Bicategorical Traces and Cotraces

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Digital Object Identifier: https://doi.org/10.13023/etd.2023.205

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Bicategorical Traces and Cotraces

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

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2023

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ABSTRACT OF DISSERTATION

Bicategorical Traces and Cotraces

Familiar constructions like the trace of a matrix and the Euler characteristic of a closed smooth manifold are generalized by a notion of trace of an endomorphism of a dualizable object in a bicategory equipped with a piece of additional structure called a shadow functor. Another example of this bicategorical trace, in the form of maps between Hochschild homology of bimodules, appears in a 1987 paper by Joseph Lipman, alongside a more mysterious "cotrace" map involving Hochschild cohomology. Putting this cotrace on the same category-theoretic footing as the trace has led us to propose a "bicategorical cotrace" in a closed bicategory with a "coshadow functor." The program of bicategorical shadows and traces aims to unify seemingly disparate pieces of mathematics underneath a common conceptual framework; by adding notions of coshadow and cotrace to this machinery, we have drawn Lipman's residues and (co)traces into this framework and made progress toward describing 2-representations and 2-characters in a way that parallels the application of traces to ordinary group representations.

We begin by reviewing the theory of duality and trace in symmetric monoidal categories and in bicategories, and we discuss the features of closed bicategories that will be needed to develop a theory of cotraces. We then motivate and define bicategorical coshadows and cotraces and proceed to establish several important properties of these constructions. We also prove a very general interplay between traces and cotraces in a closed bicategory with compatible shadows and coshadows. Finally, we discuss applications of the bicategorical cotrace to Lipman's residues and traces and to Ganter and Kapranov's study of 2-representations and 2-characters.

KEYWORDS: bicategory, trace, shadow, cotrace, topology, category theory

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May 3, 2023

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ACKNOWLEDGMENTS

I would first like to express my deep gratitute to my advisor, Dr. Kate Ponto, for pushing me to become a better mathematician, writer, and educator, for supporting me through many personal and professional transitions, for useful advice on everything from TEXing large commutative diagrams to knitting, and for teaching me the power of thinking in crazy generality.

Thank you to Jonathan Campbell for bringing to our attention the paper that motivated much of the work in this dissertation, and to Cary Malkiewich for suggesting a better approach to string diagrams and helping discover cyclicity of the cotrace. I thank Inbar Klang, Niles Johnson, and Davis Deaton for helpful conversations, and Jordan Sawdy for sharing a wealth of knowledge about category theory.

I would like to thank the many teachers who have generously shared their time and expertise, especially Bert Guillou, Nat Stapleton, Peter Bonventre, Sara Jensen, Mark Snavely, Jean Quashnock, Aaron Trautwein, Erik Brekke, Michael Olson, Janelle Bailey, Kelly Fitzgerald, Anne Schmidt, and Joe Van Erem. Thank you to Rejeana Cassady, Christine Levitt, and Sheri Rhine for all of their assistance over the last six years. I would also like to thank the many faculty at UK who have helped me grow as a teacher, especially Ben Braun, Brad Elliott, Katie Paullin, Jack Schmidt, Chloe Urbanski Wawrzyniak, and erica Whitaker.

I thank all the graduate students from UK and the Midwest topology group who have made me feel at home in the math department and the homotopy theory community, especially Shane Clark, Landon Gauthier, Derek Hanely, Shahzad Kalloo, Sarah Petersen, Josh Peterson, Cole Pospisil, Jordan Sawdy, and Matias von Bell.

Finally, I am grateful to my parents for their love and support throughout my academic journey.

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A recurring theme in mathematics is that traces provide a way to turn complicated objects into simpler ones, discarding much of the information in the original object but retaining just enough information to say something useful about it. In particular, many familiar invariants arise as traces of identity maps. The trace of the identity map on a real or complex vector space is the vector space's dimension, which classifies the vector space up to isomorphism. The trace of the identity map on a finite CW complex is its Euler characteristic (vertices plus edges minus faces, and so on in higher dimensions), which is not a complete invariant but is still a useful tool for distinguishing between topological spaces. Finally, the trace of the identity map on a group representation is its character, which forgets most of the data of the representation and retains only the trace of each group element's action on the underlying vector space; over a field of characteristic zero, the character determines the representation up to isomorphism.

The first two of these examples are generalized by a well-known notion of the trace of an endomorphism of a dualizable object in a symmetric monoidal category [2, 15] (*dualizability* generalizes the finiteness conditions in these examples). The character of a group representation, however, requires the trace in a bicategory equipped with a shadow, which was introduced by Ponto [11, 14, 13, 12] to generalize fixed-point invariants such as the Reidemeister trace. It also generalizes the Hattori-Stallings trace [5, 16], which extends the familiar linear algebra trace to endomorphisms of modules over noncommutative rings. The price we pay for including noncommutative rings is that the trace of an R-module endomorphism takes values not in R itself but only in the quotient $\overline{R}$ of R's underlying abelian group by the subgroup generated by all elements of the form $rs - sr$. The passage from R to $\overline{R}$ is an example of a *shadow*, which is the additional structure a bicategory requires in order to support a notion of trace.

There are, however, some examples of trace-like constructions which are not fully explained by categorical notions of trace; incorporating them into this perspective requires a dual notion of *cotrace*. The example that originally motivated our development of a bicategorical cotrace is the cotrace maps of [9]. Lipman provides a more elementary development of Grothendieck's residue symbol [4] by reframing it in terms of Hochschild homology, and he establishes a sort of adjointness between trace and cotrace, mediated by a pairing map on Hochschild homology and cohomology. The simplest case of this adjointness is expressed by the following commutative diagram, where R is any ring and F is a right R-module which is finitely generated and projective (this is the appropriate sort of "finiteness" for taking modules):

$$
\begin{tikzcd}
 & R^c \otimes \overline{\mathrm{Hom}_R(F,F)} \arrow[dl, "\mathrm{cotr}\otimes 1"'] \arrow[dr, "1\otimes \mathrm{tr}"] & \\
\mathrm{Hom}_R(F,F)^c \otimes \overline{\mathrm{Hom}_R(F,F)} \arrow[d, "\rho_{\mathrm{Hom}_R(F,F)}"'] & & R^c \otimes \overline{R} \arrow[d, "\rho_R"] \\
\overline{\mathrm{Hom}_R(F,F)} \arrow[dr, "\mathrm{tr}"'] & & \overline{R} \arrow[dl, equal] \\
 & \overline{R} &
\end{tikzcd}
\tag{1.1}
$$

Just as the Hattori-Stallings trace necessitates the passage from R to its quotient $\overline{R}$, a cotrace compels us, dually, to replace R with its center R^c. The trace map $\overline{\mathrm{Hom}_R(F,F)} \to \overline{R}$ is the Hattori-Stallings trace; its domain is $\overline{\mathrm{Hom}_R(F,F)}$, rather than $\mathrm{Hom}_R(F,F)$, because of cyclicity of the trace, i.e. the fact that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$. The cotrace $R^c \to \mathrm{Hom}_R(F,F)^c$ sends r to multiplication by r, i.e. the map $\mu_r : F \to F$, $x \mapsto xr$. The pairing map ρ_R is multiplication in R, and $\rho_{\mathrm{Hom}_R(F,F)}$ is composition. Thus the diagram asserts that $\mathrm{tr}(\mu_r \circ f) = r\,\mathrm{tr}(f)$, but writing this as $\mathrm{tr}(\rho(\mathrm{cotr}(r), f)) = \rho(r, \mathrm{tr}(f))$ makes it more strongly resemble an adjointness between cotr and tr.

Lipman acknowledges that his description of residues is not fully satisfactory, and he suspects that "there might well be a more fundamental approach to the subject, encompassing a great deal more than we have dealt with here" [9]. We offer a candidate for the more fundamental approach Lipman imagined, by repackaging his traces and cotraces in terms of Ponto's bicategorical trace [11, 14] and a new notion of bicategorical cotrace. In doing so, we have teased out the formal structure underlying Lipman's trace formulas, which includes (1) a notion of "coshadow" and "cotrace" dual to Ponto's shadows and traces and (2) an interplay between bicategorical traces and cotraces generalizing the results of [9].

Before we describe this interplay, some comments are in order about the shape that traces and cotraces take. The trace of a linear transformation $f : V \to V$ is often thought of as a *number* (i.e. an element of the ground field), but we prefer to think of it as a *map*, namely the linear transformation $k \to k$ sending 1 to the number which is ordinarily thought of as the trace of f. While this distinction may seem inconsequential for the example of vector spaces, it turns out to be a crucial shift in perspective that brings many different mathematical constructs underneath the umbrella of "trace." For instance, the trace of a endomorphism f of a right R-module is an abelian group homomorphism $\mathbb{Z} \to \overline{R}$ (again, sending 1 to what would typically be called the Hattori-Stallings trace of f). If we have not just a right R-module but an (S,R)-bimodule, its trace is a map $\overline{S} \to \overline{R}$; for example, a right R-module F is naturally a $(\mathrm{Hom}_R(F,F), R)$-bimodule, whose trace is the map $\overline{\mathrm{Hom}_R(F,F)} \to \overline{R}$ appearing in the diagram above. A representation of a group G can be regarded as a $(k[G], k)$-bimodule (where $k[G]$ is the group algebra), so its trace is a map $\overline{k[G]} \to k$ (since k is a field, and in particular a commutative ring, $\overline{k} = k$), and this map is nothing other than the character of the representation (Example 10). This perspective allows us to view the character as a single trace, rather than a collection of traces (one for each group element).

More generally, the trace of an (S, R)-bimodule (or a 1-cell $R \to S$ in a bicategory) is a map $\langle\!\langle S \rangle\!\rangle \to \langle\!\langle R \rangle\!\rangle$, where the shadow $\langle\!\langle - \rangle\!\rangle$ generalizes the quotienting operation $R \mapsto \overline{R}$ required for a sensible notion of trace. Dually, the replacement of a ring by its center is generalized by a coshadow $\langle\!\langle - \langle\!\langle$, which supports the formation of cotraces $\langle\!\langle R \langle\!\langle \to \langle\!\langle S \langle\!\langle$. With this notation in place, we are finally prepared to state the bicategorical generalization of the cotrace-trace adjointness illustrated by (1.1), albeit somewhat imprecisely (see Theorem 3 for the precise statement).

Theorem 1. *Given suitable maps f, g, h, i involving objects A, A', B, B', C, a coshadow $\langle\!\langle - \langle\!\langle$, shadows $\langle\!\langle - \rangle\!\rangle$ and $\langle\!\langle - \rangle\!\rangle$, and a pairing map $\langle\!\langle - \langle\!\langle \otimes \langle\!\langle - \rangle\!\rangle \xrightarrow{\rho} \langle\!\langle - \otimes - \rangle\!\rangle$, the following commutes:*

$$\begin{tikzcd}
& \langle\!\langle A \langle\!\langle \otimes \langle\!\langle B' \rangle\!\rangle \arrow[dl, "\mathrm{cotr}(f)\otimes 1"'] \arrow[dr, "1\otimes \mathrm{tr}(g)"] & \\
\langle\!\langle A' \langle\!\langle \otimes \langle\!\langle B' \rangle\!\rangle \arrow[d, "\rho"'] & & \langle\!\langle A \langle\!\langle \otimes \langle\!\langle B \rangle\!\rangle \arrow[d, "\rho"] \\
\langle\!\langle A' \otimes B' \rangle\!\rangle \arrow[dr, "\mathrm{tr}(h)"'] & & \langle\!\langle A \otimes B \rangle\!\rangle \arrow[dl, "\mathrm{tr}(i)"] \\
& \langle\!\langle C \rangle\!\rangle &
\end{tikzcd}$$

This abstract perspective on traces is valuable because it often allows us to make mathematical constructs and theorems more accessible by extracting the core ideas from the technical details of their original presentation. Moreover, we are often able to prove vastly more general versions of these results (for example, Theorem 1) and port them over to other mathematical contexts. For example, the tools that we have built to understand Lipman's work have revealed promising links to the theory of group representations and 2-representations.

The classical formulas for the characters of restricted and induced group representations are consequences of multiplicativity of trace (Example 11). Weakening the conditions defining a representation leads to *2-representations*; Ganter and Kapranov [3] defined a *2-character* and proved an induction formula for 2-representations, and we have made progress toward understanding this result from the perspective of bicategorical (co)traces. In particular, one of the key ingredients in the 2-character of [3] turns out to be an example of a coshadow.

The aim of this dissertation is to lay the foundation for a theory of bicategorical coshadows and cotraces. The main result is Theorem 1 (or Theorem 3), but along the way we establish properties of coshadows and cotraces analogous to many of the properties of trace described in [14], including cyclicity, functoriality, and Morita invariance. The overarching goal is to illuminate connections between diverse mathematical contexts, so we emphasize the application of this categorical machinery to examples such as Lipman's trace-cotrace interplay and the categorical trace of Ganter and Kapranov.

1.1 Outline

In Section 2 we begin by reviewing the theory of duality and trace in symmetric monoidal categories and in bicategories, and then we discuss closed bicategories, which will be the setting for our study of cotraces.

In Section 3 we develop the theory of bicategorical cotraces; we define coshadows and cotraces and establish their important properties (including functoriality and Morita invariance). We also demonstrate that while cotraces can also be defined in symmetric monoidal categories, they coincide with symmetric monoidal traces.

In Section 4 we prove Theorem 1, showing that traces and cotraces are compatible with the additional structure of a pairing map from a shadow and coshadow to a second shadow.

Chapter 2 Background

We begin by reviewing the theory of duality and trace in symmetric monoidal categories and bicategories, which is developed in [2, 8, 10, 11, 14].

2.1 Symmetric Monoidal Duality and Trace

If V is a finite-dimensional vector space, then any linear transformation $f : V \to V$ has a trace, which is the sum of the diagonal entries in any matrix representation of f. The trace makes sense in much more general contexts though; instead of endomorphisms vector spaces, we can take the trace of an endomorphism of an object in a symmetric monoidal category, as long as that object satisfies a "dualizability" condition which generalizes finite-dimensionality for vector spaces.

Definition 1 ([2, Theorem 1.3])**.** Let $(\mathscr{C}, \otimes, I)$ be a symmetric monoidal category. An object M of $\mathscr{C}$ is ***dualizable*** if there is an object M^* of $\mathscr{C}$ and maps

$$\eta : I \to M \otimes M^* \qquad \varepsilon : M^* \otimes M \to I$$

such that the following ***triangle identities*** hold:

$$\begin{tikzcd} M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & M \otimes M^* \otimes M \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M^* \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[dr, "\mathrm{id}"'] & M^* \otimes M \otimes M^* \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\ & M^* \end{tikzcd}$$

We say that M^* is a ***dual*** for M, and we refer to η and ε as the ***coevaluation*** and ***evaluation*** (respectively) of the dual pair (they are sometimes called the *unit* and *counit*, but we avoid that terminology so as not to overload the word "unit").

We generally omit unit isomorphisms; for instance, we write $M \xrightarrow{\eta\otimes\mathrm{id}} M \otimes M^* \otimes M$ as shorthand for $M \cong I \otimes M \xrightarrow{\eta\otimes\mathrm{id}} M \otimes M^* \otimes M$.

Example 1. A vector space V over a field k is dualizable if and only if it is finite-dimensional. If V is finite-dimensional, then $V^* := \mathrm{Hom}_k(V, k)$ is a dual for V; if $e_1, \ldots, e_n$ is a basis for V and $e_1^*, \ldots, e_n^*$ is the corresponding dual basis for V^* (so that $e_i^*(e_j) = \delta_{ij}$), then the maps

$$\begin{array}{ccc} k & \xrightarrow{\quad\eta\quad} & V \otimes V^* \\ 1 & \longmapsto & \sum_{i=1}^n e_i \otimes e_i^* \end{array} \qquad \begin{array}{ccc} V^* \otimes V & \xrightarrow{\quad\varepsilon\quad} & k \\ \phi \otimes v & \longmapsto & \phi(v) \end{array}$$

witness (V, V^*) as a dual pair. Conversely, if V is a dualizable vector space, then $\eta(1) =: \sum_{i=1}^n v_i \otimes \phi_i$ is some finite sum of simple tensors, and one of the triangle identities implies that $v_1, \ldots, v_n$ generate V, so V is finite-dimensional.

Dualizability allows us to extract useful information about an object; for example, if a vector space is dualizable (i.e. finite-dimensional), we can use the structure of a dual pair to determine its dimension:

Example 2. If V is a finite-dimensional vector space over k and V^* is its dual, with η and ε as in Example 1, then the composite

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\varepsilon} k$$

is the element of $\mathrm{Hom}_k(k, k)$ which is multiplication by $\dim V$.

While there may be many choices of V^*, η, and ε satisfying Definition 1, this map $k \to k$ is independent of such choices, since it is determined solely by the dimension of V.

