q-Polymatroids and their application to rank-metric codes.

Benjamin Jany
University of Kentucky, bja246@uky.edu
Author ORCID Identifier:
https://orcid.org/0000-0002-4273-5668
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Benjamin Jany, Student

Dr. Heide Gluesing-Luerssen, Major Professor

Dr. Benjamin Braun, Director of Graduate Studies

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DISSERTATION

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By
Benjamin Jany
Lexington, Kentucky

Director: Dr. Heide Gluesing-Luerssen, Professor of Mathematics
Lexington, Kentucky
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https://orcid.org/0000-0002-4273-5668

ABSTRACT OF DISSERTATION

q-Polymatroids and their application to rank-metric codes.

Matroid theory was first introduced to generalize the notion of linear independence. Since its introduction, the theory has found many applications in various areas of mathematics including coding theory. In recent years, q-matroids, the q-analogue of matroids, were reintroduced and found to be closely related to the theory of $\mathbb{F}_{q^m}$-linear rank metric codes. This relation was then generalized to q-polymatroids and $\mathbb{F}_q$-linear rank metric codes. This dissertation aims at developing the theory of q-(poly)matroid and its relation to the theory of rank metric codes.

In a first part, we recall and establish preliminary results for both q-polymatroids and q-matroids. We then describe how linear rank metric codes induce q-polymatroids and show how some invariants of rank-metric codes are fully determined by the induced q-polymatroid. Furthermore, we show that not all q-polymatroids arise from rank metric codes which gives rise to the class of non-representable q-polymatroids. We then define the notion of independent space for q-polymatroids and show that together with their rank values, those independents spaces fully determine the q-polymatroid.

Next, we restrict ourselves to the study of q-matroids. We start by studying the characteristic polynomial of q-matroids by relating it to the characteristic polynomial of the projectivazition matroid. We establish a deletion/contraction formula for the characteristic polynomial of q-matroids and prove a q-analogue of the Critical Theorem.

Afterwards, we study the direct-sum of q-matroids. We show the cyclic flats of the direct sum can be nicely characterized in terms of the cyclic flats of each summands. Using this characterization, we show all q-matroids can be uniquely decomposed (up to equivalence) into the direct sum of irreducible components. We furthermore show that unlike classical matroids, the direct sum of two representable q-matroids over some fixed field is not necessarily representable over that same field.

Finally we consider q-matroids from a category theory perspective to study the theoretical similarities and differences between classical matroids and q-matroids. We show the direct sum of q-matroids is a coproduct in only one of those categories which stands in contrast to categories of classical matroids. We conclude by showing the existence of a functor from categories of q-matroids to categories of matroids which provide an alternative method to study the former categories.

KEYWORDS: Rank-metric codes, Matroids, q-Matroids, q-Polymatroids

Benjamin Jany

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By
Benjamin Jany

Dr. Heide Gluesing-Luerssen
Director of Dissertation

Dr. Benjamin Braun
Director of Graduate Studies

April 29, 2023
Date

Dedicated to my parents Anne and Eric Jany.

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Chapter 1 Introduction

Rank-metric codes – originally introduced by Delsarte [18] and later independently re-discovered by Gabidulin [21] as well as Roth [43] – have been in the focus of algebraic coding theory throughout the last 15 years thanks to their suitability for communication networks. Their coding-theoretic properties have been studied in detail, and various constructions of optimal codes, such as MRD codes, have been found. For details we refer to the vast literature.

In this dissertation we focus on the algebraic and combinatorial aspects of rank-metric codes and study them with the aid of associated q-polymatroids. We will focus on linear rank-metric codes, that is, subspaces of some matrix space $\mathbb{F}_q^{n\times m}$, endowed with the rank metric. On various occasions we will consider $\mathbb{F}_{q^m}$-linear rank-metric codes, that is, codes that turn into $\mathbb{F}_{q^m}$-subspaces of $\mathbb{F}_{q^m}^n$ under a suitable identification of $\mathbb{F}_q^{n\times m}$ with $\mathbb{F}_{q^m}^n$. Not surprisingly, the algebraic and combinatorial properties of a rank-metric code depend on the 'degree of linearity'.

In [36] Jurrius/Pellikaan introduce q-matroids and show that $\mathbb{F}_{q^m}$-linear rank-metric codes give rise to q-matroids, thus providing a vast variety of examples of q-matroids. As the terminology indicates, q-matroids form the q-analogue of matroids: instead of subsets of a finite set one considers subspaces of a finite-dimensional vector space over a finite field. Furthermore, similarly to classical matroids there are various ways to define q-matroids, called cryptomorphisms. A large collection of cryptomorphisms is presented in [11]. In [11] (and the precursor [10]), Byrne and co-authors considerably extend the list of cryptomorphic definitions. As has been shown in [10, 11, 36], the theory of q-matroids nicely parallels the theory of matroids. It should be noted that q-matroids appeared already much earlier in the Ph.D. thesis [14] but remained unnoticed in the coding community until [36].

While $\mathbb{F}_{q^m}$-linear rank-metric codes give rise to q-matroids, this is not the case for $\mathbb{F}_q$-linear rank-metric codes. However, as shown by Gorla and co-authors in [29] as well as Shiromoto [44] and Ghorpade/Johnson [22], $\mathbb{F}_q$-linear rank-metric codes induce q-polymatroids. This means that the rank function attains rational values. As for classical polymatroids this seemingly slight generality in the rank function causes q-polymatroids to be much less rigid than q-matroids. An even further generalization appears in [7], where Britz and co-authors study q-demimatroids associated with rank-metric codes.

Since their reintroduction, much progress was made in the theory of q-(poly)matroids, see [9, 13, 26, 23, 24, 27, 25, 33, 34] amongst others. This dissertation aims to develop the combinatorial and algebraic theory of q-(poly)matroids and establish new relation with the theory of rank metric codes. The content of the dissertation is separated in five chapters as described below.

In Chapter 2, we introduce the notion of q-polymatroids and provide preliminary results regarding the combinatorial structure of those objects. We then recall how rank metric codes induce q-polymatroids and study their relation. We for example show that the generalized weights of a code are fully determined by the flats of the induced q-polymatroid and that deletion and contraction of q-polymatroids correspond to puncturing and shortening of rank-metric codes. Finally we consider the question of representability of q-polymatroids. That is, a q-polymatroid is representable if it is induced by a rank metric code. We will show several example of q-polymatroids that are not representable.

In Chapter 3, we introduce the notion of independent space of q-polymatroids and study their properties. We show that our notion non-trivially generalizes the notion of independent space for q-matroids. We furthermore prove the collection of independent spaces of a q-polymatroid satisfies the same axiomatic properties than the independent spaces of q-matroids. However, unlike the latter, we show the independent spaces of q-polymatroids fully determine the q-polymatroid, if the rank value of those spaces is also considered. This characterization provides a cryptomorphic definition for q-polymatroids. We conclude the chapter with a study of the spanning spaces of q-polymatroids and how their properties differ from those of spanning spaces of q-matroids.

In the remaining of the chapters, we restrict ourselves to the study of q-matroids. In Chapter 4 we study the relation between q-matroids and classical matroids via the intermediate of the projectivization matroid, introduced in [34]. The latter is a matroid defined on the projective space of the groundspace of the q-matroid together with a suitable rank function. A q-matroids and its projectivization matroid share a similar flat structure. Therefore we can use well-known results from classical matroid theory to study invariants of q-matroids that depend only on the flat structure such as the characteristic polynomial of q-matroids. We use this to establish a deletion/contraction formula for the characteristic polynomial of q-matroids and prove a q-analogue of the Critical Theorem. The latter establishes a further connection between $\mathbb{F}_{q^m}$-linear rank metric codes and q-matroids. In fact, it shows that given a rank metric code, one can determine the number of t-tuples of codewords with given support by evaluating the characteristic polynomial of a contraction of the q-matroid.

In Chapter 5, we investigate properties of the direct-sum of q-matroids, an operation between q-matroids established in [13]. We first study the rank function of the direct-sum and provide several equivalent ways to define it. Furthermore we show that the cyclic flats of the direct sum can be nicely characterized in terms of the cyclic flats of each summands. This in turn, allows us to show that every q-matroid can be uniquely decomposed (up to equivalence) into the direct sum of irreducible q-matroids. The latter are precisely the q-matroids that can't be written as the direct sum of two "smaller" q-matroids. We conclude the chapter by studying the representability of the direct sum. For classical matroids, it is known that the direct sum of two representable matroids over a fixed field is also representable over that same field. However, this does not hold true for q-matroids. We provide examples of this fact by using the notion of paving q-matroids.

Finally in Chapter 6 we introduce several types of maps between q-matroids, such as q-weak and q-strong maps. The former type of map preserves the rank structure of q-matroids whereas the latter preserves the flat structure. We use those maps to study q-matroids from a category theory approach. This approach allows to bring forward the similarities and differences between classical matroids and their q-analogue. We show a coproduct always exist in only one of the categories introduced. The category in question is that of q-matroids where the morphisms are linear q-weak maps where the coproduct is the the direct sum introduced in [13]. This holds in contrast to categories of matroids as a coproduct also exist when the maps between matroids are strong maps. Finally, we show the projectivization map is a functor from categories of q-matroids to categories of matroids. This functor provides an alternative approach to study maps between q-matroids. In fact it allows us to show an equivalent definition for q-strong maps and show that q-strong maps are also q-weak.

Notation: q is a prime power, $\mathbb{F} = \mathbb{F}_q$ is the finite field of order q. E is a finite dimensional vector space over $\mathbb{F}$ and $\mathcal{L}(E)$ is the collection of subspaces of E. A k-dimensional subspace is sometimes referred to as a k-space. The subspace generated by $x_1, \ldots, x_t \in E$ is denoted by $\langle x_1, \ldots x_t \rangle$. We write $U \leq V$ if U is a subspace of V and $U \lneq V$ if U is a proper subspace of V. Given $G \in \mathbb{F}^{n\times m}$, we use $\text{rowsp}(G), \text{colsp}(G)$ to denote the row space and column space of G respectively. Furthermore we use the abbreviation RREF for reduced row echelon form. The standard basis vectors of $\mathbb{F}^n$ are denoted by $e_1, \ldots, e_n$.

Let $[n] = \{1, \ldots, n\}$ and for some finite set S we denote its collection of subsets by 2^S. Finally q-matroid are denoted by script letter, i. e. $\mathcal{M}, \mathcal{N}$ whereas classical matroids are denoted by capital letters, i. e. M, N.

Chapter 2 An introduction to q-(poly)Matroids and their application.

In this chapter, we first introduce the notion q-polymatroids and q-matroids and establish some basic properties. We then introduce and study several classes of spaces of q-(poly)matroids. Afterwards, we proceed to relate the theory of q-polymatroids to the theory of rank-metric codes. We show how rank-metric codes induce q-polymatroids and how the two theory relate to one another. Finally we discuss the notion of representable q-polymatroids and show that not all q-polymatroids arise from rank-metric codes. Results from this chapter also appear in [24] and [27].

2.1 Preliminaries on q-Polymatroids.

q-Polymatroids can be seen as the q-analogue of polymatroids and as a generalization of q-matroids [36]. The following definition is from [29, Def. 4.1] of Gorla et al., with the sole difference that we require rank functions to assume rational values.

Definition 2.1.1. *A* q-polymatroid (q-PM) on E *is a pair* (E, ρ)*, where* $\rho : \mathcal{L}(E) \longrightarrow \mathbb{Q}_{\geq 0}$ *satisfies*

(R1) Dimension-Boundedness: $0 \leq \rho(V) \leq \dim V$ *for all* $V \in \mathcal{L}(E)$*;*

(R2) Monotonicity: $V \leq W \Longrightarrow \rho(V) \leq \rho(W)$ *for all* $V, W \in \mathcal{L}(E)$*;*

(R3) Submodularity: $\rho(V + W) + \rho(V \cap W) \leq \rho(V) + \rho(W)$ *for all* $V, W \in \mathcal{L}(E)$*.*

The function ρ *is called a* rank-function*, for all* $V \leq E$ *the value* $\rho(V)$ *is called the* rank *of* V *and the value* $\rho(E)$ *is called the* rank *of the q-PM.*

q-PM is a generalization of the notion of q-matroid as defined in [36]. In fact, the latter can be defined in the following way.

Definition 2.1.2. *A q-PM* $\mathcal{M} = (E, \rho)$ *for which* $\rho(V) \in \mathbb{N}_0$ *for all* $V \in \mathcal{L}(E)$ *is called a* q-matroid.

Because of the above, all notions introduced in this section apply to both q-matroids and q-polymatroids. Later on we will restrict ourselves to the study of q-matroids which have a more rigid structure then that of q-PM. The following additional notions for q-PMs will be useful.

Definition 2.1.3. *Let* $\mathcal{M} = (E, \rho)$ *be a q-PM. A number* $\mu \in \mathbb{Q}_{>0}$ *is a* denominator *of* ρ *(and* $\mathcal{M}$*) if* $\mu\rho(V) \in \mathbb{N}_0$ *for all* $V \in \mathcal{L}(E)$*. The smallest denominator is called the* principal denominator*. We declare* 1 *the principal denominator of the trivial q-PM.*

Let us relate our definition to the literature. First of all, a q-matroid in the sense of Jurrius/ Pellikaan [36] is exactly a q-matroid as defined above. Next, as already mentioned, our definition coincides with that in [29, Def. 4.1] by Gorla et al. except that our rank functions take rational values. As we will see in Section 2.3, this is indeed the case for the q-PMs induced by rank-metric codes. Finally, for any $r \in \mathbb{N}$ a (q, r)-polymatroid as in [44, Def. 2] by Shiromoto can be turned into a q-PM with denominator r by dividing the rank function by r. Conversely, given a q-PM (E, ρ) with denominator μ, then $(E, \mu\rho)$ is a $(q, \lceil\mu\rceil)$-polymatroid in the sense of [44]. As we will see in 3, denominators will play a crucial role in defining the independent spaces of q-PMs.

Remark 2.1.4. *(1) Every denominator μ of a q-PM (E, ρ) satisfies $\mu \geq 1$. Indeed, by (R1) $\rho(V) \leq 1$ for all 1-spaces V, and by (R3) ρ is the zero map if and only if $\rho(V) = 0$ for all 1-spaces V.*

(2) Let (E, ρ) be a non-trivial q-PM. For $V \in \mathcal{L}(E)$ write $\rho(V) = \alpha_V/\beta_V$ with α_V, $\beta_V \in \mathbb{N}$ relatively prime. Then the principal denominator is given by

$$\mu = \frac{\operatorname{lcm}\{\beta_V \mid V \in \mathcal{L}(E)\}}{\gcd\{\alpha_V \mid V \in \mathcal{L}(E)\}},$$

and $\mu\mathbb{N}$ is the set of all denominators of (E, ρ).

We now define the notion of loop and loopspace for q-polymatroids. They will be used throughout the dissertation.

Definition 2.1.5. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid. A 1-dimensional space $V \leq E$ is a* loop *of $\mathcal{M}$ if $\rho(V) = 0$.*

Lemma-Definition 2.1.6. *[36, Lemma 11] Let $\mathcal{M} = (E, \rho)$ be a q-matroid, $\{\langle v_1\rangle, \ldots, \langle v_s\rangle\}$ be the collection of loops of $\mathcal{M}$ and $L = \sum_{i=1}^{s} \langle v_i\rangle$. Then $\rho(L) = 0$. The space L the* loop space *of $\mathcal{M}$.*

The following q-matroids will occur throughout the dissertation. They can also be found at [36, Ex. 4]. One easily verifies that the map ρ is indeed a rank function.

Example 2.1.7. *Let E a vector space over $\mathbb{F}_q$ and $\dim E = n$. Fix $k \in [n]$ and define $\rho(V) = \min\{k, \dim V\}$ for $V \in \mathcal{L}(E)$. Then (E, ρ) is a q-matroid. It is called the* uniform q-matroid on E of rank k *and denoted by $\mathcal{U}_{k,n}(q)$, or $\mathcal{U}_k(E)$. Furthermore the q-matroid $\mathcal{U}_{0,n}(q)$ is called the* trivial q-matroid *and $\mathcal{U}_{n,n}(q)$ is called the* free *q-matroid.*

Some of the basic properties for q-matroids derived in [36, Sec. 3] hold true for q-PMs as well. We spell out the following ones, which we will need later on. The proofs are identical to the ones in [36, Prop. 6 and 7].

Proposition 2.1.8. *Let (E, ρ) be a q-PM.*

(a) Let $V, W \in \mathcal{L}(E)$. Suppose $\rho(V + \langle x\rangle) = \rho(V)$ for all $x \in W$. Then $\rho(V + W) = \rho(V)$.

(b) Let $V \in \mathcal{L}(E)$ and $X, Y \in \mathcal{L}(E)$ be 1-spaces such that $\rho(V) = \rho(V + X) = \rho(V + Y)$. Then $\rho(V + X + Y) = \rho(V)$.

The following notion of equivalence is from [29, Def. 4.4].

Definition 2.1.9. *Two q-PMs $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, are* equivalent*, denoted by $\mathcal{M}_1 \approx \mathcal{M}_2$, if there exists an $\mathbb{F}$-isomorphism $\alpha \in Hom_{\mathbb{F}}(E_1, E_2)$ such that $\rho_2(\alpha(V)) = \rho_1(V)$ for all $V \in \mathcal{L}(E_1)$.*

At this point we want to briefly discuss a more general notion of equivalence for q-PMs.

Remark 2.1.10. *Two q-PMs $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, are* scaling-equivalent *if there exists an $\mathbb{F}$-isomorphism $\alpha \in Hom_{\mathbb{F}}(E_1, E_2)$ and $a \in \mathbb{Q}_{>0}$ such that $\rho_2(\alpha(V)) = a\rho_1(V)$ for all $V \in \mathcal{L}(E_1)$. This notion makes sense for q-PMs because there exist non-trivial q-PMs that do not attain the upper bound in (R1) non-trivially. We briefly elaborate. Let us call a q-PM $\mathcal{M} = (E, \rho)$* exact *if there exists some nonzero space $\hat{V} \in \mathcal{L}$ such that $\rho(\hat{V}) = \dim \hat{V}$. Clearly, a non-trivial q-matroid is exact, but there exist non-trivial non-exact q-PMs; see Example 2.3.18 in the next section. It follows immediately from the submodularity in (R3) that a q-PM $\mathcal{M} = (E, \rho)$ is exact if and only if there exists a 1-space V such that $\rho(V) = 1$. This implies that any denominator of an exact q-PM is an integer. One can turn a non-exact q-PM (E, ρ) into an exact one using scaling-equivalence. Indeed, suppose $\rho(V) < \dim V$ for all $V \in \mathcal{L}(E) \setminus 0$. Let $a = \max\{\rho(V)/\dim V \mid V \in \mathcal{L}(E) \setminus 0\}$. Then $a \in \mathbb{Q}_{>0}$ and there exists $\hat{V} \in \mathcal{L}(E)$ such that $a = \rho(\hat{V})/\dim \hat{V}$. Thus $(E, a^{-1}\rho)$ is an exact q-PM.*

We now introduce the dual q-PM. It is a straightforward generalization of duality of matroids based on the rank function (see, e.g. [39, Prop. 2.1.9]), but requires more details when replacing set-theoretic complements by orthogonal spaces. Since we define q-PMs over arbitrary ground spaces, we need to specify a non-degenerate symmetric bilinear form, and, not surprisingly, the dual rank function depends on the choice of this form. But as we will see, different forms lead to equivalent dual q-PMs. This generality is needed in order to discuss deletions and contractions later on. Part of the following result is from [29, 4.5–4.7] (see also [36, Thm. 42] for q-matroids).

Theorem 2.1.11. *Let $\langle \cdot \,|\, \cdot \rangle$ be a non-degenerate symmetric bilinear form on E. For $V \in \mathcal{L}(E)$ define $V^\perp = \{w \in E \mid \langle v \,|\, w \rangle = 0$ for all $v \in V\}$. Let $\mathcal{M} = (E, \rho)$ be a q-PM and set*

$$\rho^*(V) = \dim V + \rho(V^\perp) - \rho(E). \tag{2.1}$$

Then ρ^* is a rank function on E and $\mathcal{M}^* = (E, \rho^*)$ is a q-PM. It is called the dual *of $\mathcal{M}$. Furthermore, $\mathcal{M}^{**} = \mathcal{M}$, where $\mathcal{M}^{**} = (\mathcal{M}^*)^*$ is the bidual, and $\mathcal{M}$ and $\mathcal{M}^*$ have the same set of denominators. Finally, the equivalence class of $\mathcal{M}^*$ does not depend on the choice of the non-degenerate symmetric bilinear form. More precisely, if $\langle\!\langle \cdot \,|\, \cdot \rangle\!\rangle$ is another non-degenerate symmetric bilinear form on E and $\mathcal{M}^{\hat{*}} = (E, \rho^{\hat{*}})$ is the resulting dual q-PM, then $\mathcal{M}^{\hat{*}} \approx \mathcal{M}^*$.*

Proof. The fact that ρ^* is a rank function and the identity $\rho^{**} = \rho$ have been proven in [29, Thms. 4.6, 4.7]. The statement about the denominators is obvious. It remains to show the very last statement. Thus, let $\langle\!\langle \cdot \,|\, \cdot \rangle\!\rangle$ be another non-degenerate symmetric bilinear form on E. For $V \in \mathcal{L}(E)$ denote by $V^\perp$ and $V^{\perp\!\!\perp}$ the orthogonal spaces of V with respect to $\langle\cdot\,|\,\cdot\rangle$ and $\langle\!\langle\cdot\,|\,\cdot\rangle\!\rangle$, respectively. Let $v_1, \ldots, v_\ell$ be a basis of E and $\psi : E \longrightarrow \mathbb{F}^\ell$ be the associated coordinate map. Let $Q = (\langle v_i\,|\,v_j\rangle)$, $\hat{Q} = (\langle\!\langle v_i\,|\,v_j\rangle\!\rangle) \in \mathbb{F}^{\ell\times\ell}$ be the Gram matrices associated to the bilinear forms. Then Q, $\hat{Q}$ are symmetric and nonsingular and we have $\langle v\,|\,w\rangle = \psi(v)Q\psi(w)^\mathsf{T}$ and $\langle\!\langle v\,|\,w\rangle\!\rangle = \psi(v)\hat{Q}\psi(w)^\mathsf{T}$ for all $v, w \in E$. Define the automorphism

$$\phi : E \longrightarrow E,\ v \longmapsto \psi^{-1}(\psi(v)\hat{Q}Q^{-1}). \tag{2.2}$$

Now we have for any $V \in \mathcal{L}(E)$ and $w \in E$

$$\begin{aligned} w \in \phi(V)^\perp &\iff \psi(\phi(v))Q\psi(w)^\mathsf{T} = 0 \text{ for all } v \in V \\ &\iff \psi(v)\hat{Q}Q^{-1}Q\psi(w)^\mathsf{T} = 0 \text{ for all } v \in V \\ &\iff w \in V^{\perp\!\!\perp}. \end{aligned}$$

Hence $V^{\perp\!\!\perp} = \phi(V)^\perp$ and thus $\rho^*(\phi(V)) = \rho^{\hat{*}}(V)$ for all $V \in \mathcal{L}(E)$. This shows that $\mathcal{M}^*$ and $\mathcal{M}^{\hat{*}}$ are equivalent. □

The next result has been proven in [29] for q-PMs on $\mathbb{F}^\ell$, endowed with the standard dot product. Thanks to the just proven invariance of the dual, it generalizes as follows without the need to specify bilinear forms.

Proposition 2.1.12 ([29, Prop. 4.7]). *Let $\mathcal{M} = (E, \rho)$ and $\hat{\mathcal{M}} = (\hat{E}, \hat{\rho})$ be q-PMs. Then $\mathcal{M} \approx \hat{\mathcal{M}}$ implies $\mathcal{M}^* \approx \hat{\mathcal{M}}^*$.*

Remark 2.1.13. *In the q-analogue the dual q-matroid depends on the choice of a NSBF (see Theorem 2.1.11) and therefore one needs to fix an NSBF in order to define certain notion related to the dual q-matroid. For example, in classical matroid theory, a single element set is a coloop of a matroid M if it is a loop in the dual matroid M^* (see 4.4 for more details). For q-matroids, the notion of a coloop is not well-defined, without fixing a choice of NSBF. Indeed, a 1-dimensional subspace $\langle x\rangle$ would be called a coloop of $\mathcal{M}$ if it is a loop of $\mathcal{M}^*$, that is, if $\rho^*(\langle x\rangle) = 0$. Hence we say a* coloop *of $\mathcal{M}$ w. r. t a choice of NSBF is a one-dimensional vector space $\langle x\rangle$ such that $\rho^*(\langle x\rangle) = 0$.*

Example 2.1.14 ([36, Ex. 47]). *It is easy to see that $\mathcal{U}_{k,n}(q)^* = \mathcal{U}_{n-k,n}(q)$.*

We now define deletion and contraction for q-PMs. We will see in 2.3 that deletion and contraction are closely related to shortening and puncturing of rank-metric codes.

Definition 2.1.15. *Let $\mathcal{M} = (E, \rho)$ be a q-PM and $X \in \mathcal{L}(E)$.*

(a) Define $\rho|_X : \mathcal{L}(X) \longrightarrow \mathbb{Q}_{\geq 0}$, $W \longmapsto \rho(W)$. Then $\mathcal{M}|_X := (X, \rho|_X)$ is a q-PM on X and is called the restriction *of $\mathcal{M}$ to X.*

(b) Fix a non-degenerate symmetric bilinear form $\langle\cdot\,|\,\cdot\rangle$ on E. The restriction of $\mathcal{M}$ to $X^\perp$ is called the deletion *of X from $\mathcal{M}$ w.r.t. $\langle\cdot\,|\,\cdot\rangle$, and the resulting q-PM $(\mathcal{M}|_{X^\perp}, \rho|_{X^\perp})$ is denoted by $(\mathcal{M} \setminus X)_{\langle\cdot|\cdot\rangle}$ or simply $\mathcal{M} \setminus X$ if the bilinear form is clear from the context.*

In (a), it is clear that $\rho|_X$ satisfies (R1)–(R3) from Definition 2.1.1 and thus is a rank function. In (b), the deletion $(\mathcal{M} \setminus X)_{\langle\cdot|\cdot\rangle}$ truly depends on $\langle\cdot\,|\,\cdot\rangle$: different choices of the bilinear form lead in general to non-equivalent q-PMs. However, for any bilinear forms $\langle\cdot\,|\,\cdot\rangle$ and $\langle\!\langle\cdot\,|\,\cdot\rangle\!\rangle$ on E with orthogonal spaces $V^\perp$ and $V^{\perp\!\!\!\perp}$, respectively, we obviously have $X^\perp = Y^{\perp\!\!\!\perp}$ for $Y = (X^\perp)^{\perp\!\!\!\perp}$ and thus $\mathcal{M}|_{X^\perp} = \mathcal{M}|_{Y^{\perp\!\!\!\perp}}$.

We now turn to contractions.

Theorem 2.1.16. *Let $\mathcal{M} = (E, \rho)$ be a q-PM and $X \in \mathcal{L}(E)$. Let $\pi : E \longrightarrow E/X$ be the canonical projection. We define the map*

$$\rho_{E/X} : \mathcal{L}(E/X) \longrightarrow \mathbb{Q}_{\geq 0}, \quad V \longmapsto \rho(\pi^{-1}(V)) - \rho(X).$$

Then $\mathcal{M}/X := (E/X, \rho_{E/X})$ is a q-PM, called the contraction *of X from $\mathcal{M}$.*

Proof. We have to show that $\rho_{E/X}$ satisfies (R1)–(R3).
(R1) Let $V \in \mathcal{L}(E/X)$. Then $X \leq \pi^{-1}(V)$ and thus the monotonicity of ρ implies $\rho_{E/X}(V) \geq 0$. Next, $\pi^{-1}(V) = X \oplus A$ for some $A \in \mathcal{L}(E)$. Then $\dim A = \dim V$ and submodularity of ρ yields $\rho_{E/X}(V) = \rho(\pi^{-1}(V)) - \rho(X) \leq \rho(A) \leq \dim A = \dim V$.
(R2) $V_1 \leq V_2 \leq E/X$ implies $\pi^{-1}(V_1) \leq \pi^{-1}(V_2)$ and therefore $\rho_{E/X}(V_1) \leq \rho_{E/X}(V_2)$.
(R3) Let $V, W \leq E/X$. Then $\pi^{-1}(V + W) = \pi^{-1}(V) + \pi^{-1}(W)$ and $\pi^{-1}(V \cap W) = \pi^{-1}(V) \cap \pi^{-1}(W)$ and therefore

$$\begin{aligned}\rho_{E/X}(V+W)+\rho_{E/X}(V\cap W) &= \rho\big(\pi^{-1}(V) + \pi^{-1}(W)\big) - \rho(X)\\ &\qquad + \rho\big(\pi^{-1}(V)\cap\pi^{-1}(W)\big) - \rho(X)\\ &\leq \rho(\pi^{-1}(V)) + \rho(\pi^{-1}(W)) - 2\rho(X)\\ &= \rho_{E/X}(V) + \rho_{E/X}(W).\end{aligned}$$

□

Remark 2.1.17. *Deletion and contraction of q-PM generalize the similar operations defined for q-matroids defined in [36]. In fact if $\mathcal{M} = (E, \rho)$ is a q-matroid it is easy to see by definition that for all $V \leq E$, both $\mathcal{M}/V$ and $\mathcal{M} \setminus V$ are q-matroids.*

We conclude this section by showing that deletion and contraction are mutually dual in the following sense. Since duality is involved, we need to pay special attention to the choice of the non-degenerate symmetric bilinear form. For $\dim X = 1$, the following result also appears in [36, Thm. 60]. However, the proof given there does not apply if $X \leq X^\perp$ and yields a weaker form of equivalence between the matroids. Also in [9, Lem. 12] the result below is proven for a weaker form of equivalence. Recall from Definition 2.1.9 that in this paper equivalence of q-PMs is based on linear isomorphisms between the ground spaces.

Theorem 2.1.18. *Let $\mathcal{M} = (E, \rho)$ be a q-PM and $X \in \mathcal{L}(E)$. Then*

$$(\mathcal{M} \setminus X)^* \approx \mathcal{M}^*/X \quad \text{and} \quad \mathcal{M} \setminus X \approx (\mathcal{M}^*/X)^*.$$

Proof. We will show the first equivalence. The second one follows from from biduality; see Theorem 2.1.11 and Proposition 2.1.12. We need some preparation. Recall from Theorem 2.1.11 that the dual of a q-PM depends on the choice of the non-degenerate symmetric bilinear form (NSBF) and that different choices lead to equivalent dual q-PMs. Hence for the first equivalence we need NSBFs on E and on $X^\perp$, the latter being the ground space of $\mathcal{M} \setminus X$. Note that if $\langle \cdot \,|\, \cdot \rangle$ is an NSBF on E, then the restriction of $\langle \cdot \,|\, \cdot \rangle$ to the resulting orthogonal $X^\perp$ is in general degenerate. For this reason we proceed as follows. Choose a subspace $Y \in \mathcal{L}(E)$ such that $X \oplus Y = E$. Choose NSBFs $\langle \cdot \,|\, \cdot \rangle_X$ on X and $\langle \cdot \,|\, \cdot \rangle_Y$ on Y and define

$$\langle x_1 + y_1 \,|\, x_2 + y_2 \rangle := \langle x_1 \,|\, x_2 \rangle_X + \langle y_1 \,|\, y_2 \rangle_Y \text{ for all } x_1, x_2 \in X,\, y_1, y_2 \in Y.$$

It is easy to verify that $\langle \cdot \,|\, \cdot \rangle$ is a NSBF on E. Denoting the resulting orthogonal of a subspace $V \in \mathcal{L}(E)$ by $V^\perp$, we observe $X^\perp = Y$ and thus $X \oplus X^\perp = E$. Furthermore, for any subspace $Z \leq X^\perp$ we have

$$Z^{\perp\!\!\perp} = Z^\perp \cap X^\perp, \tag{2.3}$$

where $Z^{\perp\!\!\perp}$ denotes the orthogonal of Z in $X^\perp$ w.r.t. $\langle \cdot \,|\, \cdot \rangle_Y$. Now we have compatible NSBFs on E and $X^\perp$ and can turn to the stated equivalence $(\mathcal{M} \setminus X)^* \approx \mathcal{M}^*/X$.

Note that we have a well-defined isomorphism

$$\xi : E/X \longrightarrow X^\perp, \quad v + X \longmapsto \hat{x},$$

where $v = x + \hat{x}$ is the unique decomposition of v into $x \in X$ and $\hat{x} \in X^\perp$. We show that

$$(\rho^*)_{E/X}(V) = (\rho|_{X^\perp})^*(\xi(V)) \text{ for all } V \in \mathcal{L}(E/X). \tag{2.4}$$

Let $\pi : E \longrightarrow E/X$ be the canonical projection and $V \in \mathcal{L}(E/X)$. Then $\pi^{-1}(V) = \xi(V) \oplus X$ and thus $\pi^{-1}(V)^\perp = \xi(V)^\perp \cap X^\perp$. Now we compute

$$\begin{aligned}
(\rho^*)_{E/X}(V) &= \rho^*(\pi^{-1}(V)) - \rho^*(X) \\
&= \dim \pi^{-1}(V) + \rho(\pi^{-1}(V)^\perp) - \rho(E) - \big(\dim X + \rho(X^\perp) - \rho(E)\big) \\
&= \dim \pi^{-1}(V) - \dim X + \rho(\pi^{-1}(V)^\perp) - \rho(X^\perp) \\
&= \dim \xi(V) + \rho(\xi(V)^\perp \cap X^\perp) - \rho(X^\perp) \\
&= \dim \xi(V) + \rho|_{X^\perp}(\xi(V)^\perp \cap X^\perp) - \rho|_{X^\perp}(X^\perp) \\
&= \dim \xi(V) + \rho|_{X^\perp}(\xi(V)^{\perp\!\!\perp}) - \rho|_{X^\perp}(X^\perp) \\
&= (\rho|_{X^\perp})^*(\xi(V)),
\end{aligned}$$

where the penultimate step follows from (2.3) and the last step is the very definition of $(\rho|_{X^\perp})^*(\xi(V))$ for the chosen NSBF on $X^\perp$. □

2.2 More properties of q-(poly)matroids.

In this section, we consider different classes of spaces for both q-matroids and q-polymatroids and study their properties. Many of the results from this section are generalizations of results from classical matroid theory, (see for example [39]), and serve as the groundwork results that will be used throughout the dissertation. We start by defining the closure operator and flats for q-polymatroids. We then restrict ourselves to q-matroids and define several other types of spaces. The reason for restricting ourselves to q-matroids in the latter part of this section, is that most of the spaces we define have either no existing generalization to q-polymatroids or generalize non-trivially (as we will see in Chapter 3). For flow of text, we delay examples of q-polymatroids to the next section and will predominantly use the uniform q-matroids as examples for this section. Throughout the section, some of the definitions and results hold only for q-matroids. The reader should therefore pay close attention on the objects considered in each statement.

2.2.1 The closure operator and flats of q-polymatroids.

We start with by defining the notion of flat and closure for q-polymatroids. As in (classical) matroid theory, a flat is, by definition, an inclusion-maximal space for a given rank. Flats naturally come with a closure operator. We derive some basis properties of the closure operator and flats and will illustrate that – just like for classical polymatroids – the lattice of flats is not semimodular and (without the associated rank values) does not fully determine the q-PM. In fact, a q-PM may have the same flats as a q-matroid without being a q-matroid itself. If otherwise specified, throughout the section let $\mathcal{M} = (E, \rho)$ be a q-PM.

Definition 2.2.1. *A space $F \in \mathcal{L}(E)$ is called a* flat *of $\mathcal{M}$ if*

$$\rho(F + \langle x \rangle) > \rho(F) \textit{ for all } x \in E \setminus F.$$

We denote the collection of flats by $\mathcal{F}(\mathcal{M})$, $\mathcal{F}_{\mathcal{M}}$, or simply $\mathcal{F}$. A flat H is called a hyperplane *if there is no flat strictly between H and E. Furthermore, we define the* closure operator *of $\mathcal{M}$ as*

$$\mathrm{cl} : \mathcal{L}(E) \longrightarrow \mathcal{L}(E), \quad V \longmapsto \sum_{\substack{\dim X = 1 \\ \rho(V+X)=\rho(V)}} X.$$

Clearly, $E \in \mathcal{F}(\mathcal{M})$, and $0 \in \mathcal{F}(\mathcal{M})$ if and only if $\mathcal{M}$ has no loops. With the aid of Proposition 2.1.8(a) we obtain immediately

$$\rho(V) = \rho(\mathrm{cl}(V)) \text{ for all } V \in \mathcal{L}(E). \tag{2.5}$$

Flats can be regarded as 'rank-closed' subspaces.

Proposition 2.2.2. *A subspace $F \in \mathcal{L}(E)$ is a flat of $\mathcal{M}$ if and only if $F = \mathrm{cl}(F)$.*

Proof. If $F = \mathrm{cl}(F)$ and $x \notin F$, then $\rho(F+\langle x \rangle) > \rho(F)$. Hence F is a flat. Conversely, let F be a flat and let $x \in \mathrm{cl}(F)$. Then $\rho(F + \langle x \rangle) = \rho(F)$, and thus $x \in F$. Hence $F = \mathrm{cl}(F)$. □

The closure operator satisfies the following properties. The last statement below has been proven in [36, Thm. 68].

Theorem 2.2.3. *Let $\mathcal{M} = (E, \rho)$ be a q-polymatroid and $V, W \in \mathcal{L}(E)$. Then*

(CL1) $V \leq \mathrm{cl}(V)$.

(CL2) If $V \leq W$, *then* $\mathrm{cl}(V) \leq \mathrm{cl}(W)$.

(CL3) $\mathrm{cl}(V) = \mathrm{cl}(\mathrm{cl}(V))$.

Furthermore, if $\mathcal{M}$ is a q-matroid then the following also holds:

(CL4) MacLane-Steinitz exchange axiom: For all $V \in \mathcal{L}(E)$ and all vectors $x, y \in E \setminus \mathrm{cl}(V)$ we have $\langle y \rangle \leq \mathrm{cl}(V + \langle x \rangle) \Longleftrightarrow \langle x \rangle \leq \mathrm{cl}(V + \langle y \rangle)$.

Proof. (CL1) is obvious.
(CL2) Let $V \leq W$ and let $x \in \mathrm{cl}(V)$. We want to show that $x \in \mathrm{cl}(W)$. This is clear if $x \in V$, and thus we assume that $x \notin V$. Choose $U \in \mathcal{L}(E)$ such that $W = V \oplus U$. Then

$$\begin{aligned}
\rho(W + \langle x \rangle) &= \rho((V \oplus U) + (V \oplus \langle x \rangle)) \\
&\leq \rho(V \oplus U) + \rho(V \oplus \langle x \rangle) - \rho((V \oplus U) \cap (V \oplus \langle x \rangle)) \\
&= \rho(W) + \rho(V) - \rho((V \oplus U) \cap (V \oplus \langle x \rangle)) \\
&= \rho(W) + \rho(V) - \rho(V) = \rho(W),
\end{aligned}$$

where the penultimate step follows from $V \leq (V \oplus U) \cap (V \oplus \langle x \rangle) \leq V \oplus \langle x \rangle$ together with $\rho(V \oplus \langle x \rangle) = \rho(V)$. All of this shows that $\rho(W + \langle x \rangle) = \rho(W)$ and therefore $x \in \mathrm{cl}(W)$.
(CL3) Let $V \in \mathcal{L}(E)$. Then $\mathrm{cl}(V) \leq \mathrm{cl}(\mathrm{cl}(V))$ follows from (CL1) since $\mathrm{cl}(V)$ is a subspace. For the converse let $x \in \mathrm{cl}(\mathrm{cl}(V))$. With the aid of (CL1) and (2.5) we obtain

$$\rho(V) \leq \rho(V + \langle x \rangle) \leq \rho(\mathrm{cl}(V) + \langle x \rangle) = \rho(\mathrm{cl}(V)) = \rho(V).$$

Thus we have equality across, and this shows that $x \in \mathrm{cl}(V)$.The proof of the last statement can be found in [36, Thm. 68] □

Flats satisfy the following simple properties. The last statement statement was shown in [10, Prop. 26, Thm. 28, Cor. 29, Thm. 31] and [11, Thm. 48]

Theorem 2.2.4. *Let $\mathcal{F}$ be the collection of flats of $\mathcal{M} = (E, \rho_{\mathcal{M}})$. Then*

(F1) $E \in \mathcal{F}$.

(F2) If $F_1, F_2 \in \mathcal{F}$, *then* $F_1 \cap F_2 \in \mathcal{F}$.

Furthermore, if $\mathcal{M}$ is a q-matroid then $\mathcal{F}$ also satisfies:

(F3) For all $F \in \mathcal{F}$ and vectors $x \in E \setminus F$ there exists a unique cover $F' \in \mathcal{F}$ of F such that $\langle x \rangle \leq F'$.

Furthermore a collection of spaces $\mathcal{F} \subseteq \mathcal{L}(E)$ satisfies (F1)-(F3) if and only if there exist a q-matroid $\mathcal{M} = (E, \rho)$ such that $\mathcal{F}_{\mathcal{M}} = \mathcal{F}$.

Proof. (F1) is clear from Definition 2.2.1. For (F2) let $F_1, F_2 \in \mathcal{F}$. Then (CL2) yields $\mathrm{cl}(F_1 \cap F_2) \subseteq \mathrm{cl}(F_1) \cap \mathrm{cl}(F_2) = F_1 \cap F_2$. Thus $F_1 \cap F_2 \in \mathcal{F}$ thanks to Proposition 2.2.2.The proof of the last statements can be found in [10, Prop. 26, Thm. 28, Cor. 29, Thm. 31], [11, Thm. 48]. □

From (CL2), Proposition 2.2.2, and (F2) we immediately obtain

Proposition 2.2.5. *Any $V \in \mathcal{L}(E)$ satisfies* $\mathrm{cl}(V) = \bigcap_{\substack{F \in \mathcal{F} \\ V \leq F}} F$.

In both Theorem 2.2.3 and Theorem 2.2.4, it is important to mention that in general, q-PM do not satisfy properties (CL4) and (F3). An example of such a q-PM is given later on in Example 2.3.17. Note, moreover, that for any q-PM $\mathcal{M}$ the collection $\mathcal{F}$ of flats forms a lattice under inclusion: the *meet* and *join* of two flats F_1, $F_2 \in \mathcal{F}$ are $F_1 \wedge F_2 := F_1 \cap F_2$ and $F_1 \vee F_2 := \mathrm{cl}(F_1 + F_2)$, respectively. We call this lattice the *lattice of flats* of a q-PM. The flat $\mathrm{cl}(0) = \bigcap_{F \in \mathcal{F}} F$ is the unique minimal element of the lattice $\mathcal{F}$. Furthermore, we say that F_2 *covers* F_1 if $F_1 \lneq F_2$ and there exists no $F \in \mathcal{F}$ such that $F_1 \lneq F \lneq F_2$. For instance, the hyperplanes of $\mathcal{F}$ are the flats covered by E. Furthermore given a lattice $\mathcal{F}$, a *chain of length t* is a sequence of elements $F_0 \lneq \cdots \lneq F_t$, where $F_i \in \mathcal{F}$ and such that F_i covers F_{i-1} for all $1 \leq i \leq t$. We let $h_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathbb{Z}_0$ where $h_{\mathcal{F}}(F)$ is the length of a maximal chain from $\mathrm{cl}(0)$ to F, and call $h_{\mathcal{F}}$ the *height function* of the lattice.

For q-matroids, the lattice of flats is a particularly well structured lattice. In fact, it was shown as was shown in [34] that the lattice of flats is a geometric lattice. We recall a geometric lattice is a finite, semimodular and atomistic lattice. Because none of the latter terms will be needed in this dissertation and for conciseness purposes we refer the reader to [39] for a more detail description of geometric lattices.

Theorem 2.2.6. *[11, 34, Thm. 48 /Prop. 21] Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{F}$ its lattice of flats. Then $\mathcal{F}$ is a geometric lattice. Furthermore for all $F \in \mathcal{F}$,*

$$\rho(F) = h_{\mathcal{F}}(F),$$

where $h_{\mathcal{F}}$ denotes the height function of $\mathcal{F}$.

Proof. The first statement was proven in [34, Prop. 2.1] whereas the second statement was proven in [11, Thm. 48]. □

The above result is in general not true for q-polymatroids. It will furthermore become especially useful in Chapter 4.

2.2.2 Independent spaces, basis and circuits of q-matroids.

We start by introducing the notion of independent spaces, basis and circuits of a q-matroid.

Definition 2.2.7. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid:*

- *A space $I \in \mathcal{L}(E)$ is* independent *if $\rho(I) = \dim I$. We denote the collection of independent spaces of $\mathcal{M}$ by $\mathcal{I}_{\mathcal{M}}$ or $\mathcal{I}(\mathcal{M})$.*

- *A space $B \in \mathcal{L}(E)$ is a* basis *if $\rho(B) = \dim(B) = \rho(E)$. We denote the collection of basis of $\mathcal{M}$ by $\mathcal{B}_{\mathcal{M}}$ or $\mathcal{B}(\mathcal{M})$.*

- *A space $C \in \mathcal{L}(E)$ is a* circuit *if C is dependent and all its proper subspaces are independent. We denote the collection of circuits of $\mathcal{M}$ by $\mathcal{C}_{\mathcal{M}}$ or $\mathcal{C}(\mathcal{M})$.*

Similarly to classical matroids, one can fully determine a q-matroid via any collection of spaces defined previously (see [11]). As we will see in Chapter 3, the above notions can be non-trivially generalized to q-polymatroids as well. However not all the properties presented in this section hold for q-polymatroids. Consider the following example.

Example 2.2.8. *Let $\dim E = n$, $0 \leq k \leq n$, and consider the uniform q-matroid $\mathcal{U}_{k,n}(q)$. Using the rank function of $\mathcal{U}_{k,n}(q)$ its easy to show that*

$$\begin{aligned}
\mathcal{I}_{\mathcal{U}_{k,n}(q)} &= \{I \leq E \,:\, \dim I \leq k\} \\
\mathcal{B}_{\mathcal{U}_{k,n}(q)} &= \{B \leq E \,:\, \dim B = k\} \\
\mathcal{C}_{\mathcal{U}_{k,n}(q)} &= \{C \leq E \,:\, \dim C = k+1\}
\end{aligned}$$

We now summarize some properties regarding those different spaces. First we introduce the following notation.

Definition 2.2.9. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{I}_{\mathcal{M}}$ its collection of independent spaces. For $V \leq E$ define*

$$\mathcal{I}(V) := \{I \leq V \,:\, I \in \mathcal{I}_{\mathcal{M}}\},$$

$$\mathcal{B}(V) := \{I \leq V \,:\, I \in \mathcal{I}_{\mathcal{M}} \text{ and if } I \lneq W \leq V \text{ then } W \notin \mathcal{I}\}.$$

The set $\mathcal{I}(V)$ are independent spaces of $\mathcal{M}$ contained in V and $\mathcal{B}(V)$ are the bases *of V, i.e. the maximal independent spaces of V w.r.t. inclusion.*

We start with the properties of independent spaces that allow to define a cryptomorphism of q-matroids.

Theorem 2.2.10. *[36, Thm 8] Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{I}_{\mathcal{M}}$ its collection of independent spaces. Then $\mathcal{I}$ satisfies:*

(I1) $\mathcal{I}_{\mathcal{M}} \neq \emptyset$.

(I2) If $I \leq J$ and $J \in \mathcal{I}_{\mathcal{M}}$ then $I \in \mathcal{I}_{\mathcal{M}}$.

(I3) If $I, J \in \mathcal{I}_{\mathcal{M}}$ and $\dim I < \dim J$ then there exist $x \in J \setminus I$ such that $I \oplus \langle x \rangle \in \mathcal{I}$.

(I4) Let $V, W \leq E$, $I \in \mathcal{B}(V)$ and $J \in \mathcal{B}(W)$. Then there exist $K \in \mathcal{B}(V + W)$ such that $K \leq I + J$.

Furthermore,

$$\rho(V) = \max\{\dim(I) \,:\, I \in \mathcal{I}_{\mathcal{M}}(V)\} \tag{2.6}$$

for all $V \leq E$. Finally if a collection of spaces $\mathcal{I} \subseteq \mathcal{L}(E)$ satisfies (I1)-(I4) then $\mathcal{M}' = (E, \rho')$, where $\rho'(V) = \max\{\dim(I) \,:\, I \in \mathcal{I}(V)\}$ for all $V \leq E$, is a q-matroid such that $\mathcal{I}_{\mathcal{M}'} = \mathcal{I}$.

Finally for circuits we have the following result for which the proof follows directly from (2.6).

Proposition 2.2.11. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{C}_{\mathcal{M}}$ its collection of circuits. Then for all $C \in \mathcal{C}_{\mathcal{M}}$ we have*

$$\rho(C) = \dim C - 1.$$

Of course basis and circuits satisfy many more properties which will not be used in this dissertation. We therefore refer to [11] or [36] for the interested reader.

2.2.3 Open spaces and cyclic cores of q-matroids.

We define the cyclic core of a subspace. Our definition (2.2.16) is a vector-based q-analogue of the classical cyclic core in [20, p. 395]. More precisely, our cyclic core consists of vectors in the ground space E (rather than subspaces) that behave in the desired way with respect to hyperplanes, and it is not hard to see that the so defined set is indeed a subspace. Making use of duality and the properties of the closure operator, we show that the cyclic core of a subspace V is the largest open space contained in V. This also shows immediately that the cyclic-core operator is the dual of the closure operator (Corollary 2.2.21) and idempotent (Corollary 2.2.22). Subspaces that coincide with their cyclic core are called cyclic spaces, and it turns out that these are exactly the open spaces. A theory of the cyclic core was first proposed in [2]. Here we develop an alternative approach, which is well suited for our study of the direct sum of q-matroidsin subsequent sections. We briefly describe the differences and similarities.

Remark 2.2.12. *The main difference between [2] and our work is the starting point for developing the theory. In [2], the authors start with the definition of cyclic spaces. This can easily be done without the need of a cyclic-core operator and is identical to what we call cyclic spaces. Furthermore, the authors define two concepts that may be regarded as q-analogues of the classical cyclic core: Cyc(V) is defined as a collection of certain 1-dimensional subspaces ([2, Def. 2.3]) and the cyclic-core operator cyc(V) is the sum of the cyclic subspaces contained in V ([2, Def. 2.8]). The relation between these two concepts is clarified in [2, Prop. 2.18], which then also shows that our cyclic-core operator coincides with the one in [2]. Due to the different routes taken, the authors in [2, Sec. 2.1] establish properties of cyclic spaces in order to study their cyclic-core operator, while we establish properties of our cyclic-core operator and obtain those for cyclic spaces as a consequence. Thus, while the results look almost identical, they arise in different order and their proofs differ in each single case.*

Notation 2.2.13. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{M}^* = (E, \rho^*)$ be its dual with respect to $\langle\cdot|\cdot\rangle$ We denote by* $\mathrm{cl}(\,\cdot\,)$ *and* $\mathrm{cl}^*(\,\cdot\,)$ *the closure operators of $\mathcal{M}$ and $\mathcal{M}^*$, respectively.*

Before getting to the notion of cyclic core, we introduce the notion of open space of q-matroids, defined in [11].

Definition 2.2.14. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid. A space $O \in \mathcal{L}(E)$ is* open *if O is a sum of circuits of $\mathcal{M}$. We denote the collection of open spaces of $\mathcal{M}$ by $\mathcal{O}_{\mathcal{M}}$ or $\mathcal{O}(\mathcal{M})$.*

We now list some fundamental properties of q-matroids. As one can see from those properties, open spaces and flats are dual notion, whereas circuits are dual to hyperplanes. Part (a) and (b) are in [11, Cor. 86, Cor. 79], and (c) is a consequence of (a) and (b) together with Definition 2.2.7 Part (d) follow from Definition 2.2.14.

Theorem 2.2.15. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{M}^* = (E, \rho^*)$ be its dual with respect to some NSBF $\langle\cdot|\cdot\rangle$. Let $V \in \mathcal{L}(E)$. Then*
(a) $V \in \mathcal{O}(\mathcal{M}) \iff V^\perp \in \mathcal{F}(\mathcal{M}^*)$.
(b) $V \in \mathcal{C}(\mathcal{M}) \iff V^\perp \in \mathcal{H}(\mathcal{M}^*)$.
(c) If $V \in \mathcal{F}(\mathcal{M})$, then $V = \bigcap_{\substack{F \leq H \\ H \in \mathcal{H}(\mathcal{M})}} H$.
(d) $O_1, O_2 \in \mathcal{O}(\mathcal{M}) \Longrightarrow O_1 + O_2 \in \mathcal{O}(\mathcal{M})$

With the above properties established, we now turn towards the notion of cyclic core.

Definition 2.2.16. *For $V \in \mathcal{L}(E)$ we define the* cyclic core *of V as*

$$\mathrm{cyc}(V) = \{x \in V \mid \rho(W) = \rho(V) \text{ for all } W \leq V \text{ such that } W + \langle x\rangle = V\}.$$

Note that every subspace W appearing in the definition has codimension at most 1 in V. In the context of cyclic cores, we use the following notation.

Notation 2.2.17. *For $V \leq E$ define*

$$\mathrm{Hyp}(V) = \{W \leq V \,:\, \dim W = \dim V - 1\}.$$

Clearly, if V is independent then $\mathrm{cyc}(V) = 0$. The converse is true as well as we will see in Theorem 2.2.20.

Proposition 2.2.18. *Let $V \in \mathcal{L}(E)$. Then $\mathrm{cyc}(V) \in \mathcal{L}(E)$, i.e., $\mathrm{cyc}(V)$ is a subspace of E.*

Proof. Clearly $0 \in \mathrm{cyc}(V)$ and $\mathrm{cyc}(V)$ is closed under scalar multiplication. Let $x, y \in \mathrm{cyc}(V)$ and let $W \leq V$ be such that $W + \langle x + y\rangle = V$. We want to show that $\rho(W) = \rho(V)$. Without loss of generality let $W \neq V$. Then $\dim W = \dim V - 1$ and $x + y \notin W$. Thus we may assume that $x \notin W$. Hence $W + \langle x\rangle = V$, and using that x is in $\mathrm{cyc}(V)$ we conclude $\rho(W) = \rho(V)$. $\square$

We will show below that $\mathrm{cyc}(V)$ is the largest open space contained in V. We will do so by making use of duality and the closure operator.

Lemma 2.2.19. *Let $V \in \mathcal{L}(E)$ and $C_1, \ldots, C_t \in \mathcal{C}(\mathcal{M})$ be the circuits of $\mathcal{M}$ contained in V. Then*

$$\mathrm{cl}^*(V^\perp) = \bigcap_{i=1}^{t} C_i^\perp.$$

Proof. Since $\mathrm{cl}^*(V^\perp)$ is a flat in $\mathcal{M}^*$, Theorem 2.2.15 implies

$$\mathrm{cl}^*(V^\perp) = \bigcap_{\substack{\mathrm{cl}^*(V^\perp) \leq H \\ H \in \mathcal{H}(\mathcal{M}^*)}} H = \bigcap_{\substack{\mathrm{cl}^*(V^\perp) \leq C^\perp \\ C \in \mathcal{C}(\mathcal{M})}} C^\perp = \bigcap_{\substack{C \leq V \\ C \in \mathcal{C}(\mathcal{M})}} C^\perp,$$

where the last step follows from the fact that $C^\perp$ is a flat of $\mathcal{M}^*$, which implies the equivalences $\mathrm{cl}^*(V^\perp) \leq C^\perp \iff V^\perp \leq C^\perp \iff C \leq V$. □

Now we obtain the following description of the cyclic core. Using either the theory developed in [2] or ours, it is easy to verify that it agrees with the one defined in [2, Def. 2.8].

Theorem 2.2.20. *For $V \in \mathcal{L}(E)$ we have*

$$\mathrm{cyc}(V) = \sum_{\substack{C \leq V \\ C \in \mathcal{C}(\mathcal{M})}} C.$$

Thus $\mathrm{cyc}(V)$ is an open space and in fact the largest open space contained in V. As a consequence, V is independent if and only if $\mathrm{cyc}(V) = 0$.

Proof. Let $C_1, \ldots, C_t \in \mathcal{C}(\mathcal{M})$ be the circuits of $\mathcal{M}$ contained in V. We have to show that $\mathrm{cyc}(V) = \sum_{i=1}^t C_i$. From Lemma 2.2.19 we know that $\mathrm{cl}^*(V^\perp) = \cap_{i=1}^t C_i^\perp$, and thus (2.5) implies

$$\rho^*(V^\perp) = \rho^*\Big(\bigcap_{i=1}^{t} C_i^\perp\Big). \tag{2.7}$$

We now turn to the stated identity.
"$\subseteq$" Let $x \in \mathrm{cyc}(V)$. Let $W \leq V$ be such that $\dim W = \dim V - 1$ and $W + \langle x \rangle = V$. Then $\rho(W) = \rho(V)$ and thus

$$\rho^*(W^\perp) = \dim W^\perp + \rho(W) - \rho(E) = \dim(V^\perp) + 1 + \rho(V) - \rho(E) = \rho^*(V^\perp) + 1.$$

Thus $\rho^*(W^\perp) > \rho^*\Big(\bigcap_{i=1}^t C_i^\perp\Big)$, which in turn implies that $W^\perp$ is not a subspace of $\bigcap_{i=1}^t C_i^\perp$. Hence $\sum_{i=1}^t C_i \not\leq W$. Since this is true for every $W \in \mathrm{Hyp}(V)$ not containing x, we conclude that $x \in \sum_{i=1}^t C_i$.
"$\supseteq$" It suffices to show that each C_i is in $\mathrm{cyc}(V)$. Let $x \in C_i$. We show that $\rho(W) = \rho(V)$ for all $W \in \mathrm{Hyp}(V)$ such that $W + \langle x \rangle = V$. Choose such a subspace W. Then clearly $x \notin W$ and thus $C_i \not\leq W$. Using the containment $C_i \leq V$ we obtain

$\mathrm{cl}^*(V^\perp) \leq \mathrm{cl}^*(C_i^\perp) = C_i^\perp$ and $\mathrm{cl}^*(W^\perp) \not\leq C_i^\perp$. This implies $\mathrm{cl}^*(V^\perp) \lneq \mathrm{cl}^*(W^\perp)$ and thus

$$\rho^*(V^\perp) = \rho^*(\mathrm{cl}^*(V^\perp)) < \rho^*(\mathrm{cl}^*(W^\perp)) = \rho^*(W^\perp).$$

Since $\dim V^\perp + 1 = \dim W^\perp$, submodularity of ρ^* yields $\rho^*(W^\perp) \leq \rho^*(V^\perp) + 1$. All of this leads to $\rho^*(W^\perp) = \rho^*(V^\perp) + 1$. Now we compute

$$\begin{aligned}\rho(V) &= \dim V + \rho^*(V^\perp) - \rho^*(E) = \dim V + \rho^*(W^\perp) - 1 - \rho^*(E) \\ &= \dim W + \rho^*(W^\perp) - \rho^*(E) = \rho(W),\end{aligned}$$

as desired. All of this proves that $x \in \mathrm{cyc}(V)$ and thus $\sum_{i=1}^t C_i \leq \mathrm{cyc}(V)$. Finally, $\mathrm{cyc}(V) = 0$ if and only if V contains no circuits, which means that V is independent. □

The cyclic-core operator is dual to the closure operator in the following sense. This also appears in [2, Lem. 2.23].

Corollary 2.2.21. *Let $V \in \mathcal{L}(E)$. Then* $\mathrm{cyc}(V)^\perp = \mathrm{cl}^*(V^\perp)$.

Proof. Let $C_1, \ldots, C_t \in \mathcal{C}(\mathcal{M})$ be the circuits contained in V. With the aid of Lemma 2.2.19 and Theorem 2.2.20 we compute $\mathrm{cl}^*(V^\perp) = \bigcap_{i=1}^t C_i^\perp = \big(\sum_{i=1}^t C_i\big)^\perp = \mathrm{cyc}(V)^\perp$. □

The previous results imply that open spaces of $\mathcal{M}$ form a lattice $(\mathcal{O}(\mathcal{M}), <, \wedge, \vee)$ with $V \wedge W = \mathrm{cyc}(V \cap W)$ and $V \vee W = V + W$. It is the dual of the lattice of flats of $\mathcal{M}^*$; for the latter see [10, Thm. 3.10 and 3.13]. The following is immediate with Theorem 2.2.20 or Corollary 2.2.21; see also [2, Thm. 2.10].

Corollary 2.2.22. *Let $V, W \in \mathcal{L}(E)$. Then*
(a) $V \leq W \Longrightarrow \mathrm{cyc}(V) \leq \mathrm{cyc}(W)$.
(b) $\mathrm{cyc}(\mathrm{cyc}(V)) = \mathrm{cyc}(V)$.

In the context of the cyclic core, it is natural to cast the following definition.

Definition 2.2.23. *A subspace $V \in \mathcal{L}(E)$ is* cyclic *if* $\mathrm{cyc}(V) = V$.

Theorem 2.2.20 and Definition 2.2.16 show that any $V \in \mathcal{L}(E)$ satisfies

$$V \text{ is open} \iff V \text{ is cyclic} \iff \rho(W) = \rho(V) \text{ for all } W \in \mathrm{Hyp}(V). \tag{2.8}$$

We will use "open" and "cyclic" interchangeably.

Dualizing the identity $\rho(V) = \rho(\mathrm{cl}(V))$ provides us with part (a) of the next result; see also [2, Lem. 2.16 and 2.17].

Proposition 2.2.24. *Let $V \in \mathcal{L}(E)$.*
(a) $\dim V - \rho(V) = \dim \mathrm{cyc}(V) - \rho(\mathrm{cyc}(V))$.
(b) Let $V = \mathrm{cyc}(V) \oplus W$. *Then* $\rho(V) = \rho(\mathrm{cyc}(V)) + \dim W$ *and* $\dim W = \rho(W)$, *i.e., W is independent.*

Proof. (a) follows from

$$\begin{aligned}\dim V^\perp + \rho(V) - \rho(E) &= \rho^*(V^\perp) = \rho^*(\mathrm{cl}^*(V^\perp)) = \rho^*(\mathrm{cyc}(V)^\perp)\\ &= \dim \mathrm{cyc}(V)^\perp + \rho(\mathrm{cyc}(V)) - \rho(E).\end{aligned}$$

(b) The first part is a consequence of (a) because $\dim W = \dim V - \dim \mathrm{cyc}(V)$. Now submodularity implies $\rho(\mathrm{cyc}(V)) + \dim W = \rho(V) \leq \rho(\mathrm{cyc}(V)) + \rho(W)$, and thus $\rho(W) = \dim W$. □

2.2.4 Cyclic flats of q-matroids.

We turn to the notion of cyclic flats, which are simply flats that are also cyclic spaces. Thanks to the closure operator and cyclic-core operator, the collection of cyclic flats turns into a lattice. The main result of this section states that the collection of cyclic flats along with their rank values uniquely determine the q-matroid . This section is the relatively standard q-analogue of the theory of cyclic flats for matroids as can be found in [4] or [20, Sec. 2.4]. The first two results appear also, with slightly different proofs, in [2, Sec. 2.2]. Thereafter we take a different route than [2] by focusing on the rank function.

We continue with the setting from Notation 2.2.13. Our first result states that the cyclic-core operator preserves flatness and the closure operator preserves cyclicity. The corresponding fact for classical matroids is only mentioned in passing at [20, Sec. 2.4].

Lemma 2.2.25. *Recall the collections $\mathcal{F}(\mathcal{M})$ of flats and $\mathcal{O}(\mathcal{M})$ of open (i.e., cyclic) spaces of $\mathcal{M}$.*

(a) $F \in \mathcal{F}(\mathcal{M}) \Longrightarrow \mathrm{cyc}(F) \in \mathcal{F}(\mathcal{M})$.

(b) $F \in \mathcal{O}(\mathcal{M}) \Longrightarrow \mathrm{cl}(F) \in \mathcal{O}(\mathcal{M})$.

As a consequence, every $V \in \mathcal{L}(E)$ satisfies $\mathrm{cl}(\mathrm{cyc}(V)) \leq \mathrm{cyc}(\mathrm{cl}(V))$ *and both spaces are elements of the intersection $\mathcal{F}(\mathcal{M}) \cap \mathcal{O}(\mathcal{M})$.*

The spaces $\mathrm{cl}(\mathrm{cyc}(V))$ and $\mathrm{cyc}(\mathrm{cl}(V))$ are in general not identical. See Example 2.2.27 below.

Proof. (a) Let $x \in E \setminus \mathrm{cyc}(F)$. We have to show that $\rho(\mathrm{cyc}(F) + \langle x\rangle) > \rho(\mathrm{cyc}(F))$.
i) If $x \notin F$, then $\rho(F + \langle x\rangle) = \rho(F) + 1$, since F is a flat, and the desired inequality follows from (2.5).
ii) Let $x \in F \setminus \mathrm{cyc}(F)$. The definition of $\mathrm{cyc}(F)$ implies the existence of a space $\hat{W} \in \mathrm{Hyp}(F)$ such that $x \notin \hat{W}$ and $\rho(\hat{W} + \langle x\rangle) = \rho(\hat{W}) + 1$. Thus, $\rho(F) = \rho(\hat{W}) + 1$. Assume by contradiction that $\rho(\mathrm{cyc}(F) + \langle x\rangle) = \rho(\mathrm{cyc}(F))$. Then (2.5) implies $\rho(W + \langle x\rangle) = \rho(W)$ for all subspaces W containing $\mathrm{cyc}(F)$. Hence we conclude $\mathrm{cyc}(F) \not\leq \hat{W}$. Thus there exists $y \in \mathrm{cyc}(F)\setminus\hat{W}$. Now $\hat{W} + \langle y\rangle = F$, and $\rho(\hat{W}) = \rho(F)$ since $y \in \mathrm{cyc}(F)$. This contradicts the above, and thus $\rho(\mathrm{cyc}(F) + \langle x\rangle) > \rho(\mathrm{cyc}(F))$.

(b) Let cyc^* be the cyclic-core operator of the dual q-matroid $\mathcal{M}^*$. With the aid of Theorem 2.2.15(a), Corollary 2.2.21 and Part (a) we conclude

$$F \in \mathcal{O}(\mathcal{M}) \Rightarrow F^\perp \in \mathcal{F}(\mathcal{M}^*) \Rightarrow \mathrm{cyc}^*(F^\perp) \in \mathcal{F}(\mathcal{M}^*) \Rightarrow \mathrm{cl}(F)^\perp \in \mathcal{F}(\mathcal{M}^*) \\ \Rightarrow \mathrm{cl}(F) \in \mathcal{O}(\mathcal{M}).$$

As for the consequence note that $\mathrm{cl}(V)$ is a flat and thus (a) together with Corollary 2.2.22 implies that $\mathrm{cyc}(\mathrm{cl}(V))$ is a flat containing $\mathrm{cyc}(V)$. Now the stated containment follows from Theorem 2.2.3(CL2). The rest is clear. □

Using the cyclic-core operator and the closure operator we obtain a lattice consisting of the cyclic flats. It also appears in [2, Prop. 2.24] and is the q-analogue of [19, Prop. 3] (which refers to [4]).

