\rangle\langle u\rangle^{-\frac{1}{2}+2a}}.$$

In the third iteration, one obtains $1/\langle u\rangle^{1+\min(1+\sigma,\delta,\delta')}$ from the $\rho$ integration alone, and for $a$ which is big enough ($a > 3/4$, more precisely), the iteration halts here. For $1/2 < a \leq 3/4$, continuing as many times as necessary, one eventually gets

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1+\min(1+\sigma,\delta,\delta')}}.$$

□

## 4.7 Converting $r$ decay to $t + r$ decay

In this section, we convert radial decay $1/\langle r\rangle^p, p \leq 1$ in $\{r < t/2\}$ to $1/\langle v\rangle^p$. In particular, the fundamental solution to the wave equation gives a $1/\langle r\rangle$ decay rate, which we can now convert to $1/\langle v\rangle$. This holds for both $\phi$ and vector fields of $\phi$.

**Lemma 4.7.1.** *Assume that $\phi$ satisfies the stationary LED. We have*

$$\|\phi_{\leq m}\|_{L^\infty(C_T^{<3T/4})} \lesssim \frac{1}{T^{3/2}}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1(\tilde{C}_T^{<3T/4})}.$$

*Proof.* This estimate will follow as a consequence of Corollary 4.3.2 and from proving that

$$\|\phi_{\leq m}\|_{LE^1(\tilde{C}_T^{<3T/4})} \lesssim \frac{1}{T}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1(\tilde{C}_T^{<3T/4})}. \tag{4.7.1}$$

The statement (4.7.1) hints at the fact that we will transfer (a limited amount of) $\langle r\rangle$ decay into $\langle v\rangle$ decay in the $LE^1$ local energy norm. From the $LE^1$ norm, we can recover pointwise bounds simply by explicit computation.

*Remark* 4.7.2 (It suffices to look at $\phi$ supported in $C_T^{<3T/4}$). In this proof we can assume that $\phi$ is supported in $C_T^{<3T/4}$ because we can control the commutator $[P, \chi_{C_T^{<3T/4}}]$ adequately where $\chi_{C_T^{<3T/4}}$ is a cutoff function adapted to the region $C_T^{<3T/4}$. Henceforth, we will assume that $\phi$ is supported in $C_T^{<3T/4}$, although support in $\langle r\rangle \leq \lambda T$ for any fixed $\lambda > 0$ would also be fine.

Let $m \geq 0$. Let $\gamma_{(T,x)}(t')$ denote an integral curve of $S$, parametrized by unit speed, such that $t' = 0$ corresponds to time $t = T$ and spatial position $x$. That is, it corresponds to the point $(T, x)$. By the fundamental theorem of calculus and Cauchy-Schwarz, we have

$$|\nabla_{t,x}\phi_{\leq m+1}(T,x)|^2 \lesssim \frac{1}{T}\int_0^T |(\nabla_{t,x}\phi_{\leq m})(\gamma_{(T,x)}(t'))|^2 + |(S\nabla_{t,x}\phi_{\leq m})(\gamma_{(T,x)}(t'))|^2\, dt'. \tag{4.7.2}$$

(This bound clearly works for any smooth-enough function other than $\phi_{\leq m}$ as well.) Next, integrating (7.7.6) on $\{x : r < \lambda t\}$ for some $\lambda > 0$, say, $\{x : r \leq t\}$,

$$\int_{C_T\cap\{t=T\}} |\nabla_{t,x}\phi_{\leq m}(T,x)|^2\, dx \lesssim \frac{1}{T}\iint_{C_T} |\nabla_{t,x}\phi_{\leq m}|^2 + |S\nabla_{t,x}\phi_{\leq m}|^2\, dxdt$$

A similar bound holds for $t = 2T$, where we now average over $[0, T]$ again but this time over the integral curves of $-S$, using $\gamma_{(2T,x)}(t')$ as the argument for the function $\nabla_{t,x}\phi_{\leq m}$, with $t' = 0$ corresponding to time $t = 2T$. Thus

$$|\nabla_{t,x}\phi_{\leq m}(2T,x)|^2 \lesssim \frac{1}{T}\int_0^T |(\nabla_{t,x}\phi_{\leq m})(\gamma_{(2T,x)}(t'))|^2 + |(S\nabla_{t,x}\phi_{\leq m})(\gamma_{(2T,x)}(t'))|^2\, dt'.$$

Then, integrating over $\{x : r \leq t\}$, we obtain the same upper boundas the $t = T$ case.

Hence by the solution $\phi$ satisfying Proposition 4.2.6 (stationary local energy decay for vector fields),

$$\begin{aligned}\|\phi_{\leq m}\|_{LE^1(\tilde{C}_T^{\leq 3T/4})} &\lesssim \|\nabla_{t,x}\phi_{\leq m}(T)\|_{L^2} + \|\partial_t\phi_{\leq m}\|_{LE} \\ &\lesssim \frac{1}{T^{1/2}}\|\nabla_{t,x}\phi_{\leq m+n}\|_{L^2(C_T)} + \|\partial_t\phi_{\leq m}\|_{LE}.\end{aligned} \tag{4.7.3}$$

Next, we bound $\|\nabla_{t,x}\phi_{\le m+n}\|_{L^2(C_T)}$ using Lemma 4.7.3. Intuitively, Lemma 4.7.3 "multiplies" or "boosts" all integrands in Proposition 4.2.6 by $\langle r\rangle^{1/2}$. A first naive thought that comes to mind is to multiply the equation by $r\partial_r\phi$ to achieve this boost. This works, if we add a zeroth-order correction term $\phi$ to the multiplier. Unlike the unweighted multiplier, this weighted multiplier leads to unsigned constant-time boundary terms, hence we put both energy terms in the majorizer. Lemma 4.7.3 adds new information beyond Proposition 4.2.6 only for sufficiently large values of $r$.

We will sometimes use the notation $C_{T_1}^{T_2} := [T_1, T_2] \times \{x : r \le t\}$.

**Lemma 4.7.3** ($\langle r\rangle^{1/2}$-weighted Stationary Local Energy Decay for Vector Fields)**.** *Suppose that the solution to $P\phi = f$ satisfies the stationary LED for vector fields, as proved in Proposition 4.2.6. For all $0 \le T_1 \le T_2$, we have*

$$\|\nabla_{t,x}\phi_{\le m}\|_{L^2(C_{T_1}^{T_2})} \lesssim \sum_{j=1}^{2} \|\langle r\rangle^{1/2}\nabla_{t,x}\phi_{\le m}(T_j)\|_{L^2} + \|\langle r\rangle f_{\le m}\|_{L^2[T_1,T_2]L^2} + \|\partial_t\phi_{\le m}\|_{L^2[T_1,T_2]L^2}. \tag{4.7.4}$$

*Proof.* We shall focus on proving (7.7.3), as the proof is very similar for the other estimate. We will take as assumptions those stated in part (1) of Theorem 7.1.1 and prove this result. This implies that this result also holds for part (2), because the assumptions in part (2) are stronger than those for part (1).

- (The zero multiindex case) We demonstrate the case $m = 0$ first for simplicity. In this proof we shall need $\sigma$ and $\delta$ to be strictly positive real numbers, as well as $\delta' > -1$, in contrast to the situation in Lemma 4.4.1.

We multiply $P\phi = f$ by $r\partial_r\phi + \phi$ and integrate by parts in $[T_1, T_2] \times \mathbb{R}^3$. There is a number $q' > 0$ such that

$$\begin{aligned}
&\int |\nabla_{t,x}\phi|^2 + O(\langle r\rangle^{-q'})|\nabla_{t,x}\phi|^2 + O(\langle r\rangle^{-1-q'})|\partial\!\!\!/\phi|^2 + O(\langle r\rangle^{-2-q'})|\phi|^2\,dxdt \\
&\qquad\lesssim \sum_{j=1}^{2}\int_{\mathbb{R}^3} O(\langle r\rangle)|\nabla_{t,x}\phi(T_j,x)|^2 + O(\langle r\rangle^{-1})|\phi(T_j,x)|^2\,dx + \int |rf\partial_r\phi| \\
&\qquad + |f\phi|\,dxdt \\
&\qquad\lesssim \sum_{j=1}^{2}\int_{\mathbb{R}^3} O(\langle r\rangle)|\nabla_{t,x}\phi(T_j,x)|^2\,dx + \int |rf\partial_r\phi| + |f\phi|\,dxdt
\end{aligned} \tag{4.7.5}$$

with the last statement following by a version of Hardy's inequality.

For instance, with the term $\partial_\mu A^\mu$,

$$\partial_\mu(A^\mu\phi)(r\partial_r\phi) \lesssim O(\langle r\rangle^{-\sigma})|\nabla_{t,x}\phi|^2 + O(\langle r\rangle^{-2-\sigma})\phi^2.$$

This follows by combining the assumptions on $A, \partial A$ and $\partial_t A$ as stated in part (1) of Theorem 7.1.1. Only an arbitrarily small $\sigma > 0$ is needed.

Next,

$$\iint |f||r\partial_r\phi| = \iint |fr|\,|\partial_r\phi| \leq \|rf\|_{L^2L^2}\|\partial_r\phi\|_{L^2L^2} \leq \frac{1}{\epsilon}\|rf\|^2_{L^2L^2} + \epsilon\|\partial_r\phi\|^2_{L^2L^2}$$

and we then bring $\epsilon\|\partial_r\phi\|_{L^2L^2}$ onto the other side together with $\|\nabla_{t,x}\phi\|_{L^2L^2}$. Similarly

$$\iint |f||\phi| = \iint |rf|\,|\frac{\phi}{r}| \leq \frac{1}{\epsilon}\|rf\|^2_{L^2L^2} + \epsilon\|\partial_r\phi\|^2_{L^2L^2}.$$

We remark that it is possible to place $rf$ in $L^1L^2$ if we place $\partial_r\phi$ in $L^\infty L^2$ (we can use Hardy's inequality for the zero order term). This alternate route leads to $\|rf\|_{L^1L^2}$ instead of $\|rf\|^2_{L^2L^2}$ on the right-hand side.

For small $|x|$ values, our assumption of Proposition 4.2.4 implies the bound on

$$\|\nabla_{t,x}\phi\|_{L^2[T_1,T_2]L^2(r\lesssim 1)}.$$

By using the positivity of $q'$ on the left-hand side of (7.7.4) for large $|x|$ values, we can then obtain

$$\begin{aligned}\|\nabla_{t,x}\phi\|_{L^2[T_1,T_2]L^2} \lesssim & \sum_{j=1}^{2} \|\langle r\rangle^{1/2}\nabla_{t,x}\phi(T_j)\|_{L^2} \\ & + \min\left(\|\langle r\rangle f\|_{L^1[T_1,T_2]L^2}^{1/2}, \|\langle r\rangle f\|_{L^2[T_1,T_2]L^2}\right).\end{aligned} \tag{4.7.6}$$

(7.7.5) implies (7.7.3) for $m = 0$.

- (The higher multiindex case) Next, we prove (7.7.5) but for $\phi_J, J \neq \vec{0}$. By Lemma 4.2.2, we have

  $$P\phi_J = f_J + O(\langle r\rangle^{-1-q'})\nabla_{t,x}\phi_{\leq |J|} + O(\langle r\rangle^{-2-q'})\phi_{\leq |J|-1}.$$

  We multiply this by $r\partial_r\phi_J + \phi_J$. Then we integrate in $[T_1, T_2] \times \mathbb{R}^3$.

  – For small $r$, the estimate (7.7.3) is implied by the weak local energy decay estimate for vector fields proved in Proposition 4.2.4, so to prove the desired conclusion (7.7.3) it suffices to restrict attention to the case of large $r$.

  – For large $r$, owing to the positivity of $q' > 0$, we may use the triangle inequality, the triangle inequality for integrals, Cauchy-Schwarz, and Hardy's inequality to absorb the terms

  $$\iint (r\partial_r\phi_J + \phi_J)\Big(O(\langle r\rangle^{-1-q'})\nabla_{t,x}\phi_{\leq |J|} + O(\langle r\rangle^{-2-q'})\phi_{<|J|}\Big)$$

  into the left-hand side, namely into

  $$\|\nabla_{t,x}\phi_{\leq m}\|_{L^2([T_1,T_2]\times\{r\leq t\})}.$$

The positiveness of $q'$ provides the necessary smallness for the absorption. We have explained how to take care of the extra terms arising from commutators in the higher multiindex case, namely the terms $O(\langle r\rangle^{-1-q'})\nabla_{t,x}\phi_{\leq |J|} + O(\langle r\rangle^{-2-q'})\phi_{\leq |J|-1}$. For the remaining part of the equation, namely $P\phi_J = f_J +$ (taken care of), we can just apply precisely the same procedure used to prove (7.7.5) to $\phi_J$—that is, the first bullet point. Then we sum over $|J| \leq m$.

□

Applying Lemma 4.7.3 for $\phi$, we have

$$\|\nabla_{t,x}\phi_{\leq m}\|_{L^2(C_T)} \lesssim \sum_{i=1}^{2} \|\langle r\rangle^{1/2}\nabla_{t,x}\phi_{\leq m}(iT)\|_{L^2} + \|\partial_t\phi_{\leq m}\|_{L^2(C_T)}.$$

Thus so far, we have

$$\begin{aligned}\|\phi_{\leq m}\|_{LE^1(\tilde{C}_T^{<3T/4})} &\lesssim \frac{1}{T^{1/2}}\|\nabla_{t,x}\phi_{\leq m+n}\|_{L^2(C_T)} + \|\partial_t\phi_{\leq m+n}\|_{LE} \\ &\lesssim \frac{1}{T^{1/2}}\Big(\sum_{i=1}^{2} \|\langle r\rangle^{1/2}\nabla_{t,x}\phi_{\leq m+n}(iT)\|_{L^2} + \|\partial_t\phi_{\leq m+n}\|_{L^2(C_T)}\Big) \\ &\quad + \|\partial_t\phi_{\leq m+n}\|_{LE}\end{aligned} \tag{4.7.7}$$

Now we bound these weighted energy terms by $LE^1$ norms, picking up appropriate $T$ weights along the way.

By the fundamental theorem of calculus and Cauchy-Schwarz once more,

$$\int \langle r\rangle|\nabla_{t,x}\phi_{\leq m+n}(T)|^2\,dx \lesssim \frac{1}{T^{1/2}}\int \langle r\rangle^{1/2}|\nabla_{t,x}\phi_{\leq m+n}|^2 + \frac{1}{T}\langle r\rangle^{3/2}|S\nabla_{t,x}\phi_{\leq m+n}|^2\,dxdt.$$

To bound the second term, we note that by Remark 4.7.2, we assume that $\langle r\rangle \lesssim T$, which lets us bound

$$\frac{1}{T^{3/2}}\int \langle r\rangle^{3/2}|S\nabla_{t,x}\phi_{\le m+n}|^2\,dxdt \lesssim \frac{1}{T}\int \langle r\rangle |S\nabla_{t,x}\phi_{\le m+n}|^2 dxdt$$
$$\lesssim \frac{1}{T}\|\langle r\rangle\phi_{\le m+n+n'}\|^2_{LE^1(C_T)}.$$

for some $n'$.

To bound the first term, we treat it perturbatively for small $r$ values in the norm, and for a fixed finite number of large $R$ regions, where $r\sim R$, we can make this bound. Thus let us decompose

$$T^{-1/4}\|\langle r\rangle^{1/4}\nabla_{t,x}\phi_{\le m}\|_{L^2(C_T)} = \sum_R T^{-1/4}\|\langle r\rangle^{1/4}\nabla_{t,x}\phi_{\le m}\|_{L^2([T,2T]\times A_R)}.$$

When $R \ll T$ we absorb this term into the left hand side. For all values of $R$ with $R\sim T$, we are able to directly bound by $T^{-1/2}\|\langle r\rangle\phi_{\le m+n}\|_{LE^1(C_T)}$.

Finally, using the relation $\partial_t = t^{-1}(S - r\partial_r)$ and (4.4.18), which implies that $|r\partial_r\phi_{\le m}| \lesssim |\phi_{\le m+n}|$ in $\tilde{C}_T^{<3T/4}$,

$$\begin{aligned}\|\partial_t\phi_{\le m+n}\|_{L^2(C_T)} &\lesssim T^{-1}\sum_R \|\phi_{\le m+n}\|_{L^2(C_T)}\\ &= T^{-1}\sum_R R^{1/2}\|R^{-1/2}\phi_{\le m+n}\|_{L^2(A_R)}\\ &\le T^{-1/2}\sum_R (\frac{R}{T})^{1/2}\sup_R \|R^{-1/2}\phi_{\le m+n}\|_{L^2(A_R)}\\ &\lesssim T^{-1/2}\|\phi_{\le m+n}\|_{LE} \le T^{-1/2}\|\langle r\rangle\phi_{\le m+n}\|_{LE^1}\end{aligned}$$

Similarly,

$$\|\partial_t\phi_{\le m}\|_{LE} \lesssim T^{-1}\|\phi_{\le m+n}\|_{LE}$$

Collecting our estimates, from (4.7.7) we now conclude

$$\|\phi_{\leq m}\|_{LE^1(\tilde{C}_T^{<3T/4})} \lesssim T^{-1}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1}.$$

□

**Theorem 4.7.4.** *Let $\phi$ solve the main equation (6.1.1). If*

$$\phi_{\leq M} \lesssim \langle r\rangle^{-p}\langle t\rangle^{-q}\langle t-r\rangle^{-\eta} \tag{4.7.8}$$

*for some real $p \leq 1$, $q,\eta \in \mathbb{R}$ and a (sufficiently large) fixed $M \in \mathbb{N}$. then*

$$\phi \lesssim \langle t\rangle^{-p-q}\langle t-r\rangle^{-\eta}.$$

*Proof.* For all $(t,r)$ pairs with $r$ sufficiently large relative to $t$, say $r > t/2$, the conclusion follows since $\langle r\rangle \sim \langle t\rangle$.

For the other region, $C_T^{<3T/4}$, this follows from the proof of Lemma 4.7.5, because in $C_T^{<3T/4}$, $\langle t-r\rangle \sim \langle t\rangle$.

**Lemma 4.7.5.** *Let $\phi$ solve the main equation (6.1.1). If*

$$\phi_{\leq M} \lesssim r^{-p}\langle t\rangle^{-q} \tag{4.7.9}$$

*for some real $p,q \in \mathbb{R}$ and a (sufficiently large) fixed $M \in \mathbb{N}$ where $p \leq 1$, then*

$$\phi \lesssim \langle t\rangle^{-p-q}.$$

*Proof of Lemma 4.7.5.* We compute the norms involved on the right-hand side in Lemma 4.7.1. The rest of this proof works for not only $C_T^{<3T/4}$, which is the region we compute in, but actually in $[T,2T] \times \{r \leq \lambda t\}$ for any fixed $\lambda > 0$. The right-hand side norm of Lemma 4.7.1 is

$$\begin{aligned}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1(C_T^{<3T/4})} &= \|\nabla_{t,x}(\langle r\rangle\phi_{\leq m+n})\|_{LE(C_T^{<3T/4})} + \|\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})}\\ &\lesssim \|\langle r\rangle\nabla_{t,x}\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})} + \|\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})}\\ &\lesssim \|\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})}\end{aligned}$$

where the last line is a consequence of Corollary 4.4.3 applied uniformly across the collection $\{C_T^R : 1 \leq R < 3T/8\}$ of dyadic regions. Thus $\|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim \frac{1}{T}\|\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})}$.

Next, we bound $\|\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})}$ and finish the proof. We shall use pointwise bounds on $|\phi_{\leq m}|$, and not just on $|\phi|$, here:

- For $R > 1$ we have

$$\begin{aligned}&\sup_{1<R<3T/8}\left(\int_T^{2T}\int_R^{2R}\frac{1}{\langle r\rangle}\left(\langle t\rangle^{-2q}r^{-2p}\right)r^2drdt\right)^{1/2}\\ &\sim \sup\left(T^{-2q}\int_T^{2T}\int_R^{2R}\frac{1}{\langle r\rangle}\left(r^{-2p}\right)r^2drdt\right)^{1/2}\\ &\lesssim T^{1/2-q}\sup_{1<R<3T/8}\frac{1}{R^{p-1}} \lesssim T^{1/2-q}\frac{1}{T^{p-1}} \text{ since } p \leq 1.\end{aligned}$$

- For $R = 1$ we have

$$\left(\int_0^2 \frac{1}{\langle r\rangle^{2p-1}}dr\right)^{1/2} \lesssim_p 1$$

  for any $p \in \mathbb{R}$.

Thus

$$\begin{aligned}\frac{1}{T^{3/2}}\|\phi_{\leq m+n}\|_{LE(C_T^{<3T/4})} &\lesssim \frac{1}{T^{3/2}}\frac{1}{T^{\min(0,p-1)+q-1/2}}\\ &= \frac{1}{T^{\min(1,p)+q}}\end{aligned}$$

hence $\|\phi_{\leq m}\|_{L^\infty(C_T^{<3T/4})} \lesssim T^{-p-q}$ if $p \leq 1$. ☐

This establishes the proof of Theorem 4.7.4. ☐

**Corollary 4.7.6.** *Let $k \geq 1$ be an integer. If $\phi$ solves $P\phi = f$ and*

$$\phi_{\leq M} \lesssim \sum_{j=1}^{k} \langle r\rangle^{-p_j} \langle t\rangle^{-q_j} \langle t-r\rangle^{-\eta_j}$$

*and the conditions on the exponents $p_j, q_j, \eta_j$ and $M$ in Theorem 4.7.4 above are satisfied, then*

$$\phi \lesssim \sum_{j=1}^{k} \langle t\rangle^{-p_j-q_j} \langle t-r\rangle^{-\eta_j}.$$

*Proof.* The proof is a straightforward consequence of what has already been done. One can use elementary inequalities to handle sums instead of single summands in the computations above, and the estimates still hold. ☐

## 4.8 The upper bound in $\{r < t\}$

We consider (4.6.1) with $r < t$. We now show the desired final decay rate in Theorem 7.1.1, namely

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle \langle t-r\rangle^{1+\min(1+\sigma,\delta,\delta')}}.$$

**Proposition 4.8.1.** *Assume that $r < t$. Assuming the hypotheses of part 2 of Theorem 7.1.1,*

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle \langle t-r\rangle^{1+\min(1+\sigma,\delta,\delta')}}.$$

*Assuming the hypotheses of part 1 of Theorem 7.1.1,*

$$w_{(m)} \lesssim \frac{1}{\langle r\rangle \langle t-r\rangle^{1+\min(\sigma,\delta,\delta')}}.$$

*Proof.* In $\{\rho \geq s\}$, the argument has essentially been done in Section 4.6; when integrating in $\{\rho > s\}$, one plugs in the final pointwise decay rates for vector fields of $\phi$ obtained in Section 4.6, only to get the final pointwise decay rates as output.

In $\{\rho < s\}$, we let

$$\nu := \min(1+\sigma, \delta, \delta', 1-) = \min(\delta, \delta', 1-).$$

We need $\nu < 1$ because we will be using the fact that in $\mathcal{R}_1$, we have

$$\sum_{R \in \mathcal{R}_1} \int_{D_{tr}^R} \rho |\Box w_{(m)}| ds d\rho \lesssim \frac{1}{\langle t-r \rangle^{\beta+\eta+\alpha-3}}$$

(using the notation from Lemma 7.4.4). Note that $2+\nu < 3$, allowing us to apply Lemma 7.4.4's Region 1 bound $1/\langle t-r \rangle^{\beta+\eta+\alpha-3}$ if we put $\langle v \rangle$ as $\langle t \rangle$ in (4.8.1) when applying Lemma 7.4.4. For the rest of the proof, the strategy will be to improve by increments $\nu$ which are strictly less than 1. Below in the proof, we split into the cases where $\min(\delta, \delta')$ is either $< 1$ or $\geq 1$, but the main idea in either case is really the same, since in the latter case we simply introduce an artificial decrement $\tilde{\epsilon} \ll 1$ to make $\nu$, which equals $1 - \tilde{\epsilon}$ in that case, smaller than 1.

By Lemma 4.4.1, we have

$$\Box w_{(m)} \lesssim \langle r \rangle^{-2-\nu} \langle v \rangle^{-1} \langle t-r \rangle^{1/2}. \tag{4.8.1}$$

By Lemma 7.4.4,

$$\langle r \rangle w_{(m)} \lesssim \langle t-r \rangle^{1/2-\nu}.$$

We have gained $\langle t-r \rangle^{-\nu}$. Hence by Theorem 4.7.4,

$$\Box w_{(m)} \lesssim \langle r \rangle^{-2-\nu} \langle v \rangle^{-1} \langle t-r \rangle^{1/2-\nu},$$

and this process can be continued as long as the uppermost case thresholds in the definition of $\kappa$ are not met.

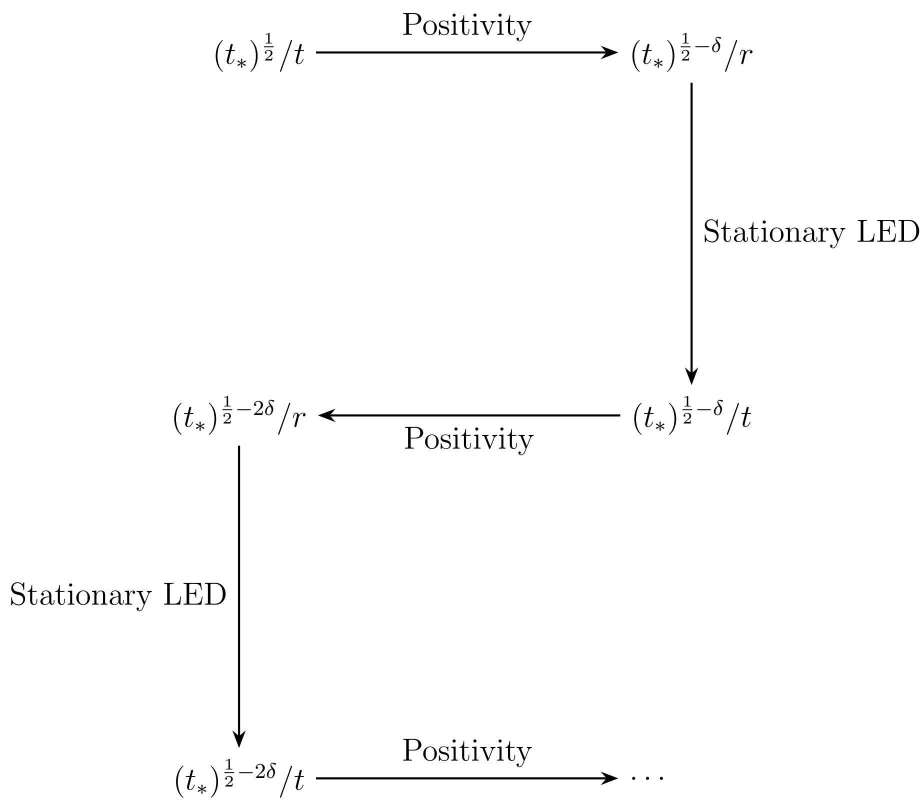


Figure 4.1: The first few steps of the proof of the upper bound in the $r < t$ region, where $t_* = t - r$. Here, $\delta > 0$ is some positive number, and "positivity" refers to the positivity of the fundamental solution in three space dimensions. Vertical arrows denote applications of the Stationary LED estimate, and horizontal arrows denote applications of the positivity property.

Suppose $n' > 0$ is an integer for which this threshold is not met; then after performing this procedure $n'$ times,

$$\Box w_{(m)} \lesssim \langle r\rangle^{-2-\nu}\langle t\rangle^{-1}\langle t-r\rangle^{1/2-n'\nu}.$$

Now we define $n'$ to be

$$n' := \max\{n \in \mathbb{N} : 1/2 + n\nu < 1\}.$$

There are two cases:

1. If

$$1 < 1/2 + (n' + 1)\nu < 1 + \nu$$

then write

$$1/2 + (n' + 1)\nu = 1 + \lambda\nu$$

where $0 < \lambda < 1$; thus

$$w_{(m)} \lesssim \langle r\rangle^{-1}\langle t - r\rangle^{-1-\lambda\nu}.$$

Then

$$\begin{aligned}\Box w_{(m)} &\lesssim \langle r\rangle^{-2-\nu}\langle v\rangle^{-1}\langle t - r\rangle^{-1-\lambda\nu} \\ &\leq \min\{\langle r\rangle^{-2-\nu}\langle v\rangle^{-1}\langle t - r\rangle^{-1-\lambda\nu}, \langle r\rangle^{-3-\nu}\langle t - r\rangle^{-1-\lambda\nu}\} \\ &=: \min\{a, b\}.\end{aligned}$$

Lemma 7.4.4 implies that in Region $\mathcal{R}_1$, we have the bound by

$$1/\langle t - r\rangle^{\beta+\eta+\alpha-3}.$$

We use $a$ to get the bound in $\mathcal{R}_1$ by

$$\langle t - r\rangle^{-1-\nu-\lambda\nu}.$$

On the other hand, we use $b$, with $\alpha = 3 + \nu$ and $\beta + \eta = 1 + \lambda\nu$ to get a Region $\mathcal{R}_2$ bound by

$$\langle t - r\rangle^{-1-\nu}.$$

Thus

$$\begin{aligned}\langle r\rangle w_{(m)} &\lesssim 1/\langle t-r\rangle^{\beta+\eta+\alpha-3} + \langle t-r\rangle^{-(\beta+\alpha-2)}\kappa(\eta, t-r)\\ &= \langle t-r\rangle^{-1-\nu-\lambda\nu} + \langle t-r\rangle^{-1-\nu}\\ &\lesssim \langle t-r\rangle^{-1-\nu}.\end{aligned}$$

2. If

$$1/2 + (n'+1)\nu = 1,$$

thus $w_{(m)} \lesssim \langle t\rangle^{-1}\langle t-r\rangle^{-1}$, we have

$$\Box w_{(m)} \lesssim \langle r\rangle^{-2-\nu}\langle t\rangle^{-1}\langle t-r\rangle^{-1}.$$

Hence

$$\begin{aligned}\langle r\rangle w_{(m)} &\lesssim \langle t-r\rangle^{-(\beta+\eta)}\kappa(\alpha-1, t-r) + \langle t-r\rangle^{-(\alpha+\beta-2)}\kappa(\eta, t-r)\\ &= \langle t-r\rangle^{-(\beta+\eta)} + \langle t-r\rangle^{-(\alpha+\beta-2)}\log\langle t-r\rangle\\ &= \langle t-r\rangle^{-2} + \langle t-r\rangle^{-1-\nu}\log\langle t-r\rangle\\ &\leq 2\langle t-r\rangle^{-1-\nu}\log\langle t-r\rangle\\ &\lesssim \langle t-r\rangle^{-1-\lambda\nu}\end{aligned}$$

for any $0 < \lambda < 1$, which now puts us in case (1).

The proof is complete when $\min(\delta, \delta') < 1$.

*Part two of the proof: The case where* $\min(\delta, \delta') \geq 1$*, that is, all three parameters* $\delta, \delta'$ *and* $1+\sigma$ *are at least 1.* Suppose that $\min(1+\sigma, \delta, \delta') \geq 1$. We shall still work with an increment $\nu$ that is less than 1. Rather than using "1−" in the definition of $\nu$, we write $\nu$ as the definite number $\nu := 1 - \tilde{\epsilon}$ where $\tilde{\epsilon} > 0$ is a small number. Then

$$\begin{aligned}\Box w_{(m)} &\lesssim \langle v\rangle^{-1}\langle t-r\rangle^{-1-\nu}\min\{\langle r\rangle^{-2-\nu}, \langle r\rangle^{-2-\min(1+\sigma,\delta,\delta')}\} \\ &= \langle v\rangle^{-1}\langle t-r\rangle^{-1-(1-\tilde{\epsilon})}\min\{\langle r\rangle^{-2-(1-\tilde{\epsilon})}, \langle r\rangle^{-2-\min(1+\sigma,\delta,\delta')}\}\end{aligned}$$

where we wrote down the trivial minimum of the two powers of $\langle r\rangle$ to emphasise the fact that we will be using $\alpha = 2 + \min(1+\sigma,\delta,\delta')$ for $\mathcal{R}_2$ but $\alpha = 2+\nu$ for $\mathcal{R}_1$. Thus by Lemma 7.4.4,

$$\begin{aligned}\langle r\rangle w_{(m)} &\lesssim \langle t-r\rangle^{-(\beta+\eta+\alpha-3)} + \frac{\kappa(\eta, t-r)}{\langle t-r\rangle^{\alpha+\beta-2}} \\ &\lesssim \langle t-r\rangle^{-(\beta+\eta+\alpha-3)} + \frac{1}{\langle t-r\rangle^{\alpha+\beta-2}} \\ &= \langle t-r\rangle^{-(\beta+\eta+\alpha-3)} + \frac{1}{\langle t-r\rangle^{(2+\min(1+\sigma,\delta,\delta'))+(1)-2}} \\ &= \langle t-r\rangle^{-1-2\nu} + \frac{1}{\langle t-r\rangle^{1+\min(1+\sigma,\delta,\delta')}}.\end{aligned}$$

It remains to prove the desired bound in Region $\mathcal{R}_1$, and it is safe to ignore the $\mathcal{R}_2$ portion of the bound henceforth because the $\beta$ and $\alpha$ exponent components of $\Box w_{(m)}$ remain stable while $\eta > 1$ will stay larger than 1, and in $\mathcal{R}_2$ we use the bound $\kappa(\eta, t-r)/\langle t-r\rangle^{\alpha+\beta-2}$. We note that no more improvement is possible in $\mathcal{R}_2$ using Lemma 7.4.4.

In $\mathcal{R}_1$, this iteration continues until

$$w_{(m)} \lesssim \langle r\rangle^{-1}\langle t-r\rangle^{-1-n''\nu}$$

where

$$n'' := \max\{n \in \mathbb{N} : n(1-\tilde{\epsilon}) < \min(1+\sigma,\delta,\delta')\},$$

e.g., $n'' = 1$ if the two numbers $\min(1+\sigma, \delta, \delta')$ and $1-\tilde{\epsilon}$ are both close to 1.

One way to view this situation is that there are two cases:

1. If

$$(n''+1)(1-\tilde{\epsilon}) > \min(1+\sigma, \delta, \delta')$$

then we obtain the bound $\langle t-r\rangle^{-1-(n''+1)\nu} = \langle t-r\rangle^{-1-(n''+1)(1-\tilde{\epsilon})}$ in $\mathcal{R}_1$ by using

$$\begin{aligned} \frac{1}{\langle t-r\rangle^{\beta+\eta+\alpha-3}} &= \frac{1}{\langle t-r\rangle^{1+(n''+1)\nu}} \\ &\leq \frac{1}{\langle t-r\rangle^{1+\min(1+\sigma,\delta,\delta')}}. \end{aligned}$$

2. If

$$(n''+1)(1-\tilde{\epsilon}) = \min(1+\sigma, \delta, \delta')$$

then we obtain the final display in item (1) but with equality rather than inequality, and we halt.