A similar story plays out in topological settings as well, though we need to pass to a category of spectra since the category of spaces has no nontrivial dual pairs. Having done so, we produce an endomorphism of the monoidal unit analogous to Example 2 and find that it recovers a familiar topological invariant:

Example 3 ([2])**.** If X is a compact CW complex, then $\Sigma^\infty_+ X$ is a dualizable spectrum with dual DX (the Spanier-Whitehead dual), and the map

$$S \xrightarrow{\eta} \Sigma^\infty_+ X \wedge DX \cong DX \wedge \Sigma^\infty_+ X \xrightarrow{\varepsilon} S$$

is the element of $[S, S] = \pi_0(S) \cong \mathbb{Z}$ which is multiplication by the Euler characteristic of X.

In either of these examples, we can insert an endomorphism f of the dualizable object in order to obtain information about f.

Example 4. Let $f : V \to V$ be an endomorphism of a finite-dimensional vector space V over k. Then the composite

$$k \xrightarrow{\eta} V \otimes V^* \xrightarrow{f \otimes 1} V \otimes V^* \xrightarrow{\cong} V^* \otimes V \xrightarrow{\varepsilon} k$$

is multiplication by the trace of f.

Example 5. If X is a compact CW complex and $f : X \to X$, then

$$S \to \Sigma^\infty_+ X \wedge DX \xrightarrow{\Sigma^\infty_+ f \wedge 1} \Sigma^\infty_+ X \wedge DX \cong DX \wedge \Sigma^\infty_+ X \to S$$

corresponds to multiplication by the Lefschetz number of f [2].

These are examples of the following general notion of trace in a symmetric monoidal category:

Definition 2 ([2])**.** If M is a dualizable object in a symmetric monoidal category, with dual M^*, then the ***trace*** of $f : M \to M$ is

$$I \xrightarrow{\eta} M \otimes M^* \xrightarrow{f \otimes 1} M \otimes M^* \xrightarrow{\cong} M^* \otimes M \xrightarrow{\varepsilon} I.$$

If f is an identity map, we call its trace the ***Euler characteristic*** of M, by analogy with Example 3. The trace of f is independent of choices of M^*, η, and ε.

2.2 Bicategorical Duality and Trace

The machinery of symmetric monoidal traces does not apply to modules over noncommutative rings, since Mod_R does not have a monoidal tensor product if R is not commutative. There is, however, a sensible notion of trace for modules over noncommutative rings, the Hattori-Stallings trace [5, 16], but it takes values in a quotient of the ring rather than the ring itself. The appropriate category-theoretic setting for describing this trace is a bicategory.

For a definition of a bicategory see [7]. We denote the bicategorical composition in a bicategory $\mathscr{B}$ by $\odot$, and if R is a 0-cell in $\mathscr{B}$, we denote the identity 1-cell for R by U_R. We write $M : R \twoheadrightarrow S$ to indicate that M is a 1-cell from R to S, i.e. $M \in \mathscr{B}(R, S)$. The most useful bicategory to keep in mind is $\mathbf{Mod}/_{\mathbf{Ring}}$, the Morita bicategory of rings, bimodules, and bimodule homomorphisms. In this bicategory, $U_R = {}_RR_R$ and $\odot$ is the tensor product. We will sometimes also work in $\mathbf{Mod}/_{\mathbf{Alg}_k}$, the bicategory of algebras over a commutative ring k, bimodules, and bimodule homomorphisms.

The following definition first appeared in [10].

Definition 3. Let M be a 1-cell in a bicategory $\mathscr{B}(R, S)$. We say M is ***right dualizable*** if there is a 1-cell M^* together with 2-cells

$$\eta : U_R \to M \odot M^* \qquad \varepsilon : M^* \odot M \to U_S$$

such that the triangle identities hold. We say that M^* is ***right dual*** to M, that (M, M^*) is a ***dual pair***, that M^* is ***left dualizable***, and that M is its ***left dual***.

Unlike dual pairs in symmetric monoidal categories, bicategorical dual pairs have a sidedness: right dualizability is not the same as left dualizability.

As in the symmetric monoidal setting, duals are unique up to isomorphism. For example, suppose that (M, N) is a dual pair with unit and counit η and ε and that (M, N') is a dual pair with unit and counit η' and ε'. Then there are maps

$$N \xrightarrow{\mathrm{id}\odot\eta'} N \odot M \odot N' \xrightarrow{\varepsilon\odot\mathrm{id}} N' \qquad\qquad N' \xrightarrow{\mathrm{id}\odot\eta} N' \odot M \odot N \xrightarrow{\varepsilon'\odot\mathrm{id}} N$$

which are mutually inverse by the triangle identities for both dual pairs.

Example 6. An (R, S)-bimodule is right dualizable in $\mathbf{Mod}/_{\mathbf{Ring}}$ if and only if it is finitely generated and projective as a right S-module. The argument is similar to Example 1 and uses the fact that a right S-module M is finitely generated and projective if and only if there are $m_1, \ldots, m_n \in M$ and $m_1^*, \ldots, m_n^* \in \mathrm{Hom}_S(M, S)$ such that $\sum_{i=1}^n m_i m_i^*(m) = m$ for all $m \in M$.

Example 7. In the bicategory of categories, functors, and natural transformations, the dual pairs are the adjoint pairs of functors, but in an unfortunate clash of nomenclature, *left* adjoints are *right* duals as a consequence of the order in which we write composition.

Proposition 1 ([10]). *If (M, M^*) and (N, N^*) are dual pairs, then so is $(M\odot N, N^*\odot M^*)$.*

Proof. Let $M \in \mathscr{B}(R,S)$ and $N \in \mathscr{B}(S,T)$. If the dual pairs (M, M^*) and (N, N^*) have units η and η' and counits ε and ε', respectively, then

$$U_R \xrightarrow{\eta} M \odot M^* \xrightarrow{1 \odot \eta' \odot 1} M \odot N \odot N^* \odot M^* \qquad N^* \odot M^* \odot M \odot N \xrightarrow{1 \odot \varepsilon \odot 1} N^* \odot N \xrightarrow{\varepsilon'} U_T$$

are a unit and a counit for $(M \odot N, N^* \odot M^*)$. □

If we attempt to write down a composite like the symmetric monoidal trace of Definition 2, we quickly encounter a problem: there is no symmetry isomorphism to get us from $M \odot M^*$ to $M^* \odot M$, and in fact these aren't even objects in the same category. One possible remedy is to apply a functor to push $M \odot M^*$ and $M^* \odot M$ into a third category, where we can ask for an isomorphism between their images. This is the notion of a *bicategorical shadow*, which was introduced in [11]:

Definition 4. A ***shadow*** for a bicategory $\mathscr{B}$ consists of functors

$$\langle\!\langle - \rangle\!\rangle : \mathscr{B}(R,R) \to \mathbf{T}$$

for each 0-cell R of $\mathscr{B}$ and some fixed category $\mathbf{T}$, equipped with a natural isomorphism

$$\theta_{M,N} : \langle\!\langle M \odot N \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle N \odot M \rangle\!\rangle$$

for each $M \in \mathscr{B}(R,S)$ and $N \in \mathscr{B}(S,R)$, such that the following diagrams commute whenever they make sense:

$$\begin{tikzcd}
\langle\!\langle (M \odot N) \odot P \rangle\!\rangle \arrow[r, "\theta", "\cong"'] \arrow[d, "\langle\!\langle a \rangle\!\rangle"', "\cong"] & \langle\!\langle P \odot (M \odot N) \rangle\!\rangle \arrow[r, "\langle\!\langle a \rangle\!\rangle", "\cong"'] & \langle\!\langle (P \odot M) \odot N \rangle\!\rangle \\
\langle\!\langle M \odot (N \odot P) \rangle\!\rangle \arrow[r, "\cong", "\theta"'] & \langle\!\langle (N \odot P) \odot M \rangle\!\rangle \arrow[r, "\cong", "\langle\!\langle a \rangle\!\rangle"'] & \langle\!\langle N \odot (P \odot M) \rangle\!\rangle \arrow[u, "\cong", "\theta"']
\end{tikzcd}$$

$$\begin{tikzcd}
\langle\!\langle M \odot U_R \rangle\!\rangle \arrow[r, "\theta", "\cong"'] \arrow[dr, "\cong", "\langle\!\langle r \rangle\!\rangle"'] & \langle\!\langle U_R \odot M \rangle\!\rangle \arrow[r, "\theta", "\cong"'] \arrow[d, "\langle\!\langle l \rangle\!\rangle"', "\cong"] & \langle\!\langle M \odot U_R \rangle\!\rangle \arrow[dl, "\cong"', "\langle\!\langle r \rangle\!\rangle"] \\
& \langle\!\langle M \rangle\!\rangle &
\end{tikzcd}$$

Here and elsewhere, the phrase "whenever they make sense" means that the 1-cells have source and target 0-cells which make all tensor products and shadows (and, later, hom objects and coshadows) valid. For example, the hexagon in Definition 4 "makes sense" if the targets of M, N, and P equal the sources of N, P, and M, respectively. The isomorphisms $\theta_{M,N}$ and $\theta_{N,M}$ are mutually inverse [14, Proposition 4.3], so we usually drop the subscripts and simply write θ.

Example 8. The zeroth Hochschild homology $\mathrm{HH}_0(R,M)$ is the quotient (of abelian groups) of M by the subgroup generated by elements of the form $mr - rm$. (This is isomorphic to $M \otimes_{R \otimes R^{\mathrm{op}}} R$.) This defines a shadow on $\mathbf{Mod}/_{\mathbf{Ring}}$ with target $\mathbf{Ab}$ (or a shadow on $\mathbf{Mod}/_{\mathbf{Alg}_k}$ with target $\mathbf{Vect}_k$). This boils down to the fact that $\mathrm{HH}_0(R, M \otimes_S N) \cong \mathrm{HH}_0(S, N \otimes_R M)$ for bimodules ${}_R M_S$ and ${}_S N_R$, since the relation $rm \otimes n \sim m \otimes nr$ imposed by Hochschild homology mirrors the relation $ms \otimes n \sim m \otimes sn$ imposed by the passage from $M \otimes_{\mathbb{Z}} N$ to $M \otimes_S N$.

Example 9. There is a bicategory $\mathbf{Ch}/_{\mathbf{Ring}}$ whose 0-cells are rings, 1-cells are non-negatively graded chain complexes of bimodules, and 2-cells are chain maps. This bicategory has a shadow, which we also call HH_0, given by $\mathrm{HH}_0(R, C) = C \otimes_{R\otimes R^{\mathrm{op}}} R[0]$; that is, $\mathrm{HH}_0(R, C)_n \cong \mathrm{HH}_0(R, C_n)$. The symmetry isomorphism $\theta : \mathrm{HH}_0(R, C \otimes_S D) \cong \mathrm{HH}_0(S, D \otimes_R C)$ has a sign: $\theta(c \otimes d) = (-1)^{|c||d|} d \otimes c$.

The following generalizes the symmetric monoidal Euler characteristic of Definition 2.

Definition 5 ([11])**.** The ***Euler characteristic*** $\chi(M)$ of a right dualizable 1-cell $M \in \mathscr{B}(R, S)$ is the map

$$\langle\!\langle U_R \rangle\!\rangle \xrightarrow{\langle\!\langle \eta \rangle\!\rangle} \langle\!\langle M \odot M^* \rangle\!\rangle \xrightarrow[\cong]{\theta} \langle\!\langle M^* \odot M \rangle\!\rangle \xrightarrow{\varepsilon} \langle\!\langle U_S \rangle\!\rangle.$$

Example 10. A k-linear representation of a group G can be viewed as module over the group algebra $k[G]$, which is a 1-cell $k[G] \nrightarrow k$ in $\mathbf{Mod}/_{\mathbf{Alg}_k}$. When the underlying vector space of V is finite-dimensional, V is right dualizable and $\chi(V)$ is a map $\langle\!\langle k[G] \rangle\!\rangle \to \langle\!\langle k \rangle\!\rangle$. If we use HH_0 as the shadow, the quotient $k[G] \to \mathrm{HH}_0(k[G], k[G])$ identifies pairs of group elements gh and hg; equivalently, it identifies conjugate elements, and thus $\mathrm{HH}_0(k[G], k[G])$ is $k[\mathrm{cl}(G)]$, the free k-vector space on the conjugacy classes of G. Since k is commutative, $\mathrm{HH}_0(k, k) \cong k$, and thus $\chi(V)$ is a map $k[\mathrm{cl}(G)] \to k$, which amounts to a class function $G \to k$. By describing a right dual, unit, and counit similar to those in Example 1, we compute that $\chi(V)$ is the character of V; that is, $\chi(V)(g) = \mathrm{tr}(V \xrightarrow{g\cdot} V)$.

Defintiion 5 generalizes in two ways. First, we can introduce an endomorphism of the dualizable object, similar to the transition from Examples 2 and 3 to Examples 4 and 5, respectively. Second, we can twist the endomorphism by 1-cells Q and P, making the trace a map $\langle\!\langle Q \rangle\!\rangle \to \langle\!\langle P \rangle\!\rangle$ rather than simply a map between shadows of unit 1-cells.

Definition 6 ([11, Definition 4.5.1])**.** Let $f : Q \odot M \to M \odot P$ be a 2-cell where M is right dualizable. The ***twisted trace*** of f is the composite

$$\begin{aligned} \langle\!\langle Q \rangle\!\rangle \cong \langle\!\langle Q \odot U_R \rangle\!\rangle &\xrightarrow{\langle\!\langle 1\odot\eta \rangle\!\rangle} \langle\!\langle Q \odot M \odot M^* \rangle\!\rangle \xrightarrow{\langle\!\langle f\odot 1 \rangle\!\rangle} \langle\!\langle M \odot P \odot M^* \rangle\!\rangle \\ &\xrightarrow[\cong]{\theta} \langle\!\langle M^* \odot M \odot P \rangle\!\rangle \xrightarrow{\langle\!\langle \varepsilon\odot 1 \rangle\!\rangle} \langle\!\langle U_S \odot P \rangle\!\rangle \cong \langle\!\langle P \rangle\!\rangle. \end{aligned}$$

The Euler characteristic for a right dualizable 1-cell $M \in \mathscr{B}(R, S)$ corresponds to the twisted trace for the canonical isomorphism 2-cell $U_R \odot M \xrightarrow{\cong} M \odot U_S$; this is the sense in which we think of the Euler characteristic as a "trace of identity map."

Suppose we have right dualizable 1-cells $M \in \mathscr{B}(R, S)$ and $N \in \mathscr{B}(S, T)$ and Q, P, L which twist endomorphisms of M and N:

$$f : Q \odot M \to M \odot P \qquad g : P \odot N \to N \odot L.$$

The twisted traces of f and g are maps $\langle\!\langle Q \rangle\!\rangle \to \langle\!\langle P \rangle\!\rangle$ and $\langle\!\langle P \rangle\!\rangle \to \langle\!\langle L \rangle\!\rangle$. The following theorem says that we can obtain the composite $\langle\!\langle Q \rangle\!\rangle \to \langle\!\langle P \rangle\!\rangle \to \langle\!\langle L \rangle\!\rangle$ as a single trace with respect to $M \odot N$, which is dualizable by Proposition 1.

Theorem 2 ([11]). *Let M, N, Q, P, L, f, g be as above. Then the trace of*

$$Q \odot M \odot N \xrightarrow{f \odot \mathrm{id}_N} M \odot P \odot N \xrightarrow{\mathrm{id}_M \odot g} M \odot N \odot L$$

is

$$\langle\!\langle Q \rangle\!\rangle \xrightarrow{\mathrm{tr}(f)} \langle\!\langle P \rangle\!\rangle \xrightarrow{\mathrm{tr}(g)} \langle\!\langle L \rangle\!\rangle.$$

When applied to the isomorphisms

$$U_R \odot M \odot N \xrightarrow{\cong} M \odot U_S \odot N \xrightarrow{\cong} M \odot N \odot U_T$$

this theorem gives the following.

Corollary 1. *If $M \in \mathscr{B}(R, S)$ and $N \in \mathscr{B}(S, T)$ are right dualizable then*

$$\chi(M \odot N) = \chi(N) \circ \chi(M).$$

Example 11. The preceding corollary can be used to recover the formulas for characters of restricted and induced representations. Given a subgroup $H \leq G$ and a G-representation V, the restricted H-representation is $\mathrm{Res}_H^G(V) := {}_\varphi k[G] \otimes_{k[H]} V$, where $\varphi : k[H] \to k[G]$ is the ring homomorphism induced by the inclusion of H into G. Now ${}_\varphi k[G]$ is right dualizable, since $({}_\varphi S, S_\varphi)$ is a dual pair for any ring homomorphism $\varphi : R \to S$ [10, Example 16.4.2]. By Corollary 1, $\chi(\mathrm{Res}_H^G(H))$ is

$$\langle\!\langle k[H] \rangle\!\rangle \xrightarrow{\chi({}_\varphi k[G])} \langle\!\langle k[G] \rangle\!\rangle \xrightarrow{\chi(V)} \langle\!\langle k \rangle\!\rangle.$$

The first map simply takes $[h] \in \mathrm{HH}_0(k[H])$ to $[h] \in \mathrm{HH}_0(k[G])$, which is well-defined since elements which are conjugate in H are also conjugate in G.

