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Digital Object Identifier: https://doi.org/10.13023/etd.2023.020

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Slone, Carissa, "Slices of C_2, Klein-4, and Quaternionic Eilenberg-Mac Lane Spectra" (2023). *Theses and Dissertations--Mathematics*. 94.
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Slices of C_2, Klein-4, and Quaternionic Eilenberg-Mac Lane Spectra

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Carissa F. Slone
Lexington, Kentucky

Director: Dr. Bertrand Guillou, Professor of Mathematics
Lexington, Kentucky
2022

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ABSTRACT OF DISSERTATION

Slices of C_2, Klein-4, and Quaternionic Eilenberg-Mac Lane Spectra

We provide the slice (co)towers of $\Sigma^V H_{C_2}\underline{M}$ for a variety of C_2-representations V and C_2-Mackey functors $\underline{M}$. We also determine a characterization of all 2-slices of equivariant spectra over the Klein four-group $C_2 \times C_2$. We then describe all slices of integral suspensions of the equivariant Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$ for the constant Mackey functor over $C_2 \times C_2$. Additionally, we compute the slices and slice spectral sequence of integral suspensions of $H\underline{\mathbb{Z}}$ for the group of equivariance Q_8. Along the way, we compute the Mackey functors $\underline{\pi}_{k\rho}H_{K_4}\underline{\mathbb{Z}}$ and $\underline{\pi}_{k\rho}H_{Q_8}\underline{\mathbb{Z}}$.

KEYWORDS: slice filtration, equivariant, homotopy, Mackey functor

Carissa F. Slone

November 17, 2022

Slices of C_2, Klein-4, and Quaternionic Eilenberg-Mac Lane Spectra

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November 17, 2022
Date

To my grandparents.
Grandpa, I'm glad I get to share this accomplishment with you.
Granny, Granddad, and Grandma, although you're no longer here, I hope I made you proud.

ACKNOWLEDGMENTS

First, I am grateful for the guidance of my advisor Bertrand Guillou. From my first year topology courses to the writing of this document, you have guided me and shaped me as a mathematician and as a member of the mathematical community. Other influential professors at UK, Heide Gluesing-Luerssen, Kate Ponto, and Nathaniel Stapleton, helped me navigate graduate school and academic and professional issues as they arose.

To my undergrad professors, Darrin Frey, Kevin Roper, Otis Wright, and Adam Hammett, thank you. I didn't know that I could pursue a PhD, let alone a masters degree, but your encouragement gave me the courage to try.

I am also grateful for the support of my family. From my parents facilitating my mathematical discovery since grade school to my brothers, sisters, grandparents, aunts, uncles, and cousins encouraging me to pursue my dreams.

I have made a great deal of friends during my time at the University of Kentucky. Rachel Petrik, Darleen Perez-Lavin, Aida Maraj, Nandita Sahajpal, Kalila Lehmann, and Julie Vega provided much needed support as older, female graduate students. Hunter Lehmann was an excellent office mate and distraction from research, and Lisa Mueller's cat sitting allowed me to attend conferences without worrying about my cats. Finally, the deep emotional support and friendship provided by my closest friends Allison Jensen, Janet Huffman, and Kathryn Hechtel helped me weather even the most tumultuous parts of graduate school.

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Chapter 1 Introduction

The Postnikov filtration is a filtration with associated tower $P^*(-)$ such that for a spectrum (or space) X, $\pi_i(P^n X) \cong \pi_i(X)$ for $i \leq n$ where $n \in \mathbb{Z}$ and the homotopy groups vanish above n. We have natural maps down the tower $P^n(-) \to P^{n-1}(-)$. We write the the fiber at each level as $P^n_n X$, and each triangle containing X commutes up to homotopy.

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
P^2_2 X \arrow[r] & P^2 X \arrow[d] \\
P^1_1 X \arrow[r] & P^1 X \arrow[d] \\
X \arrow[uur, bend left] \arrow[ur] \arrow[r] & P^0 X
\end{tikzcd}$$

The fiber at each level, $P^n_n X$, contains precisely the homotopy of X in dimension n. So we get a picture of how to construct X via its homotopy groups.

If we take X to be KU or KO, we get particularly nice towers.

$$P^n_n(KU) \simeq \begin{cases} S^n \wedge H\mathbb{Z} & n \text{ even} \\ * & \text{else.} \end{cases}$$

$$P^n_n(KO) \simeq \begin{cases} S^n \wedge H\mathbb{Z} & n \equiv 0, 4 \pmod 8 \\ S^n \wedge H\mathbb{Z}/2 & n \equiv 1, 2 \pmod 8 \\ * & \text{else.} \end{cases}$$

So the Postnikov tower for KU is 2-periodic, which reflects the 2-periodicity of its homotopy groups. Similarly, the Postnikov tower for KO is 8-periodic, which reflects the 8-periodicity of its homotopy groups.

Now the spaces that compose KU, namely, U and BU, have a C_2-action, so we can move to the genuine equivariant setting by constructing a C_2-spectrum $K\mathbb{R}$ with underlying spectrum KU. However, the Postnikov story here is not as clean. The (categorical) fixed points of $K\mathbb{R}$ are KO, and the Postnikov tower sees the information for both KU and KO, so at best we can get an 8-periodic tower for $K\mathbb{R}$. This is unsatisfying because equivariant Bott periodicity yields that $K\mathbb{R} \simeq S^{\rho_{C_2}} \wedge K\mathbb{R}$, i.e., $K\mathbb{R}$ is 2-periodic with respect to the regular representation for C_2. Since we have this 2-periodicity we'd like to see it reflected in the Postnikov tower, but this is not the case.

Thus, we consider the equivariant slice filtration. Let G be a finite group. The G-equivariant slice filtration was first defined in the context of G-equivariant stable homotopy theory by Dugger in [D]; it came to prominence as a result of its role in the proof of the Kervaire invariant conjecture by Hill, Hopkins, and Ravenel [HHR1].

The slice filtration is an analogue in the G-equivariant stable homotopy category of the classical Postnikov filtration of spectra. One can also define a G-equivariant Postnikov filtration; on passage to fixed points with respect to any subgroup $H \leq G$, this recovers the Postnikov filtration of the H-fixed point spectrum. However, there are many equivariant spectra which possess a periodicity with respect to suspension by a G-representation sphere. The slice filtration was devised by Dugger in order to display this periodicity for the case of the C_2-spectrum $K\mathbb{R}$.

Much work has been done computing the slices of certain $RO(G)$-graded suspensions of $H_G\underline{\mathbb{Z}}$ including $G = C_{p^n}$ by [HHR3] and [Y2], and $G = D_{2p}$ by [Z2]. [GY] computes the slices of $\Sigma^n H_K \underline{\mathbb{F}_2}$ where $K = C_2 \times C_2$. We primarily focus on the slices of $\Sigma^{\pm n} H_K \underline{\mathbb{Z}}$ and $\Sigma^n H_{Q_8} \underline{\mathbb{Z}}$.

1.1 Review of C_2

Let $H \leq G$ where G is finite. A key feature of the slice filtration is that any n-slice over G must restrict to an n-slice over H. That is, if X is an n-slice over G, then $i_H^* X$ is also an n-slice. This provides a useful strategy for determining slices of X over G: first determine the slices of $i_H^* X$ and then use this data to inform the calculations over G. As C_2 is a subgroup of Klein-4 and the quaternions, it is important to know the slices of various spectra over C_2. Chapter 3 covers the slices of $\Sigma^V H_{C_2} \underline{M}$ where $\underline{M}$ ranges over all Mackey functors presented in Table 2.1 and V is a C_2-representation. Additionally, we provide a characterization of 2-slices over C_2.

1.2 Klein Four 2-slices and the Slices of $\Sigma^{\pm n} H\underline{\mathbb{Z}}$

There is a complete characterization of all n-slices where $-1 \leq n \leq 1$, listed in Proposition 2.3.4. This, combined with Proposition 2.3.5, characterizes all slices in degrees congruent to -1, 0, or 1, modulo the order of G. For $G = C_2 \times C_2$, we are then only missing the $(4k+2)$-slices. In Section 4.1 we finish this characterization with the following result.

Theorem 4.1.9: Suppose the only nontrivial homotopy Mackey functors of a $(C_2 \times C_2)$-spectrum X are $\underline{\pi}_1(X)$ and $\underline{\pi}_2(X)$ where certain maps in each Mackey functor are injective. Then X is a 2-slice. Conversely, if X is a 2-slice, then its only nontrivial homotopy Mackey functors are $\underline{\pi}_1(X)$ and $\underline{\pi}_2(X)$ where $\Sigma^2 H\underline{\pi}_2(X)$ is a 2-slice and $\Sigma^1 H\underline{\pi}_1(X) \in [2,4]$.

Most of the slices in [GY] are $RO(K)$-graded suspensions of $H\underline{\pi}_{-i}(\Sigma^{-k\rho_K} H_K \underline{\mathbb{F}_2})$ for i in the range $[k+3, 4k]$. We encouter a similar phenomenon for the slices of $\Sigma^{\pm n} H_K \underline{\mathbb{Z}}$. Although we have cofiber sequences relating $\Sigma^n H_K \underline{\mathbb{Z}}$ to $\Sigma^n H_K \underline{\mathbb{F}_2}$, we can only recover some information about the former from the latter.

As for the slices of $\Sigma^{\pm n} H_K \underline{\mathbb{Z}}$, the main result can be summarized as follows:

Main Result: For $n < 0$, all nontrivial slices of $\Sigma^n H_K \underline{\mathbb{Z}}$ are given by:

$$P_i^i(\Sigma^n H_K \underline{\mathbb{Z}}) \simeq \Sigma^{-V} H_K \underline{M}$$

where i is in the range $[4n, n]$. For $0 \leq n \leq 5$, $\Sigma^n H\underline{\mathbb{Z}}$ is an n-slice. Finally, for $n > 5$,

$$P_i^i(\Sigma^n H_K\underline{\mathbb{Z}}) \simeq \Sigma^V H_K\underline{M}$$

where i is in the range $[n, 4(n-4)]$. The representations V and Mackey functors $\underline{M}$ are given in Proposition 4.3.3, Proposition 4.3.4, Proposition 4.3.8, Proposition 4.4.5, Proposition 4.4.6, and Proposition 4.4.7.

The author is grateful for the guidance of Bertrand Guillou and some helpful conversations with Vigleik Angelveit. Figures 4.6.10, 4.6.11, 4.6.12, 4.6.13, 4.6.14, and 4.6.15 were created using Hood Chatham's `spectralsequences` package.

1.3 The slices of quaternionic Eilenberg-Mac Lane spectra

Since the groundbreaking work [HHR1], a number of authors have calculated the slice filtration, as well as the associated slice spectral sequence, for G-spectra of interest. We extend the literature by considering in this article the case of the nonabelian group $G = Q_8$.

The choice of the group Q_8 is not arbitrary, as we now explain. Some of the most far-reaching applications of the slice filtration and associated spectral sequence have come in the case of cyclic p-groups of equivariance. In addition to [HHR1], this also includes [HHR2], [MSZ], [S2], and [HSWX]. In particular, in [HSWX] the authors use slice technology to understand a C_4-equivariant, height 4 Lubin-Tate theory at the prime 2. For each height n, there is a height n Lubin-Tate theory that comes equipped with a continuous action of the height n (profinite) Morava stabilizer group. The homotopy fixed points with respect to this action gives a model for the $K(n)$-local sphere, a central object of study in stable homotopy theory. However, the homotopy fixed points with respect to the entire stabilizer group are quite difficult to calculate. More approachable are the homotopy fixed points with respect to finite subgroups of the stabilizer group. At height 4, the Morava stabilizer group contains a C_4-subgroup (in fact a C_8), which gives the context for [HSWX]. On the other hand, at height $2m$, where m is odd, the Morava stabilizer group contains a Q_8-subgroup. Therefore it is reasonable to expect that Q_8-equivariant slice techniques will eventually shed light on the $K(n)$-local sphere when $n = 2m$ and m is odd.

The focus of Chapter 5 is the determination of the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$. We list the slices in Section 5.4 and describe the associated spectral sequence in Section 5.6. We rely heavily on the computation of the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ given in Chapter 4. The quotient map $Q_8 \longrightarrow K_4$ allows us to gain insight into the Q_8-equivariant slices from the K_4-case, as we now explain in greater generality.

Given a normal subgroup $N \trianglelefteq G$, there are several constructions that will produce a G-spectrum from a G/N-spectrum. First is the ordinary pullback, or inflation, functor. If $q\colon G \longrightarrow G/N$ is the quotient, then inflation is denoted $q^*\colon \mathbf{Sp}^{G/N} \longrightarrow \mathbf{Sp}^G$; it is left adjoint to the N-fixed point functor. This inflation functor plays an important role. For instance $q^*(S^0_{G/N})$ is equivalent to S^0_G. However, from our point of view, this construction has two deficiencies. First, the ordinary inflation does not

interact well with the slice filtration. Secondly, the inflation of an $H_{G/N}\underline{\mathbb{Z}}$-module does not have a canonical $H_G\underline{\mathbb{Z}}$-module structure.

On the other hand, the "geometric inflation" functor ([H, Definition 4.1], [LMSM, Section II.9])

$$\phi_N^* \colon \mathbf{Sp}^{G/N} \longrightarrow \mathbf{Sp}^G,$$

which is right adjoint to the geometric fixed points functor, interacts well with slices. Namely, if N is a normal subgroup of order d and X is a G/N-spectrum, then

$$\phi_N^* P_k^k(X) \simeq P_{dk}^{dk}\left(\phi_N^* X\right),$$

by [U1, Corollary 4-5] (see also [H, Section 4.2]). However, in general the geometric inflation of an $H_{G/N}\underline{\mathbb{Z}}$-module will not be an $H_G\underline{\mathbb{Z}}$-module.

The third variant is the $\underline{\mathbb{Z}}$-module inflation functor ([Z1, Section 3.2])

$$\Psi_N^* \colon \operatorname{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_G\underline{\mathbb{Z}}}.$$

By design, the $\underline{\mathbb{Z}}$-module inflation of an $H_{G/N}\underline{\mathbb{Z}}$-module has a canonical $H_G\underline{\mathbb{Z}}$-module structure, though in general this functor does not interact well with the slice filtration.

In some cases, these constructions agree. For instance, if the underlying spectrum of the G/N-spectrum X is contractible, then $q^*X \simeq \phi_N^* X$. If X is furthermore an $H_{G/N}\underline{\mathbb{Z}}$-module, then the three inflation functors coincide on X (Proposition 5.1.14).

The above discussion applies to the slices of $\Sigma^n H_{G/N}\underline{\mathbb{Z}}$: all slices, except for the bottom slice, have trivial underlying spectrum. It follows that these inflate to give many of the slices of $\Sigma^n H_G\underline{\mathbb{Z}}$.

Our main result along these lines, Theorem 5.1.15, describes the higher slices of such an inflated $H_G\underline{\mathbb{Z}}$-module. In the case of $G = Q_8$, $N = Z(Q_8)$, and $G/N = Q_8/Z \cong K_4$, it gives the following:

Theorem 1.3.1. *Let $n \geq 0$. Then the nontrivial slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$, above level $2n$, are*

$$P_{2k}^{2k}\left(\Sigma^n H_{Q_8}\underline{\mathbb{Z}}\right) \simeq \Psi_Z^* P_k^k\left(\Sigma^n H_{K_4}\underline{\mathbb{Z}}\right) \simeq \phi_Z^* P_k^k\left(\Sigma^n H_{K_4}\underline{\mathbb{Z}}\right)$$

for $k > n$. Furthermore,

$$P_n^{2k}\left(\Sigma^n H_{Q_8}\underline{\mathbb{Z}}\right) \simeq \Psi_Z^* P_n^k\left(\Sigma^n H_{K_4}\underline{\mathbb{Z}}\right).$$

As the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ were determined in Chapter 4, this immediately provides all of the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ above level $2n$. The remaining slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ are then given by analyzing the slice tower of $\Psi_N^*(P_n^n H_K\underline{\mathbb{Z}})$. We perform this analysis in Section 5.4.1.

The authors are very happy to thank Agnes Beaudry, Michael Geline, Cherry Ng, and Mincong Zeng for a number of helpful discussions. The spectral sequence charts in Section 5.6 were created using Hood Chatham's `spectralsequences` package.

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Chapter 2 Preliminaries

2.1 Mackey Functors and Representations

In this section we review Mackey functors for the groups C_2, C_4, and K_4. We also discuss the representations associated with these groups and Q_8. Mackey functors for Q_8 will be presented in Chapter 5.

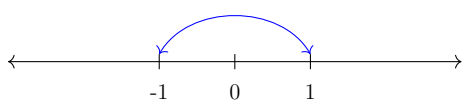
Definition 2.1.1. A Mackey functor, $\underline{M}$, for a finite group G is abelian groups $\underline{M}(G/H)$, for every $H \subseteq G$ and for all $K \subset H$, maps

$$\begin{tikzcd}
\underline{M}(G/H) \arrow[loop right, "W_G(H)"] \arrow[d, bend right, "res"'] \\
\underline{M}(G/K) \arrow[loop right, "W_G(K)"] \arrow[u, bend right, "tr"']
\end{tikzcd}$$

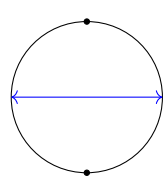
Arrows going down the diagram are called restrictions and arrows going up are called transfers. They satisfy the double-coset formula. All Mackey functors throughout will be introduced pictorially using Lewis diagrams.

2.1.1 Background for C_2

The group C_2 has one nontrivial, irreducible representation: σ, the sign representation. It acts on $\mathbb{R}$ as follows.



The one point compactification of this representation is S^σ, the signed sphere. The points 0 and ∞ are fixed, while the two hemispheres have a nontrival C_2-action.



We will write $\mathbb{Z}^\sigma$ for the C_2-module corresponding to σ.
The Lewis diagram for a C_2-Mackey functor takes the form

$$\begin{tikzcd}
\underline{M}(C_2) \arrow[d, bend right] \\
\underline{M}(e) \arrow[u, bend right]
\end{tikzcd}$$

where $\underline{M}(e)$ has a C_2/e action. For the sake of clarity in large diagrams, in a general Mackey functor $\underline{M}$ we will henceforward denote $\underline{M}(H)$ by M_H.

We will see the C_2-Mackey functors in Table 2.1.

Table 2.1: C_2-Mackey functors.

$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}^*$	$\underline{f}$	$\underline{Q}$
$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[d, bend right] \\ \mathbb{Z}_\sigma \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right] \\ \mathbb{Z}_\sigma \arrow[u, bend right, "\pi"'] \end{tikzcd}$
$\underline{\mathbb{F}_2}$	$\underline{\mathbb{F}_2}^*$	$\underline{f}$	$\underline{g}$
$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "1"'] \\ \mathbb{F}_2 \arrow[u, bend right, "0"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "0"'] \\ \mathbb{F}_2 \arrow[u, bend right, "1"'] \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[d, bend right] \\ \mathbb{F}_2 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$

Here and throughout a blue arrow will represent the identity and orange will represent multiplication by two.

2.1.2 Background for C_4

The C_4-sign representation σ_{C_4} is the inflation $p^*\sigma_{C_2}$ of the C_2-sign representation along the surjection $C_4 \longrightarrow C_2$. We will simply write σ for σ_{C_4}. Then the regular representation for C_4 splits as

$$\rho_{C_4} = 1 \oplus \sigma \oplus \lambda,$$

where λ is the irreducible 2-dimensional rotation representation of C_4.

Let $C_4 = \langle \gamma \rangle$. The representation λ then acts on $\mathbb{R}^2$ as follows.

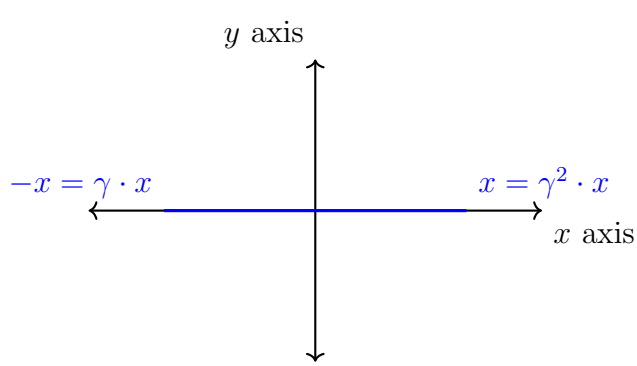
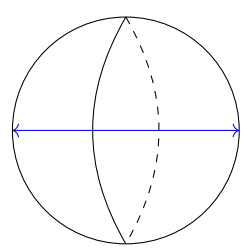


Table 2.2: Some C_4-Mackey functors

$\square = \underline{\mathbb{Z}}$	$\circledast = \underline{\mathbb{Z}}^*$	$\underline{\mathbb{Z}}(2,1)$	$\circ = \underline{B}(2,0)$
$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$	$\begin{tikzcd} \mathbb{Z}/4 \arrow[d, bend right, "1"'] \\ \mathbb{Z}/2 \arrow[u, bend right, "2"'] \\ 0 \end{tikzcd}$
$\bullet = \underline{g}$	$\bar{\bullet} = \phi^* \underline{f}$	$\blacklozenge = \phi^* \underline{\mathbb{F}_2}$	$\phi^* \underline{\mathbb{F}_2}^*$
$\begin{tikzcd} \mathbb{F}_2 \\ 0 \\ 0 \end{tikzcd}$	$\begin{tikzcd} 0 \\ \mathbb{F}_2 \\ 0 \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \arrow[d, "1"] \\ \mathbb{F}_2 \\ 0 \end{tikzcd}$	$\begin{tikzcd} \mathbb{F}_2 \\ \mathbb{F}_2 \arrow[u, "1"] \\ 0 \end{tikzcd}$

The one-point compactification of this space is S^λ. The equator is fixed while the east and west hemispheres have a nontrival $C_4/C_2 \cong C_2$ action.



Some C_4-Mackey functors that will appear are displayed in Table 2.2. All of these Mackey functors have trivial Weyl-group actions.

2.1.3 Background for K_4

The Klein 4-group $K_4 = C_2 \times C_2$ has three sign representations, obtained as the inflation along the three surjections $K_4 \longrightarrow C_2$. We denote these three surjections by p_1, m, and p_2. Then the regular representation of K_4 splits as

$$\rho_{K_4} \cong 1 \oplus p_1^*\sigma \oplus m^*\sigma \oplus p_2^*\sigma.$$

As in [GY], we will write

$$\alpha = p_1^*\sigma, \qquad \beta = p_2^*\sigma, \qquad \text{and} \qquad \gamma = m^*\sigma.$$

2.1.4 Background for Q_8

The regular representation of Q splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where $\mathbb{H}$ is the 4-dimensional irreducible Q_8-representation given by the action of the unit quaternions on the algebra of quaternions and ρ_K is the regular representation of K, inflated to Q along the quotient.

Denoting by C_4 any of the subgroups L, D, or R of Q_8, we have that

$$\downarrow_{C_4}^{Q_8} \rho_K = 2 + 2\sigma \qquad \text{and} \qquad \downarrow_{C_4}^{Q_8} \mathbb{H} = 2\lambda.$$

2.2 K-Mackey Functors

Here, we pay special attention to K-Mackey functors as they will feature prominently in Chapter 4 and Chapter 5. Let L, D, and R be the left, diagonal, and right subgroups of K, respectively. The Lewis diagram for a K-Mackey functor takes the form

$$\begin{tikzcd}
 & M_K \arrow[dl, bend right] \arrow[d, shift right] \arrow[dr, bend left] & \\
M_L \arrow[ur, bend right] \arrow[dr, bend right] & M_D \arrow[u, shift right] \arrow[d, shift right] & M_R \arrow[ul, bend left] \arrow[dl, bend left] \\
 & M_e \arrow[u, shift right] \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}$$

with a Weyl group action $W_G(H) \circlearrowright M_H$ at each level. If we do not indicate the Weyl group actions for a specific Mackey functor, then they are trivial.

Some K-Mackey functors that will appear are displayed in Table 2.3. We will also see the duals of these Mackey functors.

Proposition 2.2.1. *(*[GY, Proposition 4.8]*) There are equivalences*

$$\Sigma^{-\rho} H_K \underline{m} \simeq \Sigma^{-2} H_K \underline{mg}^* \tag{2.1}$$

$$\Sigma^{-2} H_K \underline{m}^* \simeq \Sigma^{-\rho} H_K \underline{mg} \tag{2.2}$$

We will also see the new Mackey functors in Table 2.4.

In Table 2.4, $\underline{\mathbb{Z}}^*$ is the dual to the constant Mackey functor $\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}}(2,1)^*$ is the dual to $\underline{\mathbb{Z}}(2,1)$. The Mackey functor $\underline{\mathbb{M}}$ results from the injection $\underline{\mathbb{Z}}^* \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{\mathbb{M}}$. Again, in each Mackey functor, the blue arrows are multiplication by one and the orange arrows are multiplication by two.

Proposition 2.2.2. *We have the equivalence*

$$\Sigma^{-\rho} H_K \underline{\mathbb{Z}} \simeq \Sigma^{-4} H_K \underline{\mathbb{Z}}^*.$$

Proof. This follows from Proposition 4.1.4. □

Table 2.3: Some K-Mackey functors

$\underline{\mathbb{F}_2}$	$\phi^*_{LDR}(\underline{\mathbb{F}_2})$	$\phi^*_{LDR}(\underline{f})$
$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"'] \arrow[dr, "1"] & \\ \mathbb{F}_2 \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[d, "1"'] & \mathbb{F}_2 \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$	$\begin{tikzcd} & \mathbb{F}_2^3 \arrow[dl, "p_1"'] \arrow[d, "p_2"'] \arrow[dr, "p_3"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$	$\begin{tikzcd} & 0 & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$
$\underline{mg}$	$\underline{m}$	$\underline{w}$
$\begin{tikzcd} & \mathbb{F}_2^2 \arrow[dl, "p_1"'] \arrow[d, "\nabla"'] \arrow[dr, "p_2"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$	$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"'] \arrow[dr, "1"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$	$\begin{tikzcd} & 0 & \\ \mathbb{F}_2 \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[d, "1"'] & \mathbb{F}_2 \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$
$\underline{g^n}$		
$\begin{tikzcd} & \mathbb{F}_2^n & \\ 0 & 0 & 0 \\ & 0 & \end{tikzcd}$		

2.2.1 Induced Mackey Functors

We will now give an explicit description of K-Mackey functors induced from the cyclic subgroups.

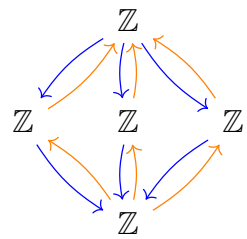
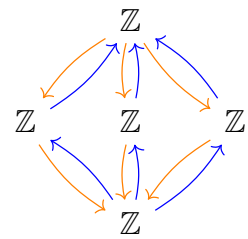
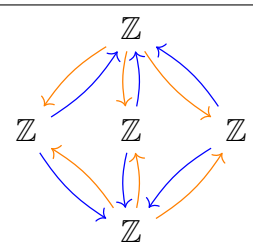
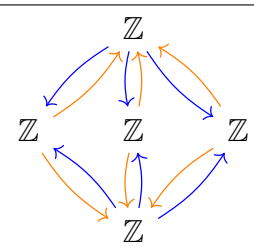
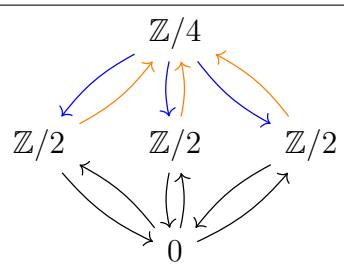
We recall the following standard result.

Lemma 2.2.3 (Shearing isomorphism)**.** *Let M be a $\mathbb{Z}[G]$-module and H a subgroup of G. Then we have an isomorphism of $\mathbb{Z}[G]$-modules,*

$$\mathbb{Z}[G] \otimes_H \underset{\underset{H}{\circlearrowleft}}{M} \cong \mathbb{Z}[G/H] \otimes \underset{\underset{G}{\circlearrowleft}}{M}$$

where G acts on the left of $\mathbb{Z}[G]$ and diagonally on $\mathbb{Z}[G/H] \otimes M$ via the map $g \otimes m \mapsto gH \otimes g \cdot m$.

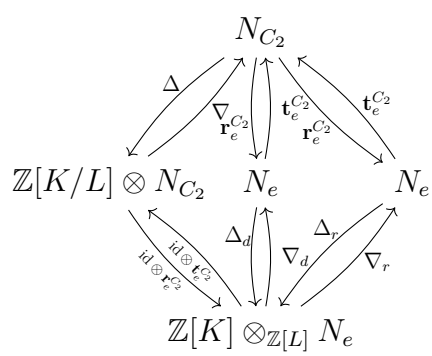
Table 2.4: New K-Mackey functors.

$\underline{\mathbb{Z}}$	$\underline{\mathbb{Z}}^*$
	
$\underline{\mathbb{Z}}(2,1)$	$\underline{\mathbb{Z}}(2,1)^*$
	
$\underline{\mathbb{M}}$	
	

Let $\underline{N}$ be a C_2-Mackey functor. Take $h \in K$ and set $\Delta_h : N_e \to \mathbb{Z}[K] \otimes_{\mathbb{Z}[L]} N_e$ and $\nabla_h : \mathbb{Z}[K] \otimes_{\mathbb{Z}[L]} N_e \to N_e$ to be $n \mapsto (e+h) \otimes n$ and $e \otimes n, h \otimes n \mapsto n$, respectively.

Then $\uparrow_L^K \underline{N}$, the induction of $\underline{N}$ from L to K, is

$$\uparrow_L^K \underline{N} \quad \cong \quad$$



The Weyl group $W_K(D)$ acts on $\uparrow_L^K \underline{N}(D) = N_e$ via the isomorphism $W_K(D) \cong C_2$ and the given action of $C_2 = W_{C_2}(e)$ on N_e. Similarly for the action of $W_K(R)$ on

N_e. As for L, $W_K(L)$ acts only on $\mathbb{Z}[K/L]$. Finally, $W_K(e) = K$ acts on the $\mathbb{Z}[K]$ factor of $\mathbb{Z}[K] \otimes_{\mathbb{Z}[L]} N_e$.

We are now going to define the unit map $\underline{M} \to \uparrow_L^K \downarrow_L^K \underline{M}$. Note that $\downarrow_L^K \underline{M}$ is the restriction of $\underline{M}$ from K to L. The pullback along $K \twoheadrightarrow K/L \cong C_2$ of $\mathbb{Z} \hookrightarrow \mathbb{Z}[C_2]$ is an inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[K/L]$. Tensoring with M_e gives

$$M_e \xhookrightarrow{i_L} \mathbb{Z}[K/L] \otimes M_e.$$

We will also use i_L to denote the inclusion of K/L fixed points

$$M_L \xhookrightarrow{i_L} \mathbb{Z}[K/L] \otimes M_L.$$

Consider

$$\nabla^r : \mathbb{Z}[K/L] \otimes M_e \to M_e \qquad \text{and} \qquad \Delta^r : M_e \to \mathbb{Z}[K/L] \otimes M_e.$$

Let ∇^r be the action map and define Δ^r by $m \mapsto e \otimes m + r \otimes r \cdot m$. Then, for $\underline{M} \in \mathrm{Mack}(K)$, the map $\underline{M} \to \uparrow_L^K \downarrow_L^K \underline{M}$ is

$$\begin{tikzcd}
 & M_K \arrow[rr, "\mathbf{r}_L^K"] \arrow[dl, bend right] \arrow[d, bend right] \arrow[d, bend left] \arrow[dr, bend left] & & M_L \arrow[dl, bend right] \arrow[d, bend right] \arrow[d, bend left] \arrow[dr, bend left] & \\
M_L \arrow[dr, bend right] \arrow[dr, bend left] & M_D \arrow[d, bend right] \arrow[d, bend left] & M_R \arrow[r, "\begin{pmatrix} i_L \\ \mathbf{r}_e^D \\ \mathbf{r}_e^R \end{pmatrix}"] \arrow[dl, bend right] \arrow[dl, bend left] & \mathbb{Z}[K/L] \otimes M_L \arrow[ul, bend right, "\Delta^r"'] \arrow[ul, bend left, "\nabla^r"] \arrow[dr, bend right, "\mathrm{id} \oplus r \cdot \mathbf{r}_e^L"'] \arrow[dr, bend left, "\mathrm{id} \oplus r \cdot \mathbf{t}_e^L"] & M_e \arrow[u, bend right, "\Delta^d"'] \arrow[u, bend left, "\nabla^d"] & M_e \arrow[ul, bend right, "\Delta^r"'] \arrow[ul, bend left, "\nabla^r"] \\
 & M_e \arrow[rr, "i_L"] & & \mathbb{Z}[K/L] \otimes M_e &
\end{tikzcd}$$

Again, $W_K(D)$ acts on M_e via the C_2-action $W_{C_2}(e) \circlearrowright M_e$. The same applies for $W_K(R) \circlearrowright M_e$. Note we have used Lemma 2.2.3 to rewrite the bottom group in $\uparrow_L^K \downarrow_L^K \underline{M}$. We now have a diagonal action K/L on $\mathbb{Z}[K/L] \otimes M_L$. Similarly, K/e acts diagonally on $\mathbb{Z}[K/L] \otimes M_e$.

Example 2.2.4. For $\underline{M} = \underline{\mathbb{Z}}$, $\uparrow_L^K \underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}} \to \uparrow_L^K \underline{\mathbb{Z}}$ are as follows.

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[rr, "1"] \arrow[dl, bend right] \arrow[d, bend right] \arrow[d, bend left] \arrow[dr, bend left] & & \mathbb{Z} \arrow[dl, bend right] \arrow[d, bend right] \arrow[d, bend left] \arrow[dr, bend left] & \\
\mathbb{Z} \arrow[dr, bend right] \arrow[dr, bend left] & \mathbb{Z} \arrow[d, bend right] \arrow[d, bend left] & \mathbb{Z} \arrow[r, "\begin{pmatrix} i_L \\ 1 \\ 1 \end{pmatrix}"] \arrow[dl, bend right] \arrow[dl, bend left] & \mathbb{Z}[K/L] \otimes \mathbb{Z} \arrow[ul, bend right, "\Delta^r"'] \arrow[ul, bend left, "\nabla^r"] \arrow[dr, bend right, "\mathrm{id} \oplus r \cdot 1"'] \arrow[dr, bend left, "\mathrm{id} \oplus r \cdot 2"] & \mathbb{Z} \arrow[u, bend right, "\Delta^d"'] \arrow[u, bend left, "\nabla^d"] & \mathbb{Z} \arrow[ul, bend right, "\Delta^r"'] \arrow[ul, bend left, "\nabla^r"] \\
 & \mathbb{Z} \arrow[rr, "i_L"] & & \mathbb{Z}[K/L] \otimes \mathbb{Z} &
\end{tikzcd}$$

Again, blue denotes multiplication by one and orange multiplication by two.

2.3 The Slice Filtration

We start with a brief review of the equivariant slice filtration. For more details see [HHR1, Section 4].

Definition 2.3.1. Let Sp^G be the category of genuine G-spectra. Let $\tau^G_{\geq n} \subseteq \mathrm{Sp}^G$ be the localizing subcategory generated by G-spectra of the form $\Sigma^\infty_G G_+ \wedge_H S^{k\rho_H}$ where $H \leq G$, ρ_H is the regular representation of H, and $k\,|H| \geq n$. We write $X \geq n$ to mean that $X \in \tau^G_{\geq n}$.

Definition 2.3.2. We say that $X < n$ if

$$[S^{k\rho_H + r}, X]^H = 0$$

for all $r \geq 0$ and all subgroups $H \leq G$ such that $k\,|H| \geq n$.

Theorem 2.3.3. [HY, Corollary 2.9, Theorem 2.10] *Let $n \geq 0$. Then $X \geq n$ if and only if*

$$\pi_k(X^H) = 0 \quad \textit{for} \quad k < \frac{n}{|H|}.$$

Proposition 2.3.4.

1. [HHR1, Proposition 4.50] *X is a 0-slice if and only if $X \simeq H\underline{M}$ for some $\underline{M} \in Mack(G)$.*
2. [HHR1, Proposition 4.50] *X is a 1-slice if and only if $X \simeq \Sigma^1 H\underline{M}$ for some $\underline{M} \in Mack(G)$ with injective restrictions.*
3. [U1, Theorem 6-4] *$\Sigma^{-1}H\underline{M}$ is a $(-n)$-slice iff $\underline{M}$ has surjective transfers for $|H| \geq n$ and $M(G/H) = 0$ for $|H| < n$.*

It is important to note that [HHR1] uses the original slice filtration whereas we employ the regular slice filtration from [U1]. Except for an indexing difference of one, the results are the same.

Proposition 2.3.5. [HHR1, Corollary 4.25] *For any $k \in \mathbb{Z}$,*

$$P^{k+|G|}_{k+|G|}(\Sigma^\rho X) \simeq \Sigma^\rho P^k_k(X).$$

That is, suspension by the regular representation commutes with the slice filtration.

Given some surjection of groups $\phi_N : G \to G/N$ where $N \trianglelefteq G$, there is a geometric pullback functor $\phi_N^* : \mathrm{Sp}^{G/N} \to \mathrm{Sp}^G$ [H, Definition 4.1].

Proposition 2.3.6. [H, Conjecture 4.11], [U1, Corollary 4-5] *Let $N \trianglelefteq G$. If the (G/N)-spectrum X is a k-slice over G/N, then $\phi_N^* X$ is a $k|N|$-slice over G.*

2.4 Brown-Comenetz and Anderson Duality

As in [HS], we write $I_{\mathbb{Q}/\mathbb{Z}}$ to indicate the representing G-spectrum of the cohomology theory $X \mapsto \mathrm{Hom}(\pi^G_{-*}X, \mathbb{Q}/\mathbb{Z})$. The Brown-Comenetz dual of X is then defined to be the function G-spectrum $F(X, I_{\mathbb{Q}/\mathbb{Z}})$. Similarly, $I_{\mathbb{Q}}$ represents $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q})$ and $I_{\mathbb{Q}}X = F(X, I_{\mathbb{Q}})$. Finally, the Anderson dual of X is $I_{\mathbb{Z}}X = F(X, I_{\mathbb{Z}})$, where $I_{\mathbb{Z}}$ is the fiber of the natural map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$.