This completes the proof for $w_{(m)}$ when $r < t$. □

*Remark* 4.8.2 (Lockstep). If $\nu := \min(\sigma, \delta, \delta', 1-)$ then essentially an identical proof follows for proving $w_{(m)} \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^{1+\min(\sigma,\delta,\delta')}}$. The case partition is then (a) part one: $\min(\sigma, \delta, \delta') < 1$, (b) part two: $\min(\sigma, \delta, \delta') \geq 1$. Everything else follows when one replaces $1+\sigma$ in the appropriate locations in the proof above by $\sigma$.

### 4.9 Cone bounds

In this section we show how we prove the final decay rate in the main theorem for the terms involving the metric coefficients $h^{\alpha\beta}$ that are supported near the cone $\{r = t\}$.

Recall that we write $\tilde{B}\partial\phi_{\leq m} = \partial(\tilde{B}\phi_{\leq m}) - (\partial\tilde{B})\phi_{\leq m}$. Let $j = 0$ (respectively $j = 1$) correspond to the hypotheses of part 1 (respectively part 2) of Theorem 7.1.1. Near the cone, we rewrite (4.5.4) as

$$\Box\phi_{\leq m} = (|\tilde{V}| + |\partial\tilde{B}| + |\tilde{g^{\omega}}|)|\phi_{\leq m+n}| + \partial_t(\chi^{\text{cone}}(\tilde{h}\partial_t + \tilde{A} + \tilde{B})\phi_{\leq m}), \tag{4.9.1}$$

$$(\phi_{\leq m}(0), \bar{N}\phi_{\leq m}(0)) = (0, 0). \tag{4.9.2}$$

and use (7.4.2). Note that

$$\begin{aligned}\Big(|\tilde{V}| + |\partial\tilde{B}| + |\tilde{g^{\omega}}|\Big)\,|_{r>t/2} &\lesssim \Big(|\tilde{V}| + |\partial_t\tilde{B}| + \langle r\rangle^{-1}|\tilde{B}_{\leq n}| + |\tilde{g^{\omega}}|\Big)\,|_{r>t/2}\\ &= O(1/\langle r\rangle^{2+\min(\sigma+(j-1),\delta,\delta')})\end{aligned}$$

assuming the hypotheses of part $j$ of Theorem 7.1.1, $j = 1, 2$.

It suffices to prove pointwise decay estimates for

$$\Box v_{(m,1)} = \chi^{\text{cone}}\tilde{h}\partial_t\phi_{\leq m}, \qquad \Box v_{(m,2)} = \chi^{\text{cone}}(\tilde{A} + \tilde{B})\phi_{\leq m},$$

Let $\tilde{v} \in \{v_{(m,j)} : j = 1, 2\}$. We now prove

**Proposition 4.9.1.** *We have*

$$\partial_t\tilde{v} \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^{1+\min(\sigma,\delta,\delta')}}$$

*under the assumptions of part 1 of the main theorem.*

*Proof.* If $\chi := \chi^{\text{cone}}$ and $f \in \{\chi(\tilde{A} + \tilde{B})\phi_{\leq m}, \chi\tilde{h}\partial_t\phi_{\leq m}\}$, then by Corollary 4.4.3 and assumptions on $h$ and $A$ we have

$$|f(s,\rho)| + |Sf| + |\langle\rho - s\rangle\partial_\rho f| \lesssim \frac{1}{\langle\rho\rangle^{1+\sigma}}|\phi_{\leq m+n}|$$

The iteration for $\tilde{v}$ is as follows. Note that $\operatorname{supp}\chi \subset \{|s-\rho| \lesssim \langle t-r\rangle\}$. One simplifying observation is that $\rho \geq c\langle t-r\rangle$ in all $f$ cases with $c \geq 1/4$; in $r < t$, $\operatorname{supp}\chi^{\text{cone}}$ for instance, $\rho \geq |t-r|/4$, which has smallest $c$ value amongst all cases (for example, if $r > t$ then $c = 1$ and if $r < t$ then in $\operatorname{supp}\chi^{\text{cone}}$, $c = 1/2$). This and the fact that the horizontal (i.e. $\rho$) diameter of $\operatorname{supp}\chi$ is $O(\langle t-r\rangle)$ leads to simpler integrations in $\rho$.

We begin with the bound (7.3.7) for the functions $\phi_{\leq m+n}$. By Corollary 4.4.3 (used to handle terms that have the operator $\langle s-\rho\rangle\partial_\rho$ in the integration inside $D_{tr}$) and Lemma 7.4.4,

$$\partial_t \tilde{v} \lesssim \frac{\langle r-t\rangle^{1/2-\sigma}}{\langle r\rangle}.$$

Thus we run the iteration with exponent $a := \min(\sigma, \delta, \delta', 1-)$—see Remark 4.8.2. By Lemma 7.4.4, after $N$ steps one gets

$$\langle r\rangle \partial_t \tilde{v} \lesssim \frac{1}{\langle t-r\rangle^{1+\sigma}} \langle t-r\rangle^{3/2-\theta}$$

where $\theta = Na < 3/2$ is the gain at the $N$-th step of the lockstep. The procedure is similar to $w_{(m)}$'s case, and in the end we get

$$\partial_t \tilde{v} \lesssim \frac{1}{\langle r\rangle\langle t-r\rangle^{1+\min(\sigma,\delta.\delta')}}.$$

We have $\partial(\tilde{A}\phi_{\leq m}) + \tilde{B}\partial\phi_{\leq m} = \partial([\tilde{A}+\tilde{B}]\phi_{\leq m}) - (\partial\tilde{B})\phi_{\leq m}$. For $\bar{v}$ solving

$$\Box\bar{v} = \chi\tilde{A}\phi_{\leq m},$$

the bound on $\partial_t\bar{v}$ is just an argument that is an application of Lemma 7.4.4 similar to what has been done. We write $B\partial\phi = \partial(B\phi) - (\partial B)\phi$; then the arguments already shown (along with the assumptions on $B$ in Theorem 7.1.1) give the bound on $\partial_t W$ for $\Box W = \chi \tilde{B} \phi_{\leq m}$. This concludes the proof. $\Box$

**Proposition 4.9.2.** *We have*

$$\partial_t \tilde{v} \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(1+\sigma,\delta,\delta')}}$$

*under the hypotheses of part 2 of the main theorem.*

*Proof.* We now prove

$$\partial_t \tilde{v} \lesssim \frac{1}{\langle r \rangle \langle t-r \rangle^{1+\min(1+\sigma,\delta,\delta')}}$$

assuming more on the time derivatives of our coefficients and also a little more on $A$ and $B$; see part (2) of Theorem 7.1.1. For the first-order terms, we again write

$$\partial(\tilde{A}\phi_{\leq m}) + \tilde{B}\partial\phi_{\leq m} = \partial([\tilde{A}+\tilde{B}]\phi_{\leq m}) - (\partial \tilde{B})\phi_{\leq m};$$

since $A$ and $B$, respectively $\partial_t A$ and $\partial_t B$, belong to the same $S^Z$ class, with one higher rate of $\langle r \rangle$ decay relative to the hypotheses of part 1 of the main theorem, we are done by the previous proof and we henceforth focus on the metric coefficients. By the product rule,

$$\partial_t(\chi \tilde{h} \partial_t \phi_{\leq m}) = \partial_t^2(\chi \tilde{h} \phi_{\leq m}) - \partial_t(\partial_t(\chi \tilde{h})\phi_{\leq m}) \tag{4.9.3}$$

where $\partial_t(\chi \tilde{h}) = O(\langle r \rangle^{-2-\sigma})$ since $\partial_t h \in S^Z_{\text{cone}}(\langle r \rangle^{-2-\sigma})$.

For

$$\Box U = -\partial_t(\partial_t(\chi \tilde{h})\phi_{\leq m})$$

the pointwise decay rates for $U$ follow from techniques already shown. The main task here is to show that we have the improved decay rate for solutions $\partial_t^2 u$ to the equation

$$\Box \partial_t^2 u = \partial_t^2(\chi \tilde{h} \phi_{\leq m}).$$

Intuitively, this means that each of the two time derivatives yields a gain of $\langle u\rangle^{-1}$ more decay in $\operatorname{supp}\chi \subset \{t/2 < r < 3t/2\}$.

For

$$\Box\partial_t^2 u = \partial_t^2(\chi\tilde{h}\phi_{\leq m}),$$

by using the relation

$$\partial_t = \frac{tS - \sum_i x_i\Omega_{0i}}{t^2 - r^2}, \qquad \Omega_{0i} = t\partial_i + x_i\partial_t$$

we have

$$\begin{aligned} |\langle r-t\rangle\partial_t(u_t)| &\lesssim (t+r)^{-1}|\sum_i x_i\Omega_{0i}(u_t)| + |S(u_t)| \\ &\lesssim |\partial_t(Lu)| + |\partial_t u| + |\partial_t(Su)| + \frac{1}{\langle r\rangle}\sum_{\bar{Z}\in\{\Omega,S\}}|\bar{Z}u|, \quad L \in \{\Omega_{0i}\}_{i=1,2,3} \end{aligned} \tag{4.9.4}$$

where we commuted $\partial_t$ with $\Omega_{0i}, S$ in the second line; this produces $|(\partial_i,\partial_t)w|$ terms, and then we noted that in the regions $\{r \sim t\}$,

$$|\partial w| \lesssim S^Z(1)|\partial_t w| + S^Z(\langle r\rangle^{-1})|(\Omega, S)w|.$$

It suffices to bound the first three terms on the right hand side by

$$O\left(\frac{\langle t-r\rangle^{3/2-\theta-q}}{\langle r\rangle}\right) \qquad \text{if } h \in S^Z(\langle r\rangle^{-q}). \tag{4.9.5}$$

To achieve this, we shall use the same one-dimensional reduction idea already employed beforehand. The $r$ decay will come from this reduction. It suffices to have the $3/2$ exponent on the right hand side because of the extra one power of $\langle u\rangle$ decay on the left-hand side of (4.9.4). Recall that $\theta$ denotes the exponent such that before this iterate, the solution obeyed the decay rate $\langle r\rangle^{-1}\langle u\rangle^{1/2-\theta}$; thus due to the positivity of

$q$, successfully showing (4.9.5) would indicate an improvement in pointwise decay by exponent $q$, because combining (4.9.4) and (4.9.5) implies

$$|\partial_t u_t| \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\theta-q}.$$

We now bound $|\partial_t u|$. Writing $\Box u = \chi\tilde{h}\phi_{\le m}$ with simplified notation henceforth as $\chi h\phi$ or $\chi h\phi_{\le m}$,

$$\begin{aligned}\langle r\rangle\langle t-r\rangle u_t &\lesssim \langle r\rangle(|Su|+|Lu|)\\ &\lesssim \int_{D_{tr}} |\chi h\phi| + |S(\chi h\phi)| + \langle s-\rho\rangle|\partial_\rho(\chi h\phi)|\ \rho ds d\rho \\ &\lesssim \frac{1}{\langle t-r\rangle^q}\langle t-r\rangle^{5/2-\theta}, \quad q = 1+\sigma\end{aligned} \tag{4.9.6}$$

where the last line follows from Lemma 7.4.4 and Corollary 4.4.3.

The same calculation shows that $\partial_t(Su)$ is also bounded by this, since by replacing $u$ by $Su$ above we still find the same upper bounds for the integrand; this is because the three functions $S^j(\chi h\phi), j = 0,1,2$ satisfy the same bounds, and the analogous integral is

$$\begin{aligned}\langle r\rangle\langle t-r\rangle(Su)_t &\lesssim \int_{D_{tr}} |\Box LSu| + \left(|\Box S^2 u|\right)\ \rho ds d\rho\\ &\lesssim \int \langle s-\rho\rangle \sum_{k=0}^{1}|\partial_\rho S^k(\chi h\phi)| + \left(\sum_{j=0}^{2}|S^j(\chi h\phi)|\right)\ \rho ds d\rho \\ &\lesssim \frac{1}{\langle t-r\rangle^q}\langle t-r\rangle^{5/2-\theta}.\end{aligned} \tag{4.9.7}$$

The function $(Lu)_t$ also obeys the same bounds as $u_t$, and the analogous integral is

$$\langle r\rangle\langle t-r\rangle(Lu)_t \lesssim \int_{D_{tr}} |\Box SLu| + |\Box LLu|\ \rho ds d\rho. \tag{4.9.8}$$

We have

$$
\begin{aligned}
\int_{D_{tr}} |\Box LLu|\ \rho ds d\rho &\leq \int (2|\Box Lu|) + |LL\Box u|\ \rho ds d\rho \\
&\lesssim \int (|S(\chi h\phi)| + \langle s-\rho\rangle |\partial_\rho(\chi h\phi)|) + |LL\Box u|\ \rho ds d\rho \\
&\lesssim \int (|S(\chi h\phi)| + \langle s-\rho\rangle |\partial_\rho(\chi h\phi)|) \\
&+ |S^2(\chi h\phi)| + |S(\langle s-\rho\rangle \partial_\rho(\chi h\phi))| + \langle s-\rho\rangle |\partial_\rho S(\chi h\phi)| \\
&+ \langle s-\rho\rangle |\partial_\rho(\langle s-\rho\rangle \partial_\rho(\chi h\phi))|\ \rho ds d\rho \\
&\lesssim \frac{1}{\langle t-r\rangle^q} \langle t-r\rangle^{5/2-\theta}, \quad q = 1+\sigma
\end{aligned} \tag{4.9.9}
$$

where we changed $\partial_\rho$ to $\partial_s$ by (7.4.2) and used the assumption on $\partial_t^2 h$. The final line follows by Lemma 7.4.4. The final integrand term (and specifically, when the two derivatives both fall on $h$),

$$
\int \langle s-\rho\rangle^2 \chi(\partial_\rho^2 h)\phi\, \rho\, dA, \quad dA = ds d\rho
$$

is the sole instance where the extra assumption on $\partial_t^2 h$ in Theorem 7.1.1,

$$
\partial_t^2 h \in S^Z_{\text{cone}}(\langle r\rangle^{-1-\sigma} \langle u\rangle^{-2}),
$$

is used.

For $\Box SLu$ we have

$$
\begin{aligned}
\Box SLu &= S\Box Lu + 2\Box Lu \\
&= SL\Box u + 2\Box Lu \\
&= LS\Box u + O(t\partial(\chi h\phi)) + 2\Box Lu
\end{aligned}
$$

where $O(t\partial(\chi h\phi))$ arises from $[S, L]$ and can be broken into three cases: this function takes one of the three forms

$$t\partial_i(\chi h\phi) = \partial_i(t\chi h\phi),$$

$$x_i\partial_t(\chi h\phi) = \partial_t(x_i\chi h\phi),$$

and

$$t\frac{x_i}{r}\partial_r(\chi h\phi) = \partial_r(t\frac{x_i}{r}\chi h\phi).$$

We may then replace $\partial_i, \partial_r$ in the first and third cases by $\partial_t$ via (7.4.2). Then all these terms on the right hand side yield the upper bound $\langle t-r\rangle^{5/2-\theta-(1+\sigma)}$ via Lemma 7.4.4; to see this, it suffices to consider a solution of $\Box w = (t+r)\chi h\phi$ and prove bounds for $\partial_t w$.

For $LS\Box u$ on the other hand,

$$\int_{D_{tr}} |S^2(\chi h\phi)| + \langle s-\rho\rangle|\partial_\rho S(\chi h\phi)| \; \rho ds d\rho \lesssim \frac{1}{\langle t-r\rangle^q}\langle t-r\rangle^{5/2-\theta}, \;\; q = 1+\sigma$$

also. For $\Box Lu$, this upper bound was proved earlier. Thus

$$\langle r\rangle\langle t-r\rangle(Lu)_t \lesssim \frac{1}{\langle t-r\rangle^q}\langle t-r\rangle^{5/2-\theta}.$$

In summary, $(Lu)_t$ and $(Su)_t$ obey the same bound as $u_t$, because $\Box(Lu)_t$ and $\Box(Su)_t$ obey the same bounds as $\Box u_t$ and the claim then follows from Lemma 7.4.4. Thus

$$u_{tt} \lesssim \frac{1}{\langle r\rangle}\langle t-r\rangle^{1/2-\theta-q}, \;\; q = 1+\sigma$$

and the iteration finishes with

$$\partial_t\tilde{v} \lesssim (\langle r\rangle\langle t-r\rangle^{1+\min(1+\sigma,\delta,\delta')})^{-1}.$$

Notice that this argument works for any positive value of $q$. $\square$

# Chapter 5 Global existence and pointwise decay for the null condition

## 5.1 Introduction

In this chapter, we study the wave equation with the classical null condition on a variety of spacetimes. The goal of this chapter is to prove global existence, and to obtain sharp pointwise decay for solutions to this wave equation satisfying the null condition.

For simplicity, in this chapter we have written down the proof of global existence and $O(\langle t+r\rangle^{-1}\langle t-r\rangle^{-1})$ upper bounds for solutions of nonlinear wave equations satisfying the semilinear null condition. However, we note that the proof provided below of the $O(\langle t+r\rangle^{-1}\langle t-r\rangle^{-1})$ upper bounds for the solution and its $Z$-vector fields, assuming sufficiently localised and regular initial data, extends nearly verbatim to solutions of quasilinear wave equations satisfying the null condition as well.

The chapter is structured as follows. Section 1 introduces the main result, and some of the history of the problem. Section 2 contains some notation, a discussion of local energy decay estimates, and the rigorous statement of the main theorem. Sections 3 and 4 contain the proof of global existence. Sections 5, 6 and 7 are dedicated to the sharp pointwise bounds.

### 5.1.1 Statement of the result

We consider the operator

$$P := \partial_\alpha g^{\alpha\beta}(t,x)\partial_\beta + g^\omega(t,x)\Delta_\omega + B^\alpha(t,x)\partial_\alpha + V(t,x). \tag{5.1.1}$$

Here $\Delta_\omega$ denotes the Laplace operator on the unit sphere, and $\alpha, \beta$ range across $0, \dots, 3$. The main assumptions on $P$ are that it is hyperbolic, asymptotically flat, and that the linear evolution satisfies strong local energy decay (and thus the Hamiltonian flow must be nontrapping); the precise conditions on the potential $V$, the coefficients $B, g^\omega$ and the Lorentzian metric $g$ are given in the main result, Theorem 7.1.1. On the other hand, we allow time-dependent coefficients, as well as large perturbations of $\Box$.

We study the nonlinear Cauchy problem

$$P\phi = S^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \quad (\phi(0), \partial_t\phi(0)) = (\phi_0, \phi_1), \tag{5.1.2}$$

where $S^{\alpha\beta} \in \mathbb{R}$ are constants such that $S^{\alpha\beta} = S^{\beta\alpha}$, and

$$S^{\alpha\beta}\xi_\alpha\xi_\beta = 0 \tag{5.1.3}$$

for all $\xi$ such that $\xi_0^2 = \sum_{j=1}^3 \xi_j^2$. We will also write $S^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = Q(\partial\phi, \partial\phi)$. We shall prove global existence and pointwise decay for (5.1.3); see Theorem 7.1.1.

In addition, we prove existence and pointwise decay for the Cauchy problem with the *quasilinear* null condition

$$P\phi = Q = Q(\partial\phi, \partial^2\phi) = C^{\alpha\beta\gamma}\partial_\alpha\phi\partial_{\beta\gamma}\phi, \quad (\phi(0), \partial_t\phi(0)) = (\phi_0, \phi_1), \tag{5.1.4}$$

with $C^{\alpha\beta\gamma} = C^{\alpha\gamma\beta}$ and $C^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0$ whenever $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. However, we assume for the quasilinear case that the metric is nontrapping. See Theorem 7.1.1 below.

Our main theorem states, informally, that if the solution to the linear wave equation $P\phi = F$ satisfies strong local energy bounds, then (7.1.2) with small initial data admits a unique global solution. Moreover, we prove global pointwise decay rates of $\langle t-r\rangle^{-1}\langle t+r\rangle^{-1}$ for the solution and vector fields applied to it. The rate of decay coincides with the one obtained by Christodoulou [19] in the case $P = \Box$ by using the conformal method; we believe this rate to be sharp. See Theorem 7.1.1 for the precise statement.

### 5.1.2 History

The semilinear wave equation in $\mathbb{R}^{1+3}$

$$\Box\phi = Q(\partial\phi, \partial\phi), \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1 \tag{5.1.5}$$

for small initial data has been studied extensively. It is known that the solution blows up in finite time if $Q(\partial\phi, \partial\phi) = (\partial_t\phi)^2$, see [42]. On the other hand, if the nonlinearity satisfies the null condition (5.1.3), first identified by Klainerman [50], it was shown independently in [19] and [51] that the solution exists globally. This result was extended to quasilinear systems with multiple speeds, as well as the case of exterior domains; see, for instance, [70], [71], [72], [36], [115], [53], [1], [56], [102], [29], as well as to systems satisfying the weak null condition, including Einstein's Equations, see [57], [58], [59].

There have also been many works for small data in the variable coefficient case. Almost global existence for nontrapping metrics was shown in [15], [105]. Global existence for stationary, small perturbations of Minkowski was shown in [121], and for nonstationary, compactly supported perturbations in [123]. See also the works [124,125] which prove global results and some pointwise decay.

In the context of black holes, global existence was shown in [65] for Kerr space-times with small angular momentum, and in [2] for the Reissner-Nordström backgrounds. See also the upcoming [66] for sharp pointwise bounds and asymptotics, given certain assumptions, for a variety of nonlinearities.

The results of this chapter can be extended to to the semilinear problem satisfying the null condition as long as weak local energy decay, also known as weak integrated local energy decay (weak ILED) holds. The weak ILED condition holds on Schwarzschild spacetimes and subextremal Kerr spacetimes.

We define

$$S^Z_{\text{radial}}(f) := \{g \in S^Z(f) : g \text{ is spherically symmetric.}\}$$

We recall the following notion of local energy decay.

**Definition 5.1.1** (Local energy decay)**.** We say that $P$ has the strong local energy decay property if the following estimate holds for all $m \geq 0$, and $0 \leq T_0 < T_1 \leq \infty$:

$$\|\phi_{\leq m}\|_{LE^1([T_0,T_1)\times\mathbb{R}^3)} \lesssim_m \|\partial\phi_{\leq m}(T_0)\|_{L^2(\mathbb{R}^3)} + \|(P\phi)_{\leq m}\|_{(L^1L^2+LE^*)([T_0,T_1)\times\mathbb{R}^3)}. \quad (5.1.6)$$

Here the implicit constant may depend on $m$, but not $T_0$ and $T_1$.

For example, Definition 5.1.1 holds if the operator $P$ is a small perturbation of $\Box$, see for instance [76]. More generally, strong local energy estimates (for $m = 0$) were shown in [77], provided there are no negative eigenvalues or real resonances. Provided that $P$ is stationary, one can extend the result to (5.1.6) by commuting with the vectors fields in $Z$.

### 5.1.3 Statement of the main theorem

Let $h = g - m$, where $m$ denotes the Minkowski metric. Let $\sigma \in (0, \infty)$ be real. We make the following assumptions on the coefficients of $P$:

$$\begin{aligned} h^{\alpha\beta}, B^\alpha &\in S^Z(\langle r\rangle^{-1-\sigma}) \\ \partial_t B^\alpha, V &\in S^Z(\langle r\rangle^{-2-\sigma}) \\ g^\omega &\in S^Z_{\text{radial}}(\langle r\rangle^{-2-\sigma}) \end{aligned} \tag{5.1.7}$$

**Theorem 5.1.2** (Main theorem)**.** *Assume that $P$ has the SLED property (Definition 5.1.1), and that the coefficients of $P$ satisfy (7.1.3).*

*(i) Assume that $(\phi(0), \partial_t\phi(0)) \in H^{13}(\mathbb{R}^3) \times H^{12}(\mathbb{R}^3)$. Then there is $\epsilon_0 > 0$ so that, if*

$$\sum_{I:|I|\leq 12} \|\partial\phi_I(0)\|_{L^2(\mathbb{R}^3)} \leq \epsilon_0,$$

*then (7.1.2) has a unique global solution.*

*(ii) Fix $m \in \mathbb{N}$. Then there is an integer $N \gg m$ so that, if we assume in addition that*

$$\sum_{I:|I|\leq N} \|\langle r\rangle^{1/2}\partial\phi_I(0)\|_{L^2(\mathbb{R}^3)} < \infty$$

*then the solution satisfies*

$$\sum_{J:|J|\leq m} |\phi_J(t,x)| \lesssim \langle v\rangle^{-1}\langle u\rangle^{-1}$$

*where* $v := t + r, u := t - r$.

We expect the rate of decay to be sharp even for the Minkowski metric.

## 5.2 Pointwise bounds from local energy

In this section we will show that local energy bounds imply certain weak pointwise bounds, see Proposition 7.3.5 and Proposition 5.2.5. Nevertheless, these bounds are sufficient to prove global existence in Section 4.

We start with the following Klainerman-Sideris type estimate for the second derivative

**Lemma 5.2.1.** *Assume* $\phi$ *is sufficiently regular. We then have for all* $r \gg 1$

$$|\partial^2\phi_J| \lesssim \left(\frac{1}{\langle r\rangle} + \frac{1}{\langle u\rangle}\right)|\partial\phi_{\leq|J|+1}| + \left(1 + \frac{t}{\langle u\rangle}\right)\langle r\rangle^{-2}|\phi_{\leq|J|+2}| + \left(1 + \frac{t}{\langle u\rangle}\right)|(P\phi)_{\leq|J|}|. \tag{5.2.1}$$

*Proof.* Note first that

$$|\partial^2\phi_J| \lesssim \left(\frac{1}{\langle r\rangle} + \frac{1}{\langle u\rangle}\right)|\partial\phi_{\leq|J|+1}| + \left(1 + \frac{t}{\langle u\rangle}\right)|(\Box\phi)_{\leq|J|}|. \tag{5.2.2}$$

The case $|J| = 0$ is an immediate consequence of Lemma 2.3 from [53]. The general case follows after commuting with vector fields.

It is thus enough to estimate the difference $P - \Box$. We write by (7.1.1)

$$P - \Box = h^{\alpha\beta}\partial_{\alpha\beta} + (\partial_\alpha h^{\alpha\beta})\partial_\beta + g^\omega(t,x)\Delta_\omega + B^\alpha(t,x)\partial_\alpha + V(t,x)$$

Using the assumptions on the coefficients in subsection 7.1.3 we have

$$(P - \Box)\phi \in S^Z(\langle r\rangle^{-1-\sigma})(\partial^2\phi + \partial\phi) + S^Z(\langle r\rangle^{-2-\sigma})\Omega^{\leq 2}\phi$$

After applying vector fields we thus obtain

$$\left|((P-\Box)\phi)_{\leq |J|}\right| \in S^Z(\langle r\rangle^{-1-\sigma})|\partial^2\phi_{\leq |J|}| + S^Z(\langle r\rangle^{-1-\sigma})|\partial\phi_{\leq |J|}| + S^Z(\langle r\rangle^{-2-\sigma})|\phi_{\leq |J|+2}| \tag{5.2.3}$$

The conclusion now follows from (7.3.2) and (7.3.3) , since the first term on the RHS of (7.3.3) can be absorbed in the LHS of (7.3.2) for $r \gg 1$. □

The main tool for turning local energy estimates into pointwise bounds is the following lemma

**Lemma 5.2.2** (Dyadically localised bounds)**.** *Let $w \in C^4$.*

- *For all $T \geq 1$ and $1 \leq U \leq 3T/8$, we have*

$$\|w\|_{L^\infty(C_T^U)} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{(T^3U)^{1/2}}\|S^i\Omega^j w\|_{L^2(C_T^U)} + \left(\frac{U}{T^3}\right)^{\frac{1}{2}} \|\partial_r S^i\Omega^j w\|_{L^2(C_T^U)}. \tag{5.2.4}$$

- *For all $T \geq 1$ and $R > T$, we have*

$$\|w\|_{L^\infty(C_R^T)} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{(R^3T)^{1/2}}\|S^i\Omega^j w\|_{L^2(C_R^T)} + \frac{1}{(RT)^{1/2}}\|\partial_t S^i\Omega^j w\|_{L^2(C_R^T)}. \tag{5.2.5}$$

- *For all $T \geq 1$ and $1 \leq R \leq 3T/8$, we have*

$$\|w\|_{L^\infty(C_T^R)} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{(R^3T)^{1/2}}\|S^i\Omega^j w\|_{L^2(C_T^R)} + \frac{1}{(RT)^{1/2}}\|\partial_r S^i\Omega^j w\|_{L^2(C_T^R)}. \tag{5.2.6}$$

- *For all $T \geq 1$ and $R > T$, we have*

$$\|w\|_{L^\infty(C_R^R)} \lesssim \sum_{i\leq 1, j\leq 2} \frac{1}{R^2}\|S^i\Omega^j w\|_{L^2(C_R^R)} + \frac{1}{R}\|\partial_t S^i\Omega^j w\|_{L^2(C_R^R)}. \tag{5.2.7}$$

The proof of this lemma can be found in [60]. For (5.2.7), note that $|C_R^R|^{1/2} \sim R^2$, which explains the $1/R^2$ factor.

We will also use the following lemma near the cone, which is a slight extension of Lemma 9.1 in [58].

**Lemma 5.2.3.** *If $f \in C^1$, then*

$$\begin{aligned}\int_{t/2}^{3t/2} \langle t-r\rangle^{-2} f(t,x)^2 dx \lesssim & \int_{t/4}^{7t/4} |\partial_r f(t,x)|^2 dx \\ & + \frac{1}{t^2}\left(\int_{t/4}^{t/2} f(t,x)^2 dx + \int_{3t/2}^{7t/4} f(t,x)^2 dx\right)\end{aligned} \tag{5.2.8}$$

*Proof.* Let $\chi : [0,\infty) \to [0,1]$ be a cutoff such that $\chi(s) = 1$ for $1/2 \leq s \leq 3/2$ and 0 when $s \leq 1/4$ and $s \geq 7/4$. We will show that, if $\gamma > -1/2$, and $\gamma \neq 1/2$, then

$$\begin{aligned}\int \langle t-r\rangle^{-2-2\gamma}\chi(r/t) f(r,\omega)^2 r^2 dr \lesssim & \int \langle t-r\rangle^{-2\gamma}|\partial_r f(r,\omega)\chi(r/t)|^2 r^2 dr \\ & + \frac{1}{t^2}\int \langle t-r\rangle^{-2\gamma}|f(r,\omega)\chi'(r/t)|^2 r^2 dr.\end{aligned}$$

The conclusion follows if we take $\gamma = 0$ and integrate over $\omega$.

We have

$$f(r,\omega)^2\chi(r/t) - f(7t/4,\omega)^2\chi((7t/4)/t) = -2\int_r^{7t/4} f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))d\rho.$$

Hence

$$f(r,\omega)^2\chi(r/t)r^2 \lesssim f(7t/4,\omega)^2\chi(3t/2)t^2 + 2\int_r^{7t/4} |f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))|d\rho$$

Recall that $\chi(7t/4) = 0$. We multiply by $\langle t-r\rangle^{-2-2\gamma}$ and integrate $r$ from $t/4$ to $7t/4$. This yields

$$\int_{t/4}^{7t/4} \langle t-r\rangle^{-2-2\gamma}\chi(r/t)f(r,\omega)^2 r^2 dr \lesssim \int_{t/4}^{7t/4} \langle t-r\rangle^{-1-2\gamma}|f(r,\omega)\chi(r/t)\partial_r(f(r,\omega)\chi(r/t))|r^2 dr$$

By the chain rule, $\partial_r(\chi(r/t)) \lesssim \chi'(r/t)\cdot\frac{1}{t}$. Thus by Cauchy-Schwarz and the chain rule

$$\int_{t/2}^{3t/2} \langle t-r\rangle^{-2-2\gamma} f(r,\omega)^2 r^2 dr \lesssim \int_{t/4}^{7t/4} \langle t-r\rangle^{-2\gamma}|\partial_r f(r,\omega)\chi(r/t)|^2 r^2 dr + \frac{1}{t^2}\int_{t/4}^{7t/4} \langle t-r\rangle^{-2\gamma}|f(r,\omega)\chi'(r/t)|^2 r^2 dr.$$

This concludes the proof of the lemma. □

Our next proposition yields global pointwise bounds for $\phi_J$ under the assumption that the local energy norms are finite. These estimates are sharp from that point of view, but can be improved for solutions to (7.1.2), see Sections 5-7.

**Proposition 5.2.4.** *Let $T$ be fixed and $\phi$ be any sufficiently regular function. There is a fixed positive integer $k$, such that for any multi-index $J$ with $|J| \le N-k$, we have:*

$$|\phi_J| \le \bar{C}_{|J|}\|\phi_{\le |J|+k}\|_{LE^1[T,2T]}\langle u\rangle^{1/2}\langle v\rangle^{-1}. \tag{5.2.9}$$

*Proof.* Away from the cone, (7.3.7) is a straightforward consequence of (5.2.5) and (5.2.6). For the $C_T^U$ region, one uses (5.2.4) in conjunction with Lemma 7.3.4:

$$\begin{aligned}\|\phi_J\|_{L^\infty(C_T^U)} &\lesssim T^{-\frac{3}{2}}U^{\frac{1}{2}}\left(\|U^{-1}\phi_{\le|J|+3}\|_{L^2(C_T^U)} + \|\partial_r\phi_{\le|J|+3}\|_{L^2(C_T^U)}\right)\\ &\lesssim T^{-\frac{3}{2}}U^{\frac{1}{2}}\left(\|\partial_r\phi_{\le|J|+3}\|_{L^2(C_T^U)} + \frac{1}{T}\|\phi_{\le|J|+3}\|_{L^2[T,2T]L^2(r\approx T)}\right)\\ &\lesssim \frac{U^{\frac{1}{2}}}{T}\|\phi_{\le|J|+3}\|_{LE^1[T,2T]}.\end{aligned}$$

□

We now obtain an improved bound on the derivatives. Note that here it is crucial that $\phi$ is a solution to (7.1.2) , since we need to use Lemma 7.3.1. Let

$$\mu := \mu(t,r) := \min(\langle t\rangle, \langle u\rangle)^{1/2}.$$

**Proposition 5.2.5.** *Let $T$ be fixed and $\phi$ solve (7.1.2) for the times $t \in I_T$. Then for any dyadic region $C \in \{C_T^R, C_R^T, C_T^U\}$ and $m \geq 0$ we have*

$$\|\partial\phi_{\leq m}\|_{L^\infty(C)} \leq \bar{C}_m \frac{1}{\mu}\left(\frac{1}{\langle r\rangle} + \|\partial\phi_{\leq \frac{m+3}{2}}\|_{L^\infty(C)}\right)\|\phi_{\leq m+5}\|_{LE^1[T,2T]} \tag{5.2.10}$$

*Proof.* Note first that if $r \lesssim 1$, the bound follows immediately from Lemma 5.2.2. Subsequently, we assume that $r \gg 1$.