Now, given an H-representation W, the induced G-representation is $\mathrm{Ind}_H^G(W) := k[G]_\varphi \otimes_{k[H]} V$. If $[G : H] < \infty$, then $k[G]_\varphi$ is right dualizable with right dual ${}_\varphi k[G]$ (this is *not* true for a general ring homomorphism φ). A coevaluation and evaluation for the dual pair $(k[G]_\varphi, {}_\varphi k[G])$ are given by

$$\begin{aligned} k[G] &\xrightarrow{\eta} k[G]_\varphi \otimes_{k[H]} {}_\varphi k[G] \\ g &\mapsto \sum_{g_i H \in G/H} g g_i \otimes g_i^{-1} \end{aligned} \qquad \begin{aligned} {}_\varphi k[G] \otimes_{k[G]} k[G]_\varphi &\xrightarrow{\varepsilon} k[H] \\ g \otimes g' &\mapsto \begin{cases} gg', & gg' \in H \\ 0, & gg' \notin H \end{cases} \end{aligned}$$

where the g_i are a choice of coset representatives for G/H. By Corollary 1, $\chi(\mathrm{Ind}_H^G(V))$ is

$$\langle\!\langle k[G] \rangle\!\rangle \xrightarrow{\chi(k[G]_\varphi)} \langle\!\langle k[H] \rangle\!\rangle \xrightarrow{\chi(V)} \langle\!\langle k \rangle\!\rangle.$$

The first map takes $[g]$ to $\sum_{g_i H \in G/H,\, g_i^{-1} g g_i \in H} [g_i^{-1} g g_i] = \frac{1}{|H|} \sum_{s \in G,\, s^{-1} g s \in H} [s^{-1} g s]$; then after applying $\chi(V)$ we get the usual character induction formula

$$\chi(\mathrm{Ind}_H^G(V))(g) = \frac{1}{|H|} \sum_{s \in G,\, s^{-1} g s \in H} \chi(V)(s^{-1} g s).$$

Definition 7. Let $f : P \odot M \to M \odot Q$ be a 2-cell where M is right dualizable with right dual M^*. The ***mate*** of F is the map $f^* : M^* \odot P \to Q \odot M^*$ given by

$$M^* \odot P \xrightarrow{\mathrm{id}\odot\eta} M^* \odot P \odot M \odot M^* \xrightarrow{\mathrm{id}\odot f\odot\mathrm{id}} M^* \odot M \odot Q \odot M^* \xrightarrow{\varepsilon\odot\mathrm{id}} Q \odot M^*.$$

There is a construction analogous to Definition 6 for a twisted endomorphism $M^* \odot Q \to P \odot M^*$ of a *left* dualizable 1-cell M^*. Using this, we obtain the following.

Proposition 2. *Let $f : Q \odot M \to M \odot P$ be a 2-cell where M is right dualizable. Then* $\mathrm{tr}(f) = \mathrm{tr}(f^*)$.

2.3 Closed Bicategories

The bicategory $\mathbf{Mod}/_{\mathbf{Ring}}$ has additional structure that we have not yet made use of, namely the hom-functors. Describing this structure is essential for generalizing notions of *cotrace* that appear in the literature, just as the symmetric monoidal and bicategorical trace generalize many familiar examples.

Definition 8 ([10])**.** A ***(right and left) closed bicategory*** is a bicategory $\mathscr{B}$ equipped with right and left internal hom functors

$$- \triangleleft - : \mathscr{B}(R,T) \times \mathscr{B}(R,S)^{\mathrm{op}} \to \mathscr{B}(S,T)$$

and

$$- \triangleright - : \mathscr{B}(S,T)^{\mathrm{op}} \times \mathscr{B}(R,T) \to \mathscr{B}(R,S)$$

for all triples of 0-cells R, S, T and natural isomorphisms

$$\mathscr{B}(S,T)(N, P \triangleleft M) \cong \mathscr{B}(R,T)(M \odot N, P) \cong \mathscr{B}(R,S)(M, N \triangleright P)$$

for all triples of 1-cells $M : R \nrightarrow S$, $N : S \nrightarrow T$, and $P : R \nrightarrow T$.

Our most frequently used example of a bicategory, $\mathbf{Mod}/_{\mathbf{Ring}}$ (or $\mathbf{Mod}/_{\mathbf{Alg}_k}$), is a closed bicategory. For bimodules ${}_RM_S$ and ${}_RP_T$, the left internal hom object is the (S,T)-bimodule $P \triangleleft M := \mathrm{Hom}_R(M,P)$. For bimodules ${}_SN_T$ and ${}_RP_T$, the right internal hom object is the (R,S)-bimodule $N \triangleright P := \mathrm{Hom}_T(N,P)$. We remember the hom functors for bimodules by noting that triangle ($\triangleleft$ or $\triangleright$) always points from source to target, and the direction it points indicates on which side the maps are linear (e.g. the *right*-pointing triangle $\triangleright$ indicates that $N \triangleright P$ is the set of *right* T-linear maps from N to P).

There are ***evaluation*** maps

$$(N \triangleright P) \odot N \xrightarrow{\mathrm{ev}} P \qquad \text{and} \qquad M \odot (P \triangleleft M) \xrightarrow{\mathrm{ev}} P$$

(the transposes of $\mathrm{id}_{N \triangleright P}$ and $\mathrm{id}_{P \triangleleft M}$, respectively), which are natural in P and extranatural in M and N (respectively). For bimodules, the first of these is the evaluation map $\mathrm{Hom}_T(N,P) \otimes_S N \to P$, $\varphi \otimes n \mapsto \varphi(n)$. Similarly, there are ***coevaluation*** maps

$$M \xrightarrow{\mathrm{coev}} N \triangleright (M \odot N) \qquad \text{and} \qquad N \xrightarrow{\mathrm{coev}} (M \odot N) \triangleleft M$$

(the transposes of $\mathrm{id}_{M\odot N}$), which are natural in M and N (respectively) and extranatural in N and M (respectively). There is a natural isomorphism

$$(M \odot N) \rhd P \xrightarrow[\cong]{t} M \rhd (N \rhd P)$$

(which we call t for "transpose" or "tensor-hom adjunction") given by the transpose of the transpose of

$$((M \odot N) \rhd P) \odot M \odot N \xrightarrow{\mathrm{ev}} P$$

and whose inverse is the transpose of

$$(M \rhd (N \rhd P)) \odot M \odot N \xrightarrow{\mathrm{ev}\odot 1} (N \rhd P) \odot N \xrightarrow{\mathrm{ev}} P.$$

There is a similar natural isomorphism for $\lhd$, as well as a natural isomorphism

$$(N \rhd P) \lhd M \xrightarrow[\cong]{a} N \rhd (P \lhd M),$$

which we call a for "associator."

One of the axioms of a bicategory is a pentagon ensuring that any two ways of reparenthesizing four composed 1-cells through associators are equal. In a closed bicategory, there are several more associativity pentagons—not axioms but rather provably commuting diagrams—since there are now three ways to put two objects together ($\odot$, $\lhd$, and $\rhd$). We describe some of them below since we will need them later; note that they come in pairs since there are both left and right internal hom functors.

Lemma 1. *In a closed bicategory, the following diagrams commute for any 1-cells W, X, Y, Z for which the diagrams make sense:*

$$\begin{tikzcd}
 & & ((W \odot X) \odot Y) \rhd Z \arrow[dll, "(a^{-1})^*"', "\cong"] \arrow[drr, "t", "\cong"'] & & \\
(W \odot (X \odot Y)) \rhd Z \arrow[dr, "t"', "\cong"] & & & & (W \odot X) \rhd (Y \rhd Z) \arrow[dl, "t", "\cong"'] \\
 & W \rhd ((X \odot Y) \rhd Z) \arrow[rr, "\cong", "t_*"'] & & W \rhd (X \rhd (Y \rhd Z)) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & ((W \lhd X) \lhd Y) \lhd Z \arrow[dll, "t_*^{-1}"', "\cong"] \arrow[drr, "t^{-1}", "\cong"'] & & \\
(W \lhd (X \odot Y)) \lhd Z \arrow[dr, "t^{-1}"', "\cong"] & & & & (W \lhd X) \lhd (Y \odot Z) \arrow[dl, "t^{-1}", "\cong"'] \\
 & W \lhd ((X \odot Y) \odot Z) \arrow[rr, "\cong", "(a^{-1})^*"'] & & W \lhd (X \odot (Y \odot Z)) &
\end{tikzcd}$$

The reader is warned that the two diagrams above will appear as squares rather than pentagons when we make use of them later on, since we typically suppress the associator for $\odot$. To verify that diagrams like these commute, the easiest approach is usually to take transposes until no hom-objects remain in the terminal object of the diagram. For example, each side of the first pentagon above is thrice transposed by tensoring with the identity map on $W \odot X \odot Y$ and then composing with

$$(W \rhd (X \rhd (Y \rhd Z))) \odot W \odot X \odot Y \xrightarrow{\mathrm{ev}\odot 1^2} (X \rhd (Y \rhd Z)) \odot X \odot Y \xrightarrow{\mathrm{ev}\odot 1} (Y \rhd Z) \odot Y \xrightarrow{\mathrm{ev}} Z.$$

Making use of naturality of ev and the definitions of a and t in terms of their transposes, one simplifies both sides of the pentagon (after transposing, that is) to the same map.

Lemma 2. *In a closed bicategory, the following diagrams commute for any 1-cells W, X, Y, Z for which the diagrams make sense:*

$$\begin{tikzcd}
& ((W \odot X) \rhd Y) \lhd Z \arrow[dl, "t_*"', "\cong"] \arrow[dr, "a", "\cong"'] & \\
(W \rhd (X \rhd Y)) \lhd Z \arrow[d, "a"', "\cong"] & & (W \odot X) \rhd (Y \lhd Z) \arrow[d, "t", "\cong"'] \\
W \rhd ((X \rhd Y) \lhd Z) \arrow[rr, "\cong", "a_*"'] & & W \rhd (X \rhd (Y \lhd Z))
\end{tikzcd}$$

$$\begin{tikzcd}
& ((W \rhd X) \lhd Y) \lhd Z) \arrow[dl, "a_*"', "\cong"] \arrow[dr, "t^{-1}", "\cong"'] & \\
(W \rhd (X \lhd Y)) \lhd Z) \arrow[d, "a"', "\cong"] & & (W \rhd X) \lhd (Y \odot Z) \arrow[d, "a", "\cong"'] \\
W \rhd ((X \lhd Y) \lhd Z) \arrow[rr, "\cong", "t_*^{-1}"'] & & W \rhd (X \lhd (Y \odot Z))
\end{tikzcd}$$

Just as there are natural isomorphisms $(M \rhd N) \lhd P \cong M \rhd (N \lhd P)$ analogous to the associators $(M \odot N) \odot P \cong M \odot (N \odot P)$, there are natural isomorphisms $U_S \rhd M \cong M \cong M \lhd U_R$ analogous to the unitors $U_R \odot M \cong M \cong M \odot U_S$.

Lemma 3. *Given a 1-cell $M : R \twoheadrightarrow S$, the transpose $\bar{l} : M \to M \lhd U_R$ of $l : U_R \odot M \xrightarrow{\cong} M$ is an isomorphism, as is the transpose $\overline{r} : M \to U_S \rhd M$ of $r : M \odot U_S \xrightarrow{\cong} M$. Moreover, these isomorphisms are natural in M.*

Proof. The inverse of $\bar{l}$ is

$$M \lhd U_R \xrightarrow[\cong]{l^{-1}} U_R \odot (M \lhd U_R) \xrightarrow{\mathrm{ev}} M.$$

The inverse of $\bar{r}$ is similar, and naturality is straightforward to check. ☐

These maps $\bar{l}$ and $\bar{r}$ satisfy properties analogous to the bicategory axiom relating the associator and unitors.

Lemma 4. *The following diagrams commute whenever they make sense:*

1.
$$\begin{tikzcd} X \rhd Y \arrow[r, "r^*", "\cong"'] \arrow[dr, "\bar{r}_*"', "\cong"] & (X \odot U_S) \rhd Y \arrow[d, "t", "\cong"'] \\ & X \rhd (U_S \rhd Y) \end{tikzcd} \qquad \begin{tikzcd} Y \lhd X \arrow[r, "l^*", "\cong"'] \arrow[dr, "\bar{l}_*"', "\cong"] & Y \lhd (U_R \odot X) \arrow[d, "t", "\cong"'] \\ & (Y \lhd U_R) \lhd X \end{tikzcd}$$

2.
$$\begin{tikzcd} X \rhd Y \arrow[r, "l^*", "\cong"'] \arrow[dr, "\bar{r}"', "\cong"] & (U_R \odot X) \rhd Y \arrow[d, "t", "\cong"'] \\ & U_R \rhd (X \rhd Y) \end{tikzcd} \qquad \begin{tikzcd} Y \lhd X \arrow[r, "r^*", "\cong"'] \arrow[dr, "\bar{l}"', "\cong"] & Y \lhd (X \odot U_S) \arrow[d, "t", "\cong"'] \\ & (Y \lhd X) \lhd U_S \end{tikzcd}$$

3.
$$\begin{tikzcd} X \rhd Y \arrow[r, "\bar{l}_*", "\cong"'] \arrow[dr, "\bar{l}"', "\cong"] & X \rhd (Y \lhd U) \arrow[d, "a^{-1}", "\cong"'] \\ & (X \rhd Y) \lhd U \end{tikzcd} \qquad \begin{tikzcd} Y \lhd X \arrow[r, "\bar{r}_*", "\cong"'] \arrow[dr, "\bar{r}"', "\cong"] & (U \rhd Y) \lhd X \arrow[d, "a", "\cong"'] \\ & U \rhd (Y \lhd X) \end{tikzcd}$$

Lemma 5. *For any 1-cell $M : R \nrightarrow S$, the maps $\bar{r}, \bar{r}_* : U_S \rhd M \to U_S \rhd (U_S \rhd M)$ are equal, as are the maps $\bar{l}, \bar{l}_* : M \lhd U_R \to (M \lhd U_R) \lhd U_R$.*

2.4 Duality in Closed Bicategories

When internal hom functors are present, they are intimately related with duality. The dual of a finite-dimensional k-vector space V, for example, is the hom space $\mathrm{Hom}_k(V, k)$; in fact, in a closed symmetric monoidal category or closed bicategory, the dual of M (when it exists) always takes the form of a hom object from M into a unit object.

There are maps, natural in M, N, and P,

$$N \odot (M \rhd P) \xrightarrow{\mu} M \rhd (N \odot P) \qquad \text{and} \qquad (N \lhd M) \odot P \xrightarrow{\nu} (N \odot P) \lhd M,$$

defined as the transposes of

$$N \odot (M \rhd P) \odot M \xrightarrow{1 \odot \mathrm{ev}} N \odot P \qquad \text{and} \qquad M \odot (N \lhd M) \odot P \xrightarrow{\mathrm{ev} \odot 1} N \odot P.$$

Proposition 3 ([10])**.** *The map $\mu : N \odot (M \triangleright P) \to M \triangleright (N \odot P)$ is an isomorphism if either M or N is right dualizable. Similarly, $\nu : (N \triangleleft M) \odot P \to (N \odot P) \triangleleft M$ is an isomorphism if either M or P is left dualizable.*

Proof. If M is right dualizable, then

$$\begin{aligned} M \triangleright (N \odot P) &\xrightarrow{1^2 \odot \eta} (M \triangleright (N \odot P)) \odot M \odot M^* \xrightarrow{\mathrm{ev} \odot 1} N \odot P \odot M^* \\ &\xrightarrow{1 \odot \mathrm{coev}} N \odot (M \triangleright (P \odot M^* \odot M)) \xrightarrow{1 \odot (1 \odot \varepsilon)_*} N \odot (M \triangleright P) \end{aligned}$$

is inverse to μ. If N is right dualizable, then

$$\begin{aligned} M \triangleright (N \odot P) &\xrightarrow{\eta \odot 1^2} N \odot N^* \odot (M \triangleright (N \odot P)) \xrightarrow{1 \odot \mu} N \odot (M \triangleright (N^* \odot N \odot P)) \\ &\xrightarrow{1 \odot (\varepsilon \odot 1)_*} N \odot (M \triangleright P) \end{aligned}$$

is inverse to μ. Inverses to ν are constructed similarly when M or P is left dualizable. □

Proposition 4. *If $M \in \mathscr{B}(R, S)$ is right dualizable, then its right dual is $M \triangleright U_S$.*

Proof. If M is right dualizable, then

$$U_R \xrightarrow{\mathrm{coev}} M \triangleright (U_R \odot M) \cong M \triangleright (M \odot U_S) \xrightarrow{\mu^{-1}} M \odot (M \triangleright U_S)$$

and $\mathrm{ev} : (M \triangleright U_S) \odot M \to U_S$ exhibit $(M, (M \triangleright U_S))$ as a dual pair. □

Similarly, if M is left dualizable, then its left dual is $U_R \triangleleft M$.