Corollary 2.2.26. *Let $\mathcal{Z}(\mathcal{M}) = \mathcal{F}(\mathcal{M}) \cap \mathcal{O}(\mathcal{M})$, that is, $\mathcal{Z}(\mathcal{M})$ is the collection of cyclic flats of $\mathcal{M}$, or, alternatively, of open and closed spaces. Then $(\mathcal{Z}(\mathcal{M}), \leq, \wedge, \vee)$ is a lattice, where the meet and join are defined as*

$$Z_1 \wedge Z_2 = \mathrm{cyc}(Z_1 \cap Z_2) \quad \textit{and} \quad Z_1 \vee Z_2 = \mathrm{cl}(Z_1 + Z_2) \quad \textit{for all } Z_1, Z_2 \in \mathcal{Z}(\mathcal{M}).$$

The rank values of the meet and join are given by

$$\rho(Z_1 \wedge Z_2) = \rho(Z_1 \cap Z_2) - \dim\big((Z_1 \cap Z_2)/Z_1 \wedge Z_2\big) \quad \textit{and} \quad \rho(Z_1 \vee Z_2) = \rho(Z_1 + Z_2).$$

As a consequence

$$\rho(Z_1) + \rho(Z_2) \geq \rho(Z_1 \vee Z_2) + \rho(Z_1 \wedge Z_2) + \dim\big((Z_1 \cap Z_2)/Z_1 \wedge Z_2\big).$$

Proof. By (F2) and Theorem 2.2.15(d) $Z_1 \cap Z_2$ is a flat and $Z_1 + Z_2$ is an open space. Hence $Z_1 \wedge Z_2$ and $Z_1 \vee Z_2$ are in $\mathcal{Z}(\mathcal{M})$ thanks to Lemma 2.2.25. Next, if $V \in \mathcal{Z}(\mathcal{M})$ satisfies $V \leq Z_i$ for $i = 1, 2$, then V is an open space in $Z_1 \cap Z_2$ and thus $V \leq \mathrm{cyc}(Z_1 \cap Z_2)$ thanks to Theorem 2.2.20. Similarly, if $W \in \mathcal{Z}(\mathcal{M})$ satisfies $Z_i \leq W$ for $i = 1, 2$, then W is a closed space containing $Z_1 + Z_2$ and thus $\mathrm{cl}(Z_1 + Z_2) \leq W$ by Theorem 2.2.3(CL2). Thus $(\mathcal{Z}(\mathcal{M}), \leq, \wedge, \vee)$ is a lattice. The rank value of $Z_1 \vee Z_2$ follows from the fact that $\rho(Z_1 \oplus Z_2) = \rho(\mathrm{cl}(Z_1 \oplus Z_2))$, while the rank value of $Z_1 \wedge Z_2$ is a consequence of Proposition 2.2.24(a). The last inequality now follows from submodularity of ρ applied to $Z_1 \cap Z_2$ and $Z_1 + Z_2$. □

Note that $\mathcal{Z} = \mathcal{Z}(\mathcal{M})$ is not empty. The least and greatest elements of the lattice $\mathcal{Z}$ are given by $0_\mathcal{Z} = \mathrm{cl}(0)$ and $1_\mathcal{Z} = \mathrm{cyc}(E)$ and their rank values are

$$\rho(0_\mathcal{Z}) = 0 \quad \text{and} \quad \rho(1_\mathcal{Z}) = \rho(E) - \dim(E/\mathrm{cyc}(E)),$$

where the second part follows from Proposition 2.2.24. In Example 2.2.31 below we will see that the lattice $\mathcal{Z}$ is in general not semi-modular, and thus not graded. But even in the case where $\mathcal{Z}$ is graded, its height function does not agree, in general, with the rank function ρ.

Example 2.2.27 (see also [2, Prop. 2.30])**.** *Let $0 < k < n = \dim E$ and consider the uniform q-matroid $\mathcal{U}_k(E)$; see Example 2.1.7. The only flats other than E are the spaces of dimension at most $k-1$, and the only nonzero cyclic spaces are the spaces of dimension at least $k+1$; see* (2.8)*. Thus $\mathcal{Z} := \mathcal{Z}(\mathcal{U}_k(E)) = \{0, E\}$. Since this is true for every k, this shows that the collection $\mathcal{Z}(\mathcal{M})$ of a q-matroid $\mathcal{M}$ does not uniquely determine $\mathcal{M}$. Furthermore, since $\rho(E) = k$, we also see that the height function of the lattice $\mathcal{Z}$ does not agree with the rank function of $\mathcal{U}_{k,n}(q)$ (unless $k = 1$). Finally, for any k-dimensional subspace V we have $\mathrm{cl}(\mathrm{cyc}(V)) = \mathrm{cl}(0) = 0$, and $\mathrm{cyc}(\mathrm{cl}(V)) = \mathrm{cyc}(E) = E$. This shows that $\mathrm{cl}(\mathrm{cyc}(V))$ and $\mathrm{cyc}(\mathrm{cl}(V))$ do not agree. For the trivial and the free q-matroid we have $\mathcal{Z}(\mathcal{U}_0(E)) = \{E\}$ and $\mathcal{Z}(\mathcal{U}_n(E)) = \{0\}$.*

Below we will show that we can reconstruct the entire q-matroid $\mathcal{M}$ by way of the cyclic flats along with their rank values. The following lemma will be needed. It is the q-analogue of [19, Lem. 5] and also appears in [2, Lem. 2.23], where it is proven with the aid of a characterization of cyclic spaces as inclusion-minimal spaces with respect to the nullity function.

Lemma 2.2.28. *Let $V \in \mathcal{L}(E)$. Then $V \cap \mathrm{cl}(\mathrm{cyc}(V)) = \mathrm{cyc}(V)$.*

Proof. "$\supseteq$" is clear. For the other containment let $x \in V \cap \mathrm{cl}(\mathrm{cyc}(V))$. Then $\langle x \rangle + \mathrm{cyc}(V) \leq V$ and $\rho(\langle x \rangle + \mathrm{cyc}(V)) = \rho(\mathrm{cyc}(V))$ because x is in the closure of $\mathrm{cyc}(V)$. Furthermore, $\mathrm{cyc}(V) = \mathrm{cyc}(\mathrm{cyc}(V)) \leq \mathrm{cyc}(\langle x \rangle + \mathrm{cyc}(V)) \leq \mathrm{cyc}(V)$, and thus $\mathrm{cyc}(V) = \mathrm{cyc}(\langle x \rangle + \mathrm{cyc}(V))$. With the aid of Proposition 2.2.24(a) we compute

$$\begin{aligned}\dim \mathrm{cyc}(V) - \rho(\mathrm{cyc}(V)) &\leq \dim(\langle x \rangle + \mathrm{cyc}(V)) - \rho(\mathrm{cyc}(V)) \\ &= \dim(\langle x \rangle + \mathrm{cyc}(V)) - \rho(\langle x \rangle + \mathrm{cyc}(V)) \\ &= \dim(\mathrm{cyc}(\langle x \rangle + \mathrm{cyc}(V))) - \rho(\mathrm{cyc}(\langle x \rangle + \mathrm{cyc}(V))) \\ &= \dim \mathrm{cyc}(V) - \rho(\mathrm{cyc}(V)).\end{aligned}$$

Hence we have equality in the first step, and this means $x \in \mathrm{cyc}(V)$. □

Now we arrive at the following q-analogue of [4, Lem. 3.1(i)] characterizing independent spaces. This characterization will be crucial because it will allow us to derive the entire rank function of the q-matroid from the cyclic flats and their rank values. This is different from the approach taken in [4] and in [2] for q-matroids, where the entire (q-)matroid is reconstructed through the lattice of flats.

Theorem 2.2.29. *Consider the collection $\mathcal{Z}(\mathcal{M})$ of cyclic flats of $\mathcal{M}$. Let $V \in \mathcal{L}(E)$. Then*

$$V \text{ is independent} \Longleftrightarrow \dim(V \cap Z) \leq \rho(Z) \text{ for all } Z \in \mathcal{Z}(\mathcal{M}).$$

Thus the cyclic flats together with their rank values fully determine the collection of independent spaces and thus the entire q-matroid $\mathcal{M}$.

Proof. "$\Rightarrow$" If V is independent, then so is $V \cap Z$ and thus $\dim(V \cap Z) = \rho(V \cap Z) \leq \rho(Z)$ for any subspace Z.
"$\Leftarrow$" Let V be dependent and $Z = \mathrm{cl}(\mathrm{cyc}(V))$, which is in $\mathcal{Z}(\mathcal{M})$. Then $V \cap Z =$

cyc(V) thanks to Lemma 2.2.28 and $\rho(Z) = \rho(\text{cyc}(V))$. Now Proposition 2.2.24(a) implies $\dim(V \cap Z) = \dim \text{cyc}(V) = \rho(Z) + \dim V - \rho(V) > \rho(Z)$, where the last step follows from the dependence of V. This establishes the equivalence. The last statement follows from the well-known fact that the independent spaces fully determine the q-matroid ; see Theorem 2.2.10. □

The previous characterization of independent spaces lets us determine the rank function of the entire q-matroid from the cyclic flats together with their rank values; see also [20, Prop. 3] for classical matroids.

Corollary 2.2.30. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{Z} = \mathcal{Z}(\mathcal{M})$ be its collection of cyclic flats. Then*

$$\rho(V) = \min_{Z \in \mathcal{Z}} \Big(\rho(Z) + \dim(V + Z)/Z\Big) \ \text{ for all } \ V \in \mathcal{L}(E).$$

Proof. Theorem 2.2.29 tells us that a space I is independent if and only if $\rho(Z) \geq \dim(I \cap Z)$ for all $Z \in \mathcal{Z}$. With the aid of the dimension formula for subspaces we may rewrite the inequality as $\dim I \leq \rho(Z) + \dim(I + Z)/Z$. Now let $V \in \mathcal{L}(E)$. Using (2.6) we obtain

$$\begin{aligned}
\rho(V) &= \max\{\dim I \mid I \leq V,\ I \text{ independent}\} \\
&= \max\{\dim I \mid I \leq V, \dim I \leq \rho(Z) + \dim(I + Z)/Z \text{ for all } Z \in \mathcal{Z}\} \\
&\leq \max\{\dim I \mid I \leq V, \dim I \leq \rho(Z) + \dim(V + Z)/Z \text{ for all } Z \in \mathcal{Z}\} \\
&\leq \min_{Z \in \mathcal{Z}} \Big(\rho(Z) + \dim(V + Z)/Z\Big).
\end{aligned}$$

For the converse consider $\hat{Z} = \text{cl}(\text{cyc}(V))$, which is in $\mathcal{Z}$. With the aid of (2.5), Proposition 2.2.24(a) and Lemma 2.2.28 we arrive at

$$\begin{aligned}
\min_{Z \in \mathcal{Z}} \Big(\rho(Z) + \dim(V + Z)/Z\Big) &\leq \rho(\hat{Z}) + \dim(V + \hat{Z})/\hat{Z} \\
&= \rho(\hat{Z}) + \dim V - \dim(V \cap \text{cl}(\text{cyc}(V)) \\
&= \rho(\text{cyc}(V)) + \dim V - \dim \text{cyc}(V) \\
&= \rho(V).
\end{aligned}$$
□

The above result does not imply that the lattice structure of $(\mathcal{Z}(\mathcal{M}), \leq, \wedge, \vee)$ together with the rank values of the cyclic flats are sufficient to determine the q-matroid up to equivalence. This will be illustrated in Example 5.3.18.

The collection of cyclic flats is often astoundingly small. The following example is inspired by [2, Sec. 3].

Example 2.2.31. *Let $\mathbb{F} = \mathbb{F}_2$ and in $\mathbb{F}^8$ consider the collection $\mathcal{Z} = \{Z_0, \ldots, Z_4\}$, where*

$$Z_0 = 0,\ Z_1 = \langle e_1, e_2\rangle,\ Z_2 = \langle e_1, e_2, e_3, e_4\rangle,\ Z_3 = \langle e_5, e_6, e_7, e_8\rangle,\ Z_4 = \mathbb{F}^8.$$

Set $\hat{\rho}(Z_i) = i$ *for* $i = 0, \ldots, 4$*. Using a computer algebra system one verifies that the map*

$$\rho : \mathcal{L}(\mathbb{F}^8) \longrightarrow \mathbb{N}_0, \quad \rho(V) = \min_{Z \in \mathcal{Z}} \Big(\hat{\rho}(Z) + \dim(V + Z)/Z \Big),$$

satisfies (R1)–(R3) from Definition 2.1.1 (see also Corollary 2.2.30). Moreover, $\rho(Z) = \hat{\rho}(Z)$ *for all* $Z \in \mathcal{Z}$ *and the q-matroid* $\mathcal{M} = (\mathbb{F}^8, \rho)$ *satisfies* $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}$*. This also follows from [2, Prop. 3.7 and Thm. 3.11]. We have the following cardinalities of flats, cyclic spaces etc.*

Table 2.1: The cardinality of flats, cyclic spaces etc. for some q-matroid.

flats	cyclic spaces	cyclic flats	ind. spaces	dep. spaces	circuits	bases
99597	105097	5	307905	109294	94079	199775

Finally, notice that the lattice $\mathcal{Z}$ *has the form*

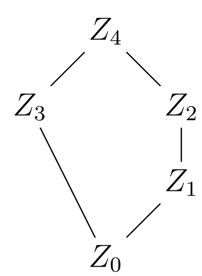


Figure 2.1: A non-semimodular lattice of cyclic flats.

and this is clearly not semi-modular (see [45, Prop. 3.3.2]).

We close this section with the most extreme case. Every q-matroid has at least one cyclic flat, namely $\mathrm{cl}(0)$. Let us consider the case where this is the sole cyclic flat. Such q-matroidsdo indeed exist, for instance, the trivial and the free q-matroid on E (see Example 2.2.27).

Proposition 2.2.32. *Let* $\mathcal{M} = (E, \rho)$ *be a q-matroid with a single cyclic flat, say* $\mathcal{Z}(\mathcal{M}) = \{\hat{Z}\}$*. Thus* $\hat{Z} = \mathrm{cl}(0) = \mathrm{cyc}(E)$ *and* $\rho(\hat{Z}) = 0$*. Then for any* $V \in \mathcal{L}(E)$

$$V \in \mathcal{F}(\mathcal{M}) \Longleftrightarrow \hat{Z} \leq V \quad \text{and} \quad V \in \mathcal{O}(\mathcal{M}) \Longleftrightarrow V \leq \hat{Z}.$$

In particular, $\mathrm{cl}(0) = E \Longleftrightarrow \mathcal{M} = \mathcal{U}_0(E)$ *and* $\mathrm{cyc}(E) = 0 \Longleftrightarrow \mathcal{M} = \mathcal{U}_{\dim E}(E)$*.*

Proof. Note first that for any $V \in \mathcal{L}(E)$ we have

$$\mathrm{cl}(\mathrm{cyc}(V)) = \mathrm{cyc}(\mathrm{cl}(V)) = \hat{Z}. \tag{2.9}$$

Thus if V is a flat, then $\hat{Z} = \mathrm{cyc}(V) \leq V$. Similarly, if V is cyclic, then $V \leq \mathrm{cl}(V) = \hat{Z}$. It remains to consider the opposite implications.

1) If $V \leq \hat{Z}$, then $\rho(V) = 0$ and V is clearly cyclic.
2) Let now $\hat{Z} \leq V$. Then (2.9) along with Lemma 2.2.28 implies $\mathrm{cyc}(V) = \hat{Z}$. Writing $V = V_1 \oplus \hat{Z}$, we obtain from Proposition 2.2.24(b)

$$\rho(V) = \rho(\hat{Z}) + \dim V_1 = \dim V_1.$$

Let now $x \in E \setminus V$. Then $\hat{Z} \leq V + \langle x \rangle$ and we may write $V + \langle x \rangle = (V_1 + \langle x_1 \rangle) \oplus \hat{Z}$ for some $x_1 \notin V_1$. But then the same reasoning as for V provides us with $\rho(V + \langle x \rangle) = \dim(V_1 + \langle x_1 \rangle)$, and hence $\rho(V + \langle x \rangle) = \dim V_1 + 1 = \rho(V) + 1$. Thus V is a flat.
The last two equivalences are clear. □

In Chapter 5, we will see that a q-matroid with a single cyclic flat is the direct sum of a trivial and a free q-matroid .

2.3 Application of q-polymatroids to coding theory.

In this subsection we study q-PMs associated to linear rank-metric codes as introduced in [29].

We furthermore establish several connections between rank-metric codes and q-PM. Throughout, let $\mathbb{F} := \mathbb{F}_q$ for a prime power q.

2.3.1 Rank-metric code and the induced q-polymatroid.

We first collect some well-known facts for codes in $\mathbb{F}^{n\times m}$.

Definition 2.3.1. *The* rank-metric *on* $\mathbb{F}^{n\times m}$ *is defined as*

$$\mathrm{d} : \mathbb{F}^{n\times m} \times \mathbb{F}^{n\times m} \longrightarrow \mathbb{N}_0, \quad \mathrm{d}(A, B) = \mathrm{rk}(A - B).$$

A linear-rank-metric code *is a subspace* $\mathcal{C}$ *of the metric space* $(\mathbb{F}^{n\times m}, \mathrm{d})$.

From the above definition note $\mathrm{d}(A, 0) = \mathrm{rk}(A)$ for all $A \in \mathbb{F}^{n\times m}$. One can also define non-linear rank-metric codes, which are simply subsets of metric spaces rather than subspaces. However, for the entirety of the dissertation we consider only linear rank-metric code. Part (a)–(c) of the following proposition is standard knowledge on rank-metric codes, see for instance [28], and Part (d) can be found in [42, Lem. 28]. For $V \leq \mathbb{F}^{\ell}$ denote by $V^{\perp} \leq \mathbb{F}^{\ell}$ the orthogonal space with respect to the standard dot product.

Proposition-Definition 2.3.2. *Let* $\mathcal{C} \leq \mathbb{F}^{n\times m}$ *be a rank-metric code.*
(a) The rank distance *of* $\mathcal{C}$ *is defined as* $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = \min\{\mathrm{rk}(M) \mid M \in \mathcal{C} \setminus 0\}$.
(b) If $d = \mathrm{d}_{\mathrm{rk}}(\mathcal{C})$*, then* $\dim(\mathcal{C}) \leq \max\{m, n\}(\min\{m, n\} - d + 1)$*, which is known as the* Singleton bound*. If* $\dim(\mathcal{C}) = \max\{m, n\}(\min\{m, n\} - d + 1)$*, then* $\mathcal{C}$ *is called an* MRD code*.*
(c) The dual code *of* $\mathcal{C}$ *is defined as* $\mathcal{C}^{\perp} = \{M \in \mathbb{F}^{n\times m} \mid \mathrm{tr}(MN^{\mathsf{T}}) = 0$ *for all* $N \in \mathcal{C}\}$*, where* $\mathrm{tr}(\cdot)$ *denotes the trace of the given matrix. If* $\mathcal{C}$ *is an MRD code with rank distance* d*, then* $\mathcal{C}^{\perp}$ *is an MRD code with rank distance* $\min\{m, n\} - d + 2$*.*

(d) *For* $V \in \mathcal{L}(\mathbb{F}^n)$ *and* $W \in \mathcal{L}(\mathbb{F}^m)$ *we define the following subspaces of* $\mathcal{C}$*, which are known as* shortenings*:*

$$\mathcal{C}(V,\mathrm{c}) = \{M \in \mathcal{C} \mid \mathrm{colsp}(M) \leq V\} \quad \textit{and} \quad \mathcal{C}(W,r) = \{M \in \mathcal{C} \mid \mathrm{rowsp}(M) \leq W\},$$

where $\mathrm{colsp}(M)$ *and* $\mathrm{rowsp}(M)$ *denote the column space and row space of* M*, respectively. Then* $\mathbb{F}^{n\times m}(V,\mathrm{c})^\perp = \mathbb{F}^{n\times m}(V^\perp,\mathcal{C})$ *and* $\mathbb{F}^{n\times m}(W,r)^\perp = \mathbb{F}^{n\times m}(W^\perp,r)$ *and*

$$\begin{aligned}\dim\mathcal{C}(V^\perp,\mathrm{c}) &= \dim\mathcal{C} - m\dim V + \dim\mathcal{C}^\perp(V,\mathrm{c}), \\ \dim\mathcal{C}(W^\perp,r) &= \dim\mathcal{C} - n\dim W + \dim\mathcal{C}^\perp(W,r).\end{aligned} \tag{2.10}$$

Now we are ready to introduce q-PMs associated to a rank-metric code. The following definition and the first statement are from [29]. The statements in (2.11) are immediate consequences of Proposition-Definition 2.3.2(d).

Proposition 2.3.3 ([29, Thm. 5.3])**.** *Let* $\mathcal{C} \leq \mathbb{F}^{n\times m}$ *be a nonzero rank-metric code. Define the maps*

$$\begin{aligned}\rho_{\mathrm{c}} :&\mathcal{L}(\mathbb{F}^n) \longrightarrow \mathbb{Q}_{\geq 0}, \quad V \longmapsto \frac{\dim\mathcal{C} - \dim\mathcal{C}(V^\perp,\mathrm{c})}{m}, \\ \rho_{\mathrm{r}} :&\mathcal{L}(\mathbb{F}^m) \longrightarrow \mathbb{Q}_{\geq 0}, \quad W \longmapsto \frac{\dim\mathcal{C} - \dim\mathcal{C}(W^\perp,r)}{n}.\end{aligned}$$

Then ρ_{c} *and* ρ_{r} *are rank functions with denominators* m *and* n*, respectively. Furthermore,*

$$\rho_{\mathrm{c}}(V) = \dim V - \frac{1}{m}\dim\mathcal{C}^\perp(V,\mathrm{c}) \quad \textit{and} \quad \rho_{\mathrm{r}}(W) = \dim W - \frac{1}{n}\dim\mathcal{C}^\perp(W,r). \tag{2.11}$$

The denominators m and n are in general not principal. Note that in the notation we suppress the dependence of these maps on the code $\mathcal{C}$.

Definition 2.3.4. *Let* $\mathcal{C} \leq \mathbb{F}^{n\times m}$ *be a nonzero rank-metric code. The* q*-PMs* $\mathcal{M}_{\mathrm{c}}(\mathcal{C}) := (\mathbb{F}^n,\rho_{\mathrm{c}})$ *and* $\mathcal{M}_r(\mathcal{C}) := (\mathbb{F}^m,\rho_{\mathrm{r}})$ *are called the* column q-polymatroid *and* row q-polymatroid *of* $\mathcal{C}$*, respectively. Their ranks are* $\dim\mathcal{C}/m$ *and* $\dim\mathcal{C}/n$*, respectively.*

The expressions in (2.11) show immediately that if $\mathcal{C}_1 \leq \mathcal{C}_2$, then $\rho_{\mathcal{C},1}(V) \leq \rho_{\mathcal{C},2}(V)$ for all $V \in \mathcal{L}(\mathbb{F}^n)$, where $\rho_{\mathcal{C},i}$ is the column rank function of $\mathcal{C}_i$. Similarly for the row q-PM. This has also been proven in [22, Lem. 22].

Equivalence of codes, in the following (standard) sense, translates into equivalence of the associated q-PMs.

Definition 2.3.5. *Let* $\mathcal{C},\mathcal{C}' \leq \mathbb{F}^{n\times m}$ *be rank-metric codes. We call* $\mathcal{C}$ *and* $\mathcal{C}'$ equivalent *if there exist matrices* $X \in \mathrm{GL}_n(\mathbb{F})$, $Y \in \mathrm{GL}_m(\mathbb{F})$ *such that* $\mathcal{C}' = X\mathcal{C}Y := \{XMY \mid M \in \mathcal{C}\}$*. If* $n = m$*, we call* $\mathcal{C},\mathcal{C}'$ transposition-equivalent *if there exist matrices* $X,Y \in \mathrm{GL}_n(\mathbb{F})$ *such that* $\mathcal{C}' = X\mathcal{C}^{\mathsf{T}}Y := \{XM^{\mathsf{T}}Y \mid M \in \mathcal{C}\}$*.*

The proof of the next result is straightforward with (2.11) by noting that $\mathcal{C}' = X\mathcal{C}Y$ implies $(\mathcal{C}')^\perp = (X^{-1})^\mathsf{T}\mathcal{C}^\perp(Y^{-1})^\mathsf{T}$ and $\mathcal{C}' = X\mathcal{C}^\mathsf{T}Y$ implies

$$(\mathcal{C}')^\perp = (X^{-1})^\mathsf{T}(\mathcal{C}^\perp)^\mathsf{T}(Y^{-1})^\mathsf{T}.$$

For an alternative proof see [29, Prop. 6.7].

Proposition 2.3.6. *Let $\mathcal{C}, \mathcal{C}' \leq \mathbb{F}^{n\times m}$ be rank-metric codes.*

(a) Suppose $\mathcal{C}, \mathcal{C}'$ are equivalent, say $\mathcal{C}' = X\mathcal{C}Y$ for some $X \in \mathrm{GL}_n(\mathbb{F})$, $Y \in \mathrm{GL}_m(\mathbb{F})$. Then $\mathcal{M}_\mathrm{c}(\mathcal{C})$ and $\mathcal{M}_\mathrm{c}(\mathcal{C}')$ are equivalent via $\beta \in Hom_\mathbb{F}(\mathbb{F}^n, \mathbb{F}^n)$ given by $x \mapsto (X^\mathsf{T})^{-1}x$. Similarly, $\mathcal{M}_r(\mathcal{C})$ and $\mathcal{M}_r(\mathcal{C}')$ are equivalent via the isomorphism $\alpha \in Hom_\mathbb{F}(\mathbb{F}^m, \mathbb{F}^m)$ given by $x \mapsto x(Y^\mathsf{T})^{-1}$.

(b) Let $n = m$ and suppose $\mathcal{C}, \mathcal{C}'$ are transposition-equivalent, say $\mathcal{C}' = X\mathcal{C}^\mathsf{T}Y$ for $X, Y \in \mathrm{GL}_n(\mathbb{F})$. Then $\mathcal{M}_\mathrm{c}(\mathcal{C})$ and $\mathcal{M}_r(\mathcal{C}')$ are equivalent via α and $\mathcal{M}_r(\mathcal{C})$ and $\mathcal{M}_\mathrm{c}(\mathcal{C}')$ are equivalent via β with α, β as in (a).

Equivalence allows us to easily introduce $\mathbb{F}_{q^m}$-linear rank-metric codes. Recall that $\mathbb{F} = \mathbb{F}_q$. Let α be a primitive element of the field extension $\mathbb{F}_{q^m}$ and $f = x^m - \sum_{i=0}^{m-1} f_i x^i \in \mathbb{F}[x]$ be its minimal polynomial over $\mathbb{F}$. We define the companion matrix

$$\Delta_f = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ f_0 & f_1 & \cdots & f_{m-1} \end{pmatrix} \in \mathrm{GL}_m(\mathbb{F}). \tag{2.12}$$

Let $\psi : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}^m$ be the coordinate map with respect to the basis $(1, \alpha, \ldots, \alpha^{m-1})$. Extending this map entry-wise, we obtain, for any n, an isomorphism

$$\Psi : \mathbb{F}_{q^m}^n \longrightarrow \mathbb{F}^{n\times m},\ \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \longmapsto \begin{pmatrix} \psi(x_1) \\ \vdots \\ \psi(x_n) \end{pmatrix}. \tag{2.13}$$

Now multiplication of $c := \sum_{i=0}^{m-1} c_i\alpha^i \in \mathbb{F}_{q^m}$ by ω corresponds to right multiplication of its coordinate vector $\psi(c) = (c_0, \ldots, c_{m-1}) \in \mathbb{F}^m$ by Δ_f. Therefore, an $\mathbb{F}$-linear subspace $\mathcal{C}$ of $\mathbb{F}_{q^m}^n$ is $\mathbb{F}_{q^m}$-linear if and only if its image $\Psi(\mathcal{C}) \leq \mathbb{F}^{n\times m}$ is invariant under right multiplication by Δ_f. Recall further that $\mathbb{F}[\Delta_f]$ is a subfield of order q^m of $\mathbb{F}^{m\times m}$, and more generally, if s is a divisor of m, then $\mathbb{F}[\Delta_f^{(q^m-1)/(q^s-1)}]$ is a subfield of order q^s. Allowing different bases for the coordinate map, we arrive at the following definition.

Definition 2.3.7. *Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a rank-metric code (hence an $\mathbb{F}$-linear subspace). Let s be a divisor of m and set $M = (q^m-1)/(q^s-1)$. Then $\mathcal{C}$ is* right $\mathbb{F}_{q^s}$-linear *if there exists an $X \in \mathrm{GL}_m(\mathbb{F})$ such that the code $\mathcal{C}X$ is invariant under right multiplication by Δ_f^M.* Left linearity *over $\mathbb{F}_{q^n}$ and its subfields is defined analogously.*

Clearly, the qualifiers right/left are needed only in the case where $\mathbb{F}_{q^s}$ is a subfield of both $\mathbb{F}_{q^m}$ and $\mathbb{F}_{q^n}$. Note that for $\tilde{\mathcal{C}} := \mathcal{C}X$ we have $\tilde{\mathcal{C}}(V, \mathrm{c}) = \mathcal{C}(V, \mathrm{c})X$ for all

$V \in \mathcal{L}(\mathbb{F}^n)$. Furthermore, if $\tilde{\mathcal{C}}$ is invariant under right multiplication by Δ_f^M, then so is $\tilde{\mathcal{C}}(V,\mathrm{c})$. Hence $\tilde{\mathcal{C}}(V,\mathrm{c})$ is right $\mathbb{F}_{q^s}$-linear (see also [36, Lem. 19] for the case $s=m$), and thus its dimension over $\mathbb{F}$ is a multiple of s. All of this leads to the following remark.

Remark 2.3.8. *Let $\mathcal{C} \le \mathbb{F}^{n\times m}$ be a right $\mathbb{F}_{q^s}$-linear rank-metric code for some subfield $\mathbb{F}_{q^s}$ of $\mathbb{F}_{q^m}$. Then $\mu = m/s$ is a denominator of the column q-PM $\mathcal{M}_\mathrm{c}(\mathcal{C})$. In particular, for $s=m$ the q-PM $\mathcal{M}_\mathrm{c}(\mathcal{C})$ is a q-matroid. These are exactly the q-matroids studied in [36]. Of course, the column q-PM of a code $\mathcal{C} \le \mathbb{F}^{n\times m}$ may be a q-matroid even if $\mathcal{C}$ is not right $\mathbb{F}_{q^m}$-linear. This is for instance the case for MRD codes in $\mathbb{F}^{n\times m}$ if $m \ge n$ (see Proposition 2.3.11). Analogous statements hold for the row q-PM.*

The above allows to consider $\mathbb{F}_{q^m}$-linear codes as subspaces of $\mathbb{F}_{q^m}^n$, which is often common and will be extremely useful when presenting examples of q-matroids. Consider the isomorphism $\Psi : \mathbb{F}_{q^m}^n \longrightarrow \mathbb{F}^{n\times m}$ from (2.13). Then with any $\mathbb{F}_{q^m}$-subspace $\mathcal{C}$ of $\mathbb{F}_{q^m}^n$ we can associate the column q-PM $\mathcal{M}_\mathrm{c}(\Psi(\mathcal{C}))$, which is in fact a q-matroid. We denote this q-matroid by $\mathcal{M}_G$, where $G \in \mathbb{F}_{q^m}^{k\times n}$ is a generator matrix of $\mathcal{C}$ (that is, its rows form a basis of $\mathcal{C}$). This is the approach taken in [36]. The following lemma has been proven in [36, Sec. 5]. It determines the rank function of $\mathcal{M}_G$ with the aid of G. For self-containment we include a short proof using our notation.

Lemma 2.3.9. *Let $\mathcal{C} \le \mathbb{F}_{q^m}^n$ be an $\mathbb{F}_{q^m}$-linear rank-metric code with generator matrix $G \in \mathbb{F}_{q^m}^{k\times n}$, where $k = \dim_{\mathbb{F}_{q^m}} \mathcal{C}$. Consider the associated q-matroid $\mathcal{M}_G = (\mathbb{F}^n, \rho_\mathrm{c})$. Let $V \in \mathcal{L}(\mathbb{F}^n)$ with $\dim V = t$, and let $Y \in \mathbb{F}^{n\times t}$ be such that $V = \mathrm{colsp}(Y)$. Then*

$$\rho_\mathrm{c}(V) = \mathrm{rk}_{\mathbb{F}_{q^m}}(GY).$$

Proof. Clearly $\mathrm{rk}_{\mathbb{F}_{q^m}}(GY)$ does not depend on the choice of Y. Since Y has entries in $\mathbb{F}$, any $x \in \mathbb{F}_{q^m}^n$ satisfies $xY = 0 \iff Y^\mathsf{T} x^\mathsf{T} = 0 \iff Y^\mathsf{T}\Psi(x) = 0 \iff \mathrm{colsp}(\Psi(x)) \le V^\perp$. Set $\tilde{\mathcal{C}} = \Psi(\mathcal{C}) \le \mathbb{F}^{n\times m}$. Then the space $\tilde{\mathcal{C}}(V^\perp, \mathcal{C})$ satisfies $\Psi^{-1}(\tilde{\mathcal{C}}(V^\perp,\mathcal{C})) = \{x \in \mathcal{C} \mid xY = 0\}$. Let $\pi_Y : \mathcal{C} \longrightarrow \mathbb{F}_{q^m}^t$ be the $\mathbb{F}_{q^m}$-linear map given by $x \longmapsto xY$. Then $\ker \pi_Y = \Psi^{-1}(\tilde{\mathcal{C}}(V^\perp,\mathcal{C}))$ and $\mathrm{im}\, \pi_Y = \mathrm{rowsp}_{\mathbb{F}_{q^m}}(GY)$. Now the desired statement follows from

$$\rho_\mathrm{c}(V) = \frac{\dim_\mathbb{F} \tilde{\mathcal{C}} - \dim_\mathbb{F} \tilde{\mathcal{C}}(V^\perp,\mathcal{C})}{m} = \dim_{\mathbb{F}_{q^m}} \tilde{\mathcal{C}} - \dim_{\mathbb{F}_{q^m}} \Psi^{-1}(\tilde{\mathcal{C}}(V^\perp,\mathcal{C})) = \mathrm{rk}_{\mathbb{F}_{q^m}}(GY).$$

□

Remark 2.3.10. *The above lemma generalizes as follows. Consider a general rank-metric code $\mathcal{C} \le \mathbb{F}^{n\times m}$ with $\mathbb{F}$-dimension k. Taking the pre-image under Ψ of a basis of $\mathcal{C}$, we obtain a matrix $G \in \mathbb{F}_{q^m}^{k\times n}$ such that $\Psi^{-1}(\mathcal{C}) = \mathrm{rowsp}_\mathbb{F}(G)$, where the latter is defined as the $\mathbb{F}$-subspace of $\mathbb{F}_{q^m}^n$ generated by the rows of G. Denote its $\mathbb{F}$-dimension by $\mathrm{rowrk}_\mathbb{F}(G)$. Then it is easy to see that for V as in Lemma 2.3.9 we have $\rho_\mathrm{c}(V) = \mathrm{rowrk}_\mathbb{F}(GY)/m$.*

Let us return to general rank-metric codes in $\mathbb{F}^{n\times m}$. From now on we will focus on the associated column q-PMs. The discussion of the row q-PMs is analogous. On several occasions we will have to pay close attention to the cases $n \leq m$ versus $n > m$. We first record the following simple fact, which is immediate with Proposition 2.3.3 (see also [29, Prop. 6.2] and [36, Lem. 30]). Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a nonzero rank-metric code with rank distance d and let $d^\perp$ be the rank distance of $\mathcal{C}^\perp$. Then for any $V \in \mathcal{L}(\mathbb{F}^n)$

$$\rho_{\rm c}(V) = \begin{cases} \dfrac{\dim \mathcal{C}}{m}, & \text{if } \dim V > n-d, \\ \dim V, & \text{if } \dim V < d^\perp. \end{cases} \tag{2.14}$$

Together with Proposition-Definition 2.3.2(b),(c) this immediately leads to the following result for MRD codes if $n \leq m$.

Proposition 2.3.11 ([29, Cor. 6.6]). *Let $n \leq m$ and $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be an MRD code with $\mathrm{d}_{\rm rk}(\mathcal{C}) = d$. Let $\rho_{\rm c}$ be the rank function of the associated column q-PM $\mathcal{M}_{\rm c}(\mathcal{C})$. Then*

$$\rho_{\rm c}(V) = \min\{n-d+1, \dim V\} \quad \textit{for all } V \in \mathcal{L}(\mathbb{F}^n).$$

Hence $\mathcal{M}_{\rm c}(\mathcal{C})$ is the uniform matroid $\mathcal{U}_{n-d+1}(\mathbb{F}^n)$. In particular, $\mathcal{M}_{\rm c}(\mathcal{C})$ is a q-matroid, i.e., its principal denominator is 1.

In order to discuss the q-PM associated to MRD codes for $m \leq n$ we need the following lemma.

Lemma 2.3.12. *Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a rank-metric code with $\mathrm{d}_{\rm rk}(\mathcal{C}) = d$. Let $V \in \mathcal{L}(\mathbb{F}^n)$ and $\dim V = v$. Then $\dim \mathcal{C}(V, \mathrm{c}) \leq \max\{m, v\}(\min\{m, v\} - d + 1)$.*

Proof. First assume that $V = \mathrm{colsp}(I_v \,|\, 0)^\mathsf{T} = \langle e_1, \ldots, e_v\rangle$, where $e_i \in \mathbb{F}^n$ is the i-th standard basis vector. Then we have a rank-preserving, thus injective, linear map

$$\pi : \mathcal{C}(V, \mathrm{c}) \longrightarrow \mathbb{F}^{v\times m}, \ \begin{pmatrix} M \\ 0 \end{pmatrix} \longmapsto M.$$

Hence $\mathrm{im}(\pi)$ is a rank-metric code in $\mathbb{F}^{v\times m}$ of rank distance at least d, and the upper bound for $\dim \mathcal{C}(V, \mathrm{c})$ follows from the Singleton bound. For the general case where $V = \langle x_1, \ldots, x_v\rangle$, choose $Y \in \mathrm{GL}_n(\mathbb{F})$ such that $Yx_i = e_i$ for $i \in [v]$ and set $\mathcal{C}' = Y\mathcal{C}$. Then $\mathcal{C}'$ has rank distance d as well and $Y\mathcal{C}(V, \mathrm{c}) = \mathcal{C}'(YV, \mathcal{C})$. Since $YV = \langle e_1, \ldots, e_v\rangle$, the upper bound on $\dim \mathcal{C}(V, \mathrm{c})$ follows from the first part of this proof. □

Now we obtain the following information about the column q-PM of an MRD code if $m \leq n$.

Theorem 2.3.13. *Let $m \leq n$ and $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be an MRD code with $\mathrm{d}_{\rm rk}(\mathcal{C}) = d$. Furthermore, denote the dimension of $V \in \mathcal{L}(\mathbb{F}^n)$ by v. Then the rank function $\rho_{\rm c}$ of the associated column q-PM $\mathcal{M}_{\rm c}(\mathcal{C})$ satisfies*

$$\rho_{\rm c}(V) = \begin{cases} v, & \textit{if } v \leq m-d+1, \\ \frac{n(m-d+1)}{m}, & \textit{if } v \geq n-d+1. \end{cases}$$

Furthermore $\rho_c(V) \geq \max\{1, v/m\}(m-d+1)$ *if* $v \in [m-d+2, n-d]$.

Proof. For $v \geq n-d+1$ we have $\rho_c(V) = m^{-1}\dim\mathcal{C} = nm^{-1}(m-d+1)$ thanks to (2.14). Next, using (2.11) and the fact that $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}^\perp) = m-d+2$ we arrive immediately at $\rho_c(V) = v$ if $v \leq m-d+1$. The remaining statement follows from (2.11) along with Lemma 2.3.12, which applied to $\mathcal{C}^\perp$ yields $\dim\mathcal{C}^\perp(V,\mathrm{c}) \leq m(v-m+d-1)$ if $v \leq m$ and $\dim\mathcal{C}^\perp(V,\mathrm{c}) \leq v(d-1)$ if $v \geq m$. □

If $m = n-1$, the interval $[m-d+2, n-d]$ is empty for any d. Thus we have

Corollary 2.3.14. *Let* $\mathcal{C} \leq \mathbb{F}^{n\times(n-1)}$ *be an MRD code. Then* $\mathcal{M}_c(\mathcal{C})$ *is fully determined by the parameters* $(n, \mathrm{d}_{\mathrm{rk}}(\mathcal{C}), |\mathbb{F}|)$ *and is not a q-matroid (unless* $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = 1$*).*

Remark 2.3.15. *In the situation of Theorem 2.3.13 the denominator* m *is not necessarily principal (even if the code is not linear over a subfield of* $\mathbb{F}_{q^m}$*). This is for instance the case for a* $[7\times 6; 4]$*-MRD code:* $n(m-d+1)/m = 7/2$*, and thus the principal denominator is* 2*.*

If $m < n-1$, the column q-PM of an MRD code $\mathcal{C}$ in $\mathbb{F}^{n\times m}$ is not fully determined by the parameters $(n, m, \mathrm{d}_{\mathrm{rk}}(\mathcal{C}), |\mathbb{F}|)$. This is illustrated by the following example.

Example 2.3.16. *In* $\mathbb{F}_2^{5\times 2}$ *consider the codes* $\mathcal{C}_1 = \langle A_1, \ldots, A_5\rangle$ *and* $\mathcal{C}_2 = \langle B_1, \ldots, B_5\rangle$*, where*

$$A_1 = \begin{pmatrix} 1 & 1\\ 1 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1\\ 1 & 1\\ 1 & 0\\ 0 & 1\\ 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0\\ 0 & 0\\ 1 & 1\\ 0 & 0\\ 0 & 1 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\\ 1 & 1 \end{pmatrix},\ A_5 = \begin{pmatrix} 1 & 0\\ 0 & 1\\ 1 & 1\\ 0 & 0\\ 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0 \end{pmatrix},\ B_3 = \begin{pmatrix} 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 0 \end{pmatrix},\ B_4 = \begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1 \end{pmatrix},\ B_5 = \begin{pmatrix} 0 & 1\\ 0 & 0\\ 0 & 1\\ 0 & 0\\ 1 & 0 \end{pmatrix}.$$

Both codes are MRD with rank distance $d = 2$*; in fact,* $\mathcal{C}_2$ *is a (*$\mathbb{F}_{2^5}$*-linear) Gabidulin code. The q-PMs* $\mathcal{M}_c(\mathcal{C}_1)$ *and* $\mathcal{M}_c(\mathcal{C}_2)$ *are not equivalent. Indeed, denote the two rank functions by* ρ_c^i*,* $i = 1, 2$*, and consider* $\rho_c^i(V)$ *for* $\dim V \in \{2, 3\} = [m-d+2, n-d]$*. For the* 155 *subspaces of* $\mathbb{F}_2^5$ *of dimension* 2 *the map* ρ_c^1 *assumes the value* 1 *exactly once and the values* 3/2 *and* 2 *exactly 28 and 126 times, respectively. On the other hand,* ρ_c^2 *assumes the values* 3/2 *and* 2 *exactly 31 and 124 times, respectively, and never takes the value* 1*.*

In the following examples we let $E_{ij} \in \mathbb{F}_q^{n\times m}$ be the matrix with entry 1 in the ij-coordinate and 0 elsewhere. The next examples show that properties (CL4) and (F3) of respectively Theorem 2.2.3 and Theorem 2.2.4 may not be satisfied for q-PM.

Example 2.3.17. *(a) Consider the code* $\mathcal{C}_1 = \langle A_1, \ldots, A_5\rangle \leq \mathbb{F}_2^{5\times 2}$ *from Example 2.3.16. It can be verified that* $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ *has 81 flats and 29 hyperplanes. Furthermore, the collection of flats does not satisfy (CL4) and (F3) (recall Theorem 2.2.3 and Theorem 2.2.4). Unsurprisingly, the hyperplane axioms known for q-matroids (see [11, Def. 12]) do not hold either, and in particular, the hyperplanes do not all have the same rank.*

(b) Let $\mathbb{F} = \mathbb{F}_2$ *and* $\mathcal{C} = \langle E_{12} + E_{23} + E_{33},\ E_{12} + E_{13} + E_{33},\ E_{12} + E_{13} + E_{21} + E_{32}\rangle \leq \mathbb{F}^{3\times 3}$*. It is straightforward to verify that the associated column q-PM* $\mathcal{M} = (\mathbb{F}^3, \rho_{\mathrm{c}})$ *is not a q-matroid but has the same collection of flats as the q-matroid* $\mathcal{M}_G$*, where*

$$G = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \alpha \end{pmatrix} \in \mathbb{F}_4^{2\times 3}.$$

As a consequence, $\mathcal{F}(\mathcal{M})$ *and the closure operator satisfy (F1)–(F3) and (CL1)–(CL4).*

We briefly return to the notion of scaling equivalence and non-exactness from Remark 2.1.10.

Example 2.3.18. *For q-PMs induced by rank-metric codes non-exactness is a very restrictive property. Indeed, the first identity in* (2.11) *shows that*

$$\mathcal{M}_{\mathrm{c}}(\mathcal{C}) \text{ is not exact } \iff \mathcal{C}^\perp(V, \mathrm{c}) \neq 0 \text{ for all } V \in \mathcal{L}(\mathbb{F}^n) \setminus 0. \tag{2.15}$$

In particular, non-exactness of $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ *implies* $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}^\perp) = 1$*. An example of a code satisfying* (2.15) *is for instance* $\mathcal{C}, \mathcal{C}^\perp \leq \mathbb{F}_2^{3\times 4}$ *given by*

$$\begin{aligned} \mathcal{C} &= \langle E_{13},\ E_{23},\ E_{33},\ E_{12} + E_{22} + E_{41} + E_{43}\rangle \\ \mathcal{C}^\perp &= \langle E_{11},\ E_{21},\ E_{31},\ E_{41} + E_{43},\ E_{12} + E_{43},\ E_{22} + E_{43},\ E_{32}, E_{42}\rangle. \end{aligned}$$

The column q-PM $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ *can be rescaled with factor* $3/2$*, leading to an exact q-PM* $\mathcal{M}'$*. It turns out that* $\mathcal{M}' = \mathcal{M}_{\mathcal{C}}(\mathcal{C}')$ *for the code* $\mathcal{C}' \leq \mathbb{F}_2^{4\times 4}$ *given by*

$$\mathcal{C}' = \langle E_{11}, E_{12}, E_{21}, E_{22}, E_{31}, E_{32}, E_{13} + E_{23} + E_{44}, E_{14} + E_{24} + E_{43} + E_{44}\rangle.$$

It is not clear to us whether representability of q-PMs by rank-metric codes (see the next section) is preserved under rescaling.

We close this subsection with the following important result showing that duality of q-PMs corresponds to trace-duality of codes. Recall duality of q-PMs from Theorem 2.1.11.

Theorem 2.3.19 ([29, Thm. 8.1])**.** *Let* $\mathcal{C} \leq \mathbb{F}^{n\times m}$ *be a rank-metric code and* $\mathcal{C}^\perp \leq \mathbb{F}^{n\times m}$ *its dual code. Then* $\mathcal{M}_{\mathrm{c}}(\mathcal{C})^* = \mathcal{M}_{\mathrm{c}}(\mathcal{C}^\perp)$*, where* $\mathcal{M}_{\mathrm{c}}(\mathcal{C})^*$ *is the dual of* $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$ *w.r.t. the standard dot product on* $\mathbb{F}^n$*. Analogously,* $\mathcal{M}_r(\mathcal{C})^* = \mathcal{M}_r(\mathcal{C}^\perp)$*.*

2.3.2 Flats of q-polymatroids and generalized rank weights.

This subsection is devoted to showing that the generalized weights of a rank-metric code in $\mathbb{F}^{n\times m}$ can be determined by the flats of the associated column q-PM if $m > n$. For $m < n$, the corresponding result is true for the row q-PM, and for $m = n$ both q-PMs are needed. In order to be aligned with most of the literature on rank-metric codes, we will assume $m \geq n$. For further details on generalized weights and anticodes we refer to [41], from which the next definition is taken.

Definition 2.3.20 ([41, Def. 22 and 23]). *Let $m \geq n$ and $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a code.*
(a) We define $\mathrm{maxrk}(\mathcal{C}) = \max\{\mathrm{rk}(M) \mid M \in \mathcal{C}\}$*. We call the code $\mathcal{C}$ an* optimal anticode *if* $\dim \mathcal{C} = m\,\mathrm{maxrk}(\mathcal{C})$.
*(b) For $i = 1, \ldots, \dim \mathcal{C}$ the i-*th generalized weight *of $\mathcal{C}$ is defined as*

$$a_i(\mathcal{C}) = \frac{1}{m}\min\{\dim(\mathcal{A}) \mid \mathcal{A} \leq \mathbb{F}^{n\times m} \textit{ is an optimal anticode}, \dim(\mathcal{C}\cap\mathcal{A}) \geq i\}.$$

In [29] it has been shown that the generalized weights can be computed from the associated q-PMs of the code. We will derive the generalized weights in a slightly different form, and, since we need the ideas of the proof later on, we also briefly sketch the proof. In [29, Thm. 7.2] the result has been used to characterize optimal anticodes with the aid of q-PMs.

Theorem 2.3.21 ([29, Thm. 7.1]). *Let $m \geq n$ and $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a code. Define*

$$a_i^{\mathcal{C}}(\mathcal{C}) = n - \max\{\dim V \mid V \in \mathcal{L}(\mathbb{F}^n),\, \rho_{\mathrm{c}}(V) \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - i/m\},$$
$$a_i^{r}(\mathcal{C}) = m - \max\{\dim W \mid W \in \mathcal{L}(\mathbb{F}^m),\, \rho_{\mathrm{r}}(W) \leq \rho_{\mathrm{r}}(\mathbb{F}^m) - i/n\}.$$

Then for all $i = 1, \ldots, \dim \mathcal{C}$

$$a_i(\mathcal{C}) = \begin{cases} a_i^{\mathcal{C}}(\mathcal{C}), & \textit{if } m > n,\\ \min\{a_i^{\mathcal{C}}(\mathcal{C}), a_i^{r}(\mathcal{C})\}, & \textit{if } m = n.\end{cases}$$

Proof. Let first $m > n$. It is well known [38, Thm. 3] that the optimal anticodes in $\mathbb{F}^{n\times m}$ are the spaces of the form $\mathbb{F}^{n\times m}(V,\mathrm{c})$ for any $V \in \mathcal{L}(\mathbb{F}^n)$. Furthermore $\dim \mathbb{F}^{n\times m}(V,\mathrm{c}) = m\dim V$. Since $\mathcal{C}\cap\mathbb{F}^{n\times m}(V^\perp,\mathcal{C}) = \mathcal{C}(V^\perp,\mathrm{c})$ and $m^{-1}\dim\mathcal{C} = \rho_{\mathrm{c}}(\mathbb{F}^n)$, the very definition of ρ_{c} leads to

$$\dim(\mathcal{C}\cap\mathbb{F}^{n\times m}(V^\perp,\mathcal{C})) \geq i \Longleftrightarrow \rho_{\mathrm{c}}(V) \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - i/m. \tag{2.16}$$

Now the statement follows from $n - \dim V = \dim V^\perp = m^{-1}\dim\mathbb{F}^{n\times m}(V^\perp,\mathcal{C})$. In the case $m = n$, the optimal anticodes are of the form $\mathbb{F}^{m\times m}(V,\mathrm{c})$ and $\mathbb{F}^{m\times m}(V,\mathrm{r})$, where $V \in \mathcal{L}(\mathbb{F}^m)$. This implies that we have to take the row and column q-PM into account, and again the statement follows. □

Let us briefly relate $a_i^{\mathcal{C}}(\mathcal{C})$ to certain invariants introduced in the literature.

Remark 2.3.22. *(a) In [22, Def. 10] the authors introduce generalized weights of q-PMs. Since their definition of q-PMs differs from ours, we need to be careful when comparing the two notions of generalized weights. Let us start with a q-PM* $\mathcal{M} = (\mathbb{F}^n, \rho)$ *in our sense and let* μ *be a denominator of* $\mathcal{M}$*. Set* $\tau := \mu\rho$*. Then* $\mathcal{M}' := (\mathbb{F}^n, \tau)$ *is a* (q, μ)*-polymatroid in the sense of [22, Def. 1]. In [22, Def. 10] the authors define the i-th generalized weight of* $\mathcal{M}'$ *as* $d_i(\mathcal{M}') = \min\{\dim X \mid \tau(\mathbb{F}^n) - \tau(X^\perp) \geq i\}$*. Suppose now that* $\mathcal{M} = \mathcal{M}_{\rm c}(\mathcal{C})$ *for some rank-metric code* $\mathcal{C} \leq \mathbb{F}^{n\times m}$*. Then* $\mu := m$ *is a denominator and the equivalence* $\rho(V) \leq \rho(\mathbb{F}^n) - i/m \iff \tau(\mathbb{F}^n) - \tau(V) \geq i$ *shows that* $d_i(\mathcal{M}') = a^i_{\mathcal{C}}(\mathcal{C})$ *for all* i*.*

(b) In [7, Thm. 8] the authors associate a q-demimatroid to a rank-metric code. A q-demimatroid is a generalization of a q-PM that does not require submodularity, but captures a certain duality relation; see [7, p. 1505]. More precisely, to the code $\mathcal{C} \leq \mathbb{F}^{n\times m}$ *they associate the q-demimatroid* $(\mathbb{F}^n, s, t)$ *defined by* $s(V) = \dim \mathcal{C}(V, {\rm c})$ *and* $t(V) = \dim \mathcal{C}^\perp(V, {\rm c})$*. Neither of these functions is submodular, but they satisfy the required duality relation thanks to* (2.10)*. In [7, p. 1507] the authors introduce various combinatorial invariants of such q-demimatroids, and it is straightforward to verify that* $a^{\mathcal{C}}_i(\mathcal{C})$ *equals their* σ_i *for all* i*. The power of their approach comes to light in particular in [7, Sec. 4.2 and 4.3], where the authors provide a very nice and short proof of the Wei duality for the generalized weights of a rank-metric code with the aid of the associated q-demimatroid.*

We return to Theorem 2.3.21 and consider the case $m = n$ in further detail. As we show next, for $i = 1$ we have $a_1(\mathcal{C}) = a^{\mathcal{C}}_1(\mathcal{C}) = a^{\rm r}_1(\mathcal{C})$.

Corollary 2.3.23. *Let* $\mathcal{C} \leq \mathbb{F}^{m\times m}$ *be a code. Then* $a^{\mathcal{C}}_1(\mathcal{C}) = a^{\rm r}_1(\mathcal{C})$ *and therefore*

$$\begin{aligned} {\rm d}_{\rm rk}(\mathcal{C}) &= m - \max\{\dim V \mid V \in \mathcal{L}(\mathbb{F}^m),\ \rho_{\rm c}(V) \leq \rho_{\rm c}(\mathbb{F}^m) - 1/m\} \\ &= m - \max\{\dim V \mid V \in \mathcal{L}(\mathbb{F}^m),\ \rho_{\rm r}(V) \leq \rho_{\rm r}(\mathbb{F}^m) - 1/m\}. \end{aligned}$$

Recall from Proposition 2.3.3 that $\rho_{\rm c}(\mathbb{F}^m) = \rho_{\rm r}(\mathbb{F}^m) = m^{-1}\dim \mathcal{C}$ for any code $\mathcal{C} \leq \mathbb{F}^{m\times m}$. Thus the above inequalities can be written as $\rho_{\rm c}(V) \leq (\dim \mathcal{C} - 1)/m$ and $\rho_{\rm r}(V) \leq (\dim \mathcal{C} - 1)/m$.

Proof. It is well known [41, Thm. 30] that ${\rm d}_{\rm rk}(\mathcal{C}) = a_1(\mathcal{C})$, the first generalized weight. Hence it suffices to show $a^{\mathcal{C}}_1(\mathcal{C}) = a^{\rm r}_1(\mathcal{C})$. Let $d = {\rm d}_{\rm rk}(\mathcal{C})$ and $M \in \mathcal{C}$ be such that ${\rm rk}(M) = d$. Let $V = {\rm colsp}(M)$ and $W = {\rm rowsp}(M)$ and set $\mathcal{A} = \mathbb{F}^{m\times m}(V, {\rm c})$ and $\mathcal{A}' = \mathbb{F}^{m\times m}(W, {\rm r})$. Then $\mathcal{A}$ and $\mathcal{A}'$ are optimal anticodes of dimension md and $\mathcal{C} \cap \mathcal{A} \neq 0 \neq \mathcal{C} \cap \mathcal{A}'$. Thus (2.16) yields $\rho_{\rm c}(V^\perp) \leq \rho_{\rm c}(\mathbb{F}^m) - 1/m$ and, similarly, $\rho_{\rm r}(W^\perp) \leq \rho_{\rm r}(\mathbb{F}^m) - 1/m$. Since V and W are clearly of minimal dimension satisfying $\mathcal{C} \cap \mathcal{A} \neq 0 \neq \mathcal{C} \cap \mathcal{A}'$, all of this shows that $a^{\mathcal{C}}_1(\mathcal{C}) = a^{\rm r}_1(\mathcal{C})$. □

For $i > 1$ and $\mathcal{C} \leq \mathbb{F}^{m\times m}$ we have in general $a^{\mathcal{C}}_i(\mathcal{C}) \neq a^{\rm r}_i(\mathcal{C})$, and it depends on i which of the two is the minimum. Our next example shows that this is even the case for *vector rank-metric codes* (that is, $\mathbb{F}_{q^m}$-linear codes in $\mathbb{F}^m_{q^m}$). As a consequence, the generalized weights in Definition 2.3.20 do not coincide with the generalized weights introduced in [37]. The example also addresses a small oversight in [41] and shows that [41, Thm. 28] is only true for codes in $\mathbb{F}^n_{q^m}$ with $n < m$.

Example 2.3.24. *Let $q = 2$ and $m = 6$. Let $\omega \in \mathbb{F}_{2^6}$ be a primitive element with minimal polynomial $f = x^6 + x^4 + x^3 + x + 1$. Consider the generator matrix*

$$G = \begin{pmatrix} 1 & 0 & 0 & \omega^9 & 0 & 0 \\ 0 & 1 & 0 & 0 & \omega^{18} & 0 \\ 0 & 0 & 1 & 0 & 0 & \omega^{18} \end{pmatrix} \in \mathbb{F}_{2^6}^{3\times 6}.$$

Note that, in fact, the entries of G are in the subfield $\mathbb{F}_{2^3}$. Using the isomorphism Ψ from (2.13), we obtain the rank-metric code

$$\Psi(\mathrm{rowsp}_{\mathbb{F}_{2^6}}(G)) = \mathcal{C} := \langle A_j \Delta_f^i \mid j = 1, 2, 3,\ i = 0, \ldots, m-1 \rangle,$$

where Δ_f is as in (2.12) and A_j is the image of the j-th row of G under Ψ. Thus $\dim_{\mathbb{F}_2} \mathcal{C} = 18$. By construction, the row spaces of the matrices A_1, A_2, A_3 are contained in the 3-dimensional subspace $\hat{V} := \Psi(\mathbb{F}_{2^3}) \leq \mathbb{F}_2^6$. It turns out that $\dim \mathcal{C}(\hat{V}, r) = 9$ (which implies $a_9^r \leq 3$). SageMath computations lead to the following data

Table 2.2: Row and column generalized weights of $\mathcal{C}$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$a_i^{\mathcal{C}}$	2	2	2	2	2	2	4	4	4	4	4	4	6	6	6	6	6	6
a_i^{r}	2	2	2	3	3	3	3	3	3	5	5	5	6	6	6	6	6	6

We see that, for instance $a_9^r < a_9^{\mathcal{C}}$, whereas $a_{10}^r > a_{10}^{\mathcal{C}}$. This implies that the generalized weights $a_i(\mathcal{C})$ in general do not coincide with $a_i^{\mathcal{C}}(\mathcal{C})$. In [37] the latter are defined to be the generalized weights of $\mathcal{C}$ (see [37, Cor. 4.4, Thm. 5.4] and [41, Thm. 18]). This shows that the definitions of generalized weights in [41, Def. 23] and [37] do not agree for $\mathbb{F}_{q^m}$-linear codes in $\mathbb{F}_{q^m}^m$.

We return to general codes in $\mathbb{F}^{n\times m}$. Our next result shows that the generalized weights of a rank-metric code are determined by the flats of the q-PM. It is straightforward to check that this is the analogue of [6, Thm. 3], where the generalized weights of a linear block code are determined via cocircuits of suitable truncations of the associated matroid. The complements of these cocircuits are, by definition, flats of a certain rank in the original matroid. The only difference to our result below is the inequality (rather than equality) in $\rho_c(V) \leq \rho_c(\mathbb{F}^n) - i/m$ below. This is needed because – in contrast to matroids and q-matroids – for q-PMs equality may not be attained.

Theorem 2.3.25. *Let $m \geq n$ and $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a code. Let $\mathcal{F}_{\mathcal{C}}$ and $\mathcal{F}_r$ be the sets of flats of the q-PMs $\mathcal{M}_c(\mathcal{C}) = (\mathbb{F}^n, \rho_c)$ and $\mathcal{M}_r(\mathcal{C}) = (\mathbb{F}^m, \rho_r)$, respectively. Define*

$$b_i^{\mathcal{C}}(\mathcal{C}) = n - \max\{\dim V \mid V \in \mathcal{F}_{\mathcal{C}},\ \rho_c(V) \leq \rho_c(\mathbb{F}^n) - i/m\},$$
$$b_i^r(\mathcal{C}) = m - \max\{\dim W \mid W \in \mathcal{F}_r,\ \rho_r(W) \leq \rho_r(\mathbb{F}^m) - i/n\}.$$

Then for all $i = 1, \ldots, \dim \mathcal{C}$

$$a_i(\mathcal{C}) = \begin{cases} b_i^{\mathcal{C}}(\mathcal{C}), & \text{if } m > n, \\ \min\{b_i^{\mathcal{C}}(\mathcal{C}), b_i^r(\mathcal{C})\}, & \text{if } m = n. \end{cases}$$

Furthermore, if $n = m$ then $b_1^{\mathcal{C}}(\mathcal{C}) = b_1^{\mathrm{r}}(\mathcal{C})$.

Proof. We will show $b_i^{\mathcal{C}}(\mathcal{C}) = a_i^{\mathcal{C}}(\mathcal{C})$ for all i. The proof for $b_i^{\mathrm{r}}(\mathcal{C}) = a_i^{\mathrm{r}}(\mathcal{C})$ is analogous. Clearly, $a_i^{\mathcal{C}}(\mathcal{C}) \leq b_i^{\mathcal{C}}(\mathcal{C})$ for all i. For the converse inequality, let $V \in \mathcal{L}(\mathbb{F}^n)$ such that $\rho_{\mathrm{c}}(V) \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - i/m$. Consider $\mathrm{cl}(V)$, the closure of V. Then $\dim V \leq \dim \mathrm{cl}(V)$ by (CL1) and $\rho_{\mathrm{c}}(\mathrm{cl}(V)) \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - i/m$ thanks to (2.5). All of this implies the desired inequality $b_i^{\mathcal{C}}(\mathcal{C}) \leq a_i^{\mathcal{C}}(\mathcal{C})$ for all i. The last statement for $i = 1$ follows from Corollary 2.3.23. $\square$

Our final result allows us to determine the rank distance of a rank-metric code via the hyperplanes of the associated q-PM.

Corollary 2.3.26. *Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a code.*
(a) If $m > n$, then $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = n - \max\{\dim H \mid H$ is a hyperplane in $\mathcal{M}_{\mathrm{c}}(\mathcal{C})\}$.
(b) If $m = n$, then $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = n - w$, where

$$\begin{aligned} w &= \max\{\dim H \,|\, H \text{ is a hyperplane in } \mathcal{M}_{\mathrm{c}}(\mathcal{C})\} \\ &= \max\{\dim H \,|\, H \text{ is a hyperplane in } \mathcal{M}_{r}(\mathcal{C})\}. \end{aligned}$$

Proof. It is well known [41, Thm. 30] that $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = a_1(\mathcal{C})$. Suppose first that $m > n$. Denote by $\mathcal{F}_{\mathcal{C}}$ the collection of flats of the q-PM $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$. Theorem 2.3.25 implies $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = n - v$, where $v = \max\{\dim V \mid V \in \mathcal{F}_{\mathcal{C}},\, \rho_{\mathrm{c}}(V) \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - 1/m\}$. Let $V \in \mathcal{F}_{\mathcal{C}}$ be such that $\dim V = v$ and $\rho_{\mathrm{c}}(V) \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - 1/m$. Suppose V is not a hyperplane. Then there exists $V' \in \mathcal{F}_{\mathcal{C}}$ such that $V \lneq V' \lneq \mathbb{F}^n$. Since all these spaces are flats we have $\rho_{\mathrm{c}}(V) < \rho_{\mathrm{c}}(V') < \rho_{\mathrm{c}}(\mathbb{F}^n)$. Using that m is a denominator of $\mathcal{M}_{\mathrm{c}}(\mathcal{C})$, this yields $\rho_{\mathrm{c}}(V') \leq \rho_{\mathrm{c}}(\mathbb{F}^n) - 1/m$. Now $\dim V' > v$ leads to a contradiction. Thus V is a hyperplane. This concludes the case $m > n$. For $m = n$ we know from Corollary 2.3.23 that $\mathrm{d}_{\mathrm{rk}}(\mathcal{C}) = a_1^{\mathcal{C}}(\mathcal{C}) = a_1^{\mathrm{r}}(\mathcal{C})$. Thus the result follows from the first part of this proof. $\square$

2.3.3 Shortening/Puncturing of codes and Deletion/Contraction of q-polymatroids.

This subsection is devoted to the relation between deletion and contraction of q-PMs induced by rank-metric codes and puncturing and shortening of the codes. We focus on row puncturing and shortening, which will correspond to deletion and contraction of the associated column q-PM. The following terminology is from [12, Sec. 3].

Definition 2.3.27. *Let $u \in [n]$ and $\pi_u : \mathbb{F}^{n\times m} \longrightarrow \mathbb{F}^{(n-u)\times m}$ be the projection onto the last $n - u$ rows. Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a rank-metric code. We define*

$$\mathcal{C}_u = \{M \in \mathcal{C} \mid M_{ij} = 0 \text{ for } i \leq u\},$$

that is, $\mathcal{C}_u$ is the subcode consisting of all matrices of $\mathcal{C}$ whose first u rows are zero. Let $A \in \mathrm{GL}_n(\mathbb{F})$.
(a) The puncturing *of $\mathcal{C}$ w.r.t. A and u is defined as the code*

$$\Pi(\mathcal{C}, A, u) = \pi_u(A\mathcal{C}) \leq \mathbb{F}^{(n-u)\times m}.$$

(b) The shortening *of $\mathcal{C}$ w.r.t. A and u is defined as the code*

$$\Sigma(\mathcal{C}, A, u) = \pi_u((A\mathcal{C})_u) \leq \mathbb{F}^{(n-u)\times m}.$$

In Proposition-Definition 2.3.2(d) we called the space $\mathcal{C}(V, \mathrm{c})$ a shortening of $\mathcal{C}$. This is consistent because $\mathcal{C}(V, \mathrm{c})$ is isomorphic (even isometric w.r.t. the rank-metric) to $\Sigma(\mathcal{C}, A, u)$, where $u = \dim V$ and $A = (A_1 \mid A_2)^\mathsf{T} \in \mathrm{GL}_n(\mathbb{F})$ such that $\mathrm{colsp}(A_1) = V^\perp$.

Now we are ready to relate deletion and contraction of q-PMs induced by rank-metric codes to puncturing and shortening of the codes, respectively. The following result is a direct q-analogue of the well-known relation between matroids and linear block codes; see [35, Sec. 1.6.4].