We first consider a non-equivariant example.

Example 2.4.1. Let M be an abelian group. If M is torsion, then

$$I_{\mathbb{Q}/\mathbb{Z}}HM \simeq HM$$

and

$$I_{\mathbb{Z}}HM \simeq \Sigma^{-1}I_{\mathbb{Q}/\mathbb{Z}}HM \simeq \Sigma^{-1}HM.$$

However, if M is torsion-free, then

$$I_{\mathbb{Z}}HM \simeq HM.$$

In the case that M contains both torsion and torsion-free subgroups, we may write the exact sequence

$$M_{tor} \hookrightarrow M \twoheadrightarrow M_{free}.$$

We then may gather information regarding the Anderson dual of M with the cofiber sequence

$$HM_{free} \simeq I_{\mathbb{Z}}HM_{free} \to I_{\mathbb{Z}}HM \to I_{\mathbb{Z}}HM_{tor} \simeq \Sigma^{-1}HM_{tor}.$$

The equivariant case takes on a similar story.

Example 2.4.2. [GM, Section 3A], [HS] For a torsion Eilenberg-MacLane spectrum $H\underline{M}$,

$$I_{\mathbb{Z}}H\underline{M} \simeq \Sigma^{-1}H\underline{M}^*$$

and

$$I_{\mathbb{Q}/\mathbb{Z}}H\underline{M} \simeq H\underline{M}^*.$$

One should note that [HS] deals with non-equivariant spectra and [GM, Section 3A] refers specifically to $H\underline{M}$ as an $\mathbb{F}_2$-torsion spectrum. This example, however, follows easily from their work and [U2, Corollary I.7.3]. See [GM, Section 3A, Section 3B] for a more detailed discussion of equivariant Anderson duality.

Example 2.4.3. The Anderson and Brown-Comenetz duals of $\Sigma^4 H_K \underline{\mathbb{F}_2}$ are

$$I_{\mathbb{Z}}\Sigma^4 H\underline{\mathbb{F}_2} \simeq \Sigma^{-5} H\underline{\mathbb{F}_2}^* \qquad \text{and} \qquad I_{\mathbb{Q}/\mathbb{Z}}\Sigma^4 H\underline{\mathbb{F}_2} \simeq \Sigma^{-4} H\underline{\mathbb{F}_2}^*.$$

However, the Anderson dual of $\Sigma^4 H\underline{\mathbb{Z}}$ is

$$I_{\mathbb{Z}}\Sigma^4 H\underline{\mathbb{Z}} \simeq \Sigma^{-4} H\underline{\mathbb{Z}}^*.$$

The Brown-Comenetz dual of $\Sigma^4 H\underline{\mathbb{Z}}$ is $\Sigma^{-4} H\underline{\mathbb{Q}/\mathbb{Z}}^*$.

Proposition 2.4.4. *(*[U2, Theorem I.7.7, Theorem I.7.8]*) For a spectrum X,*

$$X \geq n \Leftrightarrow I_{\mathbb{Q}/\mathbb{Z}} X \leq -n.$$

In particular,

$$P_k^n I_{\mathbb{Q}/\mathbb{Z}} X \simeq I_{\mathbb{Q}/\mathbb{Z}} P_{-n}^{-k} X.$$

That is, the Brown-Comenetz dualization functor dualizes slice status.

Example 2.4.5. Let G be any finite group and take $\underline{M}$ to be a torsion G-Mackey functor. By Theorem 2.3.3, $\Sigma^n H_G \underline{M} \geq n$. Then $\Sigma^{-n} H\underline{M}^* \simeq I_{\mathbb{Q}/\mathbb{Z}}\Sigma^n H\underline{M} \leq -n$.

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Chapter 3 Review of C_2

Here we will discuss the motivating example for the slice filtration – the slice spectral sequence for $k\mathbb{R}$ – and provide the slice towers for $\Sigma^V H_{C_2}\underline{M}$ where $\underline{M}$ ranges over the Mackey functors listed in Table 2.1 and V is a C_2-representation.

3.1 The Slice Spectral Sequence for $k\mathbb{R}$

For consistency with Dugger's work, we present the slice spectral sequence for $k\mathbb{R}$, the connective cover of $K\mathbb{R}$. One of the key features of the slice filtration is that the tower for $k\mathbb{R}$ is 2-periodic for positive n with respect to the regular representation for C_2. That is,

$$P_n^n(k\mathbb{R}) \simeq \begin{cases} S^{\frac{n}{2}\rho_2} \wedge H\underline{\mathbb{Z}} & , n \geq 0 \text{ even} \\ * & , \text{else} \end{cases}$$

which we can show using equivariant Bott periodicity. Here $H\underline{\mathbb{Z}}$ is the genuine C_2-spectrum with homotopy Mackey functor $\underline{\mathbb{Z}}$ in dimension zero.

If we calculate the homotopy of each of these slices, we can use it to get the slice spectral sequence for $k\mathbb{R}$.

As $\rho_{C_2} = 1 + \sigma$, it is sufficient to determine the homotopy of $S^{n\sigma} \wedge H\underline{\mathbb{Z}}$ for $n \geq 1$. As $S^{(n-1)\sigma}$ is the $(n-1)$-skeleton of $S^{n\sigma}$, we can do so by examing long exact sequences in homotopy that arise from cofiber sequences of the form

$$C_{2+} \wedge S^{n-1} \wedge H\underline{\mathbb{Z}} \to S^{(n-1)\sigma} \wedge H\underline{\mathbb{Z}} \to S^{n\sigma} \wedge H\underline{\mathbb{Z}}.$$

For $n = 1$ we get the following long exact sequence.

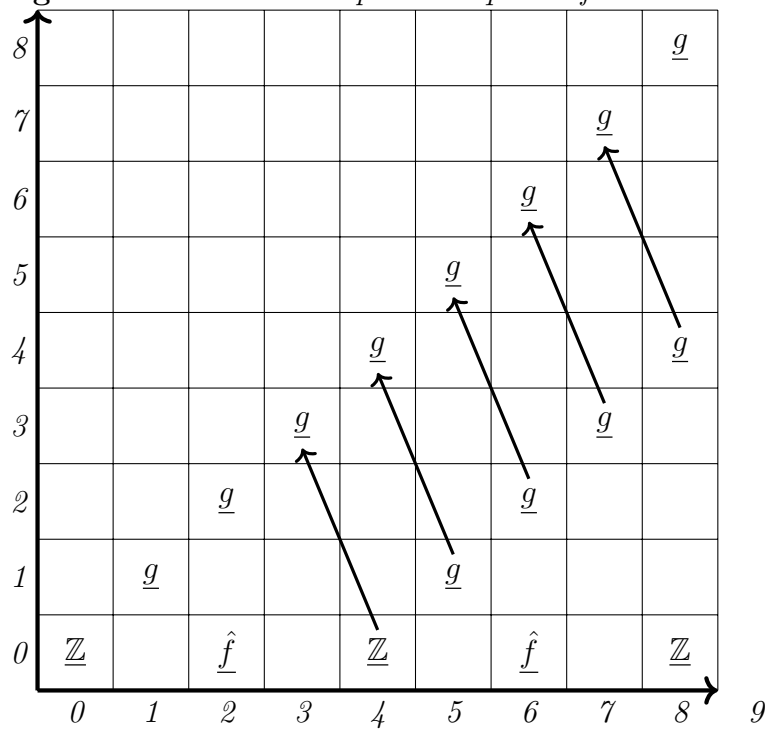
$$\begin{tikzcd}
0 \arrow[r] & \pi_1^G(S^\sigma \wedge H\underline{\mathbb{Z}}) \arrow[r] \arrow[d, bend right] & \mathbb{Z} \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[r] \arrow[d, bend right, "1"'] & \pi_0^G(S^\sigma \wedge H\underline{\mathbb{Z}}) \arrow[r] \arrow[d, bend right] & 0 \\
0 \arrow[r] & \pi_1^e(S^\sigma \wedge H\underline{\mathbb{Z}}) \arrow[r] \arrow[u, bend right] & \mathbb{Z}[C_2/e] \arrow[r, "\nabla"] \arrow[u, bend right, "\nabla"'] & \mathbb{Z} \arrow[r] \arrow[u, bend right, "2"'] & \pi_0^e(S^\sigma \wedge H\underline{\mathbb{Z}}) \arrow[r] \arrow[u, bend right] & 0
\end{tikzcd}$$

From this we find that

$$\underline{\pi}_0(S^\sigma \wedge H\underline{\mathbb{Z}}) \cong \underline{g} \cong \underline{\pi}_1(P_2^2(k\mathbb{R}))$$
$$\underline{\pi}_1(S^\sigma \wedge H\underline{\mathbb{Z}}) \cong \underline{\hat{f}} \cong \underline{\pi}_2(P_2^2(k\mathbb{R})).$$

We may then repeat this procedure to find the homotopy of the rest of the slices of $k\mathbb{R}$. From these calculations, we can write down the E_2-page of the slice spectral sequence for $k\mathbb{R}$.

Figure 3.1.1. *The slice spectral sequence for $k\mathbb{R}$*



The homotopy of $P_n^n(k\mathbb{R})$ lies on the line $y = -x + n$.

At the fixed point level we need the homotopy of ko (8-periodic in nonnegative degrees) and at the free orbit level we need the homotopy of ku (2-periodic in nonnegative degrees).

	0	1	2	3	4	5	6	7
$\underline{\pi}_n(k\mathbb{R})$	$\underline{\mathbb{Z}}$	$\underline{g}$	$\underline{Q}$	$\underline{0}$	$\underline{\mathbb{Z}}^*$	$\underline{0}$	$\underline{\hat{f}}$	$\underline{0}$

This tells us we need the differentials, most of which are isomorphisms. There is no differential to remove the $\underline{g}$ over $\underline{\hat{f}}$, so we get an extension $\underline{g} \to \underline{\pi}_2(k\mathbb{R}) \to \underline{\hat{f}}$, which turns out to be $\underline{Q}$.

The differentials each go up three, so they are d_3's. All of the $\underline{g}$'s persist until the E_3-page, at which point we take non-trivial homology and so the spectral sequence collapses on the E_4-page.

3.2 Towers over C_2

All of these towers can be derived from three short exact sequences of Mackey functors

$$\underline{g} \to \underline{\mathbb{F}_2}^* \to \underline{f} \tag{3.1}$$

$$\underline{f} \to \underline{\mathbb{F}_2} \to \underline{g} \tag{3.2}$$

$$\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \to \underline{g} \tag{3.3}$$

and a variety of homotopy equivalences illustrated in the charts below.

Figure 3.2.1. $\underline{\pi}_n(\Sigma^{k\sigma} H_{C_2}\underline{\mathbb{F}_2})$

σ								
3					$\underline{g}$	$\underline{g}$	$\underline{g}$	$\underline{\mathbb{F}_2}$
2					$\underline{g}$	$\underline{g}$	$\underline{\mathbb{F}_2}$	
1					$\underline{g}$	$\underline{\mathbb{F}_2}$		
0					$\underline{\mathbb{F}_2}$			
-1				$\underline{f}$				
-2			$\underline{\mathbb{F}_2}^*$					
-3		$\underline{\mathbb{F}_2}^*$	$\underline{g}$					
-4	$\underline{\mathbb{F}_2}^*$	$\underline{g}$	$\underline{g}$					
	-4	-3	-2	-1	0	1	2	3

Figure 3.2.2. $\underline{\pi}_n(\Sigma^{k\sigma} H_{C_2}\underline{\mathbb{Z}})$

σ								
3					$\underline{g}$		$\underline{g}$	$\underline{\hat{f}}$
2					$\underline{g}$		$\underline{\mathbb{Z}}$	
1					$\underline{g}$	$\underline{\hat{f}}$		
0					$\underline{\mathbb{Z}}$			
-1				$\underline{\hat{f}}$				
-2			$\underline{\mathbb{Z}}^*$					
-3		$\underline{Q}$						
-4	$\underline{\mathbb{Z}}^*$	$\underline{g}$						
	-4	-3	-2	-1	0	1	2	3

For example, in Figure 3.2.1, the row containing the homotopy of $\Sigma^{-\sigma} H\underline{\mathbb{F}_2}$ has one nontrivial entry, $\underline{f}$. Thus, we have Eilenberg-MacLane spectra with

$$\underline{\pi}_n(\Sigma^{-\sigma} H\underline{\mathbb{F}_2}) \cong \underline{\pi}_n(\Sigma^{-1} H\underline{f}).$$

Consequently, $\Sigma^{-\sigma} H\underline{\mathbb{F}_2} \simeq \Sigma^{-1} H\underline{f}$. Note that such a statement using $RO(G)$-graded homotopy only holds for $G = C_2$!

Proposition 3.2.3 ([GY])**.** *There are equivalences*

1. $\Sigma^4 H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{2\rho} H_{C_2}\underline{\mathbb{Z}}^*$
2. $\Sigma^2 H_{C_2}\underline{f} \simeq \Sigma^{\rho} H_{C_2}\underline{\mathbb{F}_2}^*$
3. $\Sigma^1 H_{C_2}\underline{g} \simeq \Sigma^{\rho} H_{C_2}\underline{g}$.

Note, in particular, that (3) makes $\Sigma^n H_{C_2}\underline{g}$ a $2n$-slice for any $n \in \mathbb{Z}$.

Proposition 3.2.4. *For $k, r \geq 0$,*

$$\underline{\pi}_i(\Sigma^{k\sigma+r} H_{C_2}\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & i = k+r,\ k \text{ even} \\ \underline{\hat{f}} & i = k+r,\ k \text{ odd} \\ \underline{g} & i \in [r, k+r-1],\ i \equiv r \pmod 2. \end{cases}$$

Proof. We calculate $\underline{\pi}_i(\Sigma^{k\sigma} H_{C_2}\underline{\mathbb{Z}})$ and then shift the degrees by r. The result follows by induction on $j \geq 1$ using the cofiber sequence

$$\Sigma^{(j-1)\sigma+2} H\underline{\mathbb{Z}} \simeq \Sigma^{j\sigma} H\underline{\mathbb{Z}}^* \to \Sigma^{j\sigma} H\underline{\mathbb{Z}} \to \Sigma^{j\sigma} H\underline{g} \simeq H\underline{g}.$$

□

3.2.1 Towers for $\Sigma^V H\underline{\mathbb{F}_2}$

Here we determine the slice (co)towers for $\Sigma^V H\underline{\mathbb{F}_2}$ where $V = n$ or $V = n\sigma$ for $n \in \mathbb{Z}$.

Proposition 3.2.5. *The C_2-spectrum $\Sigma^n H\underline{\mathbb{F}_2}$ is an n-slice for $0 \leq n \leq 4$.*

Proof. For $0 \leq n \leq 2$, this follows from Proposition 2.3.4 and Proposition 3.3.1. Also by Proposition 2.3.4, $\Sigma^{-1} H\underline{\mathbb{F}_2}$ is a (-1)-slice. Thus, $\Sigma^n H\underline{\mathbb{F}_2} \simeq \Sigma^{n-4+2\rho} H\underline{\mathbb{F}_2}^*$ is an n-slice for $n \in \{3, 4\}$. □

Proposition 3.2.6 ([GY])**.** *Let $n \geq 5$ and take $r \equiv n \pmod 2$ with $r \in \{3, 4\}$. The slice tower of $\Sigma^n H\underline{\mathbb{F}_2}$ is*

$$\begin{tikzcd}
P^{2n-4}_{2n-4} = \Sigma^{n-2} H\underline{g} \arrow[r] & \Sigma^n H\underline{\mathbb{F}_2} \arrow[d] \\
P^{2n-6}_{2n-6} = \Sigma^{n-3} H\underline{g} \arrow[r] & \Sigma^{\rho+(n-2)} H\underline{\mathbb{F}_2} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{n+r-2}_{n+r-2} = \Sigma^{n-\frac{n-r}{2}-1} H\underline{g} \arrow[r] & \Sigma^{(\frac{n-r}{2}-1)\rho+r+2} H\underline{\mathbb{F}_2} \arrow[d] \\
 & P^n_n = \Sigma^{\frac{n-r}{2}\rho+r} H\underline{\mathbb{F}_2}.
\end{tikzcd}$$

Proof. We take Equation (3.1) and the equivalences

$$\Sigma^{n-4+2\rho} H\underline{\mathbb{F}_2}^* \simeq \Sigma^n H\underline{\mathbb{F}_2} \simeq \Sigma^{n-2+\rho} H\underline{f}$$

to find the fiber sequence $\Sigma^{n-2} H\underline{g} \to \Sigma^n H\underline{\mathbb{F}_2} \to \Sigma^{\rho+(n-2)} H\underline{\mathbb{F}_2}$. We then augment this sequence with its appropriate ρ suspensions until $\Sigma^{\frac{n-r}{2}\rho+r} H\underline{\mathbb{F}_2}$, a slice, is reached. □

Example 3.2.7. The slice tower for $\Sigma^5 H\underline{\mathbb{F}_2}$ is

$$\begin{tikzcd}
P^6_6 = \Sigma^3 H\underline{g} \arrow[r] & \Sigma^5 H\underline{\mathbb{F}_2} \arrow[d] \\
 & P^5_5 = \Sigma^{\rho+3} H\underline{\mathbb{F}_2}.
\end{tikzcd}$$

Proposition 3.2.8. *Let $n \geq 1$ and take $r \equiv n \pmod 2$ with $r \in \{0, -1\}$. The slice cotower of $\Sigma^{-n} H\underline{\mathbb{F}_2}$ is*

$$
\begin{tikzcd}
\Sigma^{-\frac{n-r}{2}\rho-r} H\underline{\mathbb{F}_2} = P^{-n}_{-n} \arrow[d] & \\
\Sigma^{-(\frac{n-r}{2}+1)\rho-r+2} H\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{-n+\frac{n-r}{2}-1} H\underline{g} = P^{-n-r-2}_{-n-r-2} \\
\vdots \arrow[d] & \\
\Sigma^{-\rho-(n-2)} H\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{-n+1} H\underline{g} = P^{-2n+2}_{-2n+2} \\
\Sigma^{-n} H\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{-n} H\underline{g} = P^{-2n}_{-2n}.
\end{tikzcd}
$$

Proof. Equation (3.2) and the equivalence $\Sigma^{-n} H\underline{f} \simeq \Sigma^{-n+2-\rho} H\underline{\mathbb{F}_2}$ provide the cofiber sequence $\Sigma^{-n+2-\rho} H\underline{\mathbb{F}_2} \to \Sigma^{-n} H\underline{\mathbb{F}_2} \to \Sigma^{-n} H\underline{g}$. We then augment this sequence with its appropriate $-\rho$ suspensions until $\Sigma^{-\frac{n-r}{2}\rho-r} H\underline{\mathbb{F}_2}$, a slice, is reached. $\square$

Example 3.2.9. The slice cotower of $\Sigma^{-1} H\underline{\mathbb{F}_2}$ is

$$
\begin{tikzcd}
\Sigma^{-\rho+1} H\underline{\mathbb{F}_2} = P^{-1}_{-1} \arrow[d] & \\
\Sigma^{-1} H\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{-1} H\underline{g} = P^{-2}_{-2}.
\end{tikzcd}
$$

Proposition 3.2.10. *Let $n \geq 1$ and take $r \equiv n \pmod 2$ with $r \in \{0, 1\}$. The slice cotower of $\Sigma^{n\sigma} H\underline{\mathbb{F}_2}$ is*

$$
\begin{tikzcd}
\Sigma^{(n-\frac{n-r}{2})\rho+r} H\underline{\mathbb{F}_2} = P^{n}_{n} \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}-1)\rho+(r-2)} H\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{\frac{n-r}{2}-1} H\underline{g} = P^{n-r-1}_{n-r-1} \\
\vdots \arrow[d] & \\
\Sigma^{(n-1)\rho-(n-2)} H\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{1} H\underline{g} = P^{2}_{2} \\
\Sigma^{n\sigma} H\underline{\mathbb{F}_2} \arrow[r] & H\underline{g} = P^{0}_{0}.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.8 by $n\rho$. $\square$

Example 3.2.11. The slice cotower of $\Sigma^\sigma H\underline{\mathbb{F}_2}$ is

$$\begin{tikzcd}
\Sigma^1 H\underline{\mathbb{F}_2} = P_1^1 \arrow[d] & \\
\Sigma^\sigma H\underline{\mathbb{F}_2} \arrow[r] & H\underline{g} = P_0^0.
\end{tikzcd}$$

Proposition 3.2.12. *Let $n \geq 5$ and take $r \equiv n \pmod 2$ with $r \in \{3, 4\}$. The slice tower of $\Sigma^{-n\sigma} H\underline{\mathbb{F}_2}$ is*

$$\begin{tikzcd}
P_{-4}^{-4} = \Sigma^{-2} H\underline{g} \arrow[r] & \Sigma^{-n\sigma} H\underline{\mathbb{F}_2} \arrow[d] \\
P_{-6}^{-6} = \Sigma^{-3} H\underline{g} \arrow[r] & \Sigma^{(-n+1)\rho+(n-2)} H\underline{\mathbb{F}_2} \arrow[d] \\
 & \vdots \arrow[d] \\
P_{-n+r-2}^{-n+r-2} = \Sigma^{-\frac{n-r}{2}-1} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{2}-1)\rho+r+2} H\underline{\mathbb{F}_2} \arrow[d] \\
 & P_{-n}^{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{\mathbb{F}_2}.
\end{tikzcd}$$

Proof. This follows by suspending the tower in Proposition 3.2.6 by $-n\rho$. □

Example 3.2.13. The slice tower for $\Sigma^{-5\sigma} H\underline{\mathbb{F}_2}$ is

$$\begin{tikzcd}
P_{-4}^{-4} = \Sigma^{-2} H\underline{g} \arrow[r] & \Sigma^{-5\sigma} H\underline{\mathbb{F}_2} \arrow[d] \\
 & P_{-5}^{-5} = \Sigma^{-4\rho+3} H\underline{\mathbb{F}_2}.
\end{tikzcd}$$

Remark 3.2.14. *Proposition 3.2.10 and Proposition 3.2.12 illustrate a method of attaining the tower for $\Sigma^{n+m\sigma} H\underline{M}$. Take the tower for $\Sigma^{n-m} H\underline{M}$ and suspend by $m\rho$.*

3.2.2 Towers for $\Sigma^V H\underline{f}$

Here we cover the slice (co)towers of $\Sigma^V H\underline{f}$ where $V = n$ or $V = n\sigma$ for $n \in \mathbb{Z}$.

Proposition 3.2.15. *The C_2-spectrum $\Sigma^n H\underline{f}$ is a n-slice for $0 \leq n \leq 2$.*

Proof. This follows from Proposition 2.3.4 and Proposition 3.3.1. □

Proposition 3.2.16. *Let $n \geq 3$ and take $r \equiv n \pmod 2$ with $r \in \{1, 2\}$. The slice tower of $\Sigma^n H\underline{f}$ is*

$$\begin{tikzcd}
P^{2n-2}_{2n-2} = \Sigma^{n-1} H\underline{g} \arrow[r] & \Sigma^n H\underline{f} \arrow[d] \\
P^{2n-4}_{2n-4} = \Sigma^{n-2} H\underline{g} \arrow[r] & \Sigma^{\rho+n-2} H\underline{f} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{n-r+2}_{n-r+2} = \Sigma^{\frac{n-r}{2}-1} H\underline{g} \arrow[r] & \Sigma^{(\frac{n-r}{2}+1)\rho+r+2} H\underline{f} \arrow[d] \\
 & P^n_n = \Sigma^{\frac{n-r}{2}\rho+r} H\underline{f}.
\end{tikzcd}$$

Proof. We take Equation (3.1) and the equivalence $\Sigma^{n-2+\rho} H\underline{\mathbb{F}_2}^* \simeq \Sigma^n H\underline{f}$ to find the fiber sequence $\Sigma^{n-1} H\underline{g} \to \Sigma^n H\underline{\mathbb{F}_2} \to \Sigma^{\rho+(n-2)} H\underline{f}$. We then augment this sequence with its appropriate ρ suspensions until $\Sigma^{\frac{n-r}{2}\rho+r} H\underline{\mathbb{F}_2}$, a slice, is reached. $\square$

Example 3.2.17. The slice tower for $\Sigma^3 H\underline{f}$ is

$$\begin{tikzcd}
P^4_4 = \Sigma^2 H\underline{g} \arrow[r] & \Sigma^3 H\underline{f} \arrow[d] \\
 & P^3_3 = \Sigma^{\rho+1} H\underline{f}.
\end{tikzcd}$$

Proposition 3.2.18. *The C_2-spectrum $\Sigma^{-n} H\underline{f}$ is a n-slice for $0 \leq n \leq 2$.*

Proof. For $n = 0, 1$ the result follows from Proposition 2.3.4. For $n = 2$, this follows from Proposition 3.3.1 and the equivalence $\Sigma^\rho H\underline{f} \simeq \Sigma^2 H\underline{\mathbb{F}_2}$. $\square$

Proposition 3.2.19. *Let $n \geq 3$ and take $r \equiv n \pmod 2$ with $r \in \{1, 2\}$. The slice cotower of $\Sigma^{-n} H\underline{f}$ is*

$$
\begin{tikzcd}
\Sigma^{-\frac{n-r}{2}\rho-r} H\underline{f} = P^{-n}_{-n} \arrow[d] & \\
\Sigma^{(-(\frac{n-r}{2}-1)\rho-(r+2)} H\underline{f} \arrow[r] \arrow[d] & \Sigma^{-n+\frac{n-r}{2}} H\underline{g} = P^{-n-r}_{-n-r} \\
\vdots \arrow[d] & \\
\Sigma^{-n} H\underline{f} \arrow[r] & \Sigma^{-n+1} H\underline{g} = P^{-2n+2}_{-2n+2}.
\end{tikzcd}
$$

Proof. We take Equation (3.2) and the equivalence $\Sigma^{-n}H\underline{f} \simeq \Sigma^{-n+2-\rho}H\underline{\mathbb{F}_2}$ to find the cofiber sequence $\Sigma^{-n+2-\rho}H\underline{f} \to \Sigma^{-n}H\underline{f} \to \Sigma^{-n+1}H\underline{g}$. We then augment this sequence with its appropriate $-\rho$ suspensions until $\Sigma^{-\frac{n-r}{2}\rho-r}H\underline{\mathbb{F}_2}$, a slice, is reached. □

Example 3.2.20. The slice cotower for $\Sigma^{-3}H\underline{f}$ is

$$
\begin{tikzcd}
\Sigma^{-\rho-1} H\underline{f} = P^{-3}_{-3} \arrow[d] & \\
\Sigma^{-3} H\underline{f} \arrow[r] & \Sigma^{-2} H\underline{g} = P^{-4}_{-4}.
\end{tikzcd}
$$

Proposition 3.2.21. *Let $n \geq 3$ and take $r \equiv n \pmod 2$ with $r \in \{1, 2\}$. The slice cotower of $\Sigma^{n\sigma}H\underline{f}$ is*

$$
\begin{tikzcd}
\Sigma^{(n-\frac{n-r}{2})\rho-r} H\underline{f} = P^{n}_{n} \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}+1)\rho-(r+2)} H\underline{f} \arrow[r] \arrow[d] & \Sigma^{\frac{n-r}{2}} H\underline{g} = P^{n-r}_{n-r} \\
\vdots \arrow[d] & \\
\Sigma^{n\sigma} H\underline{f} \arrow[r] & \Sigma^{1} H\underline{g} = P^{2}_{2}.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.19 by $n\rho$. □

Example 3.2.22. The slice cotower for $\Sigma^{n\sigma}H\underline{f}$ is

$$
\begin{tikzcd}
\Sigma^{2\rho-r} H\underline{f} = P^{3}_{3} \arrow[d] & \\
\Sigma^{3\sigma} H\underline{f} \arrow[r] & \Sigma^{1} H\underline{g} = P^{2}_{2}.
\end{tikzcd}
$$

Proposition 3.2.23. *Let $n \geq 3$ and take $r \equiv n \pmod 2$ with $r \in \{1, 2\}$. The slice tower of $\Sigma^{-n\sigma} H\underline{f}$ is*

$$\begin{tikzcd}
P^{-2}_{-2} = \Sigma^{-1} H\underline{g} \arrow[r] & \Sigma^{-n\rho+n} H\underline{f} \arrow[d] \\
P^{-4}_{-4} = \Sigma^{-2} H\underline{g} \arrow[r] & \Sigma^{(-n+1)\rho+n-2} H\underline{f} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{-n-(r-2)}_{-n-(r-2)} = \Sigma^{-n+\frac{n-r}{2}+1} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{2}-1)\rho+r+2} H\underline{f} \arrow[d] \\
 & P^{-n}_{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{f}.
\end{tikzcd}$$

Proof. This follows by suspending the tower in Proposition 3.2.16 by $-n\rho$. □

Example 3.2.24. The slice tower for $\Sigma^{-3\sigma} H\underline{f}$ is

$$\begin{tikzcd}
P^{-2}_{-2} = \Sigma^{-1} H\underline{g} \arrow[r] & \Sigma^{-3\sigma} H\underline{f} \arrow[d] \\
 & P^{-3}_{-3} = \Sigma^{-2\rho+1} H\underline{f}.
\end{tikzcd}$$

3.2.3 Towers for $\Sigma^V H\underline{\mathbb{F}_2}^*$

Here we present the slice (co)towers for $\Sigma^V H\underline{\mathbb{F}_2}^*$ where $V = n$ or $V = n\sigma$ for $n \in \mathbb{Z}$.

Proposition 3.2.25. *Let $n \geq 1$ and take $r \equiv n \pmod 2$ with $r \in \{-1, 0\}$. The slice tower of $\Sigma^n H\underline{\mathbb{F}_2}^*$ is*

$$
\begin{tikzcd}
P_{2n}^{2n} = \Sigma^n H\underline{g} \arrow[r] & \Sigma^n H\underline{\mathbb{F}_2}^* \arrow[d] \\
P_{2n-2}^{2n-2} = \Sigma^{n-1} H\underline{g} \arrow[r] & \Sigma^{\rho+(n-2)} H\underline{\mathbb{F}_2}^* \arrow[d] \\
 & \vdots \arrow[d] \\
P_{n-r+2}^{n-r+2} = \Sigma^{n-\frac{n-r}{2}+1} H\underline{g} \arrow[r] & \Sigma^{(\frac{n-r}{2}-1)\rho+r+2} H\underline{\mathbb{F}_2}^* \arrow[d] \\
 & P_n^n = \Sigma^{\frac{n-r}{2}\rho+r} H\underline{\mathbb{F}_2}^*.
\end{tikzcd}
$$

Proof. We take Equation (3.1) and the equivalence $\Sigma^{n-2+\rho} H\underline{\mathbb{F}_2}^* \simeq \Sigma^n H\underline{f}$ to find the fiber sequence $\Sigma^n H\underline{g} \to \Sigma^n H\underline{\mathbb{F}_2}^* \to \Sigma^{\rho+(n-2)} H\underline{f}$. We then augment this sequence with its appropriate ρ suspensions until $\Sigma^{\frac{n-r}{2}\rho+r} H\underline{f}$, a slice, is reached. □

Example 3.2.26. The slice tower for $\Sigma^1 H\underline{\mathbb{F}_2}^*$ is

$$
\begin{tikzcd}
P_2^2 = \Sigma^1 H\underline{g} \arrow[r] & \Sigma^1 H\underline{\mathbb{F}_2}^* \arrow[d] \\
 & P_1^1 = \Sigma^{\rho-1} H\underline{\mathbb{F}_2}^*.
\end{tikzcd}
$$

Proposition 3.2.27. *The C_2-spectrum $\Sigma^{-n} H\underline{\mathbb{F}_2}^*$ is a $(-n)$-slice for $0 \leq n \leq 4$.*

Proof. For $0 \leq n \leq 2$, this follows from Proposition 2.3.4 and Proposition 3.3.1. For $n = 3, 4$, the result follows from Proposition 2.3.4 and the equivalence $\Sigma^\rho H\underline{\mathbb{F}_2}^* \simeq \Sigma^4 H\underline{\mathbb{F}_2}$. □

Proposition 3.2.28. *Let $n \geq 5$ and take $r \equiv n \pmod 2$ with $r \in \{3, 4\}$. The slice cotower of $\Sigma^{-n} H\underline{\mathbb{F}_2}^*$ is*

$$
\begin{tikzcd}
\Sigma^{-\frac{n-r}{2}\rho-r} H\underline{\mathbb{F}_2}^* = P_{-n}^{-n} \arrow[d] & \\
\Sigma^{-(\frac{n-r}{2}-1)\rho-(r+2)} H\underline{\mathbb{F}_2}^* \arrow[r] \arrow[d] & \Sigma^{-n+\frac{n-r}{2}+1} H\underline{g} = P_{-n-(r-2)}^{-n-(r-2)} \\
\vdots \arrow[d] & \\
\Sigma^{-n} H\underline{\mathbb{F}_2}^* \arrow[r] & \Sigma^{-n+2} H\underline{g} = P_{-2n+4}^{-2n+4}.
\end{tikzcd}
$$

Proof. We take Equation (3.2) and the equivalence $\Sigma^{-n}H\underline{f} \simeq \Sigma^{-n+2-\rho}H\underline{\mathbb{F}_2}^*$ to find the cofiber sequence $\Sigma^{-n+2-\rho}H\underline{\mathbb{F}_2}^* \to \Sigma^{-n}H\underline{\mathbb{F}_2}^* \to \Sigma^{-n}H\underline{g}$. We then augment this sequence with its appropriate $-\rho$ suspensions until $\Sigma^{-\frac{n-r}{2}\rho-r}H\underline{\mathbb{F}_2}^*$, a slice, is reached. $\square$

Example 3.2.29. The slice cotower for $\Sigma^{-5}H\underline{\mathbb{F}_2}^*$ is

$$\begin{tikzcd}
\Sigma^{-\rho-3}H\underline{\mathbb{F}_2}^* = P^{-5}_{-5} \arrow[d] & \\
\Sigma^{-5}H\underline{\mathbb{F}_2}^* \arrow[r] & \Sigma^{-3}H\underline{g} = P^{-6}_{-6}.
\end{tikzcd}$$

Proposition 3.2.30. *Let $n \geq 5$ and take $r \equiv n \pmod 2$ with $r \in \{3, 4\}$. The slice cotower of $\Sigma^{n\sigma}H\underline{\mathbb{F}_2}^*$ is*

$$\begin{tikzcd}
\Sigma^{(n-\frac{n-r}{2})\rho-r}H\underline{\mathbb{F}_2}^* = P^n_n \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}+1)\rho-(r+2)}H\underline{\mathbb{F}_2}^* \arrow[r] \arrow[d] & \Sigma^{\frac{n-r}{2}+1}H\underline{g} = P^{n-r+2}_{n-r+2} \\
\vdots \arrow[d] & \\
\Sigma^{n\sigma}H\underline{\mathbb{F}_2}^* \arrow[r] & \Sigma^2 H\underline{g} = P^4_4.
\end{tikzcd}$$

Proof. This follows by suspending the tower in Proposition 3.2.28 by $n\rho$. $\square$

Example 3.2.31. The slice cotower for $\Sigma^{5\sigma}H\underline{\mathbb{F}_2}^*$ is

$$\begin{tikzcd}
\Sigma^{4\rho-3}H\underline{\mathbb{F}_2}^* = P^5_5 \arrow[d] & \\
\Sigma^{5\sigma}H\underline{\mathbb{F}_2}^* \arrow[r] & \Sigma^2 H\underline{g} = P^4_4.
\end{tikzcd}$$

Proposition 3.2.32. *Let $n \geq 1$ and take $r \equiv n \pmod 2$ with $r \in \{-1, 0\}$. The slice tower of $\Sigma^{-n\sigma}H\underline{\mathbb{F}_2}^*$ is*

$$
\begin{tikzcd}
P_0^0 = H\underline{g} \arrow[r] & \Sigma^{-n\sigma} H\underline{\mathbb{F}_2}^* \arrow[d] \\
P_{-2}^{-2} = \Sigma^{-1} H\underline{g} \arrow[r] & \Sigma^{(-n+1)\rho+(n-2)} H\underline{\mathbb{F}_2}^* \arrow[d] \\
 & \vdots \arrow[d] \\
P_{-n+r+2}^{-n+r+2} = \Sigma^{-\frac{n-r}{2}+1} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{2}-1)\rho+r+2} H\underline{\mathbb{F}_2}^* \arrow[d] \\
 & P_{-n}^{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{\mathbb{F}_2}^*.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.25 by $-n\rho$. □

Example 3.2.33. The slice tower for $\Sigma^{-\sigma} H\underline{\mathbb{F}_2}^*$ is

$$
\begin{tikzcd}
P_0^0 = H\underline{g} \arrow[r] & \Sigma^{-\sigma} H\underline{\mathbb{F}_2}^* \arrow[d] \\
 & P_{-1}^{-1} = \Sigma^{-1} H\underline{\mathbb{F}_2}^*.
\end{tikzcd}
$$

3.2.4 Towers for $\Sigma^V H\underline{\mathbb{Z}}$

Here we determine the slice (co)towers for $\Sigma^V H\underline{\mathbb{Z}}$ where $V = n$ or $V = n\sigma$ where $n \in \mathbb{Z}$.