A map of the appropriate form for taking traces has a mate (Definition 7), and dually a map of the appropriate form for taking cotraces (Definition 11) has a mate:

Definition 9. Let (M, M^*) be a dual pair in a closed bicategory. A map $f : M \triangleright Q \to P \triangleleft M$ has a ***mate*** $f^* : Q \triangleleft M^* \to M^* \triangleright P$ given by

$$\begin{aligned} Q \triangleleft M^* &\xrightarrow[\cong]{\bar{r}_*} (U_S \triangleright Q) \triangleleft M^* \xrightarrow{(\varepsilon^*)_*} ((M^* \odot M) \triangleright Q) \triangleleft M^* \\ &\xrightarrow[\cong]{t_*} (M^* \triangleright (M \triangleright Q)) \triangleleft M^* \xrightarrow{f_{**}} (M^* \triangleright (P \triangleleft M)) \triangleleft M^* \\ &\xrightarrow[\cong]{a} M^* \triangleright ((P \triangleleft M) \triangleleft M^*) \xrightarrow[\cong]{t_*} M^* \triangleright (P \triangleleft (M \odot M^*)) \\ &\xrightarrow{(\eta^*)_*} M^* \triangleright (P \triangleleft U_R) \xrightarrow[\cong]{\bar{l}_*^{-1}} M^* \triangleright P. \end{aligned}$$

Similarly, a map $g : Q \triangleleft M^* \to M^* \triangleright P$ has a mate $g^* : M \triangleright Q \to P \triangleleft M$. Moreover, $f^{**} = f$ and $g^{**} = g$ for any $f : M \triangleright Q \to P \triangleleft M$ and $g : Q \triangleleft M^* \to M^* \triangleright P$. While this definition of f^* resembles Definition 9, there is an alternate description of f^* which is often easier to work with by virtue of involving fewer hom-objects; it is the transpose of the following:

$$(Q \triangleleft M^*) \odot M^* \cong M \odot Q \odot M^* \cong M \odot (M \triangleright Q) \xrightarrow{1 \odot f} M \odot (P \triangleleft M) \xrightarrow{\mathrm{ev}} P.$$

Chapter 3 Coshadows and Cotraces

There are certain constructions appearing in the literature under the name of *cotrace*, which resemble traces in some ways but differ in others. In this section, we develop a theory of bicategorical cotraces which generalizes these examples and draws them into the framework of bicategorical duality and trace. In this section, we present the motivating examples, define a bicategorical coshadow and cotrace, and establish key properties of coshadows and cotraces (most of them analogous to well-known properties of shadows and traces). One might also ask if there are symmetric monoidal cotraces; there are, but we show that in a symmetric monoidal setting, the notion of cotrace coincides with that of trace.

3.1 Motivation, Definitions, and Examples

The prototypical example of a shadow is Hochschild homology $\mathrm{HH}_0(R, M) \cong M \otimes_{R\otimes R^{\mathrm{op}}} R$, which has the property that $\mathrm{HH}_0(R, M \otimes_S N) \cong \mathrm{HH}_0(S, N \otimes_R M)$ for bimodules ${}_RM_S, {}_SN_R$. Hochschild cohomology $\mathrm{HH}^0(R, M) \cong \mathrm{Hom}_{R-R}(R, M)$ does not have this property, but it does have the property that $\mathrm{HH}^0(R, \mathrm{Hom}_S(M, N)) \cong \mathrm{Hom}_{R-S}(M, N) \cong \mathrm{HH}^0(S, \mathrm{Hom}_R(M, N))$ for bimodules ${}_RM_S, {}_RN_S$. This suggests that the appropriate setting for studying cotraces is a closed bicategory and that the analogue of a shadow functor should be the following.

Definition 10. Let $\mathscr{B}$ be a closed bicategory and $\mathbf{T}$ a category. A ***coshadow*** for $\mathscr{B}$ taking values in $\mathbf{T}$ is a collection of functors

$$\langle\!\langle - \langle\!\langle : \mathscr{B}(R, R) \to \mathbf{T}$$

for each object R of $\mathscr{B}$, equipped with a natural isomorphism

$$\theta : \langle\!\langle M \rhd N \langle\!\langle \xrightarrow{\cong} \langle\!\langle N \lhd M \langle\!\langle$$

for each $M, N : R \nrightarrow S$, such that the following diagrams commute whenever they make sense:

$$\begin{tikzcd}
\langle\!\langle M \rhd (N \rhd P) \langle\!\langle \arrow[r, "\theta", "\cong"'] \arrow[d, "\cong"] & \langle\!\langle (N \rhd P) \lhd M \langle\!\langle \arrow[r, "\cong"'] & \langle\!\langle N \rhd (P \lhd M) \langle\!\langle \arrow[d, "\cong"', "\theta"] \\
\langle\!\langle (M \odot N) \rhd P \langle\!\langle \arrow[r, "\cong", "\theta"'] & \langle\!\langle P \lhd (M \odot N) \langle\!\langle \arrow[r, "\cong"'] & \langle\!\langle (P \lhd M) \lhd N \langle\!\langle
\end{tikzcd}$$

$$\begin{tikzcd}
\langle\!\langle U_R \rhd M \langle\!\langle \arrow[r, "\theta", "\cong"'] \arrow[dr, "\cong", "\langle\!\langle \bar{r}^{-1} \langle\!\langle"'] & \langle\!\langle M \lhd U_R \langle\!\langle \arrow[r, "\theta", "\cong"'] \arrow[d, "\langle\!\langle \bar{l}^{-1} \langle\!\langle"', "\cong"] & \langle\!\langle U_R \rhd M \langle\!\langle \arrow[dl, "\cong"', "\langle\!\langle \bar{r}^{-1} \langle\!\langle"] \\
& \langle\!\langle M \langle\!\langle &
\end{tikzcd}$$

Example 12. Zeroth Hochschild cohomology $\mathrm{HH}^0(R, M) \cong \{m \in M : mr = rm \,\forall r \in R\} \cong \mathrm{Hom}_{R-R}(R, M)$ defines a coshadow on $\mathbf{Mod}/_{\mathbf{Ring}}$ with target $\mathbf{Ab}$.

Example 13. The *categorical trace* of [3], which sends a 1-cell $M \in \mathscr{B}(R, R)$ to $\mathrm{Hom}_{\mathscr{B}(R,R)}(U_R, M)$, is an example of a coshadow. The key observation is that for $M, N \in \mathscr{B}(R, S)$, the required isomorphism θ comes from the tensor-hom adjunctions

$$\mathrm{Hom}_{\mathscr{B}(R,R)}(U_R, M \rhd N) \cong \mathrm{Hom}_{\mathscr{B}(R,S)}(M, N) \cong \mathrm{Hom}_{\mathscr{B}(S,S)}(U_S, N \lhd M).$$

This coshadow takes values in the category of sets, but if the bicategory is enriched in a symmetric monoidal category $\mathscr{V}$ (in the sense that categories $\mathscr{B}(R, S)$ are $\mathscr{V}$-enriched categories in a way that is compatible with the horizontal composition of $\mathscr{B}$), then the categorical trace defines a $\mathscr{V}$-valued coshadow. It is a curious fact that the categorical trace provides a simple example of a coshadow on any closed bicategory whatsoever.

Definition 11. Let $\mathscr{B}$ be a closed bicategory with a coshadow and (M, M^*) a dual pair of 1-cells of $\mathscr{B}$, with $M \in \mathscr{B}(R, S)$. The ***cotrace*** of a 2-cell $f : M \rhd Q \to P \lhd M$, denoted $\mathrm{cotr}(f)$, is the composite:

$$\begin{aligned}
&\langle\!\langle Q \langle\!\langle \xrightarrow[\cong]{\langle\!\langle \bar{r} \langle\!\langle} \langle\!\langle U_S \rhd Q \langle\!\langle \xrightarrow{\langle\!\langle \varepsilon^* \langle\!\langle} \langle\!\langle (M^* \odot M) \rhd Q \langle\!\langle \xrightarrow[\cong]{} \langle\!\langle M^* \rhd (M \rhd Q) \langle\!\langle \\
&\xrightarrow{\langle\!\langle f_* \langle\!\langle} \langle\!\langle M^* \rhd (P \lhd M) \langle\!\langle \xrightarrow[\cong]{\theta} \langle\!\langle (P \lhd M) \lhd M^* \langle\!\langle \\
&\xrightarrow[\cong]{} \langle\!\langle P \lhd (M \odot M^*) \langle\!\langle \xrightarrow{\eta^*} \langle\!\langle P \lhd U_R \langle\!\langle \xrightarrow[\cong]{\langle\!\langle \bar{r}^{-1} \langle\!\langle} \langle\!\langle P \langle\!\langle
\end{aligned}$$

When there are multiple dualizable objects in play, we will sometimes subscript cotr (or tr) with the dualizable object being used for that cotrace (or trace); that is, we might write cotr_M for the cotrace in Definition 11.

This definition mirrors that of the bicategorical trace (Definition 6); the only reason it appears to composed of more maps than the trace is that in the trace we usually suppress the associators $Q \odot (M \odot M^*) \cong (Q \odot M) \odot M^*$ and $M^* \odot (M \odot P) \cong (M^* \odot M) \odot P$, whereas for the cotrace we always explicitly write out the isomorphisms $(M^* \odot M) \rhd Q \cong M^* \rhd (M \rhd Q)$ and $(P \lhd M) \lhd M^* \cong P \lhd (M \odot M^*)$.

Lemma 6. *The cotrace of f is independent of the choices of M^*, η, and ε.*

Example 14. Let $M : R \nrightarrow S$ be a right dualizable 1-cell in $\mathbf{Mod}/_{\mathbf{Ring}}$, i.e. an (R, S)-bimodule which is finitely generated and projective as a right S-module. Using HH^0 as the coshadow, the cotrace of a map $f : \mathrm{Hom}_S(M, Q) \to \mathrm{Hom}_R(M, P)$ is the map $\mathrm{HH}^0(S, Q) \to \mathrm{HH}^0(R, P)$ taking q to $\sum_i f(qe_i^*(-))(e_i)$, where $\{e_i\}$ and $\{e_i^*\}$ are a pair of dual bases for M_S.

Example 15. Given a G-representation V, there is a map $f : V \rhd k \to k[G] \lhd V$ given by $f(\phi)(v) = \sum_{g \in G} \phi(g^{-1}v)g$. In fact, this is an isomorphism, with inverse $f^{-1} : k[G] \lhd V \to V \rhd k$ given by $f^{-1}(\varphi)(v) = e^*(\varphi(v))$. If V is finite-dimensional (i.e. right dualizable), then f has a cotrace $\mathrm{HH}^0(k) \to \mathrm{HH}^0(k[G])$; since k is commutative

$\mathrm{HH}^0(k)$ is just k, and $\mathrm{HH}^0(k[G])$ is the subset of $k[G]$ consisting of linear combinations $\sum_{g\in G} a_g g$ such that $a_g = a_{g'}$ whenever g and g' are conjugate. The cotrace is

$$\mathrm{cotr}(f)(1) = \sum_{g\in G} \chi(V)(g^{-1})g,$$

where $\chi(V)$ is as in Example 10. Note that $\mathrm{cotr}(f)$ contains precisely the same information as the character $\chi(V)$: an element of $\mathrm{HH}^0(k[G])$ amounts to a scalar for each conjugacy class of G, and the scalars picked out by this cotrace are the values of $\chi(V)$.

3.2 Basic Properties

The cotrace satisfies properties analogous to those of the bicategorical trace, which are catalogued in Section 7 of [14]. Some of the diagrams proving these properties get quite large, so to make them a bit more manageable we sometimes omit the symbol $\odot$. We adopt the convention that $\triangleleft$ and $\triangleright$ bind more loosely than composition by juxtaposition, e.g. $AB \triangleright C$ is to be understood as $(A \odot B) \triangleright C$, not $A \odot (B \triangleright C)$.

The following is analogous to Proposition 7.1 in [14]:

Proposition 5. *Let M be a right dualizable 1-cell, let $f : M \triangleright Q \to P \triangleleft M$ be a 2-cell, and let $g : Q' \to Q$ and $h : P \to P'$ be 2-cells. Then*

$$\llangle h \llangle \circ \mathrm{cotr}(f) \circ \llangle g \llangle = \mathrm{cotr}(h_* \circ f \circ g_*).$$

Proof. The composite around the outside top, right, and bottom of Figure 3.1 is $\mathrm{cotr}(h_* \circ f \circ g_*)$. Each square commutes because of functoriality of the internal hom or naturality of θ, t, $\bar{r}$, or $\bar{l}$. $\square$

$$\begin{tikzcd}
\llangle Q'\llangle \arrow[r, "\llangle\bar{r}\llangle", "\cong"'] \arrow[d, "\llangle g\llangle"'] & \llangle U_S \triangleright Q'\llangle \arrow[r, "\llangle\varepsilon^*\llangle"] \arrow[d, "\llangle g_*\llangle"'] & \llangle (M^*\odot M)\triangleright Q'\llangle \arrow[r, "\llangle t\llangle", "\cong"'] \arrow[d, "\llangle g_*\llangle"'] & \llangle M^*\triangleright(M\triangleright Q')\llangle \arrow[d, "\llangle g_{**}\llangle"'] \arrow[rrdd, "\llangle (h_*\circ f\circ g_*)_*\llangle"] & & \\
\llangle Q\llangle \arrow[r, "\llangle\bar{r}\llangle", "\cong"'] \arrow[dd, "\mathrm{cotr}(f)"'] & \llangle U_S \triangleright Q\llangle \arrow[r, "\llangle\varepsilon^*\llangle"] & \llangle (M^*\odot M)\triangleright Q\llangle \arrow[r, "\llangle t\llangle", "\cong"'] & \llangle M^*\triangleright(M\triangleright Q)\llangle \arrow[dr, "\llangle f_*\llangle"] & & \\
& & & & \llangle M^*\triangleright(P\triangleleft M)\llangle \arrow[r, "\llangle h_{**}\llangle"] \arrow[dl, "\cong"', "\llangle\theta\llangle"] & \llangle M^*\triangleright(P'\triangleleft M)\llangle \arrow[ddll, "\cong"', "\llangle\theta\llangle"] \\
\llangle P\llangle \arrow[d, "\llangle h\llangle"'] & \llangle P\triangleleft U_R\llangle \arrow[l, "\cong"', "\llangle\bar{l}^{-1}\llangle"] \arrow[d, "\llangle h_*\llangle"'] & \llangle P\triangleleft(M\odot M^*)\llangle \arrow[l, "\llangle\eta^*\llangle"] \arrow[d, "\llangle h_*\llangle"'] & \llangle (P\triangleleft M)\triangleleft M^*\llangle \arrow[l, "\cong"', "\llangle t^{-1}\llangle"] \arrow[d, "\llangle h_{**}\llangle"'] & & \\
\llangle P'\llangle & \llangle P'\triangleleft U_R\llangle \arrow[l, "\cong"', "\llangle\bar{l}^{-1}\llangle"] & \llangle P'\triangleleft(M\odot M^*)\llangle \arrow[l, "\llangle\eta^*\llangle"] & \llangle (P'\triangleleft M)\triangleleft M^*\llangle \arrow[l, "\cong"', "\llangle t^{-1}\llangle"] & &
\end{tikzcd}$$

Figure 3.1: Diagram for Proposition 5

The following is analogous to Proposition 7.4 in [14]:

Proposition 6. *If $f : U_R \rhd Q \to P \lhd U_R$ is any 2-cell, then* $\operatorname{cotr}(f)$ *is*

$$\langle\!\langle Q \langle\!\langle \xrightarrow[\cong]{\langle\!\langle \bar{r} \langle\!\langle} \langle\!\langle U_R \rhd Q \langle\!\langle \xrightarrow{\langle\!\langle f \langle\!\langle} \langle\!\langle P \lhd U_R \langle\!\langle \xrightarrow[\cong]{\langle\!\langle \bar{l}^{-1} \langle\!\langle} \langle\!\langle P \langle\!\langle$$

Proof. In the diagram in Figure 3.2, the map around the outside from $\langle\!\langle Q \langle\!\langle$ to $\langle\!\langle P \langle\!\langle$ is $\operatorname{cotr}(f)$. Note that two of the arrows are labeled two different ways; this makes use of Lemma 5. ☐

$$\begin{tikzcd}
 & & \langle\!\langle (U_R \odot U_R) \rhd Q \langle\!\langle \arrow[drr, "\langle\!\langle t \langle\!\langle"] \arrow[d, phantom, "\cong \text{ Lemma 1.1 } \cong"] & & \\
\langle\!\langle Q \langle\!\langle \arrow[r, "\langle\!\langle \bar{r} \langle\!\langle", "\cong"'] & \langle\!\langle U_R \rhd Q \langle\!\langle \arrow[ur, "\langle\!\langle r^* \langle\!\langle"] \arrow[rr, "\cong", "\langle\!\langle \bar{r}_* \langle\!\langle = \langle\!\langle \bar{r} \langle\!\langle"'] \arrow[dr, "\langle\!\langle f \langle\!\langle"'] & {} & \langle\!\langle U_R \rhd (U_R \rhd Q) \langle\!\langle \arrow[dr, "\langle\!\langle f_* \langle\!\langle"] \arrow[d, phantom, "\text{Lemma 3}"] & \\
 & & \langle\!\langle P \lhd U_R \langle\!\langle \arrow[rr, "\langle\!\langle \bar{r} \langle\!\langle", "\cong"'] \arrow[dl, "1"'] & {} \arrow[d, phantom, "\text{Definition 10}"] & \langle\!\langle U_R \rhd (P \lhd U_R) \langle\!\langle \arrow[dl, "\cong"', "\langle\!\langle \theta \langle\!\langle"] \\
\langle\!\langle P \langle\!\langle & \langle\!\langle P \lhd U_R \langle\!\langle \arrow[l, "\cong"', "\langle\!\langle \bar{l}^{-1} \langle\!\langle"] & {} \arrow[d, phantom, "\cong \text{ Lemma 1.1 } \cong"] & \langle\!\langle (P \lhd U_R) \lhd U_R \langle\!\langle \arrow[ll, "\langle\!\langle \bar{l}^{-1} \langle\!\langle = \langle\!\langle (\bar{l}_*)^{-1} \langle\!\langle"', "\cong"] \arrow[dl, "\langle\!\langle t^{-1} \langle\!\langle"] & \\
 & & \langle\!\langle P \lhd (U_R \odot U_R) \langle\!\langle \arrow[ul, "\langle\!\langle (l^{-1})^* \langle\!\langle"] & &
\end{tikzcd}$$

Figure 3.2: Diagram for Proposition 6

The following is analogous to Theorem 2, which is Proposition 7.5 in [14]; it says that a composite of cotraces with respect to dualizable objects M and N is the same as a single cotrace with respect to $N \odot M$ (which is dualizable by Proposition 1).