Note first that

$$(P\phi)_{\leq m+3} \lesssim \left|\left(\partial\phi_{\leq \frac{m+3}{2}}\right)(\partial\phi_{\leq m+3})\right|$$

We also have

1. In $C_R^T$ and $C_T^R$, the bound in (7.3.1) is

$$\partial^2\phi_{\leq m+3} \lesssim R^{-1}|\partial\phi_{\leq m+4}| + R^{-2}|\phi_{\leq m+5}| + |(P\phi)_{\leq m+3}|. \tag{5.2.11}$$

2. In $C_T^U$, the bound in (7.3.1) is

$$\partial^2\phi_{\leq m+3} \lesssim U^{-1}|\partial\phi_{\leq m+4}| + \frac{1}{TU}|\phi_{\leq m+5}| + TU^{-1}|(P\phi)_{\leq m+3}|. \tag{5.2.12}$$

We now apply Lemma 5.2.2 in our region $C$. When $C = C_T^R$ we obtain, using (5.2.6) and (5.2.11):

$$\begin{aligned}\|\partial\phi_{\le m}\|_{L^\infty(C_T^R)} &\lesssim \frac{1}{(R^3T)^{1/2}}\|\partial\phi_{\le m+3}\|_{L^2(C_T^R)} + \frac{1}{(RT)^{1/2}}\|\partial^2\phi_{\le m+3}\|_{L^2(C_T^R)} \\ &\lesssim \frac{1}{RT^{1/2}}\|\phi_{\le m+5}\|_{LE^1[T,2T]} + \frac{\|\partial\phi_{\le\frac{m+3}{2}}\|_{L^\infty(C_T^R)}}{(RT)^{1/2}}\|\partial\phi_{\le m+3}\|_{L^2(C_T^R)} \\ &\lesssim \left(\frac{1}{RT^{1/2}} + T^{-1/2}\|\partial\phi_{\le\frac{m+3}{2}}\|_{L^\infty(C_T^R)}\right)\|\phi_{\le m+5}\|_{LE^1[T,2T]}\end{aligned}$$

Similar computations yield, using (5.2.5) and (5.2.11)

$$\|\partial\phi_{\le m}\|_{L^\infty(C_R^T)} \lesssim \left(\frac{1}{RT^{1/2}} + T^{-1/2}\|\partial\phi_{\le\frac{m+3}{2}}\|_{L^\infty(C_R^T)}\right)\|\phi_{\le m+5}\|_{LE^1[T,2T]}$$

When $C = C_T^U$ we obtain by (5.2.4) and (5.2.12):

$$\|\partial\phi_{\le m}\|_{L^\infty(C_T^U)} \lesssim \left(\frac{1}{TU^{1/2}} + U^{-1/2}\|\partial\phi_{\le\frac{m+3}{2}}\|_{L^\infty(C_T^U)}\right)\|\phi_{\le m+5}\|_{LE^1[T,2T]}$$

This completes the proof of Proposition 5.2.5. □

*Remark* 5.2.6. We only need to use the following estimate (5.2.10):

$$\|\partial\phi_{\le m}\|_{L^\infty(C)} \le \bar{C}_m\left(\frac{1}{\langle r\rangle\mu} + \|\partial\phi_{\le\frac{m+3}{2}}\|_{L^\infty(C)}\right)\|\phi_{\le m+5}\|_{LE^1[T,2T]} \tag{5.2.13}$$

Thus in $r \le 3t/2$ we have

$$\partial\phi_{\le m} \lesssim \langle r\rangle^{-1}\langle u\rangle^{-1/2}\|\phi_{\le m+5}\|_{LE^1[T,2T]}$$

### 5.3 The proof of small data global existence

We are now ready to prove our first theorem. For any $N \in \mathbb{N}$, define

$$\mathcal{E}_N(t) = \|\partial\phi_{\le N}\|_{L^\infty[0,t]L^2} + \|\phi_{\le N}\|_{LE^1[0,t]}$$

We also define $\tilde{N} = N - 1$, and $N_1 = N/2$.

**Theorem 5.3.1.** *Assume that $\phi$ solves (7.1.2). Then there exists a global classical solution to (7.1.2), provided that the initial data is smooth and satisfies, for some sufficiently small $\epsilon_0 \ll 1$,*

$$\mathcal{E}_N(0) \leq \epsilon_0$$

*for any natural number $N \geq 12$.*

*Moreover, write $\mathcal{E}_N(0) := \nu_N \epsilon$ where $\nu_N$ is a small constant to be determined later (see (5.3.14)). Then for any $\delta > 0$, there is some $\tilde{C} > 0$ so that*

$$\mathcal{E}_N(t) \leq \tilde{C} \langle t \rangle^{\delta} \nu_N \epsilon, \tag{5.3.1}$$

$$|\phi_{\leq N_1}| \leq \epsilon \langle u \rangle^{1/2} \langle v \rangle^{-1}, \quad |\partial \phi_{\leq N_1}| \leq \frac{\epsilon}{\langle r \rangle \mu^{\frac{1}{2}}}. \tag{5.3.2}$$

*Proof.* The proof will be by a bootstrap argument. Clearly (5.3.1) and (5.3.2) hold for small times. Assuming now that (5.3.1) and (5.3.2) hold for $0 \leq t \leq T$, we improve the constants by a factor of $1/2$. Thus by continuity the solution exists for all time.

The proof that (5.3.1) holds with a better constant uses, crucially, the smallness of $\epsilon$. For instance, as the reader can verify below, in the region $r \leq t/2$ we manage to absorb the nonlinearity to the left-hand side by way of this smallness. We note that in the course of proving (5.3.1) we can actually assume $\tilde{C}$ to be as big as desired.

The proof that (5.3.2) holds with a better constant makes use of Proposition 7.3.5 and Proposition 5.2.5 and the previous paragraph. By using the fact, now already proved, that $\mathcal{E}_{\tilde{N}}(T) = O(\nu_N \epsilon)$ (see previous paragraph) where $\nu_N$ is a constant independent of $T$ and the other constants involved in the proof, we can choose $\nu_N$ small enough so that (5.3.2) indeed holds with a better constant. This concludes our overview of the proof.

Note that

1. By Sobolev embeddings and the smallness of the initial data, the estimates in (5.3.2) hold for time 0. By local existence theory and the continuity in time of the functions in (5.3.2), the estimates in (5.3.2) hold for all sufficiently small times.

2. The estimates in (5.3.1) hold for time 0 by assumption that the initial data satisfies $\mathcal{E}_N(0) \leq \nu_N \epsilon$. The estimates in (5.3.1) hold for all sufficiently small times by continuity in time of the norms involved in $\mathcal{E}_k(t)$, $k \leq N$—where we choose $\tilde{C}$ to be big enough.

Note also that, since $Q$ satisfies the null condition, we have that

$$Q(\partial\phi, \partial\phi) \in S^Z(1)\partial\phi\overline{\partial}\phi$$

Moreover,

$$\overline{\partial}\phi \in S^Z\Big(\frac{\langle u\rangle}{r}\Big)\partial\phi + S^Z\Big(\frac{1}{r}\Big)Z\phi, \tag{5.3.3}$$

Combined with (5.3.2), (5.3.3) yields

$$\overline{\partial}\phi_{\leq N_1}(t,x)|_{\frac{1}{2}t\leq r\leq\frac{3}{2}t} \lesssim \epsilon\frac{\langle u\rangle^{1/2}}{\langle r\rangle^2}. \tag{5.3.4}$$

Assume that (5.3.1) and (5.3.2) hold for all $0 \leq t \leq T$. Let

$$\mathcal{N} := L^1L^2 + LE^*$$

be the space in which we place the nonlinearity.

We start with the bound for $\mathcal{E}_N$. We will use $L^1L^2$ near the cone, and $LE^*$ away from it.

Given the assumption of local energy decay (5.1.6), we have

$$\mathcal{E}_N(t) \leq C_N\Big(\mathcal{E}_N(0) + \|Q_{\leq N}\|_{\mathcal{N}[0,t]}\Big). \tag{5.3.5}$$

We define

$$S_1 := \{(s,x) : 0 \leq s \leq t,\ |x| \leq s/2\}$$

$$S_2 = \{(s,x) : 0 \leq s \leq t,\ s/2 \leq |x| \leq 3s/2\}$$

$$S_3 := \{(s,x) : 0 \leq s \leq t,\ |x| \geq 3s/2\}$$

In $S_1$ we have by (5.3.2)

$$\begin{aligned} \|Q_{\leq N}\|_{LE^*(S_1)} &\lesssim \|((\partial\phi)^2)_{\leq N}\|_{LE^*(S_1)} \\ &\lesssim \epsilon\|\langle r\rangle^{-3/2}\partial\phi_{\leq N}\|_{LE^*(S_1)} \\ &\lesssim \epsilon\|\langle r\rangle^{-1}\partial\phi_{\leq N}\|_{L^2_{t,x}(S_1)} \\ &\lesssim \epsilon\|\phi_{\leq N}\|_{LE^1[0,t]}. \end{aligned} \tag{5.3.6}$$

On the other hand, in $S_2 \cup S_3$ (5.3.2) implies that $|\partial\phi_{\leq N_1}| \lesssim \epsilon\langle t\rangle^{-1}$. We thus obtain

$$\|Q_{\leq N}\|_{L^1L^2(S_2\cup S_3)} \lesssim \epsilon\|\langle s\rangle^{-1}\partial\phi_{\leq N}\|_{L^1L^2(S_2\cup S_3)} \lesssim \epsilon\int_0^t \langle s\rangle^{-1}\mathcal{E}_N(s)ds \tag{5.3.7}$$

We thus obtain, by (5.3.5), (5.3.6), and (5.3.7):

$$\mathcal{E}_N(t) \leq C_N\Big(\mathcal{E}_N(0) + \epsilon\int_0^t \langle s\rangle^{-1}\mathcal{E}_N(s)ds\Big)$$

By Gronwall's inequality,

$$\mathcal{E}_N(t) \leq C_N\mathcal{E}_N(0)\exp\left(C_N\int_0^t \epsilon\langle s\rangle^{-1}ds\right) \leq C_N\mathcal{E}_N(0)\langle t\rangle^{C_N\epsilon}. \tag{5.3.8}$$

We now choose $\epsilon_0$ small enough so that $C_N\epsilon_0 < \delta$, and $\tilde{C} = 2C_N$.

We will now show that $\mathcal{E}_{\tilde{N}}$ is bounded, where $\tilde{N} := N-1$. More precisely,

$$\mathcal{E}_{\tilde{N}}(t) \leq C_{\tilde{N}} \nu_N \epsilon \tag{5.3.9}$$

We first see, similarly to (5.3.6), that

$$\|Q_{\leq \tilde{N}}\|_{LE^*(S_1)} \lesssim \epsilon \|\phi_{\leq \tilde{N}}\|_{LE^1[0,t]} \tag{5.3.10}$$

Since (5.3.2) implies that $|\partial \phi_{\leq N_1}| \lesssim \epsilon \langle t \rangle^{-3/2}$ in $S_3$, we obtain

$$\|Q_{\leq \tilde{N}}\|_{L^1 L^2(S_3)} \lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_{\tilde{N}}(s) ds \lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_N(s) ds \lesssim \epsilon \mathcal{E}_N(0) \tag{5.3.11}$$

where the last bound holds by (5.3.8).

For the bound in $S_2$, however, we proceed differently. Let $|\alpha| \leq \tilde{N}/2$ and let $|\alpha + \beta| = \tilde{N}$. We have

$$\|Q_{\leq \tilde{N}}\|_{L^1 L^2(S_2)} \lesssim \sum_{\alpha, \beta} \|\partial \phi_\alpha \bar{\partial} \phi_\beta\|_{L^1 L^2(S_2)} + \|\bar{\partial} \phi_\alpha \partial \phi_\beta\|_{L^1 L^2(S_2)}.$$

We begin with the second term. By (5.3.4) we have

$$\|\bar{\partial} \phi_\alpha \partial \phi_\beta\|_{L^1 L^2(S_2)} \lesssim \epsilon \|\langle s \rangle^{-3/2} \partial \phi_{\leq \tilde{N}}\|_{L^1 L^2(S_2)} \lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_N(s) ds \lesssim \epsilon \mathcal{E}_N(0) \tag{5.3.12}$$

where the last bound holds by (5.3.8).

For the first term, we use (5.3.3), (5.3.2), Lemma 7.3.4 and (5.3.1):

$$\begin{aligned}
&\|\partial\phi_\alpha\bar\partial\phi_\beta\|_{L^1L^2(S_2)}\\
&\lesssim \|\partial\phi_\alpha\langle s-r\rangle\langle r\rangle^{-1}\partial\phi_\beta\|_{L^1L^2(S_2)}+\|\partial\phi_\alpha\langle r\rangle^{-1}S\phi_\beta\|_{L^1L^2(S_2)}\\
&\lesssim \epsilon\int_0^t\langle s\rangle^{-3/2}\mathcal{E}_{\tilde N}(s)ds+\int_0^t\epsilon\|\langle s-r\rangle^{-1/2}\langle r\rangle^{-2}S\phi_\beta(s,\cdot)\|_{L^2(s/2\le|x|\le 3s/2)}ds\\
&\lesssim \epsilon\mathcal{E}_{\tilde N}(t)+\int_0^t\epsilon\|\langle s-r\rangle^{-1}\langle r\rangle^{-3/2}S\phi_\beta(s,\cdot)\|_{L^2(s/2\le|x|\le 3s/2)}ds \qquad (5.3.13)\\
&\lesssim \epsilon\mathcal{E}_{\tilde N}(t)+\int_0^t\epsilon\langle s\rangle^{-3/2}\|\langle s-r\rangle^{-1}S\phi_\beta(s,\cdot)\|_{L^2(s/2\le|x|\le 3s/2)}ds\\
&\lesssim \epsilon\mathcal{E}_{\tilde N}(t)+\int_0^t\epsilon\langle s\rangle^{-3/2}\left(\|\partial_r S\phi_\beta(s,\cdot)\|_{L^2}+\|r^{-1}S\phi_\beta(s,\cdot)\|_{L^2}\right)ds\\
&\lesssim \epsilon\mathcal{E}_{\tilde N}(t)+\int_0^t\epsilon\langle s\rangle^{-3/2}\mathcal{E}_{\tilde N+1}(s)ds\lesssim \epsilon\mathcal{E}_{\tilde N}(t)+\epsilon\mathcal{E}_N(0)
\end{aligned}$$

(5.3.10), (5.3.11) and (5.3.12) imply

$$\mathcal{E}_{\tilde N}(t)\lesssim \epsilon\mathcal{E}_{\tilde N}(t)+\epsilon\mathcal{E}_N(0)$$

and (5.3.9) follows for small enough $\epsilon$.

We now improve the constants in (5.3.2). We pick $\nu_N$ so that

$$2\nu_N\bar C_{N_1}C_{\tilde N}\le\frac{1}{2} \qquad (5.3.14)$$

We have by Proposition 7.3.5, (5.3.9) and (5.3.14):

$$|\phi_{\le N_1}|\le \bar C_{N_1}\langle u\rangle^{1/2}\langle v\rangle^{-1}\mathcal{E}_{\tilde N}(T)\le \bar C_{N_1}C_{\tilde N}\nu_N\epsilon\langle u\rangle^{1/2}\langle v\rangle^{-1}\le\frac{1}{2}\epsilon\langle u\rangle^{1/2}\langle v\rangle^{-1}$$

To improve the constant for the derivative, we note that, by Proposition 5.2.5's (5.2.10) and the fact that $\frac{N_1+3}{2}\le N_1$ and $N_1+5\le\tilde N$:

$$\|\partial\phi_{\le N_1}\|_{L^\infty(C)}\le \bar C_{N_1}\left(\frac{1}{\langle r\rangle}\frac{1}{\mu^{\frac12}}+\|\partial\phi_{\le N_1}\|_{L^\infty(C)}\right)\mathcal{E}_{\tilde N}(T)$$

Using (5.3.9) we see that we can absorb the second term on the right to the left as long as (5.3.14) holds. We now obtain

$$|\partial\phi_{\leq N_1}| \leq 2\bar{C}_{N_1}\frac{1}{\langle r\rangle}\frac{1}{\mu^{\frac{1}{2}}}\mathcal{E}_{\tilde{N}}(T) \leq \frac{\epsilon}{2}\frac{1}{\langle r\rangle}\frac{1}{\mu^{\frac{1}{2}}}$$

We have thus improved the constants in (5.3.2) by 1/2. This concludes the continuity argument and the proof of small data global existence. □

## 5.4 Preliminaries to the iteration

*Remark* 5.4.1 (The initial data). Let $w := S(t,0)\phi[0]$ denote the solution to the free wave equation with initial data $\phi[0]$ at time 0. Then for any multiindex $J$ with $|J| = O_N(1)$,

$$w_J(t,x) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} (\phi_0)_J(y) + \nabla_y(\phi_0)_J(y)\cdot(y-x) + t(\phi_1)_J(y)\,dS(y). \quad (5.4.1)$$

By (5.4.1) and the assumptions $(\phi(0), \partial_t\phi(0)) \in H^{13}(\mathbb{R}^3)\times H^{12}(\mathbb{R}^3)$,

$$\|\langle r\rangle^{1/2}\partial\phi_{\leq N}(0)\|_{L^2} < \infty,$$

we have

$$w_J \lesssim \langle v\rangle^{-1}\langle u\rangle^{-1}.$$

### 5.4.1 Overview of the iteration

The iteration proceeds as follows. First we note that by Remark 7.4.1, we may assume zero initial data in the following iteration. Second, note that (5.3.2) is already optimal when $t-1 < r < t+1$. Third, we distinguish the nonlinearity and the coefficients of $P - \Box$, and for both of these, we apply the fundamental solution and iterate. That is, we decompose $\phi_{\leq m}$ into $\phi_{\leq m} = f_L + f_N$ where $f_L$ solves the linear part of (5.4.4). The iterations for $f_L$ and $f_N$ proceed in lockstep with one another.

Since the iteration for $\{r < t - 1\}$ would depend on the values of the solution and its vector fields in the region where $r > t$, we shall first complete the iteration in $\{r > t + 1\}$. For the iteration in $\{r < t - 1\}$, we note that the decay rates obtained from the fundamental solution are insufficient in the region $\{r < t/2\}$. To remedy this, we prove Proposition 5.6.2. With the new decay rates obtained from Proposition 5.6.2, we are then able to obtain new decay rates for the solution and its vector fields. At every step of the iteration, Lemma 7.4.4 is used to turn the decay gained at previous steps into new decay rates.

A little more precisely: The fundamental solution gives us an improvement for $r\phi_{\leq m}$, say $r\phi_{\leq m} \lesssim \langle u \rangle^{-\alpha}$ for some real $\alpha$, and then by Proposition 5.6.2, we obtain $v\phi_{\leq m} \lesssim \langle u \rangle^{-\alpha}$. Then, using this improvement for $\phi_{\leq m}$, we improve the decay rate for the derivatives $\partial\phi_{\leq m}$. This improvement for the derivatives $\partial\phi_{\leq m}$ is then used to improve the decay rates for $r\phi_{\leq m}$ using the fundamental solution. We then again apply Proposition 5.6.2, and this cyclical iteration between these two improvements continues until we reach the final decay rate $v\phi_{\leq m} \lesssim \langle u \rangle^{-1}$.

To simplify the iteration—in particular, to avoid the appearance of logarithms—we shall reduce the value of $\sigma$ if necessary to be equal to some positive irrational number less than the original value of $\sigma$. We also take $0 < \sigma \ll 1$.

### 5.4.2 Setting up the problem

We rewrite (7.1.2) as

$$\Box\phi = (\Box - P)\phi + Q = -\partial_\alpha(h^{\alpha\beta}\partial_\beta\phi + B^\alpha\phi) - g^\omega\Delta_\omega\phi - (V - \partial_\alpha B^\alpha)\phi + Q$$

Using the assumptions (7.1.3), we can thus write

$$\Box\phi \in \partial\left(S^Z(r^{-1-\sigma})\phi_{\leq 1}\right) + S^Z(r^{-2-\sigma})\phi_{\leq 2} + Q$$

Pick any multiindex $|J| \leq N_1 - 2$. We have after commuting

$$\Box\phi_J \in \partial\left(S^Z(r^{-1-\sigma})\phi_{\leq m+1}\right) + S^Z(r^{-2-\sigma})\phi_{\leq m+2} + Q_{\leq m} \tag{5.4.2}$$

When we commute vector fields with the null form in (7.1.2), we obtain more than one null form, but for the purposes of pointwise decay iteration we may treat all of these null forms as a single null form, which by a slight abuse of notation we also denote by $Q$.

When $r \leq t/2$ or $r \geq 3t/2$ we will gain a factor of $1/r$ for the derivative. On the other hand, we only gain a factor of $1/\langle u\rangle$ for the derivative in the region $t/2 \leq r \leq 3t/2$, which causes an additional issue. To deal with it, we remark that, for any function $w$, we have

$$\partial w \in S^Z(r^{-1})w_{\leq 1} + S^Z(1)\partial_t w, \quad r \geq t/2 \tag{5.4.3}$$

This is obvious for $\partial_t$ and $\not\partial$, whereas for $\partial_r$ we write

$$\partial_r = \frac{S}{r} - \frac{t}{r}\partial_t.$$

Let $\chi_{\text{cone}}$ be a cutoff subordinated to the region $t/2 \le r \le 3t/2$. We now rewrite (7.4.1) as

$$\Box\phi_J \in S^Z(r^{-2-\sigma})\phi_{\le m+2} + (1-\chi_{\text{cone}})\left(S^Z(r^{-1-\sigma})\partial\phi_{\le m+1}\right) + \partial_t\left(\chi_{\text{cone}}S^Z(r^{-1-\sigma})\phi_{\le m+1}\right) + Q_{\le m} \tag{5.4.4}$$

We now write $\phi_J = \sum_{j=1}^{3}\phi_j$ where

$$\begin{aligned} \Box\phi_1 = G_1, \quad G_1 \in S^Z(r^{-2-\sigma})\phi_{\le m+2} + (1-\chi_{\text{cone}})\left(S^Z(r^{-1-\sigma})\partial\phi_{\le m+1}\right) \\ \Box\phi_2 = \partial_t G_2, \quad G_2 \in \chi_{\text{cone}}S^Z(r^{-1-\sigma})\phi_{\le m+1} \\ \Box\phi_3 = Q_{\le m} = G_3 \end{aligned} \tag{5.4.5}$$

Finally, from now on $n$ will represent a large constant, which does not depend on $m$, but may increase from one estimate to the next. We will not track the exact value of $n$ needed.

### 5.4.3 Estimates for the fundamental solution

We have the following result, which is similar to previous classical results, see for instance [41], [8], [109], [114].

**Lemma 5.4.2.** *Let $m \ge 0$ be an integer and suppose that $\psi : [0,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ solves*

$$\Box\psi(t,x) = g(t,x), \qquad \psi(0) = 0, \quad \partial_t\psi(0) = 0.$$

*Define*

$$h(t,r)=\sum_0^2\|\Omega^i g(t,r\omega)\|_{L^2(\mathbb{S}^2)} \tag{5.4.6}$$

*Assume that*

$$h(t,r)\lesssim\frac{1}{\langle r\rangle^{\alpha}\langle v\rangle^{\beta}\langle u\rangle^{\eta}},\quad 2<\alpha<3,\quad \beta\geq 0,\quad \eta\geq -1/2.$$

*Define*

$$\tilde{\eta}=\begin{cases}\eta-2, & \eta<1\\ -1, & \eta>1\end{cases}.$$

*We then have in both the interior region* $\{r<t-1\}$ *(without additional restrictions on* $\alpha+\beta+\eta$*), and in the exterior region* $\{r>t+1\}$ *in the case* $\alpha+\beta+\eta>3$*:*

$$\psi(t,x)\lesssim\frac{1}{\langle r\rangle\langle u\rangle^{\alpha+\beta+\tilde{\eta}-1}}. \tag{5.4.7}$$

*On the other hand, if* $\alpha+\beta+\eta<3$*,* $\eta\geq 0$[1] *and* $r\geq t+1$*, we have*

$$\psi(t,x)\lesssim r^{2-(\alpha+\beta+\eta)}. \tag{5.4.8}$$

*Proof.* A detailed proof of (7.4.6) can be found in Lemma 5.5 of [60] (see also Lemma 6.1 in [120]). The idea is to use Sobolev embedding and the positivity of the fundamental solution of $\Box$ to show that

$$r\psi\lesssim\int_{D_{tr}}\rho h(s,\rho)dsd\rho,$$

where $D_{tr}$ is the backwards light cone with vertex $(r,t)$, and use the pointwise bounds on $h$.

[1] We only use $\eta\geq 0$ in Section 7.6, wherein we iterate in $\{r>t+1\}$.

Let us now prove (7.4.7). In this case $D_{tr} \subset \{r-t \leq u' \leq r+t, \quad r-t \leq \rho \leq r+t\}$ and we obtain, using that $u' \leq \rho$ and $\rho \gtrsim \rho + s$ in $D_{tr}$:

$$r\psi \lesssim \int_{r-t}^{r+t} \int_{u'}^{r+t} \langle \rho \rangle^{1-\alpha-\beta} d\rho \, \langle u' \rangle^{-\eta} du' \lesssim \int_{r-t}^{r+t} \langle u' \rangle^{2-(\alpha+\beta+\eta)} du' \lesssim (t+r)^{3-(\alpha+\beta+\eta)}$$

where the final bound follows from the hypothesis that $\alpha + \beta + \eta < 3$. This finishes the proof. □

In view of (5.4.4), we will also need the following result for an inhomogeneity of the form $\partial_t g$ supported near the cone. The result is similar to Lemma 7.4.4, except that we gain an extra factor of $\langle u \rangle$ in the estimate.

**Lemma 5.4.3.** *Let $\psi$ solve*

$$\Box \psi = \partial_t g, \qquad \psi(0) = 0, \quad \partial_t \psi(0) = 0, \tag{5.4.9}$$

*where $g$ is supported in $\{t/2 \leq |x| \leq 3t/2\}$. Let $h$ be as in (7.4.5), and assume that*

$$|h| + |Sh| + |\Omega h| + \langle t - r \rangle |\partial h| \lesssim \frac{1}{\langle r \rangle^{\alpha} \langle u \rangle^{\eta}}, \quad 2 < \alpha < 3, \quad \eta \geq -1/2.$$

*We then have in the interior region $\{r < t - 1\}$, as well as in the exterior region $\{r > t + 1\}$ in the case $\alpha + \eta > 3$:*

$$\psi(t, x) \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{\alpha + \tilde{\eta}}}. \tag{5.4.10}$$

*Proof.* Let $\widetilde{\psi}$ be the solution to

$$\Box \widetilde{\psi} = g, \qquad \widetilde{\psi}[0] = 0.$$

Clearly $\psi = \partial_t \widetilde{\psi}$. We also note that in the support of $g$ we have

$$(t\partial_i + x_i \partial_t) h \lesssim |Sh| + |\Omega h| + \langle t - r \rangle |\partial_r h|.$$

We now apply Lemma 7.4.4 (with $\beta = 0$) to $\nabla\widetilde{\psi}$, $\Omega\widetilde{\psi}$, $S\widetilde{\psi}$, and use the fact that

$$\langle u\rangle \partial_t \widetilde{\psi} \lesssim |\nabla\widetilde{\psi}| + |S\widetilde{\psi}| + |\Omega\widetilde{\psi}| + \sum_i |(t\partial_i + x_i\partial_t)\widetilde{\psi}|.$$

□

### 5.4.4 Derivative bounds

We will now derive better bounds for the derivatives; roughly speaking, the derivative gains $1/\langle r\rangle$ away from the cone, and $1/\langle u\rangle$ near the cone. The idea is that we can use Morawetz type estimates in the various dyadic regions defined in Section 2.

**Proposition 5.4.4.** *Let $\phi$ solve (7.1.2), and assume that*

$$\phi_{\leq m+n} \lesssim \langle r\rangle^{-\alpha}\langle t\rangle^{-\beta}\langle u\rangle^{-\eta}, \tag{5.4.11}$$

*for some sufficiently large $n$. We then have*

$$\partial\phi_{\leq m} \lesssim \langle r\rangle^{-\alpha}\langle t\rangle^{-\beta}\langle u\rangle^{-\eta}\nu^{-1}, \quad \nu := \min(\langle r\rangle, \langle u\rangle)$$

*Proof.* It is enough to show that, if $\mathcal{R} \in \{C_T^R, C_R^T\}$, we have

$$\|\partial\phi_{\leq m}\|_{L^\infty(\mathcal{R})} \lesssim \frac{1}{R}\|\phi_{\leq m+n}\|_{L^\infty(\tilde{\mathcal{R}})} + R^{-1/2}\|\partial\phi_{\leq m+4}\|_{L^\infty(\tilde{\mathcal{R}})} \tag{5.4.12}$$

and if $\mathcal{R} = C_T^U$, then

$$\|\partial\phi_{\leq m}\|_{L^\infty(\mathcal{R})} \lesssim \frac{1}{U}\|\phi_{\leq m+n}\|_{L^\infty(\tilde{\mathcal{R}})} + U^{-1/2}\|\partial\phi_{\leq m+4}\|_{L^\infty(\tilde{\mathcal{R}})} \tag{5.4.13}$$

Indeed, plugging (5.4.11) into the bounds (5.4.12) and (5.4.13) yields for a large enough value of $n$:

$$\|\partial\phi_{\leq m+\frac{1}{2}n}\|_{L^\infty(\mathcal{R})} \lesssim \langle r\rangle^{-\alpha}\langle t\rangle^{-\beta}\langle u\rangle^{-\eta}\nu^{-1/2}$$

and now plugging the above estimate in (5.4.12) and (5.4.13) finishes the proof.

We now prove (5.4.12) and (5.4.13). Given a function $w$, we have

$$\|\partial w_{\leq m}\|_{L^2(\mathcal{R})} \lesssim \|\frac{w_{\leq m+n}}{\nu}\|_{L^2(\tilde{\mathcal{R}})} + \|\langle r\rangle (Pw)_{\leq m}\|_{L^2(\tilde{\mathcal{R}})}. \tag{5.4.14}$$

(We refer the reader to [60] or [74] for a proof.)

We remark that the second part of (5.3.2) can now be improved to

$$|\partial \phi_{\leq N_1}| \leq \frac{\epsilon}{\langle r\rangle \nu^{\frac{1}{2}}}. \tag{5.4.15}$$

when $r > t$. Indeed, this follows by the same arguments from Proposition 5.2.5 in the region $C_R^R$.

Note now that, by (5.3.2) and (5.4.15), we have that

$$(P\phi)_{\leq m} \lesssim \left|\left(\partial \phi_{\leq \frac{m}{2}}\right)\left(\partial \phi_{\leq m}\right)\right| \lesssim \frac{1}{\langle r\rangle \nu^{1/2}} |\partial \phi_{\leq m}| \tag{5.4.16}$$

and thus (7.3.9) and (5.4.16) imply

$$\|\partial \phi_{\leq m}\|_{L^2(\mathcal{R})} \lesssim \|\frac{\phi_{\leq m+n}}{\nu}\|_{L^2(\tilde{\mathcal{R}})} + \|\nu^{-1/2}\partial \phi_{\leq m}\|_{L^2(\tilde{\mathcal{R}})}. \tag{5.4.17}$$

We now return to Lemma 5.2.2, using Lemma 7.3.1 and (5.4.16) to bound the second-order derivatives pointwise and (5.4.17) to bound the first-order derivatives in $L^2$. We find

$$\begin{aligned}\|\partial \phi_{\leq m}\|_{L^\infty(\mathcal{R})} &\lesssim |\mathcal{R}|^{-\frac{1}{2}} \left(\|\frac{\phi_{\leq m+n}}{\nu}\|_{L^2(\tilde{\mathcal{R}})} + \|\nu^{-1/2}\partial \phi_{\leq m}\|_{L^2(\tilde{\mathcal{R}})}\right) + |\mathcal{R}|^{-\frac{1}{2}} \|\langle r\rangle (P\phi)_{\leq m+4}\|_{L^2(\tilde{\mathcal{R}})} \\ &\lesssim \|\frac{\phi_{\leq m+n}}{\nu}\|_{L^\infty(\tilde{\mathcal{R}})} + \|\frac{\partial \phi_{\leq m+4}}{\nu^{1/2}}\|_{L^\infty(\tilde{\mathcal{R}})},\end{aligned}$$

where the second line follows by (5.4.16). This finishes the proof of (5.4.12) and (5.4.13). □

## 5.5 The iteration in $\{r > t+1\}$

In this section we prove the optimal pointwise bounds in the region $r > t+1$.