Proposition 3.2.34. *The C_2-spectrum $\Sigma^n H\underline{\mathbb{Z}}$ is an n-slice for $0 \leq n \leq 6$.*

Proof. For $0 \leq n \leq 2$, this follows from Proposition 2.3.4 and Proposition 3.3.1. By the same results, $\Sigma^n H\underline{\mathbb{Z}}^*$ is an n-slice for $0 \leq n \leq 2$. Furthermore, $\Sigma^{-1} H\underline{\mathbb{Z}}^*$ is a (-1)-slice by Proposition 2.3.4. Then, by Proposition 3.3.1 and Proposition 2.3.5, $\Sigma^n H\underline{\mathbb{Z}} \simeq \Sigma^{n-4+2\rho} H\underline{\mathbb{Z}}^*$ is a n-slice for $3 \leq n \leq 6$. □

Proposition 3.2.35. *Let $n \geq 7$ and take $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. The slice tower of $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ is*

$$\begin{tikzcd}
P^{2n-6}_{2n-6} = \Sigma^{n-3} H\underline{g} \arrow[r] & \Sigma^n H\underline{\mathbb{Z}} \arrow[d] \\
P^{2n-10}_{2n-10} = \Sigma^{n-5} H\underline{g} \arrow[r] & \Sigma^{2\rho+(n-4)} H\underline{\mathbb{Z}} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{n+r-2}_{n+r-2} = \Sigma^{\frac{n+r}{2}-1} H\underline{g} \arrow[r] & \Sigma^{\frac{n-r-4}{2}\rho+(r+4)} H\underline{\mathbb{Z}} \arrow[d] \\
 & P^n_n = \Sigma^{\frac{n-r}{2}\rho+r} H\underline{\mathbb{Z}}.
\end{tikzcd}$$

Proof. The exact sequence $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \to \underline{g}$ and the homotopy equivalence $\Sigma^n H\underline{\mathbb{Z}} \simeq \Sigma^{n-4+2\rho} H\underline{\mathbb{Z}}^*$ provide the fiber sequence

$$\Sigma^{n-3} H\underline{g} \to \Sigma^n H\underline{\mathbb{Z}} \to \Sigma^{2\rho+(n-4)} H\underline{\mathbb{Z}}.$$

We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{\frac{n-r}{2}\rho+r} H\underline{\mathbb{Z}}$, a slice, is reached. $\square$

Example 3.2.36. The slice tower for $\Sigma^7 H_{C_2}\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P^8_8 = \Sigma^4 H\underline{g} \arrow[r] & \Sigma^7 H\underline{\mathbb{Z}} \arrow[d] \\
 & P^7_7 = \Sigma^{2\rho+3} H\underline{\mathbb{Z}}.
\end{tikzcd}$$

Corollary 3.2.37. *Let $n \geq 7$ and take $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. The $(2k)$-slices of $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ are*

$$P^{2k}_{2k}(\Sigma^n H_{C_2}\underline{\mathbb{Z}}) \simeq \Sigma^k H_{C_2}\underline{g}$$

for $k \equiv n+1 \pmod 2$ and $k \in \left[\frac{n+r}{2} - 1, \ldots, n-3\right]$.

Proposition 3.2.38. *Let $n \geq 1$ and take $r \equiv n \pmod 4$ with $1 \leq r \leq 4$. The slice cotower of $\Sigma^{-n} H_{C_2}\underline{\mathbb{Z}}$ is*

$$\begin{tikzcd}
\Sigma^{-\frac{n-r}{2}\rho - r} H\underline{\mathbb{Z}}^* = P^{-n}_{-n} \arrow[d] & \\
\Sigma^{-\frac{n-r}{2}\rho - r} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-\frac{n+r}{2}} H\underline{g} = P^{-n-r}_{-n-r} \\
\vdots \arrow[d] & \\
\Sigma^{-2\rho-n+4} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-n+2} H\underline{g} = P^{-2n+4}_{-2n+4} \\
\Sigma^{-n} H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-n} H\underline{g} = P^{-2n}_{-2n}.
\end{tikzcd}$$

Proof. The exact sequence $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \to \underline{g}$ and the equivalence $\Sigma^{-n} H\underline{\mathbb{Z}} \simeq \Sigma^{2\rho-n-4} H\underline{\mathbb{Z}}$ provide the cofiber sequence

$$\Sigma^{-2\rho-n+4} H\underline{\mathbb{Z}} \to \Sigma^{-n} H\underline{\mathbb{Z}} \to \Sigma^{-n} H\underline{g}.$$

We then augment this sequence with its appropriate -2ρ suspensions until $\Sigma^{-\frac{n-r}{2}\rho - r} H\underline{\mathbb{Z}}$, a slice, is reached. $\square$

Example 3.2.39. The slice cotower for $\Sigma^{-1} H_{C_2}\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
\Sigma^{-1} H\underline{\mathbb{Z}}^* = P^{-1}_{-1} \arrow[d] & \\
\Sigma^{-1} H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-1} H\underline{g} = P^{-2}_{-2}.
\end{tikzcd}$$

Corollary 3.2.40. *Let $n \geq 1$ and take $r \equiv n \pmod 4$ with $1 \leq r \leq 4$. The $(-2k)$-slices of $\Sigma^{-n} H_{C_2}\underline{\mathbb{Z}}$ are*

$$P^{-2k}_{-2k}(\Sigma^{-n} H_{C_2}\underline{\mathbb{Z}}) \simeq \Sigma^{-k} H_{C_2}\underline{g}$$

for $k \equiv n \pmod 2$ and $k \in \left[\frac{n+r}{2}, n\right]$.

Proposition 3.2.41. *Let $n \geq 1$ and take $r \equiv n \pmod 4$ with $1 \leq r \leq 4$. The slice cotower of $\Sigma^{n\sigma} H\underline{\mathbb{Z}}$ is*

$$
\begin{tikzcd}
\Sigma^{\frac{n+r}{2}\rho-r} H\underline{\mathbb{Z}}^* = P_n^n \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}+2)\rho-r-4} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{2(\frac{n-r}{4}-1)} H\underline{g} = P_{n-r-1}^{n-r-1} \\
\vdots \arrow[d] & \\
\Sigma^{(n-2)\rho-n+4} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^2 H\underline{g} = P_4^4 \\
\Sigma^{n\sigma} H\underline{\mathbb{Z}} \arrow[r] & H\underline{g} = P_0^0.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.38 by $n\rho$. □

Example 3.2.42. The slice cotower for $\Sigma^\sigma H\underline{\mathbb{Z}}$ is

$$
\begin{tikzcd}
\Sigma^{\rho-1} H\underline{\mathbb{Z}}^* = P_1^1 \arrow[d] & \\
\Sigma^\sigma H\underline{\mathbb{Z}} \arrow[r] & H\underline{g} = P_0^0.
\end{tikzcd}
$$

Proposition 3.2.43. *Let $n \geq 7$ and take $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. The slice tower of $\Sigma^{-n\sigma} H\underline{\mathbb{Z}}$ is*

$$
\begin{tikzcd}
P_{-2}^{-2} = \Sigma^{-1} H\underline{g} \arrow[r] & \Sigma^{-n\sigma} H\underline{\mathbb{Z}} \arrow[d] \\
P_{-6}^{-6} = \Sigma^{-3} H\underline{g} \arrow[r] & \Sigma^{(-n+2)\rho+(n-4)} H\underline{\mathbb{Z}} \arrow[d] \\
 & \vdots \arrow[d] \\
P_{-n+r+1}^{-n+r+1} = \Sigma^{-\frac{n-r}{2}+1} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{2}-2)\rho+(r+4)} H\underline{\mathbb{Z}} \arrow[d] \\
 & P_{-n}^{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{\mathbb{Z}}.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.35 by $-n\rho$. □

Example 3.2.44. The slice tower for $\Sigma^{-7\sigma} H\underline{\mathbb{Z}}$ is

$$
\begin{tikzcd}
P^{-2}_{-2} = \Sigma^{-1} H\underline{g} \arrow[r] & \Sigma^{-7\sigma} H\underline{\mathbb{Z}} \arrow[d] \\
P^{-6}_{-6} = \Sigma^{-3} H\underline{g} \arrow[r] & \Sigma^{(-n+2)\rho+(n-4)} H\underline{\mathbb{Z}} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{-n+r+1}_{-n+r+1} = \Sigma^{-\frac{n-r}{4}+1} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{4}-2)\rho+(r+4)} H\underline{\mathbb{Z}} \arrow[d] \\
 & P^{-7}_{-7} = \Sigma^{-5\rho+3} H\underline{\mathbb{Z}}.
\end{tikzcd}
$$

3.2.5 Towers for $\Sigma^V H\underline{\hat{f}}$

Here we determine the slice (co)towers for $\Sigma^V H\underline{\hat{f}}$ where $V = n$ or $V = n\sigma$ for $n \in \mathbb{Z}$.

Proposition 3.2.45. *The C_2-spectrum $\Sigma^n H\underline{\hat{f}}$ is a n-slice for $0 \leq n \leq 4$.*

Proof. For $0 \leq n \leq 2$, this follows from Proposition 2.3.4 and Proposition 3.3.1. For $n = 3, 4$, it follows from Proposition 2.3.4 and the equivalence $\Sigma^\rho H\underline{\mathbb{Z}}^* \simeq \Sigma^2 H\underline{\hat{f}}$. $\square$

Proposition 3.2.46. *Let $n \geq 5$ and take $r \equiv n \pmod 4$ with $-1 \leq r \leq 2$. The slice tower of $\Sigma^n H\underline{\hat{f}}$ is*

$$
\begin{tikzcd}
P^{2n-4}_{2n-4} = \Sigma^{n-2} H\underline{g} \arrow[r] & \Sigma^n H\underline{\hat{f}} \arrow[d] \\
P^{2n-12}_{2n-12} = \Sigma^{n-6} H\underline{g} \arrow[r] & \Sigma^{2\rho+(n-4)} H\underline{\hat{f}} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{2r+4}_{2r+4} = \Sigma^{r+2} H\underline{g} \arrow[r] & \Sigma^{(\frac{n-r}{4}-2)\rho+(r+4)} H\underline{\hat{f}} \arrow[d] \\
 & P^n_n = \Sigma^{\frac{n-r}{2}\rho+r} H\underline{\hat{f}}.
\end{tikzcd}
$$

Proof. We take Equation (3.3) and the equivalences

$$\Sigma^{n-2+\rho} H\underline{\mathbb{Z}}^* \simeq \Sigma^n H\underline{\hat{f}} \simeq \Sigma^{n+2-\rho} H\underline{\mathbb{Z}}$$

to find the fiber sequence $\Sigma^{n-2}H\underline{g} \to \Sigma^n H\underline{\hat{f}} \to \Sigma^{2\rho+n-4}H\underline{\hat{f}}$. We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{\frac{n-r}{4}\rho+r}H\underline{\hat{f}}$, a slice, is reached. $\square$

Example 3.2.47. The slice tower for $\Sigma^5 H\underline{\hat{f}}$ is

$$\begin{tikzcd} P_6^6 = \Sigma^3 H\underline{g} \arrow[r] & \Sigma^5 H\underline{\hat{f}} \arrow[d] \\ & P_5^5 = \Sigma^{\rho+3}H\underline{\hat{f}}. \end{tikzcd}$$

Proposition 3.2.48. *The C_2-spectrum $\Sigma^{-n}H\underline{\hat{f}}$ is a $(-n)$-slice for $0 \leq n \leq 2$.*

Proof. This follows from Proposition 2.3.4, Proposition 3.3.1, and the equivalence $\Sigma^{2\rho}H\underline{\hat{f}} \simeq \Sigma^4 H\underline{\mathbb{Z}}$. $\square$

Proposition 3.2.49. *Let $n \geq 3$ and take $r \equiv n \pmod 4$ with $-1 \leq r \leq 2$. The slice tower of $\Sigma^{-n}H\underline{\hat{f}}$ is*

$$\begin{tikzcd} \Sigma^{-\frac{n-r}{2}\rho-r}H\underline{\hat{f}} = P_{-n}^{-n} \arrow[d] & \\ \Sigma^{-2(\frac{n-r}{4}-1)\rho-r-4}H\underline{\hat{f}} \arrow[r] \arrow[d] & \Sigma^{-2m}H\underline{g} = P_{-n+r}^{-n+r} \\ \vdots \arrow[d] & \\ \Sigma^{-2\rho-n+4}H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-n+3}H\underline{g} = P_{-2n+6}^{-2n+6} \\ \Sigma^{-n}H\underline{f} \arrow[r] & \Sigma^{-n+1}H\underline{g} = P_{-2n+2}^{-2n+2}. \end{tikzcd}$$

Proof. We take Equation (3.3) and the equivalences

$$\Sigma^{-n-2+\rho}H\underline{\mathbb{Z}}^* \simeq \Sigma^{-n}H\underline{\hat{f}} \simeq \Sigma^{-n+2-\rho}H\underline{\mathbb{Z}}$$

to find the fiber sequence $\Sigma^{-n+4-2\rho}H\underline{\hat{f}} \to \Sigma^{-n}H\underline{\hat{f}} \to \Sigma^{-n+1}H\underline{g}$. We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{-\frac{n-r}{2}\rho-r}H\underline{\hat{f}}$, a slice, is reached. $\square$

Example 3.2.50. The slice cotower for $\Sigma^{-3}H\underline{\hat{f}}$ is

$$\begin{tikzcd} \Sigma^{-2\rho+1}H\underline{\hat{f}} = P_{-3}^{-3} \arrow[d] & \\ \Sigma^{-3}H\underline{f} \arrow[r] & \Sigma^{-2}H\underline{g} = P_{-4}^{-4}. \end{tikzcd}$$

Proposition 3.2.51. *Let $n \geq 3$ and take $r \equiv n \pmod 4$ with $-1 \leq r \leq 2$. The slice cotower of $\Sigma^{n\sigma} H\underline{\hat{f}}$ is*

$$\begin{tikzcd}
\Sigma^{(n-\frac{n-r}{2})\rho-r} H\underline{\hat{f}} = P^n_n \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}+2)\rho-r-4} H\underline{\hat{f}} \arrow[r] \arrow[d] & \Sigma^{n-\frac{n-r}{2}} H\underline{g} = P^{n+r}_{n+r} \\
\vdots \arrow[d] & \\
\Sigma^{(n-2)\rho-n+4} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^3 H\underline{g} = P^6_6 \\
\Sigma^{n\sigma} H\underline{\hat{f}} \arrow[r] & \Sigma^1 H\underline{g} = P^2_2.
\end{tikzcd}$$

Proof. This follows by suspending the tower in Proposition 3.2.49 by $n\rho$. □

Example 3.2.52. The slice cotower for $\Sigma^{3\sigma} H\underline{\hat{f}}$ is

$$\begin{tikzcd}
\Sigma^{\rho+1} H\underline{\hat{f}} = P^3_3 \arrow[d] & \\
\Sigma^{3\sigma} H\underline{\hat{f}} \arrow[r] & \Sigma^1 H\underline{g} = P^2_2.
\end{tikzcd}$$

Proposition 3.2.53. *Let $n \geq 5$ and take $r \equiv n \pmod 4$ with $-1 \leq r \leq 2$. The slice tower of $\Sigma^{-n\sigma} H\underline{\hat{f}}$ is*

$$\begin{tikzcd}
P^{-4}_{-4} = \Sigma^{-2} H\underline{g} \arrow[r] & \Sigma^{-n\sigma} H\underline{\hat{f}} \arrow[d] \\
P^{-12}_{-12} = \Sigma^{-6} H\underline{g} \arrow[r] & \Sigma^{(-n+2)\rho+(n-4)} H\underline{\hat{f}} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{-2n+2r+4}_{-2n+2r+4} = \Sigma^{-n+r+2} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{4}-2)\rho+(r+4)} H\underline{\hat{f}} \arrow[d] \\
 & P^{-n}_{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{\hat{f}}.
\end{tikzcd}$$

Proof. This follows by suspending the tower in Proposition 3.2.46 by $-n\rho$. $\square$

Example 3.2.54. The slice tower for $\Sigma^{-5\sigma}H\underline{\hat{f}}$ is

$$\begin{tikzcd} P_{-4}^{-4} = \Sigma^{-2}H\underline{g} \arrow[r] & \Sigma^{-4\sigma}H\underline{\hat{f}} \arrow[d] \\ & P_{-5}^{-5} = \Sigma^{-4\rho+3}H\underline{\hat{f}}. \end{tikzcd}$$

3.2.6 Towers for $\Sigma^V H\underline{\mathbb{Z}}^*$

Here we present the slice (co)towers for $\Sigma^V H\underline{\mathbb{Z}}^*$ where $V = n$ or $V = n\sigma$ for $n \in \mathbb{Z}$.

Proposition 3.2.55. *The C_2-spectrum $\Sigma^n H\underline{\mathbb{Z}}^*$ is a n-slice for $0 \leq n \leq 2$.*

Proof. The result follows from Proposition 2.3.4 and Proposition 3.3.1. $\square$

Proposition 3.2.56. *Let $n \geq 3$ and take $r \equiv n \pmod 4$ with $-1 \leq r \leq 2$. The slice tower of $\Sigma^n H\underline{\mathbb{Z}}^*$ is*

$$\begin{tikzcd} P_{2n-2}^{2n-2} = \Sigma^{n-1}H\underline{g} \arrow[r] & \Sigma^n H\underline{\mathbb{Z}}^* \arrow[d] \\ P_{2n-6}^{2n-6} = \Sigma^{n-3}H\underline{g} \arrow[r] & \Sigma^{2\rho+(n-4)}H\underline{\mathbb{Z}}^* \arrow[d] \\ & \vdots \arrow[d] \\ P_{n+r+2}^{n+r+2} = \Sigma^{n-\frac{n-r}{2}+1}H\underline{g} \arrow[r] & \Sigma^{2(\frac{n-r}{4}-1)\rho+(r+4)}H\underline{\mathbb{Z}}^* \arrow[d] \\ & P_n^n = \Sigma^{\frac{n-r}{2}\rho+r}H\underline{\mathbb{Z}}^*. \end{tikzcd}$$

Proof. We take Equation (3.3) and the homotopy equivalence $\Sigma^{n-4+2\rho}H\underline{\mathbb{Z}}^* \simeq \Sigma^n H\underline{\mathbb{Z}}$ to find the fiber sequence $\Sigma^{n-1}H\underline{g} \to \Sigma^n H\underline{\mathbb{Z}}^* \to \Sigma^{n-4+2\rho}H\underline{\mathbb{Z}}^*$. We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{\frac{n-r}{2}\rho+r}H\underline{\mathbb{Z}}^*$, a slice, is reached. $\square$

Example 3.2.57. The slice tower for $\Sigma^3 H\underline{\mathbb{Z}}^*$ is

$$\begin{tikzcd} P_4^4 = \Sigma^2 H\underline{g} \arrow[r] & \Sigma^3 H\underline{\mathbb{Z}}^* \arrow[d] \\ & P_5^5 = \Sigma^{2\rho-1}H\underline{\mathbb{Z}}^*. \end{tikzcd}$$

Proposition 3.2.58. *The C_2-spectrum $\Sigma^{-n}H\underline{\mathbb{Z}}^*$ is a $(-n)$-slice for $0 \leq n \leq 4$.*

Proof. For $n = 0, 1$, this follows from Proposition 2.3.4. For $n = 3, 4$, the result follows from Proposition 3.2.34 and the equivalence $\Sigma^{2\rho}H\underline{\mathbb{Z}}^* \simeq \Sigma^4 H\underline{\mathbb{Z}}$. □

Proposition 3.2.59. *Let $n \geq 5$ and take $r \equiv n \pmod 4$ with $1 \leq r \leq 4$. The slice cotower of $\Sigma^{-n}H\underline{\mathbb{Z}}^*$ is*

$$\begin{tikzcd}
\Sigma^{-\frac{n-r}{2}\rho-r}H\underline{\mathbb{Z}}^* = P^{-n}_{-n} \arrow[d] & \\
\Sigma^{(-\frac{n-r}{2}+2)\rho-r-4}H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-\frac{n-r}{2}-r-2}H\underline{g} = P^{-n-r-4}_{-n-r-4} \\
\Sigma^{(-\frac{n-r}{2}+4)\rho-r-8}H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-\frac{n-r}{2}-r-4}H\underline{g} = P^{-n-r-8}_{-n-r-8} \\
\vdots \arrow[d] & \\
\Sigma^{-2\rho-n+4}H\underline{\mathbb{Z}}^* \arrow[r] \arrow[d] & \Sigma^{-n+4}H\underline{g} = P^{-2n+8}_{-2n+8} \\
\Sigma^{-n}H\underline{\mathbb{Z}}^* \arrow[r] & \Sigma^{-n+2}H\underline{g} = P^{-2n+4}_{-2n+4}.
\end{tikzcd}$$

Proof. We take Equation (3.3) and the equivalence $\Sigma^{-n-4+2\rho}H\underline{\mathbb{Z}}^* \simeq \Sigma^{-n}H\underline{\mathbb{Z}}$ to find the fiber sequence $\Sigma^{-2\rho-n+4}H\underline{\mathbb{Z}}^* \to \Sigma^{-n}H\underline{\mathbb{Z}}^* \to \Sigma^{-n+2}H\underline{g}$. We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{-\frac{n-r}{2}\rho-r}H\underline{\mathbb{Z}}^*$, a slice, is reached. □

Example 3.2.60. The slice cotower for $\Sigma^{-5}H\underline{\mathbb{Z}}^*$ is

$$\begin{tikzcd}
\Sigma^{-2\rho-1}H\underline{\mathbb{Z}}^* = P^{-5}_{-5} \arrow[d] & \\
\Sigma^{-5}H\underline{\mathbb{Z}}^* \arrow[r] & \Sigma^{-3}H\underline{g} = P^{-6}_{-6}.
\end{tikzcd}$$

Proposition 3.2.61. *Let $n \geq 5$ and take $r \equiv n \pmod 4$ with $1 \leq r \leq 4$. The slice tower of $\Sigma^{n\sigma}H\underline{\mathbb{Z}}^*$ is*

$$
\begin{tikzcd}
\Sigma^{(n-\frac{n-r}{2})\rho-r} H\underline{\mathbb{Z}}^* = P^n_n \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}+2)\rho-r-4} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{n-\frac{n-r}{2}-r-2} H\underline{g} = P^{n-r-4}_{n-r-4} \\
\Sigma^{(n-\frac{n-r}{2}+4)\rho-r-8} H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{n-\frac{n-r}{2}-r-4} H\underline{g} = P^{n-r-8}_{n-r-8} \\
\vdots \arrow[d] & \\
\Sigma^{(n-2)\rho-n+4} H\underline{\mathbb{Z}}^* \arrow[r] \arrow[d] & \Sigma^4 H\underline{g} = P^8_8 \\
\Sigma^{n\sigma} H\underline{\mathbb{Z}}^* \arrow[r] & \Sigma^2 H\underline{g} = P^4_4.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.59 by $n\rho$. □

Example 3.2.62. The slice cotower for $\Sigma^{5\sigma} H\underline{\mathbb{Z}}^*$ is

$$
\begin{tikzcd}
\Sigma^{3\rho-1} H\underline{\mathbb{Z}}^* = P^5_5 \arrow[d] & \\
\Sigma^{5\sigma} H\underline{\mathbb{Z}}^* \arrow[r] & \Sigma^2 H\underline{g} = P^4_4.
\end{tikzcd}
$$

Proposition 3.2.63. *Let $n \geq 3$ and take $r \equiv n \pmod 4$ with $-1 \leq r \leq 2$. The slice tower of $\Sigma^{-n\sigma} H\underline{\mathbb{Z}}^*$ is*

$$
\begin{tikzcd}
P^{-2}_{-2} = \Sigma^{-1} H\underline{g} \arrow[r] & \Sigma^{-n\sigma} H\underline{\mathbb{Z}}^* \arrow[d] \\
P^{-6}_{-6} = \Sigma^{-3} H\underline{g} \arrow[r] & \Sigma^{(-n+2)\rho+(n-4)} H\underline{\mathbb{Z}}^* \arrow[d] \\
 & \vdots \arrow[d] \\
P^{-n+r+2}_{-n+r+2} = \Sigma^{-\frac{n-r}{2}+1} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{2}-2))\rho+(r+4)} H\underline{\mathbb{Z}}^* \arrow[d] \\
 & P^{-n}_{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{\mathbb{Z}}^*.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.56 by $-n\rho$. □

Example 3.2.64. The slice tower for $\Sigma^{-3\sigma}H\underline{\mathbb{Z}}^*$ is

$$\begin{tikzcd} P^{-2}_{-2} = \Sigma^{-1}H\underline{g} \arrow[r] & \Sigma^{-3\sigma}H\underline{\mathbb{Z}}^* \arrow[d] \\ & P^{-3}_{-3} = \Sigma^{-2\rho+1}H\underline{\mathbb{Z}}^*. \end{tikzcd}$$

3.2.7 Towers for $\Sigma^V H\underline{Q}$

Here we determine the slice (co)towers for $\Sigma^V H\underline{Q}$ where $V = n$ or $V = n\sigma$ for $n \in \mathbb{Z}$.

Proposition 3.2.65. *Let $n \geq 1$ and take $r \equiv n \pmod 4$ with $-3 \leq r \leq 0$. The slice tower of $\Sigma^n H\underline{Q}$ is*

$$\begin{tikzcd} P^{2n}_{2n} = \Sigma^n H\underline{g} \arrow[r] & \Sigma^n H\underline{Q} \arrow[d] \\ P^{2n-4}_{2n-4} = \Sigma^{n-2}H\underline{g} \arrow[r] & \Sigma^{2\rho+n-4}H\underline{Q} \arrow[d] \\ & \vdots \arrow[d] \\ P^{n+r+4}_{n+r+4} = \Sigma^{n-2(\frac{n-r}{4}-1)}H\underline{g} \arrow[r] & \Sigma^{2(\frac{n-r}{4}-1)\rho+r+4}H\underline{Q} \arrow[d] \\ & P^n_n = \Sigma^{\frac{n-r}{2}\rho+r}H\underline{Q}. \end{tikzcd}$$

Proof. We take Equation (3.3) and the homotopy equivalences

$$\Sigma^{n+2-\rho}H\underline{\mathbb{Z}}^* \simeq \Sigma^n H\underline{Q} \simeq \Sigma^{n+6-3\rho}H\underline{\mathbb{Z}}$$

to find the fiber sequence $\Sigma^n H\underline{g} \to \Sigma^n H\underline{Q} \to \Sigma^{n-4+2\rho}H\underline{g}$. We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{\frac{n-r}{2}\rho+r}H\underline{Q}$, a slice, is reached. $\square$

Example 3.2.66. The slice tower for $\Sigma^1 H\underline{Q}$ is

$$\begin{tikzcd} P^2_2 = \Sigma^1 H\underline{g} \arrow[r] & \Sigma^1 H\underline{Q} \arrow[d] \\ & P^n_n = \Sigma^{2\rho-3}H\underline{Q}. \end{tikzcd}$$

Proposition 3.2.67. *The C_2-spectrum $\Sigma^{-n}H\underline{Q}$ is a $(-n)$-slice for $0 \leq r \leq 6$.*

Proof. For $n = 0, 1$ this follows from Proposition 2.3.4. For $3 \leq r \leq 6$ the result follows from Proposition 3.2.58 and the equivalence $\Sigma^{\rho} H\underline{Q} \simeq \Sigma^{2} H\underline{\mathbb{Z}}^{*}$. $\square$

Proposition 3.2.68. *Let $n \geq 7$ and take $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. The slice cotower of $\Sigma^{-n} H\underline{Q}$ is*

$$\begin{tikzcd}
\Sigma^{-\frac{n-r}{2}\rho - r} H\underline{Q} = P^{-n}_{-n} \arrow[d] & \\
\Sigma^{-2(\frac{n-r}{4}-1)\rho - r - 4} H\underline{Q} \arrow[r] \arrow[d] & \Sigma^{-n+3+2(\frac{n-r}{4}-1)} H\underline{g} = P^{-n-r+2}_{-n-r+2} \\
\vdots \arrow[d] & \\
\Sigma^{-2\rho-n+4} H\underline{Q} \arrow[r] \arrow[d] & \Sigma^{-n+5} H\underline{g} = P^{-2n+10}_{-2n+10} \\
\Sigma^{-n} H\underline{Q} \arrow[r] & \Sigma^{-n+3} H\underline{g} = P^{-2n+6}_{-2n+6}.
\end{tikzcd}$$

Proof. We take Equation (3.3) and the equivalences

$$\Sigma^{-n+2-\rho} H\underline{\mathbb{Z}}^{*} \simeq \Sigma^{-n} H\underline{Q} \simeq \Sigma^{-n+6-3\rho} H\underline{\mathbb{Z}}$$

to find the fiber sequence $\Sigma^{-n+4-2\rho} H\underline{Q} \to \Sigma^{-n} H\underline{Q} \to \Sigma^{-n+3} H\underline{g}$. We then augment this sequence with its appropriate 2ρ suspensions until $\Sigma^{-\frac{n-r}{2}\rho - r} H\underline{Q}$, a slice, is reached. $\square$

Example 3.2.69. The slice cotower for $\Sigma^{-7} H\underline{Q}$ is

$$\begin{tikzcd}
\Sigma^{-2\rho-3} H\underline{Q} = P^{-7}_{-7} \arrow[d] & \\
\Sigma^{-7} H\underline{Q} \arrow[r] & \Sigma^{-4} H\underline{g} = P^{-8}_{-8}.
\end{tikzcd}$$

Proposition 3.2.70. *Let $n \geq 7$ and take $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. The slice cotower of $\Sigma^{n\sigma} H\underline{Q}$ is*

$$
\begin{tikzcd}
\Sigma^{(n-\frac{n-r}{2})\rho-r} H\underline{Q} = P^n_n \arrow[d] & \\
\Sigma^{(n-\frac{n-r}{2}+2))\rho-r-4} H\underline{Q} \arrow[r] \arrow[d] & \Sigma^{3+2(\frac{n-r}{4}-1)} H\underline{g} = P^{\frac{n-r}{2}+1}_{\frac{n-r}{2}+1} \\
\vdots \arrow[d] & \\
\Sigma^{(n-2)\rho-n+4} H\underline{Q} \arrow[r] \arrow[d] & \Sigma^5 H\underline{g} = P^{10}_{10} \\
\Sigma^{n\sigma} H\underline{Q} \arrow[r] & \Sigma^3 H\underline{g} = P^6_6.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.68 by $n\rho$. □

Example 3.2.71. The slice cotower for $\Sigma^{7\sigma} H\underline{Q}$ is

$$
\begin{tikzcd}
\Sigma^{5\rho-3} H\underline{Q} = P^7_7 \arrow[d] & \\
\Sigma^{7\sigma} H\underline{Q} \arrow[r] & \Sigma^3 H\underline{g} = P^6_6.
\end{tikzcd}
$$

Proposition 3.2.72. *Let $n \geq 1$ and take $r \equiv n \pmod 4$ with $-3 \leq r \leq 0$. The slice tower of $\Sigma^{-n\sigma} H\underline{Q}$ is*

$$
\begin{tikzcd}
P^0_0 = H\underline{g} \arrow[r] & \Sigma^{-n\sigma} H\underline{Q} \arrow[d] \\
P^{-4}_{-4} = \Sigma^{-2} H\underline{g} \arrow[r] & \Sigma^{(-n+2)\rho+n-4} H\underline{Q} \arrow[d] \\
 & \vdots \arrow[d] \\
P^{-\frac{n-r}{2}+2}_{-\frac{n-r}{2}+2} = \Sigma^{-2(\frac{n-r}{4}-1)} H\underline{g} \arrow[r] & \Sigma^{(-n+\frac{n-r}{2}-2)\rho+r+4} H\underline{Q} \arrow[d] \\
 & P^{-n}_{-n} = \Sigma^{(-n+\frac{n-r}{2})\rho+r} H\underline{Q}.
\end{tikzcd}
$$

Proof. This follows by suspending the tower in Proposition 3.2.65 by $-n\rho$. □

Example 3.2.73. The slice tower for $\Sigma^{-\sigma} H\underline{Q}$ is

$$\begin{tikzcd} P_0^0 = H\underline{g} \arrow[r] & \Sigma^{-\sigma} H\underline{Q} \arrow[d] \\ & P_{-1}^{-1} = \Sigma^{\rho-3} H\underline{Q}. \end{tikzcd}$$

3.3 Review of C_2 2-slices

Any 2-slices over K must restrict to a 2-slice over C_2, so we consider the characterization over C_2 first.

Proposition 3.3.1. *A C_2-spectrum Y is a 2-slice over C_2 if and only if the only nontrivial homotopy Mackey functors are $\underline{\pi}_1(Y)$ and $\underline{\pi}_2(Y)$, where*

1. *The restriction map $\boldsymbol{r}_e^{C_2} : \pi_2^{C_2}(Y) \to \pi_2^e(Y)$ is injective*
2. *$\pi_1^e(Y) = 0$.*

That is, both $\Sigma^2 H\underline{\pi}_2(Y)$ and $\Sigma^1 \underline{\pi}_1(Y)$ are 2-slices.

Proof. Let $\underline{M} = \underline{\pi}_2(Y)$ and $\underline{N} = \underline{\pi}_1(Y)$ and suppose that they are the only nontrivial homotopy Mackey functors of Y. We show that $\Sigma^2 H\underline{M}$ and $\Sigma^1 H\underline{N}$ are 2-slices if and only if Item 1 and Item 2 hold.

First, consider $\Sigma^2 H\underline{M}$. We immediately see that this spectrum is at least two. As for $\Sigma^2 H\underline{M} \leq 2$, the cofiber sequence

$$(S^{-\sigma} \to S^0 \to C_2/e_+) \wedge \Sigma^2 H\underline{M}$$

provides the homotopy

$$\underline{\pi}_2(\Sigma^{-\sigma} H\underline{M}) \quad = \quad \begin{array}{c} \ker(\mathbf{r}_e^{C_2}) \\ \Big\updownarrow \\ 0 \end{array} \quad \text{and} \quad \underline{\pi}_1(\Sigma^{-\sigma} H\underline{M}) \quad = \quad \begin{array}{c} M_e/\operatorname{im} \mathbf{r}_e^{C_2} \\ \Big\updownarrow \\ \mathbb{Z}_\sigma \otimes M_e \end{array}$$

Then the cofiber sequences

$$(S^{-k\sigma} \to S^{-(k-1)\sigma} \to C_2/e_+ \wedge S^{-(k-1)\sigma}) \wedge \Sigma^2 H\underline{M}$$

reveal that $\underline{\pi}_2^{C_2}(\Sigma^{-k\sigma}\Sigma^2 H\underline{M}) = \ker(\mathbf{r}_e^{C_2})$ for all $k \geq 2$.

By definition, $\Sigma^2 H\underline{M} \leq 2$ if and only if $\underline{\pi}_2^{C_2}(\Sigma^{-k\sigma}\Sigma^2 H\underline{M}) = 0$ for all $k \geq 2$. Consequently, $\Sigma^2 H\underline{M}$ is a 2-slice if and only if $\mathbf{r}_e^{C_2}$ is injective .

As for $\Sigma^1 H\underline{N}$, by Theorem 2.3.3, we find that $Y \geq 2$ if and only if $N_e = 0$. But then $\Sigma^1 H\underline{N}$ is the pullback $\phi_K^* \Sigma^1 H N_e$ and consequently, a 2-slice by Proposition 2.3.6.

Conversely, assume Y is a 2-slice. By Theorem 2.3.3, we know $\underline{\pi}_1^e(Y) = 0$ and $\underline{\pi}_k(Y) = \underline{0}$ for $k \leq 0$. For $k \geq 3$, the slice status of Y and Definition 2.3.2 dictate that $\underline{\pi}_k(Y) = \underline{0}$. Consequently, the only nontrivial homotopy Mackey functors of Y and $\underline{\pi}_1(Y) = \underline{N}$ and $\underline{\pi}_2(Y) = \underline{M}$. Thus, we have a fiber sequence $\Sigma^2 H\underline{M} \to Y \to \Sigma^1 H\underline{N}$.

If either $\Sigma^2 H\underline{M}$ or $\Sigma^1 H\underline{N}$ is trivial, the result follows from above. So assume that both are nontrivial. Because $N_e = 0$, $\Sigma^1 H\underline{N}$ is a 2-slice. In particular, $\Sigma^1 H\underline{N} \leq 2$, and since $Y \leq 2$, we have that $\Sigma^2 H\underline{M} \leq 2$. Consequently, as $\Sigma^2 H\underline{M} \geq 2$, we have that $\Sigma^2 H\underline{M}$ is a 2-slice. $\square$

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Chapter 4 Klein Four 2-slices and the Slices of $\Sigma^{\pm n} H\underline{\mathbb{Z}}$

The slices of $\Sigma^n H_K \underline{\mathbb{F}_2}$ for nonnegative n were determined in [GY]. A complementary project is to find all slices of $\Sigma^n H_K \underline{\mathbb{Z}}$ for nonnegative n. We do so in this chapter, as well as expand the values of n to include all negative integers. Additionally, we characterize 2-slices over K.

This chapter is organized as follows. Section 2.2 provides us with the main Mackey functors for K and some pertinent results for Section 4.1, in which we characterize all 2-slices over K. We provide some slice towers in Section 4.2 and describe the slices of $\Sigma^{-n} H_K \underline{\mathbb{Z}}$ in Section 4.3.1. In Section 4.4, we use Brown-Comenetz duality and the slices of $\Sigma^{-n} H\underline{\mathbb{Z}}$ to obtain the slices of $\Sigma^n H_K \underline{\mathbb{Z}}$. We then compute the homotopy Mackey functors of the slices of $\Sigma^{\pm n} H_K \underline{\mathbb{Z}}$ in Section 4.5. Finally, we provide some examples of the slice spectral sequence for $\Sigma^{\pm n} H_K \underline{\mathbb{Z}}$ in Section 4.6.

4.1 2-slice Characterization

Consider a K-spectrum X. Then by Proposition 2.3.5,

$$P_n^n X \simeq \Sigma^{\frac{n-r}{4}\rho} P_r^r \left(\Sigma^{-\frac{n-r}{4}\rho} X\right)$$

where $n \equiv r \pmod 4$ and $0 \leq r \leq 3$. Thus, to know the slices of X, we need only know the 0-, 1-, 2-, and 3-slices of certain suspensions of X. Proposition 2.3.4 and Proposition 2.3.5 characterize the 0-, 1-, and 3-slices. We now characterize the 2-slices.