Proposition 7. *Let M and N be right dualizable 1-cells in a closed bicategory with a coshadow. For 2-cells $f : M \rhd Q \to P \lhd M$ and $g : N \rhd P \to L \lhd N$, the composite* $\operatorname{cotr}(g) \circ \operatorname{cotr}(f) : \langle\!\langle Q \langle\!\langle \to \langle\!\langle L \langle\!\langle$ *is equal to the cotrace (with respect to $N \odot M$) of the composite*

$$\begin{aligned} (N \odot M) \rhd Q \xrightarrow[\cong]{t} N \rhd (M \rhd Q) \xrightarrow{f_*} N \rhd (P \lhd M) \xrightarrow[\cong]{a^{-1}} (N \rhd P) \lhd M \\ \xrightarrow{g_*} (L \lhd N) \lhd M \xrightarrow[\cong]{t^{-1}} L \lhd (N \odot M) \end{aligned} \tag{3.1}$$

Proof. The left and bottom sides of the diagram in Figure 3.3 are $\operatorname{cotr}(g) \circ \operatorname{cotr}(f)$, and the top and right sides are the cotrace of (3.1), except that we have deleted the

$\langle\!\langle Q \langle\!\langle \xrightarrow{\langle\!\langle \varepsilon^* \overline{r} \langle\!\langle} \langle\!\langle (M^* \odot M) \triangleright P \langle\!\langle$ at the beginning and the $\langle\!\langle L \triangleleft (N \odot N^*) \langle\!\langle \xrightarrow{\langle\!\langle \bar{l}^{-1} \eta^* \langle\!\langle} \langle\!\langle L \langle\!\langle$ at the end, since these are common to both sides. Every unlabeled square commutes because of functoriality of $- \triangleright -$ and $- \triangleleft -$ or naturality of θ, t, a, $\overline{r}$, or $\bar{l}$. $\square$

If (M, M^*) is a dual pair, a 2-cell $f : M \triangleright Q \to P \triangleleft M$ has a mate $f^* : Q \triangleleft M^* \to M^* \triangleright P$ (Definition 9), which has the same cotrace as f:

Proposition 8. *If M is right dualizable and $f : M \triangleright Q \to P \triangleleft M$ is a 2-cell, then* $\operatorname{cotr}(f) = \operatorname{cotr}(f^*)$.

Proof. The left and bottom sides of the diagram in Figure 3.4 are $\operatorname{cotr}(f)$, and the top and right sides are $\operatorname{cotr}(f^*)$. Every unlabeled square commutes because of functoriality of $- \triangleright -$ and $- \triangleleft -$ or naturality of θ, t, a, $\overline{r}$, or $\bar{l}$. $\square$

We conclude this list of properties with an analogue of cyclicity of the bicategorical trace [14, Proposition 7.2], which generalizes the familiar fact from linear algebra that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any matrices such that AB and BA are square matrices (even if A and B themselves are not square).

Proposition 9. *Let M and N be right dualizable 1-cells in a closed bicategory with a coshadow. For maps $f : M \triangleright Q_1 \to P_1 \triangleleft N$ and $g : P_2 \odot M \to N \odot Q_2$, the following diagram commutes:*

$$\begin{tikzcd}
\langle\!\langle Q_1 \triangleleft Q_2 \langle\!\langle \arrow[r, "\operatorname{cotr}(g^* f_*)"] \arrow[d, "\theta"', "\cong"] & \langle\!\langle P_1 \triangleleft P_2 \langle\!\langle \arrow[d, "\cong"', "\theta"] \\
\langle\!\langle Q_2 \triangleright Q_1 \langle\!\langle \arrow[r, "\operatorname{cotr}(f_* g^*)"'] & \langle\!\langle P_2 \triangleright P_1 \langle\!\langle
\end{tikzcd}$$

where $g^* f_*$ and $f_* g^*$ mean the following:

$$M \triangleright (Q_1 \triangleleft Q_2) \cong (M \triangleright Q_1) \triangleleft Q_2 \xrightarrow{f_*} (P_1 \triangleleft N) \triangleleft Q_2 \cong P_1 \triangleleft (N \odot Q_2) \xrightarrow{g^*} P_1 \triangleleft (P_2 \odot M) \cong (P_1 \triangleleft P_2) \triangleleft M$$

$$N \triangleright (Q_2 \triangleright Q_1) \cong (N \odot Q_2) \triangleright Q_1 \xrightarrow{g^*} (P_2 \odot M) \triangleright Q_1 \cong P_2 \triangleright (M \triangleright Q_1) \xrightarrow{f_*} P_2 \triangleright (P_1 \triangleleft N) \cong (P_2 \triangleright P_1) \triangleleft N$$

Proof. The top and right sides of the diagram in Figure 3.5 are $\operatorname{cotr}(g^* f_*) : \langle\!\langle Q_1 \triangleleft Q_2 \langle\!\langle \to \langle\!\langle P_1 \triangleleft P_2 \langle\!\langle$, while the left and bottom sides of the diagram in Figure 3.6 are

$$\langle\!\langle Q_1 \triangleleft Q_2 \langle\!\langle \xrightarrow[\cong]{\theta} \langle\!\langle Q_2 \triangleright Q_1 \langle\!\langle \xrightarrow{\operatorname{cotr}(f_* g^*)} \langle\!\langle P_2 \triangleright P_1 \langle\!\langle \xrightarrow[\cong]{\theta} \langle\!\langle P_1 \triangleleft P_2 \langle\!\langle.$$

The two diagrams glue together along their other edges. Every unlabeled square commutes because of functoriality of $- \odot -$, $- \triangleright -$, or $- \triangleleft -$ or naturality of θ, t, a, $\overline{r}$, or $\bar{l}$. $\square$