**Theorem 5.5.1.** *If $r > t+1$, then*

$$\phi_{\leq m} \lesssim \langle r\rangle^{-1}\langle u\rangle^{-1}.$$

*Proof.* We begin with the bounds (5.3.1) and (5.4.15), combined with (5.3.3), which in the outside region translate to

$$|\phi_{\leq m+n}| \lesssim \frac{\langle u\rangle^{1/2}}{\langle r\rangle}, \quad |\partial\phi_{\leq m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2}}, \quad \bar{\partial}\phi_{\leq m+n} \lesssim \frac{\langle u\rangle^{1/2}}{\langle r\rangle^{2}}. \tag{5.5.1}$$

Since $\langle u\rangle \leq \langle r\rangle$, this can be weakened to

$$|\phi_{\leq m+n}| \lesssim \frac{1}{\langle r\rangle^{1/2}}, \quad |\partial\phi_{\leq m+n}| \lesssim \frac{1}{\langle r\rangle^{1/2}\langle u\rangle}, \quad \bar{\partial}\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{3/2}}. \tag{5.5.2}$$

Recall the decomposition (7.4.4), and let

$$H_i = \sum_{k=0}^{2} \|\Omega^k (G_i)_{\leq n}(t, r\omega)\|_{L^2(\mathbb{S}^2)}.$$

We thus have, using (7.6.2) (and (7.6.1) for $H_3$):

$$H_1 \lesssim \frac{1}{\langle r\rangle^{5/2+\sigma}}, \quad \partial_t H_2 \lesssim \frac{1}{\langle r\rangle^{3/2+\sigma}\langle u\rangle}, \quad H_3 \lesssim \frac{1}{\langle r\rangle^{2+\lambda}\langle u\rangle^{1-\lambda}}, \quad \lambda \in (0,1)$$

By (7.4.7) with $\alpha = 5/2+\sigma$, $\beta = 0$, and $\eta = 0$, we obtain

$$(\phi_1)_{\leq m+n} \lesssim r^{-1/2-\sigma}$$

which gains a factor of $\langle r\rangle^{-\sigma}$ compared to (7.6.2). Similarly (7.4.7) with $\alpha = 3/2+\sigma$, $\beta = 0$, and $\eta = 1$ yields

$$(\phi_2)_{\leq m+n} \lesssim r^{-1/2-\sigma}$$

Finally, (7.4.7) with $\alpha = 2 + \sigma$, $\beta = 0$, and $\eta = 1/2$ yields

$$(\phi_3)_{\leq m+n} \lesssim r^{-1/2-\sigma}$$

The three inequalities above, combined with Proposition 7.3.7 and (5.3.3), give the following improved bounds (by a factor of $\langle r \rangle^{-\sigma}$)

$$|\phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle^{1/2+\sigma}}, \quad |\partial \phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle^{1/2+\sigma} \langle u \rangle}, \quad \overline{\partial} \phi_{\leq m+n} \lesssim \frac{1}{\langle r \rangle^{3/2+\sigma}}. \tag{5.5.3}$$

We now repeat the iteration, replacing $\alpha$ by $\alpha + \sigma$ and applying (7.4.7). The process stops after $\lfloor \frac{1}{2\sigma} \rfloor$ steps, when (7.4.7), combined with Proposition 7.3.7 and (5.3.3), yield

$$|\phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle}, \quad |\partial \phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle}, \quad \overline{\partial} \phi_{\leq m+n} \lesssim \frac{1}{\langle r \rangle^2}. \tag{5.5.4}$$

We now switch to using (7.4.6) for $\phi_1$ and $\phi_3$, and (7.4.10) for $\phi_2$. Note that (7.6.4) implies

$$H_1 \lesssim \frac{1}{\langle r \rangle^{3+\sigma}}, \quad H_2 \lesssim \frac{1}{\langle r \rangle^{2+\sigma}}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^3 \langle u \rangle}$$

By (7.4.6) with $\alpha = 2 + \sigma$, $\beta = 1$, and $\eta = 0$, we obtain

$$(\phi_1)_{\leq m+n} \lesssim r^{-1} \langle u \rangle^{-\sigma}$$

Similarly (7.4.10) with $\alpha = 2 + \sigma$, and $\eta = 0$ yields

$$(\phi_2)_{\leq m+n} \lesssim r^{-1} \langle u \rangle^{-\sigma}$$

Finally, (7.4.6) with $\alpha = 5/2$, $\beta = 1/2$, and $\eta = \sigma$ yields

$$(\phi_3)_{\leq m+n} \lesssim r^{-1} \langle u \rangle^{-\sigma}$$

The three inequalities above, combined with Proposition 7.3.7 and (5.3.3), give the following improved bounds (by a factor of $\langle u\rangle^{-\sigma}$)

$$|\phi_{\leq m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{\sigma}}, \quad |\partial\phi_{\leq m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1+\sigma}}, \quad \overline{\partial}\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{2}\langle u\rangle^{\sigma}}. \tag{5.5.5}$$

We now repeat the iteration, but we carefully observe that this is the last step where we can improve decay for $\phi_3$. Indeed, we have

$$H_3 \lesssim \frac{1}{\langle r\rangle^{3}\langle u\rangle^{1+2\sigma}}$$

and (7.4.6) now yields

$$(\phi_3)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-1}$$

which does not improve as we gain powers of $\langle u\rangle$. On the other hand, we can continue improving the decay rates of $\phi_1$ and $\phi_2$ all the way to

$$(\phi_1)_{\leq m}, (\phi_2)_{\leq m} \lesssim r^{-1}\langle u\rangle^{-1}$$

which finishes the proof.

□

## 5.6 The iteration in $\{r < t-1\}$

In the interior region the iteration is similarly based on Lemma 7.4.4, with an additional twist. It turns out that plugging in a bound of $\langle r\rangle^{-1}\langle u\rangle^{-\eta}$ on the right hand side is not enough to gain decay. Instead, we first need to turn the $r^{-1}$ factor into a $t^{-1}$ factor.

### 5.6.1 Converting $r$ decay to $t$ decay

We start with the following lemma:

**Lemma 5.6.1.** *Assume that $\phi$ solves (7.1.2). We then have*

$$\|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim T^{-1}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1(C_T^{<3T/4})} + \|Q_{\leq m+n}\|_{LE^*(C_T^{<3T/4})}. \tag{5.6.1}$$

*Proof.* One begins with (5.1.6), and we may assume that $\phi$ is supported in $C_T^{<3T/4}$ because the commutator $[P, \chi_{C_T^{<3T/4}}]$ can be controlled (here, $\chi_{C_T^{<3T/4}}$ is a smooth cutoff). Thus

$$\|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim \|\partial\phi_{\leq m}(T)\|_{L^2_x} + \|Q_{\leq m}\|_{LE^*(C_T^{<3T/4})}$$

and from here transitions the spatial norm $L^2_x$ to the $L^2_{t,x}$ norm in the usual way (averaging in time using the scaling vector field $S$) and then transitions to the $LE^1$ norm. For details, we refer the reader to [60] or [74]. □

The next proposition uses the previous lemma to turn $r$-decay in $\{r < t/2\}$ into $t$-decay.

**Proposition 5.6.2.** *Let $\phi$ solve (7.1.2). Assume that*

$$\phi_{\leq m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{-q}, \quad \partial\phi_{\leq m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{-1-q+\sigma}, \tag{5.6.2}$$

*for some $q \geq -1/2$. We then have*

$$\|\phi_{\leq m}\|_{L^\infty(C_T^{<3T/4})} \lesssim \langle t\rangle^{-1}\langle u\rangle^{-q}. \tag{5.6.3}$$

*Proof.* We estimate the right hand side of (5.6.1). Using (5.6.2) we compute

$$T^{-1}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1(C_T^{<3T/4})} \lesssim T^{-1}T^{1/2-q} = T^{-1/2-q}$$

$$\|Q_{\le m+n}\|_{LE^*(C_T^{<3T/4})} \lesssim \sum_R \|\langle r\rangle^{-3/2} t^{-2-2q+2\sigma}\|_{L^2((T,2T)\times A_R)} \lesssim T^{-3/2-2q+2\sigma} \ln T \lesssim T^{-1/2-q}$$

where the last inequality holds for all $q > -1 + 2\sigma$. Therefore Lemma 7.7.2 implies

$$\|\phi_{\le m}\|_{LE^1(C_T^{<3T/4})} \lesssim T^{-1/2-q},$$

and the conclusion follows by (5.2.6). □

### 5.6.2 The iteration

Finally, we are ready to improve the bounds in the interior region.

**Theorem 5.6.3.** *If $r < t - 1$, then*

$$\phi_{\le m} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-1}.$$

*Proof.* We have the following bound that is similar to (5.3.3):

$$\bar{\partial}\phi \in S^Z\Big(\frac{\langle u\rangle}{t}\Big)\partial\phi + S^Z\Big(\frac{1}{t}\Big)Z\phi. \tag{5.6.4}$$

As before, we begin with the bounds (5.3.1) and (5.3.2), combined with (5.6.4), which in the inside region translate to

$$|\phi_{\le m+n}| \lesssim \frac{\langle u\rangle^{1/2}}{\langle t\rangle}, \quad |\partial\phi_{\le m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2}}, \quad \bar{\partial}\phi_{\le m+n} \lesssim \frac{\langle u\rangle^{1/2}}{\langle r\rangle\langle t\rangle}. \tag{5.6.5}$$

We thus have, using (7.7.8):

$$H_1 \lesssim \frac{\langle u\rangle^{1/2}}{\langle r\rangle^{2+\sigma}\langle t\rangle}, \quad \partial_t H_2 \lesssim \frac{1}{\langle r\rangle^{1+\sigma}\langle t\rangle\langle u\rangle^{1/2}}, \quad H_3 \lesssim \frac{1}{\langle r\rangle^2\langle t\rangle}$$

By (7.4.6) with $\alpha = 2+\sigma$, $\beta = 1$, and $\eta = -1/2$, we obtain

$$(\phi_1)_{\le m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma}$$

Similarly (7.4.6) with $\alpha = 2+\sigma$, $\beta = 0$, and $\eta = 1/2$ yields

$$(\phi_2)_{\le m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma}$$

Finally, (7.4.6) with $\alpha = 2+\sigma$, $\beta = 1-\sigma$, and $\eta = 0$ yields

$$(\phi_3)_{\le m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma}$$

The three inequalities above give

$$\phi_{\le m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma}$$

Now recall that by (7.7.8) we also have that

$$|\partial\phi_{\le m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2}}$$

We can now apply Proposition 5.6.2 (with $q = -1/2+\eta$), in conjunction with Proposition 7.3.7 and (5.6.4), to obtain the following improved bounds (by a factor of $\langle u\rangle^{-\sigma}$):

$$|\phi_{\le m+n}| \lesssim \frac{\langle u\rangle^{1/2-\sigma}}{\langle t\rangle}, \quad |\partial\phi_{\le m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2+\sigma}}, \quad \bar{\partial}\phi_{\le m+n} \lesssim \frac{\langle u\rangle^{1/2-\sigma}}{\langle r\rangle\langle t\rangle}. \tag{5.6.6}$$

We now repeat the iteration, replacing $\eta$ by $\eta+\sigma$, applying (7.4.7) and then turning the $r$ decay into $t$ decay by Proposition 5.6.2. The process stops after $\lfloor \frac{1}{2\sigma} \rfloor$ steps, when (7.4.6), combined with Proposition 5.6.2, Proposition 7.3.7 and (5.6.4), yield

$$|\phi_{\le m+n}| \lesssim \frac{1}{\langle t\rangle}, \quad |\partial\phi_{\le m+n}| \lesssim \frac{1}{\langle r\rangle\langle u\rangle}, \quad \bar{\partial}\phi_{\le m+n} \lesssim \frac{1}{\langle r\rangle\langle t\rangle}. \tag{5.6.7}$$

At this point we switch to using (7.4.10) for $\phi_2$, and the iteration process follows the same pattern as in Section 6, with the extra use of Proposition 5.6.2 to turn factors of $\langle r\rangle^{-1}$ into factors of $\langle t\rangle^{-1}$.

□

## Chapter 6 Scattering for the quintic-power nonlinear wave equation

### 6.1 Introduction

In Minkowski space, solutions of the equation

$$\Box u = |u|^{p-1}u$$

with $\Box = -\partial_t^2 + \Delta$ have a conserved and positive-definite energy

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\,dx$$

and the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{2}{1-p}} u(\frac{t}{\lambda}, \frac{x}{\lambda}).$$

In three dimensions, the exponent $p = 5$ is called the energy-critical exponent, because solutions of the equation have an energy that is invariant under the scaling symmetry.

For the Cauchy problem with initial data in the energy space $\dot{H}^1 \times L^2$, local well-posedness is proven for $1 < p \leq 5$ by Strichartz estimates. Global existence for small initial data is a straightforward adaptation of the proof of local existence. In addition, there is global existence for large initial data due to the existence of a blowup criterion, which informally says that the energy cannot concentrate at any point in spacetime. Moreover, given any finite-energy initial data there is a unique global solution with finite energy lying in $L^4L^{12}([0,\infty)\times\mathbb{R}^3)$; these solutions are known as strong (or Shatah-Struwe) solutions. See [31], [34], [32], [33], [44], [83], [84], [87], [95], [98], [99], [113] for details and more. The results in [10] and [9] then combine to prove scattering of solutions with finite-energy initial data using a profile decomposition, which describes the failure of a sequence of uniformly bounded solutions to the free wave equation to be compact in the sense of Strichartz estimates. A similar result holds for the focusing equation with energy below that of the ground state: see [46].

This chapter considers the equation

$$\begin{cases} Pu(t,x) = u(t,x)^5 \quad (t,x) \in (0,\infty) \times \mathbb{R}^3, \quad P = \partial_\alpha g^{\alpha\beta} \partial_\beta \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{6.1.1}$$

Global existence and uniqueness of strong solutions (lying in $C(\mathbb{R}_t, \dot{H}^1) \cap L^5_{loc} L^{10}$) was shown in [37] in the stationary setting. A similar result for classical solutions in the non-stationary setting was shown in [54]. These results require minimal assumptions on the coefficients, as eliminating the blowup scenario only requires local-in-time arguments.

Our main theorem establishes scattering of strong solutions to (6.1.1) for certain small, asymptotically flat perturbations of the Minkowski metric. To the authors' knowledge, this is the first such result for small perturbations of the Minkowski metric $m$ with variable coefficients.

**Definition 6.1.1** (Scattering in the energy space)**.** We say that the solution $u$ to (6.1.1) *scatters in the energy space* if there exists $(f,g) \in \dot{H}^1 \times L^2$ such that

$$\lim_{t\to\infty} \|u[t] - S(t,0)(f,g)\|_{\dot{H}^1 \times L^2} = 0.$$

**Definition 6.1.2.** We define $\underline{L} := \sum_{i=1}^3 \frac{x^i}{|x|} \partial_{x_i} - \partial_t$.

**Theorem 6.1.3.** *Let $g^{\alpha\beta}(t,x)$ be a Lorentzian metric, let $P = \partial_\alpha g^{\alpha\beta}\partial_\beta$, and let $h := g - m$ denote the perturbative terms of the metric $g$. The unique global strong solution to the Cauchy problem (6.1.1) scatters in the energy space $\dot{H}^1 \times L^2$ provided that*

$$|h| \lesssim \epsilon \frac{\langle t-|x|\rangle^{1/2}}{\langle r\rangle^\gamma \langle t+|x|\rangle^{1/2}}, \tag{6.1.2}$$

$$|h^{\underline{L}\,\underline{L}}| \lesssim \epsilon \frac{\langle t-|x|\rangle}{\langle r\rangle^\gamma \langle t+|x|\rangle}, \tag{6.1.3}$$

$$|\partial^J h| \lesssim \epsilon \frac{1}{\langle r\rangle^{|J|+\gamma}} \quad \text{for } |J| = 1 \text{ and } |J| = 2 \tag{6.1.4}$$

*where $\gamma > 0$ is an arbitrarily small constant and $\epsilon > 0$ is a sufficiently small constant. In these assumptions, $\partial^J h$ denotes $\partial^J h^{\alpha\beta}$ for all multi-indices $\alpha$ and $\beta$, and $h^{\underline{L}\,\underline{L}} = h^{\alpha\beta}\underline{L}_\alpha \underline{L}_\beta$, where we lower indices with respect to the Minkowski metric.*

This says that the unique global solution of the non-linear problem on small, asymptotically flat perturbations of Minkowski space that have appropriate decay at infinity behave, in the asymptotic sense, like the solution to the linear homogeneous problem $Pu = 0$, at least in the energy space.

*Remark* 6.1.4. The assumptions (6.1.2), (6.1.3), and (6.1.4) are satisfied by metrics that arise as solutions to Einstein's Vacuum Equations when expressed in harmonic coordinates, see [**?**].

*Remark* 6.1.5. One of the key ingredients in our proof is the fact that Strichartz estimates for the linear problem hold. Assuming

$$|\partial^J h| \lesssim \epsilon \frac{1}{\langle r\rangle^{|J|+\gamma}} \quad \text{for } 0 \le |J| \le 2$$

this was proved by Metcalfe-Tataru [75]. Our assumptions are the same, except that we require more decay of $h$ (but not its derivatives) near the light cone. This is due to the fact that we need to control certain boundary terms that appear when multiplying the equation by $t\partial_t + r\partial_r$. In particular the extra decay requirement (6.1.3) is needed to control the term $|\underline{L}u|^2$ on the boundary, and geometrically it implies that the light cones of the perturbed metric are comparable to those of the Minkowski metric.

*Remark* 6.1.6. Theorem 6.1.3 also holds if we replace $P$ by the geometric wave operator

$$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\alpha \sqrt{|g|} g^{\alpha\beta}\partial_\beta, \quad |g| := |\det g^{\alpha\beta}|.$$

Indeed, in the estimates below one integrates with respect to the volume form $\sqrt{|g|}dtdx$, and uses the fact that $\sqrt{|g|}g^{\alpha\beta} \approx g^{\alpha\beta}$. There are extra error terms of the form $\partial\sqrt{|g|}u^6$ arising, which can be absorbed since by (6.1.4) we have

$$\partial\sqrt{|g|} \lesssim \frac{\epsilon}{\langle x\rangle^{1+\gamma}}$$

*Remark* 6.1.7. A key tool in proving scattering on variable-coefficient backgrounds is local energy decay. Such an estimate was proven in [79], [111], and [45] for Minkowski space and in [**?**], [76] for perturbations of Minkowski space, and became a valuable tool in the study of both linear and nonlinear problems. In particular, they imply Strichartz estimates on certain variable-coefficient backgrounds, see [75]. Our result is one of several showing that local energy decay is fruitful for understanding the long-time behavior and asymptotics of solutions to nonlinear dispersive equations on variable-coefficient backgrounds.

*Remark* 6.1.8. For the energy-critical problem on Minkowski space, global a priori estimates were proven in [83], [84], from which scattering for the wave and Klein-

Gordon equations were deduced. Analogous results in the exterior of obstacles were obtained in [45], [104], [18], [28]. Scattering on Profile decompositions akin to [9] have been shown for waves on hyperbolic space in [55]; a similar result was shown for $(\Box + a|x|^{-2})u = u^5$ in [78]. Finally, for the equation $(\Box + V(x))u = u^5$, scattering to steady states was shown in [39], [40].

For the nonlinear Schrödinger equation, the energy-critical problem for the defocusing quintic problem with initial data in the energy space for small, compactly supported perturbations of the Euclidean metric also exhibits scattering to linear solutions for all finite-energy data, as shown in [38]. There are many other known results for the energy-critical energy Schrödinger with potential, and for the exterior of a strictly convex obstacle, see [128], [48], [49] etc.

## 6.2 Notation and Preliminaries

We write $\nabla = (\partial_t, \nabla_x)$ for the spacetime gradient. Throughout the chapter, we use the Einstein summation convention, and we let Greek (resp. Latin) indices denote spacetime (resp. space) indices. We write $u[T] = (u(T,x), \partial_t u(T,x))$.

The energy of the solution $u$ is defined to be

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}u(t,x)^6 \, dx.$$

We will also use the notation

$$E_K(t) := \int_K \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}u(t,x)^6 \, dx$$

for some subset $K$ of $\mathbb{R}^3$.

In this chapter, we let $P$ denote

$$P = \partial_\alpha g^{\alpha\beta} \partial_\beta$$

where $g = g(t, x)$ is a Lorentzian metric.

For any $(f, g) \in \dot{H}^1 \times L^2$, we denote by $S(t, s)(f, g)$ the unique solution $u \in C(\mathbb{R}_t, \dot{H}^1)$ with $\partial_t u \in C(\mathbb{R}_t, L^2)$ to the equation

$$\begin{cases} Pu = 0 & (t, x) \in (s, \infty) \times \mathbb{R}^3 \\ u[s] = (f, g) \end{cases} \tag{6.2.1}$$

Let

$$X = \left( C(\mathbb{R}_t, \dot{H}^1) \cap L^5_{loc} L^{10} \right) \times C(\mathbb{R}_t, L^2)$$

and for any closed, finite interval

$$X(I) = \left( C(I, \dot{H}^1) \cap L^5(I) L^{10} \right) \times C(I, L^2)$$

Consider the Cauchy problem

$$\begin{cases} Pu(t, x) = u(t, x)^5 & (t, x) \in (0, \infty) \times \mathbb{R}^3 \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{6.2.2}$$

By Duhamel's formula, classical solutions to (6.2.2) satisfy

$$u(t) = S(t, 0)u[0] + \int_0^t \frac{1}{g^{00}} S(t, s)(0, u^5(s)) ds \tag{6.2.3}$$

We can thus define a strong solution to be a solution of (6.2.3) so that $(u, \partial_t u)$ also lies in $X$.

The results of [37] and [54] show that, for smooth initial data, there is a unique global classical solution to (6.2.2) that is also a strong solution. Moreover, this result is extended to initial data in the energy space in [37] for time-independent coefficients, and the same argument can be used to prove it in the time-dependent case. We will be interested in studying the asymptotic properties of the unique strong solution in the energy space, in particular the fact that it approaches a solution to the linear equation in the energy space.

## 6.3 Uniform energy bounds and local energy decay for the nonlinear problem on perturbations

We now come to certain key tools, analogous to the results presented in the previous section, that will be used in the proof of scattering for the non-linear problem on certain perturbations of Minkowski space that have appropriate decay at infinity.

If $g(t, x)$ is a non-stationary metric that satisfies certain decay conditions, and $u$ is a solution of the Cauchy problem (6.2.2), then we have uniform energy bounds: with the energy now defined to be

$$E(t) := \int_{\{t\}\times\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{1}{6}|u|^6 \, dx,$$

if $g$ and its derivatives satisfy certain decay conditions, then

$$E(T) \lesssim E := E(0)$$

for some implicit constant that is independent of $T$. In fact, we may prove local energy decay and uniform energy bounds in one fell swoop for (6.2.2), as the following proposition shows.

**Theorem 6.3.1** (Integrated local energy decay for the nonlinear Cauchy problem)**.** *Let $u$ be a solution of (6.2.2) and let $|J| \leq 1$ be a multi-index. If $\partial^J h^{\alpha\beta} \lesssim \epsilon\langle r\rangle^{-|J|-\gamma}$*

*where* $\gamma > 0$ *is an arbitrarily small constant and* $\epsilon > 0$ *is a sufficiently small constant then*

$$\|u\|^2_{LE^1[T_1,T_2]} + E(T_2) \lesssim E(T_1) \tag{6.3.1}$$

*for some implicit constant that is independent of* $T_1$ *and* $T_2$*, where*

$$\|u\|^2_{LE^1[T_1,T_2]} = \iint_{[T_1,T_2]\times\mathbb{R}^3} \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}} + \frac{u^2}{\langle r\rangle^{3+\gamma}} + \frac{|u|^6}{\langle r\rangle}\, dxdt$$

Let us first assume that $u$ is a classical solution to the equation

$$Pu = u^5 + F$$

Following the discussion in Chapter 3, we multiply the equation by $a(r)u + b(r)\partial_r u + C\partial_t u$, with

$$b(r) = \sum_{j=0}^{\infty} 2^{-j\gamma}\frac{r}{r+2^j}, \quad a(r) = b(r)/r.$$

Upon integrating by parts, we obtain

$$E(T_2) + \iint_{[T_1,T_2]\times\mathbb{R}^3} \frac{1}{2}b'(u_r^2 + u_t^2) - (\frac{1}{2}b' - \frac{b}{r})|\partial\!\!\!/ u|^2 - \frac{\Delta a}{2}u^2 + Err$$
$$+ (\frac{2}{3}a(r) - \frac{1}{6}b'(r))u^6\, dxdt \lesssim E(T_1) + \iint_{[T_1,T_2]\times\mathbb{R}^3} |F|(|\nabla u| + \frac{|u|}{\langle r\rangle})\, dxdt.$$

where the error satisfies

$$Err \lesssim (\frac{|h|}{\langle x\rangle} + |\nabla h|)(|\nabla u|^2 + |\nabla u|\frac{|u|}{\langle x\rangle})$$

Since $\partial^J h^{\alpha\beta} \lesssim \epsilon\langle r\rangle^{-|J|-\gamma}$, we can estimate by Cauchy-Schwarz

$$Err \lesssim \epsilon\Big(\frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}} + \frac{u^2}{\langle r\rangle^{3+\gamma}}\Big)$$

Moreover,

$$\frac{2}{3}a(r) - \frac{1}{6}b'(r) = \sum_{j=0}^{\infty} 2^{-j\gamma} \left( \frac{1}{2}\frac{1}{r+2^j} + \frac{1}{6}\frac{r}{(r+2^j)^2} \right) \gtrsim \frac{1}{\langle r \rangle}$$

Taking (3.1.2) into account, and applying Hölder and Hardy to control the inhomogeneity, we get

$$\|u\|^2_{LE^1[T_1,T_2]} + E(T_2) \lesssim E(T_1) + \|F\|_{L^1[T_1,T_2]L^2} \|\nabla u\|_{L^\infty[T_1,T_2]L^2} \tag{6.3.2}$$

Consider now a strong solution $u$, and a sequence of classical solutions $u_n$ so that $u_n(T_1) \to u(T_1)$ in the energy norm. After dividing the interval $I = [T_1, T_2]$ into finitely many intervals so that the $L^5L^{10}$ norm of $u$ is suitably small on each interval, a contraction argument shows that $u_n \to u$ in $X(I)$. In particular this implies that $u_n^5$ is a Cauchy sequence in $L^1[T_1, T_2]L^2$, and thus by (7.2.8) we must have $u_n \to u$ in $LE^1[T_1, T_2]$. The desired conclusion (7.2.7) now follows.

### 6.4 $L^6$ norm decay of solutions in Minkowski space

In order to motivate the the next section, which contains the main result and its proof, in this section we shall present the highlights of the proof of $L^6$ norm decay in Minkowski space for the non-linear problem, as done in [10].

#### 6.4.1 More notation

First, we fix some notation which will be used for the rest of the chapter. Let

$$\Gamma = \{(t, x) : |x| - c < t, t > 0\}$$

be a forward solid light cone with $c \geq 0$ to be determined and let $\Gamma(I) = \Gamma \cap (I \times \mathbb{R}^3)$ where $I \subset [0, \infty)$ is a time interval. Let

$$D(T) = \{(t,x) : t = T,\ |x| - c < t\}$$

denote its $t = T$ slices and let $L(I) = \{(t,x) : t \in I, |x| - c = t\}$ denote the lateral boundary of $\Gamma(I)$ with

$$L_c(I) := L(I)$$

also used for emphasis, but usually we shall simply write $L(I)$.

Consider next the Cauchy problem

$$\begin{cases} \Box u = u^5 & (t,x) \in (0,\infty) \times \mathbb{R}^3 \\ u[0] \in \dot{H}^1 \times L^2 \end{cases} \tag{6.4.1}$$

We now sketch a proof of the $L^6$ norm decay in the energy space for solutions to (6.4.1) (see [99], [9] for more details). We will adapt this proof to the variable-coefficient case in the next section.

Given $\delta > 0$, pick $c$ sufficiently large so that the energy in the exterior region $|x| > c$ is less than $\delta/2$. Now given any $c \geq 0$, the flux on the time interval $I$ is defined to be the integral on the lateral boundary arising from multiplying the equation $\Box u = u^5$ by $\partial_t u$, namely

$$\text{flux}(I) = \int_{L(I)} \frac{1}{2}|\not{\partial} u|^2 + \frac{1}{2}|\partial_t u + \partial_r u|^2 + \frac{1}{6}u^6 \,\frac{d\sigma}{\sqrt{2}}$$

It is clear that the flux is non-negative. As the upper and lower limits of $I$ approach infinity, the flux decays, as an application of the divergence theorem in the interior of the $\Gamma(I)$ region shows. More precisely, if $I = [T_1, T_2]$, then in Minkowski space one obtains for the arbitrary number $c \geq 0$ chosen

$$E_{|x|<c+T_2}(T_2) = E_{|x|<c+T_1}(T_1) + \text{flux}([T_1, T_2]).$$

Thus $E_{|x|<c+t}(t)$ is monotone non-decreasing; moreover, it is bounded; therefore it converges to a limit as $t \to \infty$, as claimed. In particular, for all $T_2$ such that $T_2 > T_1$, and any $c \geq 0$,

$$\lim_{T_1 \to \infty} \lim_{T_2 \to \infty} \text{flux}([T_1, T_2]) = 0.$$

We now multiply (6.4.1) by

$$Xu := (t + c)\partial_t u + x^i \partial_i u + u = Su + c\partial_t u + u$$

where

$$S := t\partial_t + \sum_{i=1}^{3} x^i \partial_i u$$

and apply the divergence theorem in $\Gamma(I)$. We obtain

$$P(T_2) + \iint_{\Gamma(I)} \frac{u^6}{3}\, dxdt = P(T_1) + \int_{L(I)} (t + c) \left(\frac{Xu}{t + c}\right)^2 \frac{d\sigma}{\sqrt{2}} \tag{6.4.2}$$

where

$$P(T) := \int_{D(T)} \frac{t + c}{2} \left[ \left(\frac{Xu}{t + c}\right)^2 + \left(|\nabla_x u|^2 - \left(\frac{x}{t + c} \cdot \nabla_x u\right)^2\right) \right] + \frac{u^2}{t + c} + \frac{t + c}{6} u^6\, dx$$

Recall that on $L(I)$ we have $r = t + c$, enabling us to write $\frac{Xu}{t+c} = \partial_t u + \partial_r u + \frac{u}{t+c}$. By Cauchy-Schwarz and Hölder we obtain

$$\begin{aligned} \int_{L(I)} (t + c) \left(\frac{Xu}{t + c}\right)^2 \frac{d\sigma}{\sqrt{2}} &\lesssim (T_2 + c) \int_{L(I)} (\partial_t u + \partial_r u)^2\, d\sigma + \int \frac{u^2}{t + c}\, d\sigma \\ &\lesssim (T_2 + c)(\|(\partial_t + \partial_r)u\|^2_{L^2(L(I))} + \|u\|^2_{L^6(L(I))}) \end{aligned}$$

In summary,

$$T_2 \int_{D(T_2)} u^6 \, dx \lesssim P(T_2) + \iint_{\Gamma(I)} u^6 \, dxdt \lesssim P(T_1) + (T_2 + c)G(\text{flux}([T_1, T_2]))$$

$$\lesssim (T_1 + c)E_{|x|<T_1+c}(T_1) + (T_2 + c)G(\text{flux}([T_1, T_2]))$$

and

$$G(\theta) := \theta + \theta^{1/3}$$

is a function which decays to zero as its argument decays to zero. Take $T_1 = \delta T_2$ to see that, since $\delta$ was arbitrary and the flux decays,

$$\limsup_{t \to \infty} \|u(t, \cdot)\|_{L^6(\mathbb{R}^3)} = 0.$$

## 6.5 $L^6$ norm decay and scattering of solutions on small asymptotically flat perturbations of Minkowski space

We now come to the main result and its proof.

**Theorem 6.5.1** (Main Theorem)**.** *Let $u$ be the unique global strong solution of* (6.2.2).

1. *We make the following assumptions on the perturbation $h$:*

$$|\partial h| \lesssim \epsilon \langle r \rangle^{-1-\gamma} \tag{6.5.1}$$

$$|h| \lesssim \epsilon \frac{\langle t - r \rangle^{1/2}}{\langle r \rangle^{\gamma} \langle t + r \rangle^{1/2}} \tag{6.5.2}$$

$$|h^{\underline{LL}}| \lesssim \epsilon \frac{\langle t - r \rangle}{\langle r \rangle^{\gamma} \langle t + r \rangle} \tag{6.5.3}$$

*where $\gamma > 0$ is an arbitrarily small constant and $\epsilon > 0$ is a sufficiently small constant.*

*Then*

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_{L^6(\mathbb{R}^3)} = 0. \tag{6.5.4}$$

*2. If in addition*

$$|\partial^J h| \lesssim \epsilon \langle r\rangle^{-2-\gamma}, \quad |J| = 2, \tag{6.5.5}$$

*then $u$ scatters in the energy space.*

Recall that we define the normal derivative to the cone

$$\underline{L} = \frac{x^i}{|x|}\partial_i - \partial_0 = \frac{x^i}{|x|}\partial_i - \partial_t$$

and

$$h^{\underline{L}\,\underline{L}} = h^{\alpha\beta}\underline{L}_\alpha \underline{L}_\beta = h^{00} - 2\sum_i h^{0i}\frac{x^i}{|x|} + \sum_{i,j} h^{ij}\frac{x^i x^j}{|x|^2}.$$

We also remark that the decay rates on $h$ and $\partial h$ are consistent with the ones required for local energy decay Theorem 7.2.7 except near the cone $t \approx |x|$, where we need better decay rates to close the argument.

We now sketch the proof our the main theorem. Let us first assume that $u$ is a classical solution to the equation

$$Pu = u^5 + F \tag{6.5.6}$$

The main estimate of the chapter is the following:

**Proposition 6.5.2.** *If $u$ solves (6.5.6), and for any $R$, $T_1$ and $T_2$ so that $R \geq 0$, $1 < T_1$ [1], and $T_1 + 2R < T_2$, we have*

$$\begin{aligned}\int_{\mathbb{R}^3} u^6(T_2, x)dx &\lesssim \frac{T_1+2R}{T_2} E_{\{|x|<T_1+2R\}}(T_1) + \frac{E}{T_2^\gamma} \\ &+ G\Big(E_{\{|x|>T_1+R\}}(T_1) + \|u\|^2_{LE^1[T_1,T_2]} + \|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}\Big)\end{aligned} \tag{6.5.7}$$

[1] We shall only be interested in certain sufficiently large values of $T_1$ and $R$.

*where* $E := E(0)$.

We now make the key observation that, unlike in the case of Minkowski, flux($I$) is not necessarily nonnegative on arbitrary light cones. Instead, by averaging we obtain that there is $c \in [R, 2R]$ so that

$$\int_{L_c([T_1,T_2])} \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}}\, d\sigma \lesssim R^{-1}\|u\|^2_{LE^1[T_1,T_2]}. \tag{6.5.8}$$

For the rest of this proof, fix $c$ as above. Note that the hypothesis $T_1 + 2R < T_2$ implies that $T_2 \approx T_2 + c$. Moreover, in this case $c$ depends on $T_1$, so we make sure that we carefully track the dependence on $c$ in our estimates. In fact, the implicit constants do not depend on $c$, $T_1$, $T_2$, or $R$.

Proposition 6.5.2 follows from the results of Sections 6.1 and 6.2. We then finish the proof of Theorem 6.5.1 in Section 6.3.

### 6.5.1 $L^6$ norm decay of solutions on spacelike slices exterior to the cone

The next lemma shows that we can control both the outside energy and the flux through $L_c$. Note that, unlike in the Minkowski case, it is not clear that this can be done for all $c$.