Proposition 4.1.1. *Let G be a finite group, H an index two subgroup of G, and σ^H the sign representation from $G \to G/H$. For a G-spectrum X, if $\downarrow_H^G \underline{\pi}_{n+1}(X) = \underline{0} = \downarrow_H^G \underline{\pi}_n(X)$, the natural map $\Sigma^{-\sigma^H} X \to X$ induces an isomorphism on $\underline{\pi}_n$. In particular, if $\pi_{n+1}^H(X) = 0 = \pi_n^G(X)$, then $\pi_n^G(\Sigma^{-\sigma^H} X) = 0$.*

Proof. Because σ^H is one-dimensional we may construct $\Sigma^{-\sigma^H} X$ with the cofiber sequence $(S^{-\sigma^H} \to S^0 \to G/H_+) \wedge X$. The resulting long exact sequence in homotopy is

$$\uparrow_H^G \downarrow_H^G \underline{\pi}_{n+1}(X) \to \underline{\pi}_n(\Sigma^{-\sigma^H} X) \to \underline{\pi}_n(X) \to \uparrow_H^G \downarrow_H^G \underline{\pi}_n(X). \tag{4.1}$$

As $\uparrow_H^G \downarrow_H^G \underline{\pi}_{n+1}(X) = \underline{0} = \uparrow_H^G \downarrow_H^G \underline{\pi}_n(X)$, we find that $\underline{\pi}_n(\Sigma^{-\sigma^H} X) \cong \underline{\pi}_n(X)$.

Now suppose that $\pi_{n+1}^H(X) = 0 = \pi_n^G(X)$. Because $\pi_{n+1}^H(X)$ is the value of $\uparrow_H^G \downarrow_H^G \underline{\pi}_{n+1}(X)$ at the orbit G/H, the left three terms of Equation (4.1) prove the exact sequence

$$0 \to \pi_n^G(\Sigma^{-\sigma^H} X) \to 0.$$

Consequently, $\pi_n^G(\Sigma^{-\sigma^H} X) = 0$. $\square$

Corollary 4.1.2. *Let $H\underline{M}$ be an Eilenberg-MacLane G-spectrum and $V \cong \mathbf{s} \oplus \bigoplus_{i=1}^{r} \sigma^{H_i}$ be a real representation with s copies of the trivial representation and each σ^{H_i} the sign representation from $G \to G/H_i$, where H_i is an index two subgroup of G. Then $\Sigma^{-V} H\underline{M}$ does not have nontrivial homotopy above degree $-s$.*

Proof. First apply Equation (4.1) to $X_1 := \Sigma^{-\sigma^{H_1}} H\underline{M}$ with $n \geq 1$. Then repeat for $X_i := \Sigma^{-\sigma^{H_i}} X_{i-1}$ where $2 \leq i \leq r$. Then $\Sigma^{-V} H\underline{M} \simeq \Sigma^{-s} X_r$ has no homotopy above degree $-s$. □

Recall that α, β, and γ are the sign representations of K.

Lemma 4.1.3. *Let $\underline{M}$ be a K-Mackey functor. The nontrivial homotopy of $\Sigma^{-\beta} H\underline{M}$ is*

$$\underline{\pi}_0 \quad = \quad \begin{tikzcd}[column sep=small] & \ker \boldsymbol{r}_L^K \arrow[dl, bend right] \arrow[d, shift right] \arrow[d, shift left] \arrow[dr, bend left] & \\ 0 \arrow[ur, bend left] \arrow[dr, bend right] & \ker \boldsymbol{r}_e^D \arrow[u, shift right] \arrow[u, shift left] \arrow[d, shift right] \arrow[d, shift left] & \ker \boldsymbol{r}_e^R \arrow[ul, bend right] \arrow[dl, bend left] \\ & 0 \arrow[ul, bend left] \arrow[ur, bend right] & \end{tikzcd}$$

$$\underline{\pi}_{-1} \quad = \quad \begin{tikzcd}[column sep=small] & M_L/\operatorname{im} \boldsymbol{r}_L^K \arrow[dl, bend right, "\varphi_r^{KL}"'] \arrow[d, shift right, "\pi"] \arrow[d, shift left] \arrow[dr, bend left] & \\ \mathbb{Z}_\sigma^L \otimes M_L \arrow[ur, bend left, "\pi"'] \arrow[dr, bend right, "1 \oplus r \cdot \boldsymbol{r}_e^L"'] & M_e/\operatorname{im} \boldsymbol{r}_e^D \arrow[u, shift right, "\varphi_r^{De}"'] \arrow[u, shift left, "1 \oplus r \cdot \boldsymbol{r}_e^L"] \arrow[d, shift right, "\pi"] \arrow[d, shift left] & M_e/\operatorname{im} \boldsymbol{r}_e^R \arrow[ul, bend right, "\varphi_r^{Re}"'] \arrow[dl, bend left, "\pi"] \\ & \mathbb{Z}_\sigma^L \otimes M_e \arrow[ul, bend left] \arrow[ur, bend right] & \end{tikzcd}$$

Here, φ_r^{KL} is induced by Δ^r in the square

$$\begin{tikzcd} M_L \arrow[r, "\pi"] \arrow[d, "\Delta^r"] & M_L/\operatorname{im} \boldsymbol{r}_L^K \arrow[d, "\varphi_r^{KL}"] \\ \mathbb{Z}[K/L] \otimes M_L \arrow[r, "q_L"] & \mathbb{Z}_\sigma^L \otimes M_L \end{tikzcd}$$

and is given by $\varphi_r^{KL}(m) = m - r \cdot m$. The maps $\varphi_r^{De} : M_e/\operatorname{im} \boldsymbol{r}_e^D \to \mathbb{Z}_\sigma^L \otimes M_e$ and $\varphi_r^{Re} : M_e/\operatorname{im} \boldsymbol{r}_e^R \to \mathbb{Z}_\sigma^L \otimes M_e$ are defined similarly.

Proof. We have the cofiber sequence $(S^{-\beta} \to S^0 \to K/L_+) \wedge H\underline{M}$. The result then follows from the description of the map $\underline{M} \to \uparrow_L^K \downarrow_L^K \underline{M}$ given in Section 2.2.1. □

Proposition 4.1.4. *Let $\underline{M}$ be a K-Mackey functor. The nontrivial homotopy of $\Sigma^{-\alpha-\beta-\gamma}H\underline{M}$ is*

$\underline{\pi}_0 \quad =$

$$\begin{tikzcd}
 & \ker(\boldsymbol{r}_L^K + \boldsymbol{r}_D^K + \boldsymbol{r}_R^K) \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\
0 \arrow[ur, bend left] \arrow[dr, bend right] & 0 \arrow[u, bend left] \arrow[d, bend right] & 0 \arrow[ul, bend right] \arrow[dl, bend left] \\
 & 0 \arrow[ul, bend left] \arrow[u, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

$\underline{\pi}_{-1} \quad =$

$$\begin{tikzcd}
 & E_2 \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\
\mathbb{Z}_\sigma^L \otimes \ker \boldsymbol{r}_e^L \arrow[ur, bend left] \arrow[dr, bend right] & \mathbb{Z}_\sigma^D \otimes \ker \boldsymbol{r}_e^D \arrow[u, bend left] \arrow[d, bend right] & \mathbb{Z}_\sigma^R \otimes \ker \boldsymbol{r}_e^R \arrow[ul, bend right] \arrow[dl, bend left] \\
 & 0 \arrow[ul, bend left] \arrow[u, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

$\underline{\pi}_{-2} \quad =$

$$\begin{tikzcd}
 & E_3 \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\
\mathbb{Z}_\sigma^L \otimes \ker \varphi_r^{Le} \arrow[ur, bend left] \arrow[dr, bend right] & \mathbb{Z}_\sigma^D \otimes \ker \varphi_r^{De} \arrow[u, bend left] \arrow[d, bend right] & \mathbb{Z}_\sigma^R \otimes \ker \varphi_r^{Re} \arrow[ul, bend right] \arrow[dl, bend left] \\
 & 0 \arrow[ul, bend left] \arrow[u, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

$\underline{\pi}_{-3} \quad =$

$$\begin{tikzcd}
 & (M_e/\operatorname{im}\varphi_r^{De})/\operatorname{im}\varphi_r^{R,L} \arrow[dl, bend right, "\varphi_d"'] \arrow[d, bend right, "\varphi_l"'] \arrow[dr, bend left, "\varphi_d"] & \\
M_e/\operatorname{im}\varphi_d^{Le} \arrow[ur, bend left, "\pi"] \arrow[dr, bend right, "\varphi_l^{dLe}"'] & M_e/\operatorname{im}\varphi_r^{De} \arrow[u, bend left, "\pi"] \arrow[d, bend right, "\varphi_d^{rDe}"'] \arrow[d, bend left, "\pi"] & M_e/\operatorname{im}\varphi_r^{Re} \arrow[ul, bend right, "\pi"'] \arrow[dl, bend left, "\varphi_l^{rRe}"] \\
 & M_e \arrow[ul, bend left, "\pi"] \arrow[u, bend right, "\pi"'] \arrow[ur, bend right, "\pi"'] &
\end{tikzcd}$$

where E_1, E_2, and E_3 are extensions

$$M_e/\operatorname{im}\varphi_r^{De} \longrightarrow E_3 \longrightarrow (M_e/\operatorname{im}\boldsymbol{r}_e^R)/\operatorname{im}\boldsymbol{r}_e^L$$

$$\ker \boldsymbol{r}_e^D \longrightarrow E_2 \longrightarrow E_1$$

$$\ker \boldsymbol{r}_e^R \longrightarrow E_1 \longrightarrow M_L/\operatorname{im}\boldsymbol{r}_L^K$$

Let $\varphi_h^* : \overline{M_e} \to \overline{M_e}'$ be one of the maps shown. Then $\varphi_h^*(m) = m - h\cdot m$. Additionally, $(M_e/\operatorname{im}\varphi_r^{De})/\operatorname{im}\varphi_r^{R,L}$ is the cokernel of the map

$$(M_e/\operatorname{im}\boldsymbol{r}_e^R)/\operatorname{im}\boldsymbol{r}_e^L \xrightarrow{\varphi_r^{R,L}} M_e/\operatorname{im}\varphi_r^{De}.$$

Proof. Lemma 4.1.3 supplies the homotopy for $\Sigma^{-\beta}H\underline{M}$. We continue constructing $\Sigma^{-\alpha-\beta-\gamma}H\underline{M}$ iteratively. The cofiber sequence

$$(S^{-\alpha} \to S^0 \to K/R_+) \wedge \Sigma^{-\beta}H\underline{M}$$

results in the homotopy of $\Sigma^{-\alpha-\beta}H\underline{M}$

$$\underline{\pi}_0 \quad = \quad \begin{tikzcd}[column sep=small] & \ker(\mathbf{r}_L^K + \mathbf{r}_R^K) \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\ 0 \arrow[ur, bend right] \arrow[d, bend right] & \ker \mathbf{r}_D^K \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[ul, bend left] \arrow[d, bend left] \\ & 0 \arrow[ul, bend left] \arrow[u, bend left] \arrow[ur, bend left] & \end{tikzcd}$$

$$\underline{\pi}_{-1} \quad = \quad \begin{tikzcd}[column sep=small] & E_1 \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\ \mathbb{Z}_\sigma^L \otimes \ker \mathbf{r}_e^L \arrow[ur, bend right] \arrow[d, bend right] & \ker \varphi_r^{De} \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}_\sigma^R \otimes \ker \mathbf{r}_e^R \arrow[ul, bend left] \arrow[d, bend left] \\ & 0 \arrow[ul, bend left] \arrow[u, bend left] \arrow[ur, bend left] & \end{tikzcd}$$

$$\underline{\pi}_{-2} \quad = \quad \begin{tikzcd}[column sep=small] & (M_e/\operatorname{im} \mathbf{r}_e^D)/\operatorname{im} \mathbf{r}_e^L \arrow[dl, bend right, "\varphi_r^{D,L}"'] \arrow[d, bend right, "\varphi_r^{D,L}"'] \arrow[dr, bend left, "\varphi_d^{D,L}"] & \\ \mathbb{Z}_\sigma^L \otimes M_e/\operatorname{im} \mathbf{r}_e^L \arrow[ur, bend right, "\pi"'] \arrow[dr, bend right, "\varphi_d^{Le}"'] & \mathbb{Z}_\sigma^D \otimes M_e/\operatorname{im} \varphi_r^{De} \arrow[u, bend right, "\pi"'] \arrow[d, bend right, "\varphi_d^{rDe}"'] & \mathbb{Z}_\sigma^R \otimes M_e/\operatorname{im} \mathbf{r}_e^R \arrow[ul, bend left, "\pi"] \arrow[dl, bend left, "1 \oplus \varphi_r^{Re}"] \\ & \mathbb{Z}_\sigma^D \otimes M_e \arrow[ul, bend left, "\pi"] \arrow[ur, bend left, "\pi"] & \end{tikzcd}$$

Finally, the cofiber sequence $(S^{-\gamma} \to S^0 \to K/D_+) \wedge \Sigma^{-\alpha-\beta} H\underline{M}$ provides the result. $\square$

Lemma 4.1.5. *Let $\phi : G \to N$ and $\psi : G \to M$ be group homomorphisms. Then*

$$\ker(\phi, \psi) = \ker(\phi) \cap \ker(\psi) = \ker(\phi|_{\ker(\psi)})$$

where $(\phi, \psi) : G \to N \oplus M$ is defined by $g \mapsto \phi(g) \oplus \psi(g)$.

Proposition 4.1.6. *Let $\underline{M}$ be a K Mackey functor where the restrictions $\boldsymbol{r}_e^L$, $\boldsymbol{r}_e^D$, and $\boldsymbol{r}_e^R$ are injective. The following are equivalent.*

A. The sequence

$$M_K \xrightarrow{\boldsymbol{r}_L^K + \boldsymbol{r}_D^K + \boldsymbol{r}_R^K} M_L \oplus M_D \oplus M_R \xrightarrow{\begin{pmatrix} -\boldsymbol{r}_e^L & \boldsymbol{r}_e^L & 0 \\ \boldsymbol{r}_e^D & 0 & -\boldsymbol{r}_e^D \\ 0 & -\boldsymbol{r}_e^R & \boldsymbol{r}_e^R \end{pmatrix}} M_e^3$$

is exact.

B. $\operatorname{im} \boldsymbol{r}_{H_1}^K = (\boldsymbol{r}_e^{H_1})^{-1}(\operatorname{im} \boldsymbol{r}_e^{H_2}) \cap (\boldsymbol{r}_e^{H_1})^{-1}(\operatorname{im} \boldsymbol{r}_e^{H_3})$ *where H_1, H_2, H_3 are distinct order two subgroups of K. This equality is represented by the diagram below.*

$$\begin{tikzcd} \operatorname{im} \boldsymbol{r}_{H_1}^K \arrow[r] \arrow[d, hook] & \operatorname{im} \boldsymbol{r}_e^{H_2} \cap \operatorname{im} \boldsymbol{r}_e^{H_3} \arrow[d, hook] \\ M_{H_1} \arrow[r, "\boldsymbol{r}_e^{H_1}"] & M_e \end{tikzcd}$$

Proof. Without loss of generality, let $H_1 = L$, $H_2 = D$, and $H_3 = R$. For convenience, set $I = (\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^D) \cap (\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^R)$.

(A.) $\Rightarrow$ (B.):

Let $x \in \operatorname{im} \mathbf{r}_L^K$. Then we have $k \in M_K$ such that $\mathbf{r}_L^K(k) = x$. Additionally, $\mathbf{r}_e^L(x) = \mathbf{r}_e^D \mathbf{r}_D^K(k) = \mathbf{r}_e^R \mathbf{r}_R^K(k)$. It follows that $x \in I$.

Now suppose $x \in I$. We then have $y \in D$ and $z \in R$ such that

$$\mathbf{r}_e^L(x) = \mathbf{r}_e^D(y) = \mathbf{r}_e^R(z).$$

Thus,

$$(x, y, z) \in \ker \begin{pmatrix} -\mathbf{r}_e^L & \mathbf{r}_e^L & 0 \\ \mathbf{r}_e^D & 0 & -\mathbf{r}_e^D \\ 0 & -\mathbf{r}_e^R & \mathbf{r}_e^R \end{pmatrix}.$$

Consequently, we have $x = \mathbf{r}_L^K(k)$ for some $k \in M_K$; that is, $x \in \operatorname{im} \mathbf{r}_L^K$.

(B.) $\Rightarrow$ (A.):

Let

$$(x, y, z) \in \ker \begin{pmatrix} -\mathbf{r}_e^L & \mathbf{r}_e^L & 0 \\ \mathbf{r}_e^D & 0 & -\mathbf{r}_e^D \\ 0 & -\mathbf{r}_e^R & \mathbf{r}_e^R \end{pmatrix}.$$

Then $\mathbf{r}_e^L(x) = \mathbf{r}_e^D(y) = \mathbf{r}_e^R(z)$, so $x \in \operatorname{im} \mathbf{r}_L^K$. So there is some $k \in M_K$ such that $x = \mathbf{r}_L^K(k)$. Because $\mathbf{r}_e^D$ and $\mathbf{r}_e^R$ are injective, it must be that $y = (\mathbf{r}_e^D)^{-1}(\mathbf{r}_e^L(x))$ and $z = (\mathbf{r}_e^R)^{-1}(\mathbf{r}_e^L(x))$. Hence, Item A. is exact. □

Proposition 4.1.7. *The K-spectrum $\Sigma^1 H\underline{N}$ is a 2-slice if and only if $N_e = 0$ and $N_K \to N_L \oplus N_D \oplus N_R$ is injective.*

Proof. By Theorem 2.3.3, we find that $\Sigma^1 H\underline{N} \geq 2$ if and only if $N_e = 0$. Consequently, going forward we may assume $N_e = 0$.

Now $\Sigma^1 H\underline{N} \leq 2$ if and only if $[S^{k\rho_H + r}, \Sigma^2 H\underline{N}]^H = 0$ for all $k \geq \frac{3}{|H|}$ and $r \geq 0$. Because $N_e = 0$, Proposition 3.3.1 implies that $i_H^*(\Sigma^1 H\underline{N})$ is a 2-slice, where H is an order two subgroup of K. Thus, to finish the equivalence, we only need consider

$$[S^{k\rho_K + r}, \Sigma^1 H\underline{N}]^K = [S^{k+r-1}, \Sigma^{-k\overline{\rho}_K} H\underline{N}]^K$$

for all $k \geq 1$ and $r \geq 0$.

From Corollary 4.1.2, we need only concern ourselves with $[S^0, \Sigma^{-k\overline{\rho}_K} H\underline{N}]$. From Proposition 4.1.4,

$$\underline{\pi}_0(\Sigma^{-\overline{\rho}} H\underline{N}) = \phi_K^*(\ker \mathbf{r}_L^K \cap \ker \mathbf{r}_D^K \cap \ker \mathbf{r}_R^K).$$

Therefore Proposition 4.1.1 shows that $\underline{\pi}_0(\Sigma^{-\overline{\rho}} H\underline{N})$ vanishes if and only if $\underline{\pi}_0(\Sigma^{-k\overline{\rho}} H\underline{N})$ vanishes for all $k \geq 1$. Hence, $\Sigma^1 H\underline{N} \leq 2$ if and only if $\ker \mathbf{r}_L^K \cap \ker \mathbf{r}_D^K \cap \ker \mathbf{r}_R^K = \{0\}$. By Lemma 4.1.5 this is equivalent to $N_K \to N_L \oplus N_D \oplus N_R$ being injective. □

Proposition 4.1.8. *The spectrum $\Sigma^2 H\underline{M}$ is a 2-slice if and only if all restrictions of $\underline{M}$ are injective and the sequence*

$$M_K \xrightarrow{\boldsymbol{r}_L^K+\boldsymbol{r}_D^K+\boldsymbol{r}_R^K} M_L \oplus M_D \oplus M_R \xrightarrow{\begin{pmatrix} -\boldsymbol{r}_e^L & \boldsymbol{r}_e^L & 0 \\ \boldsymbol{r}_e^D & 0 & -\boldsymbol{r}_e^D \\ 0 & -\boldsymbol{r}_e^R & \boldsymbol{r}_e^R \end{pmatrix}} M_e^3$$

is exact.

Proof. Note that $\Sigma^2 H\underline{M} \geq 2$, so we just need to show that $\Sigma^2 H\underline{M} \leq 2$. Now $\Sigma^2 H\underline{M} \leq 2$ if and only if $[S^{k\rho_H+r}, \Sigma^2 H\underline{M}]^H = 0$ for all $k \geq \frac{3}{|H|}$ and $r \geq 0$. For H an order two subgroup of K, $i_H^* \Sigma^2 H\underline{M} \leq 2$ if and only if $\mathbf{r}_e^H$ is injective. Consequently, going forward we may assume $\mathbf{r}_e^L$, $\mathbf{r}_e^D$, and $\mathbf{r}_e^R$ are injective.

To finish the equivalence, we only need consider

$$[S^{k\rho_K+r}, \Sigma^2 H\underline{M}]^K = [S^{k+r-2}, \Sigma^{-k\overline{\rho}_K} H\underline{M}]^K$$

for all $k \geq 1$ and $r \geq 0$. By Corollary 4.1.2, it is enough to examine $[S^0, \Sigma^{-k\overline{\rho}_K} H\underline{M}]$ and $[S^{-1}, \Sigma^{-k\overline{\rho}_K} H\underline{M}]$. From Proposition 4.1.4,

$$\underline{\pi}_0(\Sigma^{-\overline{\rho}} H\underline{M}) = \phi_K^*(\ker \mathbf{r}_L^K)$$

and

$$\underline{\pi}_{-1}(\Sigma^{-\overline{\rho}} H\underline{M}) = \phi_K^* \left(\frac{(\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^D) \cap (\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^R)}{\operatorname{im} \mathbf{r}_L^K} \right).$$

Note that Proposition 4.1.4 states

$$\underline{\pi}_0(\Sigma^{-\overline{\rho}} H\underline{M}) = \phi_K^*(\ker \mathbf{r}_L^K \cap \ker \mathbf{r}_D^K \cap \ker \mathbf{r}_R^K),$$

but because the lower restrictions are injective, these kernels coincide. Proposition 4.1.1 then yields that

$$\begin{aligned} \underline{\pi}_0^G(\Sigma^{-k\overline{\rho}} H\underline{M}) &\cong \underline{\pi}_0^G(\Sigma^{-\overline{\rho}} H\underline{M}) = \ker \mathbf{r}_L^K \\ \underline{\pi}_{-1}^G(\Sigma^{-k\overline{\rho}} H\underline{M}) &\cong \underline{\pi}_{-1}^G(\Sigma^{-\overline{\rho}} H\underline{M}) = \frac{(\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^D) \cap (\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^R)}{\operatorname{im} \mathbf{r}_L^K} \end{aligned}$$

for all $k \geq 2$. By Definition 2.3.2, we find that $\Sigma^2 H\underline{M} \leq 2$ if and only if these two homotopy groups vanish. Hence, $\Sigma^2 H\underline{M} \leq 2$ if and only if $\ker \mathbf{r}_L^K = \{0\}$ and $\operatorname{im} \mathbf{r}_L^K = (\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^D) \cap (\mathbf{r}_e^L)^{-1}(\operatorname{im} \mathbf{r}_e^R)$. Because the homotopy of $\Sigma^{-\alpha-\beta-\gamma} H\underline{M}$ is invariant under the order of construction – that is, whether $H\underline{M}$ is suspended by say $-\alpha$ or $-\beta$ first – we find that all upper restrictions must be injective and that (B.) must hold. □

Theorem 4.1.9. *Suppose the only nontrivial homotopy Mackey functors of a K-spectrum X are $\underline{\pi}_1(X)$ and $\underline{\pi}_2(X)$ where*

1. *All restrictions of* $\underline{\pi}_2(X)$ *are injective and the sequence*

$$\pi_2^K(X) \xrightarrow{r_L^K + r_D^K + r_R^K} \pi_2^L(X) \oplus \pi_2^D(X) \oplus \pi_2^R(X) \xrightarrow{\begin{pmatrix} -r_e^L & r_e^L & 0 \\ r_e^D & 0 & -r_e^D \\ 0 & -r_e^R & r_e^R \end{pmatrix}} \pi_2^e(X)^3$$

   is exact.

2. $\pi_1^e(X) = 0$.

3. $\pi_1^K(X) \to \pi_1^L(X) \oplus \pi_1^D(X) \oplus \pi_1^R(X)$ *is injective.*

Then X *is a 2-slice.*

Conversely, if a K*-spectrum* X *is a 2-slice, then its only nontrivial homotopy Mackey functors of a* K*-spectrum* X *are* $\underline{\pi}_1(X)$ *and* $\underline{\pi}_2(X)$ *where* $\Sigma^2 H\underline{\pi}_2(X)$ *is a 2-slice and* $\Sigma^1 H\underline{\pi}_1(X) \in [2,4]$*, i.e., both* (1) *and* (2) *hold.*

Proof. Let $\underline{\pi}_2(X) = \underline{M}$ and $\underline{\pi}_1(X) = \underline{N}$. If these are the only nontrivial homotopy Mackey functors of X, we have a fiber sequence

$$\Sigma^2 H\underline{M} \to X \to \Sigma^1 H\underline{N}. \tag{4.2}$$

By Proposition 4.1.7 and Proposition 4.1.8, conditions (1) - (3) show that $\Sigma^2 H\underline{M}$ and $\Sigma^1 H\underline{N}$ are 2-slices. Now if $\Sigma^2 H\underline{M}$ and $\Sigma^1 H\underline{N}$ are 2-slices, then X must be a 2-slice as well.

Conversely, suppose that X is a 2-slice. We then find that X has no homotopy above degree two and none below degree one; thus, we have the fiber sequence in Equation (4.2). Because $X \geq 2$ and $\Sigma^2 H\underline{M} \geq 2$, it follows that $\Sigma^1 H\underline{N} \geq 2$. So by Theorem 2.3.3, $N_e = 0$. That $\Sigma^1 H\underline{N} \leq 4$ follows from a similar argument as in Proposition 4.1.7.

Rotating this fiber sequence gives $H\underline{N} \to \Sigma^2 H\underline{M} \to X$. As $H\underline{N}$ is a 0-slice and X is a 2-slice, we have $\Sigma^2 H\underline{M} \in [0,2]$, so it must be that $\Sigma^2 H\underline{M} = 2$. Consequently, by Proposition 4.1.8, (1) holds. □

It is not necessary for condition (3) to hold for X to be a 2-slice as we show in the following example.

Example 4.1.10. Take $X \simeq \Sigma^{1+\beta} H\underline{E}$ where

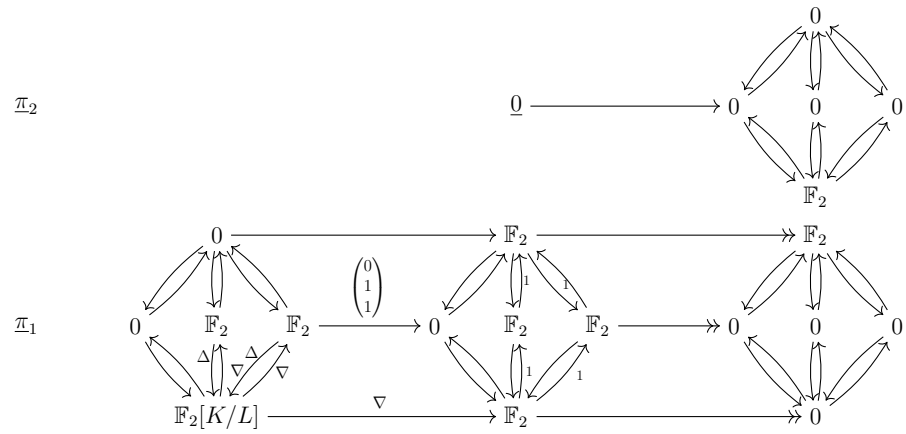
$$\underline{E} = \begin{tikzcd}[column sep=small] & \mathbb{F}_2 \arrow[dl, bend right, "1"'] \arrow[d, shift right, "0"'] & \\ 0 \arrow[ur, bend right, "0"'] \arrow[dr, shift right, "0"'] & \mathbb{F}_2 \arrow[u, shift right, "1"'] \arrow[d, shift right, "0"'] & \mathbb{F}_2 \arrow[ul, bend left, "0"] \arrow[dl, shift left, "1"] \\ & \mathbb{F}_2 \arrow[u, shift right, "1"'] \arrow[ur, shift left, "1"] & \end{tikzcd}$$

Then X is a 2-slice with $\underline{\pi}_2(X) = \underline{f}$ and $\underline{\pi}_1(X) = \underline{g}$. But $\Sigma^1 H\underline{g}$ is not a 2-slice.

We can construct $\Sigma^{1+\beta}H\underline{E}$ using the cofiber sequence

$$(K/L_+ \to S^0 \to S^\beta) \wedge \Sigma^1 H\underline{E}.$$

The resulting long exact sequence in homotopy is



So $\underline{\pi}_2(X) = \underline{f}$ and $\underline{\pi}_1(X) = \underline{g}$. From this we see that $X \geq 2$ by Theorem 4.1.9. Note that $\Sigma^2 H\underline{f}$ is a 2-slice and $\Sigma^1 H\underline{g}$ is a 4-slice.

To show that $X \leq 2$, we need $[S^{k\rho_H+r}, \Sigma^{1+\beta}H\underline{E}]^H = 0$ for all $k \geq \frac{3}{|H|}$ and $r \geq 0$. As $i_H^*(X) \simeq \Sigma^2 H_{C_2}\underline{f}$ is a 2-slice, where H is an order two subgroup of K, we only need to consider

$$[S^{k\rho_K+r}, \Sigma^{1+\beta}H\underline{E}]^K = [S^{k+r-1}, \Sigma^{-\alpha-\gamma-(k-1)\overline{\rho}_K}H\underline{E}]^K$$

for all $k \geq 1$ and $r \geq 0$.

By Corollary 4.1.2, it is sufficient to examine $[S^0, \Sigma^{-\alpha-\gamma-(k-1)\overline{\rho}_K}H\underline{E}]^K$ for all $k \geq 1$.

From Lemma 4.1.3 we find that $[S^0, \Sigma^{-\alpha-\gamma}H\underline{E}] = \underline{0}$. Consequently, given any $k \geq 1$, repeated application of Proposition 4.1.1 shows that

$$[S^0, \Sigma^{-\alpha-\gamma-(k-1)\overline{\rho}_K}H\underline{E}] = \underline{0}.$$

That is, X is a 2-slice.

4.2 Cotowers for $\Sigma^{-n}H\underline{\mathbb{Z}}$

We determine the slice towers of $\Sigma^{-n}H\underline{\mathbb{Z}}$ and $\Sigma^{-n}H\underline{m}^*$ for $1 \leq n \leq 5$.

Example 4.2.1. By Proposition 2.3.4 and Proposition 2.2.1, $\Sigma^{-1}H\underline{m}^*$ is a (-2)-slice and $\Sigma^{-2}H\underline{m}^*$ is a (-4)-slice.

Alternatively, by Proposition 2.2.1, $\Sigma^\rho\Sigma^{-1}H\underline{m}^* \simeq \Sigma^1 H\underline{mg}$, which is a 2-slice by Proposition 4.1.7. Consequently, $\Sigma^{-1}H\underline{m}^*$ is a (-2)-slice.

Example 4.2.2. By [U1, Theorem 6-4], the cotower for $\Sigma^{-1}H\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
\Sigma^{-1}H\underline{\mathbb{Z}}^* = P_{-1}^{-1} \arrow[d] & \\
\Sigma^{-1}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-1}H\underline{m}^* = P_{-2}^{-2} \\
\Sigma^{-1}H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-1}H\underline{g} = P_{-4}^{-4}
\end{tikzcd}$$

Example 4.2.3. We suspend the cotower in Example 4.2.2 by -1 to get the cotower for $\Sigma^{-2}H\underline{\mathbb{Z}}$.

$$\begin{tikzcd}
\Sigma^{-2}H\underline{\mathbb{Z}}^* = P_{-2}^{-2} \arrow[d] & \\
\Sigma^{-2}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-2}H\underline{m}^* = P_{-4}^{-4} \\
\Sigma^{-2}H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-2}H\underline{g} = P_{-8}^{-8}
\end{tikzcd}$$

Example 4.2.4. We suspend the cotower in Example 4.2.3 by -1 to get a partial cotower for $\Sigma^{-3}H\underline{\mathbb{Z}}$.

$$\begin{tikzcd}
\Sigma^{-3}H\underline{\mathbb{Z}}^* = P_{-3}^{-3} \arrow[d] & \\
\Sigma^{-3}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-3}H\underline{m}^* \\
\Sigma^{-3}H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-3}H\underline{g} = P_{-12}^{-12}
\end{tikzcd}$$

The issue here is that $\Sigma^{-3}H\underline{m}^*$ is not a slice.

However, the short exact sequence

$$\underline{g}^2 \to \phi^*_{LDR}(\underline{\mathbb{F}_2}^*) \to \underline{m}^*$$

provides the tower

$$P_{-6}^{-6} = \Sigma^{-3}H\phi^*_{LDR}(\underline{\mathbb{F}_2}^*) \to \Sigma^{-3}H\underline{m}^* \to \Sigma^{-2}H\underline{g}^2 = P_{-8}^{-8}.$$

Consequently, the remaining slices of $\Sigma^{-3}H\underline{\mathbb{Z}}$ are $P_{-6}^{-6} = \Sigma^{-2p+1}H\phi^*_{LDR}(\underline{\mathbb{F}_2})$ and $P_{-8}^{-8} = \Sigma^{-2}H\underline{g}^2$.

Example 4.2.5. We suspend the cotower in Example 4.2.4 by -1 to get a partial cotower for $\Sigma^{-4}H\underline{\mathbb{Z}}$.

$$\begin{tikzcd}
\Sigma^{-4}H\underline{\mathbb{Z}}^* = P^{-4}_{-4} \arrow[d] & \\
\Sigma^{-4}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-4}H\underline{m}^* \\
\Sigma^{-4}H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-4}H\underline{g} = P^{-16}_{-16}
\end{tikzcd}$$

Again, $\Sigma^{-4}H\underline{m}^*$ is not a slice. But suspending the cotower for $\Sigma^{-3}H\underline{m}^*$ by -1 provides the missing slices: $P^{-8}_{-8} = \Sigma^{-4}H\phi^*_{LDR}(\underline{\mathbb{F}_2}^*)$ and $P^{-12}_{-12} = \Sigma^{-3}H\underline{g}^2$.

Example 4.2.6. We suspend the cotower in Example 4.2.5 by -1 and augment with the $-\rho$ suspension of Example 4.2.2 to get a partial cotower for $\Sigma^{-5}H\underline{\mathbb{Z}}$.

$$\begin{tikzcd}
\Sigma^{-\rho-1}H\underline{\mathbb{Z}}^* = P^{-5}_{-5} \arrow[d] & \\
\Sigma^{-\rho-1}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-\rho-1}H\underline{m}^* = P^{-6}_{-6} \\
\Sigma^{-\rho-1}H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-2}H\underline{g} = P^{-8}_{-8} \\
\Sigma^{-5}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-5}H\underline{m}^* \\
\Sigma^{-5}H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-5}H\underline{g} = P^{-20}_{-20}
\end{tikzcd}$$

This time, the cotower for $\Sigma^{-5}H\underline{m}^*$ is

$$\begin{tikzcd}
\Sigma^{-3\rho+1}H\phi^*_{LDR}(\underline{\mathbb{F}_2}) = P^{-10}_{-10} \arrow[d] & \\
\Sigma^{-5}H\phi^*_{LDR}(\underline{\mathbb{F}_2}^*) \arrow[r] \arrow[d] & \Sigma^{-3}H\underline{g}^3 = P^{-12}_{-12} \\
\Sigma^{-5}H\underline{m}^* \arrow[r] & \Sigma^{-4}H\underline{g}^2 = P^{-16}_{-16}
\end{tikzcd}$$

The remaining slices of $\Sigma^{-5}H\underline{\mathbb{Z}}$ are then $P^{-10}_{-10} = \Sigma^{-3\rho+1}H\phi^*_{LDR}(\underline{\mathbb{F}_2})$, $P^{-12}_{-12} = \Sigma^{-3}H\underline{g}^3$, and $P^{-16}_{-16} = \Sigma^{-4}H\underline{g}^2$.

Example 4.2.7. The partial cotower for $\Sigma^{-7}H\underline{\mathbb{Z}}$ follows by suspending the partial cotower in Example 4.2.6 by -2.

$$\begin{tikzcd}
\Sigma^{-\rho-3}H\underline{\mathbb{Z}}^* = P_{-7}^{-7} \arrow[d] & \\
\Sigma^{-\rho-3}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-\rho-3}H\underline{m}^* \\
\Sigma^{-\rho-3}H\underline{\mathbb{Z}} \arrow[r] \arrow[d] & \Sigma^{-4}H\underline{g} = P_{-16}^{-16} \\
\Sigma^{-7}H\underline{\mathbb{Z}}(2,1) \arrow[r] \arrow[d] & \Sigma^{-7}H\underline{m}^* \\
\Sigma^{-7}H\underline{\mathbb{Z}} \arrow[r] & \Sigma^{-7}H\underline{g} = P_{-28}^{-28}
\end{tikzcd}$$

We now have a cotower for $\Sigma^{-\rho-3}H\underline{m}^*$.

$$P_{-10}^{-10} = \Sigma^{-\rho-3}H\phi^*_{LDR}(\underline{\mathbb{F}_2}^*) \to \Sigma^{-\rho-3}H\underline{m}^* \to \Sigma^{-3}H\underline{g}^2 = P_{-12}^{-12},$$

And a cotower for $\Sigma^{-7}H\underline{m}^*$.

$$\begin{tikzcd}
\Sigma^{-4\rho+1}H\phi^*_{LDR}(\underline{\mathbb{F}_2}) = P_{-14}^{-14} \arrow[d] & \\
\Sigma^{-\rho-5}H\phi^*_{LDR}(\underline{\mathbb{F}_2}^*) \arrow[r] \arrow[d] & \Sigma^{-4}H\underline{g}^3 = P_{-16}^{-16} \\
\Sigma^{-7}H\phi^*_{LDR}(\underline{\mathbb{F}_2}^*) \arrow[r] \arrow[d] & \Sigma^{-5}H\underline{g}^3 = P_{-20}^{-20} \\
\Sigma^{-7}H\underline{m}^* \arrow[r] & \Sigma^{-6}H\underline{g}^2 = P_{-24}^{-24}
\end{tikzcd}$$

We now see interference from the cotower for $\Sigma^{-7}H\underline{m}^*$. Its (-14)-slice appears below the (-16)-slice in the partial cotower for $\Sigma^{-7}H\underline{\mathbb{Z}}$. Additionally, both of these (partial) cotowers have a (-16)-slice.