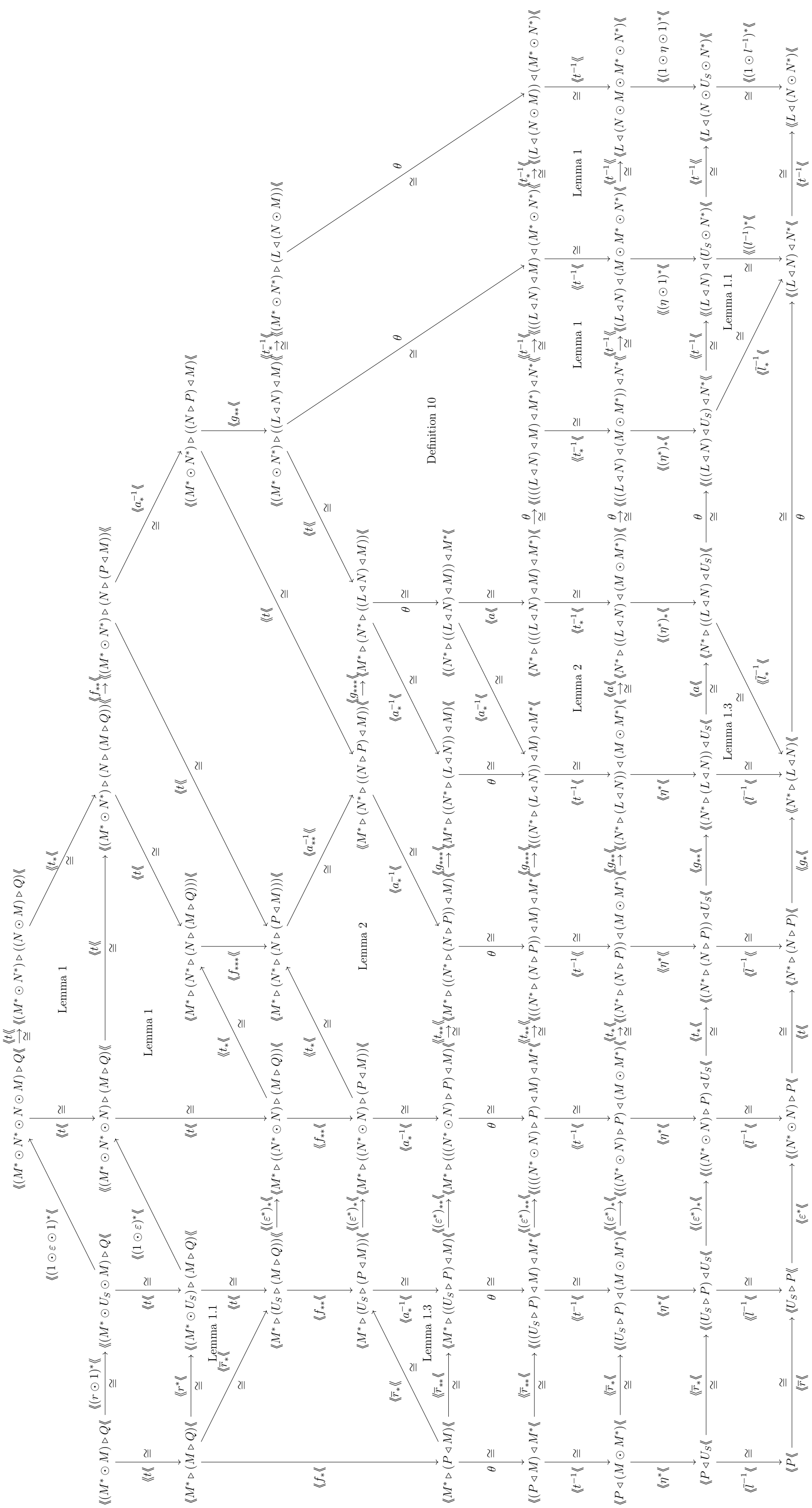

Figure 3.3: Diagram for Proposition 7

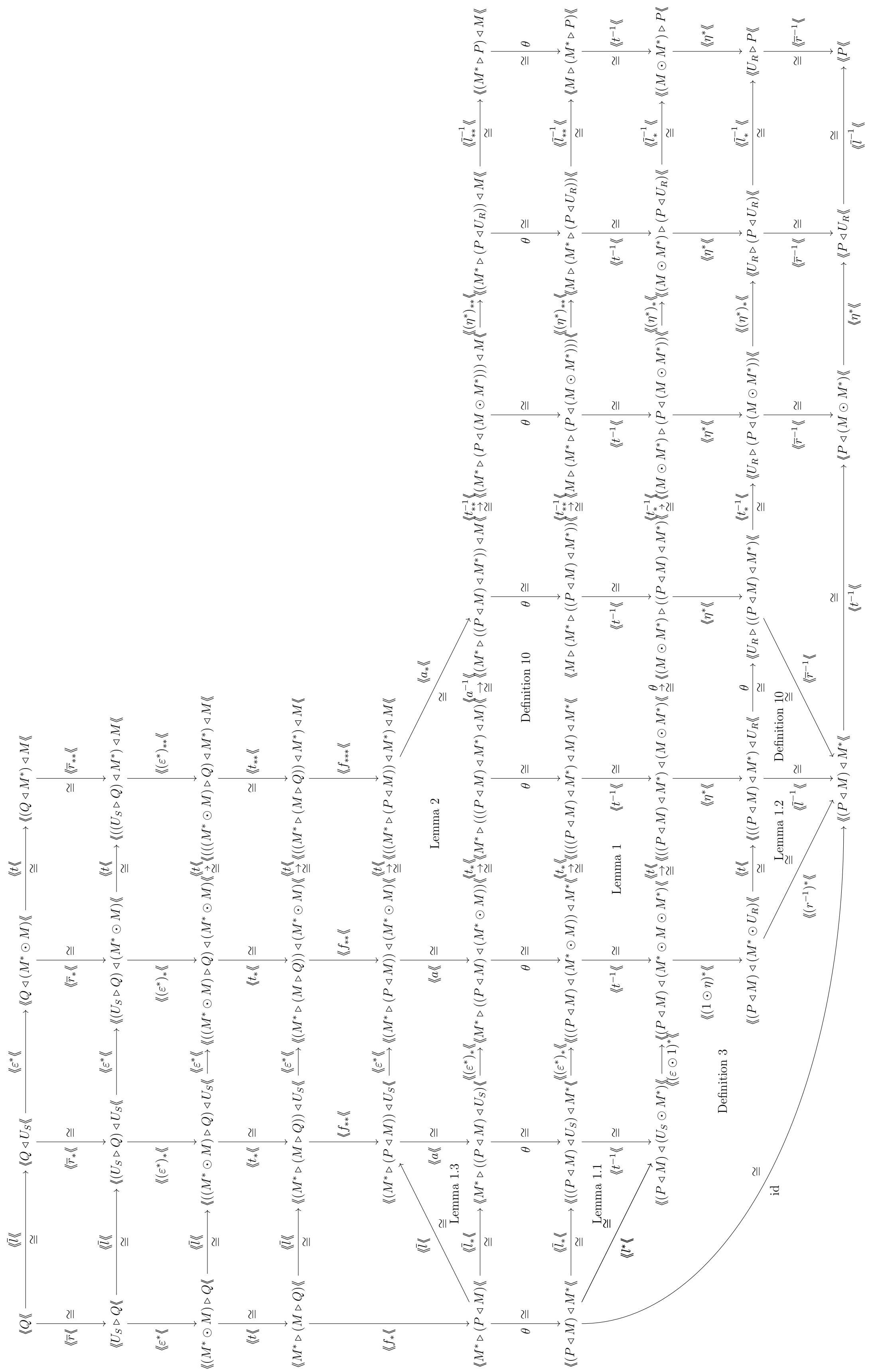

Figure 3.4: Diagram for Proposition 8

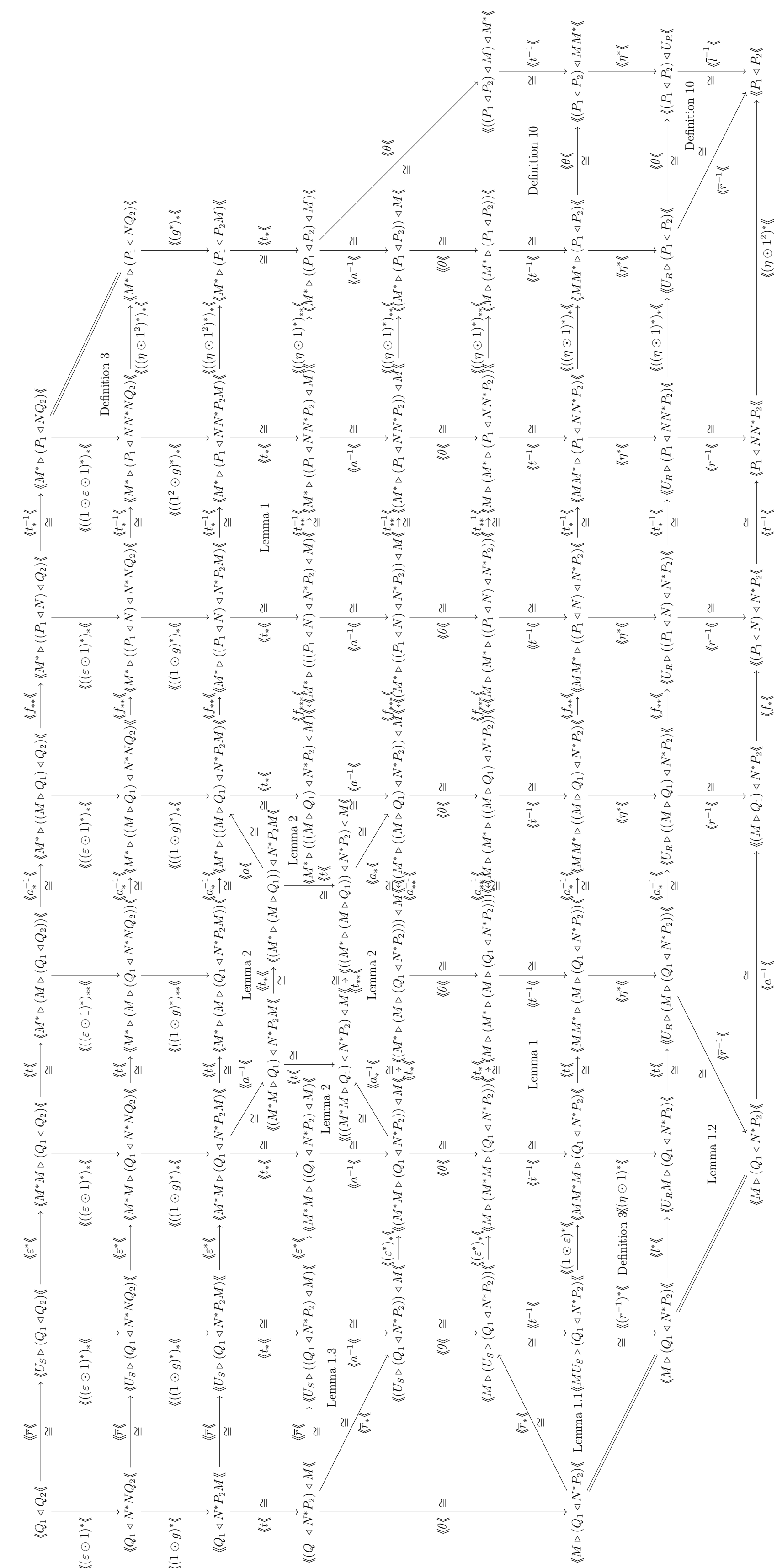

Figure 3.5: Diagram for Proposition 9

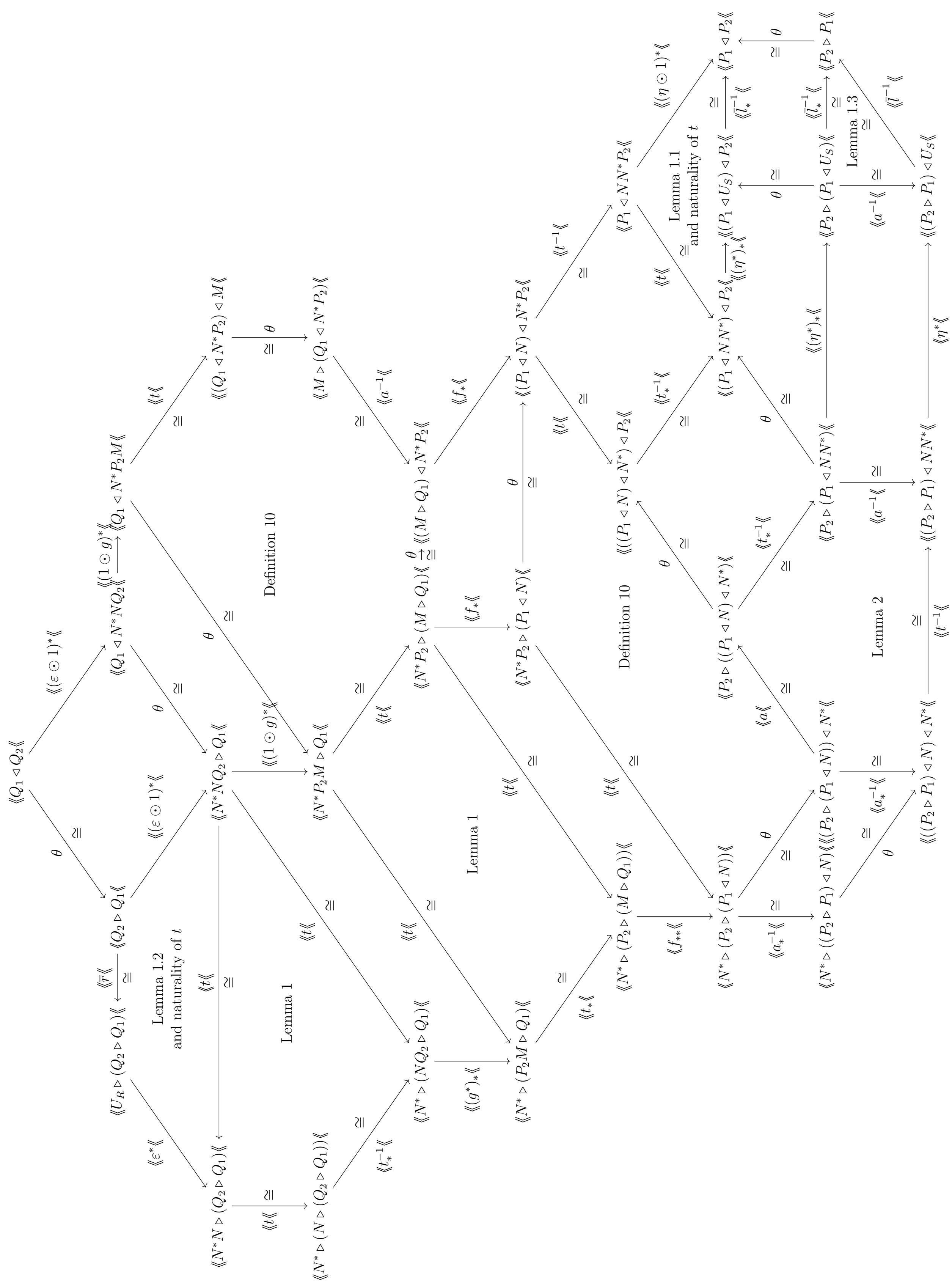

Figure 3.6: Diagram for Proposition 9

3.3 Functoriality

One of the most important properties of the trace is that it is preserved by lax functors, or at least those which preserve the dual pair relevant to the trace in question (Proposition 8.3 in [14]). We state a similar result for cotraces, after describing the structure of a lax functor between closed bicategories.

A lax functor $F : \mathscr{B} \to \mathscr{C}$ between bicategories is compatible with the horizontal composition in $\mathscr{B}$ and $\mathscr{C}$, in the sense that it comes equipped with coherence 2-cells $F(M) \odot F(N)$ in $\mathscr{C}$ for each pair of 1-cells M and N in $\mathscr{B}$. This might lead us to ask a lax functor between *closed* bicategories to include similar compatibility with the internal hom functors, but in fact this is automatic: for example, we get a map $F(M \triangleright N) \to F(M) \triangleright F(N)$ as the transpose of

$$F(M \triangleright N) \odot F(M) \to F((M \triangleright N) \odot M) \xrightarrow{F(\mathrm{ev})} F(N).$$

Thus a lax functor between closed bicategories is nothing more than a lax functor between the underlying bicategories. However, the functoriality result we want and the concomitant notion of lax coshadow functor use the coherence 2-cells for $\triangleright$ and $\triangleleft$ rather than the ones for $\odot$, so we present a definition of lax functor between closed bicategories in terms of the former.

In the following definition we make use of the transpose $\overline{\circ} : Y \triangleright Z \to ((X \triangleright Y) \triangleright (X \triangleright Z))$ of the "composition" map $\circ : (Y \triangleright Z) \odot (X \triangleright Y) \to X \triangleright Z$, which itself is is the transpose of

$$(Y \triangleright Z) \odot (X \triangleright Y) \odot X \xrightarrow{1 \odot \mathrm{ev}} (Y \triangleright Z) \odot Y \xrightarrow{\mathrm{ev}} Z.$$

Definition 12. Let $\mathscr{B}$ and $\mathscr{C}$ be closed bicategories. A ***lax closed functor*** $F : \mathscr{B} \to \mathscr{C}$ is

- A function $F_0 : \mathrm{ob}\mathscr{B} \to \mathrm{ob}\mathscr{C}$
- For each $R, S \in \mathrm{ob}\mathscr{B}$, a functor $F_{R,S} : \mathscr{B}(R,S) \to \mathscr{C}(F_0(R), F_0(S))$
- Natural transformations $c : F_{R,S}(N \triangleright P) \to F_{S,T}(N) \triangleright F_{R,T}(P)$
- Natural transformations $c : F_{S,T}(P \triangleleft M) \to F_{R,T}(P) \triangleleft F_{R,S}(M)$
- Maps $i : U_{F_0(R)} \to F_{R,R}(U_R)$

such that the following diagrams commute whenever they make sense (we usually suppress the subscripts of F when they are clear from context):

$$\begin{tikzcd}
F(N \triangleright P) \arrow[r, "\overline{\circ}"] \arrow[d, "c"'] & F((M \triangleright N) \triangleright (M \triangleright P)) \arrow[r, "c"] & F(M \triangleright N) \triangleright F(M \triangleright P) \arrow[d, "c_*"] \\
F(N) \triangleright F(P) \arrow[r, "\overline{\circ}"'] & (F(M) \triangleright F(N)) \triangleright (F(M) \triangleright F(P)) \arrow[r, "c^*"'] & F(M \triangleright N) \triangleright (F(M) \triangleright F(P))
\end{tikzcd}$$

$$\begin{tikzcd}
U_{F(R)} \arrow[r, "i"] \arrow[d, "\bar{l}"'] & F(U_R) \arrow[d, "F(\bar{l})"] \\
F(M) \rhd F(M) & F(M \rhd M) \arrow[l, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(M) \arrow[r, "F(\bar{r})", "\cong"'] \arrow[d, "\bar{r}"', "\cong"] & F(U_S \rhd M) \arrow[d, "c"] \\
U_{F(S)} \rhd F(M) & F(U_S) \rhd F(M) \arrow[l, "i^*"]
\end{tikzcd}$$

along with similar diagrams for the other hom functor $- \lhd -$, and the following diagram relating the maps c for the two hom functors:

$$\begin{tikzcd}
F((M \rhd N) \lhd P) \arrow[r, "c"] \arrow[d, "F(a)"', "\cong"] & F(M \rhd N) \lhd F(P) \arrow[r, "c_*"] & (F(M) \rhd F(N)) \lhd F(P) \arrow[d, "a"', "\cong"] \\
F(M \rhd (N \lhd P)) \arrow[r, "c"'] & F(M) \rhd F(N \lhd P) \arrow[r, "c_*"'] & F(M) \rhd (F(N) \lhd F(P))
\end{tikzcd}$$

Definition 13. Let $\mathscr{B}$ and $\mathscr{C}$ be closed bicategories equipped with coshadows with target categories $\mathbf{T}$ and $\mathbf{Z}$, respectively. A ***lax coshadow functor*** is a lax closed functor $F : \mathscr{B} \to \mathscr{C}$ together with a functor $F_{\mathrm{cotr}} : \mathbf{T} \to \mathbf{Z}$ and a natural transformation

$$\phi : F_{\mathrm{cotr}} \circ \langle\!\langle - \langle\!\langle_{\mathscr{B}} \to \langle\!\langle - \langle\!\langle_{\mathscr{C}} \circ F$$

such that the following diagram commutes whenever it makes sense:

$$\begin{tikzcd}
F_{\mathrm{cotr}}\langle\!\langle M \rhd N \langle\!\langle \arrow[r, "F_{\mathrm{cotr}}(\theta)", "\cong"'] \arrow[d, "\phi"'] & F_{\mathrm{cotr}}\langle\!\langle N \lhd M \langle\!\langle \arrow[d, "\phi"] \\
\langle\!\langle F(M \rhd N) \langle\!\langle \arrow[d, "\langle\!\langle c \langle\!\langle"'] & \langle\!\langle F(N \lhd M) \langle\!\langle \arrow[d, "\langle\!\langle c \langle\!\langle"] \\
\langle\!\langle F(M) \rhd F(N) \langle\!\langle \arrow[r, "\cong", "\theta"'] & \langle\!\langle F(N) \lhd F(M) \langle\!\langle
\end{tikzcd}$$

We will make use of the following result in proving that lax closed functors preserve cotraces.

Lemma 7. *If $F : \mathscr{B} \to \mathscr{C}$ is a lax closed functor, the following commutes:*

$$\begin{tikzcd}
F((M \odot N) \rhd P) \arrow[r, "c"] \arrow[d, "F(t)"', "\cong"] & F(M \odot N) \rhd F(P) \arrow[r, "c^*"] & (F(M) \odot F(N)) \rhd F(P) \arrow[d, "\cong"', "t"] \\
F(M \rhd (N \rhd P)) \arrow[r, "c"'] & F(M) \rhd F(N \rhd P) \arrow[r, "c_*"'] & F(M) \rhd (F(N) \rhd F(P))
\end{tikzcd}$$

Proof. Both sides are the transpose (via the tensor-hom adjunction for $F(M)$) of the transpose (via the tensor-hom adjunction for $F(N)$) of

$$F((M \odot N) \rhd P) \odot F(M) \odot F(N) \xrightarrow{c \circ (1 \odot c)} F(((M \odot N) \rhd P) \odot M \odot N) \xrightarrow{F(\mathrm{ev})} F(P).$$

□

Proposition 10. *Let $F : \mathscr{B} \to \mathscr{C}$ be a lax coshadow functor and $M \in \mathscr{B}(R,S)$ a right dualizable 1-cell with right dual M^*.*

1. *[11, Proposition 4.3.6] If $c : F(M) \odot F(M^*) \to F(M \odot M^*)$ and $i : U_{F(S)} \to F(U_S)$ are isomorphisms, then $F(M)$ is right dualizable with right dual $F(M^*)$.*
2. *If, furthermore, the map $c_{M,Q} : F(M \triangleright Q) \to F(M) \triangleright F(Q)$ is an isomorphism, then for any 2-cell $f : M \triangleright Q \to P \triangleleft M$, the following commutes:*

$$\begin{tikzcd}
F\langle\!\langle Q \langle\!\langle_{\mathscr{B}} \arrow[r, "F(\mathrm{cotr}(f))"] \arrow[d, "\phi"'] & F\langle\!\langle P \langle\!\langle_{\mathscr{B}} \arrow[d, "\phi"] \\
\langle\!\langle F(Q) \langle\!\langle_{\mathscr{C}} \arrow[r, "\mathrm{cotr}(c_{M,P} \circ F(f) \circ c_{M,Q}^{-1})"'] & \langle\!\langle F(P) \langle\!\langle_{\mathscr{C}}
\end{tikzcd}$$

Proof. Part (i) is Proposition 8.3(i) in [14]. For part (ii), the diagram in Figure 3.7 has $\phi \circ F(\mathrm{cotr}(f))$ along the left side and $\mathrm{cotr}(cF(f)c^{-1}) \circ \phi$ along the right side. Every unlabeled square commutes because of naturality of ϕ, c, or θ. $\square$

3.4 Morita Invariance

One of the most important properties of Hochschild homology is that it is Morita invariant, meaning that if R and S are Morita equivalent rings, then $\mathrm{HH}_n(R) \cong \mathrm{HH}_n(S)$. Moreover, there is a trace map which is an isomorphism between the Hochschild homologies of R and S. In fact, this is an example of a general notion of Morita invariance that all shadow functors satisfy. After reviewing the classical notion of Morita equivalence and its generalization to bicategories and shadows, we will demonstrate that coshadows also satisfy Morita invariance.