**Lemma 6.5.3.** *Let $u$ solve (6.5.6). Then*

$$\begin{aligned} E_{\{|x|>T_2+c\}}(T_2) + \mathit{flux}([T_1,T_2]) \lesssim E_{\{|x|>T_1+c\}}(T_1) + \|u\|^2_{LE^1[T_1,T_2]} \\ +\|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}. \end{aligned} \tag{6.5.9}$$

*Here,*

$$\mathit{flux}([T_1,T_2]) := \int_{L_c([T_1,T_2])} \frac{1}{2}|\bar\partial u|^2 + \frac{1}{6}u^6\, \frac{d\sigma}{\sqrt{2}}$$

*and*

$$\bar{\partial} u := \{Lu, (r\sin\phi)^{-1}\partial_\theta u, r^{-1}\partial_\phi u\}, \quad L = \frac{x^i}{|x|}\partial_i + \partial_t$$

*denote the tangential derivatives of u to the light cone.*

*Proof.* Let $I = [T_1, T_2]$. Multiplying both sides of the equation in (6.5.6) by $\partial_t u$, we obtain the identity

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta u \partial_t u) - \frac{1}{2}\partial_t(g^{\alpha\beta}\partial_\beta u\partial_\alpha u) + \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\beta u \partial_\alpha u = \frac{1}{6}\partial_t(u^6) + F\partial_t u.$$

Define

$$\Gamma^{\text{ext}}(I) := \{T_1 \le t \le T_2, |x| > t + c\}, \quad D(T)^c := \Gamma^{\text{ext}}(I) \cap \{t = T\}$$

Applying the divergence theorem within the region $\Gamma^{\text{ext}}(I)$ leads to

$$\begin{aligned}\iint_{\Gamma^{\text{ext}}(I)} F\partial_t u \, dxdt = &\iint_{\Gamma^{\text{ext}}(I)} \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\beta u \partial_\alpha u \, dxdt + \\ &\int_{\partial\Gamma^{\text{ext}}(I)} \nu_\alpha g^{\alpha\beta}\partial_\beta u \partial_t u - \frac{1}{2}\nu_0 g^{\alpha\beta}\partial_\beta u \partial_\alpha u - \frac{1}{6}\nu_0 u^6 \, d\sigma.\end{aligned} \tag{6.5.10}$$

Next, let BD denote the part of the energy density on the boundary of $\Gamma^{\text{ext}}(I)$ arising from $h$

$$\text{BD} := \nu_\alpha h^{\alpha\beta}\partial_\beta u \partial_t u - \frac{1}{2}\nu_0 h^{\alpha\beta}\partial_\beta u \partial_\alpha u.$$

Note that BD depends on the domain of integration. Expanding (6.5.10), we have

$$\begin{aligned}&E_{\{|x|>T_2+c\}}(T_2) + \int_{D(T_2)^c} \text{BD} \, dx + \text{flux}([T_1, T_2]) + \iint_{\Gamma^{\text{ext}}(I)} \frac{1}{2}\partial_t h^{\alpha\beta}\partial_\alpha u \partial_\beta u \, dxdt \\ &= \iint_{\Gamma^{\text{ext}}(I)} F\partial_t u \, dxdt + E_{\{|x|>T_1+c\}}(T_1) + \int_{D(T_1)^c} \text{BD} \, dx + \int_{L_c(I)} \text{BD} \, d\sigma\end{aligned} \tag{6.5.11}$$

The space-time term is easy to estimate by (6.5.1)

$$\iint_{\Gamma^{\text{ext}}(I)} \frac{1}{2}\partial_t h^{\alpha\beta}\partial_\alpha u \partial_\beta u \, dxdt \lesssim \iint_{\Gamma^{\text{ext}}(I)} \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}} \, dxdt \le \|u\|^2_{LE^1[T_1,T_2]}. \tag{6.5.12}$$

Similarly, using that $|h| \lesssim \epsilon$, we obtain

$$\int_{D(T_j)^c} \text{BD}\, dx \lesssim \epsilon E_{\{|x|>T_j+c\}}(T_j), \quad j = 1, 2. \tag{6.5.13}$$

which can be absorbed in $E_{\{|x|>T_j+c\}}(T_j)$ for small enough $\epsilon$.

Finally, we need to estimate the perturbative error term on the lateral boundary; this is where we will use (6.5.2) and (6.5.3). We write

$$\begin{aligned} &\nu_\alpha h^{\alpha\beta}\partial_\beta = -\frac{1}{2} h^{\underline{L}\,\underline{L}}\underline{L} + O(h)\bar{\partial} \\ &\partial_t = \frac{1}{2}(L - \underline{L}) \\ &h^{\alpha\beta}\partial_\alpha u \partial_\beta u = \frac{1}{4} h^{\underline{L}\,\underline{L}}(\underline{L}u)^2 + O(h)\bar{\partial}u\partial u \end{aligned}$$

Note that, due to (6.5.3) and (6.5.2) we have that on $L(I)$

$$h^{\underline{L}\,\underline{L}} \lesssim \frac{R}{\langle x\rangle^{1+\gamma}}, \quad h \lesssim \epsilon \frac{R^{1/2}}{\langle x\rangle^{1/2+\gamma}} \tag{6.5.14}$$

and thus by Cauchy-Schwarz

$$\int_{L_c(I)} \text{BD}\, d\sigma \lesssim \int_{L_c(I)} |h^{\underline{L}\,\underline{L}}|(\underline{L}u)^2 + |h||\bar{\partial}u||\partial u| d\sigma \lesssim \epsilon \text{flux}([T_1, T_2]) + R \int_{L_c(I)} \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}}\, d\sigma \tag{6.5.15}$$

The conclusion of the lemma now follows from (6.5.8), (6.5.11), (6.5.12), (6.5.13), and (6.5.15). □

### 6.5.2 $L^6$ norm decay of solutions on interior spacelike slices of the cone

The objective within this section is to show that solutions to (6.5.6) satisfy the following estimate.

**Lemma 6.5.4.** *Let $u$ solve (6.5.6). Then*

$$\int_{D(T_2)} u^6(T_2,x)dx \lesssim \frac{T_1+2R}{T_2}E_{\{|x|<T_1+c\}}(T_1) + \frac{E}{T_2^\gamma} + G(\mathit{flux}([T_1,T_2]))+ \\ \|u\|^2_{LE^1[T_1,T_2]} + \|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2} \tag{6.5.16}$$

We remark that Proposition 6.5.2 easily follows from Lemma 6.5.3 and Lemma 6.5.4.

*Proof.* To prove (6.5.16), we multiply both sides of (6.5.6) by $Xu$ and obtain

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta uXu) - \frac{1}{2}\partial_t((t+c)g^{\alpha\beta}\partial_\beta u\partial_\alpha u) - \frac{1}{2}\partial_i(x^i g^{\alpha\beta}\partial_\beta u\partial_\alpha u) \\ +\frac{1}{2}((X-1)g^{\alpha\beta})\partial_\alpha u\partial_\beta u = \partial_t((t+c)\frac{u^6}{6}) + \partial_i(x^i\frac{u^6}{6}) - \frac{u^6}{3} + FXu. \tag{6.5.17}$$

Indeed, (6.5.17) follows by the following computations and the symmetry of $g^{\alpha\beta}$:

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta u\partial_t u) - \frac{1}{2}\partial_t(g^{\alpha\beta}\partial_\beta u\partial_\alpha u) + \frac{1}{2}\partial_t g^{\alpha\beta}\partial_\alpha u\partial_\beta u = (Pu)\partial_t u;$$

similarly, we have

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta ut\partial_t u) - \frac{1}{2}\partial_t(tg^{\alpha\beta}\partial_\beta u\partial_\alpha u) - g^{0\beta}\partial_\beta u\partial_t u + \frac{1}{2}g^{\alpha\beta}\partial_\alpha u\partial_\beta u \\ +\frac{1}{2}t\partial_t g^{\alpha\beta}\partial_\beta u\partial_\alpha u = (Pu)t\partial_t u,$$

and

$$\partial_\alpha(g^{\alpha\beta}\partial_\beta ux^j\partial_j u) - g^{j\beta}\partial_\beta u\partial_j u - \frac{1}{2}\partial_j(g^{\alpha\beta}\partial_\beta ux^j\partial_\alpha u) + \frac{1}{2}x^j\partial_j g^{\alpha\beta}\partial_\beta u\partial_\alpha u \\ +\frac{3}{2}g^{\alpha\beta}\partial_\beta u\partial_\alpha u = (Pu)(x^j\partial_j u)$$

as well as

$$\partial_\alpha(g^{\alpha\beta}u\partial_\beta u) - g^{\alpha\beta}\partial_\beta u\partial_\alpha u = (Pu)u.$$

The nonlinear term follows in a similar manner. Upon summing these terms we obtain (6.5.17).

We now integrate (6.5.17) on $\Gamma(I)$ and apply the divergence theorem. We obtain

$$\iint_{\Gamma(I)} \frac{u^6}{3} + \frac{1}{2}((X-1)g^{\alpha\beta})\partial_\alpha u \partial_\beta u - FXu\, dxdt = -\int_{\partial\Gamma(I)} \nu_\alpha g^{\alpha\beta}\partial_\beta u Xu - \frac{1}{2}\nu\cdot(t+c,x)g^{\alpha\beta}\partial_\beta u\partial_\alpha u - \frac{1}{2}\nu\cdot(t+c,x)\frac{u^6}{6}\, d\sigma$$

Recall that on $L(I)$ the outward unit normal vector $\nu$ to $L(I)$ is $(-1, x/|x|)/\sqrt{2}$, and thus $\nu\cdot(t+c,x) = 0$ on $L(I)$. The boundary term can now be written more explicitly as

$$-P(T_2) + P(T_1) + \text{flux}([T_1,T_2]) - BDR_h$$

where the first three terms come from the Minkowski case, and $BDR_h$ denotes the part of the energy density on the boundary arising from $h$:

$$BDR_h := \int_{D(T_2)} h^{0\beta}\partial_\beta u Xu - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u\partial_\alpha u\, dx - \int_{D(T_1)} h^{0\beta}\partial_\beta u Xu - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u \partial_\alpha u\, dx + \int_{L(I)} \nu_\alpha h^{\alpha\beta}\partial_\beta u Xu\, d\sigma$$

As explained in Section 5, we know that

$$P(T_2) \gtrsim T_2 \int_{D(T_2)} u^6(T_2,x)dx$$

$$P(T_1) \lesssim (T_1+c)E_{\{|x|<T_1+c\}}(T_1)$$

We can also make the trivial estimate

$$\iint_{\Gamma(I)} FXu\, dxdt \lesssim T_2\|F\|_{L^1[T_1,T_2]L^2}\|\nabla u\|_{L^\infty[T_1,T_2]L^2}$$

Moreover, our assumptions on $h^{\alpha\beta}$ immediately imply that

$$(X-1)h^{\alpha\beta} \lesssim t\langle r\rangle^{-1-\gamma}$$

and thus

$$\iint_{\Gamma(I)} |(X-1)g^{\alpha\beta}\partial_\alpha u \partial_\beta u|\, dxdt \lesssim T_2 \iint_{\Gamma(I)} \frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}}\, dxdt \leq T_2 \|u\|^2_{LE^1[T_1,T_2]} \quad (6.5.18)$$

The conclusion (6.5.16) will follow if we show that

$$BDR_h \lesssim \epsilon(P(T_2)+P(T_1)) + T_2^{1-\gamma}E + T_2\left(G(\text{flux}([T_1,T_2])) + \|u\|^2_{LE^1[T_1,T_2]}\right)$$

Let us write

$$D(T_2) = D_{int}(T_2) \cup D_{ext}(T_2)$$

where

$$D_{int}(T_2) = D(T_2) \cap \{|x| \leq \frac{T_2+c}{2}\}, \quad D_{ext}(T_2) = D(T_2) \cap \{|x| \geq \frac{T_2+c}{2}\},$$

Since $|h| \lesssim \epsilon$ in $D_{int}$, we have that

$$\int_{D_{int}(T_2)} h^{0\beta}\partial_\beta u X u - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u \partial_\alpha u\, dx \lesssim \epsilon P(T_2)$$

On the other hand,

$$|h| \lesssim \frac{1}{T_2^\gamma}$$

in $D_{ext}$, and thus by the boundedness of energy

$$\int_{D_{ext}(T_2)} h^{0\beta}\partial_\beta u X u - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u \partial_\alpha u\, dx \lesssim T_2^{1-\gamma}E(T_2) \lesssim T_2^{1-\gamma}E$$

Adding the last two inequalities we obtain

$$\int_{D(T_2)} h^{0\beta}\partial_\beta u X u - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u \partial_\alpha u\, dx \lesssim \epsilon P(T_2) + T_2^{1-\gamma}E$$

Similarly we can show that

$$\int_{D(T_1)} h^{0\beta}\partial_\beta u Xu - \frac{1}{2}(t+c)h^{\alpha\beta}\partial_\beta u\partial_\alpha u\, dx \lesssim \epsilon P(T_1) + T_1^{1-\gamma}E$$

We are left with dealing with the lateral terms. We will show that

$$\int_{L(I)} \nu_\alpha h^{\alpha\beta}\partial_\beta u Xu\, d\sigma \lesssim T_2\left(G(\text{flux}([T_1,T_2])) + \|u\|^2_{LE^1[T_1,T_2]}\right) \tag{6.5.19}$$

We first remark that $Xu = (rL+1)u$ on $L(I)$, and we again write

$$\nu_\alpha h^{\alpha\beta}\partial_\beta u = -\frac{1}{2}h^{\underline{L}\,\underline{L}}\underline{L}u + O(h)\bar{\partial}u \tag{6.5.20}$$

Note that (6.5.14) in particular imply the weaker estimates

$$h^{\underline{L}\,\underline{L}} \lesssim \frac{R^{1/2}}{\langle x\rangle^{1/2+\gamma}}, \quad h \lesssim 1$$

We can now estimate by Cauchy-Schwarz, (6.5.14) and the fact that $r \leq T_2 + c \lesssim T_2$:

$$\begin{aligned}\int_{L(I)} |h^{\underline{L}\,\underline{L}}\underline{L}u(rLu)|\, d\sigma &\lesssim T_2\int_{L(I)} |\frac{R^{1/2}}{\langle r\rangle^{1/2+\gamma}}\underline{L}uLu|\, d\sigma \\ &\leq T_2\left(R\int_{L(I)}\frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}}\, d\sigma + \int_{L(I)}|Lu|^2\, d\sigma\right) \\ &\lesssim T_2(\|u\|^2_{LE^1[T_1,T_2]} + \text{flux}([T_1,T_2]))\end{aligned}$$

where in the last inequality we used (6.5.8).

Similarly,

$$\begin{aligned}\int_{L(I)} |h\bar{\partial}u(rLu)|\, d\sigma &\lesssim T_2\int_{L(I)} |\bar{\partial}uLu|\, d\sigma \\ &\leq T_2\int_{L(I)} |\bar{\partial}u|^2\, d\sigma \lesssim T_2\text{flux}([T_1,T_2])\end{aligned}$$

Thirdly, by an application of (6.5.14), Cauchy-Schwarz and then Hölder's inequality,

$$\begin{aligned}\int_{L(I)} |h^{\underline{L}\,\underline{L}}\underline{L}uu|\, d\sigma &\lesssim \int_{L(I)} \frac{R^{1/2}}{\langle r\rangle^{1/2+\gamma}} |\underline{L}u||u|\, d\sigma \\ &\lesssim \left(\int_{L(I)} R\frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}}\, d\sigma\right)^{1/2} \left(\int_{L(I)} u^2\, d\sigma\right)^{1/2} \\ &\lesssim \left(\int_{L(I)} R\frac{|\nabla u|^2}{\langle r\rangle^{1+\gamma}}\, d\sigma\right)^{1/2} \|u\|_{L^6(L(I))}T_2 \\ &\lesssim T_2\left(\|u\|^2_{LE^1[T_1,T_2]} + \text{flux}([T_1,T_2])^{1/3}\right)\end{aligned}$$

where again we used (6.5.8).

Finally,

$$\begin{aligned}\int_{L(I)} |h\bar{\partial}uu|\, d\sigma &\lesssim (\int_{L(I)} |\bar{\partial}u|^2\, d\sigma)^{1/2}(\int_{L(I)} u^2\, d\sigma)^{1/2} \\ &\lesssim (\int_{L(I)} |\bar{\partial}u|^2\, d\sigma)^{1/2}(\int_{L(I)} |u|^6\, d\sigma)^{1/6}T_2 \\ &\lesssim T_2 G(\text{flux}[T_1,T_2])\end{aligned}$$

The last four estimates now imply (6.5.19), which finishes the proof of (6.5.16). □

### 6.5.3 Proof of Theorem 6.5.1

*Proof.* Assume now that Proposition 6.5.2 holds. A similar argument as the one in Section 4 allows us to pass to the limit and deduce the following:

**Proposition 6.5.5.** *Let $u$ be the strong solution to (6.2.2). For any $R \geq 1$, $1 < T_1$, and $T_1 + 2R < T_2$, we have*

$$\begin{aligned}\int_{\mathbb{R}^3} u^6(T_2,x)dx \lesssim &\frac{T_1+2R}{T_2} E_{\{|x|<T_1+2R\}}(T_1) + \frac{E}{T_2^{\gamma}} \\ &+ G\Big(E_{\{|x|>T_1+R\}}(T_1) + \|u\|^2_{LE^1[T_1,T_2]}\Big)\end{aligned} \tag{6.5.21}$$

Let us now prove (6.5.4). Pick any $\tilde{\epsilon} > 0$, and let $T_1$ be large enough such that

$$\|u\|^2_{LE^1[T_1,\infty)} < \tilde{\epsilon};$$

such a number may be found because of the local energy estimate (7.2.7). Next pick $R$ large enough so that

$$E_{\{|x|>T_1+R\}}(T_1) < \tilde{\epsilon}$$

Now let $T_2 \to \infty$ in (6.5.21). We obtain

$$\limsup_{T_2\to\infty} \int_{\mathbb{R}^3} u^6(T_2, x)dx \lesssim G(\tilde{\epsilon})$$

and (6.5.4) follows by letting $\tilde{\epsilon} \to 0$.

To obtain part (2) of Theorem 6.5.1, note that if

$$\partial^J h \lesssim \epsilon \langle r\rangle^{-|J|-\gamma}$$

where $|J| \leq 2$ (which are implied by our assumptions in the main theorem), then global Strichartz estimates are implied by a refinement of the local energy decay estimates (see Theorem 2 in [75][2]). Then, for any $\eta > 0$, by choosing a sufficiently large number $T > 0$, we obtain

$$\|w\|_{L^5L^{10}([T,\infty)\times\mathbb{R}^3)} \leq \eta$$

where $w$ solves (6.1.1). For any $\tilde{w}$ with

$$\|\tilde{w}\|_{L^5L^{10}([T,\infty)\times\mathbb{R}^3)} \leq \eta,$$

let $W$ be the solution to

$$PW = (w + \tilde{w})^5$$

[2] In the current arXiv version of this paper, see Theorem 6 instead.

with

$$\lim_{t\to\infty} \|\nabla W(t,\cdot)\|_{L^2(\mathbb{R}^3)} = 0.$$

As $\eta > 0$ was arbitrary, we may select $\eta$ sufficiently small so that the map $\tilde{w} \mapsto W$ is a contraction mapping, so that for any finite energy solution $w$ of (6.1.1), there exists a unique solution to (6.2.1) such that their difference vanishes in the $\dot{H}^1 \times L^2$ norm as $t \to \infty$, and we conclude that the solution scatters in the energy space. $\square$

# Chapter 7 Pointwise decay for power-type nonlinear wave equations

## 7.1 Introduction

We study the power-type nonlinear wave equations

$$P\phi = \mu|\phi|^p\phi, \quad p \geq 2,\ \mu \in \{-1, 1\}$$

in three spatial dimensions on a variety of spacetimes. For clarity, we emphasise that these spacetimes are allowed to depend on $(t, x)$ or only on $x$. The goal is to obtain the optimal pointwise decay rate, stated in Theorem 7.1.1; this is achieved by an iteration scheme that is outlined in Section 7.4.1. Along the way, we prove an $r$-weighted integrated local energy decay estimate (Theorem 7.5.3). The results can be viewed as extensions of the results for the linear problem studied in [60], albeit under stronger assumptions on the coefficients of the wave operator $P$ than in [60]. See also Section 7.8 for how we reach analogous pointwise decay rates for both focusing and defocusing power nonlinearities that are cubic or higher order, corresponding to $p \geq 2, p \in \mathbb{N}$.

This chapter primarily focuses on the case $p = 4$. As part of the difficulties introduced by the nonlinearity, we have to establish that the integrated local energy decay norms are bounded. We have to use Strichartz estimates to handle the power-type nonlinearity. We also need to prove an $r$-weighted integrated local energy decay estimate, which we prove on spacelike foliations of the spacetime; this idea was inspired by its origin in general relativity, in the work [24] of Dafermos and Rodnianski. Our proof of the estimate on spacelike foliations differs from the original setting of [24], which used null foliations outside of compact spatial regions. Moreover, in order to complete our iteration to reach the final pointwise decay rate, which we believe to be the optimal rate, we prove all three of these results for vector fields of the solution. From the perspective of pointwise decay, the lower powers $p$ are more difficult, because the nonlinearity decays more slowly. In contrast to power nonlinearities with $p \geq 5$, the case $p = 4$ is more difficult because the nonlinearity does not decay sufficiently rapidly; the purpose of proving the $r$-weighted integrated local energy decay estimate is to overcome this difficulty.

We consider the operator

$$P := \partial_\alpha g^{\alpha\beta}(t,x)\partial_\beta + g^\omega(t,x)\Delta_\omega + B^\alpha(t,x)\partial_\alpha + V(t,x) \quad \text{on } \mathbb{R}^{1+3} \tag{7.1.1}$$

where the coefficients are allowed to depend on $t$ and we use the summation convention. Here $\Delta_\omega$ denotes the Laplace operator on the unit sphere, and $\alpha, \beta$ range across $0, \ldots, 3$. The main assumptions on $P$ are that it is hyperbolic and a small asymptotically flat perturbation of the d'Alembertian $\Box = -\partial_t^2 + \Delta$; see Section 7.1 for the precise assumptions on $P$. The precise conditions on the potential $V$, the coefficients $B, g^\omega$ and the Lorentzian metric $g$ are given in the main result, Theorem 7.1.1.

We study the nonlinear Cauchy problem

$$\begin{cases} P\phi(t,x) = \mu\phi(t,x)^5 & (t,x) \in (0,\infty)\times\mathbb{R}^3 \\ (\phi(0,x), \partial_t\phi(0,x)) = (\phi_0, \phi_1) \end{cases}, \quad \mu \in \{-1, 0, 1\}; \tag{7.1.2}$$

the convention we adopt is that $\mu = 1$ corresponds to the defocusing sign.

Our main theorem (Theorem 7.1.1) states, informally, that if the coefficients of $P - \Box$ are small and asymptotically flat, then the solution to (7.1.2), as well as its vector fields, obey the global pointwise decay rates of $\langle t - r \rangle^{-1-\min(c(P),2)} \langle t + r \rangle^{-1}$; here $c(P)$ is a constant depending on the coefficients in (7.1.1). Thus for bounded $|x|$, we have

$$|\phi(t,x)| \leq C t^{-2-\min(c(P),2)}.$$

This rate of decay coincides with the one obtained by [34] in the case $P = \Box$, but in contrast to [34] we obtain it, on these general backgrounds $P$, for the solution everywhere in spacetime for initial data in a weighted Sobolev space, rather than merely in forward light cones $|x| \leq \lambda t, \lambda < 1$ for the Minkowski background and for the narrower class of compactly supported initial data. We believe this pointwise rate of decay to be sharp. An overview of the proof is contained in Section 7.1.1.

**History of the problem**

The theory of global existence, uniqueness and scattering for the semilinear wave equation on Minkowski spacetime, in three spatial dimensions, for

$$\Box \phi = \pm |\phi|^p \phi, \qquad \phi(0,x) = \phi_0(x), \qquad \partial_t \phi(0,x) = \phi_1(x)$$

was studied extensively; for instance, in the articles [10, 34, 43, 89, 100, 108]. For small initial data, there is a unique global solution if $p > \sqrt{2}$; see [30, 41, 117]. Work has also been done for the pointwise decay of solutions; see [88, 108, 127]. In the case of compactly supported smooth data, decay rates were proved in [116] (for small data) and in [13, 34] (for large data).

We now briefly remark on other spacetimes. Much work has been done for solutions to the initial value problem

$$\Box_g \phi = |g|^{-1/2}\partial_\mu(|g|^{1/2}g^{\mu\nu}\partial_\nu\phi) = 0, \qquad \phi(0,x) = \phi_0(x), \qquad \partial_t\phi(0,x) = \phi_1(x)$$

for various Lorentzian metrics $g$. For the Schwarzschild metric, the solution to the wave equation was conjectured to decay at the rate of $t^{-3}$ on a compact region in [93]. This rate of decay was shown to hold for the Schwarzschild spacetime, the subextremal Kerr spacetime with $|a| < M$, and other spacetimes; see [5, 27, 35, 74, 118].

### Statement of the main theorem

#### Assumptions on $P$

Let $h^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$, where $m^{\alpha\beta}$ denotes the Minkowski metric. Let $\sigma \in (0,\infty)$ be real. We make the following assumptions on the coefficients of $P$:

$$\begin{aligned} h^{\alpha\beta}, B^\alpha &\in S^Z(\langle r\rangle^{-1-\sigma}) \\ \partial_t B^\alpha, V &\in S^Z(\langle r\rangle^{-2-\sigma}) \\ g^\omega &\in S^Z_{\text{radial}}(\langle r\rangle^{-2-\sigma}). \end{aligned} \tag{7.1.3}$$

(The assumption on $\partial_t B$ is satisfied if, for instance, $B$ is stationary.) In addition, suppose that for a sufficiently small $\epsilon > 0$ we have, for $A_j := \{2^j \leq |x| \leq 2^{j+1}\}$ and an arbitrary interval $I \subset \mathbb{R}_+$,

$$\begin{aligned} \sum_{j\geq 0} \sup_{I\times A_j} \langle x\rangle^2|\partial^2 h^{\alpha\beta}| + \langle x\rangle|\partial h^{\alpha\beta}| + |h^{\alpha\beta}| &\leq \epsilon \\ \sum_{j\geq 0} \sup_{I\times A_j} \langle x\rangle^2|\partial B^\alpha| + \langle x\rangle|B^\alpha| &\leq \epsilon \\ \sum_{j\geq 0} \sup_{I\times A_j} \langle x\rangle^2|V| \leq \epsilon, \quad \sup_{I\times\mathbb{R}^3} |x|^2|V| &\leq \epsilon. \end{aligned} \tag{7.1.4}$$

We use the assumptions (7.1.4) in order to apply Strichartz estimates on such variable-coefficient backgrounds. Let $N$ be the maximal number of vector fields we apply to $\phi$ in this chapter.

**Theorem 7.1.1** (Main theorem). *Let $\phi$ solve (7.1.2) with the assumptions (7.1.3) and (7.1.4) and let*

$$\kappa := \min(\sigma, 2).$$

*Fix $m \in \mathbb{N}$ and some fixed $N \gg m$. We assume $\phi_0 \in L^2(\mathbb{R}^3)$ and that up to $N$ vector fields applied to the initial data lie in an $\langle r \rangle^\alpha$-weighted Sobolev space, for some sufficiently large $\alpha = \alpha(\kappa)$.*

1. *If the initial data are of any finite size, and if $\mu \in \{0, 1\}$, then we have*

$$\sum_{|J|=0}^{m} |\phi_J(t,x)| \lesssim \frac{1}{\langle t + |x| \rangle \langle t - |x| \rangle^{1+\kappa}}, \qquad t > 0 \tag{7.1.5}$$

*where the implicit constant is allowed to depend on $\|\phi\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))}$.*[1]

2. *If*

$$\|\phi_0\|_{H^{N+1}(\mathbb{R}^3)} + \|\phi_1\|_{H^N(\mathbb{R}^3)} \ll 1,$$

*then for any $\mu \in \{-1, 0, 1\}$, we have*

$$\sum_{|J|=0}^{m} |\phi_J(t,x)| \lesssim \frac{1}{\langle t + |x| \rangle \langle t - |x| \rangle^{1+\kappa}}, \qquad t > 0. \tag{7.1.6}$$

*Remark* 7.1.2 (Commentary on the main theorem). Thus for $\sigma$ close to 0, the solution decays at the rate

$$|\phi| \lesssim \langle t + r \rangle^{-1} \langle t - r \rangle^{-(1+\sigma)};$$

[1]See Remark 7.2.2 for an explanation of this dependence.

for large $\sigma$, the solution decays at the rate

$$\langle t+r\rangle^{-1}\langle t-r\rangle^{-3}.$$

- Thus if the coefficients decay rapidly as $r\to\infty$, then the pointwise decay rate is similar to the problem on the Minkowski background,

$$\Box\phi=\phi^5.$$

- On the other hand, if the coefficients decay more slowly (for instance, suppose that the potential $V$ decays more slowly than $r^{-4}$ for $r$ large), then the pointwise decay rate is similar to the *linear* problem

$$P\phi=0.$$

Moreover, we prove these rates in (1) and (2) for vector fields of $\phi$. The parameter $\sigma>0$ can be of any size and appears in, for instance, the assumptions for proving the $r$-weighted local energy decay estimate of Theorem 7.5.3.

*Remark* 7.1.3 (The linear problem: $\mu=0$ in (7.1.2)). When $\mu=0$, Theorem 7.1.1 states that the solution to

$$P\phi=0,\quad \phi(0,x)=\phi_0(x),\ \partial_t\phi(0,x)=\phi_1(x)$$

obeys the bound

$$\phi_{\le m}\lesssim\langle v\rangle^{-1}\langle u\rangle^{-1-\sigma},\quad v:=t+r,u:=t-r; \tag{7.1.7}$$

this result is part of the main theorem in the article [60]. In [60], all coefficients were allowed to be large perturbations of the Minkowski metric (thus $\epsilon\in(0,\infty)$ in that setting), and the assumptions (7.1.4) were not needed. Here we bring in (7.1.4) to apply Strichartz estimates for the variable-coefficient backgrounds encoded in $P$, and are unneeded for $\mu = 0$ due to the absence of the nonlinearity.

In addition, for $\mu = 0$, the main theorem in [60] shows that the final decay rate (7.1.7) still holds as long as a weak local energy decay estimate is assumed to hold. As its name implies, having such an estimate is a more general assumption than having a local energy decay estimate. Weak local energy decay estimates are satisfied in a large variety of situations, such as in the study of black holes, where they are known to hold for the Schwarzschild spacetime and the subextremal Kerr spacetimes.

*Remark* 7.1.4 (Differing increments)*.* The argument shown in this chapter straightforwardly yields a proof of a more general version of Theorem 7.1.1 when the decay increments $\sigma$ differ for the coefficients. (In (7.1.3), the increments are all assumed to be equal to $\sigma$.) The proof of the main theorem in [60] demonstrates this claim in full rigour; by contrast, the present chapter does not emphasise this.

### 7.1.1 Outline of the chapter and strategy of the proof

Here we overview the proof of Theorem 7.1.1.

- In Section 7.2 we prove the boundedness of the Strichartz and integrated local energy decay norms for $\phi$ and vector fields applied to $\phi$. In Section 7.3 we connect pointwise bounds to $L^2$ estimates and norms, thereby connecting local energy decay to pointwise bounds. We show how the derivative of $\phi$ decays at a certain better rate than $\phi$ and its vector fields; this improvement depends on the distance from the light cone $\{r = t\}$ at which one is evaluating the pointwise value of the solution. In Section 7.4 we rewrite the equation in a way amenable to our pointwise decay iteration scheme. We state and prove lemmas that are used in the scheme to improve the pointwise decay rates of the solution.

- In Section 7.5 we prove an $r$-weighted local energy decay estimate in Theorem 7.5.3. To do so, we use Strichartz estimates on such variable-coefficient backgrounds that satisfy the assumptions in Section 7.1 (more precisely, only (7.1.4) is assumed[2] in order to use Strichartz estimates). This part uses the results from Section 7.2.

  This result of Theorem 7.5.3 is then used to improve the decay rate of $\phi$ (and its vector fields)—see Proposition 7.5.6.

- In Section 7.6 we prove the final decay rate for $\phi$ and its vector fields in the region exterior to the light cone $\{r = t\}$, that is, in the region $\{r \geq t\}$. In Section 7.7 we prove the final decay rate for $\phi$ and its vector fields in the region inside of the light cone, that is, $\{r \leq t\}$. In Section 7.8 we explain how the iteration applies to other power nonlinearities, reaching pointwise bounds analogous to the one stated in Theorem 7.1.1.

**Pointwise estimates and asymptotic behaviour**

We begin with works studying the Minkowski background. In [115], pointwise decay estimates were proven for linear wave equations with a source term using the comparison

[2]The Strichartz estimate is written as Theorem 2 in the published version of [75], or Theorem 6 in the current arXiv version of [75].

theorem (positivity of the fundamental solution) in $1+3$ dimensions. In [14], numerics were shown for the asymptotic behaviour of small spherically symmetric solutions of nonlinear wave equations with a potential which showed that the dominant tail results from a competition between linear and nonlinear effects. In [127], pointwise decay estimates in $\mathbb{R}^{1+3}$ for various ranges of $p$ in the defocusing nonlinearity $|\phi|^p\phi$ were shown given data in a weighted energy space; more precisely, the solution is shown to decay as rapidly as the linear case for $p+1 > (1+\sqrt{17})/2$. [126] investigated similar questions for $1+d$ dimensions where $d \geq 3$. While prior investigations along these lines of questioning used the time decay of

$$t \mapsto \int_{\mathbb{R}^{1+d}} |\phi(t,x)|^{p+2}\, dx$$

for $1 < p+1 < 5$ to study pointwise decay estimates and scattering, [127] uses the method introduced in [24] to obtain pointwise decay estimates, via a weighted spacetime energy estimate for $2 < p+1 < 5$.

We now comment on various works that study other backgrounds, or general backgrounds that include the Minkowski spacetime as a special case. For small initial data, global existence and an upper bound of $t^{-1}$ was shown for spherically symmetric solutions on the Schwarzschild background for $p > 4$ in [23]; also on Schwarzschild, a similar result was shown in [12] for $p > 3$ and without the assumption of spherical symmetry. More recently, the work [120] proved upper bounds for power type nonlinearities with small initial data on Kerr backgrounds.

The work [62] considers the null condition, also on nonstationary spacetimes similar to those in the present chapter; it proves global existence and sharp pointwise decay, assuming an integrated local energy decay estimate holds. On these nonstationary spacetimes, the work [63] proves decay rates that are sharp in many cases for wave equations with cubic and higher order nonlinearities

$$P\phi = \mathcal{N}_{\geq \text{cubic}}(\partial^2\phi, \partial\phi, \phi),$$

including quasilinear equations, assuming an elliptic-type estimate and assuming global existence. Global existence for these cubic and higher order nonlinear equations holds, for instance, when the initial data is small. The upcoming work [66] obtains sharp pointwise asymptotics, given certain assumptions, for a variety of nonlinearities.

*Remark* 7.1.5 (High and low power nonlinearities)*.* In this remark we explain how the methods of the present chapter automatically give either partial or complete (if certain known results are assumed) proofs of pointwise decay rates (7.1.8) for various other power nonlinearities. Beyond the present remark, we provide more specifics about this in Remark 7.8.1, after the iteration has been presented.