Proposition 4.3.4 and Proposition 4.3.8 tell us that $P_{-14}^{-14}(\Sigma^{-7}H\underline{\mathbb{Z}})$ is indeed $\Sigma^{-4\rho+1}H\phi^*_{LDR}(\underline{\mathbb{F}_2})$ and $P_{-16}^{-16}(\Sigma^{-7}H\underline{\mathbb{Z}})$ is $\Sigma^{-4}H\underline{g} \vee \Sigma^{-4}H\underline{g}^3 \simeq \Sigma^{-4}H\underline{g}^4$.

All partial cotowers for $\Sigma^{-n}H\underline{\mathbb{Z}}$ will follow this pattern of utilizing the homotopy equivalence $\Sigma^{-n}H\underline{\mathbb{Z}}^* \simeq \Sigma^{-\rho-n+4}H\underline{\mathbb{Z}}$ to augment the bottom of the cotower for $\Sigma^{-n}H\underline{\mathbb{Z}}$ with the cotower for $\Sigma^{-\rho-n+4}H\underline{\mathbb{Z}}$.

4.3 Slices of $\Sigma^{-n}H_K\underline{\mathbb{Z}}$

Here we determine the slices of $\Sigma^{-n}H_K\underline{\mathbb{Z}}$.

Proposition 4.3.1. *$\Sigma^n H\underline{\mathbb{Z}}$ is an n-slice for $1 \leq n \leq 5$ and $\Sigma^{-n}H\underline{\mathbb{Z}}^*$ is a $(-n)$-slice for $1 \leq n \leq 4$.*

Proof. The K-spectra $\Sigma^1 H\underline{\mathbb{Z}}$, $\Sigma^1 H\underline{\mathbb{Z}}^*$, $\Sigma^2 H\underline{\mathbb{Z}}$, and $\Sigma^{-1}H\underline{\mathbb{Z}}^*$ are 1-, 2-, and (-1)-slices by Proposition 2.3.4 and Theorem 4.1.9. The result then follows from Proposition 2.2.2 and the resulting equivalences

$$\Sigma^5 H\underline{\mathbb{Z}} \simeq \Sigma^{\rho+1}H\underline{\mathbb{Z}}^*, \quad \Sigma^4 H\underline{\mathbb{Z}} \simeq \Sigma^{\rho}H\underline{\mathbb{Z}}^*, \quad \text{and} \quad \Sigma^3 H\underline{\mathbb{Z}} \simeq \Sigma^{\rho-1}H\underline{\mathbb{Z}}^*.$$

□

4.3.1 The $(-n)$-slice

We first establish a comparison of the slices of $\Sigma^{-n}H\underline{\mathbb{Z}}$ with those of $\Sigma^{-n+4}H\underline{\mathbb{Z}}$.

Proposition 4.3.2. *Let $n \geq 5$. Then*

$$P^{-k}_{-k}(\Sigma^{-n}H\underline{\mathbb{Z}}) \simeq \Sigma^{-\rho}P^{-k+4}_{-k+4}(\Sigma^{-n+4}H\underline{\mathbb{Z}})$$

for $k \in [n, 2n-1]$.

Proof. By Proposition 2.2.2, we have

$$\Sigma^{-\rho}P^{-k+4}_{-k+4}(\Sigma^{-n+4}H\underline{\mathbb{Z}}) \simeq \Sigma^{\rho}\Sigma^{-\rho}P^{-k}_{-k}(\Sigma^{-n+4-\rho}H\underline{\mathbb{Z}}) \simeq P^{-k}_{-k}(\Sigma^{-n}H\underline{\mathbb{Z}}^*).$$

Thus, it is sufficient to compare the $(-k)$-slices of $\Sigma^{-n}H\underline{\mathbb{Z}}$ and $\Sigma^{-n}H\underline{\mathbb{Z}}^*$.

Recall the injection $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \to \underline{\mathbb{M}}$ from Section 2.2. We wish to show that

$$\Sigma^{-n}H\underline{\mathbb{Z}}^* \to \Sigma^{-n}H\underline{\mathbb{Z}} \xrightarrow{\mu} \Sigma^{-n}\underline{\mathbb{M}} \tag{4.3}$$

induces an equivalence on slices strictly above level $-2n$.

We take the Brown-Comenetz dual of Equation (4.3) to find the fiber sequence

$$\Sigma^n H\underline{\mathbb{M}} \xrightarrow{\iota} \Sigma^n I_{\mathbb{Q}/\mathbb{Z}}H\underline{\mathbb{Z}} \to \Sigma^n I_{\mathbb{Q}/\mathbb{Z}}H\underline{\mathbb{Z}}^*$$

Now $\Sigma^n H\underline{\mathbb{M}}$ is n-connective, and as its underlying spectrum is contractible, when $n \geq 1$, we know $\Sigma^n H\underline{\mathbb{M}} \geq 2n$ by Theorem 2.3.3. Then [HHR1, Lemma 4.28] provides that ι induces an equivalence on slices strictly below level $2n$. It then follows from Proposition 2.4.4 that μ in Equation (4.3) induces an equivalence on slices strictly above level $-2n$. □

This is an analogous result to [GY, Proposition 5.3]. However, the injection $\underline{\mathbb{Z}}^* \hookrightarrow \underline{\mathbb{Z}}$ allows us to simplify the argument.

Proposition 4.3.3. *Let $n \geq 1$ and take $r \equiv n \pmod 4$ with $1 \leq r \leq 4$. Then the top slice of $\Sigma^{-n}H\underline{\mathbb{Z}}$ is*

$$P^{-n}_{-n}(\Sigma^{-n}H\underline{\mathbb{Z}}) = \Sigma^{-\frac{n-r}{4}\rho-r}H\underline{\mathbb{Z}}^*$$

Proof. Section 4.2 gives the result for $1 \leq n \leq 4$, and $n \geq 5$ follows from repeated application of Proposition 4.3.2. □

4.3.2 The $(-4k)$-slices

Here we determine the $(-4k)$-slices of $\Sigma^{-n}H\underline{\mathbb{Z}}$.

Proposition 4.3.4. *For $n \geq 1$,*

$$P^{-4k}_{-4k}(\Sigma^{-n}H\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{-k\rho}H\underline{mg} & 4k = n+2 \\ \Sigma^{-k}H\underline{g}^{\frac{1}{2}(4k-n-1)} & 4k \in [n+1, 2n-2] \\ & \quad n \equiv 1 \pmod 2 \\ \Sigma^{-k\rho}H(\underline{g}^{\frac{1}{2}(4k-n+2)-3} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2}) & 4k \in [n+3, 2n], \\ & n \equiv 0 \pmod 2 \\ \Sigma^{-k}H\underline{g}^{n-k+1} & 4k \in [2n+1, 4n] \end{cases}$$

Proof. We have the equivalence

$$P^{-4k}_{-4k}(\Sigma^{-n}H\underline{\mathbb{Z}}) \simeq \Sigma^{-k\rho}H\underline{\pi}_0(\Sigma^{-n+k\rho}H\underline{\mathbb{Z}}).$$

The restriction to each cyclic subgroup agrees with the slices found in Proposition 3.2.38 and Proposition 4.3.5 gives the fixed points. All that remains is to verify the result for $4k \in [n+2, 2n]$ with n even. For $4k \in [n+2, 2n)$, this follows from Example 4.2.2, Example 4.2.3, Example 4.2.4, Example 4.2.5, and Proposition 4.3.2.

Now let $4k = 2n$. The transfer $\mathbf{t}^K_L$ fits in the cofiber sequence

$$(\Sigma^{-n+k\rho}H\underline{\mathbb{Z}})^L \xrightarrow{\mathbf{t}^K_L} (\Sigma^{-n+k\rho}H\underline{\mathbb{Z}})^K \longrightarrow (\Sigma^{-n+k\rho+\beta}H\underline{\mathbb{Z}})^K.$$

This gives the exact sequence

$$\pi^L_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}) \xrightarrow{\mathbf{t}^K_L} \pi^K_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}) \longrightarrow \pi^K_n(\Sigma^{k\rho+\beta}H\underline{\mathbb{Z}}). \tag{4.4}$$

We wish to show this transfer is trivial. By Proposition 4.3.5 and Proposition 4.3.7, Equation (4.4) becomes

$$\mathbb{F}_2 \xrightarrow{\mathbf{t}^K_L} \mathbb{F}_2^{\frac{1}{2}(n+2)} \twoheadrightarrow \mathbb{F}_2^{\frac{1}{2}(n+2)} \longrightarrow 0 = \pi^L_{n-1}(\Sigma^{k\rho}H\underline{\mathbb{Z}}).$$

Thus, $\mathbf{t}^K_L$ must be zero. A similar argument shows that $\mathbf{t}^K_R$ and $\mathbf{t}^K_D$ are trivial as well.

The restriction $\mathbf{r}^K_L$ fits into the cofiber sequence

$$(\Sigma^{-n+k\rho-\beta}H\underline{\mathbb{Z}})^K \longrightarrow (\Sigma^{-n+k\rho}H\underline{\mathbb{Z}})^K \xrightarrow{\mathbf{r}^K_L} (\Sigma^{-n+k\rho}H\underline{\mathbb{Z}})^L$$

This gives the exact sequence

$$\pi^K_n(\Sigma^{k\rho-\beta}H\underline{\mathbb{Z}}) \longrightarrow \pi^K_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}) \xrightarrow{\mathbf{r}^K_L} \pi^L_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}). \tag{4.5}$$

We wish to show this restriction is surjective. By Proposition 4.3.5 and Proposition 4.3.6, Equation (4.5) becomes

$$\pi^L_{n+1}(\Sigma^{k\rho}H\underline{\mathbb{Z}}) = 0 \longrightarrow \mathbb{F}_2^{\frac{1}{2}(n)} \hookrightarrow \mathbb{F}_2^{\frac{1}{2}(n)+1} \xrightarrow{\mathbf{r}^K_L} \mathbb{F}_2$$

Thus, $\mathbf{r}^K_L$ must be surjective. A similar argument shows that $\mathbf{r}^K_R$ and $\mathbf{r}^K_D$ are surjective as well.

All that remains to be shown is that the restrictions have distinct kernels. Consider the cofiber sequence

$$\Sigma^{-n+k\rho-\beta}H\underline{\mathbb{Z}} \longrightarrow \Sigma^{-n+k\rho}H\underline{\mathbb{Z}} \longrightarrow K/L_+ \wedge \Sigma^{-n+k\rho}H\underline{\mathbb{Z}}.$$

This results in the exact sequence

$$\underline{\pi}_n(\Sigma^{k\rho-\beta}H\underline{\mathbb{Z}}) \longrightarrow \underline{\pi}_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}) \longrightarrow \uparrow^K_L\downarrow^K_L \underline{\pi}_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}) \tag{4.6}$$

which by Proposition 3.2.4, Proposition 4.3.5, and Proposition 4.3.6 is

$$\begin{tikzcd}
 & \mathbb{F}_2^{\frac{n}{2}} \arrow[rr, "\varphi"] \arrow[dl, bend right, "p_2"'] \arrow[d, shift right, "0"'] \arrow[dr, shift right, "p_3"'] & & \mathbb{F}_2^{\frac{n}{2}+1} \arrow[rr, "\mathbf{r}^K_L"] \arrow[dl, shift right, "p_1"'] \arrow[d, shift right, "p_2"'] \arrow[dr, shift right, "p_3"'] & & \mathbb{F}_2 \arrow[d, shift right] \arrow[dl, bend right] \\
0 \arrow[ur, bend left] \arrow[dr, bend right] & \mathbb{F}_2 \arrow[u, shift right, "0"'] \arrow[d, shift right] & \mathbb{F}_2 \arrow[r, "0\ 1\ 1"] \arrow[ul, shift right] \arrow[dl, shift right] & \mathbb{F}_2 \arrow[u, shift right, "0"'] \arrow[d, shift right] & \mathbb{F}_2 \arrow[r, "\Delta\ 0\ 0"] \arrow[ul, shift right] \arrow[dl, shift right] & \mathbb{F}_2[K/L] \arrow[ur, bend left] \arrow[dr, bend right] & 0 \arrow[u, shift right] \arrow[d, shift right] & 0 \arrow[ul, bend left] \arrow[dl, bend right] \\
 & 0 \arrow[u, shift right] \arrow[rr, "0"] \arrow[ur, shift right] & & 0 \arrow[u, shift right] \arrow[rr, "0"] \arrow[ur, shift right] & & 0 \arrow[u, shift right] \arrow[ur, shift right] & &
\end{tikzcd}$$

Each restriction is surjective with kernel of rank $\frac{n}{2}$. As

$$\dim \pi^K_n(\Sigma^{k\rho}H\underline{\mathbb{Z}}) = \frac{n}{2} + 1,$$

it is sufficient to show that the kernels are pairwise distinct.

Because the diagram on the left commutes, we find that

$$\operatorname{im}\varphi \cap \ker(\mathbf{r}^K_R) = \{0\} \qquad \text{and} \qquad \operatorname{im}\varphi \cap \ker(\mathbf{r}^K_D) = \{0\}.$$

As $\operatorname{im}\varphi = \ker(\mathbf{r}^K_L)$, we have that $\ker(\mathbf{r}^K_L)$ is distinct from $\ker(\mathbf{r}^K_R)$ and $\ker(\mathbf{r}^K_D)$. Replacing β by α shows that $\ker(\mathbf{r}^K_R)$ and $\ker(\mathbf{r}^K_D)$ are distinct as well. □

Proposition 4.3.5. *For $G = K$ and $k \geq 1$,*

$$\pi^K_i(\Sigma^{k\rho}H\underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & i = 4k \\ \mathbb{F}_2^{\frac{1}{2}(4k-1-i)} & i \in [2k+1, 4k-1] \text{ odd} \\ \mathbb{F}_2^{\frac{1}{2}(4k+2-i)} & i \in [2k+2, 4k-2] \text{ even} \\ \mathbb{F}_2^{i-k+1} & i \in [k, 2k] \end{cases}$$

Proof. Note that

$$P^{-4k}_{-4k}(\Sigma^{-4k}H\underline{\mathbb{Z}}) \simeq \Sigma^{-k\rho}H\underline{\pi}_{4k}(\Sigma^{k\rho}H\underline{\mathbb{Z}}).$$

Thus, by Proposition 4.3.3, $H\underline{\pi}_{4k}(\Sigma^{k\rho}H\underline{\mathbb{Z}}) \simeq H\underline{\mathbb{Z}}$. So the result holds for $i = 4k$ and we only need consider $i \leq 4k-1$.

We will use the resulting long exact sequences in homotopy resulting from the cofiber sequences

$$(K/L_+ \to S^0 \to S^\beta) \wedge \Sigma^{k\rho+1}H\underline{\mathbb{Z}} \tag{4.7}$$

$$(K/R_+ \to S^0 \to S^\alpha) \wedge \Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}} \tag{4.8}$$

$$(K/D_+ \to S^0 \to S^\gamma) \wedge \Sigma^{k\rho+\alpha+\beta+1}H\underline{\mathbb{Z}} \tag{4.9}$$

$$\Sigma^{k\rho}H\underline{\mathbb{Z}} \xrightarrow{2} \Sigma^{k\rho}H\underline{\mathbb{Z}} \to \Sigma^{k\rho}H\underline{\mathbb{F}_2} \tag{4.10}$$

where Equation (4.10) is induced by the short exact sequence of Mackey functors $\underline{\mathbb{Z}} \xrightarrow{2} \underline{\mathbb{Z}} \to \underline{\mathbb{F}_2}$.

For $k = 0$, Equation (4.7) - Equation (4.9) provide that

$$\underline{\pi}_n(\Sigma^\rho H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & n = 4 \\ \underline{0} & n = 3 \\ \underline{mg} & n = 2 \\ \underline{g} & n = 1 \end{cases}$$

Consequently, the result holds for $k = 1$. We now proceed by induction on k. Assume the result holds for k. By restriction to L, D, and R, we find that $\underline{\pi}_i(\Sigma^{(k+1)\rho}H\underline{\mathbb{Z}})$ is a pullback over K for $i \leq 4k+3$ odd, and consequently, 2-torsion as in [Z1, Remark 2.13].

In the long exact sequence of fixed points resulting from Equation (4.7) - Equation (4.9), we have at the K/K level,

$$\pi_i^{C_2}(\Sigma^{2k\rho_{C_2}+1}H_{C_2}\underline{\mathbb{Z}}) \longrightarrow \pi_i^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}) \longrightarrow \pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}})$$

$$\pi_i^{C_2}(\Sigma^{(2k+1)\rho_{C_2}}H_{C_2}\underline{\mathbb{Z}}) \longrightarrow \pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \longrightarrow \pi_i^K(\Sigma^{k\rho+\alpha+\beta+1}H\underline{\mathbb{Z}})$$

$$\pi_i^{C_2}(\Sigma^{(2k+2)\rho_{C_2}-1}H_{C_2}\underline{\mathbb{Z}}) \longrightarrow \pi_i^K(\Sigma^{k\rho+\alpha+\beta+1}H\underline{\mathbb{Z}}) \longrightarrow \pi_i^K(\Sigma^{(k+1)\rho}H\underline{\mathbb{Z}})$$

When $2k+1 \leq i \leq 4k+1$, by Proposition 3.2.4,

$$\pi_i^{C_2}(\Sigma^{2k\sigma+2k+1}H_{C_2}\underline{\mathbb{Z}}) = \begin{cases} \mathbb{F}_2 & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

Now for i even, we have in our long exact sequence,

$$0 \longrightarrow \pi_i^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}) \hookrightarrow \pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \longrightarrow$$

$$\mathbb{F}_2 \longrightarrow \pi_{i-1}^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}) \twoheadrightarrow \pi_{i-1}^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \xrightarrow{0}\!\!\!\!\rightarrow$$

Consequently, when i is even,

$$2\text{-rk } \pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \leq 2\text{-rk } \pi_i^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}) + 1,$$

where 2-rk (M) is the rank of the 2-torsion contained in M and equality occurs if $\pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) = \pi_i^K(\Sigma^{(k\rho+1}H\underline{\mathbb{Z}}) \oplus \mathbb{F}_2$.

And, when i is odd,

$$2\text{-rk } \pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \leq 2\text{-rk } \pi_i^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}).$$

For $i \leq 2k$ we have

$$0 \longrightarrow \pi_i^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}) \hookrightarrow\!\!\!\!\rightarrow \pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \xrightarrow{0}\!\!\!\!\rightarrow$$

Thus, for $i \leq 2k$,

$$\pi_i^K(\Sigma^{k\rho+\beta+1}H\underline{\mathbb{Z}}) \cong \pi_i^K(\Sigma^{k\rho+1}H\underline{\mathbb{Z}}).$$

A similar statement holds for

$$2\text{-rk } \pi_i^K(\Sigma^{k\rho+\alpha+\beta+1}H\underline{\mathbb{Z}}) \qquad \text{and} \qquad 2\text{-rk } \pi_i^K(\Sigma^{(k+1)\rho}H\underline{\mathbb{Z}}).$$

From [GY, Corollary 7.2], in the long exact sequence resulting from Equation (4.10), we have

$$\begin{aligned}\mathbb{F}_2^l \cong \pi_{2i+1}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \xrightarrow{0} \pi_{2i+1}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \hookrightarrow \pi_{2i+1}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{F}_2}) \cong \mathbb{F}_2^{2i+1} \\ \twoheadrightarrow \pi_{2i}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \cong \pi_{2i-1}^K(\Sigma^{i\rho}H\underline{\mathbb{Z}}) \cong \mathbb{F}_2^i.\end{aligned}$$

Consequently, $l = i+1$ and $\pi_{2i+1}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \cong \mathbb{F}_2^{i+1}$. We then have in our long exact sequence

$$\pi_{2i+2}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \to \pi_{2i+2}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{F}_2}) \cong \mathbb{F}_2^{2i+3} \twoheadrightarrow \mathbb{F}_2^{i+1} \cong \pi_{2i+1}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}).$$

Thus,

$$2\text{-rk } \pi_{2i+2}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \geq i+2 = 2\text{-rk } \pi_{2i+1}^K(\Sigma^{i\rho}H\underline{\mathbb{Z}}) + 3.$$

We achieve the maximum bound for 2-rk $\pi_{2i+2}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}})$; thus,

$$\pi_{2i+2}^K(\Sigma^{(i+1)\rho}H\underline{\mathbb{Z}}) \cong \pi_{2i+1}^K(\Sigma^{i\rho}H\underline{\mathbb{Z}}) \oplus \mathbb{F}_2^3 \cong \mathbb{F}_2^{i+2}.$$

The rest of the result now follows from this long exact sequence in a similar manner. □

Proposition 4.3.6. *For $G = K$ and $k \geq 1$,*

$$\pi_i^K(\Sigma^{k\rho-\beta}H\underline{\mathbb{Z}}) = \begin{cases} \mathbb{F}_2^{\frac{1}{2}(4k-i)} & i \in [2k, 4k-2] \text{ even} \\ \mathbb{F}_2^{\frac{1}{2}(4k-i-1)} & i \in [2k-1, 4k-3] \text{ odd} \\ \mathbb{F}_2^{i-k+1} & i \in [k, 2k] \end{cases}$$

Proof. This follows from a similar argument as in Proposition 4.3.5. □

Proposition 4.3.7. *For $G = K$ and $k \geq 1$,*

$$\pi_i^K(\Sigma^{k\rho+\beta} H\underline{\mathbb{Z}}) = \begin{cases} \mathbb{F}_2^{\frac{1}{2}(4k-i)+1} & i \in [2k+2, 4k] \text{ even} \\ \mathbb{F}_2^{\frac{1}{2}(4k-i+1)} & i \in [2k+1, 4k-1] \text{ odd} \\ \mathbb{F}_2^{i-k+1} & i \in [k, 2k] \end{cases}$$

Proof. This follows from a similar argument as in Proposition 4.3.5. □

4.3.3 The $(-4k-2)$-slices

We now determine the $(-4k-2)$-slices of $\Sigma^{-n} H\underline{\mathbb{Z}}$.

Proposition 4.3.8. *For even n, $\Sigma^{-n} H\underline{\mathbb{Z}}$ has no $(-4k-2)$-slices, except for possibly the $-n$ slice. For odd $n \geq 1$,*

$$P_{-4k-2}^{-4k-2}(\Sigma^{-n} H\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{-k\rho-1} H\underline{m}^* & 4k+2 = n+1 \\ \Sigma^{-k\rho-1} H\phi_{LDR}^* \underline{\mathbb{F}_2}^* & 4k+2 \in (n+1, 2n] \end{cases}$$

Proof. When n is even, the slices in Proposition 4.3.4 restrict to the all of the appropriate slices for $\Sigma^{-n} H_{C_2}\underline{\mathbb{Z}}$. If, then, $\Sigma^{-n} H_K\underline{\mathbb{Z}}$ has a $(-4k-2)$-slice, it must be a pullback over K. But this is a contradiction as such slices are $(-4k)$-slices. Thus, for n even, $\Sigma^{-n} H_K\underline{\mathbb{Z}}$ has no $(-4k-2)$-slices.

Now let n be odd. We first handle the case $4k+2 = 2n$. By Proposition 2.3.5,

$$P_{-4k-2}^{-4k-2}(\Sigma^{-n} H_K\underline{\mathbb{Z}}) \simeq \Sigma^{-(k+1)\rho} P_2^2(\Sigma^{-n+(k+1)\rho} H_K\underline{\mathbb{Z}}).$$

For clarity, let

$$X := P_2^2(\Sigma^{-n+(k+1)\rho} H_K\underline{\mathbb{Z}}).$$

Now from Corollary 3.2.40,

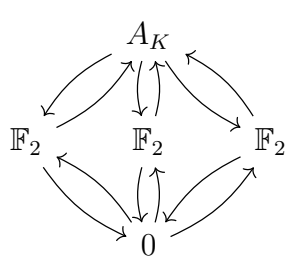
$$i_H^* X \simeq \Sigma^1 H_{C_2}\underline{g}.$$

where H is L, D, or R. Thus, by Theorem 4.1.9,

$$\underline{\pi}_2(X) = \phi_K^* B \qquad \text{and} \qquad \underline{\pi}_1(X) = \underline{A}$$

where B is some group and

$\underline{A} =$



and $A_K \to \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$ is injective. That is, $A_K = \mathbb{F}_2^n$ with $0 \leq n \leq 3$.

If $B \neq 0$, then X cannot be a 2-slice. Consequently, $X \simeq \Sigma^1 H\underline{A}$.

Because $\underline{\mathbb{Z}}$ is invariant under the automorphisms of K, the spectrum $\Sigma^{-n}H\underline{\mathbb{Z}}$ is as well. Therefore, the slices of $\Sigma^{-n}H\underline{\mathbb{Z}}$ are also invariant under the automorphisms of K.

Thus, $\underline{A}$ must be one of the following:

$$\phi^*_{LDR}\underline{f}, \quad \underline{m}, \quad \underline{mg}, \quad \text{or} \quad \phi^*_{LDR}\underline{\mathbb{F}_2}.$$

Except for degree $-k-1$, $\Sigma^{-(k+1)\rho+1}H\underline{A}$ has the same homotopy for each choice of A.

For degree $-k-1$, $\underline{\pi}_{-k-1}(\Sigma^{-(k+1)\rho+1}H\underline{A})$ is a pullback over K of dimension 3, 2, 1, or 0.

From Proposition 4.6.4 we find that $\pi^K_{-k-1}(\Sigma^{-(k+1)\rho+1}H\underline{A}) = 0$. The only choice of $\underline{A}$ that meets this requirement is $\phi^*_{LDR}\underline{\mathbb{F}_2}$.

Now let $4k+2 \in [n+1, 2n-1]$. The base cases are established in Example 4.2.2, Example 4.2.4, and Example 4.2.6. The result then follows from Proposition 4.3.2. □

4.4 Slices of $\Sigma^n H\underline{\mathbb{Z}}$

Recall from Proposition 4.3.1 that $\Sigma^n H\underline{\mathbb{Z}}$ is a slice for $1 \leq n \leq 5$.

Proposition 4.4.1. *Let $n \geq 6$. Then*

$$P^k_k(\Sigma^n H\underline{\mathbb{Z}}) \simeq \Sigma^\rho P^{k-4}_{k-4}(\Sigma^{n-4}H\underline{\mathbb{Z}})$$

for $k \in [n, 2n-7]$.

Proof. We employ a similar argument as in Proposition 4.3.2. Note that

$$P^k_k(\Sigma^n H\underline{\mathbb{Z}}) \simeq \Sigma^\rho P^{k-4}_{k-4}(\Sigma^{n-\rho}H\underline{\mathbb{Z}}) \simeq \Sigma^\rho P^{k-4}_{k-4}(\Sigma^{n-4}H\underline{\mathbb{Z}}^*).$$

Consequently, it is sufficient to compare the $(k-4)$-slices of $\Sigma^{n-4}H\underline{\mathbb{Z}}$ and $\Sigma^{n-4}H\underline{\mathbb{Z}}^*$.

The exact sequence $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}} \to \underline{\mathbb{M}}$ provides the fiber sequence

$$\Sigma^{j-1}H\underline{\mathbb{M}} \to \Sigma^j H\underline{\mathbb{Z}}^* \xrightarrow{\iota} \Sigma^j H\underline{\mathbb{Z}}.$$

Then, because $\Sigma^{j-1}H\underline{\mathbb{M}} \geq 2j-2$, by [HHR1, Lemma 4.28], ι induces an equivalence on slices strictly below level $2j-2$. Taking $j = n-4$ gives the result. □

Example 4.4.2. The tower for $\Sigma^6 H\underline{\mathbb{Z}}$ is

$$\begin{array}{ccc} P^8_8 = \Sigma^2 H\underline{g} & \longrightarrow & \Sigma^6 H\underline{\mathbb{Z}} \\ & & \downarrow \\ & & P^6_6 = \Sigma^{\rho+2}H\underline{\mathbb{Z}}(2,1)^* \end{array}$$

Proof. We have the short exact sequence $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}}(2,1)^* \to \underline{g}$. This leads to the cofiber sequence

$$P_8^8 = \Sigma^2 H\underline{g} \longrightarrow \Sigma^6 H\underline{\mathbb{Z}} \simeq \Sigma^{\rho+2} H\underline{\mathbb{Z}}^* \longrightarrow \Sigma^{\rho+2} H\underline{\mathbb{Z}}(2,1)^* = P_6^6.$$

□

Example 4.4.3. The tower for $\Sigma^7 H\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P_{12}^{12} = \Sigma^3 H\underline{g} \arrow[r] & \Sigma^7 H\underline{\mathbb{Z}} \arrow[d] \\
P_8^8 = \Sigma^{\rho+2} H\underline{m} \arrow[r] & \Sigma^{\rho+3} H\underline{\mathbb{Z}}(2,1)^* \arrow[d] \\
 & P_7^7 = \Sigma^{\rho+3} H\underline{\mathbb{Z}}
\end{tikzcd}$$

Proof. We suspend the tower in Example 4.4.2 by 1 and augment with the cofiber sequence $\Sigma^{\rho+2} H\underline{m} \to \Sigma^{\rho+3} H\underline{\mathbb{Z}}(2,1)^* \to \Sigma^{\rho+3} H\underline{\mathbb{Z}}$ which arises from the short exact sequence $\underline{\mathbb{Z}}(2,1)^* \to \underline{\mathbb{Z}} \to \underline{m}$. □

We now determine the slices of $\Sigma^n H\underline{\mathbb{Z}}$.

Theorem 4.4.4. *Let $n \geq 6$. For $k \geq n+2$,*

$$P_k^k(\Sigma^n H\underline{\mathbb{Z}}) \simeq \Sigma^\rho I_{\mathbb{Q}/\mathbb{Z}}\, P_{-k+4}^{-k+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}.$$

Proof. Take $r \equiv n-5 \pmod 4$ with $1 \leq r \leq 4$. We may map the top slice of $\Sigma^{-n+5} H\underline{\mathbb{Z}}$ into it to find the cofiber sequence

$$P_{-n+5}^{-n+5} = \Sigma^{-\frac{n-5-r}{4}\rho - r} H\underline{\mathbb{Z}}^* \to \Sigma^{-n+5} H\underline{\mathbb{Z}} \to P^{-n+5-1}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}. \tag{4.11}$$

Note that all slices of $P^{-n+4} H\underline{\mathbb{Z}}$ are torsion, so then

$$I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, H\underline{\mathbb{Z}} = \Sigma^1 I_{\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}.$$

Apply $I_{\mathbb{Z}}$ to Equation (4.11) and suspend by one to find

$$\Sigma^1 I_{\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}} \to \Sigma^{n-4} H\underline{\mathbb{Z}}^* \to \Sigma^{\frac{n-5-r}{4}\rho + r + 1} H\underline{\mathbb{Z}}.$$

We can rewrite this as

$$I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}} \to \Sigma^{n-\rho} H\underline{\mathbb{Z}} \to \Sigma^{\frac{n-5-r}{4}\rho + r + 1} H\underline{\mathbb{Z}}.$$

Finally, suspend by ρ to obtain

$$\Sigma^\rho I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}} \to \Sigma^n H\underline{\mathbb{Z}} \to \Sigma^{\frac{n-1-r}{4}\rho + r + 1} H\underline{\mathbb{Z}}. \tag{4.12}$$

Note that $\Sigma^{\frac{n-1-r}{4}\rho+r+1}H\underline{\mathbb{Z}}$ is an n-slice and

$$\Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}} \in [n, 4(n-4)].$$

Furthermore, if $n \not\equiv 2 \pmod 4$,

$$\Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}} \in [n+1, 4(n-4)].$$

From Proposition 2.4.4,

$$I_{\mathbb{Q}/\mathbb{Z}}\, P^{-k}_{-k}\, \Sigma^{-n+5} H\underline{\mathbb{Z}} \simeq P^{k}_{k} I_{\mathbb{Q}/\mathbb{Z}}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}.$$

Consequently, $\Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}$ provides all slices of $\Sigma^n H\underline{\mathbb{Z}}$.

Now suppose $n \equiv 2 \pmod 4$ so that $r = 1$. Then from Proposition 2.4.4,

$$\begin{aligned} P^n_n(\Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}) &\simeq \Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n}_{-n}(P^{-n+4}\, \Sigma^{-n+5} H\underline{\mathbb{Z}}) \\ &\simeq \Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, \Sigma^{-(\frac{n-6}{4}+1)\rho+1} H\underline{mg} \\ &\simeq \Sigma^{\frac{n-2}{4}\rho+1} H\underline{m} \end{aligned}$$

Apply $P^n_n(-)$ to Equation (4.12) to get the extension

$$\Sigma^{\frac{n-2}{4}\rho+1} H\underline{m} \to P^n_n \Sigma^n H\underline{\mathbb{Z}} \to \Sigma^{\frac{n-2}{4}\rho+2} H\underline{\mathbb{Z}}$$

and the fiber sequence

$$\Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+5} H\underline{\mathbb{Z}} \to \Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+4} H\underline{\mathbb{Z}} \to P^n_n \Sigma^n H\underline{\mathbb{Z}}.$$

Now $\Sigma^{\rho} I_{\mathbb{Q}/\mathbb{Z}}\, P^{-n+5} H\underline{\mathbb{Z}} \in [n+1, 4(n-4)]$ and thus supplies the remaining slices of $\Sigma^n H\underline{\mathbb{Z}}$. □

Proposition 4.4.5. *Let $n \geq 6$ and set $r \equiv n \pmod 4$ with $2 \leq r \leq 5$. The n-slice of $\Sigma^n H\underline{\mathbb{Z}}$ is*

$$P^n_n \Sigma^n H\underline{\mathbb{Z}} \simeq \begin{cases} \Sigma^{\frac{n-2}{4}\rho+2} H\underline{\mathbb{Z}}(2,1)^* & n \equiv 2 \pmod 4 \\ \Sigma^{\frac{n-r}{4}\rho+r} H\underline{\mathbb{Z}} & \textit{otherwise} \end{cases}$$

Proof. For $n \not\equiv 2 \pmod 4$, this follows from Equation (4.12). When $n \equiv 2 \pmod 4$ it follows from Example 4.4.2 and repeated application of Proposition 4.4.1. □

Proposition 4.4.6. *Let $n \geq 6$. The $4k$-slices of $\Sigma^n H\underline{\mathbb{Z}}$ are*

$$P^{4k}_{4k}(\Sigma^n H\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{k\rho} H\underline{mg}^* & 4k = n+1 \\ \Sigma^{k} H\underline{g}^{\frac{1}{2}(4k-n)} & 4k \in [n+2, 2n-8], \\ & \quad n \equiv 0 \pmod 2 \\ \Sigma^{k\rho} H(\underline{g}^{\frac{1}{2}(4k-n+3)-3} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2}^*) & 4k \in [n+2, 2n-6], \\ & \quad n \equiv 1 \pmod 2 \\ \Sigma^{k} H\underline{g}^{n-k-3} & 4k \in [2n-4, 4(n-4)] \end{cases}$$

Proof. This follows from Proposition 4.3.4 and Theorem 4.4.4. □

Proposition 4.4.7. *For $n \geq 6$, except for possibly the n slice, the nontrivial $(4k+2)$-slices of $\Sigma^n H\underline{\mathbb{Z}}$ are*

$$P^{4k+2}_{4k+2}(\Sigma^n H\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{k\rho+1} H\phi^*_{LDR}\underline{\mathbb{F}_2} & 4k+2 \in [n+2, 2n-6], \\ & n \equiv 0 \pmod 2 \end{cases}$$

Proof. By Corollary 3.2.37, $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ has no $(4k+2)$-slices except in the range $[n+2, 2n-6]$ when n is even. So for $4k+2$ not in this range, any such slice must be a pullback over K. But then it is a $4k$-slice. For $4k+2 \in [n+2, 2n-6]$, the result follows from Proposition 4.3.8 and Theorem 4.4.4. □

4.4.1 Comparison with the Slices of $\Sigma^n H\underline{\mathbb{F}_2}$

This work is complementary to [GY], which calculates the slices of $\Sigma^n H\underline{\mathbb{F}_2}$ for $n \geq 1$. One would hope that the exact sequence $\underline{\mathbb{Z}} \xrightarrow{2} \underline{\mathbb{Z}} \to \underline{\mathbb{F}_2}$ could play a role in recovering the slices of $\Sigma^n H\underline{\mathbb{Z}}$ from the slices of $\Sigma^n H\underline{\mathbb{F}_2}$ or vice versa, but this is not always the case.

When $G = C_2$, [GY, Theorem 3.18] shows that the slices of $\Sigma^n H_{C_2}\underline{\mathbb{F}_2}$ contain both even and odd suspensions of $H_{C_2}\underline{g}$, whereas Proposition 3.2.35 shows that $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ has only even or odd suspensions of $H_{C_2}\underline{g}$. This is illustrated in Table 4.1.