Rings R and S are ***Morita equivalent*** if their module categories Mod_R and Mod_S are equivalent. If there are bimodules ${}_RP_S$ and ${}_SQ_R$ such that $P \otimes_S Q \cong R$ (as (R,R)-bimodules) and $Q \otimes_R P \cong S$ (as (S,S)-bimodules), then

$$- \otimes_R P : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_S : - \otimes_S Q$$

are mutually inverse equivalences of categories, and it turns out that any equivalence between Mod_R and Mod_S arises this way. The leads to a notion of Morita equivalence in a bicategory:

Definition 14 ([1])**.** Two 0-cells R and S in a bicategory $\mathscr{B}$ are ***Morita equivalent*** if there are 1-cells $P \in \mathscr{B}(R,S)$ and $Q \in \mathscr{B}(S,R)$ and isomorphisms

$$\eta : U_R \xrightarrow{\cong} P \odot Q \qquad \varepsilon : Q \odot P \xrightarrow{\cong} U_S$$

satisfying the triangle identities. This means that (P,Q) is a dual pair, and η^{-1} and ε^{-1} witness (Q,P) as a dual pair as well.

$$
\begin{tikzcd}[column sep=small]
F\langle\!\langle Q\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F\langle\!\langle \bar{r}\rangle\!\rangle"', "\cong"] & \langle\!\langle FQ\rangle\!\rangle \arrow[r, "\langle\!\langle \bar{r}\rangle\!\rangle", "\cong"'] \arrow[d, "\langle\!\langle F\bar{r}\rangle\!\rangle"', "\cong"] & \langle\!\langle U_{FS} \triangleright FQ\rangle\!\rangle \arrow[r, "\langle\!\langle (i^{-1})^*\rangle\!\rangle", "\cong"'] & \langle\!\langle FU_S \triangleright FQ\rangle\!\rangle \arrow[r, "\langle\!\langle (F\varepsilon)^*\rangle\!\rangle"] & \langle\!\langle F(M^* \odot M) \triangleright FQ\rangle\!\rangle \arrow[d, "\langle\!\langle c^*\rangle\!\rangle"] \\
F\langle\!\langle U_S \triangleright Q\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F\langle\!\langle \varepsilon^*\rangle\!\rangle"'] & \langle\!\langle F(U_S \triangleright Q)\rangle\!\rangle \arrow[urr, "\langle\!\langle c\rangle\!\rangle"'] \arrow[d, "\langle\!\langle F(\varepsilon^*)\rangle\!\rangle"'] \arrow[r, phantom, "\text{Definition 12}"] & & & \langle\!\langle (FM^* \odot FM) \triangleright FQ\rangle\!\rangle \arrow[d, "\cong"', "\langle\!\langle t\rangle\!\rangle"] \\
F\langle\!\langle (M^* \odot M) \triangleright Q\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F\langle\!\langle t\rangle\!\rangle"', "\cong"] & \langle\!\langle F((M^* \odot M) \triangleright Q)\rangle\!\rangle \arrow[uurrr, "\langle\!\langle c\rangle\!\rangle"'] \arrow[d, "\langle\!\langle Ft\rangle\!\rangle"', "\cong"] & \text{Lemma 7} & & \langle\!\langle FM^* \triangleright (FM \triangleright FQ)\rangle\!\rangle \arrow[d, "\cong"', "\langle\!\langle c_*^{-1}\rangle\!\rangle"] \\
F\langle\!\langle M^* \triangleright (M \triangleright Q)\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F\langle\!\langle f_*\rangle\!\rangle"'] & \langle\!\langle F(M^* \triangleright (M \triangleright Q))\rangle\!\rangle \arrow[rrr, "\langle\!\langle c\rangle\!\rangle"] \arrow[d, "\langle\!\langle F(f_*)\rangle\!\rangle"'] & & & \langle\!\langle FM^* \triangleright F(M \triangleright Q)\rangle\!\rangle \arrow[d, "\langle\!\langle (Ff)_*\rangle\!\rangle"'] \\
F\langle\!\langle M^* \triangleright (P \triangleleft M)\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F(\theta)"', "\cong"] & \langle\!\langle F(M^* \triangleright (P \triangleleft M))\rangle\!\rangle \arrow[rrr, "\langle\!\langle c\rangle\!\rangle"] \arrow[d, phantom, "\text{Definition 13}"] & & & \langle\!\langle FM^* \triangleright F(P \triangleleft M)\rangle\!\rangle \arrow[dll, "\theta"', "\cong"] \arrow[d, "\langle\!\langle c_*\rangle\!\rangle"] \\
F\langle\!\langle (P \triangleleft M) \triangleleft M^*\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F\langle\!\langle t^{-1}\rangle\!\rangle"', "\cong"] & \langle\!\langle F((P \triangleleft M) \triangleleft M^*)\rangle\!\rangle \arrow[r, "\langle\!\langle c\rangle\!\rangle"] \arrow[d, "\langle\!\langle F(t^{-1})\rangle\!\rangle"', "\cong"] & \langle\!\langle F(P \triangleleft M) \triangleleft FM^*\rangle\!\rangle \arrow[drr, "\langle\!\langle c_*\rangle\!\rangle"'] & & \langle\!\langle FM^* \triangleright (FP \triangleleft FM)\rangle\!\rangle \arrow[d, "\cong"', "\theta"] \\
F\langle\!\langle P \triangleleft (M \odot M^*)\rangle\!\rangle \arrow[r, "\phi"] \arrow[dd, "F\langle\!\langle \eta^*\rangle\!\rangle"'] & \langle\!\langle F(P \triangleleft (M \odot M^*))\rangle\!\rangle \arrow[ddrrr, "\langle\!\langle c\rangle\!\rangle"] \arrow[dd, "\langle\!\langle F(\eta^*)\rangle\!\rangle"'] & \text{Lemma 7} & & \langle\!\langle (FP \triangleleft FM) \triangleleft FM^*\rangle\!\rangle \arrow[d, "\cong"', "\langle\!\langle t^{-1}\rangle\!\rangle"] \\
& & & & \langle\!\langle FP \triangleleft (FM \odot FM^*)\rangle\!\rangle \arrow[d, "\cong"', "\langle\!\langle (c^{-1})^*\rangle\!\rangle"] \\
F\langle\!\langle P \triangleleft U_R\rangle\!\rangle \arrow[r, "\phi"] \arrow[d, "F\langle\!\langle \bar{l}^{-1}\rangle\!\rangle"', "\cong"] & \langle\!\langle F(P \triangleleft U_R)\rangle\!\rangle \arrow[drrr, "\langle\!\langle c\rangle\!\rangle"] \arrow[d, "\langle\!\langle F(\bar{l}^{-1})\rangle\!\rangle"', "\cong"] \arrow[dr, phantom, "\text{Definition 12}"] & & & \langle\!\langle FP \triangleleft F(M \odot M^*)\rangle\!\rangle \arrow[d, "\langle\!\langle (F\eta)^*\rangle\!\rangle"'] \\
F\langle\!\langle P\rangle\!\rangle \arrow[r, "\phi"] & \langle\!\langle FP\rangle\!\rangle & \langle\!\langle FP \triangleleft U_{FR}\rangle\!\rangle \arrow[l, "\cong", "\langle\!\langle \bar{l}^{-1}\rangle\!\rangle"'] & & \langle\!\langle FP \triangleleft FU_R\rangle\!\rangle \arrow[ll, "\langle\!\langle i^*\rangle\!\rangle"']
\end{tikzcd}
$$

Figure 3.7: Diagram for Proposition 10

In the Morita bicategory $\mathbf{Mod}/_{\mathbf{Ring}}$, this recovers the classical notion of Morita equivalence. Shadows are Morita invariant:

Proposition 11 ([1, Proposition 4.8])**.** *Let (P, Q) be a Morita equivalence between 0-cells R and S in a bicategory $\mathscr{B}$ with a shadow $\langle\!\langle - \rangle\!\rangle$. Then $\langle\!\langle M \rangle\!\rangle \cong \langle\!\langle Q \odot M \odot P \rangle\!\rangle$ for any 1-cell $M \in \mathscr{B}(R, R)$, and moreover, there is a bicategorical trace witnessing this equivalence.*

Proof. The trace of $\eta \odot 1^2 : M \odot P \xrightarrow{\cong} P \odot Q \odot M \odot P$ is an isomorphism $\langle\!\langle M \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle Q \odot M \odot P \rangle\!\rangle$. ∎

In the case that M is a unit 1-cell, we have:

Corollary 2. *If R and S are Morita equivalent 0-cells in a bicategory with a shadow $\langle\!\langle - \rangle\!\rangle$, then $\langle\!\langle U_R \rangle\!\rangle \cong \langle\!\langle U_S \rangle\!\rangle$.*

Coshadows are also Morita invariant:

Proposition 12. *Let (P, Q) be a Morita equivalence between 0-cells R and S in a closed bicategory $\mathscr{B}$ with a coshadow $\langle\!\langle - \langle\!\langle$. Then $\langle\!\langle M \langle\!\langle \cong \langle\!\langle Q \odot M \odot P \langle\!\langle$ for any 1-cell $M \in \mathscr{B}(R, R)$, and moreover, there is a bicategorical cotrace witnessing this equivalence.*

Proof. The transpose of $Q \odot (Q \triangleright M) \xrightarrow{\cong} (Q \odot M \odot P)$ is an isomorphism, so its cotrace is an isomorphism $\langle\!\langle M \langle\!\langle \xrightarrow{\cong} \langle\!\langle Q \odot M \odot P \langle\!\langle$. □

Example 16. The coshadow of Proposition 12 using the categorical trace of [3] as the coshadow (Example 13) recovers the isomorphism of [3, Proposition 3.8(a)].

3.5 Symmetric Monoidal Cotraces

We can define a symmetric monoidal cotrace, but it turns out be a trace. We write $[-,-]$ for the internal hom in a closed symmetric monoidal category (i.e. $[Y, -]$ is the right adjoint to $- \otimes Y$). We also make use of the map $\mu : X \otimes [Y, Z] \to [Y, X \otimes Z]$, which is an isomorphism if X or Y is dualizable (cf. Proposition 3).

Definition 15. Let $(\mathscr{C}, \otimes, I)$ be a closed symmetric monoidal category and (M, M^*) a dual pair. The ***cotrace*** of a map $f : [M, Q] \to [M, P]$ is the composite:

$$Q \cong [I, Q] \xrightarrow{\varepsilon^*} [M^* \otimes M, Q] \xrightarrow[\cong]{t} [M^*, [M, Q]] \xrightarrow{f_*} [M^*, [M, P]]$$
$$\xrightarrow[\cong]{t} [M^* \otimes M, P] \xrightarrow[\cong]{s^*} [M \otimes M^*, P] \xrightarrow{\eta^*} [I, P] \cong P$$

This is similar to the bicategorical cotrace (Definition 11), but no coshadow is needed since there are not two different internal hom functors.

Proposition 13. *Let M be a dualizable object in a symmetric monoidal category, and let $f : [M, Q] \to [M, P]$. Then $\mathrm{cotr}(f) = \mathrm{tr}(\tilde{f})$, where $M^* := [M, I]$ and $\tilde{f}$ is the unique map making the following commute:*

$$\begin{tikzcd}
Q \otimes M^* \arrow[rr, "\mu", "\cong"'] \arrow[d, "\tilde{f}"'] & & {[M, Q]} \arrow[d, "f"] \\
M^* \otimes P \arrow[r, "\cong", "s"'] & P \otimes M^* \arrow[r, "\cong", "\mu"'] & {[M, P]}
\end{tikzcd}$$

Proof. In the diagram of Figure 3.8, the left-hand side is $\mathrm{tr}(\tilde{f})$ and the right-hand side is $\mathrm{cotr}(f)$. As in a closed bicategory, if M is dualizable its dual is isomorphic to $[M, I]$ (see Proposition 4), so we take M^* to be $[M, I]$, with coevaluation and evaluation

$$\eta : I \xrightarrow{\bar{l}} [M, M] \xrightarrow[\cong]{\mu^{-1}} M \otimes [M, I] \qquad \text{and} \qquad \varepsilon : [M, I] \otimes M \xrightarrow{\mathrm{ev}} I.$$

We let M^{**} be $[M^*, I]$ and define the coevaluation η' and evaluation ε' similarly. To verify that a subdiagram of Figure 3.8 commutes (if it's not because of something straightforward like properties of symmetric monoidal categories or naturality of μ), the simplest approach is, as usual, to take transposes of both sides until no hom objects remain in the target. $\square$

$$
\begin{tikzcd}
 & & Q \arrow[dll, "r^{-1}"', "\cong"] \arrow[drr, "\overline{r}", "\cong"'] & & \\
Q \otimes I \arrow[rrr, "\cong", "1 \otimes \overline{r}"'] \arrow[d, "1 \otimes \eta'"'] & & & Q \otimes [I,I] \arrow[r, "\cong", "\mu"'] \arrow[d, "1 \otimes \varepsilon^*"] & {[I,Q]} \arrow[d, "\varepsilon^*"] \\
Q \otimes M^* \otimes M^{**} \arrow[rr, "1 \otimes \mu", "\cong"'] \arrow[dr, "\cong", "\mu \otimes 1"'] \arrow[dddd, "\tilde{f} \otimes 1"'] & & Q \otimes [M^*, [M, I]] \arrow[r, "1 \otimes t^{-1}", "\cong"'] & Q \otimes [M^* \otimes M, I] \arrow[r, "\mu", "\cong"'] & {[M^* \otimes M, Q]} \arrow[d, "\cong"', "t"] \\
 & {[M,Q] \otimes M^{**}} \arrow[rrr, "\mu", "\cong"'] \arrow[d, "f \otimes 1"'] & & & {[M^*, [M,Q]]} \arrow[d, "f_*"] \\
 & {[M,P] \otimes M^{**}} \arrow[rrr, "\mu", "\cong"'] & & & {[M^*, [M,P]]} \arrow[d, "\cong"', "t^{-1}"] \\
M^* \otimes P \otimes M^{**} \arrow[r, "s \otimes 1", "\cong"'] \arrow[d, "s"', "\cong"] & P \otimes M^* \otimes M^{**} \arrow[r, "1 \otimes \mu", "\cong"'] \arrow[u, "\mu \otimes 1"', "\cong"] \arrow[d, "1 \otimes s"', "\cong"] & P \otimes [M^*, [M, I]] \arrow[r, "1 \otimes t^{-1}", "\cong"'] & P \otimes [M^* \otimes M, I] \arrow[r, "\mu", "\cong"'] \arrow[d, "\cong"', "1 \otimes s^*"] & {[M^* \otimes M, P]} \arrow[d, "\cong"', "s^*"] \\
M^{**} \otimes M^* \otimes P \arrow[r, "s", "\cong"'] \arrow[d, "\varepsilon' \otimes 1"'] & P \otimes M^{**} \otimes M^* \arrow[r, "1 \otimes \mu", "\cong"'] \arrow[d, "1 \otimes \varepsilon'"'] & P \otimes [M, [M^*, I]] \arrow[r, "1 \otimes t^{-1}", "\cong"'] & P \otimes [M \otimes M^*, I] \arrow[r, "\mu", "\cong"'] \arrow[d, "1 \otimes \eta^*"] & {[M \otimes M^*, P]} \arrow[d, "\eta^*"] \\
I \otimes P \arrow[r, "s", "\cong"'] \arrow[drr, "\cong", "l"'] & P \otimes I \arrow[rr, "1 \otimes \overline{r}", "\cong"'] \arrow[dr, "\cong", "r"'] & & P \otimes [I,I] \arrow[r, "\mu", "\cong"'] & {[I,P]} \arrow[dll, "\cong", "\overline{r}^{-1}"'] \\
 & & P & &
\end{tikzcd}
$$

Figure 3.8: Diagram for Proposition 13

In a bicategory, however, traces and cotraces truly are different, for the simple reason that shadows and coshadows are different; Hochschild homology and cohomology, for example, are not the same thing.

Chapter 4 Interplay Between Traces and Cotraces

The original motivation for our study of cotraces was [9], in which both traces and cotraces arise and interact with each other. This interaction is mediated by a "pairing" map from a shadow and coshadow to a second shadow.

Example 17. If M and N are (R,R)-bimodules, we have $\rho : \mathrm{HH}^0(R,M)\otimes\mathrm{HH}_0(R,N) \to \mathrm{HH}_0(R, M\otimes_R N)$ taking $m\otimes[n]$ to $[m\otimes n]$. In fact, there is a pairing $\mathrm{HH}^n(R,M)\otimes \mathrm{HH}_n(R,N) \to \mathrm{HH}_0(R, M\otimes_R N)$ for any $n \geq 0$.

The main result of [9], Proposition 4.5.4, is a relation between traces and cotraces. In the case that $n = 0$, it takes the following form, where R is a commutative ring, M is an R-module considered as an (R,R)-bimodule, and F is a finitely generated projective right R-module viewed as an (S,R)-bimodule, where $S := \mathrm{Hom}_R(F,F)$:

$$\begin{tikzcd}
& \begin{matrix}\mathrm{HH}^0(R,M)\\ \otimes\,\mathrm{HH}_0(S,S)\end{matrix} \arrow[dl, "\mathrm{cotr}\otimes 1"'] \arrow[dr, "1\otimes\mathrm{tr}"] & \\
\begin{matrix}\mathrm{HH}^0(S,\mathrm{Hom}_R(F,F\otimes_R M))\\ \otimes\,\mathrm{HH}_0(S,S)\end{matrix} \arrow[d, "\rho"'] & & \begin{matrix}\mathrm{HH}^0(R,R)\\ \otimes\,\mathrm{HH}_0(R,R)\end{matrix} \arrow[d, "\rho"] \\
\mathrm{HH}_0(S,S) \arrow[dr, "\mathrm{tr}"'] & & \mathrm{HH}_0(R,R) \arrow[dl, equal] \\
& \mathrm{HH}_0(R,R) &
\end{tikzcd} \tag{4.1}$$

If $M = R$, this reduces to the diagram of (1.1). Symbolically, this asserts that if $m \in \mathrm{HH}^0(R,M)$ and $[\varphi] \in \mathrm{HH}_0(S,S)$, then

$$\mathrm{tr}(\rho(\mathrm{cotr}(m),[\varphi])) = \rho(m, \mathrm{tr}([\varphi])),$$

which looks like a kind of "adjointness" between trace and cotrace. The cotrace takes $m \in \mathrm{HH}^0(R,M)$ to the homomorphism $x \mapsto x\otimes m$ in $\mathrm{HH}^0(S, \mathrm{Hom}_R(F, F\otimes_R M))$.