We shall distinguish between the small and large data cases:

1. For small initial data, we consider the Cauchy problems

$$P\phi = \pm\phi^{p+1}, \quad p \in \mathbb{N}_{\geq 2}$$

with smooth and compactly supported initial data that is small in a $H^{n+1} \times H^n$ norm. If $p+1 > 1+\sqrt{2}$ then for sufficiently small and smooth initial data, there exist smooth global solutions. Then the techniques in the present chapter prove the decay rate

$$|\phi(t,x)| \lesssim \frac{1}{\langle t+|x|\rangle\langle t-|x|\rangle^{1+\min(\sigma,p-2)}}. \tag{7.1.8}$$

Here we simply mention that

a) for $p \geq 5$ the estimate found in Theorem 7.5.3 is unnecessary; instead, the bound (7.3.7) and Lemmas 7.4.4 and 7.4.5 alone suffice to reach (7.1.8). Heuristically, this is because if the nonlinearity contains enough decay, then the initial global decay rate from local energy decay alone (see (7.3.7)) suffices to bootstrap the solution toward the sharp decay rate stated in (7.1.8).

b) The (energy-critical) case $p = 4$ is the subject of the present chapter (wherein both large and small data are considered).

c) For $p = 2, 3$, if one had the initial decay rates[3]

$$\phi_{\leq m}|_{r<t/2} \lesssim \langle r\rangle^{-1-\delta}, \quad \phi_{\leq m}|_{r\sim t} \lesssim \langle r\rangle^{-1+\delta}\langle u\rangle^{-2\delta}, \qquad p = 2 \tag{7.1.9}$$

$$\phi_{\leq m}|_{r<t/2} \lesssim \langle r\rangle^{-3/4-\delta}, \phi_{\leq m}|_{r\sim t} \lesssim \langle r\rangle^{-3/4+\delta}\langle u\rangle^{-2\delta}, \qquad p = 3 \tag{7.1.10}$$

for some sufficiently small $\delta$—more precisely, for $p = 2$, we need $2 - 3\delta > 1$ and thus $\delta < \frac{1}{3}$, while for $p = 3$ we need $2 - 4\delta > 1$ and thus $\delta < \frac{1}{4}$— then the iteration also follows Sections 7.6 and 7.7 nearly verbatim and one reaches the rate (7.1.8) from the method in this chapter. This means, for

[3]This is because of an integration in the radial variable $\rho$ in the backward light cone $D_{tr}$ (see Definition 2.0.3), when bounding $\int_{D_{tr}} \rho|\phi|^{p+1} dA$. More precisely, we are taking the exponent ex $> 1$ in $1/\langle\rho\rangle^{\text{ex}}$. (7.1.9) and (7.1.10) follow if Proposition 7.5.6 holds with $\gamma > 1$, $\gamma > 1/2$ respectively, and see also Remark 7.5.7.

At first glance, the rates of decay in $r < t/2$ seem insufficient. However with the aid of an auxiliary result for the $r < t/2$ region (Proposition 7.7.3) they can be converted to $\langle v\rangle^{-1}\langle r\rangle^{-\delta}, \langle v\rangle^{-3/4-\delta}$ for $p = 2, 3$ respectively. This (when combined with the $r \sim t$ pointwise decay rates) is then sufficient for an improvement over (7.1.9) and (7.1.10). One perspective on what is happening here is that one is propagating improved decay in $\{r \geq t/2\}$ into $\{r \leq t/2\}$. This auxiliary result makes use of the scaling vector field $S$ and some sort of integrated local energy decay statement or some elliptic-type estimate.

instance, that in the cubic nonlinearity case $p = 2$, a decay rate of the form

$$\phi_{\leq m} \lesssim \langle r \rangle^{-\frac{2}{3}-} \langle u \rangle^{-\frac{2}{3}+}$$

in the wave zone and a decay rate

$$\phi_{\leq m} \lesssim \langle r \rangle^{-\frac{4}{3}+}$$

in $\{r < t/2\}$ together suffice to bootstrap to the final optimal decay rate. See [120] for a proof of the small data problem on the Kerr background. In the small data case, additional lemmas become available.

2. For large initial data, we consider the (defocusing) Cauchy problems

$$P\phi = |\phi|^p \phi, \quad p \in 2\mathbb{N}_{\geq 1}$$

where we avoid the odd integer values of $p$ because of the issue regarding the smoothness of the modulus of $\phi$ close to the zero set of $\phi$. (The method presented in this chapter applies vector fields to the nonlinearity.)

The value $p = 4$ is covered in the present chapter. Concerning higher odd $p$ values and the question of global existence (but not uniqueness) on the Minkowski background, at least for weak solutions, we refer the reader to a result of Segal, [97]. If global existence and sufficient regularity are assumed for the values $p \geq 5$, then the iteration presented in Sections 7.6 and 7.7 automatically give (7.1.8), with the same remark in Item 1a above applying here. For $p = 2$, if global existence is assumed, then the remark in Item 1c above applies here as well.

## 7.2 Boundedness of the integrated local energy decay norm and Strichartz norm

After introducing Strichartz estimates, we state the version of these estimates that holds on the variable-coefficient backgrounds considered in the present chapter (Lemma 7.2.3), and then show how to control the Strichartz and integrated local energy decay norms by the initial data (Theorem 7.2.7).

*Remark* 7.2.1. A subset of the Strichartz estimates for the constant-coefficient wave equation on $\mathbb{R}^{1+3}$ can be stated as follows: For $q_j \neq \infty$,

$$\||D_x|^{-\rho_1}\partial\phi\|_{L_t^{p_1}L_x^{q_1}} \lesssim \|\partial\phi(0)\|_{L^2} + \||D_x|^{\rho_2}\Box\phi\|_{L_t^{p_2'}L_x^{q_2'}} \tag{7.2.1}$$

where $(\rho_i, p_i, q_i)$ obey the scaling relation

$$\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \rho$$

and the dispersion relation

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \qquad 2 < p \leq \infty.$$

(The $q_1, q_2 = \infty$ cases will not be relevant for this chapter.) This Strichartz estimate is now understood globally in time within the setting of operators with variable coefficients that are a small perturbation of the constant coefficients in $\Box$: see Lemma 7.2.3.

*Remark* 7.2.2 (Overview of the proof of Theorem 7.2.7). We would like to control the local energy norm and the $L^5L^{10}$ norm of not only the solution $\phi$, but also its vector fields $\phi_{\leq m}$. Prior to proving Theorem 7.5.3, we must first (see Theorem 7.2.7) control the nonlinearity $\phi^5$ and its vector fields on the right-hand side of the Strichartz estimate. We partition the time interval $\mathbb{R}_+$ into finitely many sub-intervals $I_k$ so that the nonlinearity norm is small on each $I_k$. This smallness enables us to treat the nonlinearity norm perturbatively (that is, we absorb it to the left-hand side of the Strichartz estimate). However, this perturbative argument comes with the cost of the implicit constant in the Strichartz estimate for $\phi_{\leq m}$ now being dependent on the Strichartz norm of the solution $\phi$.

**Lemma 7.2.3** (Preliminary LED and Strichartz)**.** *Let $I \in \{[T_0, T_1], [T_0, \infty)\}$ be an interval, where $T_1 \geq T_0 \geq 0$ are real numbers. If for a sufficiently small $\epsilon > 0$, (7.1.4) holds, and $\psi$ is a function, then*

$$\|\psi\|_{(LE^1 \cap L^5 L^{10})(I \times \mathbb{R}^3)} + \|\partial \psi\|_{L^\infty L^2(I \times \mathbb{R}^3)} \lesssim \|\partial \psi(T_0)\|_{L^2} + \|P\psi\|_{(L^1 L^2 + LE^*)(I \times \mathbb{R}^3)}. \quad (7.2.2)$$

*Remark* 7.2.4. We will assume but not prove Lemma 7.2.3. The statement of Lemma 7.2.3 is obtained by combining Theorem 3 in [75] and Proposition 8 in [76]. The assumptions on the operator $P$ in this chapter in (7.1.4) satisfy the assumptions in both results.

**Lemma 7.2.5** (Bound on commutator of wave operator and vector field $Z^J$)**.** *Given the assumptions on $P$ in Theorem 7.1.1, there exists a positive real number $q' > 0$ such that for any multi-index $J$,*

$$|P\phi_J| \lesssim \frac{|\phi_{\leq |J|-1}|}{\langle r\rangle^{2+q'}} + \frac{|\nabla_{t,x}\phi_{\leq |J|}|}{\langle r\rangle^{1+q'}} + |(P\phi)_{\leq |J|}|.$$

*Proof.* Let $s_a$ denote a member of $S^Z(\langle r\rangle^{-a})$. There is a constant $q' > 0$ such that the operator $P$ can be written schematically as

$$P = \Box + \partial s_{1+q'}\partial + s_{1+q'}\not\partial^2 + s_{2+q'} + s_{1+q'}\partial.$$

We have $[Z, \partial] = c_Z\partial$, for some real number $c_Z \in \mathbb{R}$ depending on $Z$. We include the terms arising from the $g^\omega\Delta_\omega$ of the operator $P$ together with the $\langle r\rangle^{-1-}|\nabla_{t,x}\phi_{\leq|J|}|$ term. □

Thus

$$|P\phi_J| \lesssim \frac{|\phi_{\leq|J|-1}|}{\langle r\rangle^{2+q'}} + \frac{|\nabla_{t,x}\phi_{\leq|J|}|}{\langle r\rangle^{1+q'}} + |(\phi^5)_{\leq|J|}|.$$

**Corollary 7.2.6** (Lemma 7.2.3 with vector fields)**.** *For any* $m \geq 0$

$$\|\phi_{\leq m}\|_{(LE^1\cap L^5L^{10})(I\times\mathbb{R}^3)} + \|\partial\phi_{\leq m}\|_{L^\infty L^2(I\times\mathbb{R}^3)} \lesssim \|\partial\phi_{\leq m}(T_0)\|_{L^2} + \|(P\phi)_{\leq m}\|_{(L^1L^2)(I\times\mathbb{R}^3)}.$$

*Proof.* Since

$$P\phi_J = (P\phi)_J + [P, Z^J]\phi,$$

by (7.2.2) we have on any fixed $I$ with left endpoint $T_0$ the following estimate

$$\begin{aligned}\|\phi_J\|_{LE^1\cap L^5L^{10}} + \|\partial\phi_J\|_{L^\infty L^2} &\lesssim \|\partial\phi_J(T_0)\|_{L^2} + \|(P\phi)_J\|_{L^1L^2} + \|[P, Z^J]\phi\|_{LE^*} \\ &\lesssim \|\partial\phi_J(T_0)\|_{L^2} + \|(P\phi)_J\|_{L^1L^2} + \epsilon\|\phi_{\leq|J|}\|_{LE^1}.\end{aligned}$$

The second line follows from assumptions on $P$ and Lemma 7.2.5. The claim follows for small $\epsilon$. □

**Theorem 7.2.7** (Corollary 7.2.6 with no nonlinearity)**.** *Let* $\phi$ *solve* (7.1.2)*. Assume the hypotheses on* $P$ *from* (7.1.4)*. For any interval* $I$ *of the form* $[T_0, T_1]$ *or* $[T_0, \infty)$*,* *the following estimate holds:*

$$\|\phi_{\leq m}\|_{(LE^1 \cap L^5 L^{10})(I\times\mathbb{R}^3)} + \|\partial\phi_{\leq m}\|_{L^\infty L^2(I\times\mathbb{R}^3)} \leq C(\|\phi\|_{L^5L^{10}(I\times\mathbb{R}^3)}, m)\|\partial\phi_{\leq m}(T_0)\|_{L^2(\mathbb{R}^3)} \tag{7.2.3}$$

*Since $\|\phi\|_{L^5L^{10}}$ is bounded by the energy of the initial data up to an implicit constant, if the initial data are sufficiently small in the energy norm, then the bound (7.2.3) holds without any specific dependence on the size of $\|\phi\|_{L^5L^{10}}$.*

*Proof.* The base case Strichartz estimate

$$\|\phi\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))} < \infty \tag{7.2.4}$$

is implied by [61, Theorem 6.1]. The result [61, Theorem 6.1] implies that the solution $\phi$ scatters in the energy space, from which we obtain (7.2.4).

The base case integrated local energy decay estimate (plus the energy estimate)

$$\|\phi\|_{LE^1(\mathbb{R}_+\times\mathbb{R}^3)} + \|\partial\phi\|_{L^\infty L^2(\mathbb{R}_+\times\mathbb{R}^3)} \lesssim \|\partial\phi(0)\|_{L^2(\mathbb{R}^3)}$$

was proven in [61, Theorem 4.1], but for completeness we include it as Theorem 7.2.8. The lower order perturbations present in $P$ were not explicitly included in [61], but we show how to include them in Theorem 7.2.8 below; see in particular the estimate on the error term Err.

For $m \geq 1$, the bounds

$$\|\phi_{\leq m}\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))\cap LE^1(\mathbb{R}_+\times\mathbb{R}^3)} < \infty$$

can be proven by induction, which we now proceed with. Suppose that for some integer $m \geq 0$, $\|\phi_{\leq m}\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))} < \infty$; we shall show that (7.2.3) holds.

There are intervals $I_0, \dots, I_n$ “almost-partitioning” $[0,\infty)$ with $t_0 := 0 \in I_0$, and for $0 \le j \le n-1$, $I_j = [t_j, t_{j+1}]$, while $I_n = [t_n, \infty)$, such that for all $j$, the following norm obeys the bound

$$\|\phi_{\le m}\|_{L^5(I_j;L^{10}(\mathbb{R}^3)) \cap LE^1(I_j \times \mathbb{R}^3)} \le 1/(4C_{\text{Stri}})^{1/4}, \qquad I_j := [t_j, t_{j+1}] \text{ unless } j = n \quad (7.2.5)$$

where $C_{\text{Stri}}$ is the constant from the Strichartz estimate from Lemma 7.2.3.

- By Lemma 7.2.3, we obtain

$$\begin{aligned}
&\|\phi_{\le m+1}\|_{LE^1(I_j)} + \|\phi_{\le m+1}\|_{L^5(I_j;L^{10}(\mathbb{R}^3))} + \|\partial\phi_{\le m+1}\|_{L^\infty(I_j;L^2(\mathbb{R}^3))} \\
&\qquad \le C_{\text{Stri}}\left(\|\partial\phi_{\le m+1}(t_j)\|_{L^2} + \|(P\phi)_{\le m+1}\|_{L^1(I_j;L^2(\mathbb{R}^3))} + \epsilon\|\phi_{\le m+1}\|_{LE^1(I_j)}\right) \\
&\qquad = C_{\text{Stri}}\left(\|\partial\phi_{\le m+1}(t_j)\|_{L^2} + \|(\phi^5)_{\le m+1}\|_{L^1(I_j;L^2(\mathbb{R}^3))} + \epsilon\|\phi_{\le m+1}\|_{LE^1(I_j)}\right).
\end{aligned}$$

  (Here, we bounded the commutator in $LE^*(I_j)$, and here we have chosen $\epsilon$ from Section 7.1 to be sufficiently small.) This estimate implies the estimate

$$\begin{aligned}
&\|\phi_{\le m+1}\|_{LE^1(I_j)} + \|\phi_{\le m+1}\|_{L^5(I_j;L^{10}(\mathbb{R}^3))} + \|\partial\phi_{\le m+1}\|_{L^\infty(I_j;L^2(\mathbb{R}^3))} \\
&\qquad \le 2C_{\text{Stri}}\left(\|\partial\phi_{\le m+1}(t_j)\|_{L^2} + \|\phi_{\le m}\|^4_{L^5(I_j;L^{10}(\mathbb{R}^3))}\|\phi_{\le m+1}\|_{L^5(I_j;L^{10}(\mathbb{R}^3))}\right).
\end{aligned}$$

- By the fact that the Strichartz norm $\|\phi_{\le m}\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))}$ is finite, we apply (7.2.5).

Thus

$$
\begin{aligned}
\|\phi_{\le m+1}\|_{LE^1[t_j,t_{j+1}]} + \|\phi_{\le m+1}\|_{L^5([t_j,t_{j+1}];L^2(\mathbb{R}^3))} + \|\partial\phi_{\le m+1}\|_{L^\infty([t_j,t_{j+1}];L^2(\mathbb{R}^3))} \\
\le C\|\partial\phi_{\le m+1}(t_j)\|_{L^2}, \quad C = C(\|\phi\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))}, m) \\
\le C\|\partial\phi_{\le m+1}\|_{L^\infty([t_{j-1},t_j];L^2(\mathbb{R}^3))} \\
\le C\cdot C\|\partial\phi_{\le m+1}(t_{j-1})\|_{L^2} \\
\le C^{j+1}\|\partial\phi_{\le m+1}(0)\|_{L^2}
\end{aligned}
\tag{7.2.6}
$$

Here when $j = n$, the interval is understood to read $[t_n, \infty)$.

- By adding these estimates together, we obtain

$$
\begin{aligned}
\|\phi_{\le m+1}\|_{LE^1(\mathbb{R}_+)} + \|\phi_{\le m+1}\|_{L^5(\mathbb{R}_+;L^{10}(\mathbb{R}^3))} + \|\partial\phi_{\le m+1}\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^3))} \\
\le C^{j+1}\|\partial\phi_{\le m+1}(0)\|_{L^2}
\end{aligned}
$$

which holds because

$$
\|\partial\phi_{\le m+1}\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^3))} \le \sum_{j=0}^{n} \|\partial\phi_{\le m+1}\|_{L^\infty(I_j;L^2(\mathbb{R}^3))}.
$$

□

**Theorem 7.2.8** (Base case integrated local energy decay, Theorem 4.1 in [61]). *Let $\phi$ be a solution of (7.1.2) and let $|J| \le 1$ be a multi-index. Let $\gamma > 0$ denote an arbitrarily small constant.*

*If (7.1.3) and (7.1.4) holds for some sufficiently small constant $\epsilon > 0$, then*

$$
\|\phi\|^2_{LE^1[T_1,T_2]} + E(T_2) \le CE(T_1) \tag{7.2.7}
$$

*where $C$ is independent of $T_1$ and $T_2$, and*

$$\|\phi\|^2_{LE^1[T_1,T_2]} = \iint_{[T_1,T_2]\times\mathbb{R}^3} \frac{|\nabla\phi|^2}{\langle r\rangle^{1+\gamma}} + \frac{\phi^2}{\langle r\rangle^{3+\gamma}} + \frac{|\phi|^6}{\langle r\rangle}\,dxdt$$

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\partial\phi(t,x)|^2 + \frac{1}{6}\phi(t,x)^6\,dx.$$

*Proof.* Suppose that $\phi$ is a classical solution to the equation

$$P\phi = \phi^5 + F.$$

We multiply the equation by $a(r)\phi + b(r)\partial_r\phi + C\partial_t\phi$, with

$$b(r) = \sum_{j=0}^{\infty} 2^{-j\gamma}\frac{r}{r+2^j}, \quad a(r) = b(r)/r.$$

We omit the discussion of how to handle the terms coming from the constant-coefficient operator $\Box$, as that is standard. Upon integrating by parts, we obtain

$$E(T_2) + \iint_{[T_1,T_2]\times\mathbb{R}^3} \frac{1}{2}b'(\phi_r^2+\phi_t^2) - (\frac{1}{2}b' - \frac{b}{r})|\not\partial\phi|^2 - \frac{\Delta a}{2}\phi^2 + \mathrm{Err}$$
$$+ (\frac{2}{3}a(r) - \frac{1}{6}b'(r))\phi^6\,dxdt \lesssim E(T_1) + \iint_{[T_1,T_2]\times\mathbb{R}^3} |F|(|\partial\phi| + \frac{|\phi|}{\langle r\rangle})\,dxdt.$$

where the error satisfies

$$\mathrm{Err} \lesssim (\frac{|h|}{\langle x\rangle} + |\partial h|)\left(|\partial\phi|^2 + |\partial\phi|\frac{|\phi|}{\langle x\rangle}\right) + (|V\phi| + |B^\alpha\partial_\alpha\phi| + |g^\omega\Delta_\omega\phi|)\left(|\partial\phi| + \frac{|\phi|}{\langle r\rangle}\right)$$
$$\lesssim \epsilon\Big(\frac{|\partial\phi|^2}{\langle r\rangle^{1+\gamma}} + \frac{\phi^2}{\langle r\rangle^{3+\gamma}}\Big)$$

The second line follows by $\partial^J h^{\alpha\beta} \lesssim \epsilon\langle r\rangle^{-|J|-\gamma}$, where $|J| \le 1$, and by the assumptions on the other coefficients. Moreover,

$$\frac{2}{3}a(r) - \frac{1}{6}b'(r) = \sum_{j=0}^{\infty} 2^{-j\gamma}\left(\frac{1}{2}\frac{1}{r+2^j} + \frac{1}{6}\frac{r}{(r+2^j)^2}\right) \gtrsim \frac{1}{\langle r\rangle}$$

and

$$\frac{1}{2}b'(\phi_r^2+\phi_t^2)+(-\frac{1}{2}b'+\frac{b}{r})|\not\partial\phi|^2-\frac{\Delta a}{2}\phi^2 \gtrsim \frac{|\partial\phi|^2}{\langle r\rangle^{1+\gamma}}+\frac{\phi^2}{\langle r\rangle^{3+\gamma}}.$$

By standard applications of the Hölder and Hardy inequalities,

$$\|\phi\|^2_{LE^1[T_1,T_2]}+E(T_2)\lesssim E(T_1)+\|F\|_{L^1[T_1,T_2]L^2}\|\partial\phi\|_{L^\infty[T_1,T_2]L^2} \tag{7.2.8}$$

Let $\phi$ be a strong solution, and let $\phi_n(T_1)\to\phi(T_1)$ in the energy norm, where $(\phi_n)$ is a sequence of classical solutions. After partitioning $I=[T_1,T_2]$ into finitely many intervals $I_n$ so that each of the numbers $\|\phi\|_{L^5(I_n;L^{10})}$ is sufficiently small, by a contraction argument we obtain convergence of $\phi_n$ to $\phi$ in the function space

$$X(I)=\Big(C^0(I;\dot H^1)\cap L^5(I;L^{10})\Big)\times C^0(I;L^2).$$

This clearly implies that $\phi_n^5$ is a Cauchy sequence in $L^1([T_1,T_2];L^2)$.

Recalling the estimate (7.2.8), we conclude $\phi_n\to\phi$ in $LE^1[T_1,T_2]$. □

*Remark* 7.2.9 (Constants can henceforth depend on the $L^5L^{10}$ norm of $\phi$). Henceforth, we always allow the implicit constant in estimates to depend on $\|\phi\|_{L^5L^{10}([0,\infty)\times\mathbb{R}^3)}$. As a consequence of Theorem 7.2.7, we have

$$\|\phi_{\le m}\|_{LE^1([0,\infty)\times\mathbb{R}^3)}\lesssim\|\partial\phi_{\le m}(0)\|_{L^2(\mathbb{R}^3)}. \tag{7.2.9}$$

### 7.3 From integrated local energy decay to pointwise bounds

In this section we will show that local energy decay bounds imply certain slow decay rates for the solution, its vector fields, and its derivatives—see Propositions 7.3.5 and 7.3.7.

We start with the following pointwise estimate for the second derivative. We shall use it, for instance, when applying Lemma 7.3.2 to the functions $w = \partial\phi_{\le m}$ (that is, when we bound the first-order derivatives pointwise); this will be done in Proposition 7.3.7.

**Lemma 7.3.1.** *Assume $\phi$ is sufficiently regular. Then for any point $(t, x)$*

$$|\partial^2\phi_J(t,x)| \lesssim \left(\frac{1}{\langle r\rangle} + \frac{1}{\langle u\rangle}\right)|\partial\phi_{\le|J|+1}| + \left(1 + \frac{t}{\langle u\rangle}\right)\langle r\rangle^{-2}|\phi_{\le|J|+2}| + \left(1 + \frac{t}{\langle u\rangle}\right)|(P\phi)_{\le|J|}|. \tag{7.3.1}$$

*Proof.* Note first that

$$|\partial^2\phi_J| \lesssim \left(\frac{1}{\langle r\rangle} + \frac{1}{\langle u\rangle}\right)|\partial\phi_{\le|J|+1}| + \left(1 + \frac{t}{\langle u\rangle}\right)|(\Box\phi)_{\le|J|}|. \tag{7.3.2}$$

The case $|J| = 0$ is an immediate consequence of a result by Klainerman and Sideris, see [53, Lemma 2.3]. The general case follows after commuting with vector fields. It suffices to bound $P - \Box$. By the definition of (7.1.1), we may write this difference as

$$P - \Box = h^{\alpha\beta}\partial_{\alpha\beta} + (\partial_\alpha h^{\alpha\beta})\partial_\beta + g^\omega(t,x)\Delta_\omega + B^\alpha(t,x)\partial_\alpha + V(t,x)$$

Using the assumptions on the coefficients in Subsection (7.1.3), we have

$$(P - \Box)\phi \in S^Z(\langle r\rangle^{-1-\sigma})(\partial^2\phi + \partial\phi) + S^Z(\langle r\rangle^{-2-\sigma})\Omega^{\le 2}\phi$$

Thus

$$\left|((P-\Box)\phi)_{\le|J|}\right| \in S^Z(\langle r\rangle^{-1-\sigma})|\partial^2\phi_{\le|J|}| + S^Z(\langle r\rangle^{-1-\sigma})|\partial\phi_{\le|J|}| + S^Z(\langle r\rangle^{-2-\sigma})|\phi_{\le|J|+2}| \tag{7.3.3}$$

The conclusion now follows from (7.3.2) and (7.3.3) , since the first term on the RHS of (7.3.3) can be absorbed in the LHS of (7.3.2) for $r \gg 1$. $\square$

By (7.3.7), (7.3.1) immediately implies for solutions to (7.1.2):

$$|\partial^2 \phi_J| \lesssim \left(\frac{1}{\langle r\rangle} + \frac{1}{\langle u\rangle}\right)|\partial\phi_{\le |J|+1}| + \left(1 + \frac{t}{\langle u\rangle}\right)\langle r\rangle^{-2}|\phi_{\le |J|+2}| \tag{7.3.4}$$

The primary estimates that let us pass from local energy decay to pointwise bounds are contained in the following lemma.

**Lemma 7.3.2.** *Let $w \in C^4$,*

$$Z_{ij} := S^i\Omega^j,$$

$$\mu := \langle \min(r, |t-r|)\rangle$$

*and*

$$\mathcal{R} \in \{C_T^R, C_T^U, C_R^T\}.$$

*Then we have*

$$\|w\|_{L^\infty(\mathcal{R})} \lesssim \sum_{i\le 1, j\le 2} \frac{1}{|\mathcal{R}|^{1/2}}\left(\|Z_{ij}w\|_{L^2(\mathcal{R})} + \|\mu\partial Z_{ij}w\|_{L^2(\mathcal{R})}\right). \tag{7.3.5}$$

*where we assume $1 \ll U \le \frac{3}{8}T$, $1 \ll R \le \frac{3}{8}T$ and $R > T \gg 1$ in the cases $C_T^U, C_T^R, C_R^T$ respectively, and $|\mathcal{R}|$ denotes the measure of $\mathcal{R}$.*

*Remark* 7.3.3. Before proving Lemma 7.3.2, we sketch its proof: One uses exponential coordinates, which results in $\mathcal{R}$ being transformed into a region of constant size in all directions. Then one uses the fundamental theorem of calculus for the $s, \rho$ variables and Sobolev embedding for the angular variables. Finally, changing coordinates to return to the original region $\mathcal{R}$ produces the $|\mathcal{R}|^{-1/2}$ factor.

*Proof.* In $C_T^U$ we make the change of coordinates $t = e^s$ and $|r - t| = e^{s+\rho}$. With this change of coordinates, we are now dealing with a region of size 1 in spherical coordinates including $s$. We have $\partial_s = t\partial_t + r\partial_r = S$ and $\partial_\rho = (r-t)\partial_r$. Then we apply the fundamental theorem of calculus in $s$ and also in $\rho$. Finally, we rescale to $C_T^U$, obtaining the result in $C_T^U$.

For $C_R^T$, we let $r = e^s$ and $r - t = e^{s+\rho}$. Thus $\partial_s = S$ and $\partial_\rho = (t-r)\partial_t$. We get

$$\|w\|_{L^\infty(C_R^T)} \lesssim \sum_{i\le 1, j\le 2} \frac{1}{(R^3T)^{1/2}} \|S^i\Omega^j w\|_{L^2(\tilde{C}_R^T)} + \frac{R-T}{(R^3T)^{1/2}} \|\partial_t S^i \Omega^j w\|_{L^2(\tilde{C}_R^T)}.$$

This implies the result in $C_R^T$ since $R - T \le R$.

For $C_T^R$, we let $t = e^s$ and $r = e^{s+\rho}$. We obtain $\partial_s = S$ and $\partial_\rho = r\partial_r$ and the result in $C_T^R$. □

**Lemma 7.3.4.** *If $f \in C^1$, then*

$$\int_{t/2}^{3t/2} \langle u\rangle^{-2} f(t,x)^2 dx \lesssim \int_{t/4}^{7t/4} |\partial_r f(t,x)|^2 dx + \frac{1}{t^2}\left(\int_{t/4}^{t/2} f(t,x)^2 dx + \int_{3t/2}^{7t/4} f(t,x)^2 dx\right). \tag{7.3.6}$$

*Proof.* Let $\chi : [0,\infty) \to [0,1]$ be a cutoff such that $\chi(s) = 1$ for $1/2 \le s \le 3/2$ and 0 when $s \le 1/4$ and $s \ge 7/4$. We will show that, if $\gamma > -1/2$, and $\gamma \ne 1/2$, then

$$\int \langle u\rangle^{-2-2\gamma} \chi(r/t) f(r,\omega)^2 r^2 dr \lesssim \int \langle u\rangle^{-2\gamma} |\partial_r f(r,\omega)\chi(r/t)|^2 r^2 dr + \frac{1}{t^2}\int \langle u\rangle^{-2\gamma} |f(r,\omega)\chi'(r/t)|^2 r^2 dr.$$

The conclusion follows if we take $\gamma = 0$ and integrate over $\omega$.

We have

$$f(r,\omega)^2\chi(r/t)-f(7t/4,\omega)^2\chi((7t/4)/t) = -2\int_r^{7t/4} f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))d\rho.$$

Hence

$$f(r,\omega)^2\chi(r/t)r^2 \lesssim f(7t/4,\omega)^2\chi(3t/2)t^2 + 2\int_r^{7t/4} |f(\rho,\omega)\chi(\rho/t)\cdot\partial_r(f(\rho,\omega)\chi(\rho/t))|d\rho$$

Recall that $\chi(7t/4) = 0$. We multiply by $\langle u\rangle^{-2-2\gamma}$ and integrate $r$ from $t/4$ to $7t/4$. This yields

$$\int_{t/4}^{7t/4} \langle u\rangle^{-2-2\gamma}\chi(r/t)f(r,\omega)^2r^2dr \lesssim \int_{t/4}^{7t/4} \langle u\rangle^{-1-2\gamma}|f(r,\omega)\chi(r/t)\partial_r(f(r,\omega)\chi(r/t))|r^2dr$$

By the chain rule,

$$\partial_r(\chi(r/t)) \lesssim \chi'(r/t)\cdot\frac{1}{t}.$$

Thus by Cauchy-Schwarz and the chain rule

$$\begin{aligned}\int_{t/2}^{3t/2} \langle u\rangle^{-2-2\gamma}f(r,\omega)^2r^2dr \lesssim &\int_{t/4}^{7t/4} \langle u\rangle^{-2\gamma}|\partial_r f(r,\omega)\chi(r/t)|^2r^2dr\\ &+\frac{1}{t^2}\int_{t/4}^{7t/4} \langle u\rangle^{-2\gamma}|f(r,\omega)\chi'(r/t)|^2r^2dr.\end{aligned}$$

□

The next proposition yields an initial global pointwise decay rate for $\phi_J$ under the assumption that the local energy decay norms are finite. We shall improve this rate of decay in future sections (see Sections 7.6 and 7.7) for solutions to (7.1.2), culminating ultimately in the final pointwise decay rate stated in the main theorem.

**Proposition 7.3.5.** *Let $T$ be fixed and $\phi$ be any sufficiently regular function. There is a fixed positive integer $k$, such that for any multi-index $J$ with $|J| \le N-k$, we have:*

$$|\phi_J| \le \bar{C}_{|J|}\|\phi_{\le|J|+k}\|_{LE^1[T,2T]}\langle u\rangle^{1/2}\langle v\rangle^{-1}. \tag{7.3.7}$$

*Proof.* Lemma 7.3.2 proves (7.3.7) immediately except in the wave zone ($C_T^U$-type) regions. For the wave zone, the inequality found in (7.3.6) is used to finish the proof. ☐

*Remark* 7.3.6. By Theorem 7.2.7, which showed the boundedness of the $LE^1$ norm for $\phi$ and its vector fields, (7.3.7) implies that $\phi$ and its vector fields obey the decay rate $\langle u\rangle^{1/2}\langle v\rangle^{-1}$.

### 7.3.1 Derivative bounds

The next proposition shows that the first-order derivative (of solutions to (7.1.2)) decays pointwise faster by a rate of $\min(\langle r\rangle, \langle t-r\rangle)$. It utilises the initial global decay rate (7.3.7). The estimates in its proof involve the nonlinearity, but for the quintic nonlinearity as in (7.1.2) it turns out that the global bounds (7.3.7) alone already suffice to make the pointwise decay of the first-order derivative similar to the linear case, which is the content of Proposition 7.3.7. The reader can find details for the linear problem in [60]. Proposition 7.3.7 will be used in the pointwise decay iteration (see later sections, Sections 7.6 and 7.7)—more precisely, for iterating upon the *linear* components of the equation, namely those having to do with the operator $P - \Box$. By contrast, the nonlinearity in (7.1.2) does not involve any derivatives, so Proposition 7.3.7 will not be involved in the iteration for the nonlinearity.

**Proposition 7.3.7.** *Let $\phi$ solve (7.1.2), and assume that*

$$\phi_{\le m+n} \lesssim \langle r\rangle^{-\alpha}\langle t\rangle^{-\beta}\langle u\rangle^{-\eta}$$

*for some sufficiently large $n$. We then have*

$$\partial\phi_{\leq m} \lesssim \langle r\rangle^{-\alpha}\langle t\rangle^{-\beta}\langle u\rangle^{-\eta}\mu^{-1}, \quad \mu := \langle \min(r, |t-r|)\rangle. \tag{7.3.8}$$

*Proof.* Let $\mathcal{R} \in \{C_T^U, C_T^R, C_R^T\}$. Given a function $w$, we have

$$\|\partial w_{\leq m}\|_{L^2(\mathcal{R})} \lesssim \|\frac{w_{\leq m+n}}{\mu}\|_{L^2(\tilde{\mathcal{R}})} + \|\langle r\rangle (Pw)_{\leq m}\|_{L^2(\tilde{\mathcal{R}})}. \tag{7.3.9}$$

(See [60] or [74] for a proof of (7.3.9).) By the initial global pointwise estimate (7.3.7), (7.3.9) with $w = \phi$ yields

$$\begin{aligned}
\|\partial\phi_{\leq m}\|_{L^2(\mathcal{R})} &\lesssim \|\frac{\phi_{\leq m+n}}{\mu}\|_{L^2(\tilde{\mathcal{R}})} + \|\langle r\rangle(\phi^5)_{\leq m}\|_{L^2(\tilde{\mathcal{R}})} \\
&\lesssim \|\frac{\phi_{\leq m+n}}{\mu}\|_{L^2(\tilde{\mathcal{R}})} + \|\langle r\rangle(\langle u\rangle^{1/2}\langle v\rangle^{-1})^4\phi_{\leq m}\|_{L^2(\tilde{\mathcal{R}})} \\
&\lesssim \|\frac{\phi_{\leq m+n}}{\mu}\|_{L^2(\tilde{\mathcal{R}})} + \|\langle r\rangle\langle v\rangle^{-2}\phi_{\leq m}\|_{L^2(\tilde{\mathcal{R}})} \\
&\lesssim \|\frac{\phi_{\leq m+n}}{\mu}\|_{L^2(\tilde{\mathcal{R}})}.
\end{aligned} \tag{7.3.10}$$

The final line follows because $\langle r\rangle\mu \leq \langle v\rangle^2$.