Table 4.1: Comparison of C_2-slices

Slices of $\Sigma^9 H_{C_2}\underline{\mathbb{Z}}$	Slices of $\Sigma^9 H_{C_2}\underline{\mathbb{F}_2}$	Slices of $\Sigma^{10} H_{C_2}\underline{\mathbb{Z}}$
	$P^{14}_{14} = \Sigma^7 H_{C_2}\underline{g}$	$P^{14}_{14} = \Sigma^7 H_{C_2}\underline{g}$
$P^{12}_{12} = \Sigma^6 H_{C_2}\underline{g}$	$P^{12}_{12} = \Sigma^6 H_{C_2}\underline{g}$	
	$P^{10}_{10} = \Sigma^5 H_{C_2}\underline{g}$	$P^{10}_{10} = \Sigma^{2\rho+2} H_{C_2}\underline{\mathbb{Z}}^*$
$P^9_9 = \Sigma^{2\rho+1} H_{C_2}\underline{\mathbb{Z}}^*$	$P^9_9 = \Sigma^{2\rho+1} H_{C_2}\underline{\mathbb{F}_2}^*$	

The $2k$-slices of $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ and $\Sigma^{n+1} H_{C_2}\underline{\mathbb{Z}}$ only combine to give the slices of $\Sigma^n H_{C_2}\underline{\mathbb{F}_2}$ when $n \equiv 3, 4 \pmod 4$. When $n \equiv 5, 6 \pmod 4$, the $\Sigma^n H_{C_2}\underline{\mathbb{Z}}$ and $\Sigma^{n+1} H_{C_2}\underline{\mathbb{Z}}$ slices miss the $(n+r)$-slice of $\Sigma^n H_{C_2}\underline{\mathbb{F}_2}$, where $r = 1, 2$, respectively. For example, neither $\Sigma^9 H_{C_2}\underline{\mathbb{Z}}$ nor $\Sigma^{10} H_{C_2}\underline{\mathbb{Z}}$ has a slice equivalent to $\Sigma^5 H_{C_2}\underline{g}$, but $\Sigma^9 H_{C_2}\underline{\mathbb{F}_2}$ does.

We can recover the $(4k)$-slices of $\Sigma^n H\underline{\mathbb{F}_2}$ from the $(4k)$-slices of $\Sigma^n H_K\underline{\mathbb{Z}}$ and $\Sigma^{n+1} H_K\underline{\mathbb{Z}}$. As in Proposition 4.3.5, we use the sequence $\underline{\mathbb{Z}} \xrightarrow{2} \underline{\mathbb{Z}} \to \underline{\mathbb{F}_2}$ to get the cofiber sequence $\Sigma^{-k\rho} H_K\underline{\mathbb{Z}} \to \Sigma^{-k\rho} H_K\underline{\mathbb{F}_2} \to \Sigma^{1-k\rho} H_K\underline{\mathbb{Z}}$. We then have the long

exact sequence in homotopy

$$\begin{aligned} &\underline{\pi}_{-n}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}) \xrightarrow{2} \underline{\pi}_{-n}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}) \longrightarrow \\ &\underline{\pi}_{-n}(\Sigma^{-k\rho}H_K\underline{\mathbb{F}_2}) \longrightarrow \underline{\pi}_{-n-1}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}) \xrightarrow{2} \underline{\pi}_{-n-1}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}). \end{aligned} \tag{4.13}$$

When $n \leq 4k-1$, all groups in Equation (4.13) are 2-torsion and the middle three terms become the exact sequence

$$\underline{\pi}_{-n}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}) \hookrightarrow \underline{\pi}_{-n}(\Sigma^{-k\rho}H_K\underline{\mathbb{F}_2}) \twoheadrightarrow \underline{\pi}_{-n-1}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}). \tag{4.14}$$

When $n = 4k$, the left four terms of Equation (4.13) become the exact sequence

$$\underline{\mathbb{Z}}^* \xrightarrow{2} \underline{\mathbb{Z}}^* \overset{2}{\twoheadrightarrow} \underline{\pi}_{-n-1}(\Sigma^{-k\rho}H_K\underline{\mathbb{Z}}) \longrightarrow \underline{0}.$$

Consequently, the $(4k)$-slices of $\Sigma^n H_K\underline{\mathbb{F}_2}$ are

$$P^{4k}_{4k}(\Sigma^n H_K\underline{\mathbb{F}_2}) \simeq \begin{cases} \Sigma^{k\rho}H_K\underline{\mathbb{F}_2}^* & n = 4k \\ \Sigma^{k\rho}\underline{E}_{-n} & n \leq 4k-1 \end{cases}$$

where $\underline{E}_{-n}$ is the middle Mackey functor in Equation (4.14). This recovery is illustrated in Table 4.2.

Table 4.2: Comparison of K-slices

Slices of $\Sigma^7 H\underline{\mathbb{Z}}$	Slices of $\Sigma^7 H\underline{\mathbb{F}_2}$	Slices of $\Sigma^8 H\underline{\mathbb{Z}}$
	$P^{16}_{16} = \Sigma^4 H\underline{g}$	$P^{16}_{16} = \Sigma^4 H\underline{g}$
$P^{12}_{12} = \Sigma^3 H\underline{g}$	$P^{12}_{12} = \Sigma^3 H\underline{g}^3$	$P^{12}_{12} = \Sigma^3 H\underline{g}^2$
	$P^{10}_{10} = \Sigma^{\rho+3} H\phi^*_{LDR}\underline{\mathbb{F}_2}$	$P^{10}_{10} = \Sigma^{\rho+3} H\phi^*_{LDR}\underline{\mathbb{F}_2}$
$P^8_8 = \Sigma^{\rho+2} H\underline{m}$	$P^8_8 = \Sigma^{\rho+2} H\underline{m}$	$P^8_8 = \Sigma^{\rho+4} H\underline{\mathbb{Z}}$
$P^7_7 = \Sigma^{\rho+3} H\underline{\mathbb{Z}}$	$P^7_7 = \Sigma^{\rho+3} H\underline{\mathbb{F}_2}$	

Except for the n-slice, all slices of $\Sigma^7 H_K\underline{\mathbb{F}_2}$ are recovered from the slices of $\Sigma^7 H_K\underline{\mathbb{Z}}$ and $\Sigma^8 H_K\underline{\mathbb{Z}}$. It is not always the case, however, that the $(4k+2)$-slices are recovered. For example, $\Sigma^{10} H_K\underline{\mathbb{F}_2}$ has a 14-slice ([GY, Example 6.14]), but neither $\Sigma^{10} H_K\underline{\mathbb{Z}}$ nor $\Sigma^{11} H_K\underline{\mathbb{Z}}$ have 14-slices.

4.5 Homotopy Mackey Functor Computations

Here we compute the homotopy Mackey functors of the slices of $\Sigma^{\pm n} H\underline{\mathbb{Z}}$.

Proposition 4.5.1. *For $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n(\Sigma^{k\rho} H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & n = 4k \\ \underline{mg} & n = 4k-2 \\ \underline{g}^{\frac{1}{2}(4k-n-1)} & n \in [2k, 4k-3], \\ & n \equiv 1 \pmod 2 \\ \underline{g}^{\frac{1}{2}(4k-n+2)-3} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2} & n \in [2k, 4k-3], \\ & n \equiv 0 \pmod 2 \\ \underline{g}^{n-k+1} & n \in [k, 2k-1] \end{cases}$$

Proof. For $n \in [k, 4k-2]$, this is a restatement of Proposition 4.3.4. For $n = 4k$, the result follows from Proposition 4.3.3 and repeated application of Proposition 4.3.2. □

Proposition 4.5.2. *For $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_{-n}(\Sigma^{-k\rho} H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}}^* & n = 4k \\ \underline{mg}^* & n = 4k-1 \\ \underline{g}^{\frac{1}{2}(4k-n)} & n \in [2k+4, 4k-2], \\ & \quad n \equiv 0 \pmod 2 \\ \underline{g}^{\frac{1}{2}(4k-n+3)-3} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2}^* & n \in [2k+3, 4k-2], \\ & \quad n \equiv 1 \pmod 2 \\ \underline{g}^{n-k-3} & n \in [k+4, 2k+2] \end{cases}$$

Proof. For $n \in [k+4, 4k-1]$, this is a restatement of Proposition 4.4.6. For $n = 4k$, the result follows from Proposition 4.4.5 and repeated application of Proposition 4.4.1. □

Proposition 4.5.3. *For $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{\mathbb{Z}}^*$ are*

$$\underline{\pi}_n(\Sigma^{k\rho} H\underline{\mathbb{Z}}^*) = \begin{cases} \underline{\mathbb{Z}} & n = 4k \\ \underline{mg} & n = 4k-2 \\ \underline{g}^{\frac{1}{2}(4k-n-1)} & n \in [2k+2, 4k-3], \\ & \quad n \equiv 1 \pmod 2 \\ \underline{g}^{\frac{1}{2}(4k-n-2)-3} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2} & n \in [2k+2, 4k-3], \\ & \quad n \equiv 0 \pmod 2 \\ \underline{g}^{n-k+2} & n \in [k+3, 2k+1] \end{cases}$$

Proof. This follows from the equivalence $\Sigma^{k\rho} H\underline{\mathbb{Z}}^* \simeq \Sigma^{(k-1)\rho+4} H\underline{\mathbb{Z}}$ and Proposition 4.5.1. □

Proposition 4.5.4. *For $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{\mathbb{Z}}(2,1)^*$ are*

$$\underline{\pi}_n(\Sigma^{k\rho} H\underline{\mathbb{Z}}(2,1)^*) = \begin{cases} \underline{\pi}_n(\Sigma^{k\rho} H\underline{\mathbb{Z}}^*) & n \in [k+3, 4k] \\ \underline{g} & n = k \end{cases}$$

Proof. The exact sequence $\underline{\mathbb{Z}}^* \to \underline{\mathbb{Z}}(2,1)^* \to \underline{g}$ and corresponding cofiber sequence $\Sigma^{k\rho} H\underline{\mathbb{Z}}^* \to \Sigma^{k\rho} H\underline{\mathbb{Z}}(2,1)^* \to \Sigma^k H\underline{g}$ provide us with a long exact sequence in homotopy. We then have that the homotopy of $\Sigma^{k\rho} H\underline{\mathbb{Z}}(2,1)^*$ is the homotopy of $\Sigma^{k\rho} H\underline{\mathbb{Z}}^*$ with an additional $\underline{g}$ in degree k. □

Proposition 4.5.5. *For $k \geq 2$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{mg}^*$ are*

$$\underline{\pi}_n(\Sigma^{k\rho} H\underline{mg}^*) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & n = 2k \\ \underline{g}^3 & n \in [k+2, 2k-1] \\ \underline{g} & n = k+1 \end{cases}$$

Proof. We have the equivalence $\Sigma^{k\rho} H\underline{mg}^* \simeq \Sigma^{(k-1)\rho+2} H\underline{m}$. The result then follows from [GY, Proposition 7.3]. □

Proposition 4.5.6. *For $k \geq 2$, the nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H\underline{mg}$ are*

$$\underline{\pi}_{-n}(\Sigma^{-k\rho} H\underline{mg}) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2}^* & n = 2k \\ \underline{g}^3 & n \in [k+2, 2k-1] \\ \underline{g} & n = k+1 \end{cases}$$

Proof. The result follows by taking the Brown-Comenetz dual of each Mackey functor in Proposition 4.5.5. □

Proposition 4.5.7. *For $k \geq 2$, the nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H\underline{m}$ are*

$$\underline{\pi}_{-n}(\Sigma^{-k\rho} H\underline{m}) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2}^* & n = 2k \\ \underline{g}^3 & n \in [k+2, 2k-1] \\ \underline{g}^2 & n = k+1 \end{cases}$$

Proof. First, take the Brown-Comenetz dual of the Mackey functors in [GY, Proposition 7.4]. The result then follows from the equivalence $\Sigma^{-\rho} H\underline{m} \simeq \Sigma^{-2} H\underline{mg}^*$. □

Proposition 4.5.8. *We have the equivalences*

$$\Sigma^{k\rho} H\phi^*_{LDR}\underline{\mathbb{F}_2}^* \simeq \begin{cases} \Sigma^2 H\phi^*_{LDR}\underline{f} & k = 1 \\ \Sigma^4 H\phi^*_{LDR}\underline{\mathbb{F}_2} & k = 2 \end{cases}$$

Then for $k \geq 3$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\phi^*_{LDR}\underline{\mathbb{F}_2}^*$ are

$$\underline{\pi}_n(\Sigma^{k\rho} H\phi^*_{LDR}\underline{\mathbb{F}_2}^*) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & n = 2k \\ \underline{g}^3 & n \in [k+2, 2k-1] \end{cases}$$

Proof. This is the pullback over L, D, and R of the Mackey functors in [GY, Proposition 3.6]. □

Proposition 4.5.9. *We have the equivalences*

$$\Sigma^{-k\rho} H\phi^*_{LDR}\underline{\mathbb{F}_2} \simeq \begin{cases} \Sigma^{-2} H\phi^*_{LDR}\underline{f} & k=1 \\ \Sigma^{-4} H\phi^*_{LDR}\underline{\mathbb{F}_2}^* & k=2 \end{cases}$$

Then for $k \geq 3$, the nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H\phi^*_{LDR}\underline{\mathbb{F}_2}$ are

$$\underline{\pi}_{-n}(\Sigma^{-k\rho} H\phi^*_{LDR}\underline{\mathbb{F}_2}) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2}^* & n = 2k \\ \underline{g}^3 & n \in [k+2, 2k-1] \end{cases}$$

Proof. This is the pullback over L, D, and R of the Mackey functors in [GY, Proposition 3.7]. □

4.6 Spectral Sequences

The slice spectral sequence for $\Sigma^{-n} H\underline{\mathbb{Z}}$ and $\Sigma^n H\underline{\mathbb{Z}}$ must recover the homotopy Mackey functors of each spectrum, that is, we must be left with $\underline{\pi}_{-n}(\Sigma^{-n} H\underline{\mathbb{Z}}) = \underline{\pi}_n(\Sigma^n H\underline{\mathbb{Z}}) = \underline{\mathbb{Z}}$ and all other homotopy Mackey functors trivial. For most of the differentials, then, there is only one choice.

We use the indexing convention from [HHR1, Section 4.4.2]. The Mackey functor $\underline{E}_2^{t-n,t}$ is $\underline{\pi}_n P_t^t(X)$. We also use the Adams convention, so that $\underline{\pi}_n P_t^t(X)$ has coordinates $(n, n-t)$ and the differential,

$$d_r : \underline{E}_r^{s,t} \to \underline{E}_r^{s+r,t+r-1},$$

points left one and up r.

The symbols in Table 1 denote the Mackey functors in the slice spectral sequences shown.

Table 4.3: Symbols for K-Mackey functors

■ $= \underline{\mathbb{Z}}$	⬟ $= \phi^*_{LDR}\underline{\mathbb{F}_2}$	▰ $= \underline{mg}$
□ $= \underline{\mathbb{Z}}^*$	⬠ $= \phi^*_{LDR}\underline{\mathbb{F}_2}^*$	▱ $= \underline{mg}^*$
◆ $= \underline{\mathbb{F}_2}$	ⓝ $= \underline{g}^n$	△ $= \underline{m}^*$

Example 4.6.1. The slices for $\Sigma^{-1} H\underline{\mathbb{Z}}$ are all a one-fold desuspension of Eilenberg-MacLane spectra (Example 4.2.2, Figure 4.6.10). Because each of these Mackey functors is in the same column, there are no differentials. Consequently, in the spectral

sequence, we find a double extension:

$$\begin{tikzcd}
\underline{\mathbb{Z}}^* \arrow[dr] & & \\
& \underline{\mathbb{Z}}(2,1) \arrow[dl] \arrow[dr] & \\
\underline{m}^* & & \underline{\mathbb{Z}} \arrow[dll] \\
\underline{g} & &
\end{tikzcd} \tag{4.15}$$

Example 4.6.2. In the spectral sequence for $\Sigma^{-5}H\underline{\mathbb{Z}}$, Figure 4.6.10, because we can only be left with $\underline{\pi}_{-5}(P^{-5}_{-5}\Sigma^{-5}H\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$ and all differentials must go left one and up at least two, all differentials are forced. Once we have evaluated each differential, we once again find ourselves with the double extension in Equation (4.15).

Example 4.6.3. In Figure 4.6.11, most of the differentials for $\Sigma^{-9}H\underline{\mathbb{Z}}$ are again forced by the fact that only $\underline{\pi}_{-9}(P^{-9}_{-9}\Sigma^{-9}H\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$ can survive the spectral sequence. For example, we have two choices for a differential from $\pi_{-8}P^{-32}_{-32}\Sigma^{-9}H\underline{\mathbb{Z}} \cong \underline{g}^2$. We find it must be

$$d_{15} : \underline{g}^2 \to \phi^*_{LDR}\underline{\mathbb{F}_2}^* \cong \underline{\pi}_{-9}(P^{-24}_{-24}\Sigma^{-9}H\underline{\mathbb{Z}})$$

so that we are left with the extension in Equation (4.15). Indeed, we will always be left with this extension once all differentials have been evaluated. Similarly, for $n \equiv 1 \pmod 4$, we will always have a

$$d_2 : \phi^*_{LDR}\underline{\mathbb{F}_2}^* \xrightarrow{\cong} \phi^*_{LDR}\underline{\mathbb{F}_2}^*$$

in the upper right corner.

In Proposition 4.3.8 we claim that for n odd,

$$P^{-2n}_{-2n}(\Sigma^{-n}H\underline{\mathbb{Z}}) \simeq \Sigma^{-(k+1)\rho+1}H\phi^*_{LDR}\underline{\mathbb{F}_2}.$$

We now prove this claim.

Proposition 4.6.4. *Let n be odd and take $\underline{A}$ to be one of the Mackey functors listed in Table 4.4. The only choice of $\underline{A}$ where the homotopy of $P^{-2n}_{-2n}(\Sigma^{-n}H\underline{\mathbb{Z}}) \simeq \Sigma^{-(k+1)\rho+1}H\underline{A}$ fits into the spectral sequence for $\Sigma^{-n}H\underline{\mathbb{Z}}$ is $\phi^*_{LDR}\underline{\mathbb{F}_2}$.*

Proof. Because $\Sigma^{-(k+1)\rho+1}H\underline{A}$ is a $(-4k-2)$-slice, its $\underline{\pi}_{-k-1}$ is located at $(-k-1, -3k-1)$. We argue that we cannot have a nonzero Mackey functor in this location.

In the spectral sequence for $\Sigma^{-n}H\underline{\mathbb{Z}}$, all slices below level $-2n$ are $\Sigma^{-k}H\underline{g}^{n-k+1}$ where $4k \in [2n+1, 4n]$. These Mackey functors, for $4k \leq 2n+1$, lie on the line $y = -3k$. Thus, for the Mackey functors in Table 4.4, the source of a differential hitting it must be $(-k, -3k)$. This is not possible.

Table 4.4: Homotopy Comparison

$\underline{A}$	$\underline{\pi}_{-k-1}(\Sigma^{-(k+1)\rho+1}H\underline{A})$
$\phi^*_{LDR}\underline{f}$	$\underline{g}^3$
$\underline{m}$	$\underline{g}^2$
$\underline{mg}$	$\underline{g}$
$\phi^*_{LDR}\underline{\mathbb{F}_2}$	$\underline{0}$

We now argue that the Mackey functor located at $(-k-1,-3k-1)$ cannot be the source of a differential.The first value of n for which we must determine the $(-2n)$-slice is $n=7$. The spectral sequence where $\underline{A}=\phi^*_{LDR}\underline{\mathbb{F}_2}$ is shown in Figure 4.6.11. This spectral sequence leaves us with the appropriate homotopy for $\Sigma^{-7}H\underline{\mathbb{Z}}$.

Note there is a copy of $\underline{g}^4$ located at $(-4,-12)$. For any other choice of $\underline{A}$, we would have a nontrivial $\underline{\pi}_{-4}$ located at $(-4,-10)$. For a differential originating from $(-4,-10)$, there are two possible targets: $\phi^*_{LDR}\underline{\mathbb{F}_2}^*$ at $(-5,-5)$ and $\underline{g}$ at $(-5,-2)$. However, these two Mackey functors must fit into the exact sequences

$$\underline{g}^2 \to \underline{g}^3 \to \underline{g} \qquad \text{and} \qquad \underline{g}^4 \to \phi^*_{LDR}\underline{\mathbb{F}_2}^* \to \underline{mg}.$$

Thus, we cannot have a nonzero Mackey functor at $(-4,-10)$.

We now consider the spectral sequence for $\Sigma^{-9}H\underline{\mathbb{Z}}$, located in Figure 4.6.11. We again use $\Sigma^{-(l+1)\rho+1}H\phi^*_{LDR}\underline{\mathbb{F}_2}$ for the $(-2n)$-slice. The resulting homotopy fits in the spectral sequence. For the other three choices of $\underline{A}$ we would have a nontrivial $\underline{\pi}_{-5}$ located at $(-5,-13)$. For a differential originating from $(-5,-13)$ there are two possible targets: $\underline{g}^3$ at $(-6,-8)$ and $\phi^*_{LDR}\underline{\mathbb{F}_2}^*$ at $(-6,-3)$.

However, we have a $d_2:\phi^*_{LDR}\underline{\mathbb{F}_2}^* \twoheadrightarrow \phi^*_{LDR}\underline{\mathbb{F}_2}^*$ and a $d_7:\underline{g}^5 \twoheadrightarrow \underline{g}^3$. Thus, there is no target for a differential from $(-5,-13)$. Consequently, the only choice of $\underline{A}$ that works is $\phi^*_{LDR}\underline{\mathbb{F}_2}$.

There is a similar story for all odd $n>9$. There will always be a d_2 hitting the $\phi^*_{LDR}\underline{\mathbb{F}_2}^*$ located at $(-k-2,-n+k+2)$. The only other possible targets for a differential from $(-k-1,-3k-1)$ are then the $\underline{g}^3$'s resulting from the homotopy of the other $(-4j-2)$-slices. All of these will be hit by a differential from the $\underline{g}^{n-k+1}$ located at $(-k-1,-3k-3)$. Thus, we cannot have a nonzero Mackey functor located at $(-k-1,-3k-1)$. The only choice of $\underline{A}$ which satisfies this requirement is $\phi^*_{LDR}\underline{\mathbb{F}_2}$. □

Example 4.6.5. For the positive, trivial suspensions of $H\underline{\mathbb{Z}}$, we find that $\Sigma^6 H\underline{\mathbb{Z}}$ has the first nontrivial slice tower. In Figure 4.6.12, we then see that there is only one possible differential. This d_3 exists because we must be left with only $\underline{\pi}_6(\Sigma^6 H\underline{\mathbb{Z}})\cong\underline{\mathbb{Z}}$.

Example 4.6.6. The spectral sequence for $\Sigma^7 H\underline{\mathbb{Z}}$, in Figure 4.6.12, is more interesting. Here we find the differentials $d_2:\phi^*_{LDR}F\to\underline{mg}$ and $d_5:\underline{mg}\to\underline{g}$. Indeed, we

will always see a $d_{n-7} : \phi^*_{LDR}\underline{\mathbb{F}_2} \to \underline{mg}$ and $d_{2n-9} : \underline{mg} \to \underline{g}$ on the right side of the spectral sequence for $\Sigma^n H_K\underline{\mathbb{Z}}$.

Example 4.6.7. Except for the homotopy of the n-slice of $\Sigma^7 H_K\underline{\mathbb{Z}}$, $\Sigma^7 H_K\underline{\mathbb{F}_2}$, and $\Sigma^8 H_K\underline{\mathbb{Z}}$, the spectral sequences for $\Sigma^7 H_K\underline{\mathbb{Z}}$ and $\Sigma^8 H_K\underline{\mathbb{Z}}$ collapse to give the spectral sequence for $\Sigma^7 H_K\underline{\mathbb{F}_2}$. We see in Figure 4.6.12 the $\underline{g}$ in $(3,9)$ and the $\underline{g}^2$ in $(3,9)$ in the spectral sequences for $\Sigma^7 H_K\underline{\mathbb{Z}}$ and $\Sigma^8 H_K\underline{\mathbb{Z}}$, respectively, combine to give the $\underline{g}^3$ in $(3,9)$ in the spectral sequence for $\Sigma^7 H_K\underline{\mathbb{F}_2}$. Off the diagonal for the n-slice for $\Sigma^7 H_K\underline{\mathbb{Z}}$ and $\Sigma^8 H_K\underline{\mathbb{Z}}$ we have a single copy of $\phi^*_{LDR}\underline{\mathbb{F}_2}$. These provide the two copies of $\phi^*_{LDR}\underline{\mathbb{F}_2}$ off the diagonal for $\Sigma^7 H_K\underline{\mathbb{F}_2}$.

Example 4.6.8. Now, in Figure 4.6.13, we have some choice of differentials in the spectral sequence for $\Sigma^{11} H\underline{\mathbb{Z}}$. Once we consider that only $\underline{\pi}_{11}(\Sigma^{11} H\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}$ can be left, there is only one choice of each differential that provides the desired result. Analogously to the spectral sequence for $\Sigma^{-n} H\underline{\mathbb{Z}}$ where $n \equiv 1 \pmod 4$, we will always have a $d_2 : \phi^*_{LDR}\underline{\mathbb{F}_2} \xrightarrow{\cong} \phi^*_{LDR}\underline{\mathbb{F}_2}^*$ in the bottom left corner when $n \equiv 3 \pmod 4$.

Example 4.6.9. Again, with the exception of the homotopy of the n-slice, the spectral sequences for $\Sigma^{10} H_K\underline{\mathbb{Z}}$ and $\Sigma^{11} H_K\underline{\mathbb{Z}}$ collapse to give the spectral sequence for $\Sigma^{10} H_K\underline{\mathbb{F}_2}$ in Figure 4.6.13. As in Example 4.6.7, the upper left diagonals in in the spectral sequences for $\Sigma^{10} H_K\underline{\mathbb{Z}}$ and $\Sigma^{11} H_K\underline{\mathbb{Z}}$ combine to even more copies of $\underline{g}$ in the upper left diagonal in the spectral sequence for $\Sigma^{10} H_K\underline{\mathbb{F}_2}$. This will always be the case.