Theorem 2.3.28. *Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be a rank-metric code and $\mathcal{M} = \mathcal{M}_\mathrm{c}(\mathcal{C})$ be the associated column q-PM. Let $X \in \mathcal{L}(\mathbb{F}^n)$ and set* $\dim X = u$. *Choose any matrix $A \in \mathrm{GL}_n(\mathbb{F})$ of the form $A = \binom{B}{D}$, where $D \in \mathbb{F}^{(n-u)\times n}$ is such that* $\mathrm{rowsp}(D) = X^\perp$. *Then*

$$\mathcal{M} \setminus X \approx \mathcal{M}_\mathrm{c}(\Pi(\mathcal{C}, A, u)) \quad \textit{and} \quad \mathcal{M}/X \approx \mathcal{M}_\mathrm{c}(\Sigma(\mathcal{C}, (A^{-1})^\mathsf{T}, u)).$$

Proof. We start with the first equivalence. Let ρ be the rank function of $\mathcal{M}_\mathrm{c}(\mathcal{C})$ and ρ' be the rank function of $\mathcal{M}_\mathrm{c}(\mathcal{C}')$, where $\mathcal{C}' = \Pi(\mathcal{C}, A, u)$. Thus

$$\rho' : \mathcal{L}(\mathbb{F}^{n-u}) \longrightarrow \mathbb{Q}_{\geq 0},\ V \longmapsto \frac{\dim \mathcal{C}' - \dim \mathcal{C}'(V^\perp, \mathrm{r})}{m}.$$

Consider the isomorphism

$$\psi : \mathbb{F}^{n-u} \longrightarrow X^\perp,\ v \longmapsto vD.$$

Then $\psi(V) \leq X^\perp$ for all $V \in \mathcal{L}(\mathbb{F}^{n-u})$ and therefore $\rho_{\mathcal{M}\setminus X}(\psi(V)) = \rho(\psi(V))$ by the very definition of deletion. Now the desired equivalence $\mathcal{M} \setminus X \approx \mathcal{M}_\mathrm{c}(\Pi(\mathcal{C}, A, u))$ follows if we can show that

$$\rho'(V) = \rho(\psi(V)) \text{ for all } V \in \mathcal{L}(\mathbb{F}^{n-u}). \tag{2.17}$$

In order to do so, let $V \in \mathcal{L}(\mathbb{F}^{n-u})$ and let $Z \in \mathbb{F}^{t\times(n-u)}$ be such that $V = \mathrm{rowsp}(Z)$. From the very definition of the matrices involved we obtain

$$\begin{aligned}
\mathcal{C}(X, \mathcal{C}) &= \{M \in \mathcal{C} \mid \mathrm{colsp}(M) \leq X\} = \{M \in \mathcal{C} \mid DM = 0\},\\
\mathcal{C}(\psi(V)^\perp, \mathcal{C}) &= \{M \in \mathcal{C} \mid \mathrm{colsp}(M) \leq \psi(V)^\perp\} = \{M \in \mathcal{C} \mid ZDM = 0\},\\
\mathcal{C}'(V^\perp, \mathcal{C}) &= \{N \in \mathcal{C}' \mid \mathrm{colsp}(N) \leq V^\perp\} = \{N \in \mathcal{C}' \mid ZN = 0\}.
\end{aligned}$$

Furthermore, we have the surjective linear map

$$\xi_D : \mathcal{C} \longrightarrow \mathcal{C}',\ M \longmapsto DM$$

and observe that $\mathcal{C}(X, \mathcal{C}) = \ker \xi_D \leq \mathcal{C}(\psi(V)^\perp, \mathcal{C})$ as well as $\xi_D\big(\mathcal{C}(\psi(V)^\perp, \mathcal{C})\big) = \mathcal{C}'(V^\perp, \mathcal{C})$. Hence we conclude

$$\dim \mathcal{C} - \dim \mathcal{C}' = \dim \mathcal{C}(X, \mathcal{C}) = \dim \mathcal{C}(\psi(V)^\perp, \mathcal{C}) - \dim \mathcal{C}'(V^\perp, \mathcal{C}).$$

As a consequence,

$$\frac{\dim \mathcal{C}' - \dim \mathcal{C}'(V^\perp, \mathcal{C})}{m} = \frac{\dim \mathcal{C} - \dim \mathcal{C}(\psi(V)^\perp, \mathcal{C})}{m},$$

and this establishes (2.17).
We now turn to the second equivalence. Applying the first equivalence to the code $\mathcal{C}^\perp$ we obtain $\mathcal{M}_c(\mathcal{C}^\perp) \setminus X \approx \mathcal{M}_c(\Pi(\mathcal{C}^\perp, A, u))$. With the aid of Theorems 2.1.18 and 2.3.19 this leads to

$$\mathcal{M}_c(\mathcal{C})/X \approx (\mathcal{M}_c(\mathcal{C}^\perp) \setminus X)^* \approx \mathcal{M}_c(\Pi(\mathcal{C}^\perp, A, u))^* = \mathcal{M}_c(\Pi(\mathcal{C}^\perp, A, u)^\perp).$$

Now the desired equivalence follows from $\Pi(\mathcal{C}^\perp, A, u)^\perp = \Sigma(\mathcal{C}, (A^{-1})^\mathsf{T}, u)$, which has been proven in [12, Thm. 3.5] (and is true for $n \le m$ and $n > m$). □

2.4 Representability of q-Polymatroids.

In this section we discuss representability of q-PMs via rank-metric codes. We will present various examples of q-matroids that are not representable via $\mathbb{F}_{q^m}$-linear rank-metric codes (thereby answering a question from [36, Sec. 11]). Later in this section we will show that a q-matroid is not even representable via any $\mathbb{F}_q$-linear rank-metric code. We first fix the following notions of representability. Recall $\mathbb{F}_{q^m}$-linearity from Definition 2.3.7. As before, $\mathbb{F} = \mathbb{F}_q$.

Definition 2.4.1. *Let E be an n-dimensional $\mathbb{F}$-vector space and $\mathcal{M} = (E, \rho)$ be a q-PM.*
(a) $\mathcal{M}$ is said to be $\mathbb{F}^{n\times m}$-representable *if there exists a rank-metric code $\mathcal{C} \le \mathbb{F}^{n\times m}$ such that $\mathcal{M} \approx \mathcal{M}_c(\mathcal{C})$.*
(b) Suppose $\mathcal{M}$ is a q-matroid. Then $\mathcal{M}$ is said to be $\mathbb{F}_{q^m}$-representable *if there exists a right $\mathbb{F}_{q^m}$-linear code $\mathcal{C} \le \mathbb{F}^{n\times m}$ such that $\mathcal{M} \approx \mathcal{M}_c(\mathcal{C})$.*

For $\mathbb{F}_{q^m}$-representability, it will be necessary to consider $\mathbb{F}_{q^m}$-linear codes as subspaces of $\mathbb{F}_{q^m}^n$, as previously discussed and consider the rank function of the induced q-matroid as in Lemma 2.3.9.

We first start with the following example that shows that uniform q-matroids are always representable over a field large enough.

Example 2.4.2. *Let $\mathcal{U}_{k,n}(q)$ be the* uniform q-matroid *of rank k with ground space $\mathbb{F}_q^n$; that is, the rank function is given by $\rho(V) = \min\{k, \dim V\}$ for all $V \in \mathcal{L}(\mathbb{F}_q^n)$. Then $\mathcal{U}_{0,n}(q)$ and $\mathcal{U}_{n,n}(q)$ are representable over $\mathbb{F}_q$. Precisely, $\mathcal{U}_{0,n}(q)$ is represented by the $1 \times n$-zero matrix and $\mathcal{U}_{n,n}(q)$ by the $n \times n$-identity matrix. For $0 < k < n$, the uniform q-matroid $\mathcal{U}_{k,n}(q)$ is representable over $\mathbb{F}_{q^m}$ if and only if $m \ge n$. Indeed, a matrix $G \in \mathbb{F}_{q^m}^{k\times n}$ represents $\mathcal{U}_{k,n}(q)$ iff $\mathrm{rk}(GY^\mathsf{T}) = k$ for all $Y \in \mathbb{F}_q^{k\times n}$ of rank k. But this is equivalent to G generating an MRD code [21, Thm. 2 and 3] and such a matrix G exists if and only if $m \ge n$; see [28, Rem. 3.10].*

In general, determining whether a q-PM, or even a q-matroid, is representable is an extremely difficult problem. For instance in classical matroid theory the notion of representable matroids has been intensively studied, yet the classification of non-representable matroids is still an open problem.

To illustrate, consider the following example which is a follow up from Example 2.2.31.

Example 2.4.3. *Let $\mathcal{M} = (\mathbb{F}_2^8, \rho)$ be the q-matroid from example Example 2.2.31. Then $\mathcal{M}$ is representable via the $\mathbb{F}_{q^m}$-linear rank-metric code generated by the following matrix. This matrix was found by a carefully crafted random search.*

$$G = \begin{pmatrix} 1 & \omega^{26772} & 0 & \omega^{43180} & 0 & 0 & \omega^{46265} & \omega^{31452} \\ 0 & 0 & 1 & \omega^{3844} & 0 & 0 & \omega^{8371} & \omega^{59093} \\ 0 & 0 & 0 & 0 & 1 & 0 & \omega^{45712} & \omega^{50716} \\ 0 & 0 & 0 & 0 & 0 & 1 & \omega^{12688} & \omega^{10916} \end{pmatrix} \in \mathbb{F}_{2^{16}}^{4\times 8}$$

where ω is a primitive element satisfying $\omega^{16} + \omega^5 + \omega^3 + \omega^2 + 1 = 0$. It is not clear to us whether $\mathcal{M}$ is representable over a smaller field.

In light of this example, we construct example of q-(poly)matroids that are not representable. We start with $\mathbb{F}_{q^m}$-representability of q-matroids. Of course, a q-matroid $\mathcal{M} = (\mathbb{F}^n, \rho)$ may be representable over some field $\mathbb{F}_{q^m}$, but not over any smaller field extension of $\mathbb{F}$. This is for instance the case for $\mathcal{M} = \mathcal{M}_G$, where

$$G = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & \alpha^2 & \alpha^4 \end{pmatrix} \in \mathbb{F}_{2^4}^{2\times 4},$$

where $\alpha^4+\alpha+1 = 0$. It can be verified with SageMath or any other computer algebra system that $\mathcal{M}_G = (\mathbb{F}_2^4, \rho_{\rm c})$ is not representable over $\mathbb{F}_{2^l}$ for $l \leq 3$. In fact, $\mathcal{M}_G$ is not even $\mathbb{F}_2^{4\times l}$-representable for any $l \leq 3$.

We now construct q-matroids that are not $\mathbb{F}_{q^m}$-representable for any $m \in \mathbb{N}$. In order to do so, we will make use of non-representable (classical) matroids. Recall that a matroid is a pair $M = (X, r)$, where X is a finite set and $r : 2^X \longrightarrow \mathbb{N}_0$ satisfies (R1)–(R3) from Definition 2.1.1 if we replace the dimension, resp. sum, of subspaces by the cardinality, resp. union, of subsets. The rank of M is defined as $r(X)$. A matroid $M = (X, r)$ is called *representable* over the field F if there exists $k \in \mathbb{N}$ and a matrix $G \in F^{k\times|X|}$ with columns indexed by the elements of X, such that for any subset $A \subseteq X$ we have $r(A) = \mathrm{rk}(G_A)$, where $G_A \in F^{k\times|A|}$ is the submatrix of G consisting of the columns with indices in A. It is easy to see that if such a matrix exists then we may choose $k = r(X)$.

The next result will provide us with a crucial link between q-matroids and matroids.

Theorem 2.4.4. *Let $\mathcal{M} = (\mathbb{F}^n, \rho)$ be a q-matroid and $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $\mathbb{F}^n$. Define*

$$r : 2^{\mathcal{B}} \longrightarrow \mathbb{N}_0, \quad A \longmapsto \rho(\langle A\rangle).$$

Then $(\mathcal{B}, r)$ is a matroid, denoted by $M(\mathcal{M}, \mathcal{B})$. Furthermore, if $\mathcal{M}$ is $\mathbb{F}_{q^m}$-representable, then $M(\mathcal{M}, \mathcal{B})$ is $\mathbb{F}_{q^m}$-representable.

Proof. 1) We show that r is indeed a rank function. (R1) and (R2) are clear. For (R3) let $A, B \subseteq \mathcal{B}$. By basic Linear Algebra $\langle A \cup B \rangle = \langle A \rangle + \langle B \rangle$ and $\langle A \cap B \rangle = \langle A \rangle \cap \langle B \rangle$, and therefore

$$\begin{aligned} r(A \cup B) + r(A \cap B) &= \rho(\langle A \cup B \rangle) + \rho(\langle A \cap B \rangle) = \rho(\langle A \rangle + \langle B \rangle) + \rho(\langle A \rangle \cap \langle B \rangle) \\ &\leq \rho(\langle A \rangle) + \rho(\langle B \rangle) = r(A) + r(B). \end{aligned}$$

2) We now turn to the statement about representability. Let $\mathcal{M} = \mathcal{M}_G$ for some $G \in \mathbb{F}_{q^m}^{k \times n}$. Define the matrix $X = (v_1, \ldots, v_n)$, which is in $\mathrm{GL}_n(\mathbb{F})$, and set $G' := GX$. Index the columns of G' by $v_1, \ldots, v_n$, and for any $A \subseteq \mathcal{B}$ denote by G'_A the submatrix of G' with columns indexed by A. Let now $A = \{v_{a_1}, \ldots, v_{a_l}\}$ be any subset of $\mathcal{B}$ and $Y \in \mathbb{F}^{n \times l}$ be the matrix with columns $v_{a_1}, \ldots, v_{a_l}$. Then $\langle A \rangle = \mathrm{colsp}(Y)$ and thus

$$r(A) = \rho(\langle A \rangle) = \mathrm{rk}(GY) = \mathrm{rk}(G'X^{-1}Y) = \mathrm{rk}(G'(e_{a_1}, \ldots, e_{a_l})) = \mathrm{rk}(G'_A).$$

This shows that the matroid $M(\mathcal{B}, r)$ is represented by the matrix $G' \in \mathbb{F}_{q^m}^{k \times n}$. □

Having established this particular relation between q-matroids and induced matroids, we now turn to a class of matroids that can be interpreted as such induced matroids. They form a special case of paving matroids. A matroid is called *paving* if no circuit has cardinality less than the rank of the matroid. The following result is well known; see [39, 1.3.10] for a more general statement. We include an elementary proof which generalizes immediately to q-matroids.

Proposition 2.4.5. *Let $\mathcal{B}$ be a set with $|\mathcal{B}| = n$ and let $k \in [n]$. Let $\mathcal{A}$ be a collection of k-subsets of $\mathcal{B}$ such that $|A \cap B| \leq k-2$ for all distinct A, $B \in \mathcal{A}$. Define the map*

$$r : 2^{\mathcal{B}} \longrightarrow \mathbb{N}_0, \;\; X \longmapsto \begin{cases} k-1, & \text{if } X \in \mathcal{A}, \\ \min\{|X|, k\}, & \text{if } X \notin \mathcal{A}. \end{cases}$$

Then r is a rank function on $\mathcal{B}$. We denote the resulting matroid by $M_{\mathcal{B},\mathcal{A}}$. The circuits are given by the subsets in $\mathcal{A}$ and all $(k+1)$-subsets that do not contain a subset in $\mathcal{A}$.

Proof. (R1) and (R2) are clear. For (R3) we have to show $r(A \cup B) \leq r(A) + r(B) - r(A \cap B)$ for all subsets $A, B \subseteq \mathcal{B}$. We may assume $A \not\subseteq B$ and $B \not\subseteq A$ for otherwise the statement is clear. Note that $r(A \cup B) \leq k$ for all subsets. Consider $S := r(A) + r(B) - r(A \cap B)$ and note that $S \geq r(A) + r(B) - |A \cap B|$. We proceed by cases.
1) If $r(A) = r(B) = k$, then $S = 2k - r(A \cap B) \geq k$.
2) If $r(A) = |A|$ and $r(B) = k$, then $S \geq |A| + k - |A \cap B| \geq |A| + k - (|A| - 1) = k+1$.
3) If $r(A) = |A|$ and $r(B) = |B|$, then $S \geq |A| + |B| - |A \cap B| = |A \cup B| \geq r(A \cup B)$.
4) If A, $B \in \mathcal{A}$, then $S \geq 2k - 2 - |A \cap B| \geq k$.
5) If $A \in \mathcal{A}$ and $r(B) = k$, then $S \geq 2k - 1 - |A \cap B| \geq 2k - 1 - (k-1) = k$.
6) If $A \in \mathcal{A}$ and $r(B) = |B|$, then $S \geq k - 1 + |B| - |A \cap B| \geq k - 1 + |B| - (|B| - 1) = k$. □

The q-analogue reads as follows. The proof is entirely analogous to the previous one: just replace cardinality and union of subsets by dimension and sum of subspaces, respectively.

Proposition 2.4.6. *Let $n \in \mathbb{N}$ and fix an integer $k \in [n]$. Let $\mathcal{S}$ be a collection of k-spaces in $\mathbb{F}^n$ such that $\dim(V \cap W) \leq k - 2$ for all distinct V, $W \in \mathcal{S}$. Define the map*

$$\rho : \mathcal{L}(\mathbb{F}^n) \longrightarrow \mathbb{N}_0,\ V \longmapsto \begin{cases} k-1, & \text{if } V \in \mathcal{S}, \\ \min\{\dim V, k\}, & \text{if } V \notin \mathcal{S}. \end{cases}$$

Then $(\mathbb{F}^n, \rho)$ is a q-matroid. We denote it by $\mathcal{M}_{n,\mathbb{F},\mathcal{S}}$.

We can easily link the two constructions.

Proposition 2.4.7. *Let $\mathcal{B}$ be a basis of $\mathbb{F}^n$ and let $\mathcal{A}$ be a collection of k-subsets of $\mathcal{B}$ such that $|A \cap B| \leq k - 2$ for all distinct A, $B \in \mathcal{A}$. Set $\mathcal{S} = \{\langle A \rangle \mid A \in \mathcal{A}\}$. Then $\dim(V \cap W) \leq k - 2$ for all V, $W \in \mathcal{S}$ and $M_{\mathcal{B},\mathcal{A}} = M(\mathcal{M}_{n,\mathbb{F},\mathcal{S}}, \mathcal{B})$.*

Proof. It is easy to see that $\dim(V \cap W) \leq k - 2$ for all V, $W \in \mathcal{S}$. For the second statement let $M_{\mathcal{B},\mathcal{A}} = (\mathcal{B}, r)$ and $\mathcal{M}_{n,\mathbb{F},\mathcal{S}} = (\mathbb{F}^n, \rho)$. Then we have for any subset $X \subseteq \mathcal{B}$

$$r(X) = \left\{ \begin{array}{cc} k-1, & \text{if } X \in \mathcal{A}, \\ \min\{|X|, k\}, & \text{if } X \notin \mathcal{A}, \end{array} \right\} \text{ and}$$
$$\rho(\langle X \rangle) = \left\{ \begin{array}{cc} k-1, & \text{if } \langle X \rangle \in \mathcal{S}, \\ \min\{\dim \langle X \rangle, k\}, & \text{if } \langle X \rangle \notin \mathcal{S}. \end{array} \right\}$$

This proves the desired statement. □

Now we are ready to present examples of q-matroids with ground space $\mathbb{F}^n$ that are not $\mathbb{F}_{q^m}$-representable for any $m \in \mathbb{N}$. In each case the non-representability follows from the non-representability of the associated matroid with the aid of Theorem 2.4.4. The following examples are universal in the sense that they apply to a large class of fields $\mathbb{F}$.

Example 2.4.8. *(a) The Vamos Matroid [39, Ex. 2.1.25]: Let $n = 8$, $k = 4$ and*

$$\mathcal{A} = \big\{\{1, 2, 3, 4\},\ \{1, 4, 5, 6\},\ \{2, 3, 5, 6\},\ \{1, 4, 7, 8\},\ \{2, 3, 7, 8\}\big\}.$$

The matroid $V_8 := M_{[8],\mathcal{A}}$ is known as the Vamos matroid. It is not representable over any field [39, p. 169, Ex. 7(e)]. In fact, it is the smallest such matroid. Choose any finite field $\mathbb{F}$ and let ξ be a bijection between $[n]$ and a fixed basis $\mathcal{B}$ of $\mathbb{F}^n$. Then Theorem 2.4.4 and Proposition 2.4.7 tell us that the q-matroid $\mathcal{M}_{8,\mathbb{F}_q,\mathcal{S}}$, where $\mathcal{S} = \{\langle \xi(A) \rangle \mid A \in \mathcal{A}\}$, is not $\mathbb{F}_{q^m}$-representable for any $m \in \mathbb{N}$. A similar example can be constructed for $n = 9$, $k = 3$ and the non-Pappus matroid [39, Ex. 1.5.15].

(b) *The Fano Matroid [39, Ex. 1.5.7]: Let* $n = 7$, $k = 3$ *and*

$$\mathcal{A} = \big\{\{1,2,4\},\ \{2,3,5\},\ \{3,4,7\},\ \{1,5,7\},\ \{2,6,7\},\ \{4,5,6\},\ \{1,3,6\}\big\},$$

which is the set of lines in the Fano plane. The matroid $F_7 := M_{[7],\mathcal{A}}$ *is called the Fano matroid. It is not representable over any field of odd characteristic [39, Prop. 6.4.8]. Thus, if* q *is odd then the corresponding* q*-matroid* $\mathcal{M}_{7,\mathbb{F}_q,\mathcal{S}}$*, where* $\mathcal{S} = \{\langle \xi(A)\rangle \mid A \in \mathcal{A}\}$*, is not* $\mathbb{F}_{q^m}$*-representable for any* m *(here* ξ *is a bijection between* $[7]$ *and a basis of* $\mathbb{F}_q^7$*).*

(c) *The non-Fano Matroid: If one omits one of the sets in the collection* $\mathcal{A}$ *in (b), one obtains the non-Fano matroid* $F_7^- := M_{[7],\mathcal{A}'}$*, which is not representable over any field of even characteristic [39, Prop. 6.4.8]. Thus we have the analogous conclusion as in (b) for* q*-matroids with even* q*.*

It remains an open question whether any of the above q-matroids is $\mathbb{F}^{n\times m}$-representable. However, we have the following smaller example of a q-matroid $\mathcal{M}$ that is not $\mathbb{F}^{n\times m}$-representable for any m. It has been shown in [13, Sec. 3.3] that $\mathcal{M}$ is not $\mathbb{F}_{q^m}$-representable. This shows that the converse of Theorem 2.4.4 is not true because every matroid over a groundset of cardinality 4 is representable over any field of size at least 3. In fact, the q-matroid of the following theorem together with all possible choices of bases $\mathcal{B}$ in Theorem 2.4.4 lead – up to isomorphism – to the uniform matroid of rank 2 or to the paving matroids $M_{[4],\mathcal{A}}$ with $\mathcal{A} = \{\{1,2\}\}$ or $\mathcal{A} = \{\{1,2\},\{3,4\}\}$ (see the notation of Proposition 2.4.5). The latter two matroids are representable over every field.

Theorem 2.4.9. *Let* $\mathbb{F} = \mathbb{F}_2$ *and consider* $\mathcal{S} = \{V_0, V_1, V_2, V_3\} \subset \mathcal{L}(\mathbb{F}^4)$*, where*

$$V_0 = \langle 1000,\ 0100\rangle, V_1 = \langle 0010,\ 0001\rangle, V_2 = \langle 1001,\ 0111\rangle, V_3 = \langle 1011,\ 0110\rangle.$$

(We may also choose any other partial spread of size 4). Let $\mathcal{M} = \mathcal{M}_{4,\mathbb{F},\mathcal{S}}$ *(see Proposition 2.4.6), that is* $\mathcal{M} = (\mathbb{F}^4, \rho)$*, where*

$$\rho(V) = 1 \text{ for } V \in \mathcal{S} \quad \text{and} \quad \rho(V) = \min\{2, \dim V\} \text{ otherwise.}$$

Then $\mathcal{M}$ *is not* $\mathbb{F}^{4\times m}$*-representable for any* $m \in \mathbb{N}$*.*

Proof. Assume to the contrary that there exists $m \in \mathbb{N}$ and a rank-metric code $\mathcal{C} \leq \mathbb{F}^{4\times m}$ such that

$$\rho(V) = \frac{\dim \mathcal{C} - \dim \mathcal{C}(V^\perp, \mathrm{c})}{m} \quad \text{for all } V \in \mathcal{L}(\mathbb{F}^4).$$

Using $V = \mathbb{F}^4$, we see that $\dim \mathcal{C} = 2m$. Furthermore, the above values of $\rho(V)$ and the identity $\mathcal{S} = \{V_0^\perp, \ldots, V_3^\perp\}$ lead to

$$\dim \mathcal{C}(V, \mathrm{c}) = \begin{cases} 0, & \text{if } \dim V \leq 1 \text{ or } [\dim V = 2 \text{ and } V \notin \mathcal{S}], \\ m, & \text{if } V \in \mathcal{S} \text{ or } \dim V = 3. \end{cases} \tag{2.18}$$

The conditions $\dim\mathcal{C}(V_0,\mathcal{C}) = \dim\mathcal{C}(V_1,\mathcal{C}) = m$ and $\dim\mathcal{C}(V,\mathrm{c}) = 0$ if $\dim V = 1$ imply that $\mathcal{C}$ has a basis of the form

$$\begin{pmatrix} A_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ B_m \end{pmatrix}, \tag{2.19}$$

where $\mathcal{A} = \langle A_1, \ldots, A_m\rangle$ and $\mathcal{B} = \langle B_1, \ldots, B_m\rangle$ are MRD codes in $\mathbb{F}^{2\times m}$. As for the spaces V_2 and V_3 note that

$$V_2 = \mathrm{rowsp}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \mathrm{rowsp}\big(I_2 \mid S^{\mathsf{T}}\big),$$
$$V_3 = \mathrm{rowsp}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \mathrm{rowsp}\big(I_2 \mid T^{\mathsf{T}}\big),$$

where

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \; T = S^2 = S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, a matrix $M \in \mathcal{C}$ is in $\mathcal{C}(V_2,\mathcal{C})$ if and only if $M = \left(\begin{smallmatrix} A \\ SA \end{smallmatrix}\right)$ for some $A \in \mathcal{A}$. Hence there exist linearly independent matrices $\left(\begin{smallmatrix} \tilde{A}_i \\ S\tilde{A}_i \end{smallmatrix}\right)$, $i = 1, \ldots, m$, in $\mathcal{C}$. But then $\tilde{A}_1, \ldots, \tilde{A}_m \in \mathcal{A}$ must be linearly independent. Thus $\mathcal{B} = S\mathcal{A}$. Using the space V_3 we obtain similarly $\mathcal{B} = T\mathcal{A}$. Hence $\mathcal{A} = T^{-1}S\mathcal{A}$, and the latter is $T\mathcal{A}$. In other words, $\mathcal{A}$ is T-invariant. Note that $\{0, I, T, T^2\}$ is the subfield $\mathbb{F}_4$, and in particular, $T^2 = I + T$. All of this shows that $\mathcal{A}$ is an $\mathbb{F}_4$-vector space (thus has even dimension over $\mathbb{F}_2$) and has an $\mathbb{F}_2$-basis of the form $A_1, \ldots, A_\ell, TA_1, \ldots, TA_\ell$, where $\ell = m/2$. Using this basis of $\mathcal{A}$, (2.19) reads as

$$\begin{pmatrix} A_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} TA_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} TA_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ TA_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ TA_\ell \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ (I+T)A_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ (I+T)A_\ell \end{pmatrix}.$$

This shows that $\mathcal{C}$ contains the matrices $\left(\begin{smallmatrix} A_i + TA_i \\ A_i + TA_i \end{smallmatrix}\right)$ for $i = 1, \ldots, \ell$, and therefore $\dim\mathcal{C}(V,\mathrm{c}) \geq \ell$ for $V = \langle 1010, 0101\rangle$. This contradicts (2.18) and we conclude that there is no code $\mathcal{C} \leq \mathbb{F}^{4\times m}$ that represents the q-matroid $\mathcal{M}$. □

It is not yet clear whether there exists a q-matroid $\mathcal{M} = (\mathbb{F}^n, \rho)$ and $m \in \mathbb{N}$ such that $\mathcal{M}$ is not $\mathbb{F}_{q^m}$-representable but $\mathbb{F}^{n\times m}$-representable.

Chapter 3 An independent space cryptomorphism for q-polymatroids.

In this chapter, we introduce the notion of independent space for q-polymatroids. We show that the collection of independent spaces satisfies properties analogous to those for q-matroids. However, different from the latter, the independent spaces do not fully determine the q-PM. Only if we also take their rank values into account, can we fully recover the q-PM. We furthermore study the notion of spanning spaces and strongly independent spaces. Supporting examples for the theory developed in this chapter are postponed to Section 3.4. Results from this chapter also appear in [23].

3.1 Independent Spaces of q-polymatroids.

We start this section by introducing the notion of independent space for q-polymatroids. Considering the theory of classical matroids and q-matroids, one may be inclined to declare a space V in a q-PM (E, ρ) independent if $\rho(V) = \dim V$. While this is indeed the right notion for q-matroids, it turns out to be too restrictive for q-PMs: in many q-PMs the only subspace satisfying $\rho(V) = \dim V$ is the zero space (e.g. Example 3.4.2). Nonetheless, the property $\rho(V) = \dim V$ turns out to play a conceptual role (see also [9]), and we will return to it in Section 3.3, where we will call such spaces strongly independent.

The following definition of independence is inspired by [39, Cor. 11.1.2], which deals with classical polymatroids.

Definition 3.1.1. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ (which need not be principal). A space $I \in \mathcal{L}(E)$ is called* μ-independent *if*

$$\rho(J) \geq \frac{\dim J}{\mu} \text{ for all subspaces } J \leq I.$$

I is called μ-dependent *if it is not μ-independent. A* μ-circuit *is a μ-dependent space for which all proper subspaces are μ-independent. A 1-dimensional μ-dependent space is called a* μ-loop*. We define $\mathcal{I}_\mu = \mathcal{I}_\mu(\mathcal{M}) = \{I \in \mathcal{L}(E) \mid I \text{ is } \mu\text{-independent}\}$. If μ is the principal denominator of $\mathcal{M}$, we may skip the quantifier μ and simply use independent, dependent, loop, circuit, and $\mathcal{I}$.*

It is easy to see that the inequality $\rho(I) \geq \dim I/\mu$ is not preserved under taking subspaces (take for instance the subspace $I = V$ from Example 3.4.3), which is why the condition for subspaces is built into our definition. Clearly, if $\hat{\mu}$ is the principal denominator of $\mathcal{M}$, then $\hat{\mu}$-independence implies μ-independence for any denominator μ of $\mathcal{M}$. Furthermore, the zero subspace of E is μ-independent, and every dependent space V contains a circuit: take any subspace W of V of smallest dimension satisfying $\rho(W) < \dim W/\mu$ (which clearly exists).

Let us consider the independent spaces of the q-PMs induced by MRD codes.

Example 3.1.2. *Let $\mathcal{C} \leq \mathbb{F}^{n\times m}$ be an MRD code with rank distance d.*

(a) If $m \geq n$ then $\mathcal{I}_1(\mathcal{M}_\mathcal{C}(\mathcal{C})) = \{V \in \mathcal{L}(\mathbb{F}^n) \mid \dim V \leq n-d+1\}$. This follows from Proposition 2.3.11 and the fact that the independent spaces of uniform q-matroid $\mathcal{U} = \mathcal{U}_k(E)$ are exactly the space of dimension at most k.

(b) If $m \leq n$ then $\mathcal{I}_m(\mathcal{M}_\mathcal{C}(\mathcal{C})) = \mathcal{L}(\mathbb{F}^n)$, which can be verified with the aid of Theorem 2.3.13.

For q-matroids our notion of independence coincides with independence in Definition 2.2.7.

Proposition 3.1.3. *Let (E, ρ) be a q-PM. Then for all $V \in \mathcal{L}(E)$*

$$\rho(V) = \dim V \implies \rho(W) = \dim W \text{ for all } W \leq V.$$

As consequence, if (E, ρ) is a q-matroid, then V is 1-independent iff $\rho(V) = \dim V$.

Proof. Writing $V = W \oplus Z$ for some complement Z of W, we obtain $\dim V = \rho(V) \leq \rho(W) + \rho(Z) \leq \dim W + \dim Z = \dim V$, and thus we have equality everywhere. □

We continue with discussing basic properties of independent spaces, thereby focusing on the differences to q-matroids. Supporting examples are given in Section 3.4.

Remark 3.1.4. *(a) While in a q-matroid a space is independent iff its rank value assumes the maximal possible value, this is not the case for q-PMs. More precisely, independent spaces of the same dimension need not have the same rank value. This is illustrated by Example 3.4.2.*

(b) Dependent spaces may have a larger rank value than independent spaces of the same dimension; see Example 3.4.3.

(c) Let $V \in \mathcal{L}(E)$ be a μ-circuit. Then $\mu\rho(V) = \dim V - 1 = \mu\rho(W)$ for all hyperplanes W in V. Indeed, independence of W along with (R2) tells us that $\dim V - 1 = \dim W \leq \mu\rho(W) \leq \mu\rho(V) < \dim V$. Thus we have equality since $\mu\rho$ takes integer values. While in a q-matroid a subspace V satisfying $\mu\rho(V) = \dim V - 1 = \mu\rho(W)$ for all its hyperplanes W is a circuit, this is not the case for q-PMs; see Example 3.4.4.

(d) A q-PM with principal denominator μ is not uniquely determined by its collection of μ-independent spaces. For instance, in either of the non-equivalent q-PMs in Example 2.3.16 all subspaces are 2-independent. This example also shows that – different from q-matroids – a q-PM in which all spaces are μ-independent need not be a uniform q-matroid.

Independence behaves well under scaling-equivalence if the denominator is taken into account.

Remark 3.1.5. *Let $\dim E_1 = \dim E_2$ and $\mathcal{M}_i = (E_i, \rho_i), i = 1, 2$, be q-PMs with principal denominators μ_i. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are scaling-equivalent, say $\rho_2(\alpha(V)) = a\rho_1(V)$ for all $V \in \mathcal{L}(E_1)$, where $a \in \mathbb{Q}_{>0}$ and $\alpha : E_1 \longrightarrow E_2$ an isomorphism. Then $a^{-1}\mu_1\rho_2(\alpha(V)) = \mu_1\rho_1(V) \in \mathbb{N}$ and thus $a^{-1}\mu_1$ is a denominator of $\mathcal{M}_2$. Hence $a^{-1}\mu_1 = k\mu_2$ for some $k \in \mathbb{N}$; see Remark 2.1.4(b). Similarly, $a\mu_2 = \hat{k}\mu_1$ for some*

$\hat{k} \in \mathbb{N}$. *Thus* $k = \hat{k} = 1$ *and* $a\mu_2 = \mu_1$. *Now we have* $\mu_2\rho_2(\alpha(V)) = \mu_1\rho_1(V)$ *for all* $V \in \mathcal{L}(E)$ *and therefore*

$$\begin{aligned} V \text{ is } \mu_1\text{-independent in } \mathcal{M}_1 &\iff \mu_1\rho_1(W) \geq \dim W \text{ for all } W \leq V \\ &\iff \mu_2\rho_2(\alpha(W)) \geq \dim \alpha(W) \text{ for all } \alpha(W) \leq \alpha(V) \\ &\iff \alpha(V) \text{ is } \mu_2\text{-independent in } \mathcal{M}_2. \end{aligned}$$

In order to derive our main result about the collection of μ-independent spaces, we will make use of an auxiliary q-matroid. The following construction mimics the corresponding one in [39, Prop. 11.1.7] for classical polymatroids.

Theorem 3.1.6. *Let* $\mathcal{M} = (E, \rho)$ *be a q-PM with denominator* μ. *Define the map*

$$r_{\rho,\mu} : \mathcal{L}(E) \longrightarrow \mathbb{N}_0, \quad V \longmapsto \min\{\mu\rho(W) + \dim V - \dim W \mid W \leq V\}.$$

Then $\mathcal{Z} := \mathcal{Z}_{\mathcal{M},\mu} := (E, r_{\rho,\mu})$ *is a q-matroid, and the independent spaces of* $\mathcal{Z}$ *coincide with the* μ*-independent spaces of* $\mathcal{M}$, *i.e.,*

$$\mathcal{I}_\mu(\mathcal{M}) = \mathcal{I}(\mathcal{Z}) = \{I \in \mathcal{L}(E) \mid r_{\rho,\mu}(I) = \dim I\}.$$

Proof. Recall the induced integer ρ-function $\tau = \mu\rho$. Thus $\tau(V) = \mu\rho(V) \leq \mu \dim V$ for all $V \in \mathcal{L}(E)$. Clearly $r := r_{\rho,\mu}$ takes integer values. We now verify (R1)–(R3) of Definition 2.1.1 for r.

(R1) Obviously $r(V) \geq 0$ for all V. Furthermore, $r(V) \leq \tau(0) + \dim(V) - \dim(0) = \dim V$.

(R2) Let $V \leq V'$. It suffices to consider the case $\dim V' = \dim V + 1$ and thus $V' = V \oplus \langle x \rangle$ for some $x \in E$. Assume by contradiction that $r(V) > r(V')$. Then there exists $W' \leq V'$ such that

$$\tau(W') + \dim V' - \dim W' < \tau(W) + \dim V - \dim W \quad \text{for all } W \leq V. \tag{3.1}$$

Clearly $W' \not\leq V$ and thus we may write $W' = X \oplus \langle y \rangle$ for some $X \leq V$ and $y \notin V$. Then $\dim X = \dim V - \dim V' + \dim W'$ and (3.1) leads to

$$\tau(W) - \dim W > \tau(W') + \dim V' - \dim W' - \dim V = \tau(W') - \dim X \text{ for all } W \leq V.$$

Choosing $W = X$, we arrive at $\tau(X) > \tau(W')$ and thus $\rho(X) > \rho(W')$. Since $X \leq W'$ this contradicts that ρ is a rank function. All of this establishes (R2) for the map r.

(R3) Let $V, V' \in \mathcal{L}(E)$. Choose $W \leq V$, $W' \leq V'$ such that

$$r(V) = \tau(W) + \dim V - \dim W \quad \text{and} \quad r(V') = \tau(W') + \dim V' - \dim W'.$$

Then $W + W' \leq V + V'$ and $W \cap W' \leq V \cap V'$ and therefore

$$\begin{aligned} r(V+V') + r(V \cap V') &\leq \tau(W + W') + \dim(V + V') - \dim(W + W') \\ &\quad + \tau(W \cap W') + \dim(V \cap V') - \dim(W \cap W') \\ &= \tau(W+W') + \tau(W \cap W') + \dim V - \dim W \\ &\qquad + \dim V' - \dim W' \\ &\leq \tau(W) + \tau(W') + \dim V - \dim W + \dim V' - \dim W' \\ &= r(V) + r(V'), \end{aligned}$$

where the second inequality follows from (R3) for ρ. This establishes (R3) for the map r.
It remains to investigate the μ-independent spaces. From Definition 3.1.1 and (R1) we obtain

$$\begin{aligned} V \text{ is } \mu\text{-independent} &\iff \tau(W) \geq \dim W \text{ for all } W \leq V \\ &\iff \tau(W) + \dim V - \dim W \geq \dim V \text{ for all } W \leq V \\ &\iff r(V) \geq \dim V \\ &\iff r(V) = \dim V. \end{aligned}$$

Together with Proposition 3.1.3 this establishes the stated result. □

It should be noted that the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$ does not uniquely determine the q-PM $\mathcal{M}$, even if μ is the principal denominator. This can be seen from Example 2.3.16: both $\mathcal{M}_1$ and $\mathcal{M}_2$ have principal denominator 2 and in either q-PM all spaces are 2-independent (see also Example 3.1.2(b)). Hence Theorem 3.1.6 implies $\mathcal{Z}_{\mathcal{M}_1,2} = \mathcal{Z}_{\mathcal{M}_2,2} = \mathcal{U}_5(\mathbb{F}_2^5)$.
As we show next, a q-matroid $\mathcal{M}$ coincides with its auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},1}$.

Remark 3.1.7. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid, thus ρ takes only integer values.*
(a) We show that $\mathcal{Z}_{\mathcal{M},1} = \mathcal{M}$. The auxiliary rank function is $r_{\rho,1}(V) = \min\{\rho(W) + \dim V - \dim W \mid W \leq V\}$ for $V \in \mathcal{L}(E)$. Choosing $W = V$ we obtain $r_{\rho,1}(V) \leq \rho(V)$. For the opposite inequality, choose $W \leq V$. Then there exists $Z \leq V$ such that $W \oplus Z = V$ and submodularity (R3) yields

$$\rho(V) = \rho(W + Z) \leq \rho(W) + \rho(Z) \leq \rho(W) + \dim Z = \rho(W) + \dim V - \dim W.$$

Since W is arbitrary, this shows $\rho(V) \leq r_{\rho,1}(V)$ and thus $\mathcal{Z}_{\mathcal{M},1} = \mathcal{M}$.
(b) If we choose $\mu > 1$, then there is in general no obvious relation between $\mathcal{Z}_{\mathcal{M},\mu}$ and $\mathcal{M}$; see Example 3.4.5.

Theorem 3.1.6 shows that the μ-independent spaces of the q-PM $\mathcal{M}$ coincide with the independent spaces of the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$. Therefore, all properties of independent spaces of q-matroids, recall 2.2.10 that do not involve the value of the rank function hold true for q-PMs as well. Before formulating our result we cast the following important notions.

Definition 3.1.8. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ. For $V \in \mathcal{L}(E)$ we define*

$$\mathcal{I}_\mu(V) = \{I \in \mathcal{I}_\mu(\mathcal{M}) \mid I \leq V\}.$$

A subspace $\hat{I} \in \mathcal{I}_\mu(V)$ is said to be a μ-basis *of V if there exists no $J \in \mathcal{I}_\mu(V)$ such that $\hat{I} \lneq J$. We denote by $\mathcal{B}_\mu(V)$ the set of all μ-bases of V. The μ-bases of E are called the* μ-bases *of $\mathcal{M}$.*

The μ-bases of V are thus the inclusion-maximal μ-independent subspaces of V (i.e., the maximal elements of the poset $(\mathcal{I}_\mu(V), \leq)$). Their rank values will be discussed in the next section. Note that the sets $\mathcal{I}_\mu(V)$ and $\mathcal{B}_\mu(V)$ are non-empty for every $V \in \mathcal{L}(E)$ since $\{0\}$ is μ-independent.

We are now ready to present the following properties of the collection of μ-independent spaces of a q-PM. The result is an immediate consequence of Theorem 3.1.6 together with [36, Thm. 8].

Corollary 3.1.9. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ and set $\mathcal{I}_\mu := \mathcal{I}_\mu(\mathcal{M})$. Then*

(I1) $\mathcal{I}_\mu \neq \emptyset$, in fact $\{0\} \in \mathcal{I}_\mu$.

(I2) If $I \in \mathcal{I}_\mu$ and $J \leq I$, then $J \in \mathcal{I}_\mu$.

(I3) If I, $J \in \mathcal{I}_\mu$ and $\dim I < \dim J$, then there exists $x \in J \setminus I$ such that $I \oplus \langle x \rangle \in \mathcal{I}_\mu$.

(I4) Let V, $W \in \mathcal{L}(E)$ and $I \in \mathcal{B}_\mu(V)$, $J \in \mathcal{B}_\mu(W)$. Then there exists a basis $K \in \mathcal{B}_\mu(V + W)$ that is contained in $I + J$.

Note that (I3) implies that for any $V \in \mathcal{L}(E)$ we have

$$\mathcal{B}_\mu(V) = \{\hat{I} \in \mathcal{I}_\mu(V) \mid \hat{I} \text{ is dimension-maximal in } \mathcal{I}_\mu(V)\}. \tag{3.2}$$

Since the independent spaces of the q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$ coincide with those of the q-PM $\mathcal{M}$, the same is true for the dependent spaces, circuits, and bases. As a consequence, any property about the collection of these spaces in q-matroids holds true for q-PMs as well – as long as it does not involve the rank value. Let us illustrate this for the dependent spaces and bases. The following properties have been established in [11, Thm. 67] and [36, Thm. 37] for q-matroids and therefore apply to q-PMs as well.

Corollary 3.1.10. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ. Let $\mathcal{D}_\mu$ and $\mathcal{B}_\mu$ be the collection of μ-dependent spaces and μ-bases of $\mathcal{M}$, respectively. Then $\mathcal{D}_\mu$ and $\mathcal{B}_\mu$ satisfy*

(D1) $\{0\} \notin \mathcal{D}_\mu$.

(D2) If $D_1 \in \mathcal{D}_\mu$ and $D_2 \in \mathcal{L}(E)$ such that $D_1 \subseteq D_2$, then $D_2 \in \mathcal{D}_\mu$.

(D3) Let D_1, $D_2 \in \mathcal{D}_\mu$ be such that $D_1 \cap D_2 \notin \mathcal{D}_\mu$. Then every subspace of $D_1 + D_2$ of codimension 1 is in $\mathcal{D}_\mu$.

(B1) $\mathcal{B}_\mu \neq \emptyset$.

(B2) Let B_1, $B_2 \in \mathcal{B}_\mu$ be such that $B_1 \leq B_2$. Then $B_1 = B_2$.

(B3) Let B_1, $B_2 \in \mathcal{B}_\mu$ and A be a subspace of B_1 of codimension 1 such that $B_1 \cap B_2 \leq A$. Then there exists a 1-dimensional subspace Y of B_2 such that $A + Y \in \mathcal{B}_\mu$.

(B4) Let A_1, $A_2 \in \mathcal{L}(E)$ and I_1, I_2 be maximal dimensional intersections of some members of $\mathcal{B}_\mu$ with A_1 and A_2, respectively. Then there exist a maximal dimensional intersection of a member of $\mathcal{B}_\mu$ with $A_1 + A_2$ that is contained in $I_1 + I_2$.

In [11, Thm. 68] and [36, Thm. 37] it has been shown that any collection of subspaces satisfying (D1)–(D3) (resp. (B1)–(B4)) is the collection of dependent spaces (resp. bases) of a unique q-matroid. Similar statements hold true for circuits in q-matroids (see [11, Cor. 72]). None of these characterizations extend to q-PMs – even if we take the rank values into account. This can be seen from the two non-equivalent q-PMs in Example 2.3.16: In both cases, the only 2-basis is $\mathbb{F}^5$ and has rank value $5/2$. Trivially, this example also shows that the circuits and dependent spaces along with their rank values do not determine the q-PM. Example 3.4.6 is a non-trivial example for the same phenomenon.

On the positive side, in the next section we will show that we can fully recover a q-PM from its independent spaces and their rank values. Recall from Remark 3.1.4(d) that the independent spaces alone (without their rank values) do not uniquely determine the q-PM.

3.2 The Rank Function on Independent Spaces.

We begin by showing that for a q-PM the rank function is fully determined by its values on the independent spaces. We then go on to prove that all bases of a given subspace have the same rank value, and this value coincides with the rank value of the subspace. This result allows us to investigate whether a collection of spaces satisfying (I1)–(I4) from Corollary 3.1.9 gives rise to a q-PM whose collection of independent spaces is exactly the initial collection. Since the rank value of independent spaces in a q-PM is not as rigid as in a q-matroid, we also need to specify a meaningful rank function on the collection of spaces. All of this results in Theorems 3.2.4 and 3.2.5.

Theorem 3.2.1. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ and let $V \in \mathcal{L}(E)$. Then*

$$\rho(V) = \max\{\rho(I) \mid I \in \mathcal{I}_\mu(V)\}.$$

Proof. Set $\rho'(V) = \max\{\rho(I) \mid I \in \mathcal{I}_\mu(V)\}$. Thanks to (R2), $\rho'(V) \leq \rho(V)$, and it remains to establish $\rho(V) \leq \rho'(V)$. Let $\hat{I} \in \mathcal{I}_\mu(V)$ be of maximal possible dimension such that $\rho(\hat{I}) = \rho'(V)$. If V is μ-independent, then $\hat{I} = V$ and we are done. Thus let V be μ-dependent.

<u>Case 1:</u> $\dim \hat{I} = \dim V - 1$.

Then $V = \hat{I} \oplus \langle x \rangle$ for any $x \in V \setminus \hat{I}$ and submodularity of ρ implies $\rho(V) \leq \rho(\hat{I}) + \rho(\langle x \rangle)$. As before, we use the integer ρ-function $\tau = \mu\rho$. Let s be minimal such that there exists an s-dimensional μ-circuit of V, say W. Such space exists by μ-dependence of V. Then Remark 3.1.4(c) implies $\tau(W) = \dim W - 1$. By (I2) W is not contained in $\hat{I}$ and thus $W \cap \hat{I}$ is a hyperplane of W thanks to $\dim \hat{I} = \dim V - 1$. Hence Remark 3.1.4(c) yields $\tau(W \cap \hat{I}) = \tau(W)$. Using that $V = W + \hat{I}$, we obtain by submodularity of τ

$$\tau(V) \leq \tau(W) + \tau(\hat{I}) - \tau(W \cap \hat{I}) = \tau(\hat{I}) = \mu\rho'(V).$$

All of this shows that $\rho(V) = \rho'(V)$, as desired.

<u>Case 2:</u> $\dim \hat{I} < \dim V - 1$.

Let $x \in V \setminus \hat{I}$. Using that $\rho'(W) \leq \rho'(Z)$ for any subspaces W, Z such that $W \leq Z$, we obtain

$$\rho(\hat{I}) = \rho'(\hat{I}) \leq \rho'(\hat{I} \oplus \langle x \rangle) \leq \rho'(V) = \rho(\hat{I}),$$

and hence $\rho(\hat{I}) = \rho'(W)$, where $W := \hat{I} \oplus \langle x \rangle$. Note that W is μ-dependent thanks to the maximality of $\hat{I}$. Furthermore, $\dim \hat{I} = \dim W - 1$. Therefore Case 1 yields $\rho'(W) = \rho(W)$. Now we arrived at $\rho(\hat{I}) = \rho(\hat{I} + \langle x \rangle)$ for all $x \in V$, and [26, Prop. 2.5(a)] (based on [36, Prop. 6]) tells us that $\rho(\hat{I}) = \rho(V)$. Since $\rho(\hat{I}) = \rho'(V)$, this concludes the proof. □

Corollary 3.1.9 and Theorem 3.2.1 generalize one direction of [36, Thm. 8] where the same properties are proven for the independent spaces of q-matroids. Our next goal is to generalize the other direction of [36, Thm. 8], namely to characterize the collections of spaces plus rank values that give rise to a q-PM having those spaces as independent spaces. The following result will be crucial. It shows that the rank value of any μ-basis of a subspace V equals the rank value of V.

Theorem 3.2.2. *Let $\mathcal{M} = (E, \rho)$ be a q-PM with denominator μ. Let $V \in \mathcal{L}(E)$. Then*

$$\rho(I) = \rho(V) \text{ for all } I \in \mathcal{B}_\mu(V).$$

In particular, all μ-bases of V have the same rank value.

Proof. Throughout the proof we will omit the subscript μ. The result is clearly true if V is independent. Thus, let V be dependent. Set $t = \dim V$. In order to avoid denominators we use again the integer ρ-function $\tau := \mu\rho$. First of all, there exists

$$J \in \mathcal{B}(V) \text{ such that } \tau(J) = \tau(V). \tag{3.3}$$

Indeed, by Theorem 3.2.1 there exists $J \in \mathcal{I}(V)$ such that $\tau(J) = \tau(V)$, and by Property (I2) along with the monotonicity of τ we may assume that $J \in \mathcal{B}(V)$. Note that by (3.2) all spaces in $\mathcal{B}(V)$ have the same dimension, which we denote by s.

Case 1: $s = t - 1$. Let $I \in \mathcal{B}(V)$. We want to show that $\tau(I) = \tau(V)$. Choose a circuit, say C, in V. Then $\tau(C) = \dim C - 1$ (see Remark 3.1.4(c)). Clearly, $C \nsubseteq I$ by Property (I2) and thus $C + I = V$ thanks to $\dim I = \dim V - 1$. Furthermore, $C \cap I$ is independent, being a subspace of I, and thus $\tau(C \cap I) \geq \dim(C \cap I)$. Using submodularity, we obtain

$$\begin{aligned}\tau(V) = \tau(C + I) &\leq \tau(C) + \tau(I) - \tau(C \cap I) \leq \dim C - 1 + \tau(I) - \dim(C \cap I) \\ &= \tau(I) + \dim(C + I) - (\dim I + 1) = \tau(I),\end{aligned}$$

where the last step follows from $C + I = V$ and $\dim I + 1 = \dim V$. All of this shows $\tau(I) \geq \tau(V)$, and thus $\tau(I) = \tau(V)$ thanks to (R2). Hence all bases of V have the same rank value.

Case 2: $s < t - 1$. We will show that

$$\tau(I) = \tau(J) \text{ for all } I \in \mathcal{B}(V), \tag{3.4}$$

where J is as in (3.3). We induct on the codimension of $I \cap J$ in I. Let $\dim(I \cap J) = s - r$, thus $0 \leq r \leq s$. The case $r = 0$ is trivial.
i) Let $r = 1$. Then $I = (I \cap J) \oplus \langle x \rangle$ for some $x \in I \setminus J$. Set $W = J \oplus \langle x \rangle$. Then $W \leq V$ and $\dim W = \dim J + 1$. Thus W is dependent by maximality of J. Hence I and J are elements of $\mathcal{B}(W)$, and Case 1 implies $\tau(I) = \tau(J)$.
ii) Assume now $\tau(I) = \tau(J)$ for all $I \in \mathcal{B}(V)$ such that $\dim(I \cap J) \geq s - (r-1)$ for some $r \geq 2$. Let $I \in \mathcal{B}(V)$ be such that $\dim(I \cap J) = s - r$. Choose $K \leq I$ and $x \in I \setminus J$ such that $I = (I \cap J) \oplus K \oplus \langle x \rangle$ and set $I_1 = (I \cap J) \oplus K$. Then I_1 is independent and $\dim I_1 = \dim I - 1 = \dim J - 1$. Thanks to Property (I3) there exists $y \in J \setminus I_1$ such that

$$I' := I_1 \oplus \langle y \rangle \in \mathcal{B}(V).$$

Now we have three bases, I', I, J, of V. We show first $\tau(I) = \tau(I')$. Since $y \notin I$ we have the subspace $W := I \oplus \langle y \rangle$ of V, which must be dependent due to maximality of I. Furthermore, $I, I' \leq W$ and $\dim I' = \dim I = \dim W - 1$, and therefore $\tau(I) = \tau(I')$ thanks to Case 1. Next, we show $\tau(I') = \tau(J)$. In order to do so, note that $I' = (I \cap J) \oplus K \oplus \langle y \rangle$, where $y \in J$. Thus $\dim(I' \cap J) \geq s - (r-1)$ and the induction hypothesis yields $\tau(I') = \tau(J)$. All of this establishes (3.4) and concludes the proof. □

Remark 3.2.3. *In a q-matroid $\mathcal{M} = (E, \rho)$ a subspace $V \in \mathcal{L}(E)$ satisfies*

$$V \text{ is independent and } \rho(V) = \rho(E) \Longleftrightarrow V \text{ is a basis of } \mathcal{M}.$$

The forward direction is the definition of basis in [36, Def. 2]. By Theorem 3.2.2 the direction "$\Longleftarrow$" holds true for q-PMs as well. However, "$\Longrightarrow$" is not true, as the q-PMs in Examples 2.3.16 and 3.4.6 show. In other words, in a q-PM not every $I \in \mathcal{I}_\mu(V)$ satisfying $\rho(I) = \rho(V)$ is a μ-basis of V.

We are now ready to provide a characterization of the pairs $(\mathcal{I}, \tilde{\rho})$ of collections $\mathcal{I}$ of subspaces and rank functions $\tilde{\rho}$ on $\mathcal{I}$ that give rise to a q-PM whose collection of independent spaces is $\mathcal{I}$ and whose rank function restricts to $\tilde{\rho}$. Clearly, $\mathcal{I}$ has to satisfy (I1)–(I4) from Corollary 3.1.9, and $\tilde{\rho}$ must satisfy (R1)–(R3). However, for independence we also need the rank condition from Definition 3.1.1. This leads to (R1′) in Theorem 3.2.4 below. Furthermore, since the sum of independent spaces need not be independent, we have to adjust (R3) and replace $\tilde{\rho}(I + J)$ by $\max\{\tilde{\rho}(K) \mid K \in \mathcal{I}, K \leq I + J\}$, thereby accounting for Theorem 3.2.1. This results in the submodularity condition (R3′) below. Since one can easily find examples showing that (R1′)–(R3′) are not sufficient to guarantee submodularity of the extended rank function (defined in (3.5) below), we also have to enforce Theorem 3.2.2. This leads to condition (R4′), which states that for any space V all maximal subspaces that are contained in $\mathcal{I}$ have the same rank value. As we will see, all these conditions together guarantee submodularity of the extended rank function, and the spaces in $\mathcal{I}$ are independent in the resulting q-PM. However, the q-PM may have additional independent subspaces; see Example 3.4.7. In order to prevent this, we need a natural closure property. This will be spelled out in Theorem 3.2.5.

Theorem 3.2.4. *Let $\mathcal{I}$ be a subset of $\mathcal{L}(E)$. For $V \in \mathcal{L}(E)$ set $\mathcal{I}(V) = \{I \in \mathcal{I} \mid I \leq V\}$ and denote by $\mathcal{I}_{max}(V)$ the set of inclusion-maximal subspaces in $\mathcal{I}(V)$. Suppose $\mathcal{I}$ satisfies the following.*

(I1) $\{0\} \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $J \leq I$, then $J \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$ and $\dim I < \dim J$, then there exists $x \in J \setminus I$ such that $I \oplus \langle x \rangle \in \mathcal{I}$.

(I4) Let $V, W \in \mathcal{L}(E)$ and $I \in \mathcal{I}_{max}(V)$, $J \in \mathcal{I}_{max}(W)$. Then there exists a space $K \in \mathcal{I}_{max}(V + W)$ that is contained in $I + J$.

Furthermore, let $\tilde{\rho} : \mathcal{I} \longrightarrow \mathbb{Q}$ and $\mu \in \mathbb{Q}_{>0}$ such that $\mu\tilde{\rho}(I) \in \mathbb{Z}$ for all $I \in \mathcal{I}$. Suppose $\tilde{\rho}$ satisfies the following.

(R1′) $0 \leq \mu^{-1} \dim I \leq \tilde{\rho}(I) \leq \dim I$ *for all* $I \in \mathcal{I}$.

(R2′) If $I, J \in \mathcal{I}$ such that $I \leq J$, then $\tilde{\rho}(I) \leq \tilde{\rho}(J)$.

(R3′) For all $I, J \in \mathcal{I}$ we have $\max\{\tilde{\rho}(K) \mid K \in \mathcal{I}(I + J)\} + \tilde{\rho}(I \cap J) \leq \tilde{\rho}(I) + \tilde{\rho}(J)$.

(R4′) For all $V \in \mathcal{L}(E)$ and $I, J \in \mathcal{I}_{max}(V)$ we have $\tilde{\rho}(I) = \tilde{\rho}(J)$.

Define the map

$$\rho : \mathcal{L}(E) \longrightarrow \mathbb{Q}, \quad V \longmapsto \max\{\tilde{\rho}(I) \mid I \in \mathcal{I}(V)\}. \tag{3.5}$$

Then $\mathcal{M} = (E, \rho)$ is a q-PM with denominator μ, and $\mathcal{I} \subseteq \mathcal{I}_\mu(\mathcal{M})$.

Note that thanks to (I3) the set $\mathcal{I}_{\max}(V)$ equals the set of maximal-dimensional spaces in $\mathcal{I}(V)$. Furthermore, by (R2′) and (R4′) every $V \in \mathcal{L}(E)$ satisfies $\rho(V) = \tilde{\rho}(I)$ for each $I \in \mathcal{I}_{\max}(V)$.

Proof. It is clear that μ is a denominator of ρ. We have to show that ρ satisfies (R1)–(R3) from Definition 2.1.1.
(R1) Let $V \in \mathcal{L}(E)$ and $I \in \mathcal{I}$ such that $I \leq V$ and $\tilde{\rho}(I) = \rho(V)$. Then $0 \leq \tilde{\rho}(I) \leq \dim I \leq \dim V$, which establishes (R1).
(R2) Let $V, W \in \mathcal{L}(E)$ be such that $V \leq W$. Let $I \in \mathcal{I}$ be such that $I \leq V$ and $\tilde{\rho}(I) = \rho(V)$. Then $I \leq W$ and the definition of ρ implies $\rho(W) \geq \tilde{\rho}(I) = \rho(V)$, as desired.
(R3) Let $V, W \in \mathcal{L}(E)$. Choose $K \in \mathcal{I}_{\max}(V \cap W)$. Then (3.5) implies $\tilde{\rho}(K) = \rho(V \cap W)$. Applying (I3) repeatedly, we can find $I \in \mathcal{I}_{\max}(V)$ and $J \in \mathcal{I}_{\max}(W)$ such that $K \leq I$ and $K \leq J$. By (I4) there exists $H \in \mathcal{I}_{\max}(V + W)$ such that $H \leq I + J$. Now (R2′) and (R4′) imply

$$\tilde{\rho}(I) = \rho(V), \quad \tilde{\rho}(J) = \rho(W), \quad \tilde{\rho}(H) = \rho(I + J) = \rho(V + W),$$

$$\tilde{\rho}(K) = \tilde{\rho}(I \cap J) = \rho(V \cap W).$$

From (R3′) we obtain $\rho(I+J)+\tilde{\rho}(I\cap J)\leq\tilde{\rho}(I)+\tilde{\rho}(J)$, and we finally arrive at

$$\rho(V+W)+\rho(V\cap W)=\tilde{\rho}(H)+\tilde{\rho}(K)=\rho(I+J)+\tilde{\rho}(I\cap J)\leq\tilde{\rho}(I)+\tilde{\rho}(J)=\rho(V)+\rho(W),$$

as desired. Finally, (R1′) shows that the spaces in $\mathcal{I}$ are μ-independent, thus $\mathcal{I}\subseteq\mathcal{I}_\mu(\mathcal{M})$. □

The q-PM $\mathcal{M}$ from the last theorem has in general more independent spaces than $\mathcal{I}$; see Example 3.4.7. We can easily force equality $\mathcal{I}=\mathcal{I}_\mu(\mathcal{M})$ by adding the following natural closure property.

Theorem 3.2.5. *Let the pair $(\mathcal{I},\tilde{\rho})$ be as in Theorem 3.2.4. Suppose $(\mathcal{I},\tilde{\rho})$ satisfies (I1)–(I4) and (R1′)–(R4′) as well as the following closure property:*

(C) If $V\in\mathcal{L}(E)$ is such that
(a) all proper subspaces of V are in $\mathcal{I}$,
(b) $\max\{\tilde{\rho}(I)\mid I\in\mathcal{I}(V)\}\geq\mu^{-1}\dim V$,
then V is in $\mathcal{I}$.

Then $\mathcal{I}=\mathcal{I}_\mu(\mathcal{M})$ for the q-PM $\mathcal{M}$ from Theorem 3.2.4.

Note that by (I2) and (R1′), any subspace $V\in\mathcal{I}$ satisfies the properties in (a) and (b).

Proof. Thanks to Theorem 3.2.4 it remains to show that any $V\in\mathcal{I}_\mu(\mathcal{M})$ is in $\mathcal{I}$. Recall that $\rho(V)=\max\{\tilde{\rho}(I)\mid I\in\mathcal{I}(V)\}$. We induct on $\dim V$.
i) Let $\dim V=1$. Then $\rho(V)\geq\mu^{-1}\dim V$ holds true by the definition of μ-independence, hence (b) is satisfied. Property (a) is trivially satisfied by (I1). Now (C) implies $V\in\mathcal{I}$.
ii) Let $\dim V=r$ and assume that all subspaces $V\in\mathcal{I}_\mu(\mathcal{M})$ of dimension at most $r-1$ are in $\mathcal{I}$. Since $V\in\mathcal{I}_\mu(\mathcal{M})$, the same is true for all its subspaces. Hence all proper subspaces are in $\mathcal{I}$ by induction hypothesis. Again, $\rho(V)\geq\mu^{-1}\dim V$ is true by μ-independence and thus Property (C) implies that $V\in\mathcal{I}$. □

3.3 Spanning Spaces and Strongly Independent Spaces.

In this section, we introduce (minimal) spanning spaces and (maximally) strongly independent subspaces. While in q-matroids the notions 'minimal spanning space', 'maximally strongly independent space', and 'basis' coincide, they are distinct for q-PMs. However, in q-PMs spanning spaces turn out to be the dual notion to strongly independent spaces. This result may be regarded as the generalization of the duality result for bases in q-matroids. The latter states that for a q-matroid $\mathcal{M}$ a space B is a basis of $\mathcal{M}$ if and only if $B^\perp$ is a basis of $\mathcal{M}^*$. We show that, in fact, this equivalence characterizes q-matroids within the class of q-PMs.

Definition 3.3.1. *Let $\mathcal{M}=(E,\rho)$ be a q-PM and let $V\in\mathcal{L}(E)$.*

(a) V is called a spanning space *if $\rho(V)=\rho(E)$ and V is a* minimal spanning space *if it is a spanning space and no proper subspace is a spanning space.*

(b) V is strongly independent *if $\rho(V)=\dim V$ and it is* maximally strongly independent *if it is strongly independent and not properly contained in a strongly independent subspace.*

Clearly, strongly independent subspaces are μ-independent for every denominator μ of $\mathcal{M}$. Furthermore, in q-matroids strong independence coincides with independence. We remark that strongly independent subspaces of q-PMs play a crucial role in [9] for the construction of subspace designs. For q-matroids the following notions coincide.

Proposition 3.3.2. *Let $\mathcal{M}=(E,\rho)$ be a q-matroid and $V\in\mathcal{L}(E)$. Then*

$$\begin{aligned} V \text{ is maximally strongly independent} &\Longleftrightarrow V \text{ is a basis}\\ &\Longleftrightarrow V \text{ is a minimal spanning space.}\end{aligned}$$

Proof. The first equivalence is clear since for q-matroids strong independence coincides with independence (see Proposition 3.1.3). We turn to the second equivalence. "$\Rightarrow$" Let V be a basis of $\mathcal{M}$. Then $\dim V=\rho(V)=\rho(E)$. For every proper subspace $W\lneq V$ we have $\rho(W)\leq\dim W<\dim V=\rho(E)$, hence W is not a spanning space. This proves minimality of V. "$\Leftarrow$" Let now V be a minimal spanning space. Then $\rho(V)=\rho(E)$. Suppose V is dependent. Then there exists a basis W of V, and Theorem 3.2.2 implies $\rho(W)=\rho(V)=\rho(E)$. This contradicts minimality of V. Hence V is independent and thus a basis thanks to Remark 3.2.3. □

The last result is not true for q-PMs. For instance, it can be verified that for either q-PM in Example 2.3.16 the basis has dimension 5, the minimal spanning spaces have dimension 3, and the maximally strongly independent spaces have dimension 2. On the other hand, there exist q-PMs that are not q-matroids and yet the bases coincide with the minimal spanning spaces (for instance the q-PM $(\mathbb{F}^3,\rho)$ in [26, Ex. 4.2]). Thus the second equivalence in Proposition 3.3.2 does not characterize q-matroids. As for the first equivalence, note that if a μ-basis of a q-PM is strongly independent, then this is true for all bases (because they all have the same dimension by (3.2) and the same rank by Theorem 3.2.2). Thus all independent spaces are strongly independent thanks to Proposition 3.1.3 and the rank function is integer-valued by Theorem 3.2.1. This shows that the first equivalence does characterize q-matroids.

The following describes the relation between bases and minimal spanning spaces in a q-PM.

Proposition 3.3.3. *Let $\mathcal{M}=(E,\rho)$ be a q-PM with denominator μ.*

(a) A minimal spanning space is μ-independent.

(b) Every μ-basis of $\mathcal{M}$ contains a minimal spanning space and every minimal spanning space is contained in a μ-basis.

Proof. (a) Let V be a minimal spanning space. If V is μ-dependent, then V contains a μ-basis W, and Theorem 3.2.2 implies $\rho(W)=\rho(V)=\rho(E)$. This contradicts minimality of V. (b) is clear. □

Recall duality from Theorem 2.1.11. Our next result shows that bases are compatible with duality in "the expected way" if and only if the q-PM is a q-matroid. Part (a) has been established in [36].

Proposition 3.3.4. *Let the q-PMs $\mathcal{M} = (E, \rho)$ and $\mathcal{M}^* = (E, \rho^*)$ be as in Theorem 2.1.11.*

(a) If $\mathcal{M}$ is a q-matroid, then for every basis B of $\mathcal{M}$ the orthogonal space $B^\perp$ is a basis of $\mathcal{M}^*$.

(b) Let μ be a denominator of $\mathcal{M}$. Suppose there exists a μ-basis B of $\mathcal{M}$ such that the orthogonal space $B^\perp$ is a μ-basis of $\mathcal{M}^*$. Then $\mathcal{M}$ is a q-matroid.

Proof. (a) has been proven in [36, Thm. 45].
(b) Let B be a μ-basis of $\mathcal{M}$ and $B^\perp$ be a μ-basis of $\mathcal{M}^*$. Then $\rho(B) = \rho(E)$ and thus $\rho^*(B^\perp) = \dim B^\perp + \rho(B) - \rho(E) = \dim B^\perp$. Theorem 3.2.2 implies that every basis $\hat{B}$ of $\mathcal{M}^*$ satisfies $\rho^*(\hat{B}) = \rho^*(B^\perp) = \dim B^\perp = \dim \hat{B}$. Now Proposition 3.1.3 yields $\rho^*(I) = \dim I$ for all μ-independent spaces I of $\mathcal{M}^*$. Hence the dual rank function ρ^* is integer-valued on the μ-independent spaces. But then the entire rank function ρ^* is integer-valued thanks to Theorem 3.2.1. Now $\rho = \rho^{**}$ is also integer-valued, which means that $\mathcal{M}$ is a q-matroid. □

The above result has an interesting consequence. Recall from Theorem 3.1.6 the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$. Part (b) above implies that if $\mathcal{M}$ is a q-PM with denominator μ and $\mathcal{M}$ is not a q-matroid, then $\mathcal{Z}_{\mathcal{M}^*,\mu} \not\approx \mathcal{Z}^*_{\mathcal{M},\mu}$. Indeed, Theorem 3.1.6 implies that a subspace $B \in \mathcal{L}(E)$ is a μ-basis in $\mathcal{M}$ if and only if it is a basis in $\mathcal{Z}_{\mathcal{M},\mu}$. Thanks to Proposition 3.3.4(a) the latter is equivalent to $B^\perp$ being in basis in $\mathcal{Z}^*_{\mathcal{M},\mu}$. But by Proposition 3.3.4(b) $B^\perp$ is not a basis of $\mathcal{M}^*$, and thus not of $\mathcal{Z}_{\mathcal{M}^*,\mu}$.

Spanning spaces and strongly independent spaces are mutually dual, as one can see immediately with (2.1). This may be regarded a generalization of [11, Prop. 87] and [36, Thm. 45] (i.e., Proposition 3.3.4(a)), where the same results have been established for q-matroids.

Proposition 3.3.5. *Let $\mathcal{M}$ and $\mathcal{M}^*$ be as in Proposition 3.3.4 and let $V \in \mathcal{L}(E)$. Then V is a (minimal) spanning space in $\mathcal{M}$ if and only if $V^\perp$ is (maximally) strongly independent in $\mathcal{M}^*$.*

We close the section with a few remarks on the properties – or rather lack thereof – of strongly independent spaces and spanning spaces in q-PMs. Neither maximally strongly independent spaces nor minimal spanning spaces are as well-behaved as bases. This is not surprising since neither collection consists of subspaces of constant dimension (which can be verified with Example 3.4.6).

Remark 3.3.6. *(a) Let $\mathcal{M}$ be a q-PM and $\tilde{\mathcal{I}}$ be its collection of strongly independent subspaces. Thanks to Proposition 3.1.3 $\tilde{\mathcal{I}}$ satisfies (I2) of Corollary 3.1.9. It is not hard to find (sufficiently large) examples showing that $\tilde{\mathcal{I}}$ does not satisfy (I3) and (I4).*

(b) Bases in a q-PM satisfy conditions (B1)–(B4) in Corollary 3.1.10. But neither the maximally strongly independent subspaces nor the minimal spanning spaces satisfy (B3) or (B4).

3.4 Examples.

Example 3.4.1. *Example 2.3.16 gives two MRD codes in* $\mathbb{F}_2^{5\times 2}$ *with the same rank distance whose column q-PMs are not equivalent. Recall* $\mathcal{C}_1 = \langle A_1, \ldots, A_5\rangle$ *and* $\mathcal{C}_2 = \langle B_1, \ldots, B_5\rangle$*, where*

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix},\ A_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},\ B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ B_5 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Both codes are MRD with rank distance $d = 2$*, and* $\mathcal{C}_2$ *is actually a (*$\mathbb{F}_{2^5}$*-linear) Gabidulin code. Consider the q-PMs* $\mathcal{M}_i := \mathcal{M}_\mathcal{C}(\mathcal{C}_i) = (\mathbb{F}^5, \rho_\mathcal{C}^i)$*,* $i = 1, 2$*. From Theorem 2.3.13 we know that* $\rho_\mathcal{C}^1(V) = \rho_\mathcal{C}^2(V) = \dim V$ *for* $\dim V \le 1$ *and* $\rho_\mathcal{C}^1(V) = \rho_\mathcal{C}^2(V) = 5/2$ *if* $\dim V \ge 4$*. As for the 2-dimensional subspaces of* $\mathbb{F}_2^5$*, it turns out that the map* $\rho_\mathcal{C}^1$ *assumes the value* 1 *exactly once and the values* $3/2$ *and* 2 *exactly 28 and 126 times, respectively, whereas* $\rho_\mathcal{C}^2$ *assumes the values* $3/2$ *and* 2 *exactly 31 and 124 times, respectively, and never takes the value* 1*. Similar differences occur for the* 3*-dimensional subspaces. Thus* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are not equivalent.*

Example 3.4.2. *Independent spaces of the same dimension need not have the same rank value. Let* $\mathbb{F} = \mathbb{F}_2$ *and consider the code* $\mathcal{C} \le \mathbb{F}^{3\times 3}$ *generated by*

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $\mathcal{M} = \mathcal{M}_\mathcal{C}(\mathcal{C}) = (\mathbb{F}^3, \rho_\mathcal{C})$ *be the associated column q-PM. Then for all* $V \in \mathcal{L}(\mathbb{F}^3) \setminus \{0\}$

$$\rho_\mathcal{C}(V) = \begin{cases} 2/3 & \text{if } \dim V = 1 \text{ or } V = \langle e_1 + e_2, e_3\rangle, \\ 1 & \text{otherwise.} \end{cases}$$

Thus 3 *is the principal denominator and all spaces are independent. In particular, all* 2*-dimensional spaces are independent, but they do not assume the same rank value.*

Example 3.4.3. *A dependent space may have a larger rank value than an independent space of the same dimension. Let $\mathbb{F} = \mathbb{F}_2$ and $\mathcal{C} \leq \mathbb{F}^{5\times 3}$ be the code generated by the standard basis matrices E_{11}, E_{12}, E_{23}, E_{32}, E_{41}, E_{42}. In the column q-PM $\mathcal{M}_{\mathcal{C}}(\mathcal{C}) = (\mathbb{F}^5, \rho_{\mathcal{C}})$ the subspace $I = \langle e_2, e_3\rangle$ is independent with $\rho_{\mathcal{C}}(I) = 2/3$, while the subspace $V = \langle e_1 + e_2, e_5\rangle$ satisfies $\rho_{\mathcal{C}}(V) = 1$ and is dependent (because $\langle e_5\rangle$ is a loop).*

Example 3.4.4. *A subspace V satisfying $\mu\rho_{\mathcal{C}}(W) = \mu\rho_{\mathcal{C}}(V) = \dim V - 1$ for all its hyperplanes W need not be a circuit. Let $\mathbb{F} = \mathbb{F}_2$ and*

$$\Delta = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&1&0&0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0&1&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&1\\ 0&0&1&1\\ 1&0&0&1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 1&0&0&1\\ 0&1&1&1\\ 0&0&0&1 \end{pmatrix},\ A_3 = \begin{pmatrix} 1&1&0&1\\ 0&0&1&0\\ 0&0&1&0\\ 1&0&0&0\\ 0&0&0&1\\ 1&0&1&0 \end{pmatrix}.$$

Set $\mathcal{C} = \langle A_1, A_2, A_3, A_1U, A_2U, A_3U\rangle$, where $U = \Delta^5$. Then $\mathcal{C}$ is a right $\mathbb{F}_{2^2}$-linear rank-metric code of dimension 6. Indeed, Δ is the companion matrix of the primitive polynomial $f := x^4+x+1 \in \mathbb{F}_2[x]$ and since $5 = (2^4-1)/(2^2-1)$, any root ω of f leads to a primitive element ω^5 of the subfield $\mathbb{F}_{2^2}$. Thus $\Psi_{\mathcal{B}}^{-1}(\mathcal{C})$, where $\mathcal{B} = (1, \omega, \omega^2, \omega^3)$, is an $\mathbb{F}_{2^2}$-subspace of $\mathbb{F}_{2^4}^6$. The principal denominator of $\mathcal{M}_{\mathcal{C}}(\mathcal{C})$ is $\mu = 2$. There exist 497 μ-circuits, one of which has dimension 1 and all others have dimension 4. An additional 169 spaces V satisfy $\mu\rho_{\mathcal{C}}(V) = \dim V - 1$, and 97 of them also satisfy $\mu\rho_{\mathcal{C}}(W) = \dim V - 1$ for all its hyperplanes W.

Example 3.4.5. *There is no obvious relation between the auxiliary q-matroid $\mathcal{Z}_{\mathcal{M},\mu}$ of a q-matroid $\mathcal{M}$ and $\mathcal{M}$ itself if $\mu > 2$. Let $n \geq 3$ and fix a 2-dimensional subspace $X \in \mathcal{L}(\mathbb{F}^n)$. Set $\rho(X) = 1$ and $\rho(V) = \min\{\dim V, 2\}$ for $V \neq X$. One can check straightforwardly that $\mathcal{M} = (\mathbb{F}^n, \rho)$ is a q-matroid (this also follows from [26, Prop. 4.7]). Choosing $\mu = 2$, one verifies that $r_{\rho,2} = \min\{\dim V, 4\}$, and thus the q-matroids $\mathcal{M}$ and $\mathcal{Z}_{\mathcal{M},2}$ are not equivalent.*

Example 3.4.6. *Let $\mathbb{F} = \mathbb{F}_2$ and consider the codes $\mathcal{C} = \langle A_1, A_2, A_3\rangle$, $\mathcal{C}' = \langle A_1, A_2, A_3'\rangle \leq \mathbb{F}^{4\times 3}$, where*

$$A_1 = \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&1\\ 0&1&0 \end{pmatrix},\quad A_2 = \begin{pmatrix} 1&0&1\\ 1&0&0\\ 0&0&0\\ 1&1&1 \end{pmatrix},\quad A_3 = \begin{pmatrix} 0&1&0\\ 0&1&1\\ 0&1&0\\ 0&1&1 \end{pmatrix},\quad A_3' = \begin{pmatrix} 1&0&0\\ 0&0&0\\ 1&0&1\\ 1&0&1 \end{pmatrix}.$$

Both the associated q-PMs $\mathcal{M} = \mathcal{M}_{\mathcal{C}}(\mathcal{C}) = (\mathbb{F}^4, \rho_{\mathcal{C}})$ and $\mathcal{M}' = \mathcal{M}_{\mathcal{C}}(\mathcal{C}') = (\mathbb{F}^4, \rho_{\mathcal{C}}')$ have principal denominator 3, and the space $\mathbb{F}^4$ is the only dependent space. Hence $\mathcal{M}$ and $\mathcal{M}'$ share the same bases, namely all 3-dimensional spaces. Moreover, $\rho_{\mathcal{C}}(V) = 1 = \rho_{\mathcal{C}}'(V)$ for all bases V. Yet, $\mathcal{M}$ and $\mathcal{M}'$ are not equivalent: in $\mathcal{M}$ the rank value 1 is assumed by 33 subspaces of dimension 2, whereas in $\mathcal{M}'$ it is assumed by 32 subspaces of dimension 2 (in both q-PMs 4 subspaces of dimension 1 have rank value 1 as well).

Example 3.4.7. *Consider the q-PM* $\mathcal{M} = (\mathbb{F}^3, \rho_{\mathcal{C}})$ *from Example 3.4.2. We have seen already that* $\mathcal{I}_3(\mathcal{M}) = \mathcal{L}(\mathbb{F}^3)$. *Define the set*

$$\mathcal{I} = \{V \in \mathcal{L}(\mathbb{F}^3) \mid V \neq \langle e_1 + e_2, e_3 \rangle \text{ and } V \neq \mathbb{F}^3\}$$

and let $\tilde{\rho} = \rho_{\mathcal{C}}|_{\mathcal{I}}$. *One easily verifies that* $(\mathcal{I}, \tilde{\rho})$ *satisfies (I1)–(I4) and (R1′)–(R4′). Furthermore, the extension* ρ *defined in Theorem 3.2.4 equals* $\rho_{\mathcal{C}}$ *and thus the induced q-PM* $(\mathbb{F}^3, \rho)$ *equals* $\mathcal{M}$. *Now we have* $\mathcal{I} \subsetneq \mathcal{I}_3(\mathcal{M})$.

Chapter 4 The Projectivization Matroid of a q-Matroid.

In this chapter, we restrict ourselves to the study of q-matroids and study their connection to classical matroids via the intermediate of the projectivization matroid. The latter is a matroid that can be associated to the q-matroid and which preserves the structure of the lattice of flats. This in turn will allows us to study the characteristic polynomial of q-matroids from a classical matroid perspective. We establish a deletion/contraction formula for the characteristic polynomial of q-matroids and then prove a q-analogue of the Critical Theorem for q-matroids and $\mathbb{F}_{q^m}$-linear rank metric codes. Result from this chapter also appear in [33].

4.1 Preliminaries on Matroids.

We start this chapter by recalling with some preliminary results on classical matroids which will be needed throughout the chapter. The reader may note that many of those properties are similar to those established in Chapter 2. Many proof regarding properties of matroid will be omitted in section and the reader may refer to [39] for more details. The following notation will be used in this chapter. S and T are finite sets, 2^S is the power set of S and $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}_0$. Furthermore given a set S and $A \subseteq S$, let $S - A := \{e \in S : e \notin A\}$. Finally, to distinguish q-matroids from matroids, the former will be denoted by the script letters $\mathcal{M}$, $\mathcal{N}$, whereas the latter will be denoted by the capital letters M, N.

Definition 4.1.1. *A* matroid *is an ordered pair $M = (S, r)$, where S is a finite set and r is a function $r : 2^S \to \mathbb{N}_0$ such that for all $A, B \in 2^S$:*

(R1) Boundedness: $0 \leq r(A) \leq |A|$.

(R2) Monotonicity: If $A \subseteq B$ *then* $r(A) \leq r(B)$.

(R3) Submodularity: $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

S is called the groundset *of M and r its* rank function.

Throughout, identify $\{e\}$ with e and $\{v\}$ with v. Two matroids $M = (S, r_M)$ and $N = (T, r_N)$ are *equivalent*, denoted $M \cong N$, if there exists a bijection between the groundsets, $\psi : S \to T$, such that $r_M(A) = r_N(\psi(A))$ for all $A \subseteq S$. Given a matroid $M = (S, r)$, $e \in S$ is a *loop* of M if $r(e) = 0$. M is said to be *loopless* if it does not contain any loops. A subset $F \subseteq S$ is a *flat* if $r(F \cup v) > r(F)$ for all $v \notin F$. It is well known that the collection of flats, denoted $\mathcal{F}_M$, forms a geometric lattice. For any $F_1, F_2 \in \mathcal{F}_M$, the meet and join are defined as follows $F_1 \wedge F_2 := F_1 \cap F_2$ and $F_1 \vee F_2 := \mathrm{cl}_M(F_1 \cup F_2)$, where $\mathrm{cl}_M(A) = \{v \in S : r(A \cup v) = r(A)\} = \bigcap\{F \in \mathcal{F}_M : A \subseteq F\}$. Given $F_1, F_2 \in \mathcal{F}_M$, we say F_2 *covers* F_1 if for all $F \in \mathcal{F}_M$ such that $F_1 \subseteq F \subseteq F_2$ then $F = F_1$ or $F = F_2$. When discussing $\mathcal{F}_M$, we interchangeably use the terms collection of flats and lattice of flats. The flats of a matroid satisfy three axiomatic properties, analogous to the properties of Theorem 2.2.4 that fully determine the matroid.

Proposition 4.1.2. *[39, Sec. 1.4 Prob 11.] Let $M = (S, r_M)$ be a matroid and $\mathcal{F}_M$ its collection of flats. Then $\mathcal{F}_M$ satisfies the following:*

(F1) $S \in \mathcal{F}_M$.

(F2) If $F_1, F_2 \in \mathcal{F}_M$ *then* $F_1 \cap F_2 \in \mathcal{F}_M$.

(F3) Let $F \in \mathcal{F}_M$ *and* $v \notin F$*, then there exists a unique* $F' \in \mathcal{F}_M$ *covering* F *such that* $F \cup v \subseteq F'$.

Furthermore, r_M *is uniquely determined by* $\mathcal{F}_M$ *and* $r_M(A) = h(\mathrm{cl}_M(A))$ *for* $A \subseteq S$*, where* $h(F)$ *denotes the height of* F *in the lattice* $\mathcal{F}_M$.

Recall, the height of an element F in a geometric lattice, is the length of a maximal chain from the minimal element of the lattice to F. The reader may refer to [39, Sec 1.7] for more details. Similarly to q-matroids (recall Theorem 2.1.11), there exist a notion of duality for matroids.

Definition 4.1.3. *Let* $M = (S, r)$ *be a matroid. The* dual matroid $M^* = (S, r^*)$ *is defined via the rank function*

$$r^*(A) = |A| - r(S) + r(S - A).$$

Using the dual matroid, a *coloop* of $M = (S, r)$ is an element $e \in S$ such that $\{e\}$ is a loop of M^*.We now define the operations of deletion and contraction for matroids.

Definition 4.1.4. *Let* $M = (S, r)$ *be a matroid and let* $A \subseteq S$.

- *The matroid* $M \setminus A = (S - A, r_{M\setminus A})$*, where* $r_{M\setminus A}(B) = r(B)$ *for all* $B \subseteq S - A$*, is called the* deletion *of* A *from* M.
- *The matroid* $M/A = (S - A, r_{M/A})$*, where* $r_{M/A}(B) = r(B \cup A) - r(A)$ *for all* $B \subseteq S - A$*, is called the* contraction *of* A *from* M.

The following well-known facts about the deletion and contraction of matroids will be needed. Refer to [39, Prop 3.1.25] for a proof.

Proposition 4.1.5. *Let* $M = (S, r)$ *be a matroid. Let* $A, B \subseteq S$ *be disjoint sets. Then*

- $(M \setminus A) \setminus B = M \setminus (A \cup B) = (M \setminus B) \setminus A$,
- $(M/A)/B = M/(A \cup B) = (M/B)/A$,
- $(M \setminus A)/B = (M/B) \setminus A$.

To avoid the surplus of parenthesis, we omit them if there is no risk of confusion. At this point we make a brief comment about the notation that is used in this chapter only. The notation $\setminus$ always denotes the deletion operation and set exclusion is denoted by the $-$ sign. However, the notation $/$ is used to denote both the contraction of (q-)matroids and quotient space (i.e E/V). The reader should therefore use context in order to differentiate between the latter two.

Similarly to q-matroids, deletion and contraction are dual operations i.e. $M^*\setminus A = (M/A)^*$. A proof of this fact can be found in [39, Sect. 3]. A matroid N (resp. q-matroid $\mathcal{N}$) is a *minor* of M (resp. $\mathcal{M}$) if it can be obtained from M (resp. $\mathcal{M}$) by a sequence of deletion and contraction.

We close this section by showing that for both matroids and q-matroids, the flats of a contraction can be characterized in terms of the flats of the original (q-)matroid. Recall Proposition 6.1.13 for the contraction of q-matroids.

Proposition 4.1.6. *Let $M = (S, r_M)$ be a matroid, $\mathcal{M} = (E, \rho_{\mathcal{M}})$ be a q-matroid and $\mathcal{F}_M$, $\mathcal{F}_{\mathcal{M}}$ their respective lattice of flats. Let $A \subseteq S$, $V \leq E$ and consider M/A and $\mathcal{M}/V$. Then*

(1) $\mathcal{F}_{M/A} = \{F \subseteq S - A \,:\, F \cup A \in \mathcal{F}_M\}$,

(2) $\mathcal{F}_{\mathcal{M}/V} = \{F \leq E/V \,:\, \pi^{-1}(F) \in \mathcal{F}_{\mathcal{M}}\}$, where $\pi : E \to E/V$.

Furthermore A (resp. V) is a flat of M (resp. $\mathcal{M}$) if and only if M/A (resp. $\mathcal{M}/V$) is loopless.

Proof. (1) is shown in [39, Prop 3.3.7]. For (2), first let $F \in \mathcal{F}_{\mathcal{M}/V}$ and consider the space $W := \pi^{-1}(F) \leq E$. Let $x \notin W$. Then $\rho_{\mathcal{M}}(W \oplus \langle x\rangle) = \rho_{\mathcal{M}/V}(F \oplus \langle \pi(x)\rangle) + \rho_{\mathcal{M}}(V) > \rho_{\mathcal{M}/V}(F) + \rho_{\mathcal{M}}(V) = \rho_{\mathcal{M}}(W)$, where the inequality holds because $F \in \mathcal{F}_{\mathcal{M}/V}$ and $\pi(x) \notin F$. Since this is true for all $x \notin W$ then $W \in \mathcal{F}_{\mathcal{M}}$.

Now let $F \leq E/V$ such that $\pi^{-1}(F) \in \mathcal{F}_{\mathcal{M}}$. Let $\langle x\rangle \leq F/V$ such that $x \notin F$. Then $\rho_{\mathcal{M}/V}(F \oplus \langle x\rangle) = \rho_{\mathcal{M}}(\pi^{-1}(F \oplus \langle x\rangle)) - \rho_{\mathcal{M}}(V) = \rho_{\mathcal{M}}(\pi^{-1}(F) + \pi^{-1}(\langle x\rangle)) - \rho_{\mathcal{M}}(V) > \rho_{\mathcal{M}}(\pi^{-1}(F)) - \rho_{\mathcal{M}}(V) = \rho_{\mathcal{M}/V}(F)$. Once again, since this is true for all $x \notin F$ then $F \in \mathcal{F}_{\mathcal{M}/V}$.

We show the second part of the statement for matroids, and note the proof for q-matroid is analogous to it. Consider M/A with $A \in \mathcal{F}_M$ and let $e \in S - A$. Then $r_{M/A}(e) = r_M(e \cup A) - r_M(A) > 0$ since A is a flat. Since this holds for all $e \in S - A$, then M/A is loopless. Now assume $A \notin \mathcal{F}_M$ then $A \subsetneq \mathrm{cl}_M(A)$ and let $e \in \mathrm{cl}_M(A) - A \subseteq S - A$. Then $r_{M/A}(e) = r_M(A \cup e) - r(A) = 0$ since $e \in \mathrm{cl}_M(A)$. Hence M/A contains a loop. □

4.2 The Projectivization Matroid.

In [34], Johnsen and co-authors showed that a q-matroid $\mathcal{M}$ with groundspace E induces a matroid $P(\mathcal{M})$ with groundset the projective space of E. This induced matroid, called the projectivization matroid of $\mathcal{M}$ turns out to be an interesting object to study. In fact, it was shown in that same paper, that the projectivization preserves the flat structure of $\mathcal{M}$. It therefore becomes a useful tool when studying properties of q-matroids that depend only on flats.

For completeness, we reintroduce the construction of the projectivization matroid. The following notation will be used. Given a finite dimensional vector space E over $\mathbb{F}_q$, let $\mathbb{P}E := \{\langle v\rangle_{\mathbb{F}_q} : v \in E - \{0\}\}$ be the *projective space of* E. The map, $\hat{P} : (E - \{0\}) \longrightarrow \mathbb{P}E\,,\, v \mapsto \langle v\rangle_{\mathbb{F}_q}$ induces a lattice map $P : \mathcal{L}(E) \longrightarrow 2^{\mathbb{P}E}$, where $P(\{0\}) = \emptyset$ and $P(V) = \{\hat{P}(v) : v \in V - \{0\}\} = \{P(\langle v\rangle) : v \in V - \{0\}\}$ for $V \leq E$. We call the lattice map P the *projectivization map*. Usually, $\hat{P}$ is called the projectivization map, however for our purposes, it is more convenient to consider the projectivization as a lattice map. Note that P is inclusion preserving and that $P(V \cap W) = P(V) \cap P(W)$ for all $V, W \in \mathcal{L}(E)$. For any $S \subseteq \mathbb{P}E$ let $P^{-1}(S) := \{v \in E : \hat{P}(v) \in S\} = \{v \in E : P(\langle v\rangle) \in S\}$. Note that $(P^{-1} \circ P)(V) = V$ for all $V \leq E$. Finally let $\langle S\rangle := \langle P^{-1}(S)\rangle_{\mathbb{F}_q}$ for any $S \subseteq \mathbb{P}E$. We say $S \subseteq \mathbb{P}E$ *contains a basis* of E if $\langle S\rangle = E$. We can now introduce the projectivization matroid.

Theorem 4.2.1. *([34, Def.14, Prop. 15]) Let $\mathcal{M} = (E, \rho)$ be a q-matroid and let $r : 2^{\mathbb{P}E} \to \mathbb{N}_0$ such that for all $S \subseteq \mathbb{P}E$,*

$$r(S) = \rho(\langle S\rangle).$$

Then $P(\mathcal{M}) := (\mathbb{P}E, r)$ is a matroid, and is called the projectivization matroid of *$\mathcal{M}$.*

We now turn towards the relation between the flats of a q-matroid $\mathcal{M}$ and those of its projectivization matroid $P(\mathcal{M})$. In the following result, the meet and join refers to those of the lattice of flats defined in Section 2.2.1 for q-matroids and Section 4.1 for matroids.

Lemma 4.2.2. *[34, Lem.16, Prop.21] Let $\mathcal{M}$ be a q-matroid, $P(\mathcal{M})$ its projectivization matroid, and $\mathcal{F}_{\mathcal{M}}$, $\mathcal{F}_{P(\mathcal{M})}$ their respective lattice of flats. Furthermore, let $P(\mathcal{F}_{\mathcal{M}}) := \{P(F) : F \in \mathcal{F}_{\mathcal{M}}\}$. Then the following hold:*

1) $\mathcal{F}_{P(\mathcal{M})} = P(\mathcal{F}_{\mathcal{M}})$.

2) $P(F_1 \vee F_2) = P(F_1) \vee P(F_2)$ and $P(F_1 \wedge F_2) = P(F_1) \wedge P(F_2)$, for all $F_1, F_2 \in \mathcal{F}_{\mathcal{M}}$.

Therefore $\mathcal{F}_{P(\mathcal{M})} \cong \mathcal{F}_{\mathcal{M}}$ as lattices.

The next result shows when a matroid with groundset $\mathbb{P}E$ is the projectivization matroid of a q-matroid with groundspace E.

Theorem 4.2.3. *Let $M = (\mathbb{P}E, r)$ be a matroid and $\mathcal{F}_M$ its lattice of flats. Furthermore let $P^{-1}(\mathcal{F}_M) := \{P^{-1}(F) \cup \{0\} : F \in \mathcal{F}_M\}$. If $P^{-1}(F) \cup \{0\}$ is a subspace of E for all $F \in \mathcal{F}_M$, then $\mathcal{M} = (E, P^{-1}(\mathcal{F}_M))$ is q-matroid. Furthermore $\mathcal{F}_M \cong \mathcal{F}_{\mathcal{M}}$.*

In the following proof, to avoid confusion between the properties of flats of matroids and q-matroids, we denote by (qF1) - (qF3) the properties satisfied by q-matroids established in Theorem 2.2.4.

Proof. We show $\mathcal{F} := P^{-1}(\mathcal{F}_M)$ is a collection of flats of a q-matroid by showing it satisfies (qF1)-(qF3) of Theorem 2.2.4. Throughout the proof we use the fact that $\mathcal{F}_M$ is the collection of flats of a matroid, and hence satisfies (F1)-(F3) of Proposition 4.1.2. Since $\mathcal{F}_M$ satisfies (F1), $\mathbb{P}E \in \mathcal{F}_M$, and therefore $P^{-1}(\mathbb{P}E) \cup \{0\} = E \in \mathcal{F}$. This shows (qF1). Let $V_1 := P^{-1}(F_1) \cup \{0\}, V_2 := P^{-1}(F_2) \cup \{0\} \in \mathcal{F}$. Since $F_1, F_2 \in \mathcal{F}_M$ then $F_1 \cap F_2 \in \mathcal{F}_M$. Furthermore, $P(V_1 \cap V_2) = P(V_1) \cap P(V_2) = F_1 \cap F_2 \in \mathcal{F}_M$. Hence $V_1 \cap V_2 = P^{-1}(F_1 \cap F_2) \cup \{0\} \in \mathcal{F}$, showing (qF2).

Finally for (qF3), fix $F \in \mathcal{F}_M$, let $V := P^{-1}(F) \cup \{0\} \in \mathcal{F}$ and $w \notin V$. Since P is inclusion preserving $P(\langle w \rangle) \notin P(V) = F$. Hence there exists a unique flat $F' \in \mathcal{F}_M$ covering F such that $F \cup P(\langle w \rangle) \subseteq F'$. Let $V' := P^{-1}(F') \cup \{0\}$. By definition $V' \in \mathcal{F}$ and since V' is a subspace containing $V \cup w$ then $V \oplus \langle w \rangle \leq V'$. To show V' covers V, assume there exists $W \in \mathcal{F}$ such that $V \lneq W \leq V'$. Applying P and using the fact that P is inclusion preserving, we get $F = P(V) \subsetneq P(W) \subseteq P(V') = F'$. However because $W \in \mathcal{F}$ then $P(W) \in \mathcal{F}_M$. But F' covers F hence we must have that $P(W) = F'$ and therefore $W = V'$. This implies V' is a cover of V and shows $\mathcal{F}$ is the collection of flats of a q-matroid.

Finally to show $\mathcal{F}_M$ and $\mathcal{F}$ are isomorphic as lattices, note that $P(\mathcal{F}) = \mathcal{F}_M$ hence by Lemma 4.2.2 the isomorphism follows. □

We now show that the lattice of flats of the q-matroid $\mathcal{M}$ contracted by a flat F is isomorphic to the lattice of flats of $P(\mathcal{M})/P(F)$.

Theorem 4.2.4. *Let $\mathcal{M}$ be a q-matroid, $P(\mathcal{M})$ its projectivization matroid and $\mathcal{F}_{\mathcal{M}}, \mathcal{F}_{P(\mathcal{M})}$ their respective lattice of flats. Then $\mathcal{F}_{\mathcal{M}/F} \cong \mathcal{F}_{P(\mathcal{M})/P(F)}$ (as lattices) for any $F \in \mathcal{F}_{\mathcal{M}}$.*

Proof. Throughout let $F_1', F_2' \in \mathcal{F}_{\mathcal{M}/F}$ and $V_i = \pi^{-1}(F_i')$, where $\pi : E \to E/F$. By Proposition 4.1.6 and Lemma 4.2.2, $F_i' \in \mathcal{F}_{\mathcal{M}/F} \Leftrightarrow V_i \in \mathcal{F}_{\mathcal{M}} \Leftrightarrow P(V_i) \in \mathcal{F}_{P(\mathcal{M})} \Leftrightarrow P(V_i) - P(F) \in \mathcal{F}_{P(\mathcal{M})/P(F)}$. Furthermore, $F_1' = F_2' \Leftrightarrow V_1 = V_2 \Leftrightarrow P(V_1) - P(F) = P(V_2) - P(F)$. Hence there is a one-to-one correspondence between $\mathcal{F}_{\mathcal{M}/F}$ and $\mathcal{F}_{P(\mathcal{M})/P(F)}$ described by the map $\psi : \mathcal{F}_{\mathcal{M}/F} \to \mathcal{F}_{P(\mathcal{M})/P(F)}$, where $\psi(F_i') = P(V_i) - P(F)$. Since the lattices of flats are finite, to show ψ is a lattice isomorphism, we need only to show ψ preserves meets. Recall that the meet of flats in either lattice is the intersection of the flats.

$$\begin{aligned}
\psi(F_1' \cap F_2') &= P(V_1 \cap V_2) - P(F) \\
&= (P(V_1) \cap P(V_2)) - P(F) \\
&= (P(V_1) - P(F)) \cap (P(V_2) - P(F)) \\
&= \psi(F_1') \cap \psi(F_2'),
\end{aligned}$$

which completes the proof. □

The next few properties about projectivization matroids, although not difficult to prove, will be useful in following sections.

Proposition 4.2.5. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid. Then $\mathcal{M}$ contains a loop if and only if $P(\mathcal{M})$ contains a loop.*

Proof. Let $\langle e\rangle \leq E$ be a 1-dimensional subspace. By definition, $r(P(\langle e\rangle)) = \rho(\langle e\rangle)$. Hence $\langle e\rangle$ is a loop in $\mathcal{M}$ if and only if $P(\langle e\rangle)$ is a loop in $P(\mathcal{M})$. □

Proposition 4.2.6. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid. Let $A \subseteq \mathbb{P}E$ be such that A contains a basis of E. Then $r(A) = r(\mathbb{P}E)$.*

Proof. Since A contains a basis of E then $\langle A\rangle = E$. Hence $r(A) = \rho(\langle A\rangle) = \rho(E) = r(\mathbb{P}E)$. □

We conclude the section by studying the relation between minors of a q-matroid and minors of its projectivization matroid. To do so we introduce the following notation.

Notation 4.2.7. *Let $V \leq E$.*

- $\mathcal{Q}_V := \{\langle w\rangle \in \mathbb{P}E : \langle w\rangle \not\leq V\} = \mathbb{P}E - P(V)$.
- $\mathcal{Q}_V^{*A} := \mathcal{Q}_V - A$ *for* $A \subseteq \mathcal{Q}_V$.

Note that $\mathbb{P}E - \mathcal{Q}_V = \mathbb{P}V$. Furthermore for spaces $W, V \leq E$ such that $W \oplus V = E$, every element in E/W can be written as $v + W$ for a unique $v \in V$. Thus the map $\psi : E/W \to V$, $v + W \mapsto v$ is a well-defined vector space isomorphism and induces a bijection on projective spaces. By slight abuse of notation we use ψ as both the vector space isomorphism and the projective space bijection. It can then easily be shown that $\langle\psi(A)\rangle = \psi(\langle A\rangle)$ for all $A \subseteq \mathbb{P}(E/W)$.

Theorem 4.2.8. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and let $W, V \leq E$ such that $W \oplus V = E$. Let $V^\perp$ be the orthogonal space of V w.r.t. a fix NSBF. Furthermore let $S = \{\langle w_1\rangle, \dots, \langle w_t\rangle\} \subseteq \mathbb{P}E$ such that $\{w_1, \dots, w_t\}$ is a basis of W.Then*

$$P(\mathcal{M}/W) \cong (P(\mathcal{M})/S) \setminus \mathcal{Q}_V^{*S}$$

$$P(\mathcal{M} \setminus V^\perp) \cong P(\mathcal{M}) \setminus \mathcal{Q}_V.$$

Proof. Let $N := P(\mathcal{M})/S \setminus \mathcal{Q}_V^{*S}$. Note N has groundset $\mathbb{P}E - \mathcal{Q}_V = \mathbb{P}V$ whereas $P(\mathcal{M}/W)$ has groundset $\mathbb{P}(E/W)$. Let $\psi : \mathbb{P}(E/W) \to \mathbb{P}V$ be the bijection described previously. To show $N \cong P(\mathcal{M}/W)$, we must show $r_{P(\mathcal{M}/W)}(A) = r_N(\psi(A))$ for all $A \subseteq \mathbb{P}(E/W)$. Let $\pi : E \to E/W$ be the canonical projection. Since $\{w_1, \dots, w_t\}$ is a basis of W and $S = \{\langle w_1\rangle, \dots, \langle w_t\rangle\}$, then $\pi^{-1}(\langle A\rangle) = \langle\psi(A)\rangle + W = \langle\psi(A) \cup S\rangle$. Furthermore by Theorem 4.2.1, $r_{P(\mathcal{M})}(S) = \rho(W)$. Hence we get:

$$\begin{aligned}
r_{P(\mathcal{M}/W)}(A) &= \rho_{\mathcal{M}/W}(\langle A\rangle) \\
&= \rho_{\mathcal{M}}(\pi^{-1}(\langle A\rangle)) - \rho_{\mathcal{M}}(W) \\
&= \rho_{\mathcal{M}}(\langle\psi(A) \cup S\rangle) - \rho_{\mathcal{M}}(\langle S\rangle) \\
&= r_{P(\mathcal{M})}(\psi(A) \cup S) - r_{P(\mathcal{M})}(S) \\
&= r_{P(\mathcal{M})/S}(\psi(A)) \\
&= r_N(\psi(A)),
\end{aligned}$$

where the last equality holds because $\psi(A) \subseteq \mathbb{P}V$ which is the groundset of N.

Moving on to the second equivalence. Both matroid $P(\mathcal{M} \setminus V^\perp)$ and $P(\mathcal{M}) \setminus \mathcal{Q}_V$ have groundset $\mathbb{P}V$. Hence we need only to show that the rank functions of both matroids are equal. Let $A \subseteq \mathbb{P}V$.

$$\begin{aligned} r_{P(\mathcal{M}\setminus V^\perp)}(A) &= \rho_{\mathcal{M}\setminus V^\perp}(\langle A \rangle) \\ &= \rho_{\mathcal{M}}(\langle A \rangle) \\ &= r_{P(\mathcal{M})}(A) \\ &= r_{P(\mathcal{M})\setminus \mathcal{Q}_V}(A), \end{aligned}$$

where the last equality follows because $A \subseteq \mathbb{P}V$ which is the groundset of $P(\mathcal{M}) \setminus \mathcal{Q}_V$. □

4.3 The Characteristic Polynomial.

The characteristic polynomial is a useful invariant for both matroids and q-matroids. For the former it was intensively studied over the years, see for example [46, 49]. The latter was more recently introduced for q-polymatroids [9], and was used to establish a weaker version of the Assmus-Mattson Theorem. However, in this paper, we are only interested in the characteristic polynomial of q-matroids.

Before defining the characteristic polynomial, we recall the definition of the Möbius function which will often be used throughout the section.

Definition 4.3.1. *Let $(P, \leq)$ be a finite partially ordered set. The Möbius function for P is defined via the recursive formula*

$$\mu_P(x,y) := \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z \lneq y} \mu_P(x,z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

We use the subscript of μ to distinguish between the Möbius functions of different posets. If the underlying poset is clear, the subscript may be omitted. We now define the characteristic polynomial of a matroid.

Definition 4.3.2. *Let $M = (S, r)$ be a matroid and $\mathcal{L}(S)$ the lattice of subsets of S. The characteristic polynomial of M is defined as follow:*

$$\begin{aligned} \chi_M(x) &:= \sum_{A \subseteq S} \mu_{\mathcal{L}(S)}(\emptyset, A) x^{r(S)-r(A)} \\ &= \sum_{A \subseteq S} (-1)^{|A|} x^{r(S)-r(A)}. \end{aligned}$$

It is well known that if a matroid M contains a loop, its characteristic polynomial is identically 0. On the other hand if M is loopless, then the characteristic polynomial of M can be rewritten using the Möbius function of the lattice of flats. Furthermore, one can recursively define the characteristic polynomial of a matroid in terms of the characteristic polynomial of its minors. We summarize this in the following theorem. Proofs can be found in [46, Sec.3] and [49, Sec.7.1].

Theorem 4.3.3. *Let $M = (S, r)$ be a matroid and $\mathcal{F}$ be its lattice of flats. If M contains a loop then $\chi_M(x) = 0$. If M has no loops, then*

$$\chi_M(x) = \sum_{F \in \mathcal{F}} \mu_{\mathcal{F}}(0, F) x^{r(S)-r(F)}.$$

Furthermore for $e \in S$,

$$\chi_M(x) = \begin{cases} \chi_{M\backslash e}(x)\chi_{M/e}(x) & \text{if } e \text{ is a coloop,} \\ \chi_{M\backslash e}(x) - \chi_{M/e}(x) & \text{otherwise.} \end{cases}$$

Similarly to matroids, the characteristic polynomial of a q-matroid $\mathcal{M}$ is identically 0 if $\mathcal{M}$ contains a loop, and can be rewritten using the Möbius function of the lattice of flats otherwise. This was in fact shown by Whittle in [48], where the author generalized the result to any weighted lattice endowed with a closure operator. However for self containment purposes proofs of those facts will be included. Furthermore, we use the projectivization matroid to find a recursive formula for the characteristic polynomial of q-matroids. The characteristic polynomial of a q-matroid is defined in a similar way than for matroids but uses the Möbius function of the lattice of subspace of the grounspace of the q-matroid.

Definition 4.3.4. *[9, Def. 22] Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\mathcal{L}(E)$ be the subspace lattice of E. The* characteristic polynomial *is defined as*

$$\begin{aligned} \chi_{\mathcal{M}}(x) &:= \sum_{V \leq E} \mu_{\mathcal{L}(E)}(0, V) x^{\rho(E)-\rho(V)} \\ &= \sum_{V \leq E} (-1)^{\dim V} q^{\binom{\dim V}{2}} x^{\rho(E)-\rho(V)} \end{aligned}$$

We state a few straightforward lemmas that will be useful later on. Recall from Lemma-Definition 2.1.6 the loop space of a q-matroid consists of all 1-dimensional loops of a q-matroids.

Lemma 4.3.5. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid, $\mathcal{F}_{\mathcal{M}}$ its lattice of flats, and L its loop space. Then $L \leq F$ for all $F \in \mathcal{F}_{\mathcal{M}}$.*

Proof. Let $V \leq E$. By the monotonicity and submodularity properties of the rank function, $\rho(V) \leq \rho(V + L) \leq \rho(V) + \rho(L) - \rho(V \cap L) \leq \rho(V)$. Hence equality holds throughout and $L \leq \mathrm{cl}_{\mathcal{M}}(V)$. Since this is true for all $V \leq E$ then $L \leq F$ for all flats F of $\mathcal{M}$. □

Lemma 4.3.6. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid, $\mathcal{F}$ its lattice of flats and $\mathcal{L} := \mathcal{L}(E)$. Then*

$$\chi_{\mathcal{M}}(x) = \sum_{F \in \mathcal{F}} \sum_{V : \mathrm{cl}_{\mathcal{M}}(V) = F} \mu_{\mathcal{L}}(0, V) x^{\rho(E)-\rho(F)}.$$

Proof. Let $V \leq E$, then $\mathrm{cl}_{\mathcal{M}}(V) \in \mathcal{F}$ and $\rho(V) = \rho(\mathrm{cl}_{\mathcal{M}}(V))$. Hence we get

$$\begin{aligned}\chi_M(x) &= \sum_{V \leq E} \mu_{\mathcal{L}}(0, V) x^{\rho(E)-\rho(V)} \\ &= \sum_{F \in \mathcal{F}} \sum_{V \,:\, \mathrm{cl}_{\mathcal{M}}(V)=F} \mu_{\mathcal{L}}(0, V) x^{\rho(E)-\rho(F)}.\end{aligned}$$

□

We can now show that if a q-matroid contains a loop its characteristic polynomial is identically 0, whereas if it is loopless then the characteristic polynomial is determined by the Möbius function of the lattice of flats.

Theorem 4.3.7. *[see also [48, Thm. 3.2,Prop. 3.4]] Let $\mathcal{M}$ be a q-matroid and $\mathcal{F}$ its lattice of flats. If $\mathcal{M}$ contains a loop then $\chi_{\mathcal{M}}(x) = 0$. If $\mathcal{M}$ is loopless then*

$$\chi_{\mathcal{M}}(x) = \sum_{F \in \mathcal{F}} \mu_{\mathcal{F}}(0, F) x^{\rho(E)-\rho(F)}$$

Proof. From Lemma 4.3.6, we know

$$\chi_{\mathcal{M}}(x) = \sum_{F \in \mathcal{F}} \sum_{V \,:\, \mathrm{cl}(V)=F} \mu_{\mathcal{L}}(0, V) x^{\rho(E)-\rho(F)}.$$

We show that for all flat $F \in \mathcal{F}$,

$$\sum_{V \,:\, \mathrm{cl}(V)=F} \mu_{\mathcal{L}}(0, V) = \begin{cases} 0 & \text{if } \mathcal{M} \text{ has a loop,} \\ \mu_{\mathcal{F}}(0, F) & \text{otherwise.} \end{cases} \tag{4.1}$$

First assume that $\mathcal{M}$ has a loop. We proceed by induction on the rank value of flats. Let $F \in \mathcal{F}_{\mathcal{M}}$ such that $\rho(F) = 0$, i.e $F = \mathrm{cl}_{\mathcal{M}}(0)$. Since $\mathcal{M}$ contains a loop, $\{0\} \subsetneq F$. Hence by Definition Definition 4.3.1,

$$\sum_{V \,:\, \mathrm{cl}(V)=F} \mu_{\mathcal{L}}(0, V) = \sum_{0 \leq V \leq F} \mu_{\mathcal{L}}(0, V) = 0.$$

Assume (Eq. (4.1)) holds for all $F \in \mathcal{F}$ such that $\rho(F) \leq k - 1$.
Fix $F \in \mathcal{F}$ such that $\rho(F) = k$. Then

$$\begin{aligned}0 &= \sum_{V \leq F} \mu_{\mathcal{L}}(0, V) \\ &= \sum_{V \,:\, \mathrm{cl}(V)=F} \mu_{\mathcal{L}}(0, V) + \sum_{F' \lneq F, F' \in \mathcal{F}} \sum_{V \,:\, \mathrm{cl}(V)=F'} \mu_{\mathcal{L}}(0, V) \\ &= \sum_{V \,:\, \mathrm{cl}(V)=F} \mu_{\mathcal{L}}(0, V),\end{aligned}$$

where the last equality follows by induction hypothesis. Therefore if $\mathcal{M}$ has a loop then $\chi_{\mathcal{M}}(x) = 0$.

Now assume $\mathcal{M}$ is loopless. We once again proceed by induction on the rank of the flats of $\mathcal{M}$. Since $\mathcal{M}$ is loopless, $\{0\} \in \mathcal{F}$. Let $F = \{0\}$, then (Eq. (4.1)) follows trivially from Definition Definition 4.3.1. Assume (Eq. (4.1)) holds true for all $F \in \mathcal{F}$ such that $\rho(F) \leq k-1$. Fix a flat $F \in \mathcal{F}$ with $\rho(F) = k$. Then

$$\begin{aligned}
\mu_{\mathcal{F}}(0, F) &= -\sum_{F' \lneq F, F' \in \mathcal{F}} \mu_{\mathcal{F}}(0, F') \\
&= -\sum_{F' \lneq F, F' \in \mathcal{F}} \sum_{V\,:\,\mathrm{cl}(V) = F'} \mu_{\mathcal{L}}(0, V) \\
&= -\sum_{V\,:\,\mathrm{cl}(V) \lneq F} \mu_{\mathcal{L}}(0, V) \\
&= \sum_{V\,:\,\mathrm{cl}(V) = F} \mu_{\mathcal{L}}(0, V),
\end{aligned}$$

where the second equality follows from the induction hypothesis, and the last equality follows from Definition Definition 4.3.1. This completes the proof. □

As the next theorem shows, defining the characteristic polynomial in terms of the lattice of flats of the q-matroid allows us to link the characteristic polynomial of a q-matroid with that of its projectivization matroid.

Theorem 4.3.8. *Let $\mathcal{M}$ be a q-matroid and $P(\mathcal{M})$ be its projectivization matroid. Then*

$$\chi_{\mathcal{M}}(x) = \chi_{P(\mathcal{M})}(x).$$

Proof. By Proposition 4.2.5 $\mathcal{M}$ contains a loop if and only if $P(\mathcal{M})$ contains a loop. Furthermore, by Lemma 4.2.2, we know $\mathcal{F}_{\mathcal{M}} \cong \mathcal{F}_{P(\mathcal{M})}$ as lattices. Due to Propositions Proposition 4.1.2 and Theorem 2.2.4, $\rho_{\mathcal{M}}(F) = h_{\mathcal{F}_{\mathcal{M}}}(F) = h_{\mathcal{F}_{P(\mathcal{M})}}(P(F)) = r_{P(\mathcal{M})}(P(F))$ and this for all flats $F \in \mathcal{F}_{\mathcal{M}}$ and $P(F) \in \mathcal{F}_{P(\mathcal{M})}$. Therefore, the result follows directly from Theorem 4.3.3 and Theorem 4.3.7. □

We furthermore get the following result when considering the contraction of $\mathcal{M}$ by a subspace $V \leq E$.

Proposition 4.3.9. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid. Then for all $V \leq E$,*

$$\chi_{\mathcal{M}/V}(x) = \chi_{P(\mathcal{M})/P(V)}(x).$$

Proof. Let $V \leq E$, then $V \in \mathcal{F}_{\mathcal{M}} \Leftrightarrow P(V) \in \mathcal{F}_{P(\mathcal{M})}$. If $V \notin \mathcal{F}_{\mathcal{M}}$ then by Proposition 4.1.6 $\mathcal{M}/V$ and $P(\mathcal{M})/P(V)$ contain loops, therefore $\chi_{\mathcal{M}/V}(x) = 0 = \chi_{P(\mathcal{M})/P(V)}(x)$. If $V \in \mathcal{F}_{\mathcal{M}}$, by Theorem 4.2.4, $\mathcal{F}_{\mathcal{M}/F} \cong \mathcal{F}_{P(\mathcal{M})/P(V)}$ as lattices and, by Proposition 4.1.6, both matroids are loopless. Hence, Theorem 4.3.3 and Theorem 4.3.7 imply, $\chi_{\mathcal{M}/V}(x) = \chi_{P(\mathcal{M})/P(V)}(x)$. □

The close connection between the characteristic polynomial of a q-matroid $\mathcal{M}$ and that of its projectivization matroid gives a new approach to study the former. In fact, we use this approach to find a recursive formula for the characteristic polynomial of a q-matroid in terms of the characteristic polynomial of its minors. Because we will be using the recursive formula defined in Theorem 4.3.3 on the projectivization matroid $P(\mathcal{M})$ and its minors, we need to pay a particular attention on whether $P(\mathcal{M})$ and its minors contain coloops. To do so, we therefore fix an NSBF, and consider coloops of q-matroids with respect to that NSBF. We thus need the following few lemmas.

Lemma 4.3.10. *Let $M = (S, r)$ be a matroid and $\mathcal{M} = (E, \rho)$ be a q-matroid. Then*

(a) $w \in S$ is a coloop of $M \iff r^*(w) = 0 \iff r(S - w) = r(S) - 1$

(b) $\langle w \rangle \leq E$ is a coloop of $\mathcal{M} \iff \rho^*(\langle w \rangle) = 0 \iff \rho(\langle w \rangle^\perp) = \rho(E) - 1$

Proof. Statement (a) can be found in [39, Section 1.6, Exercise 6]. Statement (b) follows from Theorem 2.1.11. □

Recall the notation $\mathcal{Q}_V = \{\langle w \rangle : \langle w \rangle \nleq V\}$, and $\mathcal{Q}_V^{*e} = \mathcal{Q}_V - \{\langle e \rangle\}$ for $\langle e \rangle \in \mathcal{Q}_V$ introduced in Notation Notation 4.2.7. To make the results and proofs easier to read, we may omit the brackets to denote 1-dimensional spaces and we let $v^\perp := \langle v \rangle^\perp$ for $v \in E$.

Lemma 4.3.11. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid, $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid and $e, v \in \mathbb{P}E$ such that $\langle e \rangle \oplus \langle v \rangle^\perp = E$. Let $A \subsetneq \mathcal{Q}_{v^\perp}^{*e}$, then:*

(a) for all $w \in \mathcal{Q}_{v^\perp}^{*e} - A$, the element w is not a coloop of the matroid $P(\mathcal{M}) \setminus A$.

(b) for all $w \in \mathcal{Q}_{v^\perp}^{*e}$ and $z \in \mathcal{Q}_{v^\perp} - (A \cup w)$, the element z is not a coloop of $P(\mathcal{M}) \setminus A/w$.

(c) for all $w_1, w_2 \in \mathcal{Q}_{v^\perp} - A$, the matroid $P(\mathcal{M}) \setminus A/\{w_1, w_2\}$ contains a loop.

(d) e is a coloop of the matroid $P(\mathcal{M}) \setminus \mathcal{Q}_{v^\perp}^{*e}$ if and only if $\langle v \rangle$ is a coloop of $\mathcal{M}$.

Proof. Throughout, let $\mathcal{Q} := \mathcal{Q}_{v^\perp}^{*e}$, $H := v^\perp$ and $\{h_1, \cdots h_{n-1}\}$ be a basis of H. Furthermore let $A \subsetneq \mathcal{Q}$, $w \in \mathcal{Q} - A$ and consider the matroid $N := P(\mathcal{M}) \setminus A = (\mathbb{P}E - A, r_N)$. Note that $r(S) = r_N(S)$ for all $S \subseteq \mathbb{P}E - A$.

Statement (a). Note the set $B := \{h_1, \ldots, h_{n-1}, e\}$ is a basis of E, since $e \notin H$ and $\dim H = \dim E - 1$. Moreover $B \subseteq \mathbb{P}E - \mathcal{Q} \subseteq \mathbb{P}E - (A \cup w) \subseteq \mathbb{P}E - A$. Hence by Proposition 4.2.6, $r(\mathbb{P}E - \mathcal{Q}) = r(\mathbb{P}E - (A \cup w)) = r(\mathbb{P}E - A) = r(\mathbb{P}E)$. Because $r_N(S) = r(S)$ for all $S \subseteq \mathbb{P}E - A$, we have $r_N(\mathbb{P}E - (A \cup w)) = r_N(\mathbb{P}E - A)$ so, by Lemma 4.3.10, w is not a coloop of N, proving statement (a).

Statement (b). Let $B := \{h_1, \dots, h_{n-1}, w\}$ which is a basis of E. We need only to show $r^*_{N/w}(z) \neq 0$ for all $z \in \mathcal{Q}_{v^\perp} - (A \cup w)$.

$$\begin{aligned} r^*_{N/w}(z) &= |z| + r_{N/w}(\mathbb{P}E - (A \cup w \cup z)) - r_{N/w}(\mathbb{P}E - (A \cup w)) \\ &= 1 + r_N((\mathbb{P}E - (A \cup w \cup z)) \cup w) - r_N(w) - r_N(\mathbb{P}E - A) + r_N(w) \\ &= 1 + r(\mathbb{P}E - (A \cup z)) - r(\mathbb{P}E - A) \\ &= 1 + \rho(E) - \rho(E) = 1 \end{aligned}$$

where the last equality follows from Proposition 4.2.6 because B is a subset $\mathbb{P}E - (A \cup z)$ and $\mathbb{P}E - A$.

Statement (c). Let $w_1, w_2 \in \mathcal{Q}_{v^\perp} - A$ and $W = \langle w_1, w_2 \rangle$. Clearly $\dim W = 2$ and $\dim(W \cap \langle v \rangle^\perp) = 1$. Hence there exists $z \in W$ such that $\langle z \rangle \notin \mathcal{Q}_{v^\perp}$. We show z is a loop of $N/\{w_1, w_2\}$, i.e. $r_{N/\{w_1,w_2\}}(z) = 0$.

$$\begin{aligned} r_{N/\{w_1,w_2\}}(z) &= r(z \cup \{w_1, w_2\}) - r(\{w_1, w_2\}) \\ &= \rho(\langle z, w_1, w_2 \rangle) - \rho(\langle w_1, w_2 \rangle) \\ &= \rho(W) - \rho(W) = 0. \end{aligned}$$

Statement (d). Let $N' := P(\mathcal{M}) \setminus \mathcal{Q} = (\mathbb{P}E - \mathcal{Q}, r_{N'})$. By Lemma 4.3.10 $\langle v \rangle$ is a coloop of $\mathcal{M}$ if and only if $\rho(H) = \rho(E) - 1$. Moreover, $\rho(H) = r(\mathbb{P}H) = r_{N'}(\mathbb{P}H)$ and $\rho(E) = r(\mathbb{P}E) = r(\mathbb{P}E - \mathcal{Q}) = r_{N'}(\mathbb{P}E - \mathcal{Q})$. Hence $\langle v \rangle$ is a coloop of $\mathcal{M}$ if and only if $r_{N'}(\mathbb{P}H) = r_{N'}(\mathbb{P}H \cup e) - 1$ if and only if e is a coloop of N'. □

With those results in place, we are now ready to consider the first step of our main theorem. For the next results we use the following notation. Given $\mathcal{Q}^{*e}_{v^\perp}$, fix an ordering of its elements. Define $\mathcal{S}_0 := \emptyset$ and $\mathcal{S}_i := \{w_1, \dots, w_i\}$ where w_j is the j^{th} element of $\mathcal{Q}^{*e}_{v^\perp}$. Note furthermore that $|\mathcal{S}_i| = i$ and $\mathcal{S}_{q^{n-1}-1} = \mathcal{Q}^{*e}_{v^\perp}$. Moreover, in the proofs of the remaining results in this section, Proposition 4.1.5 and Theorem 4.3.3 may be used without mention.

Proposition 4.3.12. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid, $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid, $e, v \in \mathbb{P}E$ such that $\langle e \rangle \oplus \langle v \rangle^\perp = E$. Then*

$$\chi_{\mathcal{M}}(x) = \begin{cases} \chi_{\mathcal{M}\setminus v}(x)\chi_{\mathcal{M}/e}(x) - \sum_{i=0}^{q^{n-1}-2} \chi_{P(\mathcal{M})\setminus S_i/w_{i+1}} & \text{if } v \text{ is a coloop of } \mathcal{M} \\ \chi_{\mathcal{M}\setminus v}(x) - \chi_{\mathcal{M}/e}(x) - \sum_{i=0}^{q^{n-1}-2} \chi_{P(\mathcal{M})\setminus S_i/w_{i+1}} & \text{otherwise} \end{cases}$$

Proof. We first use an induction argument on $k := |\mathcal{S}_k|$ to show that for all $1 \leq k \leq q^{n-1} - 1$,

$$\chi_{P(\mathcal{M})}(x) = \chi_{P(\mathcal{M})\setminus\mathcal{S}_k}(x) - \sum_{i=0}^{k-1} \chi_{P(\mathcal{M})\setminus\mathcal{S}_i/w_{i+1}}(x). \tag{4.2}$$

We prove the base case when $k = 1$. By Lemma 4.3.11 (a), w_1 is not a coloop of $P(\mathcal{M})$ hence, by Theorem 4.3.3, $\chi_{P(\mathcal{M})}(x) = \chi_{P(\mathcal{M})\setminus\mathcal{S}_1}(x) - \chi_{P(\mathcal{M})\setminus\mathcal{S}_0/w_1}(x)$, where recall $\mathcal{S}_0 = \emptyset$ and $\mathcal{S}_1 = \{w_1\}$.

Now assume (Eq. (4.2)) holds for $k \leq q^{n-1}-2$. Then

$$\begin{aligned}\chi_{P(\mathcal{M})}(x) &= \chi_{P(\mathcal{M})\setminus\mathcal{S}_k}(x) - \sum_{i=0}^{k-1} \chi_{P(\mathcal{M})\setminus\mathcal{S}_i/w_{i+1}}(x) \\ &= \chi_{P(\mathcal{M})\setminus(\mathcal{S}_k\cup w_{k+1})}(x) - \chi_{P(\mathcal{M})\setminus\mathcal{S}_k/w_{k+1}}(x) - \sum_{i=0}^{k-1} \chi_{P(\mathcal{M})\setminus\mathcal{S}_i/w_{i+1}}(x) \\ &= \chi_{P(\mathcal{M})\setminus\mathcal{S}_{k+1}}(x) - \sum_{i=0}^{k} \chi_{P(\mathcal{M})\setminus\mathcal{S}_i/w_{i+1}}(x).\end{aligned}$$

The second equality holds true by Theorem 4.3.3, because w_{k+1} is not a coloop of $P(\mathcal{M}) \setminus \mathcal{S}_k$ by Lemma 4.3.11 (a). This establishes (Eq. (4.2)).

Because $\mathcal{S}_{q^{n-1}-1} = \mathcal{Q}_{v^\perp}^{*e}$, we conclude the proof by using Theorem 4.3.3 on $\chi_{P(\mathcal{M})\setminus\mathcal{S}_{q^{n-1}-1}}(x)$ with the element $e \in \mathcal{Q}_{v^\perp} - \mathcal{S}_{q^{n-1}-1}$. We therefore consider two cases: when e is a coloop of $P(\mathcal{M}) \setminus \mathcal{Q}_{v^\perp}^{*e}$ or not. By Lemma 4.3.11 (d), those two cases correspond exactly to when $\langle v\rangle$ is a coloop of $\mathcal{M}$ or not. First assume e is a coloop of $P(\mathcal{M}) \setminus \mathcal{Q}_{v^\perp}^{*e}$, and therefore $\langle v\rangle$ is a coloop of $\mathcal{M}$. Then by Theorem 4.3.3

$$\chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}^{*e}}(x) = \chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}}(x)\chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}^{*e}/e}(x).$$

By Theorem 4.2.8, $P(\mathcal{M}) \setminus \mathcal{Q}_{v^\perp} = P(\mathcal{M} \setminus v)$ and $P(\mathcal{M}) \setminus \mathcal{Q}_{v^\perp}^{*e}/e \cong P(\mathcal{M}/e)$ hence their respective characteristic polynomials are equal. Furthermore by Theorem 4.3.8, $\chi_{P(\mathcal{M}\setminus v)}(x) = \chi_{\mathcal{M}\setminus v}(x)$ and $\chi_{P(\mathcal{M}/e)}(x) = \chi_{\mathcal{M}/e}(x)$. Therefore, $\chi_{P(\mathcal{M})\setminus\mathcal{S}_{q^{n-1}-1}}(x) = \chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}}(x) \cdot \chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}^{*e}/e}(x) = \chi_{\mathcal{M}\setminus v}(x) \cdot \chi_{\mathcal{M}/e}(x)$, which, when substituted in (Eq. (4.2)) gives us the wanted equality.

If e is a not coloop of $P(\mathcal{M}) \setminus \mathcal{Q}_{v^\perp}^{*e}$, and therefore $\langle v\rangle$ is not a coloop of $\mathcal{M}$. Then

$$\chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}^{*e}}(x) = \chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}}(x) - \chi_{P(\mathcal{M})\setminus\mathcal{Q}_{v^\perp}^{*e}/e}(x).$$

Once again using Theorems Theorem 4.2.8 and Theorem 4.3.8, the wanted equality follows. □

At this point, note that the characteristic polynomial $\chi_{\mathcal{M}}(x)$ depends on both the characteristic polynomial of minors of the q-matroid $\mathcal{M}$ and the characteristic polynomial of minors of $P(\mathcal{M})$. In the following Theorem, we rewrite all characteristic polynomials of minors of $P(\mathcal{M})$ in terms of characteristic polynomials of minors of $\mathcal{M}$.

Theorem 4.3.13. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $e, v \in E$ such that $\langle e\rangle \oplus \langle v\rangle^\perp = E$. Then*

$$\chi_{\mathcal{M}}(x) = \begin{cases} \chi_{\mathcal{M}\setminus v}(x)\chi_{\mathcal{M}/e}(x) - \sum_{w\in\mathcal{Q}_{v^\perp}^{*e}} \chi_{\mathcal{M}/w}(x) & \text{if } v \text{ is a coloop of } \mathcal{M}, \\ \chi_{\mathcal{M}\setminus v}(x) - \sum_{w\in\mathcal{Q}_{v^\perp}} \chi_{\mathcal{M}/w}(x) & \text{otherwise.}\end{cases}$$

Proof. Given the equation of Proposition 4.3.12, we show that

$$\sum_{i=0}^{q^{n-1}-2} \chi_{P(\mathcal{M})\setminus\mathcal{S}_i/w_{i+1}}(x) = \sum_{w\in\mathcal{Q}^{*e}_{v^\perp}} \chi_{\mathcal{M}/w}(x). \tag{4.3}$$

Fix $0 \leq i \leq q^{n-1}-2$ and consider $\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x)$. For all $A \subseteq \mathcal{Q}_{v^\perp} - (S_i \cup w_{i+1})$, we show by induction on $|A|$ that

$$\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x) = \chi_{P(\mathcal{M})\setminus(S_i\cup A)/w_{i+1}}(x). \tag{4.4}$$

First let $|A| = 1$ and let $w \in A$. By Lemma 4.3.11 (b), w is not a coloop of $P(\mathcal{M}) \setminus \mathcal{S}_i/w_{i+1}$. Therefore

$$\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x) = \chi_{P(\mathcal{M})\setminus(S_i\cup w)/w_{i+1}}(x) - \chi_{P(\mathcal{M})\setminus S_i/\{w_{i+1},w\}}(x).$$

By Lemma 4.3.11 (c), $P(\mathcal{M}) \setminus S_i\, /\{w_{i+1}, w\}$ contains a loop which implies its characteristic polynomial is 0. Hence $\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x) = \chi_{P(\mathcal{M})\setminus(S_i\cup w)/w_{i+1}}(x)$. Now let $|A| = k$ and let $w \in A$. Since $|A - w| = k - 1$ by induction hypothesis we get $\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x) = \chi_{P(\mathcal{M})\setminus(S_i\cup(A-w))/w_{i+1}}(x)$. Once again by Lemma 4.3.11 (b), w is not a coloop of $P(\mathcal{M}) \setminus (\mathcal{S}_i \cup (A - w))/w_{i+1}$ hence

$$\chi_{P(\mathcal{M})\setminus(S_i\cup(A-w))/w_{i+1}}(x) = \chi_{P(\mathcal{M})\setminus(S_i\cup A)/w_{i+1}}(x) - \chi_{P(\mathcal{M})\setminus(S_i\cup A)/\{w_{i+1},w\}}(x).$$

By Lemma 4.3.11 (c) $P(\mathcal{M}) \setminus (S_i \cup A)/\{w_{i+1}, w\}$ contains a loop and therefore its characteristic polynomial is 0. This completes the proof of (Eq. (4.4)), which if $A = \mathcal{Q}_{v^\perp} - (S_i \cup w_{i+1})$ shows that

$$\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x) = \chi_{P(\mathcal{M})\setminus\mathcal{Q}^{*w_{i+1}}_{v^\perp}/w_{i+1}}(x).$$

Finally by Theorems Theorem 4.2.8 (where $S = \{w_{i+1}\}$) and Theorem 4.3.8 we get

$$\chi_{P(\mathcal{M})\setminus S_i/w_{i+1}}(x) = \chi_{\mathcal{M}/w_{i+1}}(x). \tag{4.5}$$

Since the above induction holds true for any i chosen, then (Eq. (4.5)) holds for all $0 \leq i \leq q^{n-1} - 2$ and

$$\begin{aligned}\sum_{i=0}^{q^{n-1}-2} \chi_{P(\mathcal{M})\setminus\mathcal{S}_i/w_{i+1}}(x) &= \sum_{i=0}^{q^{n-1}-2} \chi_{\mathcal{M}/w_{i+1}}(x)\\ &= \sum_{w\in\mathcal{Q}^{*e}_{v^\perp}} \chi_{\mathcal{M}/w}(x).\end{aligned}$$

Substituting (Eq. (4.3)) into the equation of Proposition 4.3.12 gives the desired result. □

4.4 Rank Metric and Linear Block Codes.

As already seen in Section 2.3, q-matroids are closely related to the study of $\mathbb{F}_{q^m}$-linear rank metric codes. Classical matroids also have a close connection to coding theory and more precisely with the study linear block codes with the Hamming metric. In fact, any linear block code induces a matroid, and many code invariants are fully determined by that matroid. In [1], Alfarano and co-authors showed that an $\mathbb{F}_{q^m}$-linear rank metric code $\mathcal{C}$ induces a linear block code that shares similar parameters. We show in this section how the projectivization matroid of a q-matroid relates to the matroid associated to that linear block code. Furthermore we use this relation and results from Section 5 to show the q-analogue of the Critical Theorem in terms of q-matroids and $\mathbb{F}_{q^m}$-linear rank metric codes.

We start the section by recalling some coding theory concepts. As described in Section 2.3 we consider $\mathbb{F}_{q^m}$-linear rank metric code as a subspaces of $\mathbb{F}_{q^m}^n$. Furthermore let Γ be a basis of the vector space $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and $\Psi_\Gamma : \mathbb{F}_{q^m}^n \longrightarrow \mathbb{F}_q^{n\times m}$ be the coordinate map introduced in Eq. (2.13). Recall for all $v = (v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n$, let $\Psi_\Gamma(v)$ be the $n \times m$ matrix such that the i^{th} row of $\Psi_\Gamma(v)$ is the coordinate vector of v_i with respect to the basis Γ. Moreover, throughout the section, we fix the NSBF on $\mathbb{F}_q^n$ to be the standard dot product. We define the following two weight functions.

Definition 4.4.1. *For all $v \in \mathbb{F}_{q^m}^n$, the* Hamming weight ω_{H}, *and the* rank weight ω_{rk} *of v are defined as follow:*

$$\omega_{\text{H}}(v) := \#\text{non-zero entries of } v$$
$$\omega_{\text{rk}}(v) := \text{rk}_{\mathbb{F}_q}(\Psi_\Gamma(v)).$$

It is well known that the rank weight is independent of the basis Γ chosen. Both weight functions induce a metric on $\mathbb{F}_{q^m}^n$, where $\text{d}_\Delta(v, w) = \omega_\Delta(v - w)$ for $v, w \in \mathbb{F}_{q^m}$ and $\Delta \in \{H, \text{rk}\}$. A *linear block code* is a subspace of the metric space $(\mathbb{F}_{q^m}^n, \text{d}_{\text{H}})$, and an $\mathbb{F}_{q^m}$*-linear rank metric code* is a subspace of the metric space $(\mathbb{F}_{q^m}^n, \text{d}_{\text{rk}})$. Given a code $\mathcal{C} \leq \mathbb{F}_{q^m}^n$, let $\text{d}_{\text{H}}(\mathcal{C})$, respectively $\text{d}_{\text{rk}}(\mathcal{C})$, denote the minimum weight over all non zero elements of the code. If $\dim \mathcal{C} = k$, then $\text{d}_\Delta(\mathcal{C}) \leq n - k + 1$ for $\Delta \in \{H, \text{rk}\}$. For each metric, the above bound is called the *Singleton bound*. $\mathcal{C}$ is said to be *maximum distance separable*, respectively *maximum rank distance* if the Hamming-metric, respectively the rank-metric, Singleton bound is achieved. For both metrics, the weight distribution of the code is defined as follows.

Definition 4.4.2. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be a code. For $\Delta \in \{H, \text{rk}\}$, let*

$$W_\Delta^{(i)}(\mathcal{C}) := |\{v \in \mathcal{C} : \omega_\Delta(v) = i\}|.$$

Furthermore let $W_\Delta(\mathcal{C}) := (W_\Delta^{(i)}(\mathcal{C}) : 0 \leq i \leq n)$. $W_H(\mathcal{C})$ is called the Hamming weight distribution *of $\mathcal{C}$ and $W_{\text{rk}}(\mathcal{C})$ is called the* rank weight distribution *of $\mathcal{C}$.*

We now introduce two notions of support for elements of $\mathbb{F}_{q^m}^n$.

Definition 4.4.3. *Let $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$ and $V \subseteq \mathbb{F}_{q^m}^n$.*

$$S_H(v) = \{i : v_i \neq 0\} \quad \text{and} \quad S_{\mathrm{H}}(V) = \bigcup_{v \in V} S_{\mathrm{H}}(v)$$

$$S_{\mathrm{rk}}(v) = \mathrm{colsp}_{\mathbb{F}_q}(\Psi_\Gamma(v)) \quad \text{and} \quad S_{\mathrm{rk}}(V) = \sum_{v \in V} S_{\mathrm{rk}}(v)$$

S_{H}, respectively S_{rk}, are called the Hamming support *and* rank support *of $v \in \mathbb{F}_{q^m}^n$ or $V \subseteq \mathbb{F}_{q^m}^n$.*

Once again, the rank support of an element $v \in \mathbb{F}_{q^m}^n$, or subset $V \subseteq \mathbb{F}_{q^m}$, is independent of the basis Γ chosen. Furthermore, a linear block code, respectively $\mathbb{F}_{q^m}$-linear rank metric code, $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ is said to be *non-degenerate* if $S_{\mathrm{H}}(\mathcal{C}) = [n]$, respectively $S_{\mathrm{rk}}(\mathcal{C}) = \mathbb{F}_q^n$. Given a code $\mathcal{C} \leq \mathbb{F}_{q^m}^n$, it is of interest to consider the set of elements of $\mathcal{C}$ with a given support.

Definition 4.4.4. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be a code, $A \subseteq [n]$ and $V \leq \mathbb{F}_q^n$. Let*

$$\mathcal{C}_{\mathrm{H}}(A) := \{v \in \mathcal{C} : S_{\mathrm{H}}(v) = A\}$$
$$\mathcal{C}_{\mathrm{rk}}(V) := \{v \in \mathcal{C} : S_{\mathrm{rk}}(v) = V\}$$

We now make the connection between codes and (q-)matroids. Note that linear block codes and $\mathbb{F}_{q^m}$-linear rank metric codes can be represented via a generator matrix $G \in \mathbb{F}_{q^m}^{k \times n}$, where $\mathcal{C} := \mathrm{rowsp}_{\mathbb{F}_{q^m}}(G)$. Both a matroid and a q-matroid can be induced from the generator matrix. The following construction is well known for matroids (see [39, Sec.6]) and is shown in Lemma 2.3.9 for q-matroids. We recall both constructions below.

Proposition 4.4.5. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be a code and $G \in \mathbb{F}_{q^m}^{k \times n}$ a generator matrix of $\mathcal{C}$. For $i \in [n]$, let $e_i \in \mathbb{F}_q^n$ denote the i^{th} standard basis vector and for $V \leq \mathbb{F}_q^n$ let $Y_V \in \mathbb{F}_q^{n \times t}$ such that $\mathrm{colsp}_{\mathbb{F}_q}(Y_V) = V$. Define $r : [n] \to \mathbb{N}_0$ and $\rho : \mathcal{L}(\mathbb{F}_q^n) \to \mathbb{N}_0$ such that:*

$$r(A) = \mathrm{rk}_{\mathbb{F}_{q^m}} \left(G \cdot \begin{bmatrix} e_{i_1} & \cdots & e_{i_a} \end{bmatrix}\right) \text{ for all } A \subseteq [n]$$
$$\rho(V) = \mathrm{rk}_{\mathbb{F}_{q^m}} (G \cdot Y_V) \text{ for all } V \leq \mathbb{F}_q^n.$$

Then $M_{\mathcal{C}} := ([n], r)$ is a matroid and $\mathcal{M}_{\mathcal{C}} = (\mathbb{F}_q^n, \rho)$ is a q-matroid and are called the matroid (resp. q-matroid) associated with $\mathcal{C}$.

Note that neither $M_{\mathcal{C}}$ nor $\mathcal{M}_{\mathcal{C}}$ depend on the choice of generator matrix for $\mathcal{C}$. Similarly to q-matroids, there exist the notion of representable matroids. A matroid is said to be $\mathbb{F}_q$ representable if it induced by a linear block code $\mathcal{C} \leq \mathbb{F}_q^n$. FThe ($q$-)matroid induced by a code is a useful tool to determine some of the code's invariants. In the rest of this section, we consider invariants of the code that are determined by the characteristic polynomial of the induced (q-)matroid. We first recall the notion of weight enumerator of a (q-)matroid. The weight enumerator of the q-matroid was defined in [9, Def. 43] and a similar concept was established in [30] for matroids.

Definition 4.4.6. *Let $M = (S, r)$ be a matroid and $\mathcal{M} = (E, \rho)$ be a q-matroid, with $|S| = n = \dim E$. Let*

$$A_M^{(i)}(x) = \sum_{A \subseteq S, |A|=i} \chi_{M/(S-A)}(x)$$
$$A_{\mathcal{M}}^{(i)}(x) = \sum_{V \leq E, \dim V = i} \chi_{\mathcal{M}/V^\perp}(x)$$

The weight enumerator *of the matroid, respectively q-matroid, is the list $A_M := (A_M^{(i)}(x) : 1 \leq i \leq n)$, respectively $A_{\mathcal{M}} := (A_{\mathcal{M}}^{(i)}(x) : 1 \leq i \leq n)$.*

Note in the above equations that if $S - A$ or $V^\perp$ are not flats of their respective matroid or q-matroid, then $\chi_{M/(S-A)}(x) = 0 = \chi_{\mathcal{M}/V^\perp}(x)$. Hence the summands of the weight enumerator can be restricted to the complement, respectively the orthogonal space, of flats. Thus we have the following result, which is well-known for matroids (see [30, Prop 3.3]) and was hinted at in [9] for q-matroids.

Theorem 4.4.7. *Let $M = (S, r)$ be a matroid and $\mathcal{M} = (E, \rho)$ be a q-matroid, with $|S| = n = \dim E$. Then*

$$A_M^{(i)}(x) = \sum_{F \in \mathcal{F}_M, |F| = n-i} \chi_{M/F}(x),$$
$$A_{\mathcal{M}}^{(i)}(x) = \sum_{F \in \mathcal{F}_{\mathcal{M}}, \dim F = n-i} \chi_{\mathcal{M}/F}(x).$$

It was shown in [9, Lem 49] that the weight enumerator of a representable q-matroid is closely related to the weight distribution of its associated code. For matroids a similar relation holds and was established in [30, Prop 3.2].

Theorem 4.4.8. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be a code, M and $\mathcal{M}$ be, respectively, the matroid and q-matroid induced by $\mathcal{C}$. Let $A \subseteq [n]$ and $V \leq \mathbb{F}_q^n$. Then*

(1) $\chi_{M/A}(q^m) = |\mathcal{C}_{\mathrm{H}}([n] - A)|$ *and* $W_{\mathrm{H}}^{(i)}(\mathcal{C}) = A_M^{(i)}(q^m)$.

(2) $\chi_{\mathcal{M}/V}(q^m) = |\mathcal{C}_{\mathrm{rk}}(V^\perp)|$ *and* $W_{\mathrm{rk}}^{(i)}(\mathcal{C}) = A_{\mathcal{M}}^{(i)}(q^m)$.

Remark 4.4.9. *The above Theorem together with Theorem 4.4.7, tells us that if $|\mathcal{C}_{\mathrm{H}}([n] - A)|$ and $|\mathcal{C}_{\mathrm{rk}}(V^\perp)|$ are non-zero then A and V are flats in M and $\mathcal{M}$, respectively.*

Because the weight enumerator of a (q-)matroid can be expressed in terms of flats, we get the following relation between the weight enumerator of a q-matroid and that of its projectivization matroid.

Proposition 4.4.10. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid. Then*

$$A_{P(\mathcal{M})}^{(j)}(x) = \begin{cases} A_{\mathcal{M}}^{(i)}(x) & \text{if } j = \frac{q^n - q^{n-i}}{q-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Proof. First, recall by Lemma 4.2.2, $\mathcal{F}_{P(\mathcal{M})} = \{P(F) : F \in \mathcal{F}_{\mathcal{M}}\}$ and $\dim F = n - i \Leftrightarrow |P(F)| = \frac{q^{n-i}-1}{q-1} \Leftrightarrow |\mathbb{P}E - P(F)| = \frac{q^n - q^{n-i}}{q-1}$. Furthermore, by Proposition 4.3.9, if $X \subseteq \mathbb{P}E$ is not a flat, then $\chi_{P(\mathcal{M})/X}(x) = 0$. So for all $1 \leq j \leq \frac{q^n-1}{q-1}$, such that $j \neq \frac{q^n-q^{n-i}}{q-1}$ for some $1 \leq i \leq n$, we get $A^{(j)}_{P(\mathcal{M})}(x) = 0$. Now assume $j = \frac{q^n-q^{n-i}}{q-1}$ for some $1 \leq i \leq n$. Then

$$\begin{aligned} A^{(i)}_{\mathcal{M}}(x) &= \sum_{F \in \mathcal{F}_{\mathcal{M}}, \dim F = n-i} \chi_{\mathcal{M}/F}(x) \\ &= \sum_{F \in \mathcal{F}_{\mathcal{M}}, \dim F = n-i} \chi_{P(\mathcal{M})/P(F)}(x) \\ &= \sum_{P(F) \in \mathcal{F}_{P(\mathcal{M})}, |P(F)| = \frac{q^{n-i}-1}{q-1}} \chi_{P(\mathcal{M})/P(F)}(x) \\ &= A^{(j)}_{P(\mathcal{M})}(x), \end{aligned}$$

where the second equality follows from Proposition 4.3.9. □

With the above setup, we now discuss the linear block code induced by an $\mathbb{F}_{q^m}$-linear rank metric code, as introduced in [1]. In their paper, the authors use q-systems and projective systems to introduce the Hamming-metric code associated to a rank metric code. We use a slightly different approach to introduce the associated Hamming metric code that does not require the previously stated notions.

Definition 4.4.11. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be an $\mathbb{F}_{q^m}$-linear rank metric code and let $G \in \mathbb{F}_{q^m}^{k\times n}$ be a generator matrix of $\mathcal{C}$. Furthermore let $H \in \mathbb{F}_q^{n\times \frac{q^n-1}{q-1}}$ where each column of H is a representative of a distinct element of $\mathbb{P}\mathbb{F}_q^n$. We call the matrix $G^H := G \cdot H$ an* $\mathbb{F}_q$-decomposition *of G via H and $\mathcal{C}^H := \mathrm{rowsp}_{\mathbb{F}_{q^m}}(G^H)$ is called a* Hamming-metric code associated to $\mathcal{C}$ via H

Remark 4.4.12. *Given a non-degenerate $\mathbb{F}_{q^m}$-rank metric code $\mathcal{C}$, the code $\mathcal{C}^H$ of Definition Definition 4.4.11 is a Hamming-metric code associated with $\mathcal{C}$ as in [1, Def. 4.6]. In fact, it easy to show the projective system induced by the columns of G^H is a representative of the equivalence class $(\mathrm{Ext}^H \circ \Phi)([\mathcal{C}])$ as introduced in [1]. Furthermore note that unlike the construction in [1], Definition Definition 4.4.11 does not depend on q-systems and projective systems hence we do not require $\mathcal{C}$ to be non-degenerate code. Finally, as noted in [1], a Hamming metric code associated to an $\mathbb{F}_{q^m}$-linear rank metric code $\mathcal{C}$ is not unique. In our case, $\mathcal{C}^H$ depends on the choice of the matrix H of the $\mathbb{F}_q$-decomposition. However, all Hamming-metric codes associated with $\mathcal{C}$ are monomially equivalent.*

Because $\mathcal{C}^H$ is a linear block code, it induces a matroid $M_{\mathcal{C}^H}$. It turns out that $M_{\mathcal{C}^H}$ is equivalent to the projectivization matroid $P(\mathcal{M}_{\mathcal{C}})$ of the q-matroid induced by $\mathcal{C}$.

Theorem 4.4.13. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be a rank metric code, $\mathcal{M}_\mathcal{C}$ its associated q-matroid, and $P(\mathcal{M}_\mathcal{C})$ its projectivization matroid. Furthermore let $\mathcal{C}^H$ be a Hamming-metric code associated to $\mathcal{C}$ via H and $M_{\mathcal{C}^H}$ its induced matroid. Then*

$$P(\mathcal{M}_\mathcal{C}) \cong M_{\mathcal{C}^H} \quad \textit{as matroids.}$$

Proof. Let $G \in \mathbb{F}_{q^m}^{k\times n}$ be a generator matrix of $\mathcal{C}$ and let $G^H = G \cdot H$ be a $\mathbb{F}_q$-decomposition of G via H. Let h_i be the i^{th} column of H, hence $\mathbb{P}\mathbb{F}_q^n = \{\langle h_i \rangle : i \in \left[\frac{q^n-1}{q-1}\right]$. Define the bijection $\psi : \mathbb{P}\mathbb{F}_q^n \to \left[\frac{q^n-1}{q-1}\right]$, where $\psi(\langle h_i\rangle) = i$. Furthermore, let $A \subseteq \mathbb{P}\mathbb{F}_q^n$, $\psi(A) = \{i_1, \dots, i_a\}$ and $Y_A := \begin{bmatrix} e_{i_1} & \cdots & e_{i_a}\end{bmatrix} \in \mathbb{F}_q^{\frac{q^n-1}{q-1}\times|A|}$, where e_j is the j^{th} standard basis element of $\mathbb{F}_q^{\frac{q^n-1}{q-1}}$. By Proposition 4.4.5 we get

$$\begin{aligned}
r_{M_{\mathcal{C}^H}}(\psi(A)) &= \mathrm{rk}_{\mathbb{F}_{q^m}}(G^H \cdot \begin{bmatrix} e_{i_1} & \cdots & e_{i_a}\end{bmatrix}) \\
&= \mathrm{rk}_{\mathbb{F}_{q^m}}(G \cdot H \cdot \begin{bmatrix} e_{i_1} & \cdots & e_{i_a}\end{bmatrix}) \\
&= \mathrm{rk}_{\mathbb{F}_{q^m}}(G \cdot \begin{bmatrix} h_{i_1} & \cdots & h_{i_a}\end{bmatrix}) \\
&= \rho_{\mathcal{M}_\mathcal{C}}(\langle h_{i_1}, \dots, h_{i_a}\rangle_{\mathbb{F}_q}) \\
&= r_{P(\mathcal{M}_\mathcal{C})}(A),
\end{aligned}$$

where the last equality follows from Theorem 4.2.1. □

Remark 4.4.14. *The above theorem allows us to relabel the groundset $\left[\frac{q^n-1}{q-1}\right]$ of the matroid $M_{\mathcal{C}^H}$ in terms of the elements of the projective space $\mathbb{P}\mathbb{F}_q^n$. Precisely if $G \cdot H$ is the $\mathbb{F}_q$-decomposition associated to $\mathcal{C}^H$, relabel $i \in \left[\frac{q^n-1}{q-1}\right]$ by $\langle h_i\rangle \in \mathbb{P}\mathbb{F}_q^n$, where h_i is the i^{th} column of H.*

We get the following result as an immediate corollary of Theorem 4.4.13.

Corollary 4.4.15. *If $\mathcal{M}$ is $\mathbb{F}_{q^m}$-representable then its projectivization matroid $P(\mathcal{M})$ is $\mathbb{F}_{q^m}$-representable.*

In [1, Theorem 4.9], it was established that the rank-weight distribution of a rank metric code $\mathcal{C}$ is closely related to the Hamming-weight distribution of any Hamming-metric code associated to $\mathcal{C}$. By Theorem 4.4.8, Proposition 4.4.10 and Theorem 4.4.13 we arrive at the same result from a purely matroid/q-matroid approach.

Theorem 4.4.16. *[1, Thm 4.9] Let $\mathcal{C}$ be an $\mathbb{F}_{q^m}$-linear rank metric code and $\mathcal{C}^H$ be a Hamming-metric code associated to $\mathcal{C}$. Then*

$$W_{\mathrm{H}}^{(j)}(\mathcal{C}^H) = \begin{cases} W_{\mathrm{rk}}^{(i)}(\mathcal{C}) & \text{if } j = \frac{q^n - q^{n-i}}{q-1}, \\ 0 & \text{otherwise.}\end{cases}$$

We conclude this chapter by showing the q-analogue of the Critical Theorem. The Critical Theorem, introduced by Crapo and Rota [15, Thm 1], states that the characteristic polynomial of the matroid $M_{\mathcal{C}}$ induced by the linear block code $\mathcal{C}$ determines the number of multisets of codewords with a given support. It was nicely restated in [5, Thm 2] in terms of coding theory terminology (recall Definition 4.4.3).

Theorem 4.4.17. *Let $\mathcal{C} \leq \mathbb{F}_q^n$ be a linear block code and $M = ([n], r)$ its induced matroid. For all $A \subseteq [n]$, the number of ordered t-tuples $V = (v_1, \ldots, v_t)$, where $v_j \in \mathcal{C}$ for all $1 \leq j \leq t$, such that $S_{\mathrm{H}}(V) = A$ is given by $\chi_{M/([n]-A)}(q^t)$.*

For our last result, we show an analogous statement for $\mathbb{F}_{q^m}$-linear rank metric codes and q-matroids by using the projectivization matroid. It is worth mentioning that Alfarano and Byrne were able to show an analogue of the Critical Theorem for q-polymatroids and matrix rank metric codes by using a different approach involving the Möbius inversion formula [8] .

For the next results we make use of Remark 4.4.14, and relabel the elements of the groundset of the matroid induced by $\mathcal{C}^H$ in terms of elements of $\mathbb{PF}_q^n$. Following this relabeling, we can also describe the support of a codeword of $\mathcal{C}^H$ in terms of the elements of $\mathbb{PF}_q^n$. More precisely, if $\mathcal{C}^H$ is induced by the $\mathbb{F}_q$-decomposition $G \cdot H$, for any $v \in \mathcal{C}^H$, let $S_{\mathrm{H}}(v) = \{\langle h_i \rangle \in \mathbb{PF}_q^n \; : \; v_i \neq 0\}$, where h_i and v_i are respectively the i^{th} column of H and the i^{th} component of v. Furthermore, we need the following well-known result for which we include a proof for self-containment. For two vectors v, w we let $v \cdot w$ denote the standard dot-product.

Lemma 4.4.18. *Let $v \in \mathbb{F}_{q^m}^n$, $S_{\mathrm{rk}}(v) = W \leq \mathbb{F}_q^n$ and $w \in \mathbb{F}_q^n$. Then $v \cdot w = 0$ if and only if $w \in W^{\perp}$.*

Proof. Let $\Gamma := \{\gamma_1, \ldots, \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, and let $Y := \Psi_\Gamma(v) \in \mathbb{F}_q^{n\times m}$, where $W := \mathrm{colsp}_{\mathbb{F}_q}(Y)$. Then $v \cdot w = 0 \Leftrightarrow \sum_{i=1}^n v_i w_i = 0 \Leftrightarrow$
$\sum_{i=1}^n \left(\sum_{j=1}^m \gamma_j v_{ij}\right) w_i = 0 \Leftrightarrow \sum_{j=1}^m \gamma_j \left(\sum_{i=1}^n v_{ij} w_i\right) = 0$. Since Γ is a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and $v_{ij} w_i \in \mathbb{F}_q$, the previous equality holds if and only if $\sum_{i=1}^n v_{ij} w_i = 0$ for all $1 \leq j \leq m$. But note $\sum_{i=1}^n v_{ij} w_i = v^{(j)} \cdot w$, where $v^{(j)}$ is the j^{th} column of Y. Hence $v \cdot w = 0 \Leftrightarrow v^{(j)} \cdot w = 0$ for all $1 \leq j \leq m \Leftrightarrow w \in \mathrm{colsp}_{\mathbb{F}_q}(Y)^{\perp} \Leftrightarrow w \in W^{\perp}$. □

The following Lemma relates the rank support of elements of the code $\mathcal{C}$ with the Hamming support of elements of the associated Hamming-metric code $\mathcal{C}^H$.

Lemma 4.4.19. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be a rank metric code, $\mathcal{C}^H$ be a Hamming-metric code associated to $\mathcal{C}$ via H. Furthermore, let $V = \{v_1, \ldots, v_t\}$ be a subset of $\mathcal{C}$ and $V \cdot H := \{v_1 \cdot H, \ldots, v_t \cdot H\}$. Then*

$$S_{\mathrm{rk}}(V) = W \Leftrightarrow S_{\mathrm{H}}(V \cdot H) = \mathbb{PF}_q^n - P(W^{\perp}).$$

Moreover if $\mathcal{M}$ and $P(\mathcal{M})$ are the q-matroid and projectivization matroid induced respectively by $\mathcal{C}$ and $\mathcal{C}^H$ then $W^{\perp}$ and $P(W^{\perp})$ are, respectively, flats of $\mathcal{M}$ and $P(\mathcal{M})$.

Proof. Consider the subset $V := \{v_1, \ldots, v_t\} \subseteq \mathcal{C}$. By definition, $W := S_{\text{rk}}(V) = \sum_{j=1}^{t} S_{\text{rk}}(v_j)$. Let $W_j := S_{\text{rk}}(v_j)$. For all $1 \leq j \leq t$, by Lemma 4.4.18, $v_j \cdot w = 0$ if and only if $w \in W_j^{\perp}$. Hence for all columns h_i of H, it follows that $v_j \cdot h_i = 0$ if and only if $h_i \in W_j^{\perp}$. By definition, this is true if and only if $S_{\text{H}}(v_j \cdot H) = \mathbb{P}\mathbb{F}_q^n - P(W_j^{\perp})$. Hence $S_{\text{H}}(V \cdot H) = \bigcup_{i=1}^{t} S_{\text{H}}(v_j \cdot H) = \bigcup_{j=1}^{t}(\mathbb{P}\mathbb{F}_q^n - P(W_j^{\perp}))$, where the first equality follows by definition. Therefore $S_{\text{rk}}(V) = W \Leftrightarrow S_{\text{H}}(V \cdot H) = \bigcup_{j=1}^{t}(\mathbb{P}\mathbb{F}_q^n - P(W_j^{\perp})) = \mathbb{P}\mathbb{F}_q^n - (\bigcap_{j=1}^{t} P(W_j^{\perp})) = \mathbb{P}\mathbb{F}_q^n - P(W^{\perp})$. Finally, $W^{\perp}$ and $P(W^{\perp})$ are flats of $\mathcal{M}$ and $P(\mathcal{M})$ respectively because of Remark Remark 4.4.9. □

We are now ready to show the Critical Theorem for $\mathbb{F}_{q^m}$-linear rank metric codes and q-matroids.

Theorem 4.4.20. *Let $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ be an $\mathbb{F}_{q^m}$-linear rank metric code and $\mathcal{M}$ its induced q-matroid. For all $W \leq \mathbb{F}_q^n$, the number of ordered t-tuples $V = (v_1, \ldots, v_t)$, where $v_j \in \mathcal{C}$ for all $1 \leq j \leq t$, such that $S_{\text{rk}}(V) = W$ is given by $\chi_{\mathcal{M}/W^{\perp}}(q^{mt})$.*

Proof. Let $\mathcal{C}^H$ be the Hamming-metric code associated with $\mathcal{C}$ via H and $P(\mathcal{M})$ be its associated matroid. Note that every element of $\mathcal{C}^H$ is of the form $v \cdot H$ for some $v \in \mathcal{C}$. Hence every tuple of elements of $\mathcal{C}^H$ is of the form $V \cdot H$ for some $V \subseteq \mathcal{C}$. Furthermore, since H has full row-rank, there is a bijection between elements of $\mathcal{C}$ and $\mathcal{C}^H$ and hence a bijection between t-tuples $V \subseteq \mathcal{C}$ and t-tuples $V \cdot H \subseteq \mathcal{C}^H$. By Theorem 4.4.17, the number of t-tuple $V \cdot H \subseteq \mathcal{C}^H$ such that $S_{\text{H}}(V \cdot H) = \mathbb{P}E - P(W^{\perp})$ is given by $\chi_{P(\mathcal{M})/P(W^{\perp})}(q^{mt})$. Moreover, by Proposition 4.3.9, $\chi_{P(\mathcal{M})/P(W^{\perp})}(q^{mt}) = \chi_{\mathcal{M}/W^{\perp}}(q^{mt})$. Finally, by Lemma 4.4.19, $S_{\text{rk}}(V \cdot H) = \mathbb{P}E - P(W^{\perp})$ if and only if $S_{\text{rk}}(V) = W$ and therefore $\chi_{\mathcal{M}/W^{\perp}}(q^{mt})$ counts the number of t-tuples $V \subseteq \mathcal{C}$ such that $S_{\text{rk}}(V) = W$. □

The above result, and its proof, shows a close connection between the Critical Theorem for matroids and that for q-matroids. Furthermore it can easily been seen from the above approach that the critical problem for q-matroids, that is finding the smallest power of q that makes the characteristic polynomial of a q-matroid non-zero, is a specific case of the critical problem for matroids.

Chapter 5 The Direct sum of q-Matroids.

In this chapter we introduce and study the direct sum of q-matroids introduces by [13]. We study the rank function of the direct sum and show that the cyclic flats of the direct sum can easily be characterized via the cyclic flats of each summands. The characterization of the cyclic flats will then allow us to study the decomposition of q-matroids into irreducible components. We show every q-matroids can be uniquely decomposed into the direct sum of irreducible q-matroids, up to equivalence. Finally we conclude the chapter by studying the representability of the direct sum of q-matroids. More precisely we show that unlike classical matroids, the direct sum of two $\mathbb{F}_{q^m}$-representable q-matroid is not necessarily $\mathbb{F}_{q^m}$ representable. Results in this chapter also appear in [27] and [25]

5.1 Preliminaries on the Direct Sum of q-Matroids.

In this introduce the direct sum of q-matroids and study its rank function. The first definition of it has been given in [13]. We will present a different, more concise, definition, which results in the same construction. It will enable us to study its properties in more detail. In particular, we will derive various ways to compute the rank function, the most efficient one being based on the cyclic flats of the components. Moreover, we will show that the dual of a direct sum is the direct sum of the dual q-matroids.

We start with the union of two q-matroids, which is the q-analogue of the matroid union; see e.g., [39, Thm. 11.3.1].

Theorem 5.1.1 ([13, Thm. 28]). *Let $\mathcal{M}_i = (E, \rho_i)$, $i = 1, 2$, be q-matroids on the same ground space E. For $V \in \mathcal{L}(E)$ define*

$$\rho(V) = \dim V + \min\{\rho_1(X) + \rho_2(X) - \dim X \mid X \leq V\} \tag{5.1}$$

Then $\mathcal{M} = (E, \rho)$ is a q-matroid, called the union *of $\mathcal{M}_1$ and $\mathcal{M}_2$, and denoted by $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2$.*

Proof. The fact that ρ is a rank function is in [13, Thm. 29] and also follows from Theorem 3.1.6. The main argument in these proofs is that the map $\rho_1+\rho_2 : \mathcal{L}(E) \longrightarrow \mathbb{N}_0$ satisfies (R2) and (R3) from Definition 2.1.1. Hence it gives rise to a q-polymatroid. The minimization in (5.1) turns this map into a rank function of a q-matroid. □

In order to define the direct sum of two q-matroids $\mathcal{M}_i = (E_i, \rho_i)$, we need, unsurprisingly, the direct sum $E = E_1 \oplus E_2$ of $\mathbb{F}$-vector spaces E_1 and E_2. For any such direct sum we denote by

$$\pi_i : E \longrightarrow E_i \quad \text{and} \quad \iota_i : E_i \longrightarrow E$$

the corresponding projection and embedding. We will identify a subspace $V_i \in \mathcal{L}(E_i)$ with its image $\iota_i(V_i)$.

The construction of the direct sum consists of two steps:

(1) Add the ground space E_1 of the q-matroid $\mathcal{M}_1 = (E_1, \rho_1)$ as a space of rank 0 to the q-matroid $\mathcal{M}_2 = (E_2, \rho_2)$ and vice versa. This results in two q-matroids $\mathcal{M}_1'$, $\mathcal{M}_2'$ on the common ground space $E_1 \oplus E_2$ and with rank functions ρ_i'.

(2) Take the union of $\mathcal{M}_1'$ and $\mathcal{M}_2'$.

Theorem 5.1.2 ([13, Sec. 7])**.** *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids and set $E = E_1 \oplus E_2$. Define $\rho_i' : \mathcal{L}(E) \longrightarrow \mathbb{N}_0$, $V \longmapsto \rho_i(\pi_i(V))$ for $i = 1, 2$. Then $\mathcal{M}_i' = (E, \rho_i')$ is a q-matroid for $i = 1, 2$. Set $\mathcal{M} = \mathcal{M}_1' \vee \mathcal{M}_2'$, that is, $\mathcal{M} = (E, \rho)$, where*

$$\rho(V) = \dim V + \min_{X \leq V} \big(\rho_1'(X) + \rho_2'(X) - \dim X\big) \quad \textit{for } V \in \mathcal{L}(E). \tag{5.2}$$

Then $\mathcal{M}$ is a q-matroid, called the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ *and denoted by $\mathcal{M}_1 \oplus \mathcal{M}_2$.*

Proof. We show that ρ_i' is indeed a rank function for $i \in \{1, 2\}$. WLOG let $i = 1$. Using that ρ_1 is a rank function, we have $0 \leq \rho_1(\pi_1(V)) \leq \dim \pi_1(V) \leq \dim V$, and this shows (R1). (R2) is trivial. For (R3) let $V, W \in E$. Note first that $\pi_1(V \cap W) \subseteq \pi_1(V) \cap \pi_1(W)$. Thus

$$\begin{aligned}
\rho_1'(V + W) &= \rho_1(\pi_1(V + W)) = \rho_1(\pi_1(V) + \pi_1(W)) \\
&\leq \rho_1(\pi_1(V)) + \rho_1(\pi_1(W)) - \rho_1(\pi_1(V) \cap \pi_1(W)) \\
&\leq \rho_1(\pi_1(V)) + \rho_1(\pi_1(W)) - \rho_1(\pi_1(V \cap W)) \\
&= \rho_1'(V) + \rho_1'(W) - \rho_1'(V \cap W).
\end{aligned}$$

The rest follows from Theorem 5.1.1. □

We clearly have $\rho_i'(E_j) = 0$ for $i \neq j$. Even more, one can easily verify that $\mathcal{M}_i' \approx \mathcal{M}_i \oplus \mathcal{U}_0(E_j)$, where $\mathcal{U}_0(E_j)$ is the trivial q-matroid on the ground space E_j. Thus $\mathcal{M}_i'$ is a special instance of the direct sum (called "adding a loop space" in [13]). Some additional basic and to-be-expected properties of the direct sum will be presented below in Theorem 5.1.5 after deriving more convenient expressions for the rank function.

The definition of the rank function of $\mathcal{M}_1 \oplus \mathcal{M}_2$ in (5.2) becomes quickly very cumbersome as it requires computing the minimum over all subspaces of V. We now derive various more efficient ways of determining the rank values. The most convenient one is given in Corollary 5.1.8 below, which only requires the cyclic flats of $\mathcal{M}_1$ and $\mathcal{M}_2$. We start with the following simple improvements.

Proposition 5.1.3. *Consider the situation of Theorem 5.1.2. Define the sets*

$$\begin{aligned}
\mathcal{X} &= \{X \in \mathcal{L}(E) \mid \rho_1'(X) + \rho_2'(X) < \dim X\}, \\
\mathcal{T} &= \{X_1 \oplus X_2 \mid X_i \in \mathcal{L}(E_i)\}, \\
\mathcal{T}(V) &= \{X_1 \oplus X_2 \mid X_i \leq \pi_i(V)\} \quad \textit{for } \; V \in \mathcal{L}(E).
\end{aligned} \tag{5.3}$$

Then for any $V \in \mathcal{L}(E)$

$$\rho(V) = \dim V + \min_{X \in \{0\} \cup (\mathcal{X} \cap \mathcal{L}(V))} \big(\rho_1'(X) + \rho_2'(X) - \dim X\big) \tag{5.4}$$

$$= \dim V + \min_{X \in \mathcal{T}} \big(\rho_1'(X) + \rho_2'(X) - \dim(X \cap V)\big) \tag{5.5}$$

$$= \dim V + \min_{X \in \mathcal{T}(V)} \big(\rho_1'(X) + \rho_2'(X) - \dim(X \cap V)\big). \tag{5.6}$$

As a consequence, V *is independent in* $\mathcal{M}_1 \oplus \mathcal{M}_2$ *iff* $\mathcal{L}(V) \cap \mathcal{X} = \emptyset$.

Proof. 1) Using $X = 0$ in (5.2) we obtain $\rho(V) \leq \dim V$ (as it has to be). Hence we only have to test the subspaces X of V that may lead to $\rho(V) < \dim V$. This is exactly the collection $\mathcal{X}$, and thus (5.4) is established. This identity also implies the statement about the independent spaces.
2) Let $M_1 := \min_{X \leq V} \big(\rho_1'(X) + \rho_2'(X) - \dim X\big)$ and $M_2 := \min_{X \leq E} \big(\rho_1'(X) + \rho_2'(X) - \dim(X \cap V)\big)$. Clearly $M_2 \leq M_1$. For the converse inequality let $\hat{X} \leq E$ be such that $M_2 = \rho_1'(\hat{X}) + \rho_2'(\hat{X}) - \dim(\hat{X} \cap V)$. Set $\tilde{X} := \hat{X} \cap V$. Using monotonicity of the rank function we obtain $M_1 \leq \rho_1'(\tilde{X}) + \rho_2'(\tilde{X}) - \dim \tilde{X} \leq \rho_1'(\hat{X}) + \rho_2'(\hat{X}) - \dim(\hat{X} \cap V) = M_2$. Thus $M_1 = M_2$ and

$$\rho(V) = \dim V + \min_{X \leq E} \big(\rho_1'(X) + \rho_2'(X) - \dim(X \cap V)\big). \tag{5.7}$$

Actually, this expression for the rank function of the direct sum appears in [13, Thm. 25]; see also [13, Rem. 26]. Now we are ready to prove (5.5). Recall that $\rho_i'(X) = \rho_i(\pi_i(X))$. Hence $\rho_1'(X) + \rho_2'(X)$ only depends on the projections $\pi_i(X)$. Since $X \leq \pi_1(X) \oplus \pi_2(X) =: X'$ for any subspace X, we have $\rho_1'(X) + \rho_2'(X) - \dim(X \cap V) \geq \rho_1'(X') + \rho_2'(X') - \dim(X' \cap V)$. This shows that it suffices to take the minimum in (5.7) over subspaces X satisfying $X = \pi_1(X) \oplus \pi_2(X)$. But this is exactly the collection $\mathcal{T}$, and hence (5.5) is established.
3) Let $X = X_1 \oplus X_2 \in \mathcal{T}$. Then $X \cap V \subseteq Y_1 \oplus Y_2$, where $Y_i = X_i \cap \pi_i(V)$. Set $Y := Y_1 \oplus Y_2$. Then $Y \in \mathcal{T}(V)$ and $X \cap V \subseteq Y \cap V$. Thus $\rho_1(Y_1) + \rho_2(Y_2) - \dim(Y \cap V) \leq \rho_1(X_1) + \rho_2(X_2) - \dim(X \cap V)$. This shows that the minimum in (5.5) is attained by a subspace in $\mathcal{T}(V)$ and (5.6) is proven. □

The above allows an immediate characterization of the circuits of the direct sum.

Corollary 5.1.4. *Consider the direct sum* $\mathcal{M}_1 \oplus \mathcal{M}_2$ *as in Theorem 5.1.2 and the set* $\mathcal{X}$ *in* (5.3). *Then the circuits of the direct sum are given by*

$$\mathcal{C}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \{X \in \mathcal{X} \mid X \textit{ is inclusion-minimal in } \mathcal{X}\}.$$

As to be expected, the direct sum behaves well with respect to restriction to the initial ground spaces E_i and the according contraction. Recall Definitions 2.1.9, 2.1.15(a) and 2.1.16.

Theorem 5.1.5 ([13, Thm. 47, Cor. 48])**.** *Let* $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ *be as in Theorem 5.1.2.*

(a) For all $V_i \in \mathcal{L}(E_i)$ we have $\rho_i(V_i) = \rho'_i(V_i)$ and $\rho'_j(V_i) = 0$ for $j \neq i$. Furthermore, $\rho(V_1 \oplus V_2) = \rho_1(V_1) + \rho_2(V_2)$. Hence $\mathcal{M}_i \approx \mathcal{M}|_{E_i} \approx (\mathcal{M}'_i)|_{E_i}$, and the isomorphism is provided by ι_i. Moreover, for $i \neq j$ the q-matroid $(\mathcal{M}'_i)|_{E_j}$ is the zero q-matroid.

(b) $\rho(\mathcal{M}) = \rho_1(\mathcal{M}_1) + \rho_2(\mathcal{M}_2)$.

(c) $\mathcal{M}/E_i \approx \mathcal{M}_j$ for $i \neq j$.

Proof. (a) The first statement follows from $\rho'_i(V_i) = \rho_i(\pi_i(V_i)) = \rho_i(V_i)$. Let now $V = V_1 \oplus V_2$. By (5.6) there exist $X = X_1 \oplus X_2$ for some $X_i \leq V_i$ such that $\rho(V) = \dim V + \rho'_1(X_1) + \rho'_2(X_2) - \dim X$. Write $V_i = X_i \oplus Z_i$ for some $Z_i \leq V_i$. Using the properties of the rank functions ρ'_i we compute

$$\begin{aligned}\rho(V) &= \rho'_1(X_1) + \rho'_2(X_2) + \dim Z_1 + \dim Z_2 \geq \rho'_1(X_1) + \rho'_2(X_2) + \rho'_1(Z_1) + \rho'_2(Z_2)\\ &\geq \rho'_1(X_1 \oplus Z_1) + \rho'_2(X_2 \oplus Z_2) = \rho'_1(V_1) + \rho'_2(V_2)\\ &= \rho_1(V_1) + \rho_2(V_2) \geq \rho(V),\end{aligned}$$

where the last step follows from (5.6) with $X = V$. This establishes the desired identity. The isomorphisms follow from $\rho(V_i) = \rho_i(V_i) = \rho'_i(V_i)$, and the rest is clear.
(b) is a consequence of (a) because $\rho(\mathcal{M}) = \rho(E) = \rho_1(E_1) + \rho_2(E_2)$.
(c) Without loss of generality let $i = 1$. Denote the rank function of $\mathcal{M}/E_1$ by $\hat{\rho}_1$. Let $V \in \mathcal{L}(E_2)$. Then $\hat{\rho}_1(V/E_1) = \rho(E_1 \oplus V) - \rho(E_1) = \rho_1(E_1) + \rho_2(V) - \rho_1(E_1) = \rho_2(V)$, where the last identity follows from (a). Hence $\mathcal{M}_2$ and$\mathcal{M}/E_1$ are equivalent via the isomorphism $\hat{\pi} : E_2 \longrightarrow E/E_1$ induced by the canonical projection from E onto E/E_1. □

Now we obtain a very efficient way of computing the rank function of the direct sum which only requires the collections of cyclic flats of the two summands. The following notation we be convenient for the rest of this paper.

Notation 5.1.6. *Let $E = E_1 \oplus E_2$ and $\mathcal{Y}_i \subseteq \mathcal{L}(E_i)$ for $i = 1, 2$. We define*

$$\mathcal{Y}_1 \oplus \mathcal{Y}_2 = \{Y_1 \oplus Y_2 \mid Y_i \in \mathcal{Y}_i\}.$$

Theorem 5.1.7. *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids and let $\mathcal{F}_i = \mathcal{F}(\mathcal{M}_i)$ and $\mathcal{Z}_i = \mathcal{Z}(\mathcal{M}_i)$, that is, $\mathcal{F}_i$ (resp. $\mathcal{Z}_i$) are the collections of flats (resp. cyclic flats) of $\mathcal{M}_i$. Let ρ be the rank function of $\mathcal{M}_1 \oplus \mathcal{M}_2$. Then for all $V \in \mathcal{L}(E_1 \oplus E_2)$ we have*

$$\rho(V) = \dim V + \min_{F_1 \oplus F_2 \in \mathcal{F}_1 \oplus \mathcal{F}_2} \big(\rho_1(F_1) + \rho_2(F_2) - \dim((F_1 \oplus F_2) \cap V)\big) \tag{5.8}$$

$$= \dim V + \min_{Z_1 \oplus Z_2 \in \mathcal{Z}_1 \oplus \mathcal{Z}_2} \big(\rho_1(Z_1) + \rho_2(Z_2) - \dim((Z_1 \oplus Z_2) \cap V)\big). \tag{5.9}$$

Proof. 1) We use (5.5). Let $X = X_1 \oplus X_2 \in \mathcal{T}$. Clearly, $\rho'_i(X) = \rho_i(X_i)$ for $i = 1, 2$. Suppose $X_1 \notin \mathcal{F}_1$. Then there exists $X'_1 \in \mathcal{L}(E_1)$ such that $X_1 \lneq X'_1$ and $\rho_1(X_1) = \rho_1(X'_1)$. Therefore $X \lneq X'_1 \oplus X_2 =: X'$ and $\rho_1(X'_1) + \rho_2(X_2) - \dim(X' \cap V) \leq \rho_1(X_1) + \rho_2(X_2) - \dim(X \cap V)$. The same argument applies to X_2, and this shows

that the minimum in (5.5) is attained by a space in $\mathcal{F}_1 \oplus \mathcal{F}_2$. This establishes (5.8).
2) Let $V \in \mathcal{L}(E)$ and set

$$M_1 = \min_{F_1 \oplus F_2 \in \mathcal{F}_1 \oplus \mathcal{F}_2} \big(\rho_1(F_1) + \rho_2(F_2) - \dim((F_1 \oplus F_2) \cap V)\big),$$
$$M_2 = \min_{Z_1 \oplus Z_2 \in \mathcal{Z}_1 \oplus \mathcal{Z}_2} \big(\rho_1(Z_1) + \rho_2(Z_2) - \dim((Z_1 \oplus Z_2) \cap V)\big).$$

Clearly $M_1 \leq M_2$. For the converse let $\hat{F} := \hat{F}_1 \oplus \hat{F}_2 \in \mathcal{F}_1 \oplus \mathcal{F}_2$ be such that $M_1 = \rho_1(\hat{F}_1) + \rho_2(\hat{F}_2) - \dim(\hat{F} \cap V)$. Set $Z_i = \mathrm{cyc}_i(\hat{F}_i)$, where $\mathrm{cyc}_i(\,\cdot\,)$ is the cyclic core in the q-matroid $\mathcal{M}_i$. Then Z_i is in $\mathcal{Z}_i$ thanks to Lemma 2.2.25. Moreover, $Z = Z_1 \oplus Z_2 \leq \hat{F}$ and $\rho_i(\hat{F}_i) = \rho_i(Z_i) + \dim(\hat{F}_i/Z_i)$ by Proposition 2.2.24. Now we compute

$$\begin{aligned} M_1 &= \rho_1(\hat{F}_1) + \rho_2(\hat{F}_2) - \dim(\hat{F} \cap V) \\ &= \rho_1(Z_1) + \dim(\hat{F}_1/Z_1) + \rho_2(Z_2) + \dim(\hat{F}_2/Z_2) - \dim \hat{F} - \dim V + \dim(\hat{F} + V) \\ &\geq \rho_1(Z_1) + \rho_2(Z_2) - \dim Z - \dim V + \dim(Z + V) \\ &= \rho_1(Z_1) + \rho_2(Z_2) - \dim(Z \cap V) \geq M_2. \end{aligned}$$

Hence $M_1 = M_2$, which establishes (5.9). □

The identity (5.9) can be rewritten as the following convenient identity. Note its resemblance with Corollary 2.2.30. Using this identity we will prove in the next section that $\mathcal{Z}_1 \oplus \mathcal{Z}_2$ is the collection of cyclic flats of $\mathcal{M}_1 \oplus \mathcal{M}_2$, and thus it is indeed a special case of Corollary 2.2.30.

Corollary 5.1.8. *In the situation of Theorem 5.1.7 we have*

$$\rho(V) = \min_{Z \in \mathcal{Z}_1 \oplus \mathcal{Z}_2} \big(\rho(Z) + \dim(V + Z)/Z\big) \quad \textit{for all} \quad V \in \mathcal{L}(E_1 \oplus E_2).$$

Proof. Let $Z = Z_1 \oplus Z_2 \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$. With the aid of Theorem 5.1.5(a) we compute

$$\begin{aligned} \rho_1(Z_1) + \rho_2(Z_2) - \dim(Z \cap V) &= \rho(Z) - \dim Z - \dim V + \dim(Z + V) \\ &= \rho(Z) + \dim(Z + V)/Z - \dim V, \end{aligned}$$

and the result follows from (5.9). □

In the last part of this section we turn to the dual of the direct sum. As shown next, it is the direct sum of the dual q-matroids if taken with respect to compatible NSBFs. Defining duality with respect to lattice anti-isomorphisms, the result below appears also in [13, Thm. 50]. We will comment on the relation in Remark 5.1.10.

Theorem 5.1.9. *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids. Set $E = E_1 \oplus E_2$. Choose NSBFs $\langle \cdot \,|\, \cdot \rangle_i$ on E_i and set $\langle v_1 + v_2 \,|\, w_1 + w_2 \rangle = \langle v_1 \,|\, w_1 \rangle_1 + \langle v_2 \,|\, w_2 \rangle_2$ for all $v_i, w_i \in E_i$. Then $\langle \cdot \,|\, \cdot \rangle$ is an NSBF on E and*

$$(\mathcal{M}_1 \oplus \mathcal{M}_2)^* = \mathcal{M}_1^* \oplus \mathcal{M}_2^*,$$

where $\mathcal{M}_i^*$ and $(\mathcal{M}_1 \oplus \mathcal{M}_2)^*$ are the dual q-matroids with respect to the given NSBFs.

Proof. It is easy to see that $\langle\cdot\,|\,\cdot\rangle$ is an NSBF on E. We denote the corresponding orthogonal space of $V \leq E$ by $V^\perp$, while for $W \leq E_i$ we use $W^{\perp(i)}$ for the orthogonal of W in E_i with respect to $\langle\cdot\,|\,\cdot\rangle_i$. By construction we have

$$(V_1 \oplus V_2)^\perp = V_1^{\perp(1)} \oplus V_2^{\perp(2)} \text{ for all } V_i \in \mathcal{L}(E_i).$$

Let $\mathcal{M}_1^* \oplus \mathcal{M}_2^* = (E, \tilde{\rho})$ and $\mathcal{M}_1 \oplus \mathcal{M}_2 = (E, \rho)$. Then $(\mathcal{M}_1 \oplus \mathcal{M}_2)^* = (E, \rho^*)$ with ρ^* as in (2.1). We have to show that $\rho^*(V) = \tilde{\rho}(V)$ for all $V \in \mathcal{L}(E)$. We will use (5.5) for the rank function ρ (Corollary 5.1.8 does not simplify the computations.). Writing $X = X_1 \oplus X_2$ for $X \in \mathcal{T}$ we have

$$\tilde{\rho}(V) = \dim V + \min_{X \in \mathcal{T}} \big(\rho_1^*(X_1) + \rho_2^*(X_2) - \dim(X \cap V)\big).$$

With the aid of Theorem 5.1.5(a) we obtain for any $X = X_1 \oplus X_2 \in \mathcal{T}$

$$\begin{aligned}
&\rho_1^*(X_1) + \rho_2^*(X_2) - \dim(X \cap V)\\
&= \dim X_1 - \rho_1(E_1) + \rho_1(X_1^{\perp(1)}) + \dim X_2 - \rho_2(E_2) + \rho_2(X_2^{\perp(2)}) - \dim(X \cap V)\\
&= \dim X - \rho(E) + \rho(X^\perp) - \dim E + \dim(X \cap V)^\perp\\
&= \dim X - \rho(E) + \rho(X^\perp) - \dim E + \dim(X^\perp + V^\perp)\\
&= \dim X - \rho(E) + \rho(X^\perp) - \dim E + \dim X^\perp + \dim V^\perp - \dim(X^\perp \cap V^\perp)\\
&= \dim V^\perp - \rho(E) + \rho(X^\perp) - \dim(X^\perp \cap V^\perp).
\end{aligned}$$

Using $\{X^\perp \mid X \in \mathcal{T}\} = \mathcal{T}$ and again (5.5) we now arrive at

$$\begin{aligned}
\tilde{\rho}(V) &= \dim E - \rho(E) + \min_{X \in \mathcal{T}} \big(\rho(X^\perp) - \dim(X^\perp \cap V^\perp)\big)\\
&= \dim E - \rho(E) + \min_{X \in \mathcal{T}} \big(\rho(X) - \dim(X \cap V^\perp)\big)\\
&= \dim V - \rho(E) + \dim V^\perp + \min_{X \in \mathcal{T}} \big(\rho_1(X_1) + \rho_2(X_2) - \dim(X \cap V^\perp)\big)\\
&= \dim V - \rho(E) + \rho(V^\perp) = \rho^*(V),
\end{aligned}$$

as desired. □

Remark 5.1.10. *In [13, Def. 6] the authors define duality of q-matorids with respect to an involutory anti-isomorphism on the subspace lattice $\mathcal{L}(E)$. Denoting such an anti-isomorphism by $\perp$, the definition of $\mathcal{M}^*$ reads exactly as in Theorem 2.1.11. Since the orthogonal complement with respect to a chosen NSBF induces a lattice anti-isomorphism, the duality result in [13, Thm. 50] appears to be more general than Theorem 5.1.9. However, as we now briefly discuss, the two results differ only by a semi-linear isomorphism on E (if $\dim E \geq 3$). Indeed, choose an NSBF on E and denote the corresponding orthogonal space of $V \leq E$ by $V^{\perp\!\!\!\perp}$. Then $\tau : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$, $V \longmapsto V^{\perp\!\!\!\perp}$ is an anti-isomorphism on the lattice $\mathcal{L}(E)$. Let now $\perp$ be any anti-isomorphism on $\mathcal{L}(E)$. Then $\tau \circ \perp$ is a lattice isomorphism and thanks to the Fundamental Theorem of Projective Geometry (see for instance [3, Ch. II.10] or [40, Thm. 1]) there exists a semi-linear isomorphism $f : E \longrightarrow E$ such that $\tau(V^\perp) =$*

$f(V)$ for all $V \in \mathcal{L}(E)$. In other words, $(V^{\perp})^{\perp\!\!\!-} = f(V)$ or $V^{\perp} = f(V)^{\perp\!\!\!-}$ for all $V \in \mathcal{L}(E)$. This shows that the lattice anti-isomorphism $\perp$ differs from the one induced by the chosen NSBF by the semi-linear isomorphism on E. Denote the dual rank function of $\mathcal{M} = (E, \rho)$ with respect to $\perp$ and τ by $\rho^{*(\perp)}$ and $\rho^{*(\perp\!\!\!-)}$, respectively. Then $\rho^{*(\perp)}(V) = \rho^{*(\perp\!\!\!-)}(f(V))$ and thus the two dual q-matroids differ only by the semi-linear isomorphism f. This shows that [13, Thm. 50] is a consequence Theorem 5.1.9 above, which has a significantly shorter and simpler proof.

5.2 The Cyclic Flats of the Direct Sum.

In this short section we show that the cyclic flats of a direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the collection of the direct sums of the cyclic flats of the two components $\mathcal{M}_1$ and $\mathcal{M}_2$. Recall Section 2.2.4 for definition and properties of cyclic flats. The following lemma is needed.

Lemma 5.2.1.
(a) Let $\mathcal{M} = (E, \rho)$ be a q-matroid. Suppose $F \in \mathcal{F}(\mathcal{M})$ and $O \in \mathcal{O}(\mathcal{M})$ are such that $F \lneq O$. Then $0 < \rho(O) - \rho(F) < \dim O - \dim F$.
(b) Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids and ρ be the rank function of $\mathcal{M}_1 \oplus \mathcal{M}_2$. Let $F = F_1 \oplus F_2$ with $F_i \in \mathcal{F}(\mathcal{M}_i)$ and $O = O_1 \oplus O_2$ with $O_i \in \mathcal{O}(\mathcal{M}_i)$ be such that $F \lneq O$. Then $0 < \rho(O) - \rho(F) < \dim O - \dim F$.

Proof. Since F is a flat, we clearly have $0 < \rho(O) - \rho(F)$. Furthermore, let $O = F \oplus T$ for some $T \leq O$. Then (R1)–(R3) for ρ imply $\rho(O) \leq \rho(F) + \rho(T) \leq \rho(F) + \dim T = \rho(F) + \dim O - \dim F$. We show that the first inequality is strict. To do so, let $U \in \text{Hyp}(T)$. Then $F \oplus U \in \text{Hyp}(O)$. Thus cyclicity of O implies $\rho(O) = \rho(F \oplus U) \leq \rho(F) + \rho(U) < \rho(F) + \dim T$, as desired.
(b) By assumption $F_i \leq O_i$ for $i = 1, 2$. Without loss of generality we may assume $F_1 \lneq O_1$. With the aid of Theorem 5.1.5(a) and Part (a) we compute

$$\begin{aligned}\rho(O) - \rho(F) &= \rho_1(O_1) - \rho_1(F_1) + \rho_2(O_2) - \rho_2(F_2)\\ &< \dim O_1 - \dim F_1 + \dim O_2 - \dim F_2 = \dim O - \dim F.\end{aligned}$$

The second expression also shows that $\rho(O) - \rho(F) > 0$. □

Now we are ready for our main result.

Theorem 5.2.2. *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids and $\mathcal{Z}_i = \mathcal{Z}(\mathcal{M}_i)$. As in Notation 5.1.6 let $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \{Z_1 \oplus Z_2 \mid Z_i \in \mathcal{Z}_i\}$. Then*

$$\mathcal{Z}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{Z}_1 \oplus \mathcal{Z}_2.$$

Proof. "$\supseteq$" Let $V \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$. Then $V = V_1 \oplus V_2$ for some $V_i \in \mathcal{Z}_i$.
a) We show that V is a flat in $\mathcal{M}_1 \oplus \mathcal{M}_2$. Let $x \in E \setminus V$. We need to show that $\rho(V + \langle x \rangle) = \rho(V) + 1$. By Corollary 5.1.8 there exists $\hat{Z} = \hat{Z}_1 \oplus \hat{Z}_2 \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$ such that

$$\rho(V + \langle x \rangle) = \rho(\hat{Z}) + \dim((V + \langle x \rangle + \hat{Z})/\hat{Z}).$$

If $\hat{Z} = V$, then this implies $\rho(V + \langle x\rangle) = \rho(V) + 1$, as desired. Let now $\hat{Z} \neq V$ and set $F := V \cap \hat{Z} = (V_1 \cap \hat{Z}_1) \oplus (V_2 \cap \hat{Z}_2)$. Since each $V_i \cap \hat{Z}_i$ is a flat (see Theorem 2.2.15), F is of the form $F = F_1 \oplus F_2$, where $F_i \in \mathcal{F}(\mathcal{M}_i)$. By assumption V_i are cyclic spaces, and thus we may apply Lemma 5.2.1(b) to $F \lneq V$. This leads to

$$\rho(V) - \rho(\hat{Z}) \leq \rho(V) - \rho(F) < \dim V - \dim(F) = \dim(V/F), \tag{5.10}$$

which in turn implies

$$\rho(V + \langle x\rangle) = \rho(\hat{Z}) + \dim((V + \langle x\rangle + \hat{Z})/\hat{Z}) > \rho(V) - \dim(V/F) + \dim((V + \langle x\rangle + \hat{Z})/\hat{Z}).$$

Since $\dim(V/F) = \dim(V/(V \cap \hat{Z})) = \dim((V + \hat{Z})/\hat{Z}) \leq \dim((V + \langle x\rangle + \hat{Z})/\hat{Z})$, we conclude that $\rho(V + \langle x\rangle) > \rho(V)$, as desired.
b) We show that V is cyclic. Let $D \in \mathrm{Hyp}(V)$. By Corollary 5.1.8 there exists $\hat{Z} \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$ such that

$$\rho(D) = \rho(\hat{Z}) + \dim((D + \hat{Z})/\hat{Z}).$$

If $\hat{Z} = V$, then this implies $\rho(D) = \rho(V)$, as desired. Thus let $\hat{Z} \neq V$. As above, we set $F := V \cap \hat{Z}$ and apply Lemma 5.2.1(b) to $F \lneq V$. Thus we have again (5.10) and compute

$$\begin{aligned}\rho(D) &= \rho(\hat{Z}) + \dim((D + \hat{Z})/\hat{Z}) \\ &> \rho(V) - \dim(V/V \cap \hat{Z}) + \dim(D/D \cap \hat{Z}) = \rho(V) - 1 + \dim(V \cap \hat{Z}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \dim(D \cap \hat{Z}) \\ &\geq \rho(V) - 1.\end{aligned}$$

This shows $\rho(D) = \rho(V)$.
"$\subseteq$" Let $Z \in \mathcal{Z}(\mathcal{M}_1 \oplus \mathcal{M}_2)$. Again, by Corollary 5.1.8 there exists $\hat{Z} \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$ such that

$$\rho(Z) = \rho(\hat{Z}) + \dim((Z + \hat{Z})/\hat{Z}). \tag{5.11}$$

Since Z is cyclic, every space $D \in \mathrm{Hyp}(Z)$ satisfies

$$\rho(D) = \rho(Z) = \rho(\hat{Z}) + \dim((Z + \hat{Z})/\hat{Z}) \geq \rho(\hat{Z}) + \dim((D + \hat{Z})/\hat{Z}) \geq \rho(D),$$

and hence $\dim((D + \hat{Z})/\hat{Z}) = \dim((Z + \hat{Z})/\hat{Z})$. Since this is true for every $D \in \mathrm{Hyp}(Z)$, we conclude $Z \leq \hat{Z}$. Next, Z is a flat and thus every $x \in E \setminus Z$ satisfies

$$\rho(Z) < \rho(Z + \langle x\rangle) \leq \rho(\hat{Z}) + \dim((Z + \langle x\rangle + \hat{Z})/\hat{Z}).$$

Together with (5.11) this implies $\dim(Z + \hat{Z}) < \dim(Z + \langle x\rangle + \hat{Z})$, and thus $x \notin \hat{Z}$. Since this is true for every $x \in E \setminus Z$, we conclude that $\hat{Z} \leq Z$. All of this shows that $Z = \hat{Z}$ and thus $Z \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$. This concludes the proof. □

Theorem 5.2.2 in combination with Corollary 5.1.8 immediately implies associativity of the direct sum operation (which is not obvious from the very definition of the direct sum). The result will be crucial for the decomposition of q-matroids in the next section.

Corollary 5.2.3. *Let $\mathcal{M}_i$, $i = 1, 2, 3$, be q-matroids. Then $(\mathcal{M}_1 \oplus \mathcal{M}_2) \oplus \mathcal{M}_3 = \mathcal{M}_1 \oplus (\mathcal{M}_2 \oplus \mathcal{M}_3)$.*

The analogous identity as in Theorem 5.2.2 is not true for the flats, independent spaces, circuits etc.

Remark 5.2.4. *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Recall Notation 5.1.6. With the aid of Theorem 5.1.5 it is easy to see that $\mathcal{I}(\mathcal{M}_1) \oplus \mathcal{I}(\mathcal{M}_2) \subset \mathcal{I}(\mathcal{M})$, $\mathcal{F}(\mathcal{M}_1) \oplus \mathcal{F}(\mathcal{M}_2) \subset \mathcal{F}(\mathcal{M})$, $\mathcal{O}(\mathcal{M}_1) \oplus \mathcal{O}(\mathcal{M}_2) \subset \mathcal{O}(\mathcal{M})$, and $\mathcal{C}(\mathcal{M}_1) \cup \mathcal{C}(\mathcal{M}_2) \subset \mathcal{C}(\mathcal{M})$. In general equality does not hold in any of these cases.*

We conclude this section by illustrating the discrepancies in the following example.

Example 5.2.5. *Let $\mathbb{F} = \mathbb{F}_2$ and consider $\mathbb{F}_{2^3}$ with primitive element ω satisfying $\omega^3 + \omega + 1 = 0$. Let*

$$G_1 = \begin{pmatrix} 1 & 0 & \omega^3 \\ 0 & 1 & \omega \end{pmatrix},\ G_2 = \begin{pmatrix} 1 & 0 & \omega^3 & \omega \\ 0 & 1 & \omega^4 & \omega^2 \end{pmatrix},\ G = \begin{pmatrix} 1 & 0 & \omega^3 & 0 & 0 & 0 & 0 \\ 0 & 1 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \omega^3 & \omega \\ 0 & 0 & 0 & 0 & 1 & \omega^4 & \omega^2 \end{pmatrix}.$$

Note that G is the block diagonal matrix with diagonal blocks G_1 and G_2. Let $\mathcal{M}_i = \mathcal{M}_{G_i}$ and $\mathcal{N} = \mathcal{M}_G$, i.e., they are the q-matroids represented by G_1, G_2, and G, respectively. Furthermore, let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Thus both $\mathcal{M}$ and $\mathcal{N}$ have ground space $\mathbb{F}^7$. In the following table we list the number of flats, cyclic spaces, etc. of all these q-matroids.

Table 5.1: Cardinality of collection of spaces for $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}$ and $\mathcal{N}$.

	flats	cyclic spaces	cyclic flats	ind. spaces	dep. spaces	circuits	bases
$\mathcal{M}_1$	7	2	2	14	2	1	6
$\mathcal{M}_2$	11	11	5	48	19	9	32
$\mathcal{N}$	2201	124	40	24108	5104	73	9792
$\mathcal{M}$	7541	412	10	24861	4351	355	10416

Note that the 10 cyclic flats of $\mathcal{M}$ are consistent with Theorem 5.2.2. It is remarkable that $\mathcal{M}$ has significantly more flats and cyclic spaces than $\mathcal{N}$, yet fewer cyclic flats. Furthermore, one can verify that the cyclic flats of $\mathcal{M}$ are also cyclic flats of $\mathcal{N}$. Finally, all independent spaces of $\mathcal{N}$ are also independent spaces of $\mathcal{M}$, which will also be studied in We will prove those latter facts in generality in Section 5.4.

5.3 Decomposition of q-Matroids into Irreducible Components.

We introduce the notion of irreducibility for q-matroids and show that every q-matroid can be decomposed as a direct sum of irreducible q-matroids, whose summands are unique up to equivalence. Our main tool are cyclic flats, in particular Theorem 5.2.2.

This makes our approach substantially different from classical matroid theory, where decompositions are usually based on connected components. As to our knowledge there is no notion of connectedness for q-matroids that may be used for decompositions into direct sums; see also [13, Sec. 8].

Throughout, let $\mathcal{M} = (E, \rho)$ be a q-matroid. In order to simplify the discussion of irreducibility and decompositions we start with the following simple fact concerning equivalence in the sense of Definition 2.1.9. It can easily be checked with the definition in Theorem 5.1.2.

Remark 5.3.1. *Suppose $\mathcal{M} \approx \hat{\mathcal{M}}_1 \oplus \hat{\mathcal{M}}_2$. Then there exists a decomposition $E = E_1 \oplus E_2$ and q-matroids $\mathcal{M}_i$ such that*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2, \quad \mathcal{M}_i = \mathcal{M}|_{E_i}, \quad \mathcal{M}_i \approx \hat{\mathcal{M}}_i.$$

As a consequence, we do not need to take equivalence into account when discussing decomposability into direct sums.

Definition 5.3.2. *The q-matroid $\mathcal{M}$ is called* reducible *if there exists q-matroids $\mathcal{M}_1$, $\mathcal{M}_2$ with nonzero ground spaces such that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Otherwise $\mathcal{M}$ is called* irreducible*.*

Clearly, a q-matroid on a 1-dimensional ground space is irreducible. Furthermore, thanks to Theorem 5.1.9 $\mathcal{M}$ is irreducible if and only if $\mathcal{M}^*$ is. We collect some facts about the uniform q-matroids.

Example 5.3.3. *(a) The trivial and the free q-matroids $\mathcal{U}_{0,n}$ and $\mathcal{U}_{n,n}$ are irreducible if and only if $n = 1$. Indeed, Theorem 5.1.2 implies $\mathcal{U}_{0,n_1} \oplus \mathcal{U}_{0,n_2} = \mathcal{U}_{0,n}$ and likewise $\mathcal{U}_{n_1,n_1} \oplus \mathcal{U}_{n_2,n_2} = \mathcal{U}_{n,n}$, where $n = n_1 + n_2$.*

(b) For $0 < k < n := \dim E$ the uniform q-matroid $\mathcal{M} := \mathcal{U}_k(E)$ is irreducible. To see this, note first that $\mathcal{Z}(\mathcal{M}) = \{0, E\}$ thanks to Example 2.2.27. Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ for some q-matroids $\mathcal{M}_i = (E_i, \rho_i)$. Then the identity $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{M}_1) \oplus \mathcal{Z}(\mathcal{M}_2)$ implies that, without loss of generality, $\mathcal{Z}(\mathcal{M}_1) = \{Z_1\}$ and $\mathcal{Z}(\mathcal{M}_2) = \{Z_2, Z_2'\}$. Thus $0 = Z_1 \oplus Z_2$ and $E = Z_1 \oplus Z_2'$, and therefore $Z_1 = 0$, $Z_2 = 0$, $Z_2' = E$. Since $E = E_1 \oplus E_2$ and $Z_2' \leq E_2$, this leads to $E_2 = E$ and thus $E_1 = 0$. Hence $\mathcal{M}_1$ has a zero-dimensional ground space and $\mathcal{M}$ is irreducible.

(c) Conversely, if $\mathcal{M} = (E, \rho)$ is such that $\mathcal{Z}(\mathcal{M}) = \{0, E\}$ and $\rho(E) = k \in \{1, \ldots, n-1\}$, then $\mathcal{M} = \mathcal{U}_k(E)$. Indeed, suppose there exists a space V such that $\rho(V) < \min\{k, \dim V\}$. Let $l = \dim V$ be minimal subject to this condition. Then $\rho(V) = l-1$ and V is cyclic. Thus $\mathrm{cl}(V) = E$ and $\rho(V) = \rho(E) = k$, contradicting the choice of V. This shows $\rho(V) = \min\{k, \dim V\}$ for all $V \in \mathcal{L}(E)$.

The goal of this section is (a) to provide a criterion for irreducibility and (b) to show that every q-matroid decomposes into a direct sum of irreducible q-matroids, whose summands are unique up to ordering and equivalence. The next lemma will be needed throughout.

Lemma 5.3.4. *Let $\hat{Z} \in \mathcal{Z}(\mathcal{M})$ and consider the restriction $\mathcal{M}|_{\hat{Z}}$. Then*

$$\mathcal{Z}(\mathcal{M}|_{\hat{Z}}) = \{Z \in \mathcal{Z}(\mathcal{M}) \mid Z \leq \hat{Z}\}.$$

Proof. Set $\mathcal{Z} = \{Z \in \mathcal{Z}(\mathcal{M}) \mid Z \leq \hat{Z}\}$. Denote the rank function of $\mathcal{M}|_{\hat{Z}}$ by $\hat{\rho}$.
"$\supseteq$" is obvious since $\hat{\rho}(V) = \rho(V)$ for all $V \leq \hat{Z}$.
"$\subseteq$" Let $Z \in \mathcal{Z}(\mathcal{M}|_{\hat{Z}})$. Clearly Z is cyclic in $\mathcal{M}$ because any $D \in \text{Hyp}(Z)$ is a subspace of $\hat{Z}$ and thus satisfies $\rho(D) = \hat{\rho}(D) = \hat{\rho}(Z) = \rho(Z)$. To show that Z is a flat in $\mathcal{M}$, let $x \in E \setminus Z$. If $x \in \hat{Z} \setminus Z$, then $\rho(Z + \langle x \rangle) = \hat{\rho}(Z + \langle x \rangle) > \hat{\rho}(Z) = \rho(Z)$. If $x \in E \setminus \hat{Z}$, then $x \notin \text{cl}(Z)$ because $\text{cl}(Z) \leq \text{cl}(\hat{Z}) = \hat{Z}$ (where $\text{cl}(\cdot)$ denotes the closure in $\mathcal{M}$). Thus $\rho(Z + \langle x \rangle) > \rho(Z)$. All of this shows that Z is a flat in $\mathcal{M}$. □

Our first result shows that whenever E is not a cyclic flat of $\mathcal{M}$, then for any direct complement E_2 of $\text{cyc}(E)$ in E we may split off the free q-matroid on E_2 from $\mathcal{M}$.