Recalling Lemma 7.3.2, we have

$$\begin{aligned}
\|\partial\phi_{\leq m}\|_{L^\infty(\mathcal{R})} &\lesssim |\mathcal{R}|^{-\frac{1}{2}}\sum_Z \|Z\partial\phi_{\leq m}\|_{L^2(\mathcal{R})} + \|\mu\partial Z\partial\phi_{\leq m}\|_{L^2(\mathcal{R})} \\
&\lesssim |\mathcal{R}|^{-\frac{1}{2}}\left(\|\partial\phi_{\leq m+n}\|_{L^2(\mathcal{R})} + \|\mu\partial^2\phi_{\leq m+n}\|_{L^2(\mathcal{R})}\right) \\
&\lesssim |\mathcal{R}|^{-\frac{1}{2}}\left(\|\mu^{-1}\phi_{\leq m+n}\|_{L^2(\tilde{\mathcal{R}})} + \|\mu\partial^2\phi_{\leq m+n}\|_{L^2(\mathcal{R})}\right) \\
&\lesssim |\mathcal{R}|^{-\frac{1}{2}}\Big(\|\mu^{-1}\phi_{\leq m+n}\|_{L^2(\tilde{\mathcal{R}})} \\
&\quad + \|\mu\left(\frac{1}{\mu}|\partial\phi_{\leq m+n}| + (1+\frac{t}{\langle u\rangle})\langle r\rangle^{-2}|\phi_{\leq m+n}|\right)\|_{L^2(\mathcal{R})}\Big) \\
&\lesssim |\mathcal{R}|^{-\frac{1}{2}}\left(\|\mu^{-1}\phi_{\leq m+n}\|_{L^2(\tilde{\mathcal{R}})} + \|\mu(1+\frac{t}{\langle u\rangle})\langle r\rangle^{-2}\phi_{\leq m+n}\|_{L^2(\mathcal{R})}\right) \\
&\lesssim |\mathcal{R}|^{-\frac{1}{2}}\|\mu^{-1}\phi_{\leq m+n}\|_{L^2(\tilde{\mathcal{R}})}
\end{aligned}$$

which follows by (7.3.4) and (7.3.10). The final line follows because

$$\mu^2(1+t/\langle u\rangle) \lesssim \langle r\rangle^2.$$

Finally, the claim (7.3.8) follows because

$$\|\mu^{-1}\phi_{\leq m+n}\|_{L^2(\tilde{\mathcal{R}})} \lesssim |\mathcal{R}|^{\frac{1}{2}}\|\mu^{-1}\phi_{\leq m+n}\|_{L^\infty(\tilde{\mathcal{R}})},$$

thus

$$\|\partial\phi_{\leq m}\|_{L^\infty(\mathcal{R})} \lesssim \|\mu^{-1}\phi_{\leq m+n}\|_{L^\infty(\tilde{\mathcal{R}})}.$$

□

### 7.4 Preliminaries for the iteration

*Remark* 7.4.1 (The initial data). Let $w := S(t,0)\phi[0]$ denote the soluation to the free wave equation with initial data $\phi[0] = (\phi_0, \phi_1)$ at time 0. Then for any $|J| = O_N(1)$, the bound

$$w_J \lesssim \langle v\rangle^{-1}\langle u\rangle^{-1-\kappa}$$

holds for initial data that lie in a sufficiently high weighted Sobolev space.

#### 7.4.1 Summary of the iteration

By Remark 7.4.1, we may assume zero initial data in the following iteration. Second, note that it suffices to prove bounds in $|u| \geq 1$, because the desired final decay rate in $|u| < 1$ already holds by (7.3.7). Third, we distinguish the nonlinearity and the coefficients of $P - \Box$, and for both of these, we apply the fundamental solution. We iterate these two components in lockstep with one another.

Due to the domain of dependence properties of the wave equation, we shall first complete the iteration in $\{u < -1\}$. For the iteration in $\{u > 1\}$, the decay rates obtained from the fundamental solution are insufficient in the region $\{r < t/2\}$. To remedy this, we prove Proposition 7.7.3. With the new decay rates obtained from Proposition 7.7.3, we are then able to obtain new decay rates for the solution and its vector fields. At every step of the iteration, Lemma 7.4.4 is used to turn the decay gained at previous steps into new decay rates.

### 7.4.2 Setting up the problem

We note that (7.1.2), with the sign $\mu = 1$, can be written as

$$\begin{aligned}\Box\phi &= (\Box - P)\phi + F, \quad F := \phi^5 \\ &= -\partial_\alpha(h^{\alpha\beta}\partial_\beta\phi) - B^\alpha\partial_\alpha\phi - g^\omega\Delta_\omega\phi - V\phi + F.\end{aligned}$$

**The rewriting of the equation**

Given a nonnegative real number $a \geq 0$, let

$$\rho^{(a)}$$

denote a member of $S^Z(\langle r\rangle^{-a})$. Using the assumptions (7.1.3), we can write this as

$$\Box\phi \in \partial\left(\rho^{(1+\sigma)}\partial\phi\right) + B^\alpha\partial_\alpha\phi + \rho^{(2+\sigma)}\phi_{\leq 2} + F.$$

(By assumption (7.1.3), $\partial_t B$ decays according to $\rho^{(2+\sigma)}$.) Pick any multiindex $J$ with $|J| \lesssim m$. After commuting, we have

$$\Box\phi_J \in \partial\left(\rho^{(1+\sigma)}\partial\phi_{\leq m}\right) + B_{\leq m}\partial\phi_{\leq m} + \rho^{(2+\sigma)}\phi_{\leq m+2} + F_{\leq m} \tag{7.4.1}$$

- (The leftmost term in (7.4.1)) We shall decompose the derivative to obtain the desired decay rate, as follows: Note that for any function $w$, we may split generic derivatives in the union of the wave zone and exterior region, namely, $\{r \geq t/2\}$:

$$\partial w \in \rho^{(1)} w_{\leq 1} + \rho^{(0)} \partial_t w, \quad r \geq t/2. \tag{7.4.2}$$

Moreover, we shall go beyond this to use the specific structure of $\rho^{(0)}$, which we record: for $\partial_t$, we have $(\rho^{(1)}, \rho^{(0)}) = (0, 1)$; for $\not\partial$ we have $\rho^{(0)} = 0$; and for $\partial_r$ we use $\partial_r = \frac{S}{r} - \frac{t}{r}\partial_t$ to obtain

$$\rho^{(0)} = -t/r. \tag{7.4.3}$$

We now partition unity as follows: let $\chi := \chi_{\text{cone}}$ be a smooth cutoff function adapted to the region $\{t/2 \leq r \leq 3t/2\}$; let $\chi_{\text{in}}$ be supported in a slight enlargement of $\{r \leq t/2\}$ and let $\chi_{\text{out}}$ be supported in a slight enlargement of $\{r \geq 3t/2\}$, such that these three sum to equal 1. Then

$$\partial \left(\rho^{(1+\sigma)} \partial \phi_{\leq m}\right) = \partial \left((\chi_{\text{in}} + \chi_{\text{cone}} + \chi_{\text{out}}) \rho^{(1+\sigma)} \partial \phi_{\leq m}\right).$$

By Proposition 7.3.7,

$$\partial(\chi_{\text{in}} \rho^{(1+\sigma)} \partial \phi_{\leq m}) \lesssim \rho^{(2+\sigma)} \phi_{\leq m+2}$$

and so we group this term with a similar term in (7.4.1). Similarly, by Proposition 7.3.7

$$\partial(\chi_{\text{out}} \rho^{(1+\sigma)} \partial \phi_{\leq m}) \lesssim \rho^{(2+\sigma)} \phi_{\leq m+2}.$$

For the term in the support of $\chi_{\text{cone}}$: if the derivative in front is $\partial_t$, we leave the term alone; if the derivative is $\not\partial$, we use (7.4.2) to obtain $\rho^{(2+\sigma)} \phi_{\leq m+2}$ and we group that term with a similar existing term in (7.4.1); if the derivative is $\partial_r$ we use $\partial_r = \frac{S}{r} - \frac{t}{r}\partial_t$ to write

$$\begin{aligned}\partial_r\left(\chi_{\text{cone}}\rho^{(1+\sigma)}\partial\phi_{\le m}\right) &= \rho^{(2+\sigma)}\phi_{\le m+2} - \frac{t}{r}\partial_t(\chi_{\text{cone}}\rho^{(1+\sigma)}\partial\phi_{\le m}) \\ &= \rho^{(2+\sigma)}\phi_{\le m+2} - \partial_t\left(\frac{t}{r}\chi_{\text{cone}}\rho^{(1+\sigma)}\partial\phi_{\le m}\right) \\ &\quad + \frac{1}{r}\chi_{\text{cone}}\rho^{(1+\sigma)}\partial\phi_{\le m}\end{aligned}$$

which is of the form

$$\rho^{(2+\sigma)}\phi_{\le m+2} - \partial_t\left(\frac{t}{r}\chi_{\text{cone}}\rho^{(1+\sigma)}\partial\phi_{\le m}\right).$$

We group this first term with a similar term in (7.4.1).

- (The second term in (7.4.1)) We partition unity as $\chi_{\text{in}} + \chi_{\text{cone}} + \chi_{\text{out}}$ again, and note that

$$\chi_{\text{in}}B_{\le m}\partial\phi_{\le m} \lesssim \chi_{\text{in}}\rho^{(1+\sigma)}|\partial\phi_{\le m}| \lesssim \chi_{\text{in}}\rho^{(2+\sigma)}|\phi_{\le m+n}|,$$

with the second bound following from Proposition 7.3.7; we group this term with a similar term in (7.4.1). A similar bound holds in the support of $\chi_{\text{out}}$, again by Proposition 7.3.7, and we also group the resulting term with a similar term in (7.4.1). For the term supported in $\operatorname{supp}\chi_{\text{cone}}$, we note that if the derivative is angular, then we rewrite it in terms of the rotation vector fields to conclude that the term $\chi_{\text{cone}}B_{\le m}\not\partial\phi_{\le m}$ is of the form $\rho^{(2+\sigma)}\phi_{\le m+2}$. If the derivative is $\partial_t$,

$$\chi_{\text{cone}}B_{\le m}\partial_t\phi_{\le m} = \partial_t(\chi_{\text{cone}}B_{\le m}\phi_{\le m}) - \partial_t(\chi_{\text{cone}}B_{\le m})\phi_{\le m}$$

and note that

$$\partial_t(\chi_{\text{cone}}B_{\le m}) \lesssim \langle r\rangle^{-2-\sigma}$$

by (7.1.3)'s assumption on $\partial_t B$, thus we may group this term with the $\rho^{(2+\sigma)}\phi_{\le m+2}$ term in (7.4.1). If the derivative is $\partial_r$, we obtain as usual a term of the form $\rho^{(2+\sigma)}\phi_{\le m+2}$ and then a term of the form

$$-\chi_{\text{cone}} B_{\le m}\frac{t}{r}\partial_t\phi_{\le m} = \partial_t(-\chi_{\text{cone}} B_{\le m}\frac{t}{r}\phi_{\le m}) + \partial_t(\chi_{\text{cone}} B_{\le m}\frac{t}{r})\phi_{\le m}.$$

We note that

$$\partial_t(\chi_{\text{cone}} B_{\le m}\frac{t}{r}) \lesssim \chi_{\text{cone}}\rho^{(2+\sigma)}$$

where we used (7.1.3)'s assumption on $\partial_t B$. Thus we group this term with a similar term in (7.4.1). This concludes our rewriting of the equation.

Thus the extra analysis in supp $\chi_{\text{cone}}$ is due to the derivative gaining only $\langle u\rangle^{-1}$ more decay than the original wave (recall the main estimate in Proposition 7.3.7).

**The decomposition of $\phi$**

We now write $\phi_J = \sum_{j=1}^{3}\phi_j$ where

$$\begin{aligned}
&\Box\phi_1 = G_1, \quad G_1 \in \rho^{(2+\sigma)}\phi_{\le m+2}\\
&\Box\phi_2 = \partial_t G_2, \quad G_2 \in \chi_{\text{cone}} B_{\le m}\phi_{\le m} + \chi_{\text{cone}} h_{\le m}\partial\phi_{\le m} + \frac{t}{r}\chi_{\text{cone}} B_{\le m}\phi_{\le m}\\
&\qquad\qquad\qquad + \frac{t}{r}\chi_{\text{cone}} h_{\le m}\partial\phi_{\le m}\\
&\Box\phi_3 = G_3 = F_{\le m}.
\end{aligned} \tag{7.4.4}$$

In $G_2$, the terms without the $t/r$ coefficient arise from terms with outside derivative $\partial_t$, and those terms with this coefficient arise from terms that originally having outside derivative $\partial_r$; this was explained in the two items above. Henceforth the symbol $n$ may increase from each line to the next.

*Remark* 7.4.2. From the perspective of obtaining upper bounds on $\phi_2$ (see Lemma 7.4.5), the presence of $(\frac{t}{r})^n, n = 1$ coefficients in $G_2$ does not change the proof significantly from the $n = 0$ case. Thus in the iterations below in Sections 7.6 and 7.7 the reader may take $n = 0$ during a first reading.

*Remark* 7.4.3 ((Temporarily) reduced, irrational parameter $\sigma$). To simplify the iteration, we shall reduce the value of $\sigma$ if necessary to be equal to some positive irrational number less than the original value of $\sigma$. We do this to avoid the appearance of logarithms in the iterations for $\phi_1$ and $\phi_2$ (see the decomposition (7.4.4)). We take $0 < \sigma \ll 1$. In the sections spelling out the details of the iteration, namely Sections 7.6 and 7.7, we explain how we reach the final decay rate in Theorem 7.1.1 (wherein the *original* value of $\sigma$ is included in the final decay rate).

### 7.4.3 Estimates for the fundamental solution

We have the following result, which is similar to previous classical results, see for instance [41], [8], [109], [114].

**Lemma 7.4.4.** *Let $m \geq 0$ be an integer and suppose that $\psi : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ solves*

$$\Box\psi(t, x) = g(t, x), \qquad \psi(0) = 0, \quad \partial_t\psi(0) = 0.$$

*Define*

$$h(t, r) = \sum_{i=0}^{2} \|\Omega^i g(t, r\omega)\|_{L^2(\mathbb{S}^2)} \tag{7.4.5}$$

*Assume that*

$$h(t, r) \lesssim \frac{1}{\langle r\rangle^\alpha \langle v\rangle^\beta \langle u\rangle^\eta}, \quad \alpha \in (2, 3) \cup (3, \infty), \quad \beta \geq 0, \quad \eta \geq -1/2.$$

*Define*

$$\tilde{\eta} = \begin{cases} \eta - 2, & \eta < 1 \\ -1, & \eta > 1 \end{cases}.$$

*We then have in both $\{u > 1\}$, and $\{u < -1\}$ in the case $\alpha + \beta + \eta > 3$:*

$$\psi(t,x) \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{\alpha+\beta+\tilde{\eta}-1}}. \tag{7.4.6}$$

*On the other hand, if $\alpha + \beta + \eta < 3$ and $u < -1$, we have*

$$\psi(t,x) \lesssim r^{2-(\alpha+\beta+\eta)}. \tag{7.4.7}$$

*Proof.* A detailed proof of (7.4.6) can be found in Lemma 6.5 of [60]. The idea is to use Sobolev embedding and the positivity of the fundamental solution of $\Box$ to show that

$$r\psi \lesssim \int_{D_{tr}} \rho h(s,\rho) ds d\rho, \tag{7.4.8}$$

where $D_{tr}$ is the backwards light cone with vertex $(r,t)$, and use (7.4.5).

Let us now prove (7.4.7), which was subject to the hypotheses $\alpha + \beta + \eta < 3$ and $u < -1$. In this case $D_{tr} \subset \{r - t \leq u' \leq r + t, \quad r - t \leq \rho \leq r + t\}$ and we obtain, using that $u' \leq \rho$ and $\rho \gtrsim \rho + s$ in $D_{tr}$:

$$r\psi \lesssim \int_{r-t}^{r+t} \int_{u'}^{r+t} \langle \rho \rangle^{1-\alpha-\beta} d\rho \, \langle u' \rangle^{-\eta} du' \lesssim \int_{r-t}^{r+t} \langle u' \rangle^{2-(\alpha+\beta+\eta)} du' \lesssim (t+r)^{3-(\alpha+\beta+\eta)}$$

where the final bound follows from the hypothesis that $\alpha + \beta + \eta < 3$. This finishes the proof because $t + r \leq 2r$ when $u < -1$. $\Box$

For the function $\phi_2$ we will use the following result for an inhomogeneity of the form $\partial_t g$ supported near the cone. The result is similar to Lemma 7.4.4, except that we gain an extra factor of $\langle u \rangle$ in the estimate.

**Lemma 7.4.5.** *Let $\psi$ solve*

$$\Box\psi = \partial_t g, \qquad \psi(0) = 0, \quad \partial_t\psi(0) = 0, \tag{7.4.9}$$

*where $g$ is supported in $\{\frac{1}{2} \le \frac{|x|}{t} \le \frac{3}{2}\}$. Let $h$ be as in (7.4.5), and assume that*

$$|h| + |Sh| + |\Omega h| + \langle t-r\rangle|\partial h| \lesssim \frac{1}{\langle r\rangle^\alpha \langle u\rangle^\eta}, \quad 2 < \alpha < 3, \quad \eta \ge -1/2.$$

*Then in $\{u > 1\}$, and $\{u < -1\}$ when $\alpha + \eta > 3$*

$$\psi(t,x) \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{\alpha+\tilde{\eta}}}. \tag{7.4.10}$$

*Proof.* Let $\widetilde{\psi}$ solve $\Box\widetilde{\psi} = g, \widetilde{\psi}(0) = 0, \partial_t\widetilde{\psi}(0) = 0$. In the support of $g$ we have

$$(t\partial_i + x_i\partial_t)h \lesssim |Sh| + |\Omega h| + \langle t-r\rangle|\partial_r h|.$$

By Lemma 7.4.4 (with $\beta = 0$) applied to $\nabla\widetilde{\psi}$, $\Omega\widetilde{\psi}$, $S\widetilde{\psi}$, and the fact

$$\langle u\rangle\partial_t\widetilde{\psi} \lesssim |\nabla\widetilde{\psi}| + |S\widetilde{\psi}| + |\Omega\widetilde{\psi}| + \sum_i |(t\partial_i + x_i\partial_t)\widetilde{\psi}|$$

the claim follows. ∎

## 7.5 $r^\gamma$ integrated local energy decay, and thus improved pointwise decay for $\phi_{\le m}$

In the following theorem and its subsequent application (Proposition 7.5.6) to the pointwise decay problem at hand, we have in mind only an arbitrarily small $\gamma > 0$. This estimate was inspired by [24], but unlike the result in [24], we partition the spacetime into $t =$ constant slices. See also [112] for a similar approach.

*Remark* 7.5.1. While Theorem 7.5.3 and its proof can be read independently of Section 7.4, we note that the subsequent application Proposition 7.5.6 does use a result from Section 7.4, which is why we chose to place this theorem after that section.

*Remark* 7.5.2. In proving Theorem 7.5.3, for the small data problem the use of (7.3.7) suffices to control the nonlinearity and finish the proof. This is because the bound (7.3.7) then comes with a small factor, and we can immediately treat the nonlinearity perturbatively by bounding four of the five functions in the nonlinearity using (7.3.7).[4] Thus for instance, at the level of zero vector fields,

$$\iint \phi^5(r^\gamma \partial_v \phi + r^{\gamma-1}\phi)\, dxdt \lesssim \epsilon \iint r^{\gamma-3}\phi^2 + r^{\gamma-1}(\partial_v\phi)^2\, dxdt$$

and thus we can absorb the interaction of the nonlinearity and the multiplier. In contrast, for the large data problem this factor can be large; nonetheless the goal in our proof's strategy will still be to treat the nonlinearity perturbatively. To achieve this, we make use of an inductive argument that takes advantage of the defocusing nature of the nonlinearity. More precisely, even though the defocusing structure is lost upon application of one or more vector fields to the equation, we are able to make use of the zeroth-order $r$-weighted estimate (wherein *no* vector fields have been applied to (7.1.2)) to prove higher order $r$-weighted estimates. Compared to the higher order case, the zeroth-order case controls an additional type of term, namely the nonlinearity term: see (7.5.5). The estimate (7.5.5) is then used to prove higher order estimates. The idea is that control of some lower order norms allows one to treat the higher order norm perturbatively.

[4]The reader is encouraged to compare this with the analysis in Remark 7.1.5 on the sextic and higher powers and to see why this approach fails for the lower power nonlinearities (cubic and quartic).

**Theorem 7.5.3** (The $r^\gamma$ estimate)**.** *Let $\phi$ solve (7.1.2). Let $\gamma < 2\sigma, \gamma < 1$ and let the potential $V$ satisfy*

$$V \in S^Z(\epsilon/r^2);$$

*assume that the solution to this equation exists. Note that $S^Z(\epsilon\langle r\rangle^{-2-\eta}) \subset S^Z(\epsilon/r^2)$ for any $\eta \geq 0$.*[5] *Let $T_2 > T_1 \geq 0$.*

*For any integer $m \geq 0$, we have*

$$A_{\gamma,m} + E^\gamma_{\phi_{\leq m}}(T_2) \lesssim_{\|\phi\|_{L^5L^{10}}} E^\gamma_{\phi_{\leq m}}(T_1) + \epsilon\|\partial\phi_{\leq m}\|^2_{LE(T_1,T_2)} + \epsilon\|\partial^2\phi_{\leq m}\|^2_{LE(T_1,T_2)} \quad (7.5.1)$$

*where the $A, E$ norms are:*

$$A_{\gamma,m} := \int_{T_1}^{T_2}\int_{\mathbb{R}^3} (\phi_{\leq m})^2 r^{\gamma-3} + |\bar\partial\phi_{\leq m}|^2 r^{\gamma-1}\, dxdt, \qquad \bar\partial := (\partial_t + \partial_r, \not\partial) \quad (7.5.2)$$

$$E^\gamma_{\phi_{\leq m}}(T_1) := \|r^{\gamma/2}(\not\partial\phi_{\leq m}, (\partial_t + \partial_r + \frac{1}{2r})\phi_{\leq m}, \frac{\phi_{\leq m}}{r})(T_1)\|^2_{L^2(\mathbb{R}^3)},$$

*where*

$$\|r^\alpha(f_1, \dots, f_n)\| := \sum_{j=1}^n \|r^\alpha f_j\|.$$

*Proof.* Fix $m \geq 0$. Let $|J| \leq m$. Fix $0 \leq T_1 < T_2$. Let

$$A_{\gamma,J} := \int_{T_1}^{T_2}\int_{\mathbb{R}^3} \phi_J^2 r^{\gamma-3} + |\bar\partial\phi_J|^2 r^{\gamma-1} dxdt;$$

note that the $m$ appearing in (7.5.2) denotes integers, while $J$ here denotes multi-indices, which distinguishes these two pieces of notation.

[5]Recall from the main theorem, Theorem 7.1.1, that the potential $V$ lies in $S^Z(\epsilon\langle r\rangle^{-2-\eta})$ with $\eta = \sigma$, but we prove Theorem 7.5.3 for a slightly more general potential. The reader can verify that the potential from Theorem 7.1.1 lies in $S^Z(\epsilon\langle r\rangle^{-2-\sigma})$ by combining the hypotheses mentioned in (7.1.3) and (7.1.4).

Integrating by parts, and recalling that $\partial_v$ denotes $\partial_t + \partial_r$,

$$\iint_{[T_1,T_2]\times\mathbb{R}^3} \Box\phi_J(r^\gamma\partial_v\phi_J + r^{\gamma-1}\phi_J)\,dx\,dt = \iint_{[T_1,T_2]\times\mathbb{R}^3} -\frac{\gamma r^{\gamma-1}}{2}(\partial_v\phi_J)^2$$
$$-\frac{1}{2}(2-\gamma)r^{\gamma-1}|\partial\!\!\!/\phi_J|^2 - \frac{\gamma(1-\gamma)r^{\gamma-3}}{2}\phi_J^2\,dxdt \tag{7.5.3}$$
$$+\int_{\mathbb{R}^3} -r^\gamma\left[\frac{1}{2}|\partial\phi_J|^2 + \partial_r\phi_J\partial_t\phi_J + \frac{1}{2}\frac{\phi_J}{r}\partial_t\phi_J\right]_0^T dx$$

- We now manipulate the boundary terms to obtain positive definite terms: we have

$$\begin{aligned}
\int_{\mathbb{R}^3} -r^\gamma\frac{1}{2}\frac{\phi_J}{r}\partial_t\phi_J\,dx &= \int -r^\gamma\frac{1}{2}\frac{\phi_J}{r}(\partial_v-\partial_r)\phi_J\,dx \\
&= \int -r^\gamma\frac{1}{2}\frac{\phi_J}{r}\partial_v\phi_J\,dx + \int_{S^2}\int_0^\infty r^\gamma\frac{1}{2}\frac{\phi_J}{r}\partial_r\phi_J\,r^2\,drd\omega \\
&= \int -r^\gamma\frac{1}{2}\frac{\phi_J}{r}\partial_v\phi_J\,dx + \int_{S^2}\int_0^\infty r^{\gamma+1}\frac{1}{4}\partial_r{\phi_J}^2\,drd\omega \\
&= \int -r^\gamma\frac{1}{2}\frac{\phi_J}{r}\partial_v\phi_J\,dx - \int_{S^2}\int_0^\infty \frac{\gamma+1}{4}r^\gamma{\phi_J}^2\,drd\omega \\
&= \int -r^\gamma\frac{1}{2}\frac{\phi_J}{r}\partial_v\phi_J\,dx - \frac{\gamma+1}{4}\int_{\mathbb{R}^3} r^\gamma\frac{{\phi_J}^2}{r^2}\,dx
\end{aligned}$$

Thus,

$$\begin{aligned}
&-\int_{\mathbb{R}^3} r^\gamma\left(\frac{1}{2}|\partial\!\!\!/\phi_J|^2 + \frac{1}{2}(\partial_v\phi_J)^2 + \frac{\gamma+1}{4}\frac{\phi_J^2}{r^2} + \frac{1}{2}\frac{\phi_J}{r}\partial_v\phi_J\right)_{T_1}^{T_2} dx \\
&\quad = -\int_{\mathbb{R}^3} r^\gamma\left(\frac{1}{2}|\partial\!\!\!/\phi_J|^2 + \left[\frac{1}{2}(\partial_v\phi_J)^2 + \frac{1}{8}\frac{\phi_J^2}{r^2} + \frac{1}{2}\frac{\phi_J}{r}\partial_v\phi_J\right] + (\frac{\gamma}{4}+\frac{1}{8})\frac{\phi_J^2}{r^2}\right)_{T_1}^{T_2} dx \\
&\quad = -\int_{\mathbb{R}^3} r^\gamma\left(\frac{1}{2}|\partial\!\!\!/\phi_J|^2 + \frac{1}{2}\left[\partial_v\phi_J + \frac{\phi_J}{2r}\right]^2 + (\frac{\gamma}{4}+\frac{1}{8})\frac{\phi_J^2}{r^2}\right)_{T_1}^{T_2} dx
\end{aligned} \tag{7.5.4}$$

- We shall now prove by induction the claim that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}(\phi_{\leq m})^2 r^{\gamma-3}dxdt \leq C$$

where $C$ is a constant depending on the initial data. We make use of the defocusing sign of the nonlinearity. We have

$$\iint \phi^5(r^\gamma \partial_v \phi + r^{\gamma-1}\phi)\, dxdt = \int_{\mathbb{R}^3} r^\gamma \frac{\phi^6}{6}|_0^T\, dx + \iint \left(\frac{2}{3} - \frac{\gamma}{6}\right) r^{\gamma-1}\phi^6\, dxdt$$

Note that both terms are nonnegative for our range of small $\gamma$ (indeed any $\gamma < 4$). This implies (for our range of small $\gamma$)

$$\int_0^\infty \int_{\mathbb{R}^3} r^{\gamma-1}\phi^6\, dxdt < \infty \tag{7.5.5}$$

(because $T_1$ and $T_2$ were arbitrary); and it also establishes the claim for the base case value $m = 0$.

Suppose that for some $m \geq 0$,

$$\int_0^\infty \int_{\mathbb{R}^3} (\phi_{\leq m})^2 r^{\gamma-3} dxdt < \infty.$$

Then, letting $M\phi_J = r^\gamma(\partial_v \phi_J + r^{-1}\phi_J)$ and $|J| = m+1$,

$$\int_{T_1}^{T_2} \int_{\mathbb{R}^3} (\phi^5)_J M\phi_J dxdt = \int r^{(\gamma-1)/2} M\phi_J \cdot r^{(\gamma+1)/2}(\phi^5)_J dxdt \tag{7.5.6}$$

$$\lesssim \epsilon' \int r^{\gamma-1}\Big((\partial_v\phi_J)^2 + (r^{-1}\phi_J)^2\Big) dxdt \tag{7.5.7}$$

$$+ \frac{1}{\epsilon'} \int r^{\gamma+1}((\phi^5)_J)^2 dxdt \tag{7.5.8}$$

for some small $\epsilon' > 0$. We treat that term with the $\epsilon'$ factor perturbatively and absorb it to the left-hand side. Then we note that:

- In the case when there is no single factor in the nonlinearity that has $m+1$ vector fields falling on it, we have by (7.3.7)

$$\int r^{\gamma+1}((\phi^5)_J)^2 dxdt \lesssim \int r^{\gamma-3} A^8 (\phi_{\leq m})^2 dxdt, \quad A := \|\phi_{\leq m+n}\|_{LE^1(\mathbb{R}_+)}$$

which is bounded by a constant depending on the initial data by the induction hypothesis.

– If there is a factor with $m+1$ fields falling on it, we return to (7.5.6) and use (7.5.5).

$$\int \phi^4 r^{\gamma-1}\phi_J^2 dxdt \le \epsilon' \int r^{\gamma-2}\phi_J^4 + \frac{1}{\epsilon'}\int r^{\gamma}\phi^8 \le \epsilon' A^2 \int r^{\gamma-3}\phi_J^2 + \frac{1}{\epsilon'}A^2 \int r^{\gamma-1}\phi^6$$

for some small $\epsilon' > 0$. We again used (7.3.7). We absorb the small term to the left hand side, and the other term is bounded by a constant depending on the initial data, from (7.5.5).

- 1. Here, in dealing with the potential $V$, we assume only that $V \in S^Z(\epsilon r^{-2})$.

We have:

$$\begin{aligned}\int_{T_1}^{T_2}\int_{\mathbb{R}^3} |V_{\le m}\phi_{\le m} r^{\gamma}(\partial_v \phi_J + \frac{\phi_J}{r})|\, dxdt &\lesssim \epsilon \int \frac{1}{r^{2-\gamma}}|\phi_{\le m}|(|\partial_v\phi_J| + |\frac{\phi_J}{r}|)\, dxdt \\ &\lesssim \epsilon \int \frac{(\phi_{\le m})^2 + \phi_J^2}{r^{3-\gamma}} + \frac{|\partial_v \phi_J|^2}{r^{1-\gamma}}\, dxdt \\ &\lesssim \epsilon A_{\gamma,m}\end{aligned} \tag{7.5.9}$$

If $B \in S^Z(\langle r\rangle^{-1-\sigma_B})$ and $2\sigma_B > \gamma$:

$$\begin{aligned}\int_{T_1}^{T_2}\int_{\mathbb{R}^3} |B_{\le m}\partial\phi_{\le m} r^{\gamma-1}\phi_J| dxdt &\lesssim \int \frac{1}{\langle r\rangle^{1+\sigma_B}}|\partial\phi_{\le m} r^{\gamma-1}\phi_J| \\ &\lesssim \frac{1}{\epsilon}\int \frac{|\partial\phi_{\le m}|^2}{\langle r\rangle^{1+2\sigma_B-\gamma}} + \epsilon\int r^{\gamma-3}\phi_J^2 \\ &\lesssim \frac{1}{\epsilon}\|\phi_{\le m}\|^2_{LE^1(T_1,T_2)} + \epsilon A_{\gamma,J}\end{aligned} \tag{7.5.10}$$

The bound on $\int |B_{\le m}\partial\phi_{\le m} r^{\gamma}| \cdot |\bar{\partial}\phi_J| dxdt$ is similar.

If $B \in S^Z(\epsilon\langle r\rangle^{-1-\sigma_B})$ then this is bounded by $\epsilon\|\partial\phi_{\le m}\|^2_{LE(T_1,T_2)} + \epsilon A_{\gamma,J}$.

2. We consider now all of the terms that involve the metric $h^{\alpha\beta}$. We may schematically write this as

$$\int (|\partial h_{\leq m}\partial\phi_{\leq m}| + |h_{\leq m}\partial^2\phi_{\leq m}|)r^\gamma(\frac{|\phi_J|}{r} + |\partial_v\phi_J|)dxdt$$

where $|J| = m$.

$$\begin{aligned}
&\int_{T_1}^{T_2}\int_{\mathbb{R}^3}(|\partial h_{\leq m}\partial\phi_{\leq m}| + |h_{\leq m}\partial^2\phi_{\leq m}|)r^\gamma(\frac{|\phi_J|}{r} + |\partial_v\phi_J|)dxdt \\
&\qquad \lesssim \epsilon\int \frac{1}{\langle r\rangle^{1+\sigma}}r^\gamma\left(|\partial\phi_{\leq m}| + |\partial^2\phi_{\leq m}|\right)\left(|r^{-1}\phi_J| + |\partial_v\phi_J|\right)dxdt \\
&\qquad \lesssim \epsilon\int \frac{r^{\frac{\gamma+1}{2}}}{\langle r\rangle^{1+\sigma}}\left(|\partial\phi_{\leq m}| + |\partial^2\phi_{\leq m}|\right)\cdot r^{\frac{\gamma-1}{2}}\left(|r^{-1}\phi_J| + |\partial_v\phi_J|\right)dxdt \\
&\qquad \lesssim \epsilon\int \frac{1}{\langle r\rangle^{2+2\sigma}}r^{\gamma+1}\left(|\partial\phi_{\leq m}| + |\partial^2\phi_{\leq m}|\right)^2 dxdt + \epsilon A_{\gamma,J} \\
&\qquad \lesssim \epsilon\|\partial\phi_{\leq m}\|^2_{LE(T_1,T_2)} + \epsilon\|\partial^2\phi_{\leq m}\|^2_{LE(T_1,T_2)} + \epsilon A_{\gamma,J} \text{ if } 2\sigma > \gamma
\end{aligned} \tag{7.5.11}$$

which we can control using (7.2.9) if we assume $2\sigma > \gamma$.