Figure 4.6.10. *The slice spectral sequence over K, $n = -1, -3, -5$.*

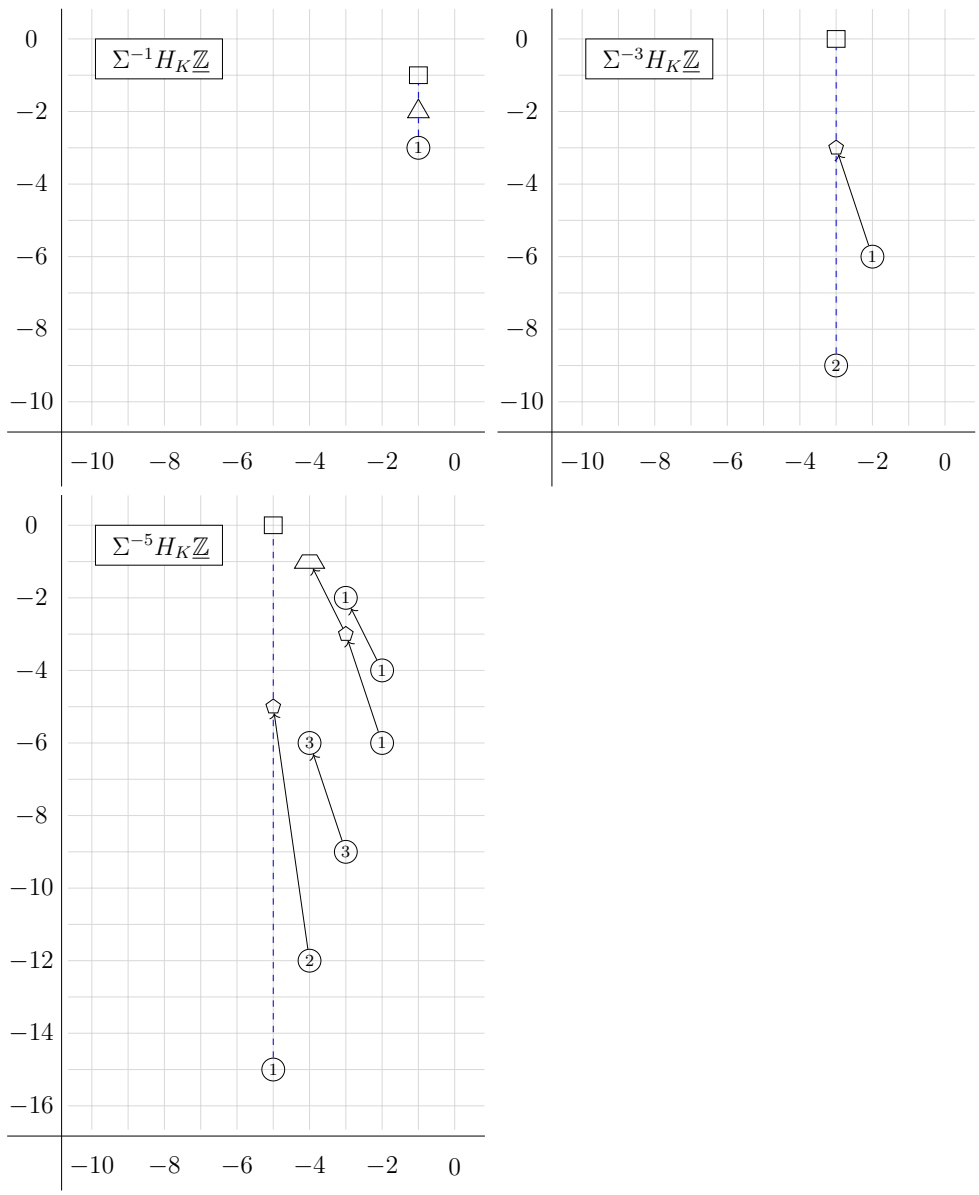


Figure 4.6.11. *The slice spectral sequence over K, $n = -7, -9$.*

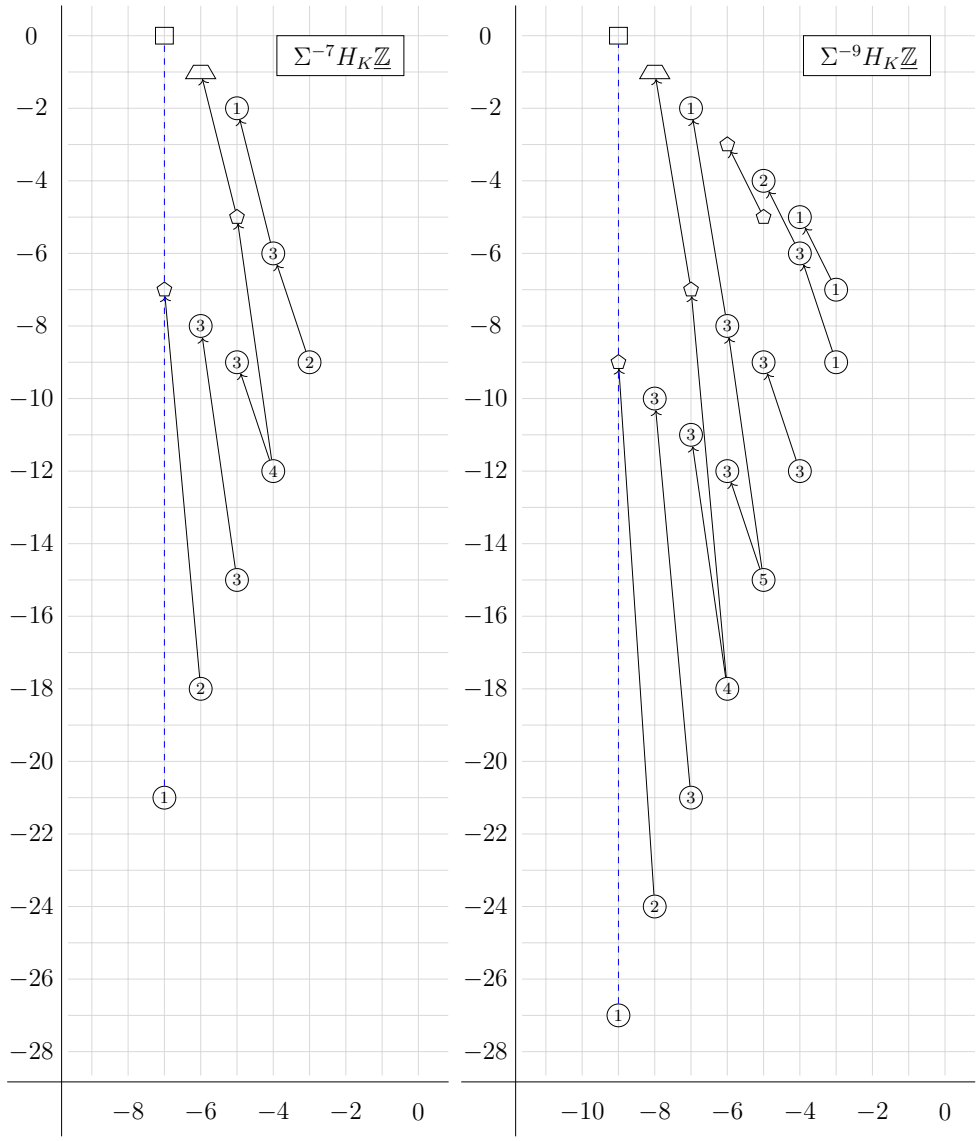


Figure 4.6.12. *The slice spectral sequence over K, $n = 6, 7, 8$.*

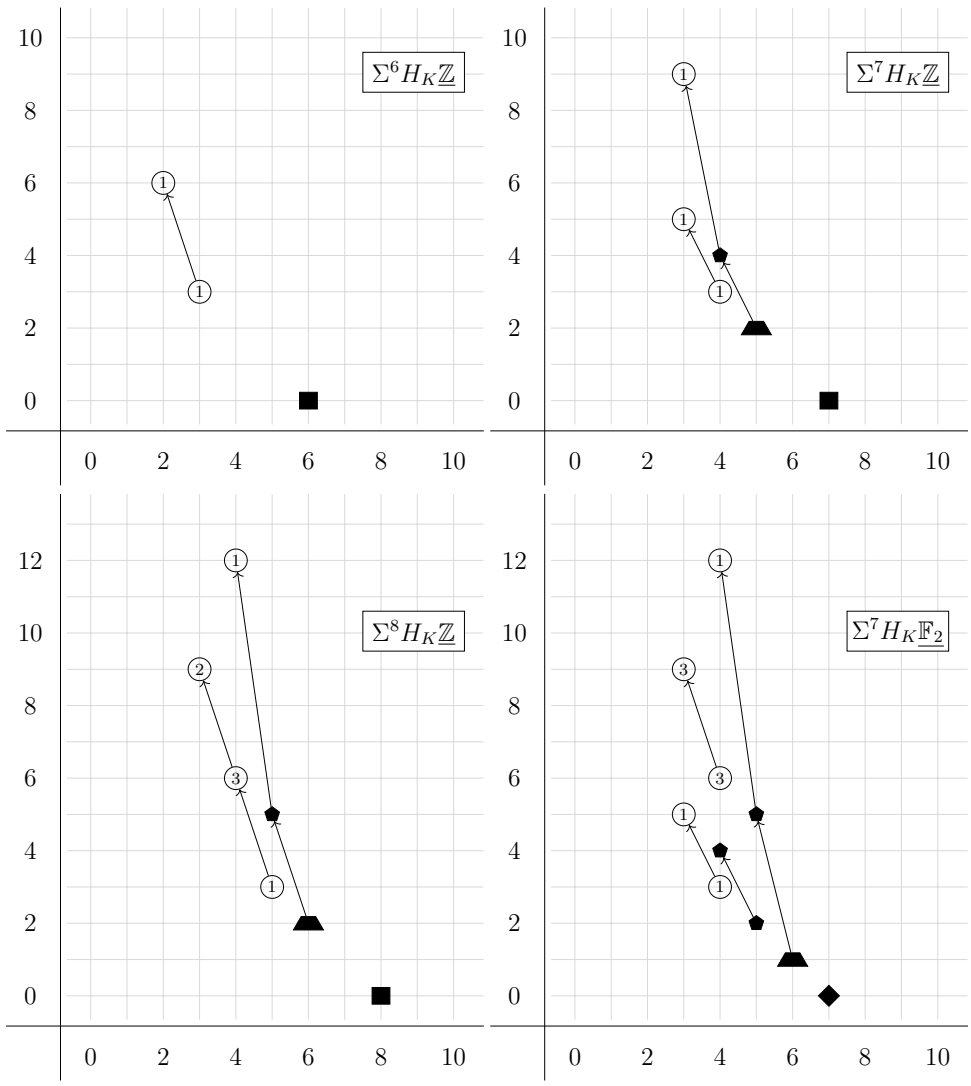


Figure 4.6.13. *The slice spectral sequence over K, $n = 10, 11$.*

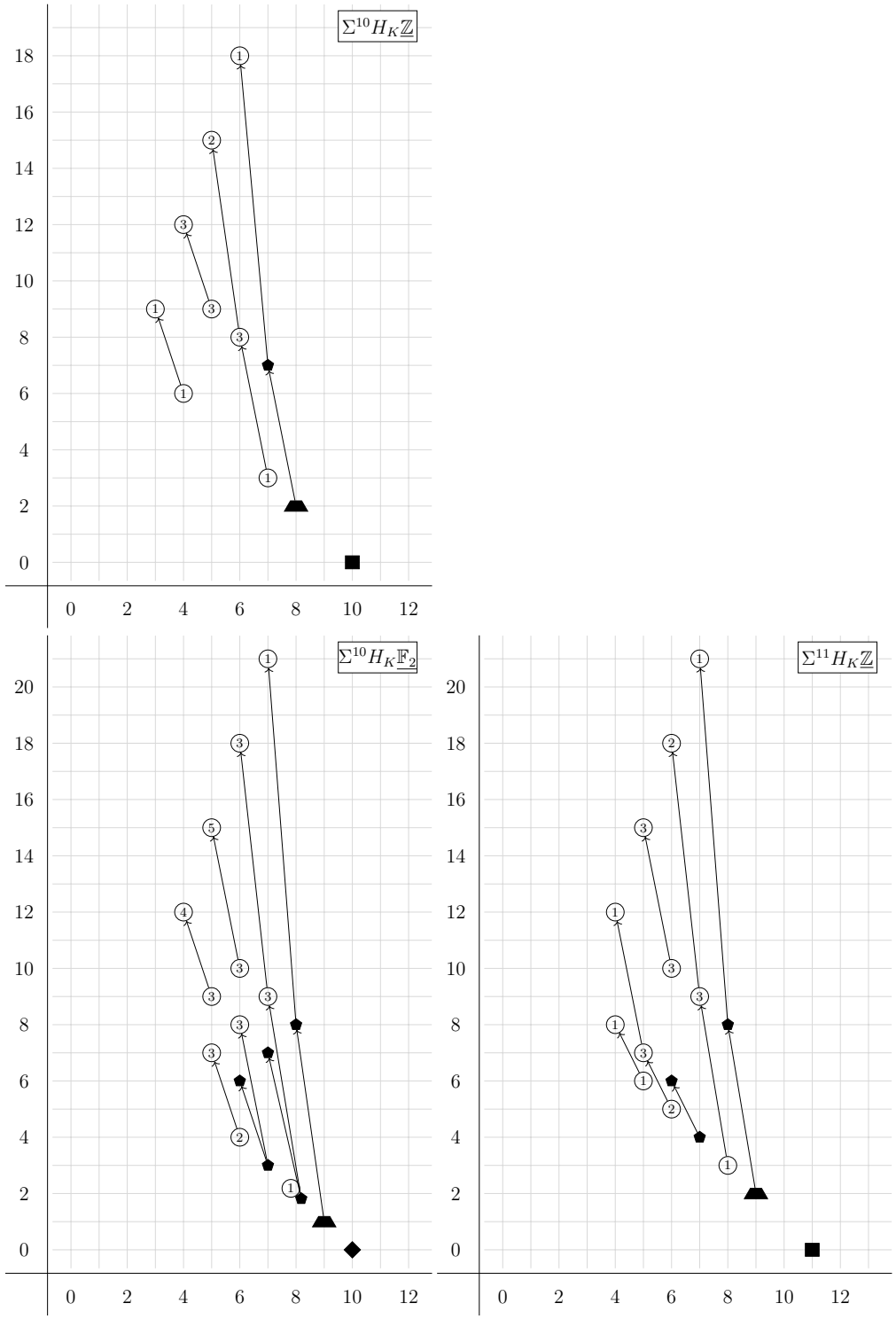


Figure 4.6.14. *The slice spectral sequence over K, $n = 12$.*

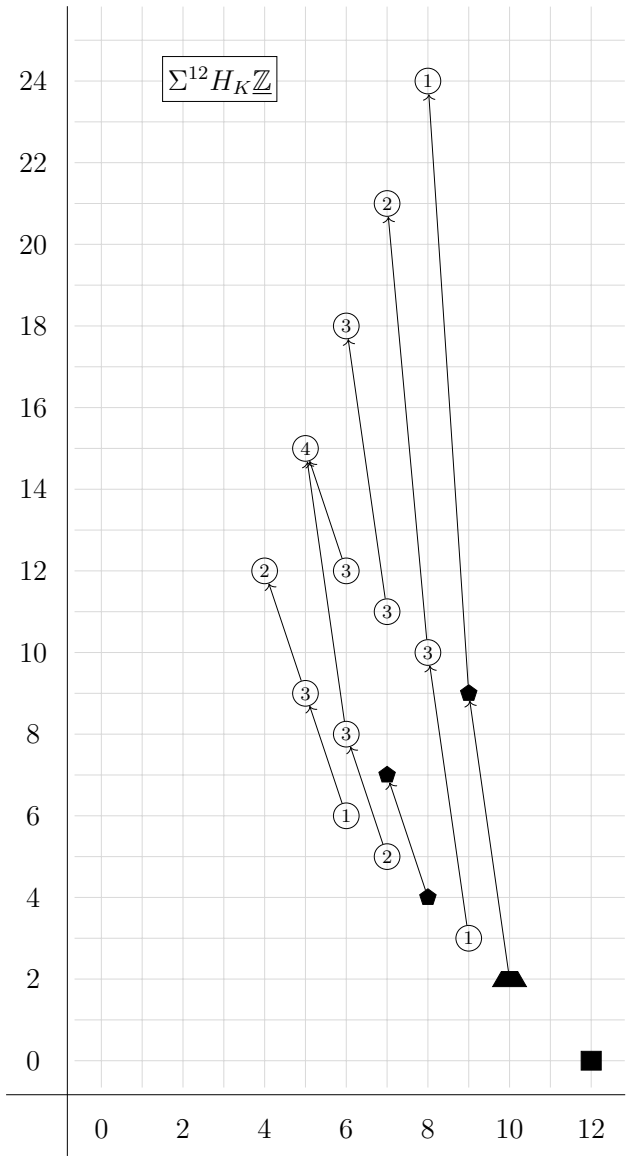
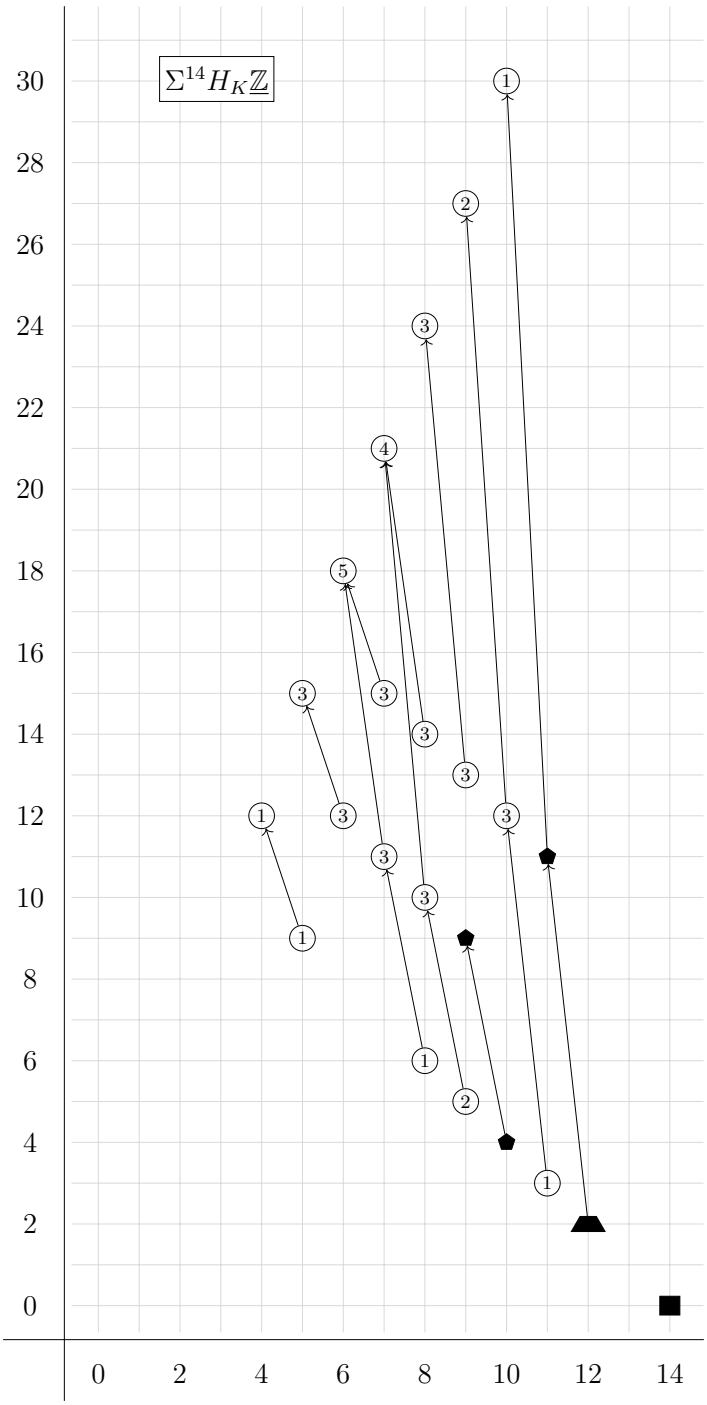


Figure 4.6.15. *The slice spectral sequence over K, $n = 14$.*



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Chapter 5 The slices of quaternionic Eilenberg-Mac Lane spectra

This chapter is joint work with Bertrand Guillou. It is organized as follows. In Section 5.1, we introduce three inflation functors from a quotient group G/N of some finite group G as well as several results that will aid in the calculation of the slices of $\Sigma^n H_{Q_8} H\underline{\mathbb{Z}}$. The relevant Q_8-Mackey functors and the homology of $\Sigma^{k\rho_{Q_8}} H_{Q_8}\underline{\mathbb{Z}}$ are found in Section 5.2. The slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ must restrict to the appropriate slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$; thus, we review this information in Section 5.3. We provide some slice towers and describe all slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ in Section 5.4. We then compute the homotopy Mackey functors of the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ in Section 5.5. Finally, we provide some examples of the slice spectral sequence for $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ and $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ in Section 5.6.

Throughout, whenever referencing the slice filtration, we will always mean the "regular" slice filtration of [U1].

We will often write simply Q and K to denote the quaternion group Q_8 and Klein four group K_4, respectively. We write Z for the central subgroup of Q of order two generated by $z = -1$. We write

$$L = \langle i \rangle, \qquad D = \langle k \rangle, \qquad \text{and} \qquad R = \langle j \rangle$$

for the normal, cyclic subgroups of Q of order 4. We also use the same names for the images of these subgroups in $Q/Z \cong K$. In other words, the subgroup lattices of Q_8 and K_4 are

$$\begin{tikzcd}[column sep=small]
 & Q_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & D \arrow[d, dash] & R \arrow[dl, dash] \\
 & Z \arrow[d, dash] & \\
 & e &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & K_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & D \arrow[d, dash] & R \arrow[dl, dash] \\
 & e &
\end{tikzcd}$$

Our nomenclature for the order 4 subgroups of Q_8 amounts to a choice of isomorphism $Q/Z \cong K$.

5.1 Inflation functors

5.1.1 Inflation and the projection formula

Let $N \trianglelefteq G$ be a normal subgroup and $q\colon G \longrightarrow G/N$ the quotient map. Recall that there is an induced adjunction

$$\mathbf{Sp}^{G/N} \underset{(-)^N}{\overset{q^*}{\rightleftarrows}} \mathbf{Sp}^G$$

where the pullback functor q^*, called inflation, is strong symmetric monoidal. We will also need a description of the N-fixed points of an Eilenberg-Mac Lane G-spectrum.

First note that there is a functor

$$\mathbf{Mack}(G) \xrightarrow{q_*} \mathbf{Mack}(G/N) \tag{5.1}$$

given by

$$q_*(\underline{M})(\overline{H}) = \underline{M}(H),$$

where $\overline{H} = H/N \leq G/N$ whenever $N \leq H$. The functor q_* is denoted $\beta^!$ in [TW, Lemma 5.4]. Then the homotopy Mackey functors of the N-fixed points of a G-spectrum X are given by

$$\underline{\pi}_n(X^N) \cong q_*\underline{\pi}_n(X). \tag{5.2}$$

In the case of an Eilenberg-Mac Lane spectrum this yields an equivalence

$$(H_G\underline{M})^N \simeq H_{G/N}(q_*\underline{M}).$$

The following result will be quite useful.

Proposition 5.1.1. [HK, Lemma 2.13] [BDS, Proposition 2.15] *(Projection formula) Let $N \trianglelefteq G$ be a normal subgroup and $q\colon G \longrightarrow G/N$ be the quotient map. Then for $X \in \mathbf{Sp}^{G/N}$ and $Y \in \mathbf{Sp}^G$, there is a natural equivalence of G/N-spectra*

$$(q^*X \wedge Y)^N \simeq X \wedge Y^N.$$

We will frequently employ this in the case that $X = S^V$ for some G/N-representation V and $Y = H_G\underline{M}$ for some G-Mackey functor $\underline{M}$. Then the projection formula reads

$$(S^{q^*V} \wedge H_G\underline{M})^N \simeq S^V \wedge H_{G/N}(q_*\underline{M}). \tag{5.3}$$

See also [Z1, Corollary 5.8]

5.1.2 Geometric fixed points

For a normal subgroup $N \trianglelefteq G$, we define the family of subgroups $\mathcal{F}[N]$ of G to consist of those subgroups that do not contain N. Recall that the N-geometric fixed points spectrum of a G-spectrum is defined as

$$\Phi^N(X) = \left(\widetilde{E\mathcal{F}[N]} \wedge X\right)^N.$$

This notation is simultaneously used to denote the resulting G/N-spectrum as well as the underlying spectrum. The N-geometric fixed points has a right adjoint, given by the geometric inflation functor

$$\phi_N^*(Z) = \widetilde{E\mathcal{F}[N]} \wedge q^*Z.$$

To sum up, we have an adjunction

$$\mathbf{Sp}^G \underset{\phi_N^*}{\overset{\Phi^N}{\rightleftarrows}} \mathbf{Sp}^{G/N}.$$

5.1.3 Bottleneck subgroups

The subgroup $Z \trianglelefteq Q$ plays an important role in this article. The primary reason is that it satisfies the following property.

Definition 5.1.2. We say that $N \trianglelefteq G$ is a **bottleneck** subgroup if it is a nontrivial, proper subgroup such that, for any subgroup $H \leq G$, either H contains N or N contains H.

We now demonstrate that bottleneck subgroups only occur in cyclic p-groups or quaternion groups. The following argument was sketched to us by Mike Geline.

Proposition 5.1.3. *Let $N \trianglelefteq G$ be a bottleneck subgroup of G. Then N is cyclic, and G is either a cyclic p-group or a generalized quaternion group.*

Proof. We will refer to a subgroup $H \leq G$ which neither contains N nor is contained in N as "adjacent" to N. The assumption that N is a bottleneck subgroup means precisely that G has no subgroups that are adjacent to N. To see that N must be cyclic, note that if g is not in N, then $N \leq \langle g \rangle$, which implies that N is cyclic.

We next observe that G is necessarily a p-group. This is because if N is contained in some Sylow p-subgroup, then any Sylow q-subgroup, for a different prime q, would be adjacent. It follows that N contains all of the Sylow subgroups and therefore is all of G.

Next, we recall [B, Theorem 4.3] that for a p-group G, the group contains a unique subgroup of order p if and only if G is either cyclic or generalized quaternion. So we will argue that G contains a unique subgroup of order p. The first step is to note that G cannot contain a subgroup isomorphic to $C_p \times C_p$. This is because such a subgroup would necessarily contain N. This would imply that $N \cong C_p$, and then N would have a complement in $C_p \times C_p$, which would be a subgroup adjacent to N in G.

Finally, note that the center $Z(G)$ contains a subgroup of order p. If G has another subgroup of order p, these two would generate a $C_p \times C_p$, contradicting the previous step. $\square$

Remark 5.1.4. *It follows from Proposition 5.1.3 that if $N \trianglelefteq G$ is a bottleneck subgroup, then G/N is either a cyclic p-group or a dihedral 2-group.*

If $N \trianglelefteq G$ is a bottleneck subgroup, then geometric fixed points with respect to G can be computed in terms of geometric fixed points with respect to the quotient group G/N.

Proposition 5.1.5. *Let $N \trianglelefteq G$ be a bottleneck subgroup. Then $\Phi^G X \simeq \Phi^{G/N} X^N$ for any $X \in \mathbf{Sp}^G$.*

Proof. If $N \trianglelefteq G$ is a bottleneck subgroup, then $q^* \widetilde{E\mathcal{P}_{G/N}} \simeq \widetilde{E\mathcal{P}_G}$. Thus

$$\Phi^G X = (\widetilde{E\mathcal{P}_G} \wedge X)^G \simeq ((q^* \widetilde{E\mathcal{P}_{G/N}} \wedge X)^N)^{G/N}.$$

By the Projection Formula (Proposition 5.1.1), this is equivalent to

$$(\widetilde{E\mathcal{P}_{G/N}} \wedge X^N)^{G/N} = \Phi^{G/N} X^N. \qquad \square$$

Proposition 5.1.5 also follows from the more general [K, Proposition 9].

5.1.4 Inflation for $\underline{\mathbb{Z}}$-modules

Given a surjection $q\colon G \longrightarrow G/N$, the inflation functor

$$\phi_N^*\colon \mathbf{Mack}(G/N) \longrightarrow \mathbf{Mack}(G)$$

does not send $\underline{\mathbb{Z}}$-modules for G/N to $\underline{\mathbb{Z}}$-modules for G. We now describe a modified inflation functor that exists at the level of $\underline{\mathbb{Z}}$-modules. This functor previously appeared in [Z1, Section 3.2] and [BG, Section 3.10].

Definition 5.1.6. Let $\mathcal{B}\underline{\mathbb{Z}}_G \subset \mathrm{Mod}_{\mathbb{Z}[G]}$ denote the full subcategory of permutation G-modules. Recall [Z1, Proposition 2.15] that $\underline{\mathbb{Z}}_G$-modules correspond to additive functors $\mathcal{B}\underline{\mathbb{Z}}_G^{op} \longrightarrow \mathrm{Ab}$. Then the $\underline{\mathbb{Z}}$-module inflation functor

$$\Psi_N^*\colon \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}} \longrightarrow \mathrm{Mod}_{\underline{\mathbb{Z}}_G}$$

is defined to be the left Kan extension along the inflation functor $\mathcal{B}\underline{\mathbb{Z}}_{G/N} \longrightarrow \mathcal{B}\underline{\mathbb{Z}}_G$.

The following is an immediate corollary of the definition as a left Kan extension.

Proposition 5.1.7. *The functor Ψ_N^* is left adjoint to the functor $q_*\colon \mathrm{Mod}_{\underline{\mathbb{Z}}_G} \longrightarrow \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, defined as in* (5.1).

Proposition 5.1.8 ([BG, (3.11)])**.** *For $\underline{M} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, the $\underline{\mathbb{Z}}_G$-module $\Psi_N^*(\underline{M})$ satisfies*

1. *$q_*\left(\Psi_N^*(\underline{M})\right)$ is $\underline{M}$ and*
2. *$\downarrow_N^G \left(\Psi_N^*(\underline{M})\right)$ is the constant Mackey funtor at $\underline{M}(e)$.*

Note that Proposition 5.1.8 completely describes $\Psi_N^*(\underline{M})$ if N is a bottleneck subgroup. The following result states that $\underline{\mathbb{Z}}$-module inflation agrees with ordinary inflation on geometric Mackey functors.

Proposition 5.1.9. *Let $\underline{M} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, and let $N \trianglelefteq G$ be a bottleneck subgroup. If $\underline{M}(e) = 0$, then $\Psi_N^*\underline{M} \cong \phi_N^*\underline{M}$.*

Proof. This follows immediately from Proposition 5.1.8. $\square$

Remark 5.1.10. *Note that Proposition 5.1.9 is not true without the bottleneck hypothesis. For instance, in the case $N = C_3 \trianglelefteq \Sigma_3$, then $\downarrow_{C_2}^{\Sigma_3} \left(\Psi_{C_3}^*\underline{M}\right) \cong \underline{M}$. In particular, it is not true that $\Psi_{C_3}^*\underline{M}$ is concentrated over $N = C_3$.*

We now discuss the extension to equivariant spectra.

Proposition 5.1.11. *The N-fixed points functor*

$$(-)^N\colon \operatorname{Mod}_{H_G\underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_{G/N}\underline{\mathbb{Z}}}$$

for $H\underline{\mathbb{Z}}$-modules has a left adjoint

$$\Psi_N^*\colon \operatorname{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_G\underline{\mathbb{Z}}}.$$

If $N \trianglelefteq G$ is a bottleneck subgroup, then the spectrum-level functor Ψ_N^* extends the functor Ψ_N^* of Definition 5.1.6, in the sense that

$$\Psi_N^* H_{G/N}\underline{M} \simeq H_G(\Psi_N^*\underline{M}) \tag{5.4}$$

for $\underline{M}$ in $\operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}}$.

Proof. For an $H_{G/N}\underline{\mathbb{Z}}$-module X, the inflation q^*X is canonically a module over $q^*H_{G/N}\underline{\mathbb{Z}}$. We then define the spectrum-level functor Ψ_N^* by the formula

$$\Psi_N^* X = H\underline{\mathbb{Z}} \wedge_{q^*H\underline{\mathbb{Z}}} (q^*X).$$

We leave it to the reader to verify that this is indeed left adjoint to the N-fixed points functor.

To see that (5.4) holds, we show first that this holds on the indecomposable projective $\underline{\mathbb{Z}}_{G/N}$-modules. These are of the form $\uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}$, and the diagram of commuting adjoint functors

$$\begin{tikzcd}
\operatorname{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \arrow[r, shift left, "\Psi_N^*"] \arrow[d, shift left, "\downarrow_{K/N}^{G/N}"] & \operatorname{Mod}_{H_G\underline{\mathbb{Z}}} \arrow[l, shift left, "(-)^N"] \arrow[d, shift left, "\downarrow_K^G"] \\
\operatorname{Mod}_{H_{K/N}\underline{\mathbb{Z}}} \arrow[r, shift left, "\Psi_N^*"] \arrow[u, shift left, "\uparrow_{K/N}^{G/N}"] & \operatorname{Mod}_{H_K\underline{\mathbb{Z}}} \arrow[l, shift left, "(-)^N"] \arrow[u, shift left, "\uparrow_K^G"]
\end{tikzcd}$$

shows that

$$\Psi_N^*\left(H_{G/N} \uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}\right) \simeq \uparrow_K^G \Psi_N^*(H_{K/N}\underline{\mathbb{Z}}) \simeq \uparrow_K^G H_K\underline{\mathbb{Z}} \simeq H_G \uparrow_K^G \underline{\mathbb{Z}} \simeq H_G \Psi_N^*\left(\uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}\right).$$

Since the functor $\Psi_N^*\colon \operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}} \longrightarrow \operatorname{Mod}_{\underline{\mathbb{Z}}_G}$ is exact [Z1, Lemma 3.14], it follows that if $\operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}}$ has finite global projective dimension, then (5.4) will hold for any $\underline{\mathbb{Z}}_{G/N}$-module $\underline{M}$. By [BSW, Theorem 1.7], this is the case precisely when G/N is as described in Remark 5.1.4. □

Example 5.1.12. Let $X \in \mathbf{Sp}^{G/N}$ and $\underline{M} \in \mathbf{Mack}(G/N)$, with $\underline{M}(e) = 0$. Again assume that N is a bottleneck subgroup. Then Proposition 5.1.9 and Proposition 5.1.11 give that

$$\begin{aligned}\Psi_N^*(X \wedge H_{G/N}\underline{M}) &\simeq q^*(X) \wedge \Psi_N^*(H_{G/N}\underline{M}) \simeq q^*(X) \wedge \phi_N^* H_{G/N}\underline{M} \\ &\simeq \phi_N^*(X \wedge H_{G/N}\underline{M}).\end{aligned}$$

We will employ this equivalence when X is a representation sphere.

Proposition 5.1.13. *Let $N \trianglelefteq G$ be a bottleneck subgroup. Then for any G/N-representation V and $\underline{\mathbb{Z}}_{G/N}$-module $\underline{L}$, we have*

$$\underline{\pi}_n \left(\Psi_N^* \Sigma^V H_{G/N} \underline{L}\right) \cong \Psi_N^* \underline{\pi}_n \left(\Sigma^V H_{G/N} \underline{L}\right).$$

Proof. Let us write $X = \Psi_N^* \Sigma^V H_{G/N} \underline{L} \simeq \Sigma^{q^*V} H_G \Psi_N^* \underline{L}$. Since N is a bottleneck subgroup, it is enough to describe $\downarrow_N^G \underline{\pi}_n X$ and $q_* \underline{\pi}_n X$. Now

$$\downarrow_N^G \underline{\pi}_n X \cong \underline{\pi}_n \downarrow_N^G X = \underline{\pi}_n \Sigma^{\dim V} H_N \underline{L}(N/N).$$

This is a constant Mackey functor. On the other hand, by Equation (5.2) and Equation (5.3), we have

$$q_* \underline{\pi}_n X \cong \underline{\pi}_n (X^N) \cong \underline{\pi}_n (\Sigma^V H_{G/N} \underline{L}).$$

By Proposition 5.1.8, this agrees with $\Psi_N^* \underline{\pi}_n \left(\Sigma^V H_{G/N} \underline{L}\right)$. □

More generally, we have an extension of Proposition 5.1.9 to $H\underline{\mathbb{Z}}$-modules:

Proposition 5.1.14. *Let $X \in \mathrm{Mod}_{H\underline{\mathbb{Z}}_{G/N}}$ and let $N \trianglelefteq G$ be a bottleneck subgroup. If the underlying spectrum $\downarrow_e^{G/N} X$ is contractible, then $\Psi_N^*(X) \simeq \phi_N^* X$.*

Proof. If the underlying spectrum of X is contractible, then $X \simeq \widetilde{E(G/N)} \wedge X$. The assumption that N is a bottleneck subgroup implies that $E(G/N) = q^*(E(G/N))$ is the universal space for the family of subgroups of N, so that $\widetilde{E(G/N)} \wedge \widetilde{E\mathcal{F}[N]} \simeq \widetilde{E(G/N)}$ and it follows that

$$q^* X \simeq \widetilde{E(G/N)} \wedge q^* X \simeq \widetilde{E(G/N)} \wedge \phi_N^*(X) \simeq \phi_N^* X.$$

Now

$$\begin{aligned} \Psi_N^*(X) &= H_G \underline{\mathbb{Z}} \wedge_{q^* H_{G/N} \underline{\mathbb{Z}}} q^*(X) \\ &\simeq H_G \underline{\mathbb{Z}} \wedge_{q^* H_{G/N} \underline{\mathbb{Z}}} (\widetilde{E(G/N)} \wedge q^*(X)). \end{aligned}$$

Since $\widetilde{E(G/N)}$ is smash idempotent, this can be rewritten as

$$\Psi_N^*(X) \simeq \widetilde{E(G/N)} \wedge H_G \underline{\mathbb{Z}} \wedge_{\widetilde{E(G/N)} \wedge q^* H_{G/N} \underline{\mathbb{Z}}} \widetilde{E(G/N)} \wedge q^*(X).$$

It remains only to show that

$$\widetilde{E(G/N)} \wedge H_G \underline{\mathbb{Z}} \simeq \widetilde{E(G/N)} \wedge q^* H_{G/N} \underline{\mathbb{Z}}.$$

Both sides restrict trivially to an N-equivariant spectrum, so it suffices to show an equivalence on Φ^H, where H properly contains N. Without loss of generality, we may suppose $H = G$. Since $\Phi^G(\widetilde{E(G/N)}) \simeq S^0$, it suffices to show that

$$\Phi^G H_G \underline{\mathbb{Z}} \simeq \Phi^G q^* H_{G/N} \underline{\mathbb{Z}}.$$

According to Proposition 5.1.5, the left side is $\Phi^{G/N} H_{G/N}\underline{\mathbb{Z}}$. Similarly, Proposition 5.1.5 and the Projection Formula (Proposition 5.1.1) show that the right side is

$$\begin{aligned}\Phi^G q^* H_{G/N}\underline{\mathbb{Z}} &\simeq \Phi^{G/N}\left(H_{G/N}\underline{\mathbb{Z}} \wedge (S^0_G)^N\right)\\ &\simeq \Phi^{G/N} H_{G/N}\underline{\mathbb{Z}} \wedge \Phi^{G/N}(S^0_G)^N\\ &\simeq \Phi^{G/N} H_{G/N}\underline{\mathbb{Z}}.\end{aligned}$$

□

Theorem 5.1.15. *Let $n \geq 0$ and let $N \trianglelefteq G$ be a bottleneck subgroup of order p, a prime. Let $\underline{M} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$ such that $P^n_n \Sigma^n H_{G/N}\underline{M}$ is of the form $\Sigma^V H_{G/N}\underline{L}$, for some G/N-representation V and $\underline{L} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$. Then the nontrivial slices of the Eilenberg-Mac Lane G-spectrum $\Sigma^n H_G(\Psi^*_N \underline{M})$, above level pn, are*

$$P^{pk}_{pk}\left(\Sigma^n H_G(\Psi^*_N\underline{M})\right) \simeq \Psi^*_N P^k_k\left(\Sigma^n H_{G/N}\underline{M}\right) \simeq \phi^*_N P^k_k\left(\Sigma^n H_{G/N}\underline{M}\right)$$

for $k > n$. Furthermore,

$$P^{pk}_n\left(\Sigma^n H_G(\Psi^*_N\underline{M})\right) \simeq \Psi^*_N P^k_n\left(\Sigma^n H_{G/N}\underline{M}\right).$$

Proof. Applying the functor Ψ^*_N to the slice tower for $\Sigma^n H_{G/N}\underline{M}$ produces a tower of fibrations whose layers are $\Psi^*_N P^k_k\left(\Sigma^n H_{G/N}\underline{M}\right)$ for $k \geq n$. We wish to show that this is a partial slice tower for $\Sigma^n H_G(\Psi^*_N\underline{M})$. For $k > n$, the k-slice $P^k_k\left(\Sigma^n H_{G/N}\underline{M}\right)$ has trivial underlying spectrum. It follows from Proposition 5.1.14 that

$$\Psi^*_N P^k_k\left(\Sigma^n H_{G/N}\underline{M}\right) \simeq \phi^*_N P^k_k\left(\Sigma^n H_{G/N}\underline{M}\right)$$

for $k > n$. As the geometric inflation of a k-slice, this is a pk-slice.

It remains to show that

$$\Psi^*_N P^n_n\left(\Sigma^n H_{G/N}\underline{M}\right) \simeq \Psi^*_N \Sigma^V H_{G/N}\underline{L} \simeq \Sigma^V H_G \Psi^*_N \underline{L}$$

has no slices above level pn. First, note that the restriction of $\Sigma^V H_G \Psi^*_N \underline{L}$ to N is the N-spectrum $\Sigma^n H_N \underline{L(N)}$, where $\underline{L(N)}$ is being considered as a constant N-Mackey functor at the value $\underline{L}(G/N)$. It follows that this N-spectrum has no slices above dimension $|N| \cdot n = pn$. Therefore, to show that $\Sigma^V H_G \Psi^*_N \underline{L}$ is less than pn, it suffices to show that

$$[G_+ \wedge_H S^{k\rho_H + r}, \Sigma^V H_G \Psi^*_N \underline{L}]^G = 0$$

for any $N < H \leq G$ and integers $r \geq 0$ and k such that $k|H| > pn$. Without loss of generality we consider the case $H = G$.

Denote by U a complement of $\rho_{G/N}$ in ρ_G, so that

$$\rho_G \cong \rho_{G/N} \oplus U.$$

We then have a cofiber sequence

$$S(kU)_+ \wedge S^{k\rho_{G/N}} \longrightarrow S^{k\rho_{G/N}} \longrightarrow S^{k\rho_G}$$

and a resulting exact sequence

$$\begin{aligned}[\Sigma^1 S(kU)_+ \wedge S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G &\longrightarrow [S^{k\rho_G+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G \\ &\longrightarrow [S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G = 0.\end{aligned}$$

We must show that the left term vanishes. Note that the G-action on $S(kU)$ is free, since N is order p. Then the desired vanishing follows from the fact that $\Sigma^1 S(kU)_+ \wedge S^{k\rho_{G/N}-V}$ is G-connected, since $\dim k\rho_{G/N} > \dim V = n$. $\square$

5.2 Q_8-Mackey functors and Bredon homology

We display a number of the Q_8-Mackey functors that will be relevant in Table 5.1. In these Lewis diagrams, we are using the subgroup lattice of Q_8 as displayed at the start of Chapter 5. We will also often abuse notation and write the name for a K_4-Mackey functor, such as $\underline{m}$ or $\underline{mg}$, to denote the resulting inflated Q_8-Mackey functor. We will only write the symbol ϕ_Z^* when it is necessary to resolve an ambiguity, for instance between $\phi_Z^* \underline{\mathbb{F}_2}$ and $\underline{\mathbb{F}_2}$.

In [HHR3, Section 2.1], the authors introduce "forms of $\underline{\mathbb{Z}}$" Mackey functors $\underline{\mathbb{Z}}(i,j)$, where $i \geq j \geq 0$, in the case of $G = C_{p^n}$. From our point of view, Q_8 behaves very similarly to C_8, and we similarly write $\underline{\mathbb{Z}}(i,j)$ for the Mackey functor that looks like $\underline{\mathbb{Z}}^*$ between the subgroups of order 2^i and 2^j and looks like $\underline{\mathbb{Z}}$ outside of this range. We will at times follow [HHR3] in denoting by $\underline{B}(i,j)$ the cokernel of $\underline{\mathbb{Z}}(i,j) \hookrightarrow \underline{\mathbb{Z}}$, although we will often instead use the descriptions given in Proposition 5.2.1.

These Mackey functors fit together in exact sequences as follows:

Proposition 5.2.1. *There are exact sequences of Mackey functors*

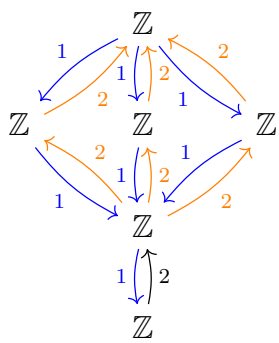
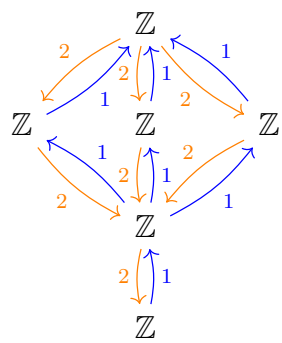
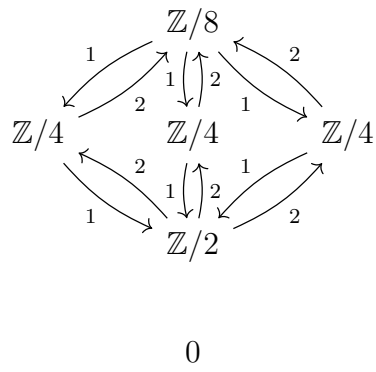
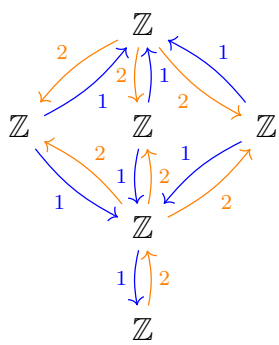
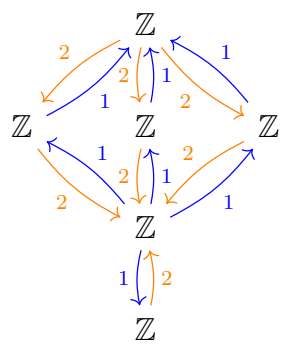
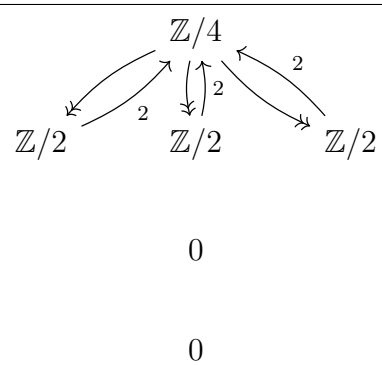
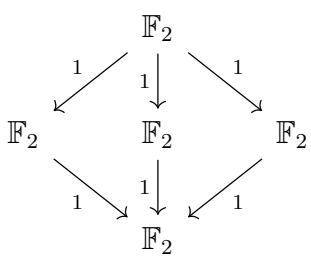
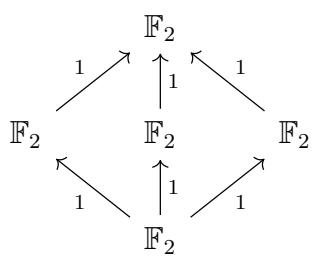
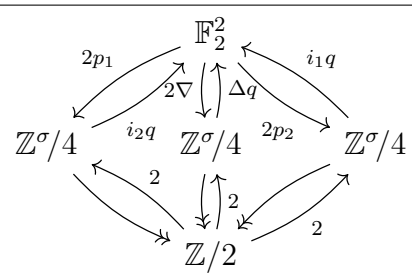
1. $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$
2. $\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{B}(2,0)$
3. $\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \twoheadrightarrow \underline{m}^*$
4. $\underline{\mathbb{Z}}(2,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{m}$
5. $\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{\mathbb{F}_2}$
6. $\underline{\mathbb{Z}}^* \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{B}(3,0)$
7. $\underline{mg} \hookrightarrow \underline{\mathbb{M}g} \twoheadrightarrow \underline{w}$.

5.2.1 $RO(Q_8)$-graded Mackey functor $\underline{\mathbb{Z}}$-homology of a point

We will now compute the homology of $S^{k\rho_Q}$, with coefficients in $\underline{\mathbb{Z}}$, as a Mackey functor. The starting point is that the regular representation of Q splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

Table 5.1: Some Q_8-Mackey functors

$\square = \underline{\mathbb{Z}}$	$\circledast = \underline{\mathbb{Z}}^*$	$\circ = \underline{B}(3,0)$
		
$\underline{\mathbb{Z}}(3,2) = \Psi_Z^* \underline{\mathbb{Z}}(2,1)$	$\underline{\mathbb{Z}}(3,1) = \Psi_Z^* \underline{\mathbb{Z}}^*$	$\circledcirc = \phi_Z^*(\underline{B}(2,0))$
		
$\blacklozenge = \phi_Z^* \underline{\mathbb{F}_2}$	$\circledast = \phi_Z^* \underline{\mathbb{F}_2}^*$	$\pentagon = \underline{\mathbb{M}g}$
		

where $\mathbb{H}$ is the 4-dimensional irreducible Q-representation given by the action of the unit quaternions on the algebra of quaternions and ρ_K is the regular representation of K, inflated to Q along the quotient. We begin by computing the homology of $S^{k\mathbb{H}}$. See also [L, Section 2] for an alternative viewpoint.

First, Proposition 5.1.1 and Proposition 4.5.1 combine to yield the following.

Proposition 5.2.2. *For $k \geq 0$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n\left(\Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 4k \\ \underline{mg} & n = 4k-2 \\ \underline{g}^{\frac{1}{2}(4k-n-1)} & n \in [2k, 4k-3], n \text{ odd} \\ \underline{g}^{\frac{1}{2}(4k-n-4)} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2} & n \in [2k, 4k-3], n \text{ even} \\ \underline{g}^{n-k+1} & n \in [k, 2k-1]. \end{cases}$$

Next, we employ the cofiber sequence

$$S(\mathbb{H})_+ \longrightarrow S^0 \longrightarrow S^{\mathbb{H}} \tag{5.5}$$

to obtain the homology of S^{ρ_Q} from that of S^{ρ_K}.

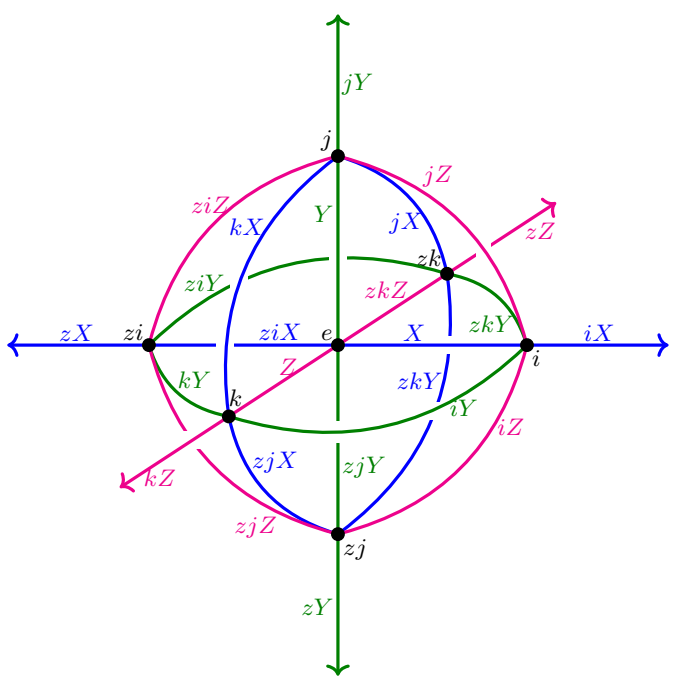


Figure 5.1: The 1-skeleton of $S(\mathbb{H})$.

Proposition 5.2.3. *The nontrivial homotopy Mackey functors of $S(\mathbb{H}) \wedge H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n\left(S(\mathbb{H})_+ \wedge H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 3 \\ \underline{\mathbb{M}g} & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

Proof. Since the action of Q on $S(\mathbb{H})$ is free, we can write down an equivariant cell structure using only free cells. Viewing $S(\mathbb{H})$ as the one-point compactification of $\mathbb{R}^3$, there is a straight-forward cell structure in which the subgroups L, D, and R act freely on the x, y, and z-axes, respectively. We display the 1-skeleton in Figure 5.1, and the cell structure is described by the following complex of $\mathbb{Z}[Q]$-modules:

$$\mathbb{Z}[Q]^2 \xrightarrow{\begin{pmatrix} e & j \\ -e & -i \\ e & k \\ -e & -e \end{pmatrix}} \mathbb{Z}[Q]^4 \xrightarrow{\begin{pmatrix} k & e & e & k \\ -e & -e & i & i \\ e & -j & -e & j \end{pmatrix}} \mathbb{Z}[Q]^3 \xrightarrow{(i-e \;\; j-e \;\; k-e)} \mathbb{Z}[Q].$$

This yields an associated complex of induced Mackey functors

$$\underline{\mathbb{Z}[Q]}^2 \longrightarrow \underline{\mathbb{Z}[Q]}^4 \longrightarrow \underline{\mathbb{Z}[Q]}^3 \longrightarrow \underline{\mathbb{Z}[Q]}$$

leading to the claimed homology Mackey functors. □

Remark 5.2.4. *A smaller chain complex for computing the homology of $S(\mathbb{H})$ is given by*

$$\mathbb{Z}[Q] \xrightarrow{\begin{pmatrix} i-e \\ e-k \end{pmatrix}} \mathbb{Z}[Q]^2 \xrightarrow{\begin{pmatrix} e+i & e+k \\ -e-j & -e+i \end{pmatrix}} \mathbb{Z}[Q]^2 \xrightarrow{(i-e \;\; j-e)} \mathbb{Z}[Q].$$

We gave a less efficient chain complex in the proof of Proposition 5.2.3 for geometric reasons.

Using Equation (5.5), this immediately yields the following.

Corollary 5.2.5. *The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n\left(\Sigma^{\mathbb{H}} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 4 \\ \underline{\mathbb{M}g} & n = 2 \\ \underline{B}(3,0) & n = 0. \end{cases}$$

We will use this to compute the homology of S^{ρ_Q}, using the following periodicity result.

Proposition 5.2.6 ([W, Proposition 4.1])**.** *For any orientable representation V of dimension d and free Q-space X, the orientation $u_V \in \mathrm{H}_d(S^V; \underline{\mathbb{Z}})$ induces an equivalence*

$$\Sigma^d X_+ \wedge H_Q\underline{\mathbb{Z}} \simeq \Sigma^V X_+ \wedge H_Q\underline{\mathbb{Z}}$$

We now compute the homology of S^{ρ_Q}.

Proposition 5.2.7. *The nontrivial homotopy Mackey functors of* $\Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}$ *are*

$$\underline{\pi}_n\left(\Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 8 \\ \underline{\mathbb{M}g} & n = 6 \\ \underline{B}(3,0) & n = 4 \\ \underline{mg} & n = 2 \\ \underline{g} & n = 1. \end{cases}$$

Proof. The representation ρ_K is orientable. For example, using the basis $\{1, i, j, k\}$ for $\rho_K = \mathbb{R}[K]$, the matrix $\rho_K(i)$ is given by

$$\rho_K(i) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has determinant equal to 1. By Proposition 5.2.6, we have

$$\underline{\pi}_n\left(S(\mathbb{H})_+ \wedge \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 7 \\ \underline{\mathbb{M}g} & n = 5 \\ \underline{\mathbb{Z}}^* & n = 4. \end{cases}$$

The result then follows from the cofiber sequence

$$S(\mathbb{H})_+ \wedge \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}.$$

□

Corollary 5.2.5 generalizes as follows.

Proposition 5.2.8. *The nontrivial homotopy Mackey functors of* $\Sigma^{k\mathbb{H}} H_Q\underline{\mathbb{Z}}$*, for* $k > 0$ *are*

$$\underline{\pi}_n\left(\Sigma^{k\mathbb{H}} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 4k \\ \underline{\mathbb{M}g} & 0 < n < 4k, n \equiv 2 \pmod 4 \\ \underline{B}(3,0) & 0 \leq n < 4k, n \equiv 0 \pmod 4. \end{cases}$$

Proof. This follows by induction, using the cofiber sequence

$$S(\mathbb{H})_+ \wedge S^{(k-1)\mathbb{H}} \longrightarrow S^{(k-1)\mathbb{H}} \longrightarrow S^{k\mathbb{H}}$$

and Proposition 5.2.6. The latter applies since $\mathbb{H}$, and therefore also $(k-1)\mathbb{H}$, is orientable. □

Combining this with the cofiber sequence

$$S(k\mathbb{H})_+ \wedge \Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}$$

and Proposition 5.2.6 gives the following result.

Proposition 5.2.9. *The nontrivial homotopy Mackey functors of $\Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}$, for $k > 0$, are*

$$\underline{\pi}_n\left(\Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 8k \\ \underline{\mathbb{M}g} & 4k < n < 8k, n \equiv 2 \pmod 4 \\ \underline{B}(3,0) & 4k \leq n < 8k, n \equiv 0 \pmod 4 \\ \phi_Z^* \underline{\pi}_n\left(\Sigma^{k\rho_K} H_K\underline{\mathbb{Z}}\right) & n < 4k, \end{cases}$$

where the latter Mackey functors are listed in Proposition 5.2.2.

The homotopy Mackey functors of $\Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}$ are displayed in Figure 5.2. When k is negative, the computation follows the same strategy. The initial input, which can again be computed using the chain complex given in Proposition 5.2.3, is that

$$\underline{\mathrm{H}}^n(S(\mathbb{H}); \underline{\mathbb{Z}}) \cong \underline{\pi}_{-n}\left(F\left(S(\mathbb{H})_+, H_Q\underline{\mathbb{Z}}\right)\right) \cong \begin{cases} \underline{\mathbb{Z}}^* & n = 3 \\ \underline{\mathbb{M}g} & n = 2 \\ \underline{\mathbb{Z}} & n = 0. \end{cases} \tag{5.6}$$

Using this and Proposition 4.5.2 leads to the following answer.

Proposition 5.2.10. *The nontrivial homotopy Mackey functors of $\Sigma^{-k\rho_Q} H_Q\underline{\mathbb{Z}}$, for $k > 0$, are*

$$\underline{\pi}_{-n}\left(\Sigma^{-k\rho_Q} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}}^* & n = 8k \\ \underline{\mathbb{M}g} & n \in [4k, 8k], n \equiv 3 \pmod 4 \\ \underline{B}(3,0) & n \in [4k+5, 8k], n \equiv 1 \pmod 4 \\ \phi_Z^* \underline{B}(2,0) & n = 4k+1 \\ \underline{mg}^* & n = 4k-1 \\ \underline{g}^{\frac{4k-n}{2}} & n \in [2k+4, 4k-2], n \equiv 0 \pmod 2 \\ \underline{g}^{\frac{4k-n-3}{2}} \oplus \phi_{LDR}^* \underline{\mathbb{F}_2}^* & n \in [2k+3, 4k-2], n \equiv 1 \pmod 2 \\ \underline{g}^{n-k-3} & n \in [k+4, 2k+2]. \end{cases}$$

Remark 5.2.11. *The "Gap Theorem"* [HHR1, Proposition 3.20] *predicts that the groups $\pi_n^Q \Sigma^{-k\rho} H\underline{\mathbb{Z}}$ vanish for $k \geq 0$ and $n \in [-3, -1]$, as indicated in Figure 5.2. Actually, for $k \geq 2$ the argument there proves more. It tells us that for $k \geq 2$, the cohomology groups $\mathrm{H}_Q^n(S^{k\rho}; \underline{M})$ vanish for positive $n \leq k+1$. This is equivalent to saying that $\pi_{-n}^Q \Sigma^{-k\rho} H\underline{M}$ vanishes, with the same conditions on k and n.*

5.2.2 Additional homology calculations

We will also need the following auxiliary calculations in Section 5.4.

Proposition 5.2.12. *The nontrivial homotopy Mackey functors of $\Sigma^{\rho_K - \mathbb{H}} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n\left(\Sigma^{\rho_K - \mathbb{H}} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

Proof. The fiber sequence

$$\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K}H_Q\underline{\mathbb{Z}} \longrightarrow F(S(\mathbb{H})_+, \Sigma^{\rho_K}H_Q\underline{\mathbb{Z}}) \simeq \Sigma^4 F(S(\mathbb{H})_+, H_Q\underline{\mathbb{Z}})$$

yields an isomorphism $\underline{\pi}_0\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}\right) \cong \underline{\mathbb{Z}}^*$ and shows that the homotopy vanishes for n outside of $[0,2]$. Given that the restriction to any C_4, which is the C_4-spectrum $\Sigma^{2+2\sigma-2\lambda}H_{C_4}\underline{\mathbb{Z}}$, has a trivial $\underline{\pi}_2$ [Z1, Theorem 6.10], the long exact sequence further shows that $\underline{\pi}_2$ vanishes as well, and it implies that we have an extension

$$\underline{w} \hookrightarrow \underline{\pi}_1\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}\right) \twoheadrightarrow \underline{g}.$$

It remains to show this is not the split extension. The fiber sequence

$$\uparrow_D^Q \Sigma^{1+2\sigma-2\lambda}H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{1+p_1^*\sigma+p_2^*\sigma-\mathbb{H}}H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}$$

shows that $\underline{\pi}_1\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}\right)$ injects into

$$\underline{\pi}_0\left(\uparrow_D^Q \Sigma^{1+2\sigma-2\lambda}H_{C_4}\underline{\mathbb{Z}}\right) \cong \uparrow_D^Q \phi_{C_2}^*\underline{\mathbb{F}_2}.$$

It follows that $\underline{\pi}_1\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}\right) \cong \phi_Z^*\underline{\mathbb{F}_2}$ □

Proposition 5.2.13. *The nontrivial homotopy Mackey functors of* $\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}(3,2)$ *are*

$$\underline{\pi}_n\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}(3,2)\right) \cong \begin{cases} \underline{w} & n=1 \\ \underline{\mathbb{Z}}^* & n=0. \end{cases}$$

Proof. The short exact sequence

$$\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$$

gives rise to a cofiber sequence

$$\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}(3,2) \longrightarrow \Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K-\mathbb{H}}H_Q\underline{g} \simeq \Sigma^1 H_Q\underline{g}.$$

Using a naturality square, the second map factors as

$$\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K}H_Q\underline{\mathbb{Z}} \longrightarrow \Sigma^1 H_Q\underline{g},$$

where the first map is an epimorphism on $\underline{\pi}_1$ by the proof of Proposition 5.2.12 and the second is an isomorphism on $\underline{\pi}_1$. The conclusion follows. □

Proposition 5.2.14. *The nontrivial homotopy Mackey functors of* $\Sigma^{\mathbb{H}-\rho_K}H_Q\underline{\mathbb{Z}}(2,0)$ *are*

$$\underline{\pi}_n\left(\Sigma^{\mathbb{H}-\rho_K}H_Q\underline{\mathbb{Z}}(2,0)\right) \cong \begin{cases} \underline{\mathbb{Z}} & n=0 \\ \underline{w}^* & n=-2. \end{cases}$$

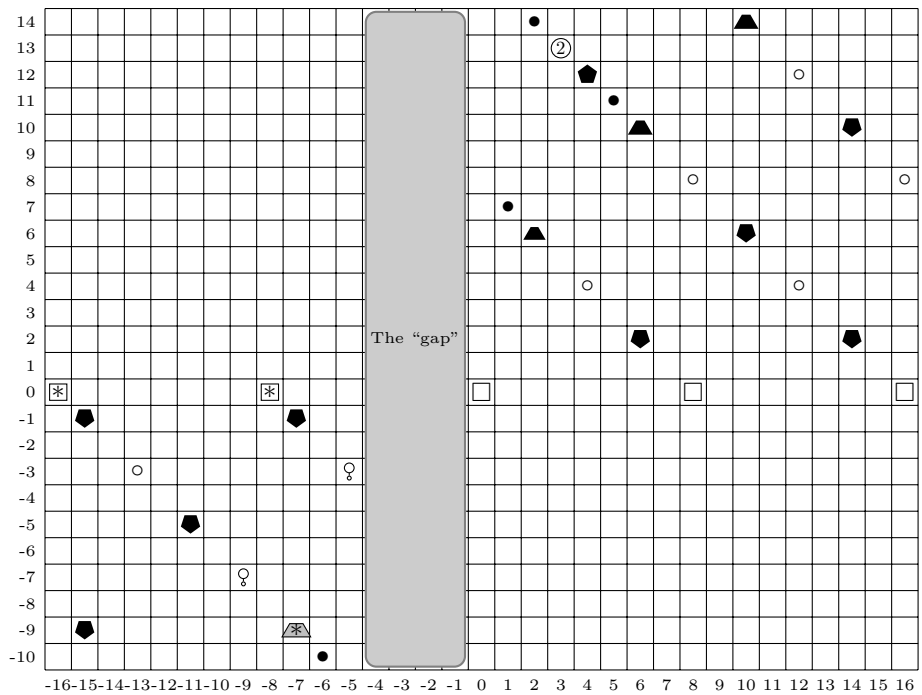


Figure 5.2: The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_Q\underline{\mathbb{Z}}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_Q\underline{\mathbb{Z}}$ appears in position $(k, 8n-k)$.

Proof. This follows from Proposition 5.2.13 by duality. In more detail, Proposition 5.2.13 gives a fiber sequence

$$\Sigma^1 H_Q\underline{w} \longrightarrow \Sigma^{\rho_K - \mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2) \longrightarrow H_Q\underline{\mathbb{Z}}^*.$$

Applying Anderson duality (see Section 2.4) gives a fiber sequence

$$I(\Sigma^1 H_Q\underline{w}) \longleftarrow I\left(\Sigma^{\rho_K - \mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2)\right) \longleftarrow I(H_Q\underline{\mathbb{Z}}^*),$$

or in other words

$$\Sigma^{-1} I(H_Q\underline{w}) \longleftarrow \Sigma^{\mathbb{H} - \rho_K} H_Q\underline{\mathbb{Z}}(2,0) \longleftarrow H_Q\underline{\mathbb{Z}}.$$

But as the Mackey functor $\underline{w}$ is torsion, the Anderson dual is the desuspension of the Brown-Comenetz dual. In other words, $I(H_Q\underline{w}) \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} H_Q\underline{w} \simeq \Sigma^{-1} H_Q\underline{w}^*$. $\square$

5.3 Review of the C_4-slices of $\Sigma^n H\mathbb{Z}$

In this section, we review the slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ from [Y1]. Note that the slices as listed in [Y1] are written using the classical slice filtration, whereas we use the regular slice filtration. The only difference is a suspension by one. The Mackey functors that appear here were introduced in Table 2.2.

According to [Y1, Section 4.2], the C_4-spectrum $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is an n-slice for $0 \leq n \leq 4$. For $n \geq 5$, $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ has a nontrivial slice tower. Yarnall's method for determining these slice towers is to splice together suspensions of the cofiber sequences

$$\Sigma^{-1} H_{C_4}\underline{g} \longrightarrow \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{2\sigma} H_{C_4}\underline{\mathbb{Z}},$$

$$\Sigma^{-1} H_{C_4}\phi^*_{C_2}\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{\lambda} H_{C_4}\underline{\mathbb{Z}}(2,1),$$

and

$$\Sigma^{-1} H_{C_4}\underline{B}(2,0) \longrightarrow \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{\lambda} H_{C_4}\underline{\mathbb{Z}}$$

in combination with the equivalences

$$\Sigma^2 H_{C_4}\underline{\mathbb{Z}} \simeq \Sigma^{2\sigma} H_{C_4}\underline{\mathbb{Z}}(2,1)$$

and

$$\Sigma^{-1} H_{C_4}\phi^*_{C_2}\underline{\mathbb{F}_2}^* \simeq \Sigma^{-\sigma} H_{C_4}\phi^*_{C_2}\underline{f} \simeq \Sigma^{1-2\sigma} H_{C_4}\phi^*_{C_2}\underline{\mathbb{F}_2}.$$

We first review these slices for odd n.

Proposition 5.3.1. [Y1, Theorem 4.2.6] *Let $n \geq 5$ be odd. The bottom slice of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is*

$$P^n_n(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{n-5}{4}\rho+4+\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 1 \pmod 8 \\ \Sigma^{\frac{n-3}{4}\rho+3} H_{C_4}\underline{\mathbb{Z}} & n \equiv 3 \pmod 8 \\ \Sigma^{\frac{n-5}{4}\rho+3+2\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 5 \pmod 8 \\ \Sigma^{\frac{n-3}{4}\rho+2+\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 7 \pmod 8. \end{cases}$$

Proposition 5.3.2. [Y1, Lemma 4.2.5] *Let $n \geq 5$ be odd. The nontrivial $4k$-slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ are*

$$P^{4k}_{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{k\rho} H_{C_4}\underline{B}(2,0) & 4k \in [n+1, 2(n-3)],\ k \text{ even} \\ \Sigma^{k\rho} H_{C_4}\phi^*\underline{f} & 4k \in [n+1, 2(n-3)],\ k \text{ odd} \\ \Sigma^{k\rho} H_{C_4}\underline{g} & 4k \in [2(n-1), 4(n-3)],\ k \text{ even.} \end{cases}$$

The $4k$-slices can also be read off of [HHR2, Figure 3]. When n is odd, these are the only nontrivial slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$.

We now recall the slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ for even n.

Proposition 5.3.3. [Y1, Theorem 4.2.9] *Let $n \geq 6$ be even. The bottom slice of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is*

$$P^n_n(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{n-4}{4}\rho+3+\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 0 \pmod 8 \\ \Sigma^{\frac{n-6}{4}\rho+3+3\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 2 \pmod 8 \\ \Sigma^{\frac{n-4}{4}\rho+4} H_{C_4}\underline{\mathbb{Z}} & n \equiv 4 \pmod 8 \\ \Sigma^{\frac{n-6}{4}\rho+4+2\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 6 \pmod 8. \end{cases}$$

Proposition 5.3.4. [Y1, Lemma 4.2.7] *Let $n \geq 6$ be even. The nontrivial $4k$-slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ are*

$$P^{4k}_{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^k H_{C_4}\underline{g}, \quad k \text{ odd}$$

for $4k$ in the range $[n+2, 4n-12]$.

Again, the $4k$-slices can also be read off of [HHR2, Figure 3].

Proposition 5.3.5. [Y1, Theorem 4.2.9] *Let $n \geq 6$ be even. The $(4k+2)$-slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ are*

$$P^{8k+2}_{8k+2}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^{1+2k\rho} H\phi^*\underline{\mathbb{F}_2}$$
$$P^{8k+6}_{8k+6}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^{3+2k\rho} H\phi^*\underline{\mathbb{F}_2}.$$

for $8k+2$ or $8k+6$ in the range $[n+2, 2n-6]$

We may also view these slices through the perspective of the $\underline{\mathbb{Z}}$-module inflation functor. By Theorem 5.1.15,

$$\Psi^*_{C_2} : \mathrm{Mod}_{H_{C_2}\underline{\mathbb{Z}}} \longrightarrow \mathrm{Mod}_{H_{C_4}\underline{\mathbb{Z}}}$$

will provide all slices of $\Sigma^n H_{C_4}$ above level $2n$. Let $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. It follows from Proposition 3.2.35 that the slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ in level at least $2n+2r-4$ are

$$P^{4k}_{4k}\left(\Sigma^n H_{C_4}\underline{\mathbb{Z}}\right) \simeq \Psi^*_{C_2}\Sigma^k H_{C_2}\underline{g} \simeq \Sigma^k H_{C_4}\underline{g}$$

for $4k \in [2n+2r-4, 4(n-3)]$. The rest of the slices then follow from determining the slices of

$$\Psi^*_{C_2}\Sigma^{\frac{n-r}{2}\rho_{C_2}+r} H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{\frac{n+r}{2}+\frac{n-r}{2}\sigma} H_{C_4}\underline{\mathbb{Z}}.$$

The slice tower for this C_4-spectrum can be found by splicing together the cofiber sequences listed at the start of this section.

5.4 Q_8-slices

The slices of $\Sigma^n H_K\underline{\mathbb{Z}}$ were determined by the second author in Section 4.4. As stated in Theorem 5.1.15, it follows that the $\underline{\mathbb{Z}}$-module inflation functor

$$\Psi^*_Z\colon \mathrm{Mod}_{H_K\underline{\mathbb{Z}}} \longrightarrow \mathrm{Mod}_{H_Q\underline{\mathbb{Z}}}$$

of Proposition 5.1.11 will produce all slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ in degree larger than $2n$, as the inflation of the slices of $\Sigma^n H_K\underline{\mathbb{Z}}$ above degree n.

The remaining slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ will be given as the slices of $\Psi^*_Z\left(P^n_n(\Sigma^n H_K\underline{\mathbb{Z}})\right)$. By Proposition 4.4.5, these are of the form

$$\Psi^*_Z\Big(\Sigma^{r+j\rho_K} H_K\underline{\mathbb{Z}}\Big) \simeq \Sigma^{r+j\rho_K} H_Q\underline{\mathbb{Z}},$$

where $r \in \{3,4,5\}$, if $n \not\equiv 2 \pmod 4$. In the case $n \equiv 2 \pmod 4$, the same result states that this is

$$\Psi_Z^*\Big(\Sigma^{2+j\rho_K} H_K\underline{\mathbb{Z}}(1,0)\Big) \simeq \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1).$$

But the cofiber sequence (Proposition 5.2.1)

$$\Sigma^{1+j\rho_K} H_Q\underline{m} \longrightarrow \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}} \tag{5.7}$$

reduces the computation of slices of $\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1)$ to the question of the slice tower for $\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}$, given that $\Sigma^{1+j\rho_K} H_Q\underline{m} \simeq \phi_Z^*(\Sigma^{1+j\rho_K} H_K\underline{m})$ is an $8j+4$-slice Proposition 4.1.7. We determine the slices of $\Sigma^{r+j\rho_K} H_Q\underline{\mathbb{Z}}$, for $r \in \{2,\dots,5\}$ in Section 5.4.1.

5.4.1 Slice towers for $\Sigma^{r+j\rho_K} H_Q\underline{\mathbb{Z}}$

The K_4-spectrum $\Sigma^{r+j\rho_K} H_K\underline{\mathbb{Z}}$ is an n-slice for $r \in \{2,\dots,5\}$ Proposition 4.3.1. However, the inflation of this to Q_8 is no longer a slice. We here determine the slice towers of these inflations. Throughout, we will implicitly use Proposition 5.4.5, which does not rely on the following material.

$(r = 2)$

First, we observe that $\Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}$ is a 6-slice. To see this we first note that it restricts to a 6-slice at every proper subgroup by Proposition 5.3.3. It therefore remains only to show that it does not have any $8k$-slices for $k \geq 1$. This is equivalent to showing that $\underline{\pi}_{-2}\left(\Sigma^{\rho_K - k\rho_Q} H_Q\underline{\mathbb{Z}}\right)$ vanishes for $k \geq 1$. In the case $k = 1$, (5.6) shows that $\Sigma^{-\mathbb{H}} H_Q\underline{\mathbb{Z}}$ is (-3)-truncated, in the sense that it has no homotopy Mackey functors above dimension -3. This remains true after further desuspending by copies of ρ_Q.

Next, the tower for $\Sigma^{2+2\rho_K} H_Q\underline{\mathbb{Z}}$ is given by

$$\begin{tikzcd}
P_{14}^{14} = \Sigma^{-1+2\rho_Q} H_Q\underline{w}^* \arrow[r] & \Sigma^{2+2\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
P_{12}^{12} = \Sigma^{1+\rho_Q} H_Q\underline{m} \arrow[r] & \Sigma^{2+\rho_Q} H_Q\underline{\mathbb{Z}}(2,0) \arrow[d] \\
& P_{10}^{10} = \Sigma^{2+\rho_Q} H_Q\underline{\mathbb{Z}}(1,0).
\end{tikzcd}$$

This uses the computation (see Proposition 5.2.14)

$$\underline{\pi}_n\left(\Sigma^{\mathbb{H}-\rho_K} H_Q\underline{\mathbb{Z}}(2,0)\right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 0 \\ \underline{w}^* & n = -2 \end{cases}$$

to produce the first cofiber sequence.

Finally, for $j \geq 3$, the tower may be obtained by recursively using

$$\begin{tikzcd}
P^{8j-2}_{8j-2} = \Sigma^{-1+j\rho_Q} H_Q \underline{w}^* \arrow[r] & \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}} \arrow[d] \\
P^{8j-4}_{8j-4} = \Sigma^{1+(j-1)\rho_Q} H_Q \underline{m} \arrow[r] & \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}(2,0) \arrow[d] \\
P^{8j-6}_{8j-6} = \Sigma^{1+(j-1)\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2} \arrow[r] & \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}(1,0) \arrow[d] \\
& \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}.
\end{tikzcd}$$

We have proved the following result.

Proposition 5.4.1. *Let $j \geq 1$. The bottom slice of $\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}$ is*

$$P^{2+4j}_{2+4j}\left(\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+\rho_K+\frac{j-1}{2}\rho_Q} H_Q \underline{\mathbb{Z}}^* & j \text{ odd} \\ \Sigma^{2+\frac{j}{2}\rho_Q} H_Q \underline{\mathbb{Z}} & j \text{ even.} \end{cases}$$

$(r = 3)$

By (5.6), the cohomology of $S^{\mathbb{H}}$ is given by

$$\underline{\widetilde{\mathrm{H}}}^n(S^{\mathbb{H}}; \underline{\mathbb{Z}}) \cong \underline{\pi}_{-n}\left(\Sigma^{-\mathbb{H}} H_Q \underline{\mathbb{Z}}\right) \cong \begin{cases} \underline{\mathbb{Z}}^* & n = 4 \\ \underline{\mathbb{M}g} & n = 3. \end{cases}$$

Suspending by $3 + \rho_Q$ leads to the cofiber sequence

$$\begin{tikzcd}
P^8_8 = \Sigma^{\rho_Q} H_Q \underline{\mathbb{M}g} \arrow[r] & \Sigma^{3+\rho_K} H_Q \underline{\mathbb{Z}} \arrow[d] \\
& P^7_7 = \Sigma^{\rho_Q - 1} H_Q \underline{\mathbb{Z}}^*.
\end{tikzcd}$$

The tower for $\Sigma^{3+j\rho_K} H_Q \underline{\mathbb{Z}}$, where $j \geq 2$, is then given recursively by

$$\begin{tikzcd}
P^{8j}_{8j} = \Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g} \arrow[r] & \Sigma^{3+j\rho_K} H_Q \underline{\mathbb{Z}} \arrow[d] \\
& \Sigma^{(j-1)\rho_K+\rho_Q-1} H_Q \underline{\mathbb{Z}}^* \arrow[d, equal] \\
P^{8j-4}_{8j-4} = \Sigma^{2+(j-1)\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2} \arrow[r] & \Sigma^{3+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}(1,0) \arrow[d] \\
& \Sigma^{3+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}.
\end{tikzcd}$$

The last cofiber sequence arises from Proposition 5.2.1. We have proved the following result.

Proposition 5.4.2. *Let $j \geq 1$. The bottom slice of $\Sigma^{3+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P^{3+4j}_{3+4j}\left(\Sigma^{3+j\rho_K} H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{-1+\frac{j+1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & j \text{ odd} \\ \Sigma^{3+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & j \text{ even.} \end{cases}$$

$(r = 4)$

The tower for $\Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}}$ is given by

$$\begin{tikzcd}
P^{12}_{12} = \Sigma^{\rho_Q+1} H_Q\underline{mg} \arrow[r] & \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{10}_{10} = \Sigma^{\rho_Q+1}\underline{w} \arrow[r] & \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,2) \arrow[d] \\
 & P^8_8 = \Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}^*.
\end{tikzcd}$$

This uses the short exact sequence (Proposition 5.2.1)

$$\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \twoheadrightarrow \underline{m}^*,$$

the equivalence $\Sigma^{\rho_K} H_K\underline{m}^* \simeq \Sigma^2 H_K\underline{mg}$ ([GY, Proposition 4.8]), and the computation (see Proposition 5.2.13)

$$\underline{\pi}_n\left(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2)\right) \cong \begin{cases} \underline{w} & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

The tower for $\Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}}$, where $j \geq 2$, may then be obtained recursively from

$$\begin{tikzcd}
P^{8j+4}_{8j+4} = \Sigma^{1+j\rho_Q} H_Q\underline{mg} \arrow[r] & \Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{(j+1)\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{8j+2}_{8j+2} = \Sigma^{1+j\rho_Q}\underline{w} \arrow[r] & \Sigma^{(j+1)\rho_K} H_Q\underline{\mathbb{Z}}(3,2) \arrow[d] \\
 & \Sigma^{(j-1)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}^* \arrow[d, equal] \\
P^{8j-2}_{8j-2} = \Sigma^{3+(j-1)\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{4+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) \arrow[d] \\
 & \Sigma^{4+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}.
\end{tikzcd}$$

Proposition 5.4.3. *Let $j \geq 1$. The bottom slice of $\Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P^{4+4j}_{4+4j}\left(\Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{\frac{j+1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & j \text{ odd} \\ \Sigma^{4+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & j \text{ even.} \end{cases}$$

$(r = 5)$

Here, we start with the slice tower for $\Sigma^5 H_Q\underline{\mathbb{Z}}$, as this is not a slice. The short exact sequence

$$\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^*\underline{B}(2,0)$$

gives rise to a cofiber sequence

$$P^8_8 = \Sigma^{\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) \longrightarrow \Sigma^5 H_Q\underline{\mathbb{Z}} \simeq \Sigma^{1+\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \longrightarrow \Sigma^{1+\rho_K} H_Q\underline{\mathbb{Z}}.$$

Now the argument showing that $\Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}$ is a 6-slice, given above in Section 5.4.1, also applies to show that $\Sigma^{1+\rho_K} H_Q\underline{\mathbb{Z}}$ is a 5-slice. Thus, this cofiber sequence is the slice tower for $\Sigma^5 H_Q\underline{\mathbb{Z}}$.

Next, the tower for $\Sigma^{5+\rho_K} H_Q\underline{\mathbb{Z}}$ is given by

$$\begin{tikzcd}
P^{16}_{16} = \Sigma^{2\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) \arrow[r] & \Sigma^{5+\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{1+2\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{12}_{12} = \Sigma^{2+\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{1+2\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
& P^9_9 = \Sigma^{1+\rho_Q} H_Q\underline{\mathbb{Z}}^*,
\end{tikzcd}$$

where the bottom cofiber sequence arises from the computation (Proposition 5.2.12)

$$\underline{\pi}_n\left(\Sigma^{\rho_K - \mathbb{H}} H_Q\underline{\mathbb{Z}}\right) \cong \begin{cases} \phi_Z^*\underline{\mathbb{F}_2} & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

The tower for $\Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}}$, where $j \geq 2$, may then be obtained recursively from

$$\begin{tikzcd}
P^{8j+8}_{8j+8} = \Sigma^{(j+1)\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) \arrow[r] & \Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d, equal] \\
& \Sigma^{1+(j+1)\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{8j+4}_{8j+4} = \Sigma^{2+j\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{1+(j+1)\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
P^{8j}_{8j} = \Sigma^{j\rho_Q} H_Q\underline{B}(3,0) \arrow[r] & \Sigma^{1+(j-1)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}^* \arrow[d] \\
& \Sigma^{1+(j-1)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}.
\end{tikzcd}$$

Proposition 5.4.4. *Let $j \geq 1$. The bottom slice of $\Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P^{5+4j}_{5+4j}\left(\Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+\frac{j+1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & j \text{ odd} \\ \Sigma^{1+\rho_K+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & j \text{ even.} \end{cases}$$

5.4.2 Slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$

In this section, we describe all slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ for $n \geq 0$.

Proposition 5.4.5. *The Q_8-spectrum $\Sigma^n H_Q\underline{\mathbb{Z}}$ is an n-slice for $0 \leq n \leq 4$.*

Proof. Since this is true after restricting to any C_4 (see Section 5.3), any higher slices would necessarily be geometric and therefore occurring in slice dimension at least 8. But we can show directly that $\Sigma^n H_Q\underline{\mathbb{Z}} < 8$ if $n \in [0,4]$. This follows from the vanishing of $\pi_{\rho_Q}\Sigma^n H_Q\underline{\mathbb{Z}} \cong \pi_{-n}\Sigma^{-\rho_Q} H_Q\underline{\mathbb{Z}}$ as displayed in Figure 5.2. □

It remains to determine the slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ when $n \geq 5$. Note that Theorem 5.1.15 applies by Proposition 4.4.5. We first describe the bottom slice.

Proposition 5.4.6 (The n-slice)**.** *For $n \geq 5$, write $n = 8k + r$, where $r \in [5, 12]$. Then the n-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is*

$$P^n_n\left(\Sigma^n H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+\rho_K+k\rho_Q} H_Q\underline{\mathbb{Z}} & r = 5 \\ \Sigma^{2+\rho_K+k\rho_Q} H_Q\underline{\mathbb{Z}}(3,2) & r = 6 \\ \Sigma^{-1+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & r = 7 \\ \Sigma^{(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & r = 8 \\ \Sigma^{1+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & r = 9 \\ \Sigma^{2+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) & r = 10 \\ \Sigma^{3+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}} & r = 11 \\ \Sigma^{4+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}} & r = 12. \end{cases}$$

Proof. By Theorem 5.1.15, the n-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is the n-slice of the $\underline{\mathbb{Z}}$-module inflation of the n-slice of $\Sigma^n H_K\underline{\mathbb{Z}}$. By Proposition 4.4.5, writing $n = 4j + r_4$ with $r_4 \in \{2,3,4,5\}$, we have

$$\Psi^*_Z P^n_n\left(\Sigma^n H_{K_4}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1) & n \equiv 2 \pmod 4 \\ \Sigma^{r_4+j\rho_K} H_Q\underline{\mathbb{Z}} & \text{else.} \end{cases}$$

If $n \not\equiv 2 \pmod 4$, the slice tower was given in Section 5.4.1. For the case of $n \equiv 2$, since $\Sigma^{1+j\rho_K} H_Q\underline{m} \simeq \phi^*_Z(\Sigma^{1+j\rho_K} H_K\underline{m})$ is an $8j+4$-slice Proposition 4.1.7, the cofiber sequence (Proposition 5.2.1)

$$\Sigma^{1+j\rho_K} H_Q\underline{m} \longrightarrow \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}, \tag{5.8}$$

combines with the work of Section 5.4.1 to to show that

$$P^n_n\left(\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1)\right) \simeq P^n_n\left(\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}\right).$$

The latter is given in Proposition 5.4.1. □

Proposition 5.4.7 (The $8k$-slices)**.** *For $n \geq 5$ and $8k > n$, the $8k$-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is*

$$P_{8k}^{8k}\left(\Sigma^n H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^k H_Q \underline{g}^{n-k-3} & 8k \in [4n-8, 8n-32] \\ \Sigma^{k\rho_