Proposition 5.3.5. *Let $E_1 = \text{cyc}(E)$ and choose $E_2 \leq E$ such that $E_1 \oplus E_2 = E$. Consider the restrictions $\mathcal{M}_i = \mathcal{M}|_{E_i}$ for $i = 1, 2$. Then*
(a) $\mathcal{M}_2$ is the free q-matroid on E_2.
(b) $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{M}_1)$.
(c) $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$.

Proof. (a) Proposition 2.2.24(b) implies that E_2 is independent in $\mathcal{M}$ and thus in $\mathcal{M}_2$. Since every subspace of an independent space is independent, the result follows.
(b) follows from Lemma 5.3.4 together with the fact that $\mathcal{Z}(\mathcal{M})$ is a lattice with greatest element $\text{cyc}(E)$.
(c) Part (a) and Example 2.2.27 tell us that $\mathcal{Z}(\mathcal{M}_2) = \{0\}$, and thus (b) and Theorem 5.2.2 imply

$$\mathcal{Z}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{Z}(\mathcal{M}_1) = \mathcal{Z}(\mathcal{M}).$$

Denote the rank function of $\mathcal{M}_1 \oplus \mathcal{M}_2$ by $\hat{\rho}$. Then Theorem 5.1.5(a) implies $\hat{\rho}(Z_1) = \rho_1(Z_1) = \rho(Z_1)$ for all $Z_1 \in \mathcal{Z}(\mathcal{M}_1)$. Hence the cyclic flats in $\mathcal{Z}(\mathcal{M})$ have the same rank value in the q-matroids $\mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}$. The result follows from Corollary 2.2.30. □

Dually, we may split off the trivial q-matroid on $\text{cl}(0)$ from $\mathcal{M}$.

Proposition 5.3.6. *Let $E_1 = \text{cl}(0)$ and choose $E_2 \leq E$ such that $E = E_1 \oplus E_2$. Consider the restrictions $\mathcal{M}_i = \mathcal{M}|_{E_i} = (E_i, \rho_i)$ for $i = 1, 2$ and let $\pi_i : E \longrightarrow E_i$ be the projections.*

(a) $\mathcal{M}_1$ is the trivial q-matroid on E_1.

(b) $\rho(V) = \rho_2(\pi_2(V))$ for all $V \in \mathcal{L}(E)$.

(c) $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$.

We call $\text{cl}(0)$ *the* loop space *of $\mathcal{M}$. It consists of all vectors $x \in E$ such that $\rho(\langle x \rangle) = 0$.*

Proof. (a) is clear and so is the very last part about the vectors in the loop space.
(b) Let $V \in \mathcal{L}(E)$. Since $V \leq \pi_1(V) \oplus \pi_2(V)$ we have $\rho(V) \leq \rho(\pi_1(V)) + \rho(\pi_2(V)) = \rho(\pi_2(V)) = \rho_2(\pi_2(V))$. For the converse inequality, let $y \in \pi_2(V)$. Then there exists $x \in E_1$ such that $x + y \in V$. As a consequence, $V + \langle y\rangle = V + \langle x\rangle$ and $\rho(V) \leq \rho(V + \langle x\rangle) \leq \rho(V) + \rho(\langle x\rangle) = \rho(V)$. Hence we have equality across and this shows that $\rho(V + \langle y\rangle) = \rho(V)$ for all $y \in \pi_2(V)$. Now Proposition 2.1.8(a) leads to $\rho(V + \pi_2(V)) = \rho(V)$ and thus $\rho(\pi_2(V)) \leq \rho(V)$.
(c) Let $\hat{\rho}$ be the rank function of $\mathcal{M}_1 \oplus \mathcal{M}_2$. Then

$$\begin{aligned}\hat{\rho}(V) &= \dim V + \min_{X\leq V}\big(\rho_2(\pi_2(X)) - \dim X\big) = \dim V + \min_{X\leq V}\big(\rho(X) - \dim X\big)\\ &= \dim V + \rho(V) - \dim V = \rho(V),\end{aligned}$$

where the third step follows from the inequality $\dim A - \rho(A) \leq \dim B - \rho(B)$ for any $A \leq B$ (which is a simple consequence of submodularity). □

The following terminology will be convenient.

Definition 5.3.7. *$\mathcal{M}$ is called* full *if* $\mathrm{cl}(0) = 0$ *and* $\mathrm{cyc}(E) = E$.

Remark 5.3.8. *The notion of a full matroid does not exist in classical matroid theory (as to our knowledge) because a matroid satisfying* $\mathrm{cl}(0) = 0$ *and* $\mathrm{cyc}(E) = E$ *is simply called loopless and coloopless. As mentioned previously the notion of coloop depends on the choice of NSBF for the dual q-matroid. For this reason we will not use the notion of coloops.*

Now we can present a first step toward a decomposition of $\mathcal{M}$.

Theorem 5.3.9. *Given $\mathcal{M} = (E, \rho)$. Set $l = \dim \mathrm{cl}(0)$ and $f = \dim E - \dim \mathrm{cyc}(E)$.*

(a) $\mathcal{M}$ is the direct sum of a trivial, a free, and a full q-matroid. Precisely, there exists a full q-matroid $\mathcal{M}'$ such that

$$\mathcal{M} \approx \mathcal{U}_{0,l} \oplus \mathcal{U}_{f,f} \oplus \mathcal{M}'.$$

(b) If $\mathcal{M} \approx \mathcal{U}_{0,a} \oplus \mathcal{U}_{b,b} \oplus \mathcal{N}$, where $\mathcal{N}$ is full, then $a = l$, $b = f$ and $\mathcal{N} \approx \mathcal{M}'$.

We call $\mathcal{U}_{0,l}$, $\mathcal{U}_{f,f}$, and $\mathcal{M}'$ the trivial, free and full component *of $\mathcal{M}$, respectively.*

Proof. Let $E = \mathrm{cyc}(E) \oplus \Delta$ and $\mathrm{cyc}(E) = \mathrm{cl}(0) \oplus \Gamma$. Then $\dim \Delta = f$.
(a) By Proposition 5.3.5 we have $\mathcal{M} = \mathcal{U}_f(\Delta) \oplus \mathcal{M}|_{\mathrm{cyc}(E)}$, and Proposition 5.3.6 implies $\mathcal{M}|_{\mathrm{cyc}(E)} = \mathcal{U}_0(\mathrm{cl}(0)) \oplus \mathcal{M}|_\Gamma$. This proves the stated decomposition of $\mathcal{M}$, and it remains to show that $\mathcal{M}' := \mathcal{M}|_\Gamma$ is full. Again Propositions 5.3.5 and 5.3.6 give us $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{M}|_{\mathrm{cyc}(E)}) = \{\mathrm{cl}(0) \oplus Z \mid Z \in \mathcal{Z}(\mathcal{M}')\}$. Since $\mathrm{cl}(0)$ and $\mathrm{cyc}(E) = \mathrm{cl}(0) \oplus \Gamma$ are the least and greatest element of the lattice $\mathcal{Z}(\mathcal{M})$, we conclude that 0 and Γ are the least and greatest element of the lattice $\mathcal{Z}(\mathcal{M}')$. Thus $\mathcal{M}'$ is full.
(b) Using Remark 5.3.1 we have

$$\mathcal{M} = \mathcal{U}_0(A) \oplus \mathcal{U}_b(B) \oplus \mathcal{N}$$

for some $A, B, N \leq E$ such that $A \oplus B \oplus N = E$, $a = \dim A$, $b = \dim B$, and where N is the ground space of $\mathcal{N}$. Moreover, by (a)

$$\mathcal{M} = \mathcal{U}_0(\mathrm{cl}(0)) \oplus \mathcal{U}_f(\Delta) \oplus \mathcal{M}'.$$

From Theorem 5.2.2 and Example 2.2.27 we obtain $\mathcal{Z}(\mathcal{M}) = \{A\} \oplus \mathcal{Z}(\mathcal{N})$. Since $\mathcal{N}$ is full, the least element of the lattice $\mathcal{Z}(\mathcal{N})$ is 0 and thus A is the least element of $\mathcal{Z}(\mathcal{M})$. But the latter is $\mathrm{cl}(0)$ and thus we arrive at $A = \mathrm{cl}(0)$ and $a = l$. In the same way, the greatest element of $\mathcal{Z}(\mathcal{N})$ is N and thus $A \oplus N$ is the greatest element of $\mathcal{Z}(\mathcal{M})$. Hence $A \oplus N = \mathrm{cyc}(E)$ and $a + \dim N = \dim \mathrm{cyc}(E)$, which implies $f = \dim E - \dim \mathrm{cyc}(E) = b$, as desired. In order to show that $\mathcal{N} \approx \mathcal{M}'$ note that $\mathrm{cyc}(E) = A \oplus \Gamma = A \oplus N$ and $\mathcal{M}|_{\mathrm{cyc}(E)} = \mathcal{U}_0(A) \oplus \mathcal{M}' = \mathcal{U}_0(A) \oplus \mathcal{N}$ (see Theorem 5.1.5(a)). Now Theorem 5.1.5(c) yields $\mathcal{N} \approx (\mathcal{M}|_{\mathrm{cyc}(E)})/A \approx \mathcal{M}'$. This concludes the proof. □

We have the following special case.

Corollary 5.3.10. $|\mathcal{Z}(\mathcal{M})| = 1$ *if and only if* $\mathcal{M}$ *is the direct sum of a trivial and a free q-matroid. Precisely, let* $l = \dim \mathrm{cl}(0)$ *and* $f = \dim E - l$. *Then* $\mathcal{Z}(\mathcal{M}) = \{\mathrm{cl}(0)\} \Longleftrightarrow \mathcal{M} \approx \mathcal{U}_{0,l} \oplus \mathcal{U}_{f,f}$.

Proof. The backward direction follows immediately from Theorem 5.2.2 together with $|\mathcal{Z}(\mathcal{U}_{0,l})| = |\mathcal{Z}(\mathcal{U}_{f,f})| = 1$ for any l, f. The forward direction is a consequence of Theorem 5.3.9 because the assumption implies $\mathrm{cl}(0) = \mathrm{cyc}(E)$. □

In order to derive a criterion for irreducibility we need the following two lemmas. Recall, for any subspace $V \in \mathcal{L}(E)$ we use $\mathcal{B}(V)$ for the collection of bases of V, i.e., $\mathcal{B}(V) = \{I \leq V \mid \dim I = \rho(I) = \rho(V)\}$.

Lemma 5.3.11. *Suppose there exist flats* F_1, F_2 *of* $\mathcal{M}$ *such that* $F_1 \cap F_2 = 0$ *and* $\rho(F_1 \oplus F_2) = \rho(F_1) + \rho(F_2)$.
(a) Let $B_i \in \mathcal{B}(F_i)$. *Then* $B_1 \oplus B_2 \in \mathcal{B}(F_1 \oplus F_2)$.
(b) Let $V_i \in \mathcal{L}(F_i)$. *Then* $\rho(V_1 \oplus V_2) = \rho(V_1) + \rho(V_2)$.

Proof. (a) $B_i \leq F_i = \mathrm{cl}(F_i)$ together with $\rho(B_i) = \rho(F_i)$ implies $\mathrm{cl}(B_i) = F_i$. Since $\mathrm{cl}(B_i) \leq \mathrm{cl}(B_1 \oplus B_2)$, this leads to $B_1 \oplus B_2 \leq F_1 \oplus F_2 = \mathrm{cl}(B_1) \oplus \mathrm{cl}(B_2) \leq \mathrm{cl}(B_1 \oplus B_2)$, and thus $\rho(B_1 \oplus B_2) = \rho(F_1 \oplus F_2)$ thanks to (2.5). Now we have $\rho(B_1 \oplus B_2) = \rho(F_1) + \rho(F_2) = \dim B_1 + \dim B_2 = \dim(B_1 \oplus B_2)$, which shows that $B_1 \oplus B_2$ is a basis of $F_1 \oplus F_2$.
(b) Let $B_i \in \mathcal{B}(V_i)$. Then B_i is an independent space in F_i and thus contained in a basis of F_i, say B_i' (see [36, Thm. 37]). Thanks to part (a) $B_1' \oplus B_2'$ is independent and hence so is $B_1 \oplus B_2$. Putting everything together, we obtain

$$\rho(V_1 \oplus V_2) \leq \rho(V_1) + \rho(V_2) = \dim B_1 + \dim B_2 = \dim(B_1 \oplus B_2) = \rho(B_1 \oplus B_2) \leq \rho(V_1 \oplus V_2).$$

which proves the stated identity. □

Lemma 5.3.12. *Let* $\mathcal{M}$ *be full. Suppose* $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ *for some q-matroids* $\mathcal{M}_i = (E_i, \rho_i)$. *Then* $E_i \in \mathcal{Z}(\mathcal{M})$ *and* $\mathcal{M}_1$, $\mathcal{M}_2$ *are full.*

Proof. Since $\mathcal{M}$ is full, E is in $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{M}_1) \oplus \mathcal{Z}(\mathcal{M}_2)$. Thus there exist $Z_i \in \mathcal{Z}(\mathcal{M}_i)$ such that $E = Z_1 \oplus Z_2$. But then $E = E_1 \oplus E_2$ together with $Z_i \leq E_i$ implies $Z_i = E_i$, and we conclude that $E_i \in \mathcal{Z}(\mathcal{M}_i)$. In the same way, $0 \in \mathcal{Z}(\mathcal{M})$ implies $0 \in \mathcal{Z}(\mathcal{M}_i)$ for $i = 1, 2$, and hence $\mathcal{M}_1$ and $\mathcal{M}_2$ are full. Finally, $E_1 = E_1 \oplus 0$ is in $\mathcal{Z}(\mathcal{M}_1) \oplus \mathcal{Z}(\mathcal{M}_2) = \mathcal{Z}(\mathcal{M})$ and similarly for E_2. □

Now we are ready for our first main result of this section, a characterization of irreducibility.

Theorem 5.3.13. *Let $\mathcal{M} = (E, \rho)$ and $\dim E \geq 2$. The following are equivalent.*
(i) $\mathcal{M}$ is irreducible.
(ii)$\mathcal{M}$ is full and there exist no nonzero spaces $Z_1, Z_2 \in \mathcal{Z}(\mathcal{M})$ such that

$$Z_1 \oplus Z_2 = E, \quad \rho(Z_1) + \rho(Z_2) = \rho(E), \quad \text{and} \;\; \mathcal{Z}_1 \oplus \mathcal{Z}_2 = \mathcal{Z}(\mathcal{M}), \tag{5.12}$$

where $\mathcal{Z}_i = \{Z \in \mathcal{Z}(\mathcal{M}) \mid Z \leq Z_i\}$.

Note that $\mathcal{Z}_i = \mathcal{Z}(\mathcal{M}|_{Z_i})$ thanks to Lemma 5.3.4.

Proof. "(i) ⇒ (ii)" Let $\mathcal{M}$ be irreducible. Then Example 5.3.3(a) and Theorem 5.3.9 together with $\dim E \geq 2$ imply that $\mathcal{M}$ is full. Suppose there do exist nonzero spaces $Z_1, Z_2 \in \mathcal{Z}(\mathcal{M})$ satisfying (5.12). We show that

$$\mathcal{M} = \mathcal{M}|_{Z_1} \oplus \mathcal{M}|_{Z_2}.$$

Set $\mathcal{Z} = \mathcal{Z}(\mathcal{M}|_{Z_1}) \oplus \mathcal{Z}(\mathcal{M}|_{Z_2})$ and denote the rank functions of $\mathcal{M}|_{Z_1} \oplus \mathcal{M}|_{Z_2}$ and $\mathcal{M}|_{Z_i}$ by ρ' and ρ_i, respectively. With the aid of Theorem 5.1.7 and Lemma 5.3.11(b) we obtain for all $V \in \mathcal{L}(E)$

$$\begin{aligned}
\rho'(V) &= \dim V + \min_{Y_1 \oplus Y_2 \in \mathcal{Z}} \Big(\rho_1(Y_1) + \rho_2(Y_2) - \dim((Y_1 \oplus Y_2) \cap V)\Big) \\
&= \dim V + \min_{Y_1 \oplus Y_2 \in \mathcal{Z}} \Big(\rho(Y_1) + \rho(Y_2) - \dim((Y_1 \oplus Y_2) \cap V)\Big) \\
&= \min_{Y_1 \oplus Y_2 \in \mathcal{Z}} \Big(\rho(Y_1 \oplus Y_2) + \dim((V + (Y_1 \oplus Y_2))/(Y_1 \oplus Y_2))\Big) \\
&= \rho(V),
\end{aligned}$$

where the very last step follows from Corollary 2.2.30 and the identity $\mathcal{Z} = \mathcal{Z}(\mathcal{M})$. This establishes the stated direct sum and thus contradicts the irreducibility of $\mathcal{M}$.
"(ii) ⇒ (i)" Suppose $\mathcal{M}$ is full. By contradiction assume that $\mathcal{M}$ is reducible, say $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ for some $\mathcal{M}_i = (E_i, \rho_i)$ with nonzero ground spaces E_i. Then $E = E_1 \oplus E_2$ and $\rho(E) = \rho(E_1) + \rho(E_2)$; see Theorem 5.1.5(a). Moreover, $\mathcal{M}_i = \mathcal{M}|_{E_i}$. Now Lemma 5.3.12 implies that $E_i \in \mathcal{Z}(\mathcal{M})$ and Theorem 5.2.2 shows that $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{M}|_{E_1}) \oplus \mathcal{Z}(\mathcal{M}|_{E_2})$. Lemma 5.3.4 tells us that $\mathcal{Z}(\mathcal{M}|_{E_i}) = \{Z \in \mathcal{Z}(\mathcal{M}) \mid Z \leq E_i\}$, and all of this gives us cyclic flats E_1, E_2 satisfying the conditions in (5.12). □

Since the collection of cyclic flats is in general quite small, the just presented criterion for irreducibility is in fact very convenient. For instance, simple inspection shows that the q-matroid in Example 2.2.31 is irreducible.

Example 5.3.14. *Let us consider the q-matroids $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, and $\mathcal{N}$ from Example 5.2.5.*
(1) We start with $\mathcal{M}_2 = \mathcal{M}_{G_2}$, which has ground space $\mathbb{F}^4$. Its cyclic flats are

$$0, \langle e_1 + e_3,\, e_2 + e_3 + e_4\rangle,\ \langle e_1 + e_3 + e_4,\, e_2\rangle,\ \langle e_3,\, e_4\rangle,\ \mathbb{F}^4$$

with rank values $0, 1, 1, 1, 2$. Hence $\mathcal{M}_2$ is full and in fact irreducible.
(2) Consider now $\mathcal{M}_1 = \mathcal{M}_{G_1}$, which has ground space $\mathbb{F}^3$. Its cyclic flats are

$$0,\ \langle e_1 + e_3,\, e_2 + e_3\rangle.$$

Hence $\mathcal{M}_1$ is not full and since $\mathrm{cyc}(\mathbb{F}^3) = \langle e_1 + e_3,\, e_2 + e_3\rangle$, we can split off a free q-matroid over a 1-dimensional ground space; see Theorem 5.3.9. The other summand is $\mathcal{M}_1|_{\mathrm{cyc}(\mathbb{F}^3)}$. Straightforward verification shows that this q-matroid is the uniform q-matroid of rank 1 on $\mathrm{cyc}(\mathbb{F}^3)$. Since $\mathcal{U}_1(\mathbb{F}^2)$ is the only full q-matroid on $\mathbb{F}^2$ of rank 1, we obtain $\mathcal{M}_1 \approx \mathcal{U}_1(\mathbb{F}) \oplus \mathcal{U}_1(\mathbb{F}^2)$.
(3) As a consequence, the q-matroid $\mathcal{M}$ has irreducible decomposition $\mathcal{M} \approx \mathcal{U}_1(\mathbb{F}) \oplus \mathcal{U}_1(\mathbb{F}^2) \oplus \mathcal{M}_2$.
(4) As for the q-matroid $\mathcal{N} = \mathcal{M}_G$, we have $\mathrm{cl}(0) = 0$ and $\dim \mathrm{cyc}(\mathbb{F}^7) = 6$, thus $\mathrm{cyc}(\mathbb{F}^7) = \mathrm{rowsp}(A)$ for some $A \in \mathbb{F}^{6\times 7}$. The q-matroid $\mathcal{N}|_{\mathrm{cyc}(\mathbb{F}^7)}$ is equivalent to the q-matroid $\mathcal{N}'$ represented by the matrix $GA^{\mathsf{T}} \in \mathbb{F}_{2^3}^{4\times 6}$, which has rank 3. By construction $\mathcal{N}'$ is full (see Theorem 5.3.9) and $\mathcal{N} \approx \mathcal{U}_1(\mathbb{F}) \oplus \mathcal{N}'$. Since $\mathcal{U}_1(\mathbb{F})$ has exactly one cyclic flat, $\mathcal{N}'$ has 40 cyclic flats just like $\mathcal{N}$; see Theorem 5.2.2. Apart from $\mathrm{cl}(0) = 0$ and $\mathrm{cyc}(\mathbb{F}^6) = \mathbb{F}^6$ with rank 3, the cyclic flats of $\mathcal{N}'$ are as follows:

7 cyclic flats of dimension 2 and rank 1,
24 cyclic flats of dimension 3 and rank 2,
7 cyclic flats of dimension 4 and rank 2.

In order to check $\mathcal{N}'$ for irreducibility it thus suffices to test whether any pair (Z_1, Z_2) of cyclic flats with $\dim Z_1 = 2$ and $\dim Z_2 = 4$ satisfies (5.12). It turns out that there are 28 pairs (Z_1, Z_2) such that $Z_1 \oplus Z_2 = \mathbb{F}^6$. By Lemma 5.3.4 the collection $\mathcal{Z}(\mathcal{N}'|_{Z_i})$ consists exactly of the cyclic flats contained in Z_i. Now one easily finds that for each pair (Z_1, Z_2) one has $|\mathcal{Z}(\mathcal{N}'|_{Z_1})| = 2$ and $|\mathcal{Z}(\mathcal{N}'|_{Z_2})| = 5$, and thus the third condition of (5.12) is not satisfied. All of this shows that $\mathcal{N} \approx \mathcal{U}_1(\mathbb{F}) \oplus \mathcal{N}'$ is a decomposition of $\mathcal{N}$ into irreducible q-matroids.

We continue with our second main result, the decomposition of q-matroids into irreducible summands.

Theorem 5.3.15. *Any q-matroid $\mathcal{M} = (E, \rho)$ is a direct sum of irreducible q-matroids, whose summands are unique up to equivalence.*

Proof. It is clear that every q-matroid is the direct sum of irreducible q-matroids. For the uniqueness, note first that thanks to Theorem 5.3.9 we may disregard trivial and free summands. Thus we may assume that $\mathcal{M}$ is full and, using once more Remark 5.3.1, that

$$\mathcal{M} = \bigoplus_{i=1}^{t} \mathcal{M}_i = \bigoplus_{i=1}^{s} \mathcal{N}_i, \tag{5.13}$$

where each summand $\mathcal{M}_i$ and $\mathcal{N}_i$ is irreducible (and full by Lemma 5.3.12). We will show that $s = t$ and, after suitable ordering, $\mathcal{N}_i = \mathcal{M}_i$ for all $i = 1, \ldots, t$. Thanks to Lemma 5.3.12 we have

$$\mathcal{M}_i = (Z_i, \rho_i),\ \mathcal{N}_i = (Z'_i, \rho'_i), \ \text{ where } \ E = \bigoplus_{i=1}^{t} Z_i = \bigoplus_{i=1}^{s} Z'_i \ \text{ and } \ Z_i,\ Z'_i \in \mathcal{Z}(\mathcal{M}).$$

Thus $\mathcal{M}_i = \mathcal{M}|_{Z_i}$ and $\mathcal{N}_i = \mathcal{M}|_{Z'_i}$ for all i. Moreover, $\mathcal{Z}(\mathcal{M}) = \bigoplus_{i=1}^{t} \mathcal{Z}(\mathcal{M}_i) = \bigoplus_{i=1}^{s} \mathcal{Z}(\mathcal{N}_i)$.
Consider now Z'_1. Then

$$Z'_1 = \bigoplus_{i=1}^{t} V_i \ \text{ for some } V_i \in \mathcal{Z}(\mathcal{M}_i). \tag{5.14}$$

Hence $V_i \leq Z_i$ for all i and by Theorem 5.1.5(a)

$$\rho(Z'_1) = \sum_{i=1}^{t} \rho_i(V_i) = \sum_{i=1}^{t} \rho(V_i). \tag{5.15}$$

We show next that

$$\mathcal{Z}(\mathcal{N}_1) = \bigoplus_{i=1}^{t} \mathcal{Z}(V_i), \text{ where } \ \mathcal{Z}(V_i) = \{Z \in \mathcal{Z}(\mathcal{M}_i) \mid Z \leq V_i\}. \tag{5.16}$$

For "$\supseteq$" choose $W_i \in \mathcal{Z}(V_i)$. Then $W := \bigoplus_{i=1}^{t} W_i \leq \bigoplus_{i=1}^{t} V_i = Z'_1$. Furthermore, W is in $\mathcal{Z}(\mathcal{M})$, and thus $W \in \{Z \in \mathcal{Z}(\mathcal{M}) \mid Z \leq V_i\} \subseteq \mathcal{Z}(\mathcal{N}_1)$, where the last containment is a consequence of Lemma 5.3.4.
For "$\subseteq$" let $W \in \mathcal{Z}(\mathcal{N}_1)$. Then $W \in \mathcal{Z}(\mathcal{M})$ and thus $W = \bigoplus_{i=1}^{t} W_i$ for some $W_i \in \mathcal{Z}(\mathcal{M}_i)$, which means in particular that $W_i \leq Z_i$. But since $W \leq Z'_1 = \bigoplus_{i=1}^{t} V_i$ and $V_i \leq Z_i$, we conclude $W_i \leq V_i$ for all i. This establishes (5.16).
Now (5.14)–(5.16) with Theorem 5.3.13 tell us that $\mathcal{N}_1$ is reducible unless exactly one subspace V_i is nonzero. In other words, irreducibility of $\mathcal{N}_1$ implies that, without loss of generality, $V_2 = \ldots = V_t = 0$ and thus $Z'_1 = V_1 \leq Z_1$.
With the same argument we obtain $Z_1 \leq Z'_j$ for some $j \in \{1, \ldots, s\}$. Hence $Z'_1 \leq Z'_j$ and the directness of the sum $\bigoplus_{i=1}^{s} Z'_i$ implies $j = 1$ and $Z'_1 = Z_1$. This shows $\mathcal{N}_1 = \mathcal{M}|_{Z_1} = \mathcal{M}_1$.
Continuing in this way, we see that every summand $\mathcal{N}_j$ appears as a summand $\mathcal{M}_l$, and thus $s \leq t$. By symmetry $s = t$ and, after reindexing, $\mathcal{M}_i = \mathcal{N}_i$ for all i. □

Now we can easily classify all q-matroids on a 3-dimensional ground space.

Example 5.3.16. *The only irreducible q-matroids on a 3-dimensional ground space are (up to equivalence) are $\mathcal{U}_{1,3}$ and $\mathcal{U}_{2,3}$. Indeed, since any irreducible q-matroid on a 3-dimensional ground space is full, all 1-dimensional spaces must have rank* 1 *and all* 2*-dimensional spaces must have the same rank as the entire space. With the aid of Theorem 5.3.9 and the decomposition of trivial and free q-matroids in Example 5.3.3(a) we obtain now all q-matroids (up to equivalence) on a 3-dimensional ground space. They are as follows. Note that this is independent of the field size q.*

Table 5.2: Classification of q-matroids over a 3-dimensional ground space.

rank =0	$\mathcal{U}_{0,1} \oplus \mathcal{U}_{0,1} \oplus \mathcal{U}_{0,1}$
rank =1	$\mathcal{U}_{0,1} \oplus \mathcal{U}_{0,1} \oplus \mathcal{U}_{1,1}$, $\mathcal{U}_{0,1} \oplus \mathcal{U}_{1,2}$, $\mathcal{U}_{1,3}$
rank =2	$\mathcal{U}_{0,1} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{1,1}$, $\mathcal{U}_{1,1} \oplus \mathcal{U}_{1,2}$, $\mathcal{U}_{2,3}$
rank =3	$\mathcal{U}_{1,1} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{1,1}$

This classification has been derived earlier by different methods in [13, Appendix A.4].

On a 4-dimensional ground space the description of the irreducible q-matroids is much harder. With the same arguments as above we see that the uniform q-matroid $\mathcal{U}_{1,4}$ is the only irreducible q-matroid of rank 1, and thus by duality $\mathcal{U}_{3,4}$ is the only one of rank 3.

In order to discuss q-matroids of rank 2 we need the notion of a partial k-spread. Recall that a subset $\mathcal{V}$ of $\mathcal{L}(E)$ consisting of k-dimensional subspaces is called a *partial k-spread* if $V \cap W = 0$ for all $V \neq W$ in $\mathcal{V}$. It is called a *k-spread* if $|\mathcal{V}| = (q^n - 1)/(q^k - 1)$, where $\dim E = n$. This is the maximum possible size of a partial k-spread and achievable if and only if k divides n. Hence if $k \mid n$, a partial k-spread of size s exists for all $1 \leq s \leq (q^n - 1)/(q^k - 1)$.

Proposition 5.3.17. *Let* $\dim E = 4$.

(a) Let $t \in \{2, \ldots, q^2 + 3\} \setminus \{4\}$ and $\mathcal{V}$ be a partial 2-spread of size $t - 2$. On $\mathcal{L}(E)$ define

$$\rho_{\mathcal{V}}(V) = \begin{cases} 1, & \text{if } V \in \mathcal{V}, \\ \min\{2, \dim V\}, & \text{otherwise.} \end{cases} \tag{5.17}$$

Then $\mathcal{M}_{\mathcal{V}} = (E, \rho_{\mathcal{V}})$ is an irreducible q-matroid with $\mathcal{Z}(\mathcal{M}_{\mathcal{V}}) = \mathcal{V} \cup \{0, E\}$, thus $|\mathcal{Z}(\mathcal{M})| = t$. Moreover, $Z_1 \wedge Z_2 = Z_1 \cap Z_2$ and $Z_1 \vee Z_2 = Z_1 + Z_2$ for all $Z_i \in \mathcal{Z}(\mathcal{M})$.

(b) Each irreducible q-matroid of rank 2 *on E is of the form $\mathcal{M}_{\mathcal{V}} = (E, \rho_{\mathcal{V}})$ for some partial 2-spread $\mathcal{V}$. Its number of cyclic flats is $|\mathcal{V}| + 2$, which is at most $q^2 + 3$.*

(c) $\mathcal{U}_2(E)$ is the unique irreducible q-matroid of rank 2 *with exactly 2 cyclic flats.*

(d) Let $t \geq 3$ and $\mathcal{V}, \mathcal{W}$ be partial 2-spreads of size $t - 2$ and $\mathcal{M}_{\mathcal{V}}, \mathcal{M}_{\mathcal{W}}$ be the associated q-matroids. Then there exists a bijection α on $\mathcal{L}(E)$ such that $\rho_{\mathcal{V}}(V) = \rho_{\mathcal{W}}(\alpha(V))$ for all $V \in \mathcal{L}(E)$. In particular, there exists a rank-preserving and dimension-preserving lattice isomorphism between $\mathcal{Z}(\mathcal{M}_{\mathcal{V}})$ and $\mathcal{Z}(\mathcal{M}_{\mathcal{W}})$. For this reason we may call $\mathcal{M}_{\mathcal{V}}$ and $\mathcal{M}_{\mathcal{W}}$ "bijectively equivalent".

Proof. (a) First of all, Proposition 2.4.6 tells us that $\mathcal{M}_{\mathcal{V}} := (E, \rho_{\mathcal{V}})$ is indeed a q-matroid. One easily checks that $\mathcal{Z}(\mathcal{M}_{\mathcal{V}}) = \mathcal{V} \cup \{0, E\}$. In particular, $|\mathcal{Z}(\mathcal{M}_{\mathcal{V}})| = t$. Since all spaces in $\mathcal{V}$ have dimension 2, it is obvious that $\mathrm{cyc}(Z_1 \cap Z_2) = Z_1 \cap Z_2$ and $\mathrm{cl}(Z_1 + Z_2) = Z_1 + Z_2$ for all $Z_1, Z_2 \in \mathcal{Z}(\mathcal{M})$. For any $V \in \mathcal{V}$ we have $\{Z \in \mathcal{Z}(\mathcal{M}_{\mathcal{V}}) \mid Z \leq V\} = \{0, V\}$, and therefore Theorem 5.3.13 shows that $\mathcal{M}_{\mathcal{V}}$ is irreducible if and only if $|\mathcal{V}| \neq 2$, which means $t \neq 4$; for $\mathcal{V} = \{V_1, V_2\}$ we obtain the reducible q-matroid $\mathcal{M}_{\mathcal{V}} = \mathcal{M}|_{V_1} \oplus \mathcal{M}|_{V_2} \approx \mathcal{U}_{1,2} \oplus \mathcal{U}_{1,2}$.
(b) Let $\mathcal{M} = (E, \rho)$ be irreducible and of rank 2. Set $\mathcal{V} = \{V \in \mathcal{L}(E) \mid \dim V = 2,\ \rho(V) = 1\}$. Using $\mathrm{cl}(0) = 0$ and submodularity of ρ one obtains that $\mathcal{V}$ is a partial 2-spread. Since $\rho(E) = 2$, the set $\mathcal{V}$ is exactly the collection of subspaces V for which $\rho(V) \neq \min\{k, \dim V\}$, and we conclude that $\mathcal{M} = \mathcal{M}_{\mathcal{V}}$. The rest follows from (a) and the fact that any partial 2-spread in E has size at most $q^2 + 1$.
(c) An irreducible q-matroid $\mathcal{M}$ with exactly 2 cyclic flats must have $\mathcal{Z}(\mathcal{M}) = \{0, E\}$, and Example 5.3.3(c) establishes the result.
(d) Choose any dimension-preserving bijection α on $\mathcal{L}(E)$ such that $\alpha(\mathcal{V}) = \mathcal{W}$. Then α is also rank-preserving thanks to (5.17). □

We have the following interesting example, where the bijection in Proposition 5.3.17 (d) is not induced by a linear (or semi-linear) isomorphism on E. Thus, we obtain non-equivalent q-matroids whose lattices of flats are related by a rank-preserving lattice isomorphism.

Example 5.3.18. *This example is inspired by some facts from finite geometry. Consider $\mathbb{F} = \mathbb{F}_3$. The two sets $\mathcal{A}_1$ and $\mathcal{A}_2$ defined below are spread sets in $\mathbb{F}^2$, that is, subsets of $\mathbb{F}^{2\times 2}$ of order 9 such that $A - B \in \mathrm{GL}_2(\mathbb{F})$ for all distinct A, B in $\mathcal{A}_i$. To any spread set one can associate a right quasifield. This is $\mathbb{F}_9$ for $\mathcal{A}_1$ and a right quasifield of Hall type for $\mathcal{A}_2$. Since these two right quasifields are not isotopic we will obtain two 2-spreads of the same size that are not related by a vector-space isomorphism. This background is not needed since everything below can also be checked straightforwardly.*
In $\mathbb{F}^{2\times 2}$ let

$$\mathcal{A}_1 = \Big\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \Big\},$$

$$\mathcal{A}_2 = \Big\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Big\}.$$

Note that $\mathcal{A}_1$ is a subspace of $\mathbb{F}^{2\times 2}$, whereas $\mathcal{A}_2$ is not. Define the set of matrices

$$\mathcal{B}_i = \big\{ (0 \mid I) \big\} \cup \big\{ (I \mid A) \mid A \in \mathcal{A}_i \big\} \subseteq \mathbb{F}^{2\times 4} \quad \textit{for} \quad i = 1, 2.$$

Then

$$\mathcal{V}_i = \{\mathrm{rowsp}(M) \mid M \in \mathcal{B}_i\},\ i = 1, 2,$$

are collections of 2-dimensional subspaces in $\mathcal{L}(\mathbb{F}^4)$*. One easily verifies that* $V \cap W = 0$ *for all distinct* $V, W \in \mathcal{V}_i$ *(or argues that* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are spread sets). Hence* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are* 2*-spreads in* $\mathbb{F}^4$ *because their cardinality is* $10 = q^2 + 1$*. By Proposition 5.3.17, we obtain two irreducible q-matroids* $\mathcal{M}_{\mathcal{V}_i} =: \mathcal{M}_i = (\mathbb{F}^4, \rho_i)$*. We have the following properties.*
(1) $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are not equivalent in the sense of Definition 2.1.9. Indeed, there exists no vector space automorphism* α *on* $\mathbb{F}^4$ *that maps the set* $\mathcal{V}_1$ *to* $\mathcal{V}_2$*. This follows from the fact that the field* $\mathbb{F}_9$ *is not isotopic to any Hall quasifield or can be checked directly using a computer algebra system by showing that no matrix in* $\mathrm{GL}_4(\mathbb{F})$ *maps the set* $\mathcal{B}_1$ *to the set* $\mathcal{B}_2$ *(after taking the reduced row echelon forms). Now it is clear that the q-matroids* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are also not lattice-equivalent either because by the Fundamental Theorem of Projective Geometry any lattice isomorphism is given by a semi-linear isomorphism, which over a prime field is linear.*
(2) $\mathcal{F}(\mathcal{M}_i) = \mathcal{O}(\mathcal{M}_i) = \mathcal{Z}(\mathcal{M}_i) = \mathcal{V}_i \cup \{0, \mathbb{F}^4\}$*, and in each of these lattices meet and join are simply intersection and sum. Moreover, there exits a rank-preserving lattice isomorphism between* $\mathcal{F}(\mathcal{M}_1)$ *and* $\mathcal{F}(\mathcal{M}_2)$*. The identities follow from the fact that* $\mathcal{V}_i$ *is a* 2*-spread (as opposed to just a partial spread), which means that every* 1*-dimensional space* $\langle x \rangle$ *is contained in a subspace of* $\mathcal{V}_i$ *and therefore* $\langle x \rangle$ *is not a flat. The rest of the identities is easily verified, and the remaining statements follow from Proposition 5.3.17.*
(3) It is remarkable that $\mathcal{M}_1$ *is representable over* $\mathbb{F}_9$*, whereas* $\mathcal{M}_2$ *is not representable over any field extension of* $\mathbb{F}$*. Indeed, one easily verifies that* $\mathcal{M}_1 = \mathcal{M}_G$*, where*

$$G = \begin{pmatrix} 1 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & \omega^2 \end{pmatrix} \in \mathbb{F}_9^{2\times 4},$$

where $\omega \in \mathbb{F}_9$ *satisfies* $\omega^2 + 2\omega + 2 = 0$*. To see that* $\mathcal{M}_2$ *is not representable, suppose that* $\mathcal{M}_2 = \mathcal{M}_{G'}$ *for some* $G' \in \mathbb{F}_{3^m}^{2\times 4}$ *(for some m). Then* $\mathrm{rk}G' = 2$ *and we may assume* G' *to be in reduced row echelon form. Next, any such* G' *must satisfy* $\mathrm{rk}\big(G'(0 \mid I)^{\mathsf{T}}\big) = \mathrm{rk}\big(G'(I \mid 0)^{\mathsf{T}}\big) = \mathrm{rk}\big(G'(I \mid I)^{\mathsf{T}}\big) = 1$ *(see the set* $\mathcal{A}_2$ *and Lemma 2.3.9), and therefore must be of the form*

$$G' = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \text{ for some } a \in \mathbb{F}_{3^m}.$$

Using now $\mathrm{rk}\big(G'(I \mid A)^{\mathsf{T}}\big) = 1$ *for A being the fourth and the sixth matrix in the set* $\mathcal{A}_2$ *leads to* $a = 1$*. But then the q-matroid* $\mathcal{M}_{G'}$ *has loops, and thus* $\mathcal{M}_{G'} \neq \mathcal{M}_2$*. Thus* $\mathcal{M}_2$ *is not representable.*

The above discussion suggests that a classification of the irreducible q-matroids on a 4-dimensional ground space appears to be quite challenging.

5.4 The representability of the direct sum.

In this section we turn to representability of the direct sum of representable q-matroids. We will provide an example of $\mathbb{F}_4$-representable q-matroids over the ground space $\mathbb{F}_2^4$ for which the direct sum is not representable over any field extension $\mathbb{F}_{2^m}$. A crucial ingredient will be paving q-matroids defined below. We show that if $\mathcal{M}_1$, $\mathcal{M}_2$ are paving q-matroids of the same rank and represented by matrices G_i satisfying a certain condition, then the direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ is not represented by the block diagonal matrix $\text{diag}(G_1, G_2)$. This will be used to create the desired example. A second example will be provided where the direct sum is representable only over larger fields than the summands.

Note that in the context of representability, it suffices to consider q-matroids with ground space $\mathbb{F}^n$.

Definition 5.4.1. *A q-matroid $\mathcal{M} = (\mathbb{F}^n, \rho)$ is* paving *if $\dim C \geq \rho(\mathcal{M})$ for all circuits C of $\mathcal{M}$.*

A large class of paving q-matroids is given by Proposition 2.4.6. For the following result recall Notation 2.2.13.

Proposition 5.4.2. *Let $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i), i = 1, 2,$ be paving q-matroids with $\rho_1(\mathcal{M}_1) = \rho_2(\mathcal{M}_2) = k$. Let $n = n_1 + n_2$ and $\mathcal{M} = (\mathbb{F}^n, \rho) = \mathcal{M}_1 \oplus \mathcal{M}_2$. Denote by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}$ the collections of circuits of $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}$, respectively. Then every $C \in \mathcal{C}$ has dimension at least k and*

$$\{C \in \mathcal{C} \mid \dim C = k\} = \{C_1 \oplus 0 \mid C_1 \in \mathcal{C}_1, \dim C_1 = k\} \cup \{0 \oplus C_2 \mid C_2 \in \mathcal{C}_2, \dim C_2 = k\}.$$

For the proof recall that in any q-matroid (E, ρ) a k-dimensional circuit C satisfies $\rho(C) = k - 1$.

Proof. We start with the stated identity. Denote the set on the right hand side by $\mathcal{V}$.
"$\supseteq$" Let $C_1 \oplus 0 \in \mathcal{V}$. From Theorem 5.1.5 we obtain $\rho(C_1 \oplus 0) = \rho_1(C_1) = k - 1$, and thus $C_1 \oplus 0$ is dependent in $\mathcal{M}$. Clearly, every subspace of $C_1 \oplus 0$ is of the form $I_1 \oplus 0$ where I_1 is independent in $\mathcal{M}_1$. Using again Theorem 5.1.5 we conclude that $I_1 \oplus 0$ is independent in $\mathcal{M}$ and thus $C_1 \oplus 0$ is a circuit of $\mathcal{M}$ of dimension k. The same reasoning holds for $0 \oplus C_2 \in \mathcal{V}$.
"$\subseteq$" By Theorem 5.1.2 a subspace $C \in \mathcal{L}(\mathbb{F}^n)$ is a circuit of $\mathcal{M}$ if and only if it is inclusion-minimal subject to

$$\rho_1(\pi_1(C)) + \rho_2(\pi_2(C)) \leq \dim C - 1, \tag{5.18}$$

where π_1, π_2 are the projections from $\mathbb{F}^n$ to the first n_1 and last n_2 coordinates, respectively. Let $C \in \mathcal{C}$ and dim $C \leq k$.
1) Let $\dim \pi_1(C) = k$ (which implies $\dim C = k$). Then (5.18) implies that $\pi_1(C)$ is a dependent space of $\mathcal{M}_1$, and thus a circuit thanks to the paving property. Hence $\rho_1(\pi_1(C)) = k - 1$ and thus $\rho_2(\pi_2(C)) = 0$ by (5.18). But then $\pi_2(C) = 0$ by the paving property of $\mathcal{M}_2$, and thus $C = \pi_1(C) \oplus 0$. In the same way we have

$C = 0 \oplus \pi_2(C)$ if $\dim \pi_2(C) = k$.
2) Let $\dim \pi_i(C) = \ell_i < k$ for $i = 1, 2$. Then $\pi_i(C)$ is independent in $\mathcal{M}_i$ and $\rho_i(\pi_i(C)) = \ell_i$. Now we obtain $\dim C \leq \dim \pi_1(C) + \dim \pi_2(C) = \ell_1 + \ell_2 = \rho_1(\pi_1(C)) + \rho_2(\pi_2(C))$ in contradiction to (5.18). Hence this case does not arise. This also shows that all circuits have dimension at least k, and the proof is complete. □

We now turn to representability of the direct sum and start with the following unsurprising result.

Theorem 5.4.3. *Let $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i), i = 1, 2$, be q-matroids of rank k_i. Let $n = n_1 + n_2$ and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Suppose $\mathcal{M}$ is representable over $\mathbb{F}_{q^m}$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are representable over $\mathbb{F}_{q^m}$ and $\mathcal{M} = \mathcal{M}_G$ for a matrix G of the form*

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix},$$

where $G_i \in \mathbb{F}_{q^m}^{k_i \times n_i}$ are such that $\mathcal{M}_{G_i} = \mathcal{M}_i$.

Proof. Let $\mathcal{M} = (\mathbb{F}^n, \rho)$ and suppose $\mathcal{M} = \mathcal{M}_G$ for some matrix G over $\mathbb{F}_{q^m}$. Since $\rho(\mathcal{M}) = k := k_1 + k_2$ by Theorem 5.1.5, we may assume that G is in $\mathbb{F}_{q^m}^{k \times n}$ and has rank k. Furthermore, without loss of generality let G be in RREF. Then we can write G as

$$G = \begin{pmatrix} G_1 & G' \\ 0 & G_2 \end{pmatrix}$$

for some matrices $G_1 \in \mathbb{F}_{q^m}^{t_1 \times n_1}$, $G_2 \in \mathbb{F}_{q^m}^{t_2 \times n_2}$ of full row rank and some matrix $G' \in \mathbb{F}_{q^m}^{t_1 \times n_2}$. Hence $t_1 + t_2 = k_1 + k_2$. We show that $\mathcal{M}_1 = \mathcal{M}_{G_1}$. To do so, let $Y \in \mathbb{F}^{\bullet \times n_1}$ and set $\hat{Y} = (Y \mid 0)$, which is in $\mathbb{F}^{\bullet \times n}$. Then $V := \text{rowsp}(Y)$ is in $\mathcal{L}(\mathbb{F}^{n_1})$ and $V \oplus 0 = \text{rowsp}(\hat{Y})$. With the aid of Lemma 2.3.9 and Theorem 5.1.5 we compute

$$\text{rk}(G_1 Y^\mathsf{T}) = \text{rk}(G\hat{Y}^\mathsf{T}) = \rho(\text{rowsp}(\hat{Y})) = \rho(V \oplus 0) = \rho_1(V).$$

This shows that $\mathcal{M}_1 = \mathcal{M}_{G_1}$. As a consequence, $t_1 = k_1$ and $t_2 = k_2$. Next, $k_2 = \rho_2(\mathbb{F}^{n_2}) = \rho(0 \oplus \mathbb{F}^{n_2}) = \rho(\text{rowsp}(0 \mid I_{n_2})) = \text{rk}\begin{pmatrix} G' \\ G_2 \end{pmatrix}$. Since $\text{rk} G_2 = k_2$ this implies $\text{rowsp}(G') \subseteq \text{rowsp}(G_2)$. Using that G is in RREF, we conclude $G' = 0$ and hence G is block diagonal. In the same way as above we obtain $\mathcal{M}_2 = \mathcal{M}_{G_2}$. This concludes the proof. □

We will now make use of some notions in the theory of rank-metric code, previously introduced in Section 4.4. The proof of Theorem 5.4.5 below illustrates the well known fact that for a representable q-matroid $\mathcal{M}_G$ the dimension of a dependent space equals the rank-weight of a suitable codeword in the dual code $\text{rowsp}(G)^\perp = \ker G$ (where the dual is defined with respect to the standard inner product). For ease of notation in the proof of Theorem 5.4.5 we introduce a different, yet equivalent way to define the notion of rank support, previously introduced in Definition 4.4.3.

Definition 5.4.4. *For a vector $v = (v_1, \dots, v_n) \in \mathbb{F}_{q^m}^n$ we define the* $\mathbb{F}$-support *of v as the subspace*

$$S_{\mathrm{rk}}(v) = \langle v_1, \dots, v_n \rangle_{\mathbb{F}} \leq \mathbb{F}_{q^m}.$$

Furthermore, recall the rank-weight *of v is $\omega_{\mathrm{rk}}(v) := \dim_{\mathbb{F}_q}(S_{\mathrm{rk}}(v))$.*

Theorem 5.4.5. *For $i = 1, 2$ let $G_i \in \mathbb{F}_{q^m}^{k \times n_i}$ be of rank k and $\mathcal{M}_i = \mathcal{M}_{G_i}$ be the associated q-matroids. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are both paving. Suppose furthermore that there exist vectors $v_i \in \ker G_i$ such that $\omega_{\mathrm{rk}}(v_1) = \omega_{\mathrm{rk}}(v_2) = k$ and $S_{\mathrm{rk}}(v_1) = S_{\mathrm{rk}}(v_2)$. Then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is not represented by*

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}. \tag{5.19}$$

Proof. Let $\mathcal{N} = \mathcal{M}_G$, that is, $\mathcal{N}$ is the q-matroid generated by G. We will show that $\mathcal{N}$ and $\mathcal{M}$ do not have the same circuits of dimension at most k.
First of all, Proposition 5.4.2 implies that all circuits of $\mathcal{M}$ have dimension at least k, and the k-dimensional ones are also circuits of $\mathcal{N}$ thanks to their form described in that proposition.
We will show the existence of a circuit of $\mathcal{N}$ of dimension at most k that is not a circuit of $\mathcal{M}$. To do so, let $S_{\mathrm{rk}}(v_1) = \langle \alpha_1, \dots, \alpha_k \rangle$ for some ($\mathbb{F}$-linearly independent) $\alpha_i \in \mathbb{F}_{q^m}$. Set $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{F}_{q^m}^k$. Then there exist matrices $Y_i \in \mathbb{F}^{k \times n_i}$ of rank k such that $\alpha Y_i = v_i$ for $i = 1, 2$. Hence $0 = G_i {Y_i}^{\mathsf{T}} \alpha^{\mathsf{T}}$, and thus $\mathrm{rk}(G_i {Y_i}^{\mathsf{T}}) < k$ for $i = 1, 2$. This shows that $V_i := \mathrm{rowsp}(Y_i) \in \mathcal{L}(\mathbb{F}^{n_i})$ is a dependent space of $\mathcal{M}_i$. Since $\dim V_i = k$, the paving property implies that V_i is a circuit of $\mathcal{M}_i$. Now we have

$$G \begin{pmatrix} {Y_1}^{\mathsf{T}} \\ {Y_2}^{\mathsf{T}} \end{pmatrix} \alpha^{\mathsf{T}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means that $G(Y_1 \mid Y_2)^{\mathsf{T}}$ has rank less than k. Therefore $W := \mathrm{rowsp}(Y_1 \mid Y_2) \in \mathcal{L}(\mathbb{F}^{n_1+n_2})$ is a k-dimensional dependent space of $\mathcal{N}$. Since Y_1 and Y_2 are both nonzero, W is not a circuit of the direct sum $\mathcal{M}$ thanks to Proposition 5.4.2. Since $\mathcal{M}$ does not have any circuits of dimension less than k, we conclude that W contains a circuit of $\mathcal{N}$ of dimension at most k that is not a circuit of $\mathcal{M}$. This implies that $\mathcal{N} \neq \mathcal{M}$, and G does not represent $\mathcal{M}$. □

Now we are ready to provide an example of a direct sum of representable q-matroids that is not representable over any field extension.

Proposition 5.4.6. *Let $\mathbb{F} = \mathbb{F}_2$ and $\mathbb{F}_4 = \{0, 1, \omega, \omega + 1\}$. Consider the matrix*

$$G_1 = \begin{pmatrix} 1 & \omega & 0 & \omega + 1 \\ 0 & 0 & 1 & \omega \end{pmatrix} \in \mathbb{F}_4^{2 \times 4}$$

and set $\mathcal{M}_1 := \mathcal{M}_{G_1} = (\mathbb{F}^4, \rho_1)$. Then $\mathcal{M}_1 \oplus \mathcal{M}_1$ is not representable over any field extension $\mathbb{F}_{2^m}$.

Proof. 1) We first show that $\mathcal{M}_1$ is of the form as in Proposition 2.4.6 and thus paving of rank 2. Set $\mathcal{V} = \{\text{rowsp}(Y_1), \text{rowsp}(Y_2), \text{rowsp}(Y_3), \text{rowsp}(Y_4), \text{rowsp}(Y_5)\}$, where

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, Y_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$
$$Y_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Y_5 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.20}$$

One can verify, by hand or with for instance SageMath, that

$$\rho_1(V) = \begin{cases} 1, & \text{for } V \in \mathcal{V}, \\ \min\{2, \dim V\}, & \text{otherwise.} \end{cases} \tag{5.21}$$

For instance, $\rho_1(V) = 1$ for all 1-dimensional subspaces simply reflects that the columns of G_1 are linearly independent over $\mathbb{F}_2$. Furthermore, $G_1 {Y_2}^{\mathsf{T}} = \left(\begin{smallmatrix} \omega & 1 \\ \omega+1 & \omega \end{smallmatrix}\right)$, which has rank 1. Similarly, the rank of all subspaces can easily be verified. As a consequence, $\mathcal{M}_1$ is of the form as in Proposition 2.4.6 and thus paving of rank 2. Clearly, the matrix

$$\hat{G}_1 = \begin{pmatrix} 1 & \omega+1 & 0 & \omega \\ 0 & 0 & 1 & \omega+1 \end{pmatrix},$$

obtained from G by replacing the primitive element ω by its conjugate $\omega + 1$, also represents $\mathcal{M}_1$.

2) We show that G_1 and $\hat{G}_1$ are the only matrices over any field extension $\mathbb{F}_{2^m}$, $m \geq 1$, that represent $\mathcal{M}_1$. To do so, let $m \geq 1$ and $H = (h_1, h_2, h_3, h_4) \in \mathbb{F}_{2^m}^{2\times 4}$ be such that $\mathcal{M}_H = \mathcal{M}_1$. Thus $\text{rk}(HY^{\mathsf{T}}) = \text{rk}(G_1 Y^{\mathsf{T}})$ for all matrices $Y \in \mathbb{F}^{\bullet \times 4}$. Without loss of generality we may assume that H is in RREF. Clearly $\text{rk}(H) = 2$. Moreover, $h_1, \ldots, h_4$ are linearly independent over $\mathbb{F}_2$ because $\mathcal{M}_1$ has no loops (a loop is a 1-dimensional space of rank 0). Next, $\text{rk}(H{Y_1}^{\mathsf{T}}) = 1 = \text{rk}(H{Y_4}^{\mathsf{T}})$ shows that $h_2 \in \langle h_1 \rangle_{\mathbb{F}_{2^m}}$ and $h_4 \in \langle h_2 + h_3 \rangle_{\mathbb{F}_{2^m}}$. Hence h_2 and h_4 are not pivot columns of H. All of this implies that H must be of the form

$$H = \begin{pmatrix} 1 & \alpha & 0 & \alpha\beta \\ 0 & 0 & 1 & \beta \end{pmatrix} \text{ for some } \alpha, \beta \in \mathbb{F}_{2^m}.$$

The $\mathbb{F}_2$-linear independence of the columns of H implies that $\alpha, \beta \notin \mathbb{F}_2$. Next,

$$1 = \text{rk}(H{Y_3}^{\mathsf{T}}) = \text{rk}\begin{pmatrix} 1+\alpha\beta & \alpha\beta \\ \beta & 1+\beta \end{pmatrix} = \text{rk}\begin{pmatrix} 1 & \alpha\beta \\ 1 & 1+\beta \end{pmatrix},$$

and this results in $\alpha\beta = 1 + \beta$. Using this, we continue with

$$\begin{aligned} 1 = \text{rk}(H{Y_2}^{\mathsf{T}}) &= \text{rk}\begin{pmatrix} 1+\alpha\beta & \alpha+\alpha\beta \\ 1+\beta & \beta \end{pmatrix} \\ &= \text{rk}\begin{pmatrix} \beta & \alpha+1+\beta \\ 1+\beta & \beta \end{pmatrix} = \text{rk}\begin{pmatrix} \beta & \alpha+1 \\ 1+\beta & 1 \end{pmatrix}. \end{aligned}$$

Using the determinant and $\alpha\beta = 1 + \beta$ we conclude that $\alpha + \beta = 0$, hence $\alpha = \beta$, and with $\alpha\beta = 1 + \beta$ we arrive at $\beta^2 + \beta + 1 = 0$. Thus $\beta \in \mathbb{F}_4 \setminus \mathbb{F}_2$ (and m is even), and the two choices $\beta \in \{\omega, \omega + 1\}$ lead to $H \in \{G_1, \hat{G}_1\}$.
3) Consider now $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_1$. From 2) and Theorem 5.4.3 we know that if $\mathcal{M}$ is $\mathbb{F}_{2^m}$-representable for some $m \in \mathbb{N}$, then m is even and any representing matrix is of the form

$$G = \begin{pmatrix} G_1' & 0 \\ 0 & G_2' \end{pmatrix}, \quad \text{where } G_i' \in \{G_1, \hat{G}_1\}.$$

Since $v = (1, 1, \omega, 1) \in \ker G_1$, $\hat{v} = (1, 1, \omega + 1, 1) \in \ker \hat{G}_1$ satisfy $S_{\text{rk}}(v) = S_{\text{rk}}(\hat{v}) = \langle 1, \omega \rangle$ we may apply Theorem 5.4.5 and conclude that $\mathcal{M}$ is not representable over any field extension $\mathbb{F}_{2^m}$. □

In the following case, the direct sum is not representable over the same field as the summands, but over a field extension.

Proposition 5.4.7. *Let $\mathcal{M}_1 = \mathcal{U}_{1,2}(q)$. Then $\mathcal{M}_1$ is representable over $\mathbb{F}_{q^2}$, whereas $\mathcal{M}_1 \oplus \mathcal{M}_1$ is representable over $\mathbb{F}_{q^m}$ iff $m \geq 4$.*

Proof. We know already, by Example 2.4.2, that $\mathcal{M}_1$ is representable over $\mathbb{F}_{q^2}$. Even more, for any $m \geq 2$ any matrix $G = \begin{pmatrix} 1 & \alpha \end{pmatrix}$ with $\alpha \in \mathbb{F}_{q^m} \setminus \mathbb{F}_q$ represents $\mathcal{M}_1$. Consider now $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_1 = (\mathbb{F}^4, \rho)$. With the aid of the very definition of the direct sum one easily verifies that $\rho(V) = 1$ for all 1-dimensional spaces $V \leq \mathbb{F}^4$, while for the 2-dimensional spaces

$$\rho(V) = \begin{cases} 1, & \text{if } V = \langle e_1, e_2 \rangle \text{ or } V = \langle e_3, e_4 \rangle, \\ 2, & \text{otherwise,} \end{cases} \tag{5.22}$$

where $e_1, \ldots, e_4$ are the standard basis vectors of $\mathbb{F}^4$. Suppose $\mathcal{M}$ is representable over $\mathbb{F}_{q^m}$. Theorem 5.4.3 tells us that $\mathcal{M}$ has a representing matrix of the form

$$G = \begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & 0 & 1 & \gamma \end{pmatrix} \quad \text{for some } \beta, \gamma \in \mathbb{F}_{q^m} \setminus \mathbb{F}.$$

1) Let $m = 2$. Then $\mathbb{F}_{q^2} = \langle 1, \beta \rangle$. Thus $\gamma = a + b\beta$ for some $a, b \in \mathbb{F}$. Set

$$V = \text{rowsp}(Y), \text{ where } Y = \begin{pmatrix} a & b & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Then $\dim V = 2$ and $\text{rk}(GY^\mathsf{T}) = 1$. This contradicts (5.22), and thus G does not represent $\mathcal{M}$.
2) Let $m = 3$. Then $\mathbb{F}_{q^3} = \langle 1, \beta, \beta^2 \rangle$ and

$$\gamma = c_0 + c_1\beta + c_2\beta^2, \quad \beta^3 = b_0 + b_1\beta + b_2\beta^2 \text{ for some } b_i,\, c_i \in \mathbb{F}.$$

This implies $\beta\gamma = c_2b_0 + (c_0 + c_2b_1)\beta + (c_1 + c_2b_2)\beta^2$. Now we have the following cases:
i) If $c_2 = 0$, then $\text{rk}(GY^\mathsf{T}) = 1$ for

$$Y = \begin{pmatrix} c_0 & c_1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

ii) If $c_2 \neq 0 = c_1 + c_2 b_2$, then $\mathrm{rk}(GY^\top) = 1$ for

$$Y = \begin{pmatrix} c_2 b_0 & c_0 + c_2 b_1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

iii) If $c_2 \neq 0 \neq c_1 + c_2 b_2$, we have $(c_1 + c_2 b_2)\gamma - c_2 \beta\gamma = f_0 + f_1 \beta$ for some $f_i \in \mathbb{F}$ and thus $\mathrm{rk}(GY^\top) = 1$ for

$$Y = \begin{pmatrix} f_0 & f_1 & 0 & 1 \\ c_1 + c_2 b_2 & -c_2 & 1 & 0 \end{pmatrix}.$$

In all cases we obtain a contradiction to (5.22) and conclude that $\mathcal{M}$ is not representable over $\mathbb{F}_{q^3}$.
3) Let $m \geq 4$. We show that $\mathcal{M}$ is represented by the matrix

$$G = \begin{pmatrix} 1 & z & 0 & 0 \\ 0 & 0 & 1 & z^2 \end{pmatrix} \quad \text{for any } z \in \mathbb{F}_{q^m} \text{ of degree at least 4.}$$

Denote by $\mathcal{D}(\mathcal{M})$ and $\mathcal{D}(\mathcal{M}_G)$ the collections of dependent spaces of $\mathcal{M}$ and $\mathcal{M}_G$, respectively. We will show that

$$\mathcal{D}(\mathcal{M}_G) = \mathcal{D}(\mathcal{M}). \tag{5.23}$$

To do so, we first determine the set $\mathcal{X} = \{X \in \mathcal{L}(\mathbb{F}^4) \mid \rho_1(\pi_1(X)) + \rho_1(\pi_2(X)) < \dim X\}$ from Theorem 5.1.2. Since $\mathcal{M}_1$ is the uniform q-matroid of rank 1, we have $\rho_1(\pi_1(X)) + \rho_1(\pi_2(X)) = \min\{1, \dim \pi_1(X)\} + \min\{1, \dim \pi_2(X)\} \leq 2$ for all $X \in \mathcal{L}(\mathbb{F}^4)$, which together with (5.22) implies

$$\mathcal{X} = \{X \leq \mathbb{F}^4 \mid \dim X \geq 3\} \cup \{\langle e_1, e_2\rangle, \langle e_3, e_4\rangle\}.$$

The form of $\mathcal{X}$ shows that every subspace of $\mathbb{F}^4$ containing a space in $\mathcal{X}$ is itself in $\mathcal{X}$. Now it follows from Theorem 5.1.2 that

$$\mathcal{X} = \mathcal{D}(\mathcal{M}).$$

We now turn to the q-matroid $\mathcal{M}_G = (\mathbb{F}^4, \hat{\rho})$. By definition $\hat{\rho}(\mathrm{rowsp}(Y)) = \mathrm{rk}(GY^\top)$ for any matrix $Y \in \mathbb{F}^{\bullet \times 4}$. Set $G_1' = (1 \ \ z)$ and $G_2' = (1 \ \ z^2)$, thus

$$G = \begin{pmatrix} G_1' & 0 \\ 0 & G_2' \end{pmatrix}.$$

Since both G_1' and G_2' represent the q-matroid $\mathcal{M}_1$, any matrix $Y = (Y_1 \mid Y_2) \in \mathbb{F}^{\bullet \times 4}$ satisfies

$$\mathrm{rk}(G_i' {Y_i}^\top) = \rho_1(\pi_i(\mathrm{rowsp}(Y))).$$

Let now $Y = (Y_1 \mid Y_2) \in \mathbb{F}^{\bullet \times 4}$ be such that $\mathrm{rowsp}(Y) \in \mathcal{D}(\mathcal{M}) = \mathcal{X}$. Then

$$\begin{aligned} \hat{\rho}(\mathrm{rowsp}(Y)) &= \mathrm{rk}(GY^\top) \\ &\leq \mathrm{rk}(G_1' {Y_1}^\top) + \mathrm{rk}(G_2' {Y_2}^\top) \\ &= \rho_1(\pi_1(\mathrm{rowsp}(Y))) + \rho_1(\pi_2(\mathrm{rowsp}(Y))) \\ &< \mathrm{rk} Y. \end{aligned}$$

This shows that $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{D}(\mathcal{M}_G)$. This can also be concluded from the coproduct property of $\mathcal{M} \oplus \mathcal{M}$; see [26, Thm. 5.5]. It remains to show the converse containment. Let now $Y \in \mathbb{F}^{t\times 4}$ be a matrix of rank t such that $\mathrm{rowsp}(Y) \in \mathcal{D}(\mathcal{M}_G)$. Thus

$$\mathrm{rk}(GY^{\mathsf{T}}) < t. \tag{5.24}$$

We will show that $\mathrm{rowsp}(Y)$ is in $\mathcal{X}$. For $t \geq 3$ every Y satisfies (5.24) and $\mathrm{rowsp}(Y)$ is contained in $\mathcal{X}$, while for $t = 1$ no matrix Y satisfies (5.24). It remains to consider the case $t = 2$. Let

$$Y = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$$

be of rank 2 and satisfying (5.24). Without loss of generality let Y be in RREF. We compute

$$GY^{\mathsf{T}} = \begin{pmatrix} a_1 + b_1 z & a_2 + b_2 z \\ c_1 + d_1 z^2 & c_2 + d_2 z^2 \end{pmatrix}$$

and

$$\det(GY^{\mathsf{T}}) = (a_1c_2 - a_2c_1) + (b_1c_2 - b_2c_1)z + (a_1d_2 - a_2d_1)z^2 + (b_1d_2 - b_2d_1)z^3.$$

Recall that $1, z, z^2, z^3$ are linearly independent over $\mathbb{F}_q$ by choice of z. Thanks to the RREF of Y we only have to consider the following cases.
i) If $a_1 = b_1 = a_2 = b_2 = 0$, then $\mathrm{rowsp}(Y) = \langle e_3, e_4\rangle$ is in $\mathcal{X}$.
ii) Let $a_1 = 1, a_2 = 0, b_1 = 0, b_2 = 1$. Then $\det(GY^{\mathsf{T}}) = c_2 - c_1 z + d_2 z^2 - d_1 z^3$ and (5.24) implies $c_1 = c_2 = d_1 = d_2 = 0$. Thus $\mathrm{rowsp}(Y) = \langle e_1, e_2\rangle$, which is in $\mathcal{X}$.
iii) Let $a_1 = 1$ and $a_2 = b_2 = 0$. Then $\det(GY^{\mathsf{T}}) = c_2 + b_1c_2 z + d_2 z^2 + b_1 d_2 z^3$ and (5.24) implies $c_2 = d_2 = 0$. But this contradicts that Y has rank 2 and thus this case does not occur.
iv) Let $a_1 = a_2 = 0 = b_2$ and $b_1 = 1$. Then $\det(GY^{\mathsf{T}}) = c_2 z + d_2 z^3$ and (5.24) implies $c_2 = d_2 = 0$, which contradicts $\mathrm{rk}Y = 2$. Hence this case does not occur either.
All of this establishes (5.23). Thanks to Definition 2.1.1, we arrive at $\mathcal{M} = \mathcal{M}_G$, and this shows the $\mathbb{F}_{q^m}$-representability of $\mathcal{M}$. □

Thus far we provided two examples. In the first one, the direct sum of representable q-matroids is not representable, and in the second one it requires a larger field size for a representation than the summands. In fact, almost all examples that we were able to compute fall into one of the two categories. The only examples known to us where the direct sum is representable over the same field as both summands, are such that one summand is representable over $\mathbb{F}_q$. While we are not able to prove a general statement about these specific direct sums, we can discuss two extreme cases. If $\mathcal{M}_1$ is the trivial q-matroid $\mathcal{U}_{0,n_1}(q)$ it can easily be seen that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is representable over the same field as $\mathcal{M}_2$ via the matrix $G' = (0 \,|\, G)$ where G represents $\mathcal{M}_2$. The other extreme, where $\mathcal{M}_1$ is the free q-matroid, is dealt with in the following proposition.

Proposition 5.4.8. *Let $\mathcal{M}_1 = \mathcal{U}_{n_1,n_1}(q)$, which is representable by I_{n_1}, and let the q-matroid $\mathcal{M}_2 = (\mathbb{F}_q^{n_2}, \rho_2)$ be represented by $G_2 \in \mathbb{F}_{q^m}^{k_2 \times n_2}$. Then $\mathcal{M}_1 \oplus \mathcal{M}_2$ is representable over $\mathbb{F}_{q^m}$ and a representing matrix is given by*

$$G = \begin{pmatrix} I_{n_1} & 0 \\ 0 & G_2 \end{pmatrix}.$$

Proof. Set $n = n_1 + n_2$. We first consider the q-matroid $\mathcal{M}_G$ and determine its rank function, which we denote by ρ_G. Let $V \in \mathcal{L}(\mathbb{F}^n)$ and $V = \text{rowsp}(Y)$ where $Y \in \mathbb{F}^{\ell \times n}$ of rank ℓ. Without loss of generality, we may assume that Y is in reduced row echelon. Then Y partitions as

$$Y = \begin{pmatrix} Y_1 & \hat{Y} \\ 0 & Y_2 \end{pmatrix}, \tag{5.25}$$

for some $Y_i \in \mathbb{F}^{\ell_i \times n_i}$ and $\hat{Y}$ of corresponding size, and where $\text{rk}(Y_i) = \ell_i$ for $i = 1, 2$. This means that $\text{rowsp}(Y_1) = \pi_1(V)$, which has dimension ℓ_1, and $\text{rowsp}(Y_2) = \pi_2(V \cap (0 \oplus \mathbb{F}^{n_2}))$, which has dimension ℓ_2. Now we have

$$GY^\mathsf{T} = \begin{pmatrix} {Y_1}^\mathsf{T} & 0 \\ G_2\hat{Y}^\mathsf{T} & G_2{Y_2}^\mathsf{T} \end{pmatrix},$$

and since ${Y_1}^\mathsf{T}$ has full column rank, we obtain the rank value

$$\rho_G(V) = \text{rk}(GY^\mathsf{T}) = \text{rk}(V_1) + \text{rk}(G_2{Y_2}^\mathsf{T}) = \dim \pi_1(V) + \rho_2(\pi_2(V \cap (0 \oplus \mathbb{F}^{n_2}))). \tag{5.26}$$

We now turn to $\mathcal{M}_1 \oplus \mathcal{M}_2$ and denote its rank function by ρ. We use (5.4) to evaluate ρ, thus

$$\rho(V) = \dim V + \min_{X \le V} \sigma(X), \quad \text{where } \sigma(X) = \dim \pi_1(X) + \rho_2(\pi_2(X)) - \dim X. \tag{5.27}$$

Consider again $V = \text{rowsp}(Y)$ with Y as in (5.25). Let X be a subspace of V of dimension x. Then X is of the form $X = \text{rowsp}(SY)$ for some $S \in \mathbb{F}^{x \times \ell}$ of rank x. Again, we may assume S in reduced row echelon form and can partition the matrix as

$$S = \begin{pmatrix} S_1 & \hat{S} \\ 0 & S_2 \end{pmatrix},$$

where $S_i \in \mathbb{F}^{x_i \times \ell_i}$ of rank x_i and $\hat{S}$ accordingly. Then

$$SY = \begin{pmatrix} S_1Y_1 & S_1\hat{Y} + \hat{S}Y_2 \\ 0 & S_2Y_2 \end{pmatrix}.$$

The two diagonal blocks have full row rank and $\pi_1(X) = \text{rowsp}(S_1Y_1)$ while $\pi_2(X) = \text{rowsp}\begin{pmatrix} S_1\hat{Y}+\hat{S}Y_2 \\ S_2Y_2 \end{pmatrix}$. Now we have

$$\sigma(X) = \text{rk}(S_1Y_1) + \rho_2(\pi_2(X)) - \dim X = \rho_2(\pi_2(X)) - \text{rk}(S_2Y_2),$$

which we want to minimize over all subspaces X of V. Note that this expression does not depend on $\pi_1(X)$. Furthermore, since $\mathrm{rk}(S_2Y_2)$ does not depend on $\hat{S}$ and ρ_2 is non-decreasing, the minimum is attained by a space X of the form $X = \mathrm{rowsp}(0 \mid S_2Y_2)$. Thus we arrive at

$$\sigma(X) = \rho_2(\pi_2(X)) - \dim(\pi_2(X)),$$

which we have to minimize over all subspaces $\pi_2(X)$ of $\mathrm{rowsp}(Y_2)$. It is easy to see that the rank function ρ of any q-matroid satisfies $\rho(V) - \dim V \leq \rho(W) - \dim W$ whenever $W \leq V$ (see for instance [13, Lem. 2]), and thus we conclude that the minimum is attained by a subspace X such that $\pi_2(X) = \mathrm{rowsp}(Y_2)$. Choosing $X = \mathrm{rowsp}(0 \mid Y_2)$, we may rewrite (5.27) as

$$\rho(V) = \dim V + \rho_2(\mathrm{rowsp}(Y_2)) - \dim \mathrm{rowsp}(Y_2) = \mathrm{rk}(Y_1) + \rho_2(\mathrm{rowsp}(Y_2)),$$

which agrees with (5.26). All of this shows that $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}_G$, as stated. □

We conclude this chapter with the following result that if $\mathcal{M}_1$, $\mathcal{M}_2$ are represented respectively by G_1, G_2 then the cyclic flats of $\mathcal{M}_1 \oplus \mathcal{M}_2$ are also cyclic flats of the q-matroid $\mathcal{N}$ induced by $\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$

Theorem 5.4.9. *Let $\mathbb{F}_{q^m}$ be a field extension of $\mathbb{F} = \mathbb{F}_q$ and $G_i \in \mathbb{F}_{q^m}^{a_i \times n_i}$, $i = 1, 2$, be matrices of full row rank. Set*

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \in \mathbb{F}_{q^m}^{(a_1+a_2)\times(n_1+n_2)}.$$

Denote by $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i)$, $i = 1, 2$, and $\mathcal{N} = (\mathbb{F}^{n_1+n_2}, \hat{\rho})$ the q-matroids represented by G_1, G_2, and G, thus $\rho_i(\mathrm{rowsp}(Y)) = \mathrm{rk}(G_iY^{\mathsf{T}})$ for $Y \in \mathbb{F}^{y\times n_i}$ and $\hat{\rho}(\mathrm{rowsp}(Y)) = \mathrm{rk}(GY^{\mathsf{T}})$ for $Y \in \mathbb{F}^{y\times(n_1+n_2)}$.

(a) If $F \in \mathcal{F}(\mathcal{M}_1) \oplus \mathcal{F}(\mathcal{M}_2)$, then $F \in \mathcal{F}(\mathcal{N})$.

(b) If $O \in \mathcal{O}(\mathcal{M}_1) \oplus \mathcal{O}(\mathcal{M}_2)$, then $O \in \mathcal{O}(\mathcal{N})$.

As a consequence, $\mathcal{Z}(\mathcal{M}_1 \oplus \mathcal{M}_2) \subseteq \mathcal{Z}(\mathcal{N})$ and thus $\mathcal{I}(\mathcal{N}) \subseteq \mathcal{I}(\mathcal{M}_1 \oplus \mathcal{M}_2)$.

Note that the very last statement about the independent spaces also follows from the coproduct property of $\mathcal{M}_1 \oplus \mathcal{M}_2$, discussed in Theorem 6.3.3.

Proof. The first part of the consequence follows from Theorem 5.2.2 and the second part from Theorem 2.2.29. For the proof of (a) and (b) we will make use of the fact

$$\hat{\rho}(V_1 \oplus V_2) = \rho_1(V_1) + \rho_2(V_2) \quad \text{for all } V_i \in \mathcal{L}(\mathbb{F}^{n_i}). \tag{5.28}$$

This follows directly from the block diagonal form of G and the fact that every space $V_1 \oplus V_2$ is the row space of a block diagonal matrix as well.

(a) Let $F \in \mathcal{F}(\mathcal{M}_1) \oplus \mathcal{F}(\mathcal{M}_2)$ and $x \in \mathbb{F}^{n_1+n_2} \setminus F$. We want to show that $\hat{\rho}(F + \langle x \rangle) > \hat{\rho}(F)$. By assumption,

$$F = \text{rowsp}\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \quad \text{for some } Y_i \in \mathbb{F}^{y_i \times n_i} \text{ such that } \text{rowsp}(Y_i) \in \mathcal{F}(\mathcal{M}_i).$$

Write $x = (x_1 \mid x_2)$ with $x_i \in \mathbb{F}^{n_i}$. Without loss of generality let $x_1 \notin \text{rowsp}(Y_1)$. Since $\text{rowsp}(Y_1)$ is a flat in $\mathcal{M}_1$, this implies

$$\text{rk}(G_1[Y_1^{\mathsf{T}}, x_1^{\mathsf{T}}]) = \rho_1(\text{rowsp}(Y_1) + \langle x_1 \rangle) > \rho_1(\text{rowsp}(Y_1)) = \text{rk}(G_1 Y_1^{\mathsf{T}}).$$

Hence $G_1 x_1^{\mathsf{T}}$ is not in the column space of $G_1 Y_1^{\mathsf{T}}$ and we obtain

$$\begin{aligned}\hat{\rho}(F + \langle x \rangle) &= \text{rk}\left[\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \\ x_1 & x_2 \end{pmatrix}^{\mathsf{T}}\right] \\ &= \text{rk}\begin{pmatrix} G_1 Y_1^{\mathsf{T}} & 0 & G_1 x_1^{\mathsf{T}} \\ 0 & G_2 Y_2^{\mathsf{T}} & G_2 x_2^{\mathsf{T}} \end{pmatrix} \\ &> \text{rk}\begin{pmatrix} G_1 Y_1^{\mathsf{T}} & 0 \\ 0 & G_2 Y_2^{\mathsf{T}} \end{pmatrix} = \hat{\rho}(F).\end{aligned}$$

Since x was arbitrary, this proves that F is a flat of $\mathcal{N}$.
(b) Let $O \in \mathcal{O}(\mathcal{M}_1) \oplus \mathcal{O}(\mathcal{M}_2)$ and let $D \in \text{Hyp}(O)$. We want to show that $\hat{\rho}(D) = \hat{\rho}(O)$. By assumption $O = O_1 \oplus O_2$ with $O_i \in \mathcal{O}(\mathcal{M}_i)$. Recall the projections π_i from $\mathbb{F}^{n_1+n_2}$ to $\mathbb{F}^{n_i}$ and set $\hat{D} = \pi_1(D) \oplus \pi_2(D)$. Then $D \leq \hat{D} \leq O$ and since $\dim D = \dim O - 1$, we have $\hat{D} = D$ or $\hat{D} = O$.
(b1) If $\hat{D} = D$ we may assume $\pi_1(D) = O_1$ and $\pi_2(D) \in \text{Hyp}(O_2)$. Using cyclicity of O_2 and (6.15) we arrive at

$$\hat{\rho}(D) = \hat{\rho}(\hat{D}) = \rho_1(\pi_1(D)) + \rho_2(\pi_2(D)) = \rho_1(O_1) + \rho_2(O_2) = \hat{\rho}(O),$$

which is what we wanted.
(b2) Let $\hat{D} = O$. Set $\dim O = k+1$ and thus $\dim D = k$. Write $D = \text{rowsp}(M_1 \mid M_2)$, where $M_i \in \mathbb{F}^{k \times n_i}$. Then

$$\hat{D} = O = \text{rowsp}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

Hence $O_i = \text{rowsp}(M_i)$. Let $\text{rk}M_1 = k_1$. Then $\text{rk}M_2 = k - k_1 + 1$. Using elementary row operations we may assume that $(M_1 \mid M_2)$ is of the form

$$\begin{pmatrix} m_1 & m_2 \\ M_{21} & 0 \\ 0 & M_{22} \end{pmatrix} \quad \text{for some } m_i \in \mathbb{F}^{n_i},\ M_{21} \in \mathbb{F}^{(k_1-1)\times n_1},\ M_{22} \in \mathbb{F}^{(k-k_1)\times n_2}.$$

Since $\text{rk}M_{2i} = \text{rk}M_i - 1$ it follows that $\text{rowsp}(M_{2i}) \in \text{Hyp}(\text{rowsp}(M_i))$. Using the cyclicity of $\text{rowsp}(M_i)$ we conclude that $\text{rk}(G_i M_{2i}^{\mathsf{T}}) = \text{rk}(G_i M_i^{\mathsf{T}}) = \text{rk}(G_i[m_i^{\mathsf{T}}, M_{2i}^{\mathsf{T}}])$

for $i = 1, 2$. This means that $G_i m_i^\top$ is in the column space of $G_i M_{2i}^\top$, and thus

$$\begin{aligned}\hat{\rho}(D) &= \mathrm{rk}\begin{pmatrix} G_1 m_1^\top & G_1 M_{21}^\top & 0 \\ G_2 m_2^\top & 0 & G_2 M_{22}^\top \end{pmatrix} \\ &= \mathrm{rk}\begin{pmatrix} G_1 M_{21}^\top & 0 \\ 0 & G_2 M_{22}^\top \end{pmatrix} \\ &= \mathrm{rk}\begin{pmatrix} G_1 M_1^\top & 0 \\ 0 & G_2 M_2^\top \end{pmatrix} = \hat{\rho}(O).\end{aligned}$$

All of this shows $\hat{\rho}(D) = \hat{\rho}(O)$ for every $D \in \mathrm{Hyp}(O)$ and thus $O \in \mathcal{O}(\mathcal{N})$. This concludes the proof. □

We believe that studying representability of q-matroids, especially finding obstructions to representability, will be a challenging, but highly instructive topic for a better understanding of q-matroids and their differences to matroids. A characterization of representability of the direct sum in terms of the summands may be a first step in this direction. Note that Theorem 5.4.5 provides already an obstruction to representability in a special case.

Chapter 6 Categories of q-Matroids.

In this chapter we take a category approach towards the study of q-matroids. This approach allows to establish differences between the structure of classical matroids and their q-analogue. The main result of this chapter is the proof that a coproduct always exist only in the category of q-matroids where maps are linear q-weak maps. This holds in contrast to classical matroid for which a coproduct also exist in the category of matroids with strong maps as morphisms. Furthermore, we show there exist a functor from categories of q-matroids to categories of matroids which provide an alternative approach the study the former category. Results in this chapter also appear in [26] and [33].

6.1 Maps Between q-Matroids.

In this section we introduce maps between q-matroids. The candidates for such maps are (possibly nonlinear) maps between the ground spaces of the q-matroids with the property that they map subspaces to subspaces. Such maps will be called $\mathcal{L}$-maps. By definition, $\mathcal{L}$-maps induce order-preserving maps between the associated subspace lattices. As a consequence, one may choose $\mathcal{L}$-maps or their induced maps as maps between q-matroids. In Section 6.4 we will briefly discuss the second option, while for now we focus on $\mathcal{L}$-maps themselves. As maps between q-matroids, they should respect the q-matroid structure. This can be achieved in various ways, and we will introduce the options later in this section.

Throughout this section let E_1, E_2 be finite-dimensional $\mathbb{F}$-vector spaces.

Definition 6.1.1. *Let $\phi : E_1 \longrightarrow E_2$ be a map. We call ϕ an* $\mathcal{L}$-map *if $\phi(V) \in \mathcal{L}(E_2)$ for all $V \in \mathcal{L}(E_1)$. The induced map from $\mathcal{L}(E_1)$ to $\mathcal{L}(E_2)$ is denoted by $\phi_{\mathcal{L}}$. A bijective $\mathcal{L}$-map is called an* $\mathcal{L}$-isomorphism. *Finally, $\mathcal{L}$-maps ϕ, ψ from E_1 to E_2 are* $\mathcal{L}$-equivalent, *denoted by $\phi \sim_{\mathcal{L}} \psi$, if $\phi_{\mathcal{L}} = \psi_{\mathcal{L}}$.*

An $\mathcal{L}$-map $\phi : E_1 \longrightarrow E_2$ is thus a possibly nonlinear map that maps subspaces of E_1 to subspaces of E_2. It satisfies $\phi(0) = 0$ and

$$\phi(\langle v\rangle) = \langle \phi(v)\rangle \quad \text{for all} \quad v \in E_1. \tag{6.1}$$

This follows from the fact that $\phi(\langle v\rangle)$ is a subspace of cardinality at most q containing 0 and $\phi(v)$. Our definition of $\mathcal{L}$-isomorphisms is justified by the following simple fact.

Remark 6.1.2. *Let $\phi : E_1 \longrightarrow E_2$ be a bijective $\mathcal{L}$-map. Then ϕ^{-1} is also an $\mathcal{L}$-map. To see this, note that $\dim E_1 = \dim E_2$ by bijectivity of ϕ and thus the subspace lattices $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are isomorphic. Hence ϕ^{-1} is also an $\mathcal{L}$-map.*

Recall that a map $\phi : E_1 \longrightarrow E_2$ is $\mathbb{F}$-semilinear if ϕ is additive and there exists $\sigma \in \mathrm{Aut}(\mathbb{F})$ such that $\phi(cv) = \sigma(c)\phi(v)$ for all $v \in E_1$ and $c \in \mathbb{F}$. Clearly, any semilinear map $\phi : E_1 \longrightarrow E_2$ is an $\mathcal{L}$-map. Here are examples of non-semi-linear $\mathcal{L}$-maps and of $\mathcal{L}$-equivalent maps. A more general construction of $\mathcal{L}$-equivalent $\mathcal{L}$-maps will be given in Proposition 6.1.9(b).

Example 6.1.3. *(a) Let $X \lneq E_1$ be any subspace and let $v_1, \ldots, v_\ell \in E_1$ be such that $\langle v_1\rangle, \ldots, \langle v_\ell\rangle$ are the distinct lines in E_1 that are not contained in X. Choose $z \in E_2 \setminus \{0\}$. Set*

$$\phi : E_1 \longrightarrow E_2, \quad v \longmapsto \begin{cases} 0, & \text{if } v \in X, \\ \lambda z, & \text{if } v = \lambda v_i \text{ for some } i \in [\ell]. \end{cases}$$

Then $\phi(V) = \{0\}$ for all $V \in \mathcal{L}(X)$ and $\phi(V) = \langle z\rangle$ for all $V \in \mathcal{L}(E_1) \setminus \mathcal{L}(X)$. Thus, ϕ is an $\mathcal{L}$-map. Furthermore, the pre-image of any subspace is a subspace. Indeed, let $W \leq E_2$. Then $\phi^{-1}(W) = X$ if $z \notin W$ and $\phi^{-1}(W) = E_1$ if $z \in W$. Note that ϕ depends on the choice of the representatives v_i for the distinct lines. One can easily create examples where ϕ is not semi-linear. (In fact, one can show that there always exist choices such that ϕ is not semi-linear unless $\dim E_1 = 1$ or $[\dim X = \dim E_1 - 1$ and $\mathbb{F}_q = \mathbb{F}_2]$.)

(b) Let $\phi : \mathbb{F}_2^3 \longrightarrow \mathbb{F}_2^2$ be given by $\phi(v_1, v_2, 0) = (v_1, v_2)$ and $\phi(v_1, v_2, 1) = (0, 0)$ for all $v_1, v_2 \in \mathbb{F}_2$. Then ϕ is a nonlinear $\mathcal{L}$-map. In this case, the pre-image of a subspace is not necessarily a subspace, for instance $\phi^{-1}(\{(0, 0)\})$.

(c) Clearly, if $\phi : E_1 \longrightarrow E_2$ is an $\mathcal{L}$-map, and $\psi = \lambda\phi$ for some $\lambda \in \mathbb{F}^*$, then ϕ and ψ are $\mathcal{L}$-equivalent $\mathcal{L}$-maps.

(d) Let $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ and consider the semi-linear map $\phi : \mathbb{F}_4^2 \longrightarrow \mathbb{F}_4^2$, $(v_1, v_2) \longmapsto (v_1^2, v_2^2)$. Furthermore, let

$$\psi : \mathbb{F}_4^2 \longrightarrow \mathbb{F}_4^2, \quad (v_1, v_2) \longmapsto \begin{cases} \alpha(v_1^2, v_2^2), & \text{if } (v_1, v_2) \in \langle(1, 1)\rangle, \\ (v_1^2, v_2^2), & \text{otherwise.} \end{cases}$$

One easily verifies that ψ is an $\mathcal{L}$-map and ϕ and ψ are $\mathcal{L}$-equivalent.

We now discuss the relation between $\mathcal{L}$-maps from E_1 to E_2 and lattice homomorphism from $(\mathcal{L}(E_1), \leq, +, \cap)$ to $(\mathcal{L}(E_2), \leq, +, \cap)$. To do so, recall the following notions and simple facts. For further details see for instance [17, Ch. 2].

Remark 6.1.4. *Let $(\mathcal{L}_i, \leq_i, \vee_i, \wedge_i)$, $i = 1, 2$, be two lattices and $\phi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ be a map. Then ϕ is a* lattice homomorphism *if it is meet- and join-preserving, that is, $\phi(a \wedge_1 b) = \phi(a) \wedge_2 \phi(b)$ and $\phi(a \vee_1 b) = \phi(a) \vee_2 \phi(b)$ for all $a, b \in \mathcal{L}_1$. If each $\mathcal{L}_i$ has a least element 0_i and a greatest element 1_i, we call ϕ a* $\{0, 1\}$-lattice homomorphism *if it is a lattice homomorphism satisfying $\phi(0_1) = 0_2$ and $\phi(1_1) = 1_2$. Finally, we call ϕ* order-preserving *if $a \leq_1 b$ implies $\phi(a) \leq_2 \phi(b)$ for all $a, b \in \mathcal{L}_1$. We have the following facts.*

(a) Every lattice homomorphism is order-preserving [17, Prop. 2.19]. Clearly, every $\mathcal{L}$-map ϕ induces an order-preserving map $\phi_{\mathcal{L}}$.

(b) Let $\phi : E_1 \longrightarrow E_2$ be an $\mathcal{L}$-map and $\phi(v_1) = \phi(v_2) \neq 0$ for some distinct $v_1, v_2 \in E_1$. Then $\phi_{\mathcal{L}}$ is not meet-preserving and thus not a lattice homomorphism. An example of such a map ϕ is in Example 6.1.3(a) if $\dim E_1 > 1$.

(c) A lattice homomorphism need not be a $\{0,1\}$-lattice homomorphism. Consider for instance the embedding $\tau : \mathbb{F}_2^2 \longrightarrow \mathbb{F}_2^3$, $(x,y) \longmapsto (x,y,0)$, and the map $\phi : \mathcal{L}(\mathbb{F}_2^2) \longrightarrow \mathcal{L}(\mathbb{F}_2^3)$ given by $\phi(V) = \tau(V)$.

(d) A lattice isomorphism is a $\{0,1\}$-lattice isomorphism.

(e) A $\{0,1\}$-lattice homomorphism need not be a lattice isomorphism. Consider, for instance, $\phi : \mathcal{L}(\mathbb{F}_2^2) \longrightarrow \mathcal{L}(\mathbb{F}_2^4)$, which maps 0 to 0 and $\mathbb{F}_2^2$ to $\mathbb{F}_2^4$ and the three 1-spaces in $\mathbb{F}_2^2$ to the subspaces $\langle 1000, 0100\rangle$, $\langle 0010, 0001\rangle$, and $\langle 1010, 0101\rangle$.

Note that the map in Remark 6.1.4(e) is not induced by an $\mathcal{L}$-map (because $\dim V < \dim \phi(V)$). For maps induced by $\mathcal{L}$-maps we have some stronger statements.

Proposition 6.1.5. *Let $\phi : E_1 \longrightarrow E_2$ be an $\mathcal{L}$-map and $\phi_{\mathcal{L}} : \mathcal{L}(E_1) \longrightarrow \mathcal{L}(E_2)$ be the induced map.*

(a) If ϕ is injective, then $\phi_{\mathcal{L}}$ is a lattice homomorphism.

(b) Suppose $E_2 \neq 0$. Then ϕ is an $\mathcal{L}$-isomorphism $\Longleftrightarrow$ $\phi_{\mathcal{L}}$ is a $\{0,1\}$-lattice homomorphism $\Longleftrightarrow$ $\phi_{\mathcal{L}}$ is a lattice isomorphism.

Note that if $E_2 = 0$, then the zero map induces a $\{0,1\}$-lattice homomorphism; thus the exceptional case in (b).

Proof. (a) Let $V_1, V_2 \in \mathcal{L}(E_1)$ and $\hat{V}_i = \phi(V_i)$. Then clearly $\phi(V_1 \cap V_2) \subseteq \hat{V}_1 \cap \hat{V}_2$. For the converse containment let $\hat{v} \in \hat{V}_1 \cap \hat{V}_2$. Then there exist $v_i \in V_i$ such that $\phi(v_1) = \hat{v} = \phi(v_2)$ and injectivity implies $v_1 = v_2 \in V_1 \cap V_2$. This shows $\phi(V_1 \cap V_2) = \hat{V}_1 \cap \hat{V}_2$ and thus $\phi_{\mathcal{L}}$ is meet-preserving. Next, we clearly have $\hat{V}_1 + \hat{V}_2 \subseteq \phi(V_1 + V_2)$ and equality follows from $\dim(\hat{V}_1 + \hat{V}_2) = \dim \hat{V}_1 + \dim \hat{V}_2 - \dim(\hat{V}_1 \cap \hat{V}_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) = \dim(V_1 + V_2) = \dim \phi(V_1 + V_2)$. Thus ϕ is join-preserving.
(b) Let $E_2 \neq 0$. The only implication that remains to be proven is that if $\phi_{\mathcal{L}}$ is a $\{0,1\}$-lattice homomorphism then ϕ is bijective. Thus, let $\phi_{\mathcal{L}}$ be a $\{0,1\}$-lattice homomorphism. By assumption $\phi(E_1) = E_2$, and thus ϕ is surjective and $\dim E_2 \leq \dim E_1$. Let $e_i = \dim E_i$. We have to show that $e_1 = e_2$. Assume by contradiction that $e_2 < e_1$. We proceed in several steps.
i) Clearly ϕ maps 1-spaces of E_1 to subspaces of E_2 of dimension 1 or 0. Suppose $\phi(V) = \phi(W)$ for some 1-spaces $V \neq W$. Then the properties of a $\{0,1\}$-lattice homomorphism imply $\phi(V) = \phi(V) \cap \phi(W) = \phi(V \cap W) = \phi(0) = 0$. Since E_i has $(q^{e_i} - 1)/(q-1)$ 1-spaces, we conclude that at most $(q^{e_2} - 1)/(q-1)$ 1-spaces of E_1 are mapped to 1-spaces, and thus ϕ maps at least $(q^{e_1} - q^{e_2})/(q-1)$ 1-spaces to 0.
ii) Let now $V \in \mathcal{L}(E_1)$ be a maximal-dimensional subspace such that $\phi(V) = 0$. Then $V \neq E_1$ since $E_2 \neq 0$. Let $\dim V = v$. The number of 1-spaces in V is $(q^v - 1)/(q-1)$. Using $e_2 \leq e_1 - 1$ as well as $v \leq e_1 - 1$, we compute

$$\frac{q^{e_1} - q^{e_2}}{q-1} - \frac{q^v - 1}{q-1} \geq \frac{q^{e_1} - q^{e_2}}{q-1} - \frac{q^{e_1 - 1} - 1}{q-1} = q^{e_1 - 1} - \frac{q^{e_2} - 1}{q-1} \geq 1.$$

Now i) tells us that there exists at least one 1-space $W \in \mathcal{L}(E_1)$ such that $W \not\leq V$ and $\phi(W) = 0$. Now the join-preserving property implies $\phi(V + W) = \phi(V) + \phi(W) = 0$, in contradiction to the maximality of V. All of this shows that $e_1 = e_2$ and thus ϕ is bijective. □

The $\mathcal{L}$-isomorphisms from Proposition 6.1.5(b) are, up to $\mathcal{L}$-equivalence, semi-linear maps. This is a consequence of the Fundamental Theorem of Projective Geometry; see for instance [3, Ch. II.10] or [40, Thm. 1] (for the interesting original source of this theorem, which is attributed to von Staudt, one may also consult [31]).

Theorem 6.1.6. *Let* $\dim E_1 = \dim E_2 = n$*, where* $n \geq 3$*. Let* $\tau : \mathcal{L}(E_1) \longrightarrow \mathcal{L}(E_2)$ *be a lattice isomorphism. Then there exists a semi-linear map* $\phi : E_1 \longrightarrow E_2$ *such that* $\tau = \phi_{\mathcal{L}}$*.*

Corollary 6.1.7. *Let* $\dim E_1 = \dim E_2 = n$*, where* $n \geq 3$ *or* $q = n = 2$*. Let* $\tau : E_1 \longrightarrow E_2$ *be an* $\mathcal{L}$*-map and suppose there exist vectors* $v_1, \ldots, v_n$ *of* E_1 *such that* $\tau(v_1), \ldots, \tau(v_n)$ *are linearly independent. Then* τ *is bijective and there exists a semi-linear isomorphism* $\phi : E_1 \longrightarrow E_2$ *such that* $\tau \sim_{\mathcal{L}} \phi$*. In particular, if* $q = 2$*, then* $\tau = \phi$ *is a linear isomorphism.*

Proof. Since $\tau(E_1)$ is a subspace and contains $\tau(v_1), \ldots, \tau(v_n)$, we conclude that $\tau(E_1) = E_2$. Hence τ is a bijection. For $n = 2 = q$ it is clear that τ is a linear isomorphism (any nonzero vector in E_1 or E_2 is the sum of the other two nonzero vectors). For $n \geq 3$ we may use Theorem 6.1.6. □

We return to general $\mathcal{L}$-maps and list some basic facts.

Proposition 6.1.8. *Let* $\phi, \psi : E_1 \longrightarrow E_2$ *be* $\mathcal{L}$*-maps such that* $\phi \sim_{\mathcal{L}} \psi$*.*
(a) For all $v \in E_1$ *there exist* $\lambda_v \in \mathbb{F}^*$ *such that* $\phi(v) = \lambda_v \psi(v)$*.*
(b) $\phi^{-1}(V) = \psi^{-1}(V)$ *for all* $V \in \mathcal{L}(E_2)$*.*
(c) If ψ *is an* $\mathcal{L}$*-isomorphism, then so is* ϕ*, and* $\phi^{-1} \sim_{\mathcal{L}} \psi^{-1}$*.*
(d) If ψ *is injective (resp. surjective), then so is* ϕ*.*
(e) If ϕ *and* ψ *are linear, then* $\phi = \lambda\psi$ *for some* $\lambda \in \mathbb{F}^*$*.*

Proof. (a) It suffices to consider $v \neq 0$. Note that $\phi(v) = 0 \Longleftrightarrow \psi(v) = 0$, and in this case we may choose $\lambda_v = 1$. In the case $\psi(v) \neq 0 \neq \phi(v)$ the result follows from (6.1).
(b) For $v \in E_1$ let $\lambda_v \in \mathbb{F}^*$ be as in (a). For any $V \in \mathcal{L}(E_2)$ we have $\phi^{-1}(V) = \{v \in E_1 \mid \phi(v) \in V\} = \{v \in E_1 \mid \lambda_v\psi(v) \in V\} = \{v \in E_1 \mid \psi(v) \in V\} = \psi^{-1}(V)$.
(c) If ψ is an $\mathcal{L}$-isomorphism, then the lattices $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$ are isomorphic (see Remark 6.1.2). Next, $E_2 = \psi(E_1) = \phi(E_1)$, and thus ϕ is bijective as well. Furthermore, $\psi_{\mathcal{L}}$ and $\phi_{\mathcal{L}}$ are lattice isomorphisms from $\mathcal{L}(E_1)$ to $\mathcal{L}(E_2)$ satisfying $(\psi^{-1})_{\mathcal{L}} = (\psi_{\mathcal{L}})^{-1} = (\phi_{\mathcal{L}})^{-1} = (\phi^{-1})_{\mathcal{L}}$.
(d) The statement about surjectivity is clear since $\phi(E_1) = \psi(E_1)$. For injectivity use (c) with $E_2 = \psi(E_1)$.
(e) Let $W = \ker\psi$, hence also $W = \ker\phi$, and let $E_1 = V \oplus W$. Let $v_1, \ldots, v_r$ be a basis of V. By (a) there exist $\lambda_i \in \mathbb{F}^*$ such that $\phi(v_i) = \lambda_i\psi(v_i)$ for $i \in [r]$. Furthermore, $\phi(v_1 + v_i) = \hat{\lambda}\psi(v_1 + v_i)$ for some $\hat{\lambda} \in \mathbb{F}^*$. Linearity implies $\psi(\lambda_1 v_1 + \lambda_i v_i) = \phi(v_1 + v_i) = \hat{\lambda}\psi(v_1 + v_i) = \psi(\hat{\lambda}v_1 + \hat{\lambda}v_i)$, and injectivity of ψ on V yields $\lambda_1 = \lambda_i = \hat{\lambda}$ for all $i \in [r]$. Hence $\phi|_V = \hat{\lambda}\psi|_V$. Now we obtain $\phi(v + w) = \phi(v) + 0 = \hat{\lambda}\psi(v) + 0 = \hat{\lambda}\psi(v) + \hat{\lambda}\psi(w) = \hat{\lambda}\psi(v + w)$ for all $v \in V$ and $w \in W$, and this proves the desired statement. □

The following results show how we may alter an $\mathcal{L}$-map without changing the induced lattice homomorphism. Part (a) also shows that $\mathcal{L}$-maps are $\mathcal{L}$-equivalent if they agree on the 1-spaces.

Proposition 6.1.9. *Let $\psi : E_1 \longrightarrow E_2$ be an $\mathcal{L}$-map.*

(a) Let $\phi : E_1 \longrightarrow E_2$ be a map such that $\phi(0) = 0$ and $\phi(\langle v\rangle) = \psi(\langle v\rangle)$ for all $v \in E_1$. Then ϕ is an $\mathcal{L}$-map and $\phi \sim_{\mathcal{L}} \psi$.

(b) Suppose ψ is an $\mathcal{L}$-isomorphism. Fix $w \in E_1 \setminus 0$ and $\tau \in \mathbb{F}^*$ and set $\hat{w} = \psi^{-1}(\tau\psi(w))$. Then $\langle\hat{w}\rangle = \langle w\rangle$. Define the map $\phi : E_1 \longrightarrow E_2$ via

$$\phi(v) = \psi(v) \ \text{ for } v \in E_1 \setminus \langle w\rangle, \quad \phi(\mu w) = \psi(\mu\hat{w}) \ \text{ for } \mu \in \mathbb{F}.$$

Then ϕ is an $\mathcal{L}$-isomorphism and $\phi \sim_{\mathcal{L}} \psi$. If $\tau = 1$, then $\phi = \psi$.

Proof. (a) is immediate from $\phi(V) = \bigcup_{v\in V} \phi(\langle v\rangle) = \bigcup_{v\in V} \psi(\langle v\rangle) = \psi(V)$ for all $V \in \mathcal{L}(E_1)$.
(b) Bijectivity of ψ implies $\hat{w} \neq 0$ and $\psi(\hat{w}) = \tau\psi(w)$, thus $\psi(\langle w\rangle) = \langle\psi(w)\rangle = \langle\psi(\hat{w})\rangle = \psi(\langle\hat{w}\rangle)$. This in turn implies $\langle w\rangle = \langle\hat{w}\rangle$. The map ϕ is clearly bijective and satisfies $\phi(0) = 0$. We show that ϕ satisfies the condition of (a). First let $v \notin \langle w\rangle$. Then also $\lambda v \notin \langle w\rangle$ for all $\lambda \in \mathbb{F}^*$ and $\phi(\langle v\rangle) = \{\phi(\lambda v) \mid \lambda \in \mathbb{F}\} = \{\psi(\lambda v) \mid \lambda \in \mathbb{F}\} = \psi(\langle v\rangle)$. Next, for $v = \mu w$ with $\mu \neq 0$ we have $\phi(\langle v\rangle) = \{\phi(\lambda\mu w) \mid \lambda \in \mathbb{F}\} = \{\psi(\lambda\mu\hat{w}) \mid \lambda \in \mathbb{F}\} = \psi(\langle\hat{w}\rangle) = \psi(\langle w\rangle) = \psi(\langle v\rangle)$. Thus we may apply (a) and the statement follows. □

We now turn to $\mathcal{L}$-maps between q-matroids. There are different options for such a map to respect the q-matroid structure. For (a) and (b) below we adopt the terminology known for classical matroids; see, e.g. [47, Def. 8.1.1 and Def. 9.1.1]. Our notion of rank-preserving maps in (c), however, is different from rank-preserving weak maps for classical matroids: the latter are weak maps that preserve the rank of the matroid; see [47, p. 260]. The definition below will be convenient for us.

Definition 6.1.10. *Let $\mathcal{M}_i = (E_i, \rho_i)$ be q-matroids with flats $\mathcal{F}_i := \mathcal{F}(\mathcal{M}_i)$. Let $\phi : E_1 \longrightarrow E_2$ be an $\mathcal{L}$-map. We define the following* types.

*(a) ϕ is a q-*strong map *from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $\phi^{-1}(F) \in \mathcal{F}(\mathcal{M}_1)$ for all $F \in \mathcal{F}(\mathcal{M}_2)$ (this implies in particular that $\phi^{-1}(F)$ is a subspace of E_1).*

*(b) ϕ is a q-*weak map *from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $\rho_2(\phi(V)) \leq \rho_1(V)$ for all $V \in \mathcal{L}(E_1)$.*

(c) ϕ is rank-preserving *from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $\rho_2(\phi(V)) = \rho_1(V)$ for all $V \in \mathcal{L}(E_1)$.*

For any $\mathcal{L}$-map $\phi : E_1 \longrightarrow E_2$ we will also use the notation $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$. This allows us to discuss its type.

In Section 6.5, we will consider both maps between matroids and maps between q-matroids. To avoid confusion, we therefore use the term q-weak and q-strong for maps between q-matroids. However, the "q-" may be omitted if there is no risk of confusion.

Note that each of the types above are actually properties of the induced map $\phi_{\mathcal{L}}$. This raises the question as to whether one should define maps between q-matroids (E_1, ρ_1) and (E_2, ρ_2) as maps (with certain properties) between the underlying subspace lattices $\mathcal{L}(E_1)$ and $\mathcal{L}(E_2)$. However lattice homomorphisms are too restrictive as they exclude some non-injective maps (see also Remark 6.1.4(b)), while simply order-preserving maps appear to be too general. In Section 6.4 we will briefly consider the setting where the maps are those induced by $\mathcal{L}$-maps. Note that the distinction of maps between the ground spaces versus maps between the subspace lattices does not occur for classical matroids because a map on a set S is uniquely determined by its induced map on the subset lattice of S.

We return to Definition 6.1.10. Clearly, the composition of maps of the same type is again a map of that type. Furthermore, $\mathcal{L}$-equivalent $\mathcal{L}$-maps are of the same type (see Proposition 6.1.8(b) for q-strong maps). Note, however, that if $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is a bijective strong (resp. weak) map, then $\phi^{-1} : \mathcal{M}_2 \longrightarrow \mathcal{M}_1$ may not be strong (resp. weak): take for instance the identity map $\mathcal{U}_k(\mathbb{F}^n) \longrightarrow \mathcal{U}_{k-1}(\mathbb{F}^n)$. Being rank-preserving and being strong are not related: there exist q-strong maps that are not rank-preserving (e.g., the identity map from any nontrivial q-matroid $\mathcal{M} = (E, \rho)$ to the trivial q-matroid on E) and there exist rank-preserving maps that are not strong (e.g., $\phi : \mathbb{F}_2^2 \longrightarrow \mathbb{F}_2^2,\ (x, y) \longmapsto (x, 0)$ and where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the q-matroids on $\mathbb{F}_2^2$ of rank 1 with $\langle e_2 \rangle$ and $\langle e_1 + e_2 \rangle$ as the unique flat of rank 0, respectively). Unsurprisingly, q-weak maps are in general not strong: take for instance the identity on $\mathbb{F}_2^4$, which induces a weak, but not q-strong map from $\mathcal{U}_2(\mathbb{F}_2^4)$ to the q-matroid $\mathcal{M}$ from Example 2.4.8. However, it can be shown that – just like in the classical case [47, Lemma 8.1.7] – q-strong maps are q-weak. This will be done later on in Section 6.5 For now, we show the following simple result.

Proposition 6.1.11. *Let $\mathcal{M}_i = (E_i, \rho_i)$ be q-matroids and $\phi : E_1 \longrightarrow E_2$ be an $\mathcal{L}$-isomorphism. Consider ϕ as a map from $\mathcal{M}_1$ to $\mathcal{M}_2$. Then*

$$\phi \text{ and } \phi^{-1} \text{ are } q\text{-weak maps} \Longleftrightarrow \phi \text{ is rank-preserving} \Longleftrightarrow \phi \text{ and } \phi^{-1} \text{ are } q\text{-strong maps.}$$

Proof. Recall from Remark 6.1.2 that ϕ^{-1} is also an $\mathcal{L}$-map. The first equivalence is clear. Consider the second equivalence.
"$\Rightarrow$" Let ϕ be a rank-preserving isomorphism. Then $\rho_2(\phi(V)) = \rho_1(V)$ and $\dim \phi(V) = \dim V$ for all $V \in \mathcal{L}(E_1)$. Using Theorem 2.2.4 we conclude that V is a flat in $\mathcal{M}_1$ iff $\phi(V)$ is a flat in $\mathcal{M}_2$.
"$\Leftarrow$" Let now ϕ and ϕ^{-1} be q-strong maps. Then $\phi(\mathcal{F}(\mathcal{M}_1)) = \mathcal{F}(\mathcal{M}_2)$, that is, ϕ induces an isomorphism between the lattices of flats of $\mathcal{M}_1$ and $\mathcal{M}_2$. Using the height function on these lattices (see Theorem 2.2.4 we conclude that $\rho_1(F) = \rho_2(\phi(F))$ for all $F \in \mathcal{F}(\mathcal{M}_1)$ and thus $\rho_1(V) = \rho_2(\phi(V))$ for all $V \in \mathcal{L}(E_1)$. Hence ϕ is rank-preserving. □

Just like there exist linear maps between any vector spaces (over the same field), there exist weak and q-strong maps between any q-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$.

Example 6.1.12. *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids.*

(a) The zero map ϕ_0 from E_1 to E_2 is a q-weak map from $\mathcal{M}_1$ to $\mathcal{M}_2$. It is also a q-strong map since $\phi_0^{-1}(W) = E_1 \in \mathcal{F}(\mathcal{M}_1)$ for all $W \leq E_2$.

(b) Let $X = \{x \in E_1 \mid \rho_1(\langle x\rangle) = 0\}$, that is, X is the closure of $\{0\}$ in $\mathcal{M}_1$ in the sense of Theorem 2.2.3. In particular, $X \leq E_1$. Let $z \in E_2 \setminus \{0\}$. As in Example 6.1.3(a) choose vectors $v_1, \ldots, v_\ell \in E_1 \setminus X$ such that $\langle v_1\rangle, \ldots, \langle v_\ell\rangle$ are the distinct lines in E that are not in X and consider the map ϕ as in that example. Then ϕ is a q-weak map. Indeed, for any $V \leq X$ we have $\rho_1(V) = 0 = \rho_2(\{0\}) = \rho_2(\phi(V))$, while for $V \in \mathcal{L}(E_1) \setminus \mathcal{L}(X)$ we have $\rho_1(V) \geq 1 \geq \rho_2(\langle z\rangle) = \rho_2(\phi(V))$. Moreover, the pre-images in Example 6.1.3(a) show that ϕ is strong. Note that if $X = E_1$, i.e., $\mathcal{M}_1$ is the trivial q-matroid, then ϕ is the zero map.

We now turn to minors of q-matroids and determine the type of the corresponding maps.

Proposition 6.1.13. *Let $\mathcal{M} = (E, \rho)$ be a q-matroid and let $X \leq E$.*

(a) Let $\mathcal{M}|_X$ be the restriction of $\mathcal{M}$ to X. Then the embedding $\iota : X \longrightarrow E$, $x \longmapsto x$, is a linear strong and rank-preserving (hence weak) map from $\mathcal{M}|_X$ to $\mathcal{M}$.

(b) Let $\mathcal{M}/X$ be the contraction of X from $\mathcal{M}$. Then the projection $\pi : E \longrightarrow E/X$, $x \longmapsto x + X$ is a linear strong and q-weak map from $\mathcal{M}$ to $\mathcal{M}/X$.

Proof. (a) Recall that $\mathcal{M}|_X = (X, \hat{\rho})$, where $\hat{\rho}(V) = \rho(V)$ for all $V \leq X$. This shows that ι is rank-preserving. Let $F \in \mathcal{F}(\mathcal{M})$. Then $\iota^{-1}(F) = F \cap X$ and $\mathrm{cl}_{\mathcal{M}}(F \cap X) \subseteq \mathrm{cl}_{\mathcal{M}}(F) = F$, where $\mathrm{cl}_{\mathcal{M}}(A)$ denotes the closure of the space A in the q-matroid $\mathcal{M}$ (for the closure see Theorem 2.2.4). Thus for any $v \in X$ the identity $\rho((F \cap X) + \langle v\rangle) = \rho(F \cap X)$ implies $v \in F$. Hence $\mathrm{cl}_{\mathcal{M}|_X}(F \cap X) = F \cap X$ and thus $\iota^{-1}(F) \in \mathcal{F}(\mathcal{M}|_X)$.
(b) Recall that $\mathcal{M}/X = (E/X, \tilde{\rho})$, where $\tilde{\rho}(\pi(V)) = \rho(V + X) - \rho(X)$. Thus by submodularity $\tilde{\rho}(\pi(V)) \leq \rho(V) - \rho(V \cap X)$, showing that π is weak. In order to show that π is strong, let $F \in \mathcal{F}(\mathcal{M}/X)$. Thus $\tilde{\rho}(F + \langle v + X\rangle) > \tilde{\rho}(F)$ for all $v + X \in (E/X) \setminus F$. Since $\pi^{-1}(F + \langle v + X\rangle) = \pi^{-1}(F) + \langle v\rangle$, this implies $\rho(\pi^{-1}(F) + \langle v\rangle) > \rho(\pi^{-1}(F))$ for all $v \in E \setminus \pi^{-1}(F)$. Thus $\pi^{-1}(F) \in \mathcal{F}(\mathcal{M})$. □

Restricting an $\mathcal{L}$-map to its image does not change its type.

Proposition 6.1.14. *Let $\mathcal{M}_i = (E_i, \rho_i)$ be q-matroids and $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a strong (resp. weak or rank-preserving) map. Let $X := \mathrm{im}\phi$. Then X is a subspace of E_2 and we call the restriction $\mathcal{M}_2|_X$ the* image of ϕ. *The map $\hat{\phi} : E_1 \longrightarrow X$, $v \longmapsto \phi(v)$, is a strong (resp. weak or rank-preserving) map from $\mathcal{M}_1$ to $\mathcal{M}_2|_X$. In other words, ϕ restricts to a map $\mathcal{M}_1 \longrightarrow \mathcal{M}_2|_X$ of the same type.*

Proof. The statement is clear for weak and rank-preserving maps. Let now ϕ be strong and $F \in \mathcal{F}(\mathcal{M}_2|_X)$. Then $\rho_2(F + \langle v\rangle) > \rho_2(F)$ for all $v \in X \setminus F$. Hence the closure $\mathrm{cl}_{\mathcal{M}_2}(F)$ satisfies $\mathrm{cl}_{\mathcal{M}_2}(F) \setminus F \subseteq E_2 \setminus X$. Using that $\mathrm{im}\phi = X$, we obtain $\phi^{-1}(\mathrm{cl}_{\mathcal{M}_2}(F)) = \hat{\phi}^{-1}(F)$. Since the former is a flat in $\mathcal{M}_1$, we conclude that $\hat{\phi}$ is a q-strong map. □

Finally we record the simple observation that representability is not preserved under strong or weak bijective maps. Take for instance the identity map on $\mathbb{F}_2^4$. It induces a bijective strong and q-weak map from the representable q-matroid $\mathcal{U}_4(\mathbb{F}_2^4)$ to the non-representable q-matroid in Example 2.4.8.

6.2 Non-Existence of Coproducts in Categories of q-Matroids.

In this section we consider categories of q-matroids with various types of morphisms. We will show that – with one exception – none of these categories has a coproduct.

Definition 6.2.1. *We denote by* q-$\mathbf{Mat}^{s}$, q-$\mathbf{Mat}^{rp}$, q-$\mathbf{Mat}^{w}$, q-$\mathbf{Mat}^{l\text{-}s}$, q-$\mathbf{Mat}^{l\text{-}rp}$, *and* q-$\mathbf{Mat}^{l\text{-}w}$ *the categories with q-matroids as objects and where the morphisms are the strong, rank-preserving, weak, linear strong, linear rank-preserving and linear q-weak maps, respectively.*

In this section we show that none of the first 5 categories has a coproduct, while in the next section we establish the existence of a coproduct in q-$\mathbf{Mat}^{\text{l-w}}$. It is in fact the direct sum as introduced recently in [13]. The non-existence of a coproduct in q-$\mathbf{Mat}^{\text{s}}$ and q-$\mathbf{Mat}^{\text{l-s}}$ stands in contrast to the case of classical matroids, where the direct sum (see [39, Sec. 4.2]) forms a coproduct in the category with q-strong maps as morphisms, see [47, Ex. 8.6, p. 244] (which goes back to [16]). We know from Proposition 6.1.11 that isomorphisms in the first three categories coincide, and so do those in the second three categories. This gives rise to the following notions of isomorphic q-matroids.

Definition 6.2.2. *We call q-matroids* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ isomorphic *if they are isomorphic in the category* q-$\mathbf{Mat}^{rp}$*, that is, there exists a rank-preserving* $\mathcal{L}$*-isomorphism* $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ *(equivalently,* ϕ *and* ϕ^{-1} *are strong).* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are* linearly isomorphic*, denoted by* $\mathcal{M}_1 \cong \mathcal{M}_2$*, if they are isomorphic in the category* q-$\mathbf{Mat}^{l\text{-}rp}$.

Due to Theorem 6.1.6 the above notion of isomorphism coincides with lattice-equivalence in [13, Def. 5]. The same theorem tells us that any rank-preserving $\mathcal{L}$-isomorphism is induced by a semi-linear map.

Remark 6.2.3. *Let* $\phi : E_1 \longrightarrow E_2$ *be an* $\mathcal{L}$*-isomorphism and* $\mathcal{M}_1 = (E_1, \rho_1)$ *be a q-matroid. Define* $\mathcal{M}_2 = (E_2, \rho_2)$ *via* $\rho_2(V) := \rho_1(\phi^{-1}(V))$ *for all* $V \in \mathcal{L}(E_2)$*. Then* $\mathcal{M}_2$ *is a q-matroid and isomorphic to* $\mathcal{M}_1$*. The flats of* $\mathcal{M}_2$ *are given by* $\mathcal{F}(\mathcal{M}_2) = \{\phi(F) \mid F \in \mathcal{F}(\mathcal{M}_1)\}$.

The following linear maps will be used throughout this paper. For $E = E_1 \oplus E_2$ let

$$\iota_i : E_i \longrightarrow E,\ x \longrightarrow x \tag{6.2}$$

be the natural embeddings. If $E_i = \mathbb{F}^{n_i}$ and $E = \mathbb{F}^{n_1+n_2}$ we define the maps as

$$\iota_1 : \mathbb{F}^{n_1} :\longrightarrow \mathbb{F}^{n_1+n_2},\ x \longrightarrow (x, 0), \qquad \iota_2 : \mathbb{F}^{n_2} :\longrightarrow \mathbb{F}^{n_1+n_2},\ y \longrightarrow (0, y). \tag{6.3}$$

Next we present a simple construction that will be crucial later on. It shows that representable q-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ can be embedded in a q-matroid $\mathcal{M}$ in such a way that the ground spaces of $\mathcal{M}_i$ form a direct sum of the ground spaces of $\mathcal{M}$.

However, as we will illustrate by an example below, the resulting q-matroid $\mathcal{M}$ is not uniquely determined by the q-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, but rather depends on the representing matrices (which has also been observed in [13, Sec. 3.2]). It is exactly this non-uniqueness that allows us to prove the non-existence of coproducts.

Proposition 6.2.4. *Let $\mathbb{F}_{q^m}$ be a field extension of $\mathbb{F} = \mathbb{F}_q$ and $G_i \in \mathbb{F}_{q^m}^{a_i \times n_i}$, $i = 1, 2$, be matrices of full row rank. Set*

$$G = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \in \mathbb{F}_{q^m}^{(a_1+a_2)\times(n_1+n_2)}.$$

Denote by $\mathcal{M}_i = (\mathbb{F}^{n_i}, \rho_i)$, $i = 1, 2$, and $\mathcal{N} = (\mathbb{F}^{n_1+n_2}, \rho)$ the q-matroids represented by G_1, G_2, and G, thus $\rho_i(\mathrm{rowsp}(Y)) = \mathrm{rk}(G_i Y^{\mathsf{T}})$ for $Y \in \mathbb{F}^{y \times n_i}$ and $\rho(\mathrm{rowsp}(Y)) = \mathrm{rk}(GY^{\mathsf{T}})$ for $Y \in \mathbb{F}^{y\times(n_1+n_2)}$. Then $\iota_i : \mathcal{M}_i \longrightarrow \mathcal{N}$, $i = 1, 2$, is a linear, rank-preserving, and q-strong map with image $\mathcal{N}|_{\mathbb{F}^{n_i}}$. Thus $\mathcal{N}|_{\mathbb{F}^{n_i}}$ is linearly isomorphic to $\mathcal{M}_i$ for $i = 1, 2$.

Proof. Let $Y \in \mathbb{F}^{y\times n_1}$ be a matrix of rank y. Then $\mathrm{rowsp}(Y) \leq \mathbb{F}^{n_1}$ and $\iota_1(\mathrm{rowsp}(Y)) = \mathrm{rowsp}(Y \mid 0)$. Moreover,

$$\rho(\mathrm{rowsp}(Y \mid 0)) = \mathrm{rk}\left(\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}\begin{pmatrix} Y^{\mathsf{T}} \\ 0 \end{pmatrix}\right) = \mathrm{rk}(G_1 Y^{\mathsf{T}}) = \rho_1(\mathrm{rowsp}(Y)).$$

Thus ι_1 is rank-preserving. Similarly, ι_2 is rank-preserving. In order to show that $\iota_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ is a q-strong map, let us first consider the pre-images of a subspace $\mathrm{rowsp}(Y)$, where $Y \in \mathbb{F}^{y\times(n_1+n_2)}$. They can be computed as follows. There exist $U_i \in \mathrm{GL}_y(\mathbb{F})$ such that

$$U_1 Y = \begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}, \quad U_2 Y = \begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where the first block column consists of n_1 columns and the second one of n_2 columns, and where A_4 and B_3 have full row rank. Then $\iota_1^{-1}(\mathrm{rowsp}(Y)) = \mathrm{rowsp}(A_1)$ and $\iota_2^{-1}(\mathrm{rowsp}(Y)) = \mathrm{rowsp}(B_2)$. Suppose now that $\mathrm{rowsp}(Y) \in \mathcal{F}(\mathcal{M})$. By symmetry it suffices to show that $\mathrm{rowsp}(A_1) \in \mathcal{F}(\mathcal{M}_1)$. To this end let $v_1 \in \mathbb{F}^{n_1}$ such that $\rho_1(\mathrm{rowsp}(A_1) + \langle v_1 \rangle) = \rho_1(\mathrm{rowsp}(A_1))$. Setting $v = \iota_1(v_1) = (v_1, 0)$, we have

$$\begin{aligned} \rho_1(\mathrm{rowsp}(A_1) + \langle v_1 \rangle) &= \mathrm{rk}\big(G_1({A_1}^{\mathsf{T}}\ {v_1}^{\mathsf{T}})\big) \quad \text{and} \\ \rho(\mathrm{rowsp}(Y) + \langle v \rangle) &= \mathrm{rk}\begin{pmatrix} G_1{A_1}^{\mathsf{T}} & G_1{v_1}^{\mathsf{T}} & G_1{A_3}^{\mathsf{T}} \\ 0 & 0 & G_2{A_4}^{\mathsf{T}} \end{pmatrix}. \end{aligned}$$

We conclude that $\rho(\mathrm{rowsp}(Y) + \langle v \rangle) = \rho(\mathrm{rowsp}(Y))$ and hence $v \in \mathrm{rowsp}(Y)$ since the latter is a flat. But then $v_1 \in \mathrm{rowsp}(A_1)$, and this shows that $\mathrm{rowsp}(A_1)$ is a flat. This shows that ι_i are linear, injective, strong, and rank-preserving maps. Clearly, $\mathcal{N}|_{\mathbb{F}^{n_i}}$ is the image of ι_i, and thus ι_i induces an isomorphism between $\mathcal{M}_i$ and $\mathcal{N}|_{\mathbb{F}^{n_i}}$ in q-$\mathbf{Mat}^{\text{l-rp}}$. □

The next proposition shows for a special case that the q-matroid $\mathcal{N}$ of the last proposition depends on the representing matrices. This stands in contrast to the classical case, where the block diagonal matrix of every choice of representing matrices is a representing matrix of the direct sum; see [39, p. 126, Ex. 7].

Proposition 6.2.5. *Let $\mathbb{F}_{q^m}$ be a field extension of $\mathbb{F} = \mathbb{F}_q$ with primitive element ω and let $\Omega = \{1, \ldots, q^m - 2\} \setminus \{k(q^m - 1)/(q - 1) \mid k \in \mathbb{N}\}$ (hence $\omega^i \notin \mathbb{F}$ for $i \in \Omega$). For $i \in \Omega$ define*

$$G^{(i)} = \begin{pmatrix} 1 & \omega & 0 & 0 \\ 0 & 0 & 1 & \omega^i \end{pmatrix},$$

and let $\mathcal{N}^{(i)} = (\mathbb{F}^4, \rho^{(i)})$ be the q-matroid represented by $G^{(i)}$. Define the subspaces $T_1 = \langle 1000, 0100\rangle$ and $T_2 = \langle 0010, 0001\rangle$. Then for all $i \in \Omega$
(a) $\rho^{(i)}(T_1) = \rho^{(i)}(T_2) = 1$ and $\rho^{(i)}(\mathbb{F}^4) = 2$.
(b) $\rho^{(i)}(V) = 1$ for all 1-spaces V and $\rho^{(i)}(V) = 2$ for all 3-spaces V.
(c) Let $\mathcal{L}_2 = \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V = 2,\ T_1 \neq V \neq T_2\}$. Then

$$\rho^{(i)}(V) = 2 \text{ for all } V \in \mathcal{L}_2 \Longleftrightarrow 1, \omega, \omega^i, \omega^{i+1} \text{ are linearly independent over } \mathbb{F}.$$

(d) The flats of $\mathcal{N}^{(i)}$ are given by $\mathcal{F}(\mathcal{N}^{(i)}) = \{0, \mathbb{F}^4\} \cup \mathcal{F}_1^{(i)} \cup \mathcal{F}_2^{(i)}$, where

$$\left.\begin{aligned} \mathcal{F}_2^{(i)} &= \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V = 2,\ \rho^{(i)}(V) = 1\}, \\ \mathcal{F}_1^{(i)} &= \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V = 1, V \not\leq W \text{ for all } W \in \mathcal{F}_2^{(i)}\}. \end{aligned}\right\} \tag{6.4}$$

Thus for $m > 3$ there exist at least two non-isomorphic q-matroids of the form $\mathcal{N}^{(i)}$.

Proof. (a) Clearly $\rho^{(i)}(T_1) = \rho^{(i)}(T_2) = 1$ and $\rho^{(i)}(\mathbb{F}^4) = \mathrm{rk}(G) = 2$.
(b) By assumption on i the elements $1, \omega^i$ are linearly independent over $\mathbb{F}$. Thus, $G^{(i)}x \neq 0$ for any nonzero vector $x \in \mathbb{F}^4$ and hence $\rho^{(i)}(V) = 1$ for all 1-spaces V. Suppose there exists a 3-space V such that $\rho^{(i)}(V) = 1$. Clearly, V does not contain both T_1 and T_2. Let $T_1 \not\leq V$. Then $\dim(V \cap T_1) = 1$, and submodularity of $\rho^{(i)}$ implies $2 = \rho^{(i)}(\mathbb{F}^4) = \rho^{(i)}(V + T_1) \leq 1 + 1 - \rho^{(i)}(V \cap T_1) = 1$, which is a contradiction.
(c) Consider now an arbitrary 2-space $V = \langle (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3)\rangle$. Since $\rho^{(i)}(V)$ is the rank of the matrix

$$\begin{pmatrix} 1 & \omega & 0 & 0 \\ 0 & 0 & 1 & \omega^i \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} a_0 + a_1\omega & b_0 + b_1\omega \\ a_2 + a_3\omega^i & b_2 + b_3\omega^i \end{pmatrix}, \tag{6.5}$$

we conclude that $\rho^{(i)}(V) = 1$ if and only if its determinant is zero, thus

$$\rho^{(i)}(V) = 1 \Longleftrightarrow (a_0b_2 - a_2b_0) + (a_1b_2 - a_2b_1)\omega + (a_0b_3 - a_3b_0)\omega^i + (a_1b_3 - a_3b_1)\omega^{i+1} = 0. \tag{6.6}$$

Now we can prove the stated equivalence. "$\Leftarrow$" Suppose $1, \omega, \omega^i, \omega^{i+1}$ are linearly independent. Then $\rho^{(i)}(V) = 1$ if and only if all coefficients in (6.6) are zero. We

consider the following cases. (i) If $a_0 \neq 0 = b_0$, then $b_2 = b_3 = 0$. Thus $b_1 \neq 0$ since $\dim V = 2$, and subsequently $a_2 = a_3 = 0$. But then $V = T_1$. (ii) Suppose $a_0 = b_0 = 0$. If $a_1 = 0 = b_1$, then $V = T_2$. Thus assume without loss of generality that $a_1 \neq 0$. But then $b_2 = (b_1/a_1)a_2$ and $b_3 = (b_1/a_1)a_3$ and thus $\dim V = 1$, which is a contradiction. (iii) If $a_0 \neq 0 \neq b_0$, then without loss of generality $a_0 = b_0$ and as in (ii) we obtain $\dim V = 1$ or $V = T_1$.
"$\Rightarrow$" Suppose $1, \omega, \omega^i, \omega^{i+1}$ are linearly dependent, say $f_0 + f_1\omega + f_2\omega^i + f_3\omega^{i+1} = 0$ with $f_0, \ldots, f_3 \in \mathbb{F}$, not all zero. Using (6.5) one obtains $\rho(V) = 1$ for $V = \langle (1, 0, -f_1, -f_3), (0, 1, f_0, f_2) \rangle \in \mathcal{L}_2$.
(d) The statement about the flats is clear from the rank values, and the statement about non-isomorphic q-matroids $\mathcal{N}^{(i)}$ follows from (c) because $1, \omega, \omega^i, \omega^{i+1}$ are linearly dependent for $i = 1$ and linearly independent for $i = 2$ if $m > 3$. □

For later use we record the following fact.

Lemma 6.2.6. *Let $m \geq 4$ and the data be as in Proposition 6.2.5. Define $\mathcal{F}' = \cup_{i\in\Omega}\mathcal{F}(\mathcal{N}^{(i)})$. Then*

$$\mathcal{F}' = \{0, \mathbb{F}^4, T_1, T_2\} \cup \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V \leq 2, V \cap T_1 = 0 = V \cap T_2\}.$$

Proof. Recall the sets $\mathcal{F}_1^{(i)}, \mathcal{F}_2^{(i)}$ from (6.4). Note that $T_1, T_2 \in \mathcal{F}_2^{(i)}$ for all $i \in \Omega$. Denote the set on the right hand side of the stated identity by $\mathcal{F}''$. "$\subseteq$" Let $V \in \mathcal{F}'$. If $\dim V = 1$, then $V \in \mathcal{F}_1^{(i)}$ for some i and thus $V \not\leq T_\ell$ for $\ell = 1, 2$. Hence $V \in \mathcal{F}''$. Let now $\dim V = 2$ and thus $\rho^{(i)}(V) = 1$ for some i. The statement is clear for $V \in \{T_1, T_2\}$, and thus let $V \notin \{T_1, T_2\}$. Suppose $V \cap T_1 = \langle v \rangle$ for some $v \neq 0$. Then $\dim(V + T_1) = 3$ and thus $2 = \rho^{(i)}(V + T_1) \leq \rho^{(i)}(V) + \rho^{(i)}(T_1) - \rho^{(i)}(\langle v \rangle) = 1$, a contradiction. Thus, $V \in \mathcal{F}''$.
'$\supseteq$" It is clear that the spaces $0, \mathbb{F}^4, T_1, T_2$ are in $\mathcal{F}'$. We consider the 1-spaces and 2-spaces separately.
i) Let $V \in \mathcal{L}(\mathbb{F}^4)$ with $\dim V = 2$ and $V \cap T_1 = V \cap T_2 = 0$. Choosing the matrix $M \in \mathbb{F}^{2\times 4}$ in reduced row echelon form such that $V = \mathrm{rowsp}(M)$ we conclude that

$$M = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}, \quad \text{where } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0. \tag{6.7}$$

Now (6.6) reads as

$$\rho^{(i)}(V) = 1 \Longleftrightarrow c - a\omega + d\omega^i - b\omega^{i+1} = 0 \Longleftrightarrow \omega^i = \frac{a\omega - c}{d - b\omega}.$$

Note that the denominator is indeed nonzero thanks to the determinant condition in (6.7). Since the fraction is in $\mathbb{F}_{q^m}$ and ω is a primitive element, $(a\omega - c)/(d - b\omega) = \omega^i$ for some $i \in \{0, \ldots, q^m - 2\}$. Furthermore, the determinant condition in (6.7) implies that $(a\omega - c)/(d - b\omega) \notin \mathbb{F}$ and thus $i \in \Omega$. All of this shows that $\rho^{(i)}(V) = 1$ for some $i \in \Omega$ and thus $V \in \mathcal{F}_2^{(i)} \subseteq \mathcal{F}'$.
ii) Let $\dim V = 1$ and $V \not\leq T_\ell$ for $\ell = 1, 2$. Let W be a 2-space containing V. Then W is distinct from T_1 and T_2, and Proposition 6.2.5(c) for $i = 2$ implies $\rho^{(2)}(W) = 2$. Hence $V \in \mathcal{F}_1^{(2)} \subseteq \mathcal{F}'$, as desired. □

We now turn to coproducts in the above defined categories of q-matroids. Let us recall the definition.

Definition 6.2.7. *Let* $\mathbf{C}$ *be a category and* M_1, M_2 *be objects in* $\mathbf{C}$. *A coproduct of* M_1 *and* M_2 *in* $\mathbf{C}$ *is a triple* (M, ξ_1, ξ_2), *where* M *is an object in* $\mathbf{C}$ *and* $\xi_i : M_i \longrightarrow M$ *are morphisms, such that for all objects* N *and all morphisms* $\tau_i : M_i \longrightarrow N$ *there exists a unique morphism* $\epsilon : M \longrightarrow N$ *such that* $\epsilon \circ \xi_i = \tau_i$ *for* $i = 1, 2$.

It is well-known (and straightforward to verify) that if (M, ξ_1, ξ_2) is a coproduct of M_1 and M_2 and $\phi : M \longrightarrow \hat{M}$ is an isomorphism (i.e., a bijective morphism whose inverse is also a morphism), then $(\hat{M}, \phi \circ \xi_1, \phi \circ \xi_2)$ is also a coproduct of M_1 and M_2. Furthermore, every coproduct of M_1 and M_2 is of this form.

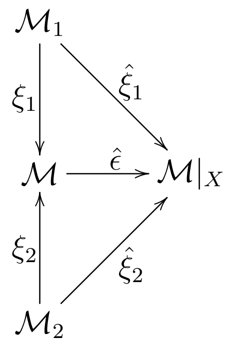
The rest of this section is devoted to the non-existence of coproducts in the first 5 categories of Definition 6.2.1. Our first result narrows down the ground space of a putative coproduct in an expected way. Furthermore, it shows that the accompanying morphisms are injective and can be chosen as linear maps. We introduce the following notation. Let $\mathcal{T} = \{\textsf{s}, \textsf{rp}, \textsf{w}, \textsf{l-s}, \textsf{l-rp}, \textsf{l-w}\}$ be the set of types of morphisms. For any $\Delta \in \mathcal{T}$ we denote by q-$\mathbf{Mat}^{\Delta}$ the corresponding category, and the morphisms in this category are called type-Δ maps. Note that the maps in Proposition 6.2.4 and Proposition 6.1.13(a) are type-Δ for each $\Delta \in \mathcal{T}$.

Theorem 6.2.8. *Let* $\Delta \in \mathcal{T}$. *Let* $\mathcal{M}_i$, $i = 1, 2$, *be representable q-matroids with ground spaces* E_i. *Suppose* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *have a coproduct* $(\mathcal{M}, \xi_1, \xi_2)$ *in* q-$\mathbf{Mat}^{\Delta}$. *Then they have a coproduct of the form* $(\tilde{\mathcal{M}}, \iota_1, \iota_2)$ *where* $\tilde{\mathcal{M}}$ *has ground space* $E_1 \oplus E_2$ *and* $\iota_i : E_i \longrightarrow E_1 \oplus E_2$ *are the natural embeddings as in* (6.2) *(hence linear).*

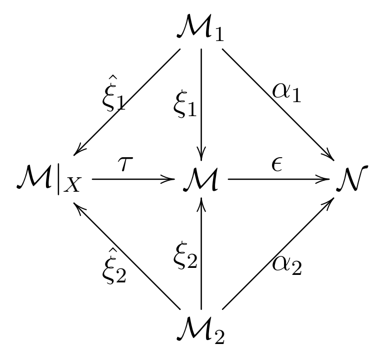
Proof. Suppose without loss of generality that $\mathcal{M}_i$ is a q-matroid on the ground space $\mathbb{F}^{n_i}$. Let $(\mathcal{M}, \xi_1, \xi_2)$ be a coproduct of $\mathcal{M}_1, \mathcal{M}_2$. We may also assume that the ground space of $\mathcal{M}$ is $\mathbb{F}^n$ for some n. Recall that ξ_i may not be semi-linear maps, but the images $\xi_i(\mathbb{F}^{n_i})$ are subspaces of $\mathbb{F}^n$. We proceed in several steps.

<u>Claim 1:</u> ξ_i is injective for $i = 1, 2$ and $\xi_1(\mathbb{F}^{n_1}) \cap \xi_2(\mathbb{F}^{n_2}) = \{0\}$.
Since each $\mathcal{M}_i$ is representable, we may apply Proposition 6.2.4 and obtain the existence of a q-matroid $\mathcal{N}$ on $\mathbb{F}^{n_1+n_2}$ and linear maps $\alpha_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ where $\alpha_1(v_1) = (v_1, 0)$ and $\alpha_2(v_2) = (0, v_2)$ for all $v_i \in \mathbb{F}^{n_i}$. Thanks to Proposition 6.2.4 the maps α_i are type-Δ. Hence the universal property of the coproduct implies the existence of a type-Δ map $\epsilon : \mathcal{M} \longrightarrow \mathcal{N}$ such that $\epsilon \circ \xi_i = \alpha_i$, $i = 1, 2$. Now injectivity of ξ_i follows from injectivity of α_i. Suppose $\xi_1(v_1) = \xi_2(v_2)$ for some $v_i \in \mathbb{F}^{n_i}$. Then $\alpha_1(v_1) = \alpha_2(v_2)$, which means $(v_1, 0) = (0, v_2)$. Thus $v_1 = 0$ and $v_2 = 0$. This implies $\xi_1(v_1) = \xi_2(v_2) = 0$, and the claim is proved.

<u>Claim 2:</u> $\mathbb{F}^n = \xi_1(\mathbb{F}^{n_1}) \oplus \xi_2(\mathbb{F}^{n_2})$ and thus $n = n_1 + n_2$.
Set $X_i = \xi_i(\mathbb{F}^{n_i})$, which is a subspace of $\mathbb{F}^n$, and $X = X_1 \oplus X_2$. Consider the restriction $\mathcal{M}|_X$. The maps $\hat{\xi}_i : \mathcal{M}_i \longrightarrow \mathcal{M}|_X$, $v \longrightarrow \xi_i(v)$, are clearly type-$\Delta$ as well; see Proposition 6.1.14. We want to show that $(\mathcal{M}|_X, \hat{\xi}_1, \hat{\xi}_2)$ is a coproduct as well. To do so, consider first the diagram



Since $\mathcal{M}$ is a coproduct, there is a unique type-Δ map $\hat{\epsilon}$ satisfying $\hat{\epsilon} \circ \xi_i = \hat{\xi}_i$ for $i = 1, 2$. Define $\tau : \mathcal{M}|_X \longrightarrow \mathcal{M}$ via $x \longrightarrow x$. From Proposition 6.1.13(a) we know that τ is type-Δ. Consider the map $\hat{\epsilon} \circ \tau : \mathcal{M}|_X \longrightarrow \mathcal{M}|_X$. It satisfies $\hat{\epsilon} \circ \tau|_{X_i} = \mathrm{id}_{X_i}$ for $i = 1, 2$. Hence Corollary 6.1.7 implies that $\hat{\epsilon} \circ \tau : X \longrightarrow X$ is a bijective map (and equivalent to a semi-linear isomorphism).
Let now $\mathcal{N}$ be any q-matroid and $\alpha_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ be type-Δ maps. Then there exists a type-Δ map $\epsilon : \mathcal{M} \longrightarrow \mathcal{N}$ resulting in the commutative diagram



Hence the map $\gamma := \epsilon \circ \tau : \mathcal{M}|_X \longrightarrow \mathcal{N}$ is type-Δ and satisfies $\gamma \circ \hat{\xi}_i = \alpha_i$. It remains to show the uniqueness of γ. Suppose there is also a type-Δ map $\delta : \mathcal{M}|_X \longrightarrow \mathcal{N}$ such that $\delta \circ \hat{\xi}_i = \alpha_i$ for $i = 1, 2$. Set $\tilde{\gamma} = \gamma \circ \hat{\epsilon} \circ \tau$ and $\tilde{\delta} = \delta \circ \hat{\epsilon} \circ \tau$. Now we have $\gamma \circ \hat{\epsilon} \circ \xi_i = \gamma \circ \hat{\xi}_i = \alpha_i$ and $\delta \circ \hat{\epsilon} \circ \xi_i = \delta \circ \hat{\xi}_i = \alpha_i$. Since $\mathcal{M}$ is a coproduct, we conclude that $\gamma \circ \hat{\epsilon} = \delta \circ \hat{\epsilon}$. This implies $\tilde{\gamma} = \tilde{\delta}$ and since $\hat{\epsilon} \circ \tau$ is a bijection, this in turn yields $\delta = \gamma = \epsilon \circ \tau$. All of this shows that $(\mathcal{M}|_X, \hat{\xi}_1, \hat{\xi}_2)$ is a coproduct. Hence $\mathcal{M}$ and $\mathcal{M}|_X$ are isomorphic in q-$\mathbf{Mat}^{\Delta}$, and this means $X = \mathbb{F}^n$.

Claim 3: $\mathcal{M}_1$, $\mathcal{M}_2$ have a coproduct of the form $(\tilde{\mathcal{M}}, \iota_1, \iota_2)$, where $\tilde{\mathcal{M}}$ has ground space $\mathbb{F}^{n_1+n_2}$ and $\iota_i : \mathcal{M}_i \longrightarrow \tilde{\mathcal{M}}$ are as in (6.3).
We show first that there exists an $\mathcal{L}$-isomorphism $\beta : X \longrightarrow \mathbb{F}^{n_1+n_2}$ such that $\beta \circ \hat{\xi}_i = \iota_i$, $i = 1, 2$. In a second step this map will be turned into the desired type-Δ isomorphism between $\mathcal{M}|_X$ and the new coproduct $\tilde{\mathcal{M}}$. To show the existence of β, use again the construction in Proposition 6.2.4: consider any q-matroid $\mathcal{N}$ on $\mathbb{F}^{n_1+n_2}$ and the type-Δ maps $\iota_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ with ι_i as in (6.3). Since $\mathcal{M}|_X$ is a coproduct, there exists a type-Δ map $\beta : \mathcal{M}|_X \longrightarrow \mathcal{N}$ such that $\beta \circ \hat{\xi}_i = \iota_i$, $i = 1, 2$. Hence $e_1, \ldots, e_n$ are in the image of β and thus Corollary 6.1.7 implies that β is bijective. All of this provides us with the desired $\mathcal{L}$-isomorphism $\beta : X \longrightarrow \mathbb{F}^{n_1+n_2}$.

Note that β is linear if $\Delta \in \{$l-s, l-rp, l-w$\}$. Now we use Remark 6.2.3 to define a new q-matroid structure on $\mathbb{F}^{n_1+n_2}$. Set $\tilde{\mathcal{M}} = (\mathbb{F}^{n_1+n_2}, \tilde{\rho})$ via $\tilde{\rho}(V) := \rho(\beta^{-1}(V))$ with ρ being the rank function of $\mathcal{M}|_X$. This makes sense because the inverse of an $\mathcal{L}$-isomorphism is again an $\mathcal{L}$-map. Thanks to Remark 6.2.3 the flats of $\tilde{\mathcal{M}}$ are given by $\tilde{\mathcal{F}} := \{\beta(F) \mid F \in \mathcal{F}(\mathcal{M}|_X)\}$. This turns β into an isomorphism in q-$\mathbf{Mat}^{\Delta}$ from $\mathcal{M}|_X$ to $\tilde{\mathcal{M}}$, and the maps $\iota_i = \beta \circ \hat{\xi}_i : \mathcal{M}_i \longrightarrow \tilde{\mathcal{M}}$ are type-Δ. Thus $(\tilde{\mathcal{M}}, \beta \circ \hat{\xi}_1, \beta \circ \hat{\xi}_2) = (\tilde{\mathcal{M}}, \iota_1, \iota_2)$ is a coproduct, as desired. □

Now we are ready to show the non-existence of a coproduct in the following linear cases.

Theorem 6.2.9. *Let $q \geq 2$ and $\Delta \in \{$l-s, l-rp$\}$. There exist representable q-matroids that do not have a coproduct in q-$\mathbf{Mat}^{\Delta}$.*

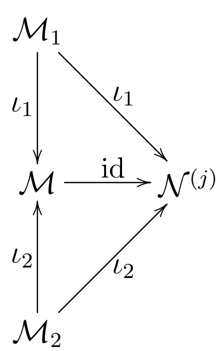
Proof. Let $\mathbb{F} = \mathbb{F}_q$ and $\mathcal{M}_1 = \mathcal{U}_1(\mathbb{F}^2) = \mathcal{M}_2$, that is $\mathcal{M}_1$ and $\mathcal{M}_2$ are the uniform q-matroids on $\mathbb{F}^2$ with rank 1. Their collections of flats are

$$\mathcal{F}(\mathcal{M}_1) = \mathcal{F}(\mathcal{M}_2) = \{\{0\}, \mathbb{F}^2\}. \tag{6.8}$$

Assume by contradiction that $\mathcal{M}_1$ and $\mathcal{M}_2$ have a coproduct. From Theorem 6.2.8 we know that it is without loss of generality of the form $(\mathcal{M}, \iota_1, \iota_2)$, where $\mathcal{M}$ has ground space $\mathbb{F}^4$ and ι_i are as in (6.3). We will show that such $\mathcal{M}$ does not exist. To do so, we construct various q-matroids $\mathcal{N}^{(j)}$ along with type-Δ maps $\tau_i : \mathcal{M}_i \longrightarrow \mathcal{N}^{(j)}$, $i = 1, 2$. Consider the construction in Proposition 6.2.5 for $m = 4$. Let $\omega \in \mathbb{F}_{q^4}$ be a primitive element and let Ω and $G^{(j)}$, $j \in \Omega$, be as in that proposition. For every $j \in \Omega$ let $\mathcal{N}^{(j)} = (\mathbb{F}^4, \rho^{(j)})$ be the associated q-matroid, thus $\rho^{(j)}(\text{rowsp}(Y)) = \text{rk}(G^{(j)} Y^{\mathsf{T}})$ for $Y \in \mathbb{F}^{y \times 4}$. Note that the uniform q-matroids $\mathcal{M}_1 = \mathcal{M}_2$ are represented by every matrix $(1\ \omega^j) \in \mathbb{F}_{q^4}^{1\times 2}, j \in \Omega$. Therefore Proposition 6.2.4 tells us that for every $j \in \Omega$ the maps $\iota_i : \mathcal{M}_i \longrightarrow \mathcal{N}^{(j)}$, $i = 1, 2$, are type-Δ for either Δ under consideration. Since $(\mathcal{M}, \iota_1, \iota_2)$ is a coproduct in q-$\mathbf{Mat}^{\Delta}$, this implies

$$\text{for all } j \in \Omega \text{ there exists a type-}\Delta \text{ map } \epsilon_j : \mathcal{M} \longrightarrow \mathcal{N}^{(j)} \text{ such that } \epsilon_j \circ \iota_i = \iota_i \text{ for } i = 1, 2. \tag{6.9}$$

Hence $\epsilon_j(v_1, 0) = (v_1, 0)$ and $\epsilon_j(0, v_2) = (0, v_2)$ for all $v_1, v_2 \in \mathbb{F}^2$. Then linearity of ϵ_j implies $\epsilon_j = \text{id}_{\mathbb{F}^4}$ for all $j \in \Omega$ and we arrive at the commutative diagrams



(6.10)

where id : $\mathcal{M} \longrightarrow \mathcal{N}^{(j)}$ is type-Δ.

a) For $\Delta =$ l-rp this implies that $\mathcal{M}$ is isomorphic to each $\mathcal{N}^{(j)}$, which contradicts

Proposition 6.2.5(d). Hence $\mathcal{M}_1$ and $\mathcal{M}_2$ do not have a coproduct in q-$\mathbf{Mat}^{\text{l-rp}}$.
b) Let $\Delta = \text{l-s}$. Since $\mathrm{id} : \mathcal{M} \longrightarrow \mathcal{N}^{(j)}$ is a q-strong map for all $j \in \Omega$, we conclude that the elements of

$$\mathcal{F}' := \bigcup_{j\in\Omega} \mathcal{F}(\mathcal{N}^{(j)}).$$

have to be flats of $\mathcal{M}$. From Lemma 6.2.6 we know that

$$\mathcal{F}' = \{0, \mathbb{F}^4, T_1, T_2\} \cup \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V \leq 2, V \cap T_1 = 0 = V \cap T_2\}, \tag{6.11}$$

where $T_1 = \langle 1000,\ 0100\rangle$ and $T_2 = \langle 0010,\ 0001\rangle$. Let $\mathcal{F} := \mathcal{F}(\mathcal{M})$. Then $\mathcal{F}$ satisfies (F1)–(F3) from Theorem 2.2.4. This implies in particular that (F3) has to be true for the flats in $\mathcal{F}'$. To investigate this further we define for $V \in \mathcal{F}'$

$$\mathrm{Cov}_{\mathcal{F}'}(V) = \{F \in \mathcal{F}' \mid V < F \text{ and there is no } Z \in \mathcal{F}' \text{ such that } V < Z < F\}.$$

We call the elements of $\mathrm{Cov}_{\mathcal{F}'}(V)$ covers of V in $\mathcal{F}'$. Of course, the covers of V in $\mathcal{F}'$ need not be covers of V in the lattice $\mathcal{F}$. Using (6.11) we can determine the covers in $\mathcal{F}'$ explicitly. For ease of notation set

$$\mathcal{F}'_1 = \{V \in \mathcal{F}' \mid \dim V = 1\} = \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V = 1,\ V \not\subseteq T_1 \cup T_2\},$$
$$\mathcal{F}'_2 = \{V \in \mathcal{F}' \mid \dim V = 2,\ T_1 \neq V \neq T_2\} = \{V \in \mathcal{L}(\mathbb{F}^4) \mid \dim V = 2,\ V \cap T_1 = 0 = V \cap T_2\}.$$

Note that

$$\mathcal{F}'_2 = \left\{ \mathrm{rowsp}\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{F},\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right\}.$$

Then

$$\mathrm{Cov}_{\mathcal{F}'}(V) = \begin{cases} \{T_1, T_2\} \cup \mathcal{F}'_1 & \text{if } V = 0, \\ \{W \in \mathcal{F}'_2 \mid V \leq W\} & \text{if } V \in \mathcal{F}'_1, \\ \{\mathbb{F}^4\} & \text{if } V \in \mathcal{F}'_2 \cup \{T_1, T_2\}. \end{cases} \tag{6.12}$$

Note that for $V \in \mathcal{F}'_1$ the set $\{W \in \mathcal{F}'_2 \mid V \leq W\}$ is indeed nonempty. Choose now $V = \langle (1,1,1,0)\rangle \in \mathcal{F}'_1$, and $v = (1,0,0,0)$. By property (F3) the space $V + \langle v\rangle$ has to be in a unique cover, say $\hat{F}$, of V in $\mathcal{F}$. Since $v \in T_1$, it is clear that $\hat{F} \notin \mathrm{Cov}_{\mathcal{F}'}(V)$. Note that $\iota_1^{-1}(\hat{F})$ contains $\iota_1^{-1}(v) = (1,0)$, and therefore $\iota_1^{-1}(\hat{F}) = \mathbb{F}^2$ thanks to (6.8) since ι_1 is a q-strong map. Hence $\hat{F}$ contains the subspace $\langle (1,1,1,0),\ (1,0,0,0), (0,1,0,0)\rangle = \langle (0,0,1,0),\ (1,0,0,0), (0,1,0,0)\rangle$. Using now ι_2 we conclude that $\hat{F} = \mathbb{F}^4$. But now the gradedness of the lattice $\mathcal{F}$ (see Theorem 2.2.4 together with $\mathcal{F}' \subseteq \mathcal{F}$ and (6.12) shows that $\mathbb{F}^4$ is not a cover of V in $\mathcal{F}$. Hence we arrive at a contradiction and thus there is no coproduct of $\mathcal{M}_1$ and $\mathcal{M}_2$ in q-$\mathbf{Mat}^{\text{l-s}}$ □

With $\Delta = \text{l-w}$ being discussed in the next section, it remains to consider the nonlinear cases.

Theorem 6.2.10. *Let $q \geq 2$ and $\Delta \in \{\text{w, s, rp}\}$. There exist representable q-matroids that do not have a coproduct in q-$\mathbf{Mat}^{\Delta}$.*

Proof. Let $\mathbb{F} = \mathbb{F}_q$. As in the proof of Theorem 6.2.9 let $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{U}_1(\mathbb{F}^2)$. Consider the uniform q-matroid $\mathcal{N} = \mathcal{U}_3(\mathbb{F}^3)$. Denote the rank functions of these q-matroids by $\rho_{\mathcal{M}_1}, \rho_{\mathcal{M}_2}$, and $\rho_{\mathcal{N}}$. For any nonzero vector $v \in \mathbb{F}^2$ let $\lambda_v \in \mathbb{F}^*$ be its leftmost nonzero entry. For $i = 1, 2$ define $\alpha_i : \mathbb{F}^2 \longrightarrow \mathbb{F}^3$ via $\alpha_i(0) = 0$ and

$$\alpha_1(v) = \lambda_v e_1 \text{ and } \alpha_2(v) = \lambda_v e_2 \text{ for all } v \in \mathbb{F}^2 \setminus 0,$$

where e_1, e_2 are the first two standard basis vectors in $\mathbb{F}^3$. Then α_1, α_2 are $\mathcal{L}$-maps. In fact $\alpha_i(V) = \langle e_i \rangle$ if $V \neq 0$ and $\alpha_i(0) = 0$. We claim that $\alpha_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ is type-Δ for each $\Delta \in \{\mathsf{w}, \mathsf{s}, \mathsf{rp}\}$. Indeed, every nonzero subspace $V \in \mathcal{L}(\mathbb{F}^2)$ satisfies $\rho_{\mathcal{M}_i}(V) = 1 = \rho_{\mathcal{N}}(\alpha_i(V))$ and hence α_i is rank-preserving (and thus weak). Furthermore, note that every subspace $V \leq \mathbb{F}^3$ is a flat of $\mathcal{N}$ and satisfies $\alpha_i^{-1}(V) \in \{0, \mathbb{F}^2\} = \mathcal{F}(\mathcal{M}_i)$ (with $\alpha_i^{-1}(V) = 0$ iff $e_i \notin V$). Thus α_i is strong.
Now we are ready to show that $\mathcal{M}_1$ and $\mathcal{M}_2$ do not have a coproduct in q-$\mathbf{Mat}^\Delta$. Recall from Theorem 6.2.8 that if they do have a coproduct then they have one of the form $(\mathcal{M}, \iota_1, \iota_2)$ with ground space $\mathbb{F}^4$ and ι_i as in (6.3). We will establish the non-existence of such a coproduct by showing that there is no $\mathcal{L}$-map $\epsilon : \mathbb{F}^4 \longrightarrow \mathbb{F}^3$ such that $\epsilon \circ \iota_i = \alpha_i$ for $i = 1, 2$.
Assume that there does exist an $\mathcal{L}$-map $\epsilon : \mathbb{F}^4 \longrightarrow \mathbb{F}^3$ such that $\epsilon \circ \iota_i = \alpha_i$ for $i = 1, 2$. Consider the subspaces $V = \langle (1,0,0,0), (0,0,1,0) \rangle$ and $W = \langle (1,0,0,0), (0,0,0,1) \rangle$ of $\mathbb{F}^4$. Then $\epsilon(V)$ and $\epsilon(W)$ are subspaces of $\mathbb{F}^3$ of cardinality at most q^2 and contain $\langle e_1, e_2 \rangle$. Hence $\epsilon(W) = \epsilon(V) = \langle e_1, e_2 \rangle$, and $\epsilon|_V$ and $\epsilon|_W$ are injective. Since $e_1 + e_2 \in \epsilon(V) = \epsilon(W)$, there exist vectors $v \in V$ and $w \in W$ such that $\epsilon(v) = \epsilon(w) = e_1 + e_2$. These vectors must be of the form $v = \lambda(1, 0, \mu, 0)$ and $w = \lambda'(1, 0, 0, \mu')$ for some $\lambda, \lambda', \mu, \mu' \in \mathbb{F}^*$. Set $U = \langle (1, 0, \mu, 0), (1, 0, 0, \mu') \rangle$. Then $\epsilon(0, 0, \mu, -\mu') = \alpha_2(\mu, -\mu') = \mu e_2 \in \epsilon(U)$, and since $\epsilon(U)$ is a subspace of cardinality at most q^2 which also contains $e_1 + e_2$, we conclude that $\epsilon(U) = \langle e_1, e_2 \rangle$. Thus $\epsilon|_U$ is injective. But this contradicts $\epsilon(v) = \epsilon(w) = e_1 + e_2$. All of this shows that there is no $\mathcal{L}$-map ϵ with the desired properties and thus $\mathcal{M}_1$ and $\mathcal{M}_2$ do not have a coproduct in q-$\mathbf{Mat}^\Delta$. □

6.3 A Coproduct in q-$\mathbf{Mat}^{\mathsf{l\text{-}w}}$.

In this section we establish the existence of a coproduct in the category q-$\mathbf{Mat}^{\mathsf{l\text{-}w}}$. In fact, we will show that the direct sum is such a coproduct.

We first show that the q-matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are naturally embedded in the direct via rank-preserving isomorphisms.

Theorem 6.3.1 ([13, Sec. 7])**.** *Let the data be as in Theorem 5.1.2 and in particular $E = E_1 \oplus E_2$. Let ι_i be as in* (6.2)*. Let $V \in \mathcal{L}(E_i)$. Then*

$$\rho'_i(\iota_i(V)) = \rho_i(V) = \rho(\iota_i(V)) \quad \textit{and} \quad \rho'_j(\iota_i(V)) = 0 \textit{ for } j \neq i.$$

As a consequence, ι_i induces linear rank-preserving isomorphisms between $\mathcal{M}_i$ and $\mathcal{M}|_{\iota_i(E_i)}$ and $(\mathcal{M}'_i)|_{\iota_i(E_i)}$.

Proof. $\rho'_i(\iota_i(V)) = \rho_i(V)$ follows from $\pi_i(\iota_i(V)) = V$ and $\rho'_j(\iota_i(V)) = 0$ is clear because $\pi_j(\iota_i(V)) = 0$. We show next the identity $\rho'_i(\iota_i(V)) = \rho(\iota_i(V))$ for $i = 1$.

Choose $X \leq \iota_1(V)$ and write $\iota_1(V) = X \oplus Y$. Using that ρ_1' is a rank function, we obtain $\rho_1'(\iota_1(V)) \leq \rho_1'(X) + \rho_1'(Y) \leq \rho_1'(X) + \dim Y = \rho_1'(X) + \dim \iota_1(V) - \dim X = \rho_1'(X) + \rho_2'(X) + \dim \iota_1(V) - \dim X$, where the last step follows from the fact that $X \in \iota_1(E_1)$, hence $\rho_2'(X) = 0$. All of this shows that the minimum in (5.2) is attained by $\rho_1'(\iota_1(V))$, and therefore $\rho_1'(\iota_1(V)) = \rho(\iota_1(V))$. The consequence is clear. □

Thanks to the above we may and will from now on identify subspaces V in E_i with their image $\iota_i(V)$.

We now turn to our main result stating that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a coproduct in q-$\mathbf{Mat}^{\text{l-w}}$. We need the following lemma.

Lemma 6.3.2. *Let $\mathcal{M}_i = (E_i, \rho_i), i = 1, 2$, be q-matroids and $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a linear map. Suppose ϕ is not a q-weak map. Then there exists a circuit C of $\mathcal{M}_1$ such that*

$$\rho_2(\phi(C)) > \rho_1(C), \quad \dim C = \dim \phi(C), \quad \phi(C) \text{ is an independent space in } \mathcal{M}_2. \tag{6.13}$$

Proof. Since ϕ is not weak there exists an inclusion-minimal subspace $V \in \mathcal{L}(E_1)$ such that $\rho_2(\phi(V)) > \rho_1(V)$. Clearly $V \neq 0$. We will show that V is the desired circuit and proceed in several steps.
1) We first establish the following identities

$$\rho_1(W) = \rho_1(V) = \rho_2(\phi(W)) = \rho_2(\phi(V)) - 1 \text{ for all } W \leq V \text{ with } \dim W = \dim V - 1. \tag{6.14}$$

Let $W \leq V$ with $\dim W = \dim V - 1$. Write $V = W \oplus X$, thus X is a 1-space. Then $\phi(V) = \phi(W) + \phi(X)$ and therefore $\dim \phi(V) \leq \dim \phi(W) + 1$. Furthermore, by minimality of V we have $\rho_2(\phi(W)) \leq \rho_1(W)$. Using the properties of rank functions, we obtain

$$\rho_2(\phi(W)) \leq \rho_1(W) \leq \rho_1(V) < \rho_2(\phi(V)), \tag{6.15}$$

and thus $\phi(W) \lneq \phi(V)$, which means $\dim \phi(V) = \dim \phi(W) + 1$, and in fact $\phi(V) = \phi(W) \oplus \phi(X)$. This implies $\rho_2(\phi(V)) \leq \rho_2(\phi(W)) + 1$. Together with (6.15) this yields $\rho_2(\phi(V)) = \rho_2(\phi(W)) + 1$ as well as $\rho_2(\phi(W)) = \rho_1(W) = \rho_1(V)$. This establishes (6.14).
2) We show that $\phi|_V$ is injective. Assume to the contrary that there exists $v \in V \setminus 0$ such that $\phi(v) = 0$. Then $V = W \oplus \langle v \rangle$ for some subspace W of V and thus $\phi(V) = \phi(W)$. But this contradicts (6.14). Hence $\phi|_V$ is injective and $\dim V = \dim \phi(V)$.
3) We show that $\phi(V)$ is independent in $\mathcal{M}_2$. Assume to the contrary that $\rho_2(\phi(V)) < \dim \phi(V)$. Then there exists an independent subspace $I \lneq \phi(V)$ such that $\rho_2(\phi(V)) = \rho_2(I) = \dim I$. Since $\phi|_V$ is injective, there exists a subspace $J \lneq V$ such that $\phi(J) = I$ and $\dim J = \dim I$. Now we have $\dim J = \dim I = \rho_2(I) \leq \rho_1(J) \leq \dim J$, where the first inequality follows from the minimality of V subject to $\rho_2(\phi(V)) > \rho_1(V)$. Thus we have equality throughout, which shows that J is independent in $\mathcal{M}_1$. Furthermore, since $J \lneq V$ there exists a subspace $W \lneq V$ with $\dim W = \dim V - 1$ such that $J \leq W \lneq V$. Applying ϕ we arrive at $\rho_2(I) \leq \rho_2(\phi(W)) \leq \rho_2(\phi(V)) = \rho_2(I)$, and we have equality throughout. But this contradicts (6.14) and therefore $\phi(V)$ is

independent in $\mathcal{M}_2$.
4) It remains to show that V is a circuit. (6.14) together with 2) and 3) implies that for every hyperplane W of V we have $\rho_1(W) = \rho_1(V) = \rho_2(\phi(V)) - 1 = \dim \phi(V) - 1 = \dim V - 1 = \dim W$. This shows that V is dependent and W is independent in $\mathcal{M}_1$. Since W was an arbitrary hyperplane of V, we conclude that V is a circuit. □

Theorem 6.3.3. *Let $\mathcal{M}_i = (E_i, \rho_i)$, $i = 1, 2$, be q-matroids and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 = (E, \rho)$ be the direct sum as defined in Theorem 5.1.2. Let ι_i be as in* (6.2). *Then $(\mathcal{M}, \iota_1, \iota_2)$ is a coproduct of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the category q-$\mathbf{Mat}^{l\text{-}w}$.*

Proof. First of all, thanks to Theorem 6.3.1 the maps $\iota_i : \mathcal{M}_i \longrightarrow \mathcal{M}$ are rank-preserving, thus weak. Let $\mathcal{N} = (\tilde{E}, \tilde{\rho})$ be a q-matroid and $\alpha_i : \mathcal{M}_i \longrightarrow \mathcal{N}$ be linear q-weak maps. We have to show the existence of a unique linear q-weak map $\epsilon : \mathcal{M} \longrightarrow \mathcal{N}$ such that $\epsilon \circ \iota_i = \alpha_i$ for $i = 1, 2$. Since $\mathcal{M}$ has ground space $E_1 \oplus E_2$ it is clear that the only linear map satisfying $\epsilon \circ \iota_i = \alpha_i$ is given by $\epsilon(v_1 + v_2) = \alpha_1(v_1) + \alpha_2(v_2)$ for all $v_i \in E_i$; recall that we identify E_i with $\iota_i(E_i)$. Thus it remains to show that this map ϵ is weak. We will use Lemma 6.3.2. Choose any circuit C in $\mathcal{M}$. Denote by ρ_i' and π_i the maps as in Theorem 5.1.2 and set $X_i = \pi_i(C)$. Then $C \leq X_1 \oplus X_2$ and, since $X_i \leq E_i$, we obtain $\epsilon(C) \leq \epsilon(X_1 \oplus X_2) = \epsilon(X_1) + \epsilon(X_2) = \alpha_1(X_1) + \alpha_2(X_2)$. Applying the rank function $\tilde{\rho}$ and using the weakness of the maps α_i, we compute

$$\begin{aligned}\tilde{\rho}(\epsilon(C)) &\leq \tilde{\rho}(\alpha_1(X_1) + \alpha_2(X_2)) \leq \tilde{\rho}(\alpha_1(X_1)) + \tilde{\rho}(\alpha_2(X_2)) \\ &\leq \rho_1(X_1) + \rho_2(X_2) = \rho_1'(C) + \rho_2'(C) \leq \dim C - 1,\end{aligned}$$

where the last step follows from Corollary 5.1.4 because C is a circuit. This shows that $\epsilon(C)$ is not an independent space of $\mathcal{N}$ with the same dimension as C. Thus, no circuit in $\mathcal{M}$ satisfies (6.13), and this shows that ϵ is weak. □

Example 6.3.4. *Let $\mathbb{F} = \mathbb{F}_q$ and consider the q-matroids $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{U}_1(\mathbb{F}^2)$, i.e., $\rho_1(V) = \rho_2(V) = \min\{1, \dim V\}$ for all $V \in \mathcal{L}(\mathbb{F}^2)$. The direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ has been determined in [13, Ex. 48] using the definition of its rank function in* (5.2). *In this example we will derive the same result by making use of the fact that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a coproduct in q-$\mathbf{Mat}^{l\text{-}w}$. Let ω be a primitive element of $\mathbb{F}_{q^4}$. Then $G = (1\ \omega^2) \in \mathbb{F}_{q^4}^{1\times 2}$ represents $\mathcal{M}_1 = \mathcal{M}_2$. Consider $G^{(2)}$ as in Proposition 6.2.5 and let $\mathcal{N}^{(2)} = (\mathbb{F}^4, \rho^{(2)})$ be the q-matroid generated by $G^{(2)}$. Thanks to Proposition 6.2.5 we have*

$$\rho^{(2)}(V) = \min\{2, \dim V\} \text{ for } V \in \mathcal{L}(\mathbb{F}^4) \setminus \{T_1, T_2\} \text{ and } \rho^{(2)}(T_1) = \rho^{(2)}(T_2) = 1,$$

where T_1, T_2 are as in Proposition 6.2.5. Furthermore, Proposition 6.2.4 provides us with the linear rank-preserving, hence weak, maps $\iota_i : \mathcal{M}_i \longrightarrow \mathcal{N}^{(2)}$ for $i = 1, 2$. As a consequence, the map $\epsilon : \mathcal{M}_1 \oplus \mathcal{M}_2 \longrightarrow \mathcal{N}^{(2)}$ from the previous proof is the identity map on $\mathbb{F}^4$. It thus induces a q-weak map from $\mathcal{M}_1 \oplus \mathcal{M}_2$ to $\mathcal{N}^{(2)}$, and this means that the rank function ρ of $\mathcal{M}_1 \oplus \mathcal{M}_2$ satisfies

$$\rho(V) \geq \min\{2, \dim V\} \text{ for } V \in \mathcal{L}(\mathbb{F}^4) \setminus \{T_1, T_2\} \text{ and } \rho(T_1) \geq 1,\ \rho(T_2) \geq 1.$$

Using that ι_i are also rank-preserving maps from $\mathcal{M}_i$ to $\mathcal{M}_1 \oplus \mathcal{M}_2$, see Theorem 6.3.1, we obtain that $\rho(T_1) = \rho(T_1) = 1$ and $\rho(\mathbb{F}^4) = \rho(\iota_1(\mathbb{F}^2) \oplus \iota_2(\mathbb{F}^2)) \leq \rho(\iota_1(\mathbb{F}^2)) + \rho(\iota_2(\mathbb{F}^2)) = \rho_1(\mathbb{F}^2) + \rho_2(\mathbb{F}^2) = 2$. This implies that $\rho = \rho^{(2)}$ and thus $\mathcal{N}^{(2)} = \mathcal{M}_1 \oplus \mathcal{M}_2$.

We close this section with the following characterization of the direct sum.

Remark 6.3.5. *The fact that* $\mathcal{M}_1 \oplus \mathcal{M}_2$ *is a coproduct in* q-$\mathbf{Mat}^{\textsf{l-w}}$ *can be translated into the following characterization of the direct sum. For any finite-dimensional* $\mathbb{F}$-*vector space* E *define the set* $\mathcal{S}_E = \{\rho \mid \rho \text{ is a rank function on } E\}$ *and the partial order* $\rho \leq \rho' :\Longleftrightarrow \rho(V) \leq \rho'(V)$ *for all* $V \in \mathcal{L}(E)$*. Let now* $\mathcal{M}_i = (E_i, \rho_i)$*,* $i = 1, 2$*, be q-matroids and set* $E = E_1 \oplus E_2$*. Then the set* $\hat{\mathcal{S}} = \{\rho \in \mathcal{S}_E \mid \rho|_{E_i} \leq \rho_i\}$ *has a unique maximum, say* $\hat{\rho}$*, and* $\mathcal{M}_1 \oplus \mathcal{M}_2 = (E, \hat{\rho})$*.*

6.4 q-Matroids with $\mathcal{L}$-Classes.

In this short section we discuss a different approach to maps between q-matroids. Since the q-matroid structure is based on subspaces and different $\mathcal{L}$-maps may induce the same map on subspaces (i.e., are $\mathcal{L}$-equivalent), one may define maps between q-matroids as maps between subspace lattices induced by $\mathcal{L}$-maps. Precisely, for an $\mathcal{L}$-map $\phi : E_1 \longrightarrow E_2$ define the $\mathcal{L}$-*class* as $[\phi] = \{\psi : E_1 \longrightarrow E_2 \mid \psi \sim_{\mathcal{L}} \phi\}$. Then

$$[\phi] : \mathcal{L}(E_1) \longrightarrow \mathcal{L}(E_2),\ V \longmapsto \phi(V), \tag{6.16}$$

is well-defined (and equals $\phi_{\mathcal{L}}$). Since the type of a map (see Definition 6.1.10) is invariant under $\mathcal{L}$-equivalence, this gives rise to strong, weak, and rank-preserving $\mathcal{L}$-classes between q-matroids. Furthermore, we call an $\mathcal{L}$-class *linear*, if it contains a linear $\mathcal{L}$-map (not all maps in a linear $\mathcal{L}$-class are linear since by Proposition 6.1.9 we may tweak a linear map into an $\mathcal{L}$-equivalent nonlinear $\mathcal{L}$-map). Setting $[\phi_1] \circ [\phi_2] := [\phi_1 \circ \phi_2]$, which is indeed well-defined, we obtain categories

$$q\text{-}\mathbf{Mat}^{[\Delta]} \text{ for } \Delta \in \{\textsf{s, rp, w, l-s, l-rp, l-w}\},$$

in which the morphisms are $\mathcal{L}$-classes of the specified type. Being isomorphic or linearly isomorphic in the sense of Definition 6.2.2 does not change when moving to $\mathcal{L}$-classes.

It turns out that $\mathcal{L}$-classes are not better behaved than $\mathcal{L}$-maps. The following result suggests that in fact $\mathcal{L}$-maps, rather than $\mathcal{L}$-classes, are the appropriate notion of maps between q-matroids.

Theorem 6.4.1. *None of the categories* q-$\mathbf{Mat}^{[\Delta]}$*,* $\Delta \in \{\textsf{s, rp, w, l-s, l-rp, l-w}\}$*, has coproduct except for the case* $(q, [\Delta]) = (2, [\textsf{l-w}])$*.*

The proof is for the most part straightforward, but tedious. We provide a sketch.

Sketch of Proof. In essence one follows the proofs of Sections 6.2 and 6.3 and replaces $\mathcal{L}$-maps by their $\mathcal{L}$-classes. Consequently, equality of maps (which is pointwise) needs to be replaced by equality of $\mathcal{L}$-classes on all 1-spaces (see Proposition 6.1.9(a)). With this in mind, one verifies the following.

1) The proof of Theorem 6.2.8 reducing coproducts to the form $(\mathcal{M}, [\iota_1], [\iota_2])$ carries through without additional changes.

2) $\Delta \in \{\textsf{s, rp, w}\}$. The proof of Theorem 6.2.10 also generalizes without additional

changes.

3) $\Delta \in \{\textsf{l-s}, \textsf{l-rp}\}$. The proof of Theorem 6.2.9 needs a bit more care. With the strategy described above we arrive at statement (6.9), which now reads as

for all $j \in \Omega$ there exists a type-$[\Delta]$ map $[\epsilon_j]$ such that $[\epsilon_j \circ \iota_i] = [\iota_i]$ for $i = 1, 2$.

Thus $[\epsilon_j](\langle(v_1, 0)\rangle) = \langle(v_1, 0)\rangle$ and $[\epsilon_j](\langle(0, v_2)\rangle) = \langle(0, v_2)\rangle$ for all $v_1, v_2 \in \mathbb{F}^2$ and all $j \in \Omega$. Since the $\mathcal{L}$-class $[\epsilon_j]$ is linear, Proposition 6.1.8(e) implies that, without loss of generality, there exist $a_j \in \mathbb{F}^*$ such that

$$\epsilon_j(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)\ \mathrm{diag}(1, 1, a_j, a_j) \text{ for all } (x_1, x_2, x_3, x_4) \in \mathbb{F}^4,$$

where $\mathrm{diag}(1, 1, a_j, a_j)$ is the 4×4-diagonal matrix with the specified diagonal entries. In particular, ϵ_j need not be the identity and in Diagram (6.10) we have potentially different maps $[\epsilon_j] : \mathcal{M} \longrightarrow \mathcal{N}^{(j)}$ for each $j \in \Omega$. Note that each ϵ_j is an $\mathcal{L}$-isomorphism. Now we consider the two types of Theorem 6.2.9.

a) Let $\Delta = \textsf{l-rp}$. Then $\mathcal{M}$ is linearly isomorphic to each $\mathcal{N}^{(j)}$, contradicting Proposition 6.2.5(d).

b) Let $\Delta = \textsf{l-s}$. Consider the q-matroids $\mathcal{N}^{(j)} = (\mathbb{F}^4, \rho^{(j)})$. We claim that $\rho^{(j)}(V) = \rho^{(j)}([\epsilon_j](V))$. Indeed, let $V = \mathrm{rowsp}(Y)$ for some $Y \in \mathbb{F}^{y \times 4}$. Then $\epsilon^{(j)}(V) = \mathrm{rowsp}(Y\ \mathrm{diag}(1, 1, a_j, a_j))$. The definition of $G^{(j)}$ implies that $G^{(j)}\ \mathrm{diag}(1, 1, a_j, a_j) = \mathrm{diag}(1, a_j)G^{(j)}$, and thus

$$\begin{aligned}\rho^{(j)}(\epsilon_j(V)) &= \mathrm{rk}\big(G^{(j)}\ \mathrm{diag}(1, 1, a_j, a_j)Y^\mathsf{T}\big) \\ &= \mathrm{rk}\big(\ \mathrm{diag}(1, a_j)G^{(j)}Y^\mathsf{T}\big) \\ &= \mathrm{rk}\big(G^{(j)}Y^\mathsf{T}\big) = \rho^{(j)}(V).\end{aligned}$$

With the aid of (6.4) we conclude that $[\epsilon_j](\mathcal{F}(\mathcal{N}^{(j)})) = \mathcal{F}(\mathcal{N}^{(j)})$ for all $j \in \Omega$. Now we are ready to return to the proof of Theorem 6.2.9 and specifically to the set $\mathcal{F}' = \bigcup_{j\in\Omega} \mathcal{F}(\mathcal{N}^{(j)})$. We claim that $\mathcal{F}' \subseteq \mathcal{F}(\mathcal{M})$. Indeed, let $F \in \mathcal{F}'$. Then $F \in \mathcal{F}(\mathcal{N}^{(j)})$ for some $j \in \Omega$ and thus $[\epsilon_j](F) \in \mathcal{F}(\mathcal{N}^{(j)})$. Since $[\epsilon_j]$ is strong, we conclude that $F = [\epsilon_j^{-1}]([\epsilon_j](F))$ is a flat in $\mathcal{M}$. Now the rest of the proof of Theorem 6.2.9, starting at (6.11), generalizes to $\mathcal{L}$-classes as described above, and leads to a contradiction.

4) $(q, \Delta) = (2, \textsf{l-w})$. Note that for $q = 2$ linear maps are $\mathcal{L}$-equivalent if and only if they are equal (see Proposition 6.1.8(e)). Therefore the result from Theorem 6.3.3 remains valid for q-$\mathbf{Mat}^{[\textsf{l-w}]}$ if $q = 2$.

5) $q > 2$ and $\Delta = \textsf{l-w}$. Following the proof of Theorem 6.3.3, we see that the existence of a coproduct implies the existence of a unique $\mathcal{L}$-class $[\epsilon]$ satisfying $[\epsilon] \circ [\iota_i] = [\alpha_i]$ for $i = 1, 2$. However, choosing the linear q-weak map ϵ as in that proof, we can now take any linear map ϵ' such that $\epsilon'|_{E_1} = \lambda_1\epsilon|_{E_1}$ and $\epsilon'|_{E_2} = \lambda_2\epsilon|_{E_2}$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$ and obtain a linear q-weak map satisfying $[\epsilon'] \circ [\iota_i] = [\alpha_i]$ for $i = 1, 2$. Choosing $\lambda_1 \neq \lambda_2$ we obtain $[\epsilon'] \neq [\epsilon]$, which proves that the uniqueness of the map $[\epsilon]$ fails. Therefore $\mathcal{M}_1 \oplus \mathcal{M}_2$ is not a coproduct of $\mathcal{M}_1$ and $\mathcal{M}_2$ in q-$\mathbf{Mat}^{[\textsf{l-w}]}$. A similar reasoning shows that the two q-matroids do not have any coproduct in q-$\mathbf{Mat}^{[\textsf{l-w}]}$. □

6.5 A functor from category of q-matroids to category of matroids.

Maps between matroids have been found useful to study matroids from a category theory approach. The reader may refer to [32, 47] for more details. In this section we focus on the relation between maps of q-matroids and maps of matroids, and show that the projectivization map is a functor from categories of q-matroids to categories of matroids. This, in turn, provides a new approach to study maps between q-matroids.

Similarly to q-matroids, one can define the notion of weak and strong maps between matroids. Before introducing these maps matroids, we must define the matroid operation of a single element extension by adjoining a loop, which we refer to as loop extension. The reader may refer to [39, Sect. 7.2] and [47, Chap. 8] for proofs and a more detailed discussion of the single element extension.

Proposition 6.5.1. *Let $M = (S, r)$ be a matroid and $\{o_M\}$ denotes a symbol disjoint from S. Let $S_o := S \cup \{o_M\}$ and $r_o : 2^{S_o} \to \mathbb{N}_0$ be such that $r_o(A) = r(A - \{o_M\})$, for all $A \subseteq S_o$. Then $M_o := (S_o, r_o)$ is a matroid, and $\{o_M\}$ is a loop in M_o. Furthermore M_o is called a* loop extension *of M.*

The subscript of the added loop may be omitted if it is clear from context in which matroid the loop is contained. The next proposition relates the flats $\mathcal{F}_{M_o}$ and $\mathcal{F}_M$. Furthermore, we recall that two lattices are *isomorphic* (denoted by $\cong$) if there exists an order preserving bijection between the lattices that preserves meets and joins.

Proposition 6.5.2. *Let M be a matroid, M_o a loop extension of M, and $\mathcal{F}_M$, $\mathcal{F}_{M_o}$ their respective collection of flats. Then*

$$\mathcal{F}_{M_o} = \{F \cup \{o\} \,:\, F \in \mathcal{F}_M\}.$$

and $\mathcal{F}_M \cong \mathcal{F}_{M_o}$ as lattices.

Remark 6.5.3. *Note that $M_o \setminus \{o\} = M$. This deletion can be seen as identifying the element $\{o\}$ with the empty set of M, and does not change the overall structure of the matroid.*

As the next definition will show a map between matroids is a map defined on the groundset of the loop extension matroids. By remark 6.5.3, the added loop can be seen as an element representing the empty set of the matroid. Hence, mapping an element to the added loop of the codomain can be seen as mapping an element to the empty set.

Definition 6.5.4. *Let $M = (S, r_M)$ and $N = (T, r_N)$ be matroids and M_o, N_o be their respective loop extension matroids. A map $\sigma : M \to N$ is a map between the groundsets of the loop extension matroids, i.e. $\sigma : S_o \to T_o$, such that $\sigma(o_M) = o_N$. Furthermore σ is said to be:*

- weak *if $r_{N_o}(\sigma(A)) \leq r_{M_o}(A)$ for all $A \subseteq S_o$.*
- strong *if $\sigma^{-1}(F) \in \mathcal{F}_{M_o}$ for all $F \in \mathcal{F}_{N_o}$.*

It is well known (see [47, Chap. 8, Lem. 8.1.7]) that strong maps are weak maps. Furthermore a map $\sigma : M \to N$ induces a map $\sigma^{\#} : \mathcal{F}_{M_o} \to \mathcal{F}_{N_o}$, where $\sigma^{\#}(F) = \mathrm{cl}_{N_o}(\sigma(F))$ for all $F \in \mathcal{F}_{M_o}$. Using Proposition 6.5.2, one can alternatively define $\sigma^{\#} : \mathcal{F}_M \to \mathcal{F}_N$. As the following theorem shows, the induced map $\sigma^{\#}$ provides an alternative definition for strong maps.

Theorem 6.5.5. *([47, Prop 8.1.3])*
A map $\sigma : M \to N$ is a strong map if and only if the following hold:

(1) for all $F_1, F_2 \in \mathcal{F}_M$,

$$\sigma^{\#}(F_1 \vee F_2) = \sigma^{\#}(F_1) \vee \sigma^{\#}(F_2)$$

(2) $\sigma^{\#}$ sends atoms of $\mathcal{F}_M$ to atoms or to the zero of $\mathcal{F}_N$.

The main result of this section is the analogue of Theorem 6.5.5 for q-matroids. We turn to the definitions of maps between q-matroids, as introduced in [26]. Similarly to matroids, maps between q-matroids are maps between groundspaces that send subspaces to subspaces.

To study the relation between maps of matroids and maps of q-matroids we need the following notation. Given a vector space E, define the *extended projective space of E* as $\mathbb{P}_oE = \mathbb{P}E \cup \{o\}$, where $\{o\}$ is an arbitrary element disjoint from $\mathbb{P}E$. Let $P_o : E \to \mathbb{P}_oE$, where $P_o(0) = o$ and $P_o(v) = \hat{P}(v)$ for $v \neq 0$ and $\hat{P} : E - \{0\} \to \mathbb{P}E$ is as introduced in the previous section. We call P_o the *extended projectivization map*. Note, unlike the projectivization map, we do not consider P_o as lattice map but as a map between a vector space to its extended projective space. Given a q-matroid $\mathcal{M} = (E, \rho)$ and the loop extension of its projectivization matroid $P(\mathcal{M})_o = (\mathbb{P}_oE, r_o)$, the map P_o can be viewed as a map between the groundspace E to the groundset $\mathbb{P}_oE$ such that $\rho(V) = r_o(P_o(V))$ for all $V \leq E$. Furthermore for any $A \subseteq \mathbb{P}_oE$ let $\langle A \rangle := \langle P_o^{-1}(A) \rangle_{\mathbb{F}_q}$. It can easily be shown that $r_o(A) = \rho(\langle A \rangle)$.

Recall from Definition 6.1.1 that an $\mathcal{L}$-map $\sigma : E_1 \to E_2$ induces a map on the lattices of subspaces $\sigma_{\mathcal{L}} : \mathcal{L}(E_1) \to \mathcal{L}(E_2)$. By restricting $\sigma_{\mathcal{L}}$ to the 1-dimensional spaces and the 0 of E_1, $\sigma_{\mathcal{L}}$ can be viewed as map between the extended projective spaces $\mathbb{P}_oE_1$ and $\mathbb{P}_oE_2$, i.e $\sigma_{\mathcal{L}} : \mathbb{P}_oE_1 \to \mathbb{P}_oE_2$. As the next proposition shows, σ and $\sigma_{\mathcal{L}}$ commute with the extended projectivization map.

Proposition 6.5.6. *Let $\sigma : E_1 \to E_2$ be an $\mathcal{L}$-map, $\sigma_{\mathcal{L}} : \mathbb{P}_oE_1 \to \mathbb{P}_oE_2$ its induced map on the extended projective spaces, and $P_o : E_i \to \mathbb{P}_oE_i$ be the extended projectivization map. Then*

$$P_o \circ \sigma = \sigma_{\mathcal{L}} \circ P_o.$$

Proof. Let $v \in E_1$. Since σ is an $\mathcal{L}$-map then $\langle \sigma(v) \rangle = \sigma(\langle v \rangle) = \sigma_{\mathcal{L}}(\langle v \rangle)$. But note $\langle \sigma(v) \rangle = P_o(\sigma(v))$ and $\langle v \rangle = P_o(v)$. Hence the wanted equality follows. □

We now consider the case when an $\mathcal{L}$-map σ is a map between q-matroids. The induced map $\sigma_{\mathcal{L}}$ between the extended projective spaces turns out to be a map between projectivization matroids. Furthermore σ being q-weak or q-strong is fully determined by whether $\sigma_{\mathcal{L}}$ is weak or strong, and vice versa.

Theorem 6.5.7. *Let $\mathcal{M} = (E_1, \rho_{\mathcal{M}})$, $\mathcal{N} = (E_2, \rho_{\mathcal{N}})$ be q-matroids and $P(\mathcal{M}) = (\mathbb{P}E_1, r_{P(\mathcal{M})})$, $P(\mathcal{N}) = (\mathbb{P}E_2, r_{P(\mathcal{N})})$ be their projectivization matroids. Let $\sigma : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{L}$-map. Then $\sigma_{\mathcal{L}} : P(\mathcal{M}) \to P(\mathcal{N})$ is a map between the projectivization matroids and the following holds:*

- *σ is q-weak if and only if $\sigma_{\mathcal{L}}$ is weak*
- *σ is q-strong if and only if $\sigma_{\mathcal{L}}$ is strong.*

Proof. To start note that $\sigma_{\mathcal{L}}$ is a map between the groundsets of the loop extension matroid $P(\mathcal{M})_o$ and $P(\mathcal{N})_o$ with $\sigma_{\mathcal{L}}(o_{P(\mathcal{M})}) = o_{P(\mathcal{N})}$. Thus $\sigma_{\mathcal{L}} : P(\mathcal{M}) \to P(\mathcal{N})$ is well defined.

We first prove σ is q-weak if and only if $\sigma_{\mathcal{L}}$ is weak. Assume σ is weak. Let $A \subseteq \mathbb{P}_o E_1$ and $V := \langle A \rangle$. Both P_o and σ are inclusion preserving, hence, $(P_o \circ \sigma)(P_o^{-1}(A)) \subseteq (P_o \circ \sigma)(V)$. Using Proposition 6.5.6 on the first term of the previous inclusion gives us $(\sigma_{\mathcal{L}} \circ P_o)(P_o^{-1}(A)) = \sigma_{\mathcal{L}}(A) \subseteq (P_o \circ \sigma)(V)$. Furthermore, by the monotonicity property of the rank functions and because σ is weak, we get

$$r_{P(\mathcal{N})_o}(\sigma_{\mathcal{L}}(A)) \leq r_{P(\mathcal{N})_o}((P_o \circ \sigma)(V))) = \rho_{\mathcal{N}}(\sigma(V)) \leq \rho_{\mathcal{M}}(V) = r_{P(\mathcal{M})_o}(A).$$

Because $A \subseteq \mathbb{P}_o E_1$ was arbitrarily chosen, then $\sigma_{\mathcal{L}}$ is weak.

Now assume $\sigma_{\mathcal{L}}$ is weak. Let $V \leq E_1$ and recall $\rho_{\mathcal{M}}(V) = r_{P(\mathcal{M})_o}(P_o(V))$. Since $\sigma_{\mathcal{L}}$ is weak $r_{P(\mathcal{N})_o}((\sigma_{\mathcal{L}} \circ P_o)(V)) \leq r_{P(\mathcal{M})_o}(P_o(V))$. Hence by Proposition 6.5.6, $r_{P(\mathcal{N})_o}((P_o \circ \sigma)(V)) \leq r_{P(\mathcal{M})_o}(P_o(V))$. This implies $\rho_{\mathcal{N}}(\sigma(V)) \leq \rho_{\mathcal{M}}(V)$ and shows σ is q-weak.

We now show that σ is q-strong if and only if $\sigma_{\mathcal{L}}$ is strong. From Proposition 6.5.2 and Lemma 4.2.2, $F \in \mathcal{F}_{\mathcal{M}} \Leftrightarrow P(F) \in \mathcal{F}_{P(\mathcal{M})} \Leftrightarrow P(F) \cup \{o\} \in \mathcal{F}_{P(\mathcal{M})_o}$. A similar chain of equivalence holds for $\mathcal{F}_{\mathcal{N}}$ and $\mathcal{F}_{P(\mathcal{N})_o}$. Furthermore all flats of $P(\mathcal{N})_o$ are of the form $P_o(F) = P(F) \cup \{o\}$ for some flat in $\mathcal{N}$. Therefore $\sigma_{\mathcal{L}}$ is strong iff $\sigma_{\mathcal{L}}^{-1}(P_o(F)) \in \mathcal{F}_{P(\mathcal{M})_o}$ for all $P_o(F) \in \mathcal{F}_{P_o(\mathcal{N})}$ iff $(\sigma_{\mathcal{L}} \circ P_o)^{-1}(P_o(F)) \in \mathcal{F}_{\mathcal{M}}$ for all $P_o(F) \in \mathcal{F}_{P_o(\mathcal{N})}$ iff $(P_o \circ \sigma)^{-1}(P_o(F)) = \sigma^{-1}(F) \in \mathcal{F}_{\mathcal{M}}$ for all $F \in \mathcal{F}_{\mathcal{N}}$ iff σ is q-strong. □

From the above theorem it can easily be seen that the projectivization map is a functor from the category of q-matroids with q-weak (resp. q-strong) map to the category of matroids with weak (resp. strong) maps.

Corollary 6.5.8. *Let $\mathcal{M} = (E_1, \rho_{\mathcal{M}})$, $\mathcal{N} = (E_2, \rho_{\mathcal{N}})$ be q-matroids and $\sigma : \mathcal{M} \to \mathcal{N}$ be a q-strong map. Then σ is a q-weak map.*

Proof. If $\sigma : \mathcal{M} \to \mathcal{N}$ is a q-strong then $\sigma_{\mathcal{L}} : P(\mathcal{M}) \to P(\mathcal{N})$ is a strong map by Theorem 6.5.7. Furthermore by [47, Chap. 8, Lem. 8.1.7], this implies $\sigma_{\mathcal{L}}$ is a weak map and hence σ is a q-weak map, once again by Theorem 6.5.7. □

Furthermore, Theorem 6.5.7 can also be used to prove an analogue of Theorem 6.5.5. To do so, we first define the analogue of the map $\sigma^{\#}$.

Definition 6.5.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be q-matroids with respective groundspaces E_1, E_2 and $\sigma : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{L}$-map. Define $\sigma^{\#} : \mathcal{F}_{\mathcal{M}} \to \mathcal{F}_{\mathcal{N}}$ such that*

$$\sigma^{\#}(F) = \mathrm{cl}_{\mathcal{N}}(\sigma(F)).$$

The next useful Lemma shows that the induced maps $\sigma^{\#}$ and $\sigma_{\mathcal{L}}^{\#}$ commute with the extended projectivization map.

Lemma 6.5.10. *Let $\mathcal{M}, \mathcal{N}$ be q-matroids, $\mathcal{F}_{\mathcal{M}}, \mathcal{F}_{\mathcal{N}}$ their lattices of flats and $P(\mathcal{M})$, $P(\mathcal{N})$ their projectivization matroids. Furthermore let $\sigma : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{L}$-map, $\sigma_{\mathcal{L}} : P(\mathcal{M}) \to P(\mathcal{N})$ its induced map and $P_o : E_i \to \mathbb{P}_o E_i$ the extended projectivization map. Then*

$$P_o \circ \sigma^{\#} = \sigma_{\mathcal{L}}^{\#} \circ P_o$$

Proof. First recall, $F \in \mathcal{F}_{\mathcal{M}} \Leftrightarrow P_o(F) \in \mathcal{F}_{P(\mathcal{M})_o}$. Let $F \in \mathcal{F}_M$, then $\sigma(F) \subseteq \sigma^{\#}(F)$ and since P_o is inclusion preserving $(P_o \circ \sigma)(F) \subseteq (P_o \circ \sigma^{\#})(F)$. By Proposition 6.5.6, the above containment implies $(\sigma_{\mathcal{L}} \circ P_o)(F) \subseteq (P_o \circ \sigma^{\#})(F)$. Applying the closure operator of $P(\mathcal{N})_o$, we get

$$(\sigma_{\mathcal{L}}^{\#} \circ P_o)(F) = \mathrm{cl}_{P(\mathcal{N})_o}((\sigma_{\mathcal{L}} \circ P_o)(F)) \subseteq \mathrm{cl}_{P(\mathcal{N})_o}((P_o \circ \sigma^{\#})(F)) = (P_o \circ \sigma^{\#})(F),$$

where the final equality holds because $\sigma^{\#}(F) \in \mathcal{F}_{\mathcal{N}}$ and therefore $(P_o \circ \sigma^{\#})(F) \in \mathcal{F}_{P(\mathcal{N})_o}$. Assume, for sake of contradiction, that $(\sigma_{\mathcal{L}}^{\#} \circ P_o)(F) \subsetneq (P_o \circ \sigma^{\#})(F)$. Let $F' := (\sigma_{\mathcal{L}}^{\#} \circ P_o)(F)$. Then

$$(\sigma_{\mathcal{L}} \circ P_o)(F) \subseteq F' \subsetneq (P_o \circ \sigma^{\#})(F).$$

By considering their preimage under P_o and because $\sigma_{\mathcal{L}} \circ P_o = P_o \circ \sigma$, we get

$$\sigma(F) \subseteq P_o^{-1}(F') \subsetneq \sigma^{\#}(F).$$

However since $F' \in \mathcal{F}_{P(\mathcal{N})_o}$ then $P_o^{-1}(F') \in \mathcal{F}_{\mathcal{N}}$. Therefore $P_o^{-1}(F')$ must contain $\mathrm{cl}_{\mathcal{N}}(\sigma(F)) = \sigma^{\#}(F)$, a contradiction. Hence

$$(\sigma_{\mathcal{L}}^{\#} \circ P_o)(F) = (P_o \circ \sigma^{\#})(F).$$

□

In the statement of the previous Lemma, one can replace the extended projectivization map P_o by the projectivization map P introduced in the previous section. In fact, as previously remarked, the map $\sigma_{\mathcal{L}}^{\#}$ can be considered as map between the lattices of flats $\mathcal{F}_{P(\mathcal{M})}$ to $\mathcal{F}_{P(\mathcal{N})}$. Furthermore the projectivization map can also be restricted to a map between the lattice of flats of a q-matroid and its projectivization matroid. In the following Lemma, P refers to the projectivization map restricted to the lattice of flats $\mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{N}}$.

Lemma 6.5.11. *Let the data be as in Lemma 6.5.10, and let P be the projectivization map on the lattices of flats $\mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{N}}$. Then*

$$P \circ \sigma^{\#} = \sigma_{\mathcal{L}}^{\#} \circ P.$$

Proof. Recall $\mathcal{F}_{P(\mathcal{M})} = \{F' - \{o\} : F' \in P(\mathcal{M})_o\} = \{P_o(F) - \{o\} : F \in \mathcal{F}_{\mathcal{M}}\} = \{P(F) : F \in \mathcal{F}_{\mathcal{M}}\}$ and that the same holds for $P(\mathcal{N})$. From the above chain of equality and Lemma 6.5.10, equality follows straightforwardly. □

We conclude the chapter by showing the analogue of Theorem 6.5.5 for q-strong maps.

Theorem 6.5.12. *Let $\mathcal{M}$, $\mathcal{N}$ be q-matroids. An $\mathcal{L}$-map $\sigma : \mathcal{M} \to \mathcal{N}$ is a q-strong map if and only if the following holds:*

(1) for all $F_1, F_2 \in \mathcal{F}_{\mathcal{M}}$,

$$\sigma^{\#}(F_1 \vee F_2) = \sigma^{\#}(F_1) \vee \sigma^{\#}(F_2)$$

(2) $\sigma^{\#}$ sends of $\mathcal{F}_{\mathcal{M}}$ atoms to atoms or to the zero of $\mathcal{F}_{\mathcal{N}}$.

Proof. ($\Rightarrow$) Let $\sigma : \mathcal{M} \to \mathcal{N}$ be a q-strong map, which implies by Theorem 6.5.7 that $\sigma_{\mathcal{L}} : P(\mathcal{M}) \to P(\mathcal{N})$ is a strong map. By Lemma 4.2.2 (1), $F \in \mathcal{F}_{\mathcal{M}} \Leftrightarrow P(F) \in \mathcal{F}_{P(\mathcal{M})}$. Furthermore, from Theorem 6.5.5 we obtain $\sigma^{\#}_{\mathcal{L}}(P(F_1) \vee P(F_2)) = \sigma^{\#}_{\mathcal{L}}(P(F_1)) \vee \sigma^{\#}_{\mathcal{L}}(P(F_2))$ for all $F_1, F_2 \in \mathcal{F}_{P(\mathcal{M})}$. By Lemma 4.2.2 (2), $P(F_1) \vee P(F_2) = P(F_1 \vee F_2)$, hence $\sigma^{\#}_{\mathcal{L}}(P(F_1 \vee F_2)) = \sigma^{\#}_{\mathcal{L}}(P(F_1)) \vee \sigma^{\#}_{\mathcal{L}}(P(F_2))$. Applying Lemma 6.5.11 on the above equalities gives us

$$\begin{aligned}(P \circ \sigma^{\#})(F_1 \vee F_2) &= (P \circ \sigma^{\#})(F_1) \vee (P \circ \sigma^{\#})(F_2) \\ &= P(\sigma^{\#}(F_1) \vee \sigma^{\#}(F_2)).\end{aligned}$$

Finally since P is an isomorphism on the lattice of flat, the above equality implies

$$\sigma^{\#}(F_1 \vee F_2) = \sigma^{\#}(F_1) \vee \sigma^{\#}(F_2),$$

which shows σ satisfies property (1) for all $F_1, F_2 \in \mathcal{F}_{\mathcal{M}}$.

To show σ satisfies property (2), let $F \in \mathcal{F}_{\mathcal{M}}$ be an atom. Since P is a lattice isomorphism then $P(F)$ is an atom of $\mathcal{F}_{P(\mathcal{M})}$. Moreover $\sigma_{\mathcal{L}}$ is a strong map, hence by Theorem 6.5.5, $(\sigma^{\#}_{\mathcal{L}} \circ P)(F)$ must be an atom or the zero of $\mathcal{F}_{P(\mathcal{N})}$. But by Lemma 6.5.11, $(\sigma^{\#}_{\mathcal{L}} \circ P)(F) = (P \circ \sigma^{\#})(F)$, which implies $\sigma^{\#}(F)$ must be an atom or the zero of $\mathcal{F}_{\mathcal{N}}$ because, once again, P is a lattice isomorphism. This concludes that $\sigma^{\#}$ satisfies the wanting properties.

($\Leftarrow$) Let $\sigma^{\#}$ satisfy properties (1) and (2). We show that σ is a q-strong map by showing that $\sigma_{\mathcal{L}}$ is a strong map. To do so we show that $\sigma^{\#}_{\mathcal{L}}$ satisfies Theorem 6.5.5.

Let $P(F_1), P(F_2) \in \mathcal{F}_{P(\mathcal{M})}$. By Lemma 4.2.2 (2) $\sigma^{\#}_{\mathcal{L}}(P(F_1) \vee P(F_2)) = \sigma^{\#}_{\mathcal{L}}(P(F_1 \vee F_2))$. Using Lemma 6.5.11 and the fact that $\sigma^{\#}$ satisfies property (1), we get

$$\begin{aligned}(\sigma^{\#}_{\mathcal{L}} \circ P)(F_1 \vee F_2) &= (P \circ \sigma^{\#})(F_1 \vee F_2) \\ &= P(\sigma^{\#}(F_1) \vee \sigma^{\#}(F_2)) \\ &= (P \circ \sigma^{\#})(F_1) \vee (P \circ \sigma^{\#})(F_2) \\ &= (\sigma^{\#}_{\mathcal{L}} \circ P)(F_1) \vee (\sigma^{\#}_{\mathcal{L}} \circ P)(F_2),\end{aligned}$$

where the second to last equality follows from Lemma 4.2.2 (2). Hence $\sigma_{\mathcal{L}}^{\#}$ satisfies property (1) of Prop 6.5.5.

Let $P(F) \in \mathcal{F}_{P(\mathcal{M})}$ be an atom which, since P is a lattice isomorphism, implies F is an atom of $\mathcal{F}_{\mathcal{M}}$. Once again we use $(\sigma_{\mathcal{L}}^{\#} \circ P)(F) = (P \circ \sigma^{\#})(F)$. Because $\sigma^{\#}$ satisfies property (2) then $\sigma^{\#}(F)$ is an atom or the zero of $\mathcal{F}_{\mathcal{N}}$. Finally since P is a lattice isomorphism between $\mathcal{F}_{\mathcal{N}}$ and $\mathcal{F}_{P(\mathcal{N})}$ then $P(\sigma^{\#}(F)) = \sigma_{\mathcal{L}}^{\#}(P(F))$ must be an atom or the zero of $\mathcal{F}_{P(\mathcal{N})}$ which show $\sigma_{\mathcal{L}}^{\#}$ satisfies property (2) of Theorem 6.5.5. Therefore $\sigma_{\mathcal{L}}$ is a strong map and by Theorem 6.5.7 we get that σ is q-strong, concluding the proof. □

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Vita

Benjamin Jany

Place of Birth:

- Toulouse, FRANCE

Education:

- University of Kentucky, Lexington, KY
  M.S. in Mathematics, May. 2020

- University of Notre Dame, Notre Dame, IN
  B.B.A in Finance and Mathematics, May 2018

Professional Positions:

- Graduate Teaching Assistant, University of Kentucky Fall 2018–Spring 2023

Honors

- Enochs Scholarship in Algebra, University of Kentucky, Summer 2022

- Clifford J. Swauger, Jr. Graduate Fellowship, University of Kentucky, Summer 2022

- Eaves Summer Fellowship, University of Kentucky, Summer 2021

- Max Steckler Fellow, University of Kentucky, Aug. 2018–May 2021

- Diversity, Equity and Inclusion Award, University of Kentucky, May 2022

Publications & Preprints:

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