Taking the sum of (7.5.3), (7.5.4) and (7.5.9) to (7.5.11) over all $|J| \leq m$, i.e. $\sum_{|J|\leq m}$ ((7.5.3), (7.5.4) and (7.5.9) to (7.5.11)), and taking into account our argument for the nonlinear terms as well, we get

$$A_{\gamma,m} + E^\gamma_{\phi_{\leq m}}(T) \lesssim E^\gamma_{\phi_{\leq m}}(0) + \epsilon\|\phi_{\leq m}\|^2_{LE^1(T_1,T_2)} + \epsilon\|\partial^2\phi_{\leq m}\|^2_{LE(T_1,T_2)}.$$

□

Thus we used Section 7.1's assumptions to obtain finiteness of the local energy norms $\|\phi_{\leq m}\|_{LE^1(\mathbb{R}_+\times\mathbb{R}^3)}$, and now we have, in particular, showed that

$$\int_0^\infty\int_{\mathbb{R}^3}(\phi_{\leq m})^2 r^{\gamma-3}dxdt \lesssim E^\gamma_{\phi_{\leq m}}(0) + \|\partial\phi_{\leq m+1}(0)\|^2_{L^2} \leq C_0$$

where $C_0$ is a constant depending only on initial data.

*Remark* 7.5.4. The small factor $\epsilon$ in the $\epsilon\|(\partial\phi_{\leq m}, \partial^2\phi_{\leq m})\|_{LE}^2$ terms of (7.5.1) will not be needed or used in the rest of the chapter, and instead reflects the fact that we assumed that $P$ was a small perturbation of $\Box$.

**Lemma 7.5.5.** *Let* $v_+ := \langle s+\rho\rangle$ *where* $(\rho, s) \in D_{tr}$*. Then*

$$\|v_+^{-1}\|_{L^2(D_{tr}^{R\in\mathcal{R}_1})} \lesssim 1, \tag{7.5.12}$$

$$\|v_+^{-1}\|_{L^2(D_{tr}^{R\in\mathcal{R}_2})} \lesssim \left(\frac{\langle u\rangle}{R}\right)^{\frac{1}{2}}. \tag{7.5.13}$$

*Proof.* (7.5.12) and (7.5.13) follow from Lemma 7.4.4. □

**Proposition 7.5.6** (Application of the $r^\gamma$ estimate)**.** *Let* $\phi$ *solve* $P\phi = \phi^5$*. Assume the hypotheses on* $\gamma$ *in Theorem 7.5.3 and also (7.1.3).*

$$|\langle r\rangle(\phi_3)_{\leq m}| \lesssim \langle u\rangle^{1/2-\gamma/2}, \quad u > 1 \tag{7.5.14}$$

$$|(\phi_3)_{\leq m}| \lesssim r^{-\frac{1}{2}-\frac{\gamma}{2}}, \quad u < -1 \tag{7.5.15}$$

*Remark* 7.5.7. If $\gamma \geq 1$, then for $u < -1$ this theorem would instead conclude

$$(\phi_3)_{\leq m} \lesssim r^{-1}\langle u\rangle^{-\frac{1}{2}(\gamma-1)}.$$

*Proof.*

- Let $u > 1$. We now show

$$\int_{D_{tr}} \rho H_3 \, dA \lesssim \langle u\rangle^{1/2-\gamma/2}, \quad H_3(t,r) := \sum_{k=0}^{2} \|\Omega^k(\phi^5)_{\leq m}(t, r\omega)\|_{L^2(S^2)}.$$

We have

$$\begin{aligned}\int_{D^R_{tr}} \rho H_3 \, dA &\lesssim \int_{D^R_{tr}} \rho \frac{1}{\langle s+\rho\rangle^2} \|\phi_{\leq m+n}\|_{L^2(S^2)} \, dA \\ &\lesssim \int_{D^R_{tr}} \frac{1}{v_+} \|\phi_{\leq m+n}\|_{L^2(S^2)} dA \\ &\lesssim \|v_+^{-1}\|_{L^2(D^R_{tr})} \|\phi_{\leq m+n}\|_{L^2_{\rho,s,\omega}} \\ &\lesssim \|v_+^{-1}\|_{L^2(D^R_{tr})} \cdot \frac{1}{R} \frac{1}{R^{(\gamma-3)/2}} C_0\end{aligned} \tag{7.5.16}$$

where $C_0$ is a constant depending on the initial data. The first line follows by (7.3.7). The last line follows by Theorem 7.5.3 because

$$\begin{aligned}\int_{D^R_{tr}} \int_{\mathbb{S}^2} |\phi_{\leq m+n}|^2 d\omega ds d\rho &\sim R^{-2} \int_{D^R_{tr}} \int_{\mathbb{S}^2} |\phi_{\leq m+n}|^2 \rho^2 d\omega ds d\rho \\ &\sim R^{-2} R^{-(\gamma-3)} \int_{D^R_{tr}} \int_{\mathbb{S}^2} |\phi_{\leq m+n}|^2 \rho^{\gamma-3} \rho^2 d\omega ds d\rho.\end{aligned}$$

1. Let RHS denote "the right-hand side of." By Lemma 7.5.5,

$$\sum_{R \in \mathcal{R}_1} \text{RHS}(7.5.16) \lesssim \langle u\rangle^{1/2-\gamma/2}, \quad \gamma \in (0,1).$$

2. Fix $R \in \mathcal{R}_2$. By Lemma 7.5.5 we have

$$\text{RHS}(7.5.16) \lesssim \left(\frac{\langle u\rangle}{R}\right)^{\frac{1}{2}} \frac{1}{R} \frac{1}{R^{(\gamma-3)/2}} = \langle u\rangle^{1/2} R^{-\gamma/2}.$$

   Then we have

$$\sum_{R \in \mathcal{R}_2} \text{RHS}(7.5.16) \lesssim \langle u\rangle^{1/2-\gamma/2}, \quad \text{valid for } \gamma \in (0,\infty).$$

This finishes the proof of (7.5.14).

- Let $u < -1$.

$$\begin{aligned}\int_{D_{tr}} \rho H_3 \, dA &\lesssim \int_{D_{tr}} \rho^{\frac{\gamma-1}{2}} \|\phi_{\leq m+n}\|_{L^2_\omega} \cdot \rho^{\frac{3-\gamma}{2}} \|\phi_{\leq m+n}\|^4_{L^\infty_\omega} \, dA \\ &\lesssim C_1 \left( \int \rho^{3-\gamma} \|\phi_{\leq m+n}\|^8_{L^\infty_\omega} dA \right)^{\frac{1}{2}} \\ &\lesssim \left(r^{1-\gamma}\right)^{\frac{1}{2}}.\end{aligned}$$

where $C_1$ is a constant depending on the initial data (and on $\|\phi\|_{L^5 L^{10}}$, but recall that Remark 7.2.9 was made). The second line follows from Theorem 7.5.3 because

$$\begin{aligned}\int_{D_{tr}} \rho^{\gamma-1} \int_{\mathbb{S}^2} |\phi_{\leq m+n}|^2 d\omega ds d\rho &= \int_{D_{tr}} \rho^{\gamma-3} \int_{\mathbb{S}^2} |\phi_{\leq m+n}|^2 \rho^2 d\omega ds d\rho \\ &\leq (C_1)^2 \quad \text{by } \textit{Theorem } 7.5.3.\end{aligned}$$

The third line follows from (7.3.7), Lemma 7.4.4 and the assumption $\gamma < 1$.

□

*Remark* 7.5.8. Alternatively, by Theorem 7.5.3 and Lemma 7.3.2 (the latter applied to $C_T^R$) we conclude that

$$\|\phi_{\leq m}\|_{L^\infty(C_T^R)} \lesssim T^{-1/2} R^{-\gamma/2}. \tag{7.5.17}$$

This can be used to provide an alternate proof of the iteration in Section 7.7 below.

## 7.6 The iteration in $\{u < -1\}$

**Theorem 7.6.1.** *If $u < -1$, then*

$$\phi_{\leq m} \lesssim \langle r \rangle^{-1} \langle u \rangle^{-1-\min(\sigma,2)},$$

*Here $\sigma$ denotes the original value of $\sigma$ taken from Theorem 7.1.1.*

*Proof.* We begin with the bounds in (7.3.7) and Propositions 7.3.7 and 7.5.6, which in the outside region translate to

$$\phi_{\leq m+n} \lesssim \frac{\langle u\rangle^{1/2}}{\langle r\rangle}, \quad \partial\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2}}, \quad \phi_{\leq m+n} \lesssim \langle r\rangle^{-\frac{1}{2}-\frac{\gamma}{2}} \tag{7.6.1}$$

For simplicity, we shall use the first (far left) $\phi_{\leq m+n}$ bound for $\phi_1$ and the other (far right) $\phi_{\leq m+n}$ bound for $\phi_3$. Since $\langle u\rangle \leq \langle r\rangle$, (7.6.1) can be weakened to

$$\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{1/2}}, \quad \partial\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{1/2}\langle u\rangle}, \quad \phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{1/2+\gamma/2}}. \tag{7.6.2}$$

Recall the decomposition (7.4.4), and let

$$H_i = \sum_{k=0}^{2} \|\Omega^k (G_i)_{\leq n}(t, r\omega)\|_{L^2(\mathbb{S}^2)}.$$

Let $\sigma$ denote the reduced, irrational number mentioned in Remark 7.4.3 until stated otherwise. We thus have, using (7.6.2):

$$H_1 \lesssim \frac{1}{\langle r\rangle^{5/2+\sigma}}, \quad \partial_t H_2 \lesssim \frac{1}{\langle r\rangle^{3/2+\sigma}\langle u\rangle}, \quad H_3 \lesssim \frac{1}{\langle r\rangle^{5/2+5/2\gamma}}.$$

By (7.4.7) with $\alpha = 5/2+\sigma$, $\beta = 0$, and $\eta = 0$, we obtain

$$(\phi_1)_{\leq m+n} \lesssim r^{-1/2-\sigma}$$

which gains a factor of $\langle r\rangle^{-\sigma}$ compared to (7.6.2). Similarly (7.4.7) with $\alpha = 3/2+\sigma$, $\beta = 0$, and $\eta = 1$ yields

$$(\phi_2)_{\leq m+n} \lesssim r^{-1/2-\sigma}$$

Finaly, (7.4.7) with $\alpha = 2+2\gamma, \beta = 0, \eta = 1/2$ yields

$$(\phi_3)_{\leq m+n} \lesssim r^{-1/2-2\gamma}.$$

The three inequalities above, combined with Proposition 7.3.7, give the following improved bounds (by a factor of $\langle r\rangle^{-\sigma'}$ where $\sigma' := \min(2\gamma, \sigma)$).

$$\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{1/2+\sigma'}}, \quad \partial\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle^{1/2+\sigma'}\langle u\rangle}. \tag{7.6.3}$$

We now repeat the iteration, replacing $\alpha$ by $\alpha + \sigma'$ and applying (7.4.7). The process stops after $\lfloor \frac{1}{2\sigma'} \rfloor$ steps, when (7.4.7), combined with Proposition 7.3.7 yield

$$\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle}, \quad \partial\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle\langle u\rangle}. \tag{7.6.4}$$

We now switch to using (7.4.6) for $\phi_1$ and $\phi_3$, and (7.4.10) for $\phi_2$. Note that (7.6.4) implies

$$H_1 \lesssim \frac{1}{\langle r\rangle^{3+\sigma}}, \quad H_2 \lesssim \frac{1}{\langle r\rangle^{2+\sigma}}, \quad H_3 \lesssim \frac{1}{\langle r\rangle^{5}}.$$

By (7.4.6) with $\alpha = 2 + \sigma$, $\beta = 1$, and $\eta = 0$, we obtain

$$(\phi_1)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-\sigma}$$

Similarly (7.4.10) with $\alpha = 2 + \sigma$, and $\eta = 0$ yields

$$(\phi_2)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-\sigma}$$

Finally, (7.4.6) with $\alpha = 5$, $\beta = 0$, and $\eta = 0$ yields

$$(\phi_3)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-2}$$

We now repeat the iteration. We can continue improving the decay rates of $\phi_1$ and $\phi_2$ all the way to

$$(\phi_1)_{\leq m}, (\phi_2)_{\leq m} \lesssim r^{-1}\langle u\rangle^{-1-\sigma}. \tag{7.6.5}$$

For $\phi_3$, we note that after the bounds $(\phi_1)_{\leq m+n}, (\phi_2)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-1/5-}$ are obtained, we have

$$(\phi_3)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-3}. \tag{7.6.6}$$

By (7.6.5) and (7.6.6) we now have, for the *original* value of $\sigma$ from Theorem 7.1.1,

$$H_1 \lesssim \frac{1}{\langle r\rangle^{3+\sigma}\langle u\rangle^{1+}}, \quad H_2 \lesssim \frac{1}{\langle r\rangle^{2+\sigma}\langle u\rangle^{1+}}, \quad H_3 \lesssim \frac{1}{\langle r\rangle^{5}\langle u\rangle^{1+}}.$$

Using (7.4.6) and (7.4.10) now completes the proof. □

## 7.7 The iteration in $\{u > 1\}$

### 7.7.1 Converting $r$ decay to $t$ decay

The pointwise decay rates for the solution and its vector fields obtained from the estimates for the fundamental solution (see Section 7.4.3) are by themselves insufficient for completing the iteration. We show below that if the $\langle r\rangle^{-1}$ decay from the fundamental solution is converted into $\langle t\rangle^{-1}$, then the iteration does work.

#### Outline of the entire conversion argument

First we note that in this proof we can assume that $\phi$ is supported in $C_T^{<3T/4}$, because we can control the commutator $[P, \chi]$ adequately where $\chi$ is a cutoff function adapted to the region $C_T^{<3T/4}$. Next, by the embeddings proved earlier in Proposition 7.3.5,

$$\|\phi_{\leq m}\|_{L^\infty(C_T^{<3T/4})} \lesssim T^{-\frac{1}{2}}\|\phi_{\leq m+n}\|_{LE^1(\tilde{C}_T^{<3T/4})}.$$

By Theorem 7.2.7, this is bounded by

$$T^{-\frac{1}{2}}\|\phi_{\leq m+n}\|_{LE^1(\tilde{C}_T^{<3T/4})} \lesssim T^{-\frac{1}{2}}\|\partial\phi_{\leq m}(T)\|_{L^2_x}.$$

By a certain averaging process which we write out below,

$$T^{-\frac{1}{2}}\|\partial\phi_{\leq m}(T)\|_{L^2_x} \lesssim T^{-1}\|\partial\phi_{\leq m+n}\|_{L^2_{x,t}[T,2T]}.$$

By a Morawetz multiplier argument in Lemma 7.7.1, we obtain

$$T^{-1}\|\partial\phi_{\leq m}\|_{L^2_{x,t}} \lesssim T^{-1}\left(\sum_{j=1}^{2}\|\langle r\rangle^{\frac{1}{2}}\partial\phi_{\leq m}(jT)\|_{L^2} + \|\langle r\rangle(P\phi)_{\leq m}\|_{L^2_{x,t}}\right).$$

By another averaging process,

$$\|\langle r\rangle^{\frac{1}{2}}\partial\phi_{\leq m}(jT)\|_{L^2} \lesssim T^{-\frac{1}{4}}\|\langle r\rangle^{\frac{1}{4}}\partial\phi_{\leq m}\|_{L^2_{x,t}} + T^{-\frac{3}{4}}\|\langle r\rangle^{\frac{3}{4}}S\partial\phi_{\leq m}\|_{L^2_{x,t}}, \quad j \in \{1,2\}.$$

Hence

$$\begin{aligned} T^{-1}\|\partial\phi_{\leq m}\|_{L^2_{x,t}} &\lesssim T^{-1}\Big(T^{-\frac{1}{4}}\|\langle r\rangle^{\frac{1}{4}}\partial\phi_{\leq m}\|_{L^2_{x,t}} + T^{-\frac{3}{4}}\|\langle r\rangle^{\frac{3}{4}}S\partial\phi_{\leq m}\|_{L^2_{x,t}} \\ &\qquad + \|\langle r\rangle(P\phi)_{\leq m}\|_{L^2_{x,t}}\Big) \\ &\lesssim T^{-1}\left(\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1} + \|\langle r\rangle(P\phi)_{\leq m}\|_{L^2_{x,t}}\right) \end{aligned} \tag{7.7.1}$$

where

$$T^{-\frac{3}{4}}\|\langle r\rangle^{\frac{3}{4}}S\partial\phi_{\leq m}\|_{L^2_{x,t}} \lesssim \|\langle r\rangle\phi_{\leq m+n}\|_{LE^1}$$

because we commute $\partial$ and $S$, pass from the $L^2$ norm to the $LE^1$ norm, and in the support of $\phi$, $\langle r\rangle \lesssim T$; on the other hand, for small $r$ values we absorbed $T^{-\frac{1}{4}}\|\langle r\rangle^{\frac{1}{4}}\partial\phi_{\leq m}\|_{L^2_{x,t}}$ into the left-hand side of (7.7.1), which completes our explanation of (7.7.1).

In summary,

$$\|\phi_{\leq m}\|_{L^\infty(C_T^{<3T/4})} \lesssim T^{-1}\left(\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1} + \|\langle r\rangle(P\phi)_{\leq m}\|_{L^2_{x,t}}\right).$$

In the sequel, first we record the aforementioned Morawetz multiplier argument, and then we prove Lemma 7.7.2, which is a more detailed version of the outline just provided.

**Lemma 7.7.1.** *Suppose that $\phi$ satisfies the bound proved in (7.2.3), thus*

$$\|\phi_{\le m}\|_{LE^1(I\times\mathbb{R}^3)} \lesssim_{m,\|\phi\|_{L^5(I;L^{10}(\mathbb{R}^3))}} \|\partial\phi_{\le m}(T_1)\|_{L^2(\mathbb{R}^3)}, \qquad I=[T_1,T_2]. \tag{7.7.2}$$

*Then for all $0\le T_1\le T_2$, we have*

$$\|\nabla_{t,x}\phi_{\le m}\|_{L^2(C_{T_1}^{T_2})} \lesssim \sum_{j=1}^{2}\|\langle r\rangle^{1/2}\nabla_{t,x}\phi_{\le m}(T_j)\|_{L^2} + \|\langle r\rangle (P\phi)_{\le m}\|_{L^2}. \tag{7.7.3}$$

*Proof.* We demonstrate the case $m=0$ first for simplicity. We multiply the equation by $r\partial_r\phi+\phi$ and integrate by parts in $[T_1,T_2]\times\mathbb{R}^3$. There is a number $q'>0$ such that

$$\begin{aligned}\int |\nabla_{t,x}\phi|^2 + O(\langle r\rangle^{-q'})|\nabla_{t,x}\phi|^2 + O(\langle r\rangle^{-1-q'})|\partial\!\!\!/\phi|^2 + O(\langle r\rangle^{-2-q'})|\phi|^2\,dxdt \\ \lesssim \sum_{j=1}^{2}\int_{\mathbb{R}^3} O(\langle r\rangle)|\nabla_{t,x}\phi(T_j,x)|^2 + O(\langle r\rangle^{-1})|\phi(T_j,x)|^2\,dx + \int |r(P\phi)\partial_r\phi| \\ + |(P\phi)\phi|\,dxdt \\ \lesssim \sum_{j=1}^{2}\int_{\mathbb{R}^3} O(\langle r\rangle)|\nabla_{t,x}\phi(T_j,x)|^2\,dx + \int |r(P\phi)\partial_r\phi| + |(P\phi)\phi|\,dxdt\end{aligned} \tag{7.7.4}$$

with the last statement following by a version of Hardy's inequality. Next, by Cauchy-Schwarz and Hardy's inequality we can bound all the terms involving $P\phi$ by

$$\frac{1}{\epsilon'}\|rP\phi\|^2_{L^2L^2} + \epsilon'\|\partial_r\phi\|^2_{L^2L^2}.$$

By using the positivity of $q'$ on the left-hand side of (7.7.4) for large $|x|$ values, and using (7.7.2) for small $|x|$ values, we can then obtain

$$\|\nabla_{t,x}\phi\|_{L^2[T_1,T_2]L^2} \lesssim \sum_{j=1}^{2}\|\langle r\rangle^{1/2}\nabla_{t,x}\phi(T_j)\|_{L^2} + \|\langle r\rangle P\phi\|_{L^2[T_1,T_2]L^2}. \tag{7.7.5}$$

(7.7.5) implies (7.7.3) for $m=0$.

(The higher multiindex case) We now prove (7.7.5) but for $\phi_J, J \neq \vec{0}$. We have

$$P\phi_J = (P\phi)_J + O(\langle r\rangle^{-1-q'})\nabla_{t,x}\phi_{\leq |J|} + O(\langle r\rangle^{-2-q'})\phi_{\leq |J|-1}.$$

We multiply this by $r\partial_r\phi_J + \phi_J$. Then we integrate in $[T_1, T_2] \times \mathbb{R}^3$. The rest of the proof is then similar. □

Thus for the small data case, for instance,

$$\|\nabla_{t,x}\phi_{\leq m}\|_{L^2(C_{T_1}^{T_2})} \lesssim \sum_{j=1}^{2} \|\langle r\rangle^{1/2}\nabla_{t,x}\phi_{\leq m}(T_j)\|_{L^2}.$$

This follows by (7.3.7): one can take the $\|\langle r\rangle(P\phi)_{\leq m}\|_{L^2}$ term in (7.7.3) and bound it as follows

$$\begin{aligned}\|\langle r\rangle(\phi^5)_{\leq m}\|_{L^2} &\lesssim \epsilon'\|\langle t\rangle^{-1}\phi_{\leq m}\|_{L^2} \\ &\lesssim \epsilon'\|\partial_r\phi_{\leq m}\|_{L^2}\end{aligned}$$

for some sufficiently small $\epsilon' > 0$, so we absorb this term to the left-hand side of (7.7.3).

**Lemma 7.7.2.** *Assume that $\phi$ satisfies (7.7.2). Then*

$$\|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim T^{-1}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1(C_T^{<3T/4})} + T^{-\frac{1}{2}}\|\langle r\rangle(P\phi)_{\leq m+n}\|_{L^2(C_T^{<3T/4})}.$$

*Proof.* Fix a dyadic number $T$. Recall (7.2.3), which states that we have ILED for vector fields. We may assume that $\phi$ is supported in $C_T^{<3T/4}$ because we can control $[P, \chi]$, where $\chi$ is a purely spatial cutoff localised to the interior region $\{r < 3t/4\}$, in the $LE^*$ norm. We need not perform any cutoffs in the time variable.

Let $m \geq 0$. Let $\gamma_{(T,x)}(t')$ denote an integral curve of $S$, parametrized by unit speed, such that $t' = 0$ corresponds to the point $(T, x)$. By the fundamental theorem of calculus and Cauchy-Schwarz, we have

$$|\nabla_{t,x}\phi_{\leq m}(T,x)|^2 \lesssim \frac{1}{T}\int_0^T |(\nabla_{t,x}\phi_{\leq m})(\gamma_{(T,x)}(t'))|^2 + |(S\nabla_{t,x}\phi_{\leq m})(\gamma_{(T,x)}(t'))|^2\, dt' \tag{7.7.6}$$

A similar bound holds for $t = 2T$. Thus, after we integrate in $x$, we control the energy terms by

$$T^{-1/2}\|\partial\phi_{\leq m+n}\|_{L^2_{t,x}}.$$

By the fundamental theorem of calculus and Cauchy-Schwarz we have

$$\|\langle r\rangle^{1/2}\partial\phi_{\leq k}(T)\|_{L^2} \lesssim T^{-1/4}\|\langle r\rangle^{1/4}\phi_{\leq k}\|_{L^2} + T^{-1/2}\|\langle r\rangle\phi_{\leq k+1}\|_{LE^1}$$

and similarly for the $t = 2T$ energy norm. We decompose

$$\|\langle r\rangle^{1/4}\nabla_{t,x}\phi_{\leq k}\|_{L^2} = \sum_{R<T}\|R^{1/4}\nabla_{t,x}\phi_{\leq k}\|_{L^2(r\sim R)}$$

and note that for all large $R$,

$$\|R^{1/4}\nabla_{t,x}\phi_{\leq k}\|_{L^2(r\sim R)} \lesssim \|\langle r\rangle\phi_{\leq k+n}\|_{LE^1}$$

while for all sufficiently small $R$, we may absorb this to the left-hand side. The proof is complete. □

The next proposition uses Lemma 7.7.2 to obtain better pointwise decay for $\phi_{\leq m}$ in the region $\{r < t/2\}$.

**Proposition 7.7.3.** *Let $\phi$ solve (7.1.2). Let $\delta > 0$. Assume that*

$$\phi_{\leq M}|_{r\leq 3t/4} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-n\delta}, \quad \phi_{\leq M}|_{r\leq 3t/4} \lesssim \langle t\rangle^{-1}\langle u\rangle^{1/2-(n-1)\delta}, \quad n \geq 1 \tag{7.7.7}$$

*for an $M$ that is sufficiently larger than $m$. Then we have*

$$\phi_{\leq m}|_{C_T^{<3T/4}} \lesssim \langle t\rangle^{-1}\langle u\rangle^{1/2-n\delta}.$$

*Proof.* By Proposition 7.3.7 and (7.7.7),

$$T^{-1}\|\langle r\rangle\phi_{\leq m+n}\|_{LE^1} \lesssim T^{-1}\|\phi_{\leq m+n}\|_{LE} \lesssim T^{-q}, \quad q = n\delta.$$

Fix $n \geq 1$. For $A_{R=1}$, we use the latter bound in (7.7.7) to obtain

$$\|(\phi^5)_{\leq m+n}\|_{L^2([T,2T];L^2(A_{R=1}))} \lesssim T^{-2-5(n-1)\delta}.$$

For $A_R, R > 1$, we use the $\langle t\rangle^{-1}\langle u\rangle^{1/2-(n-1)\delta}$ bound and the $\langle r\rangle^{-1}\langle u\rangle^{1/2-n\delta}$ bound in (7.7.7) in a three-to-two ratio (respectively). This yields, for $R > 1$,

$$\begin{aligned}
T^{-\frac{1}{2}}\|\langle r\rangle(\phi^5)_{\leq m+n}\|_{L^2([T,2T];L^2(A_R))} &\lesssim \|\langle r\rangle^{\frac{1}{2}}(\phi^5)_{\leq m+n}\|_{L^2([T,2T];L^2(A_R))} \\
&\lesssim \|R^{1/2}(R^{-1}T^{1/2-q})^2(T^{-1/2-q+\delta})^3\| \\
&\lesssim \left(T^{-10n\delta+6\delta}\right)^{\frac{1}{2}} \\
&= T^{-5n\delta+3\delta}
\end{aligned}$$

Therefore Lemma 7.7.2 implies, after the dyadic sum,

$$\|\phi_{\leq m}\|_{LE^1(C_T^{<3T/4})} \lesssim T^{-n\delta}$$

and the conclusion now follows by Lemma 7.3.2. ☐

### 7.7.2 The iteration

**Theorem 7.7.4.** *If $u > 1$, then*

$$\phi_{\leq m} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-1-\min(\sigma,2)}.$$

*Here $\sigma$ denotes the original value of $\sigma$ taken from Theorem 7.1.1.*

*Proof.* As before, we begin with the bounds in (7.3.7) and Propositions 7.3.7 and 7.5.6, which in the inside region translate to

$$\phi_{\leq m+n} \lesssim \frac{\langle u\rangle^{1/2}}{\langle t\rangle}, \quad \partial\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2}}, \quad \phi_{\leq m+n} \lesssim \frac{\langle u\rangle^{1/2-\gamma/2}}{\langle t\rangle} \tag{7.7.8}$$

where again for simplicity we use the far left $\phi_{\leq m+n}$ bound for $\phi_1$ and the far right $\phi_{\leq m+n}$ bound for $\phi_3$.

Let $\sigma$ denote the reduced, irrational number mentioned in Remark 7.4.3 until stated otherwise. We thus have, using (7.7.8):

$$H_1 \lesssim \frac{\langle u\rangle^{1/2}}{\langle r\rangle^{2+\sigma}\langle t\rangle}, \quad \partial_t H_2 \lesssim \frac{1}{\langle r\rangle^{1+\sigma}\langle t\rangle\langle u\rangle^{1/2}}, \quad H_3 \lesssim \frac{\langle u\rangle^{5(1/2-\gamma/2)}}{\langle t\rangle^5} \lesssim \frac{\langle u\rangle^{1/2}}{\langle t\rangle^{3+5\gamma/2}}.$$

By (7.4.6) with $\alpha = 2+\sigma$, $\beta = 1$, and $\eta = -1/2$, we obtain

$$(\phi_1)_{\leq m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma}$$

Similarly (7.4.6) with $\alpha = 2+\sigma$, $\beta = 0$, and $\eta = 1/2$ yields

$$(\phi_2)_{\leq m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma}$$

Finally, (7.4.6) with $\alpha = 0$, $\beta = 3+5\gamma/2$, and $\eta = -1/2$ yields

$$(\phi_3)_{\leq m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-5\gamma/2}$$

The three inequalities above give

$$\phi_{\leq m+n} \lesssim \langle r\rangle^{-1}\langle u\rangle^{1/2-\sigma'}, \quad \sigma' := \min(2\gamma, \sigma).$$

By Proposition 7.7.3 we obtain the following improved bounds (by a factor of $\langle u\rangle^{-\sigma'}$):

$$\phi_{\leq m+n} \lesssim \frac{\langle u\rangle^{1/2-\sigma'}}{\langle t\rangle}, \quad \partial\phi_{\leq m+n} \lesssim \frac{1}{\langle r\rangle\langle u\rangle^{1/2+\sigma'}}. \tag{7.7.9}$$

We now repeat the iteration, replacing $\eta$ by $\eta + \sigma'$, applying (7.4.7) and then improving decay in $\{t/r > 2\}$ by Proposition 7.7.3. The process stops after $\lfloor \frac{1}{2\sigma'} \rfloor$ steps, when (7.4.6) and Proposition 7.7.3 yield

$$|\phi_{\leq m+n}| \lesssim \frac{1}{\langle t \rangle}, \quad |\partial \phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle}. \tag{7.7.10}$$

At this point we switch to using (7.4.10) for $\phi_2$, and the iteration process follows the same pattern as in Section 7.6, with the extra use of Proposition 7.7.3. Like before in Theorem 7.6.1, we make the final iterate involving the *original* value of $\sigma$ from Theorem 7.1.1. □

## 7.8 Other integral powers $p \in \mathbb{N}_{\geq 2} \setminus \{4\}$

*Remark* 7.8.1 (Other integral powers $p \in \mathbb{N}_{\geq 2} \setminus \{4\}$). For $\phi$ solving $P\phi = G_3$ with

$$G_3 = |\phi|^p \phi, \quad p \geq 5 \text{ with large initial data}$$

or

$$G_3 = \pm \phi^{p+1}, \quad p \geq 5 \text{ with small initial data}$$

we remark that *if* global existence holds for the large data case (as this is clear for the small data case), then just by the initial global decay rate (7.3.7) alone, the decay rates in Proposition 7.5.6 are immediately achieved for some $\gamma$ (and hence there is no need to prove Theorem 7.5.3), and letting $\phi_3$ solve $\Box \phi_3 = G_3$ as before in Section 7.4.2, the iterations in Sections 7.6 and 7.7 follow nearly verbatim, with the natural modification that in the end we reach the final decay rate

$$(\phi_3)_{\leq m+n} \lesssim r^{-1} \langle u \rangle^{-(p-1)}, \quad \text{for } u < -1, \qquad (\phi_3)_{\leq m+n} \lesssim v^{-1} \langle u \rangle^{-(p-1)}, \quad \text{for } u > 1.$$

Thus

$$\phi_{\leq m} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-\min\{1+\sigma,p-1\}}.$$

We explain this modification: One way to view the estimates for the fundamental solution in Lemmas 7.4.4 and 7.4.5 is that they involve integrating in the $v$ and $u$ directions. For nonlinearities that are $O(|\phi|^{p+1})$, in the relevant domain $D_{tr}$ for the spherically symmetric model (which majorizes the solution $\phi$ by the positivity of the fundamental solution in three space dimensions), they are bounded along the $\rho + s$ direction by $O((\rho+s)^{-p-1})$ and since the integration involves one power of $\rho$ (see (7.4.8)), near the cone $\{\rho \approx s\}$, this is an integral of $\rho(\rho+s)^{-p-1} = O((\rho+s)^{-p})$, which integrates out to $u^{-p+1}$; on the other hand, integrations in the other null $(\rho - s)$ direction do not gain after a certain number of steps, and so this explains the final $-p+1$ exponent above.

For $\phi$ solving $P\phi = G_3$ with

$$G_3 = |\phi|^2\phi$$

with *large initial data*, we remark here that *if* global existence holds and if Proposition 7.5.6 holds with $\gamma > 1$, then the iteration also follows Sections 7.6 and 7.7 nearly verbatim, and we reach the final decay rates

$$(\phi_3)_{\leq m+n} \lesssim r^{-1}\langle u\rangle^{-1} \text{ for } u < -1, \qquad (\phi_3)_{\leq m+n} \lesssim v^{-1}\langle u\rangle^{-1} \text{ for } u > 1$$

and thus

$$\phi_{\leq m} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-1}.$$

Similarly we have global existence for $P\phi = \pm\phi^3$ with small initial data, and if we had Proposition 7.5.6 with $\gamma > 1$ then this remark holds as well.

For $\phi$ solving $P\phi = G_3$ with

$$G_3 = \pm\phi^4$$

with small initial data, if we had Proposition 7.5.6 with $\gamma > 1/2$ then after setting $\Box\phi_3 = G_3$ in Section 7.4.2, the iterations in Sections 7.6 and 7.7 also hold nearly verbatim and we reach

$$(\phi_3)_{\leq m+n} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-2}.$$

thus

$$\phi_{\leq m} \lesssim \langle v\rangle^{-1}\langle u\rangle^{-\min\{1+\sigma,2\}}.$$

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