More generally, suppose that we have shadows $\langle\!\langle - \rangle\!\rangle$ and $\langle\!\langle - \rangle\!\rangle$ and a coshadow $\langle\!\langle - \langle\!\langle$, all taking values in a common monoidal category. Suppose also that there are maps

$$\rho_{M,N} : \langle\!\langle M \langle\!\langle \otimes \langle\!\langle N \rangle\!\rangle \to \langle\!\langle M \odot N \rangle\!\rangle$$

which are natural in M and N. If ρ is appropriately compatible with the isomorphisms θ for the coshadow and shadows, then cotraces with respect to $\langle\!\langle - \langle\!\langle$ and traces with respect to $\langle\!\langle - \rangle\!\rangle$ satisfy the same sort of adjointness that Lipman's traces and cotraces do. We record the necessary compatibility in the following definition.

Definition 16. A ***pairing*** ρ between a coshadow ${\color{blue}\langle\!\langle - \langle\!\langle}$, a shadow ${\color{blue}\langle\!\langle - \rangle\!\rangle}$, and another shadow $\langle\!\langle - \rangle\!\rangle$, all taking values in the same monoidal category, is a family of maps $\rho_{M,N} : {\color{blue}\langle\!\langle M \langle\!\langle} \otimes {\color{blue}\langle\!\langle N \rangle\!\rangle} \to \langle\!\langle M \odot N \rangle\!\rangle$ which are natural in M and N and such that the following diagram commutes whenever it makes sense:

$$\begin{tikzcd}
{\color{blue}\langle\!\langle Z \lhd X \langle\!\langle} \otimes {\color{blue}\langle\!\langle Y \odot X \rangle\!\rangle} \arrow[r, "\theta\otimes\theta", "\cong"'] \arrow[d, "\rho"'] & {\color{blue}\langle\!\langle X \rhd Z \langle\!\langle} \otimes {\color{blue}\langle\!\langle X \odot Y \rangle\!\rangle} \arrow[d, "\rho"] \\
\langle\!\langle (Z \lhd X) \odot Y \odot X \rangle\!\rangle \arrow[d, "\theta"', "\cong"] & \langle\!\langle (X \rhd Z) \odot X \odot Y \rangle\!\rangle \arrow[d, "\cong"', "\langle\!\langle \mathrm{ev}\odot 1 \rangle\!\rangle"] \\
\langle\!\langle X \odot (Z \lhd X) \odot Y \rangle\!\rangle \arrow[r, "\langle\!\langle \mathrm{ev}\odot 1 \rangle\!\rangle"'] & \langle\!\langle Z \odot Y \rangle\!\rangle
\end{tikzcd}$$

With HH^0 as the coshadow and HH_0 playing the role of both shadows, the maps of Example 17 form a pairing. We are finally in a position to state and prove a precise version of Theorem 1.

Theorem 3. *Let* $\mathbf{T}$ *be a monoidal category and* $\mathscr{B}$ *a closed bicategory with shadow functors* $\langle\!\langle - \rangle\!\rangle$ *and* ${\color{blue}\langle\!\langle - \rangle\!\rangle}$ *and coshadow* ${\color{blue}\langle\!\langle - \langle\!\langle}$*, all with target* $\mathbf{T}$*. Suppose also that there is a pairing* $\rho : {\color{blue}\langle\!\langle - \langle\!\langle} \otimes {\color{blue}\langle\!\langle - \rangle\!\rangle} \to \langle\!\langle - \odot - \rangle\!\rangle$*. Let* F *and* H *be right dualizable 1-cells in* $\mathscr{B}$*, and let* $\xi : Q \odot F \to F \odot P$*,* $\gamma : F \rhd M \to N \lhd F$*,* $\zeta : N \odot Q \odot F \odot H \to F \odot H \odot Z$*, and* $\delta : M \odot P \odot H \to H \odot Z$ *be 2-cells in* $\mathscr{B}$ *such that the following commutes:*

$$\begin{tikzcd}
F \odot (F \rhd M) \odot Q \odot F \odot H \arrow[r, "1\odot\gamma\odot 1^3"] \arrow[d, "1^2\odot\xi\odot 1"'] & F \odot (N \lhd F) \odot Q \odot F \odot H \arrow[d, "\mathrm{ev}\odot 1^3"] \\
F \odot (F \rhd M) \odot F \odot P \odot H \arrow[d, "1\odot\mathrm{ev}\odot 1^2"'] & N \odot Q \odot F \odot H \arrow[d, "\zeta"] \\
F \odot M \odot P \odot H \arrow[r, "1\odot\delta"'] & F \odot H \odot Z
\end{tikzcd}$$

Then the following commutes:

$$\begin{tikzcd}
& {\color{blue}\langle\!\langle M \langle\!\langle} \otimes {\color{blue}\langle\!\langle Q \rangle\!\rangle} \arrow[dl, "\mathrm{cotr}_F(\gamma)\otimes 1"'] \arrow[dr, "1\otimes \mathrm{tr}_F(\xi)"] & \\
{\color{blue}\langle\!\langle N \langle\!\langle} \otimes {\color{blue}\langle\!\langle Q \rangle\!\rangle} \arrow[d, "\rho"'] & & {\color{blue}\langle\!\langle M \langle\!\langle} \otimes {\color{blue}\langle\!\langle P \rangle\!\rangle} \arrow[d, "\rho"] \\
\langle\!\langle N \otimes Q \rangle\!\rangle \arrow[dr, "\mathrm{tr}_{F\odot H}(\zeta)"'] & & \langle\!\langle M \odot P \rangle\!\rangle \arrow[dl, "\mathrm{tr}_H(\delta)"] \\
& \langle\!\langle Z \rangle\!\rangle &
\end{tikzcd}$$

Proof. In the diagram in Figure 4.1, the left-hand side is $\mathrm{tr}_H(\delta) \circ \rho \circ (1 \otimes \mathrm{tr}_F(\xi))$ and the right-hand side is $\mathrm{tr}_{F\odot H}(\zeta) \circ \rho \circ (\mathrm{cotr}_F(\gamma) \otimes 1)$. We follow the convention discussed at the start of Section 3.2, omitting the symbol $\odot$. Every unlabeled square commutes because of naturality of ρ, θ, or ev or functoriality of $-\odot-$ or $-\otimes-$. $\square$

The situation in which Lipman's result arises is as follows. If we let Q, P, and H be unit 1-cells, and let $Z = M$, $\xi = \mathrm{id}_F$, and $\delta = \mathrm{id}_M$, then the hypothesis of Theorem 3 reduces to the following:

$$\begin{tikzcd}
F \odot (F \triangleright M) \odot F \arrow[r, "1\odot\gamma\odot 1"] \arrow[d, "1\odot\mathrm{ev}"'] & F \odot (N \triangleleft F) \odot F \arrow[d, "\mathrm{ev}\odot 1"] \\
F \odot M & N \odot F \arrow[l, "\zeta"]
\end{tikzcd}$$

For an example of a collection of objects and maps making this diagram commute, start with any 1-cells $F : S \nrightarrow R$ and $M : R \nrightarrow R$ (with F right dualizable), and let $N = F \triangleright (F \odot M)$. Then let ζ be $\mathrm{ev} : (F \triangleright (F \odot M)) \odot F \to F \odot M$ and let

$$\gamma : F \triangleright M \to (F \triangleright (F \odot M)) \triangleleft F$$

be the adjoint of $\mu : F \odot (F \triangleright M) \to F \triangleright (F \odot M)$. The hypothesis of Theorem 3 is satisfied since the following diagram commutes:

$$\begin{tikzcd}
F \odot (F \triangleright M) \odot F \arrow[r, "1\odot\overline{\mu}\odot 1"] \arrow[d, "1\odot\mathrm{ev}"'] \arrow[dr, "\mu\odot 1"] & F \odot ((F \triangleright (F \odot M)) \triangleleft F) \odot F \arrow[d, "\mathrm{ev}\odot 1"] \\
F \odot M & (F \triangleright (F \odot M)) \odot F \arrow[l, "\mathrm{ev}"']
\end{tikzcd}$$

Example 18. Start with a ring R, a finitely generated projective right R-module F, and an (R, R)-bimodule M. Using HH^0 and HH_0 as the (co)shadows, the pairing of Example 17, and the setup above ($N = F \triangleright (F \odot M)$, $\zeta = \mathrm{ev}$, $\gamma = \overline{\mu}$, $\zeta = \mathrm{id}_F$, $\delta = \mathrm{id}_M$), Theorem 3 recovers (4.1), which is [9, Proposition 4.5.4] in the case $n = 0$.

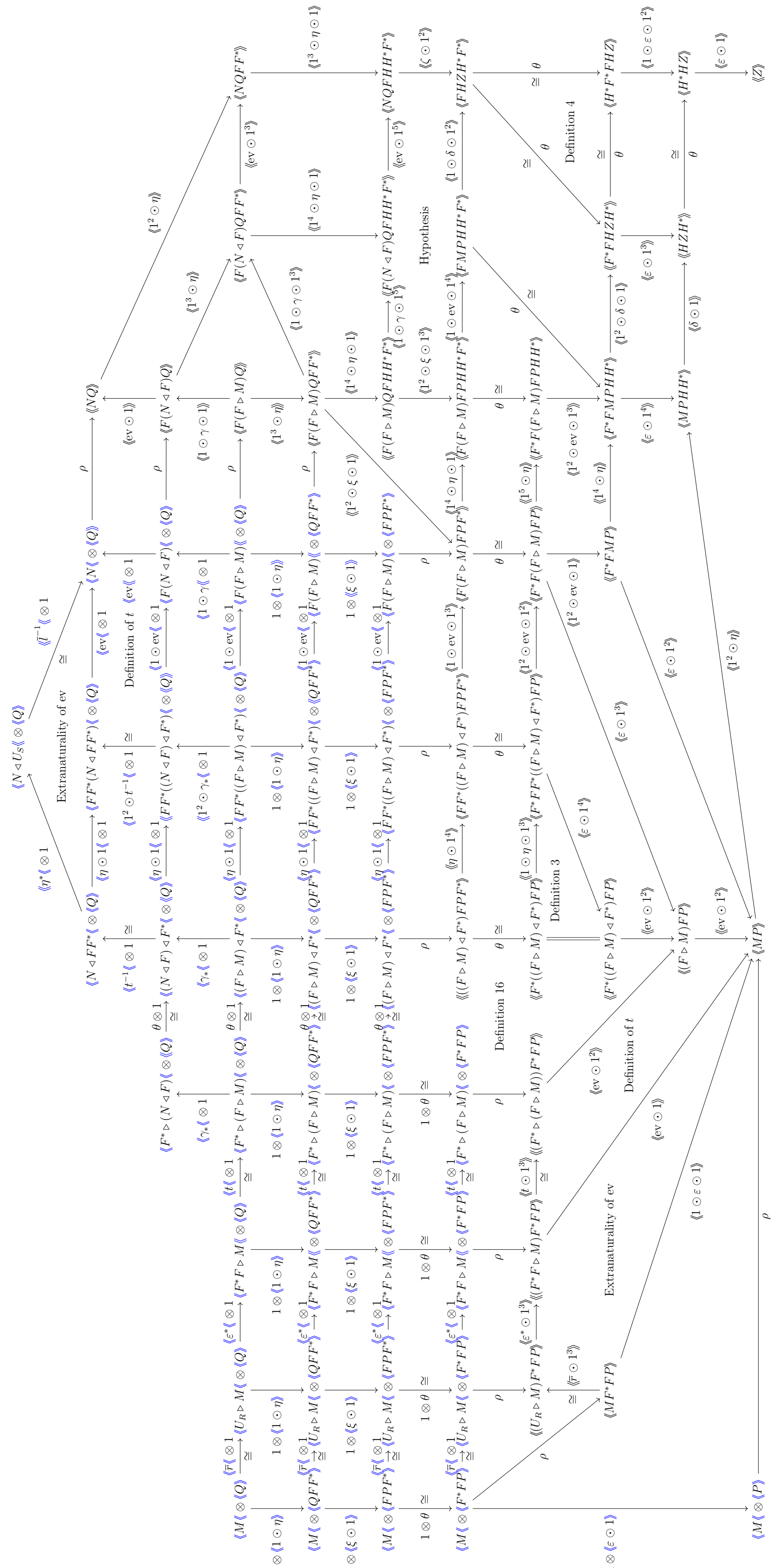

Figure 4.1: Diagram for Theorem 3

Chapter 5 Future Work

The program of bicategorical shadows and traces aims to unify seemingly disparate pieces of mathematics underneath a common conceptual framework. By adding notions of coshadow and cotrace to this machinery, we have drawn Lipman's residues and (co)traces into this framework and opened the road to incorporating Ganter and Kapranov's 2-characters in a way that parallels the application of traces to ordinary group representations. In this section, we give an overview of several promising directions for future work along these lines.

5.1 2-Representations and 2-Characters

With Travis Wheeler, we aim to describe Ganter's and Kapranov's categorical character as a bicategorical cotrace. Having done so, we expect to be able to deduce their main result, which generalizes the induction formula for characters, from formal properties of cotraces (though we suspect that this may actually require a theory of *tricategorical* coshadows and cotraces, rather than bicategorical traces and cotraces).

Conjecture 1. *The 2-character induction formula of [3] is an example of properties of the cotrace.*

A further outcome of identifying the right bicategorical setting for 2-representations is that we can compute traces of 2-representations, with the hope that these traces will recover the same 2-character which is an example of a cotrace:

Conjecture 2. *The 2-character can be computed with either a trace or a cotrace as a consequence of a version of Theorem 3.*

5.2 Understanding Coshadows

Hess and Rasekh established an equivalence between functors out of the Hochschild homology of a bicategory and shadows on that bicategory [6]. We plan to investigate whether there is an analogous equivalence between functors out of the Hochschild cohomology of a closed bicategory and coshadows on that bicategory.

5.3 Residues and Traces of Differential Forms

Lipman [9] studied residues and traces of differential forms, translating ideas from algebraic geometry into commutative algebra through the use of Hochschild (co)homology. Having reinterpreted Lipman's (co)traces as bicategorical (co)traces, We would like to reverse-engineer this translation in order to apply these trace methods to the original algebro-geometric situation.

5.4 Fixed Point Theory

Bicategorical traces have applications to the study of fixed points, such as the Lefschetz-Hopf theorem and analogues for the Reidemeister trace. We would like to investigate whether cotraces have similar applications to fixed point theory.

Bibliography

[1] J. A. Campbell and K. Ponto. Topological Hochschild homology and higher characteristics. *Algebr. Geom. Topol.*, 19(2):965–1017, 2019.

[2] A. Dold and D. Puppe. Duality, trace, and transfer. In *Proceedings of the International Conference on Geometric Topology (Warsaw, 1978)*, pages 81–102. PWN, Warsaw, 1980.

[3] N. Ganter and M. Kapranov. Representation and character theory in 2-categories. *Adv. Math.*, 217(5):2268–2300, 2008.

[4] R. Hartshorne. *Residues and duality*. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne.

[5] A. Hattori. Rank element of a projective module. *Nagoya Math. J.*, 25:113–120, 1965.

[6] K. Hess and N. Rasekh. Shadows are bicategorical traces, 2023.

[7] T. Leinster. Basic bicategories, 1998.

[8] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[9] J. Lipman. *Residues and traces of differential forms via Hochschild homology*, volume 61 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1987.

[10] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[11] K. Ponto. Fixed point theory and trace for bicategories. *Astérisque*, (333):xii+102, 2010.

[12] K. Ponto. Relative fixed point theory. *Algebr. Geom. Topol.*, 11(2):839–886, 2011.

[13] K. Ponto. Equivariant fixed-point theory, 2015.

[14] K. Ponto and M. Shulman. Shadows and traces in bicategories. *J. Homotopy Relat. Struct.*, 8(2):151–200, 2013.

[15] K. Ponto and M. Shulman. Traces in symmetric monoidal categories. *Expo. Math.*, 32(3):248–273, 2014.

[16] J. Stallings. Centerless groups—an algebraic formulation of Gottlieb's theorem. *Topology*, 4:129–134, 1965.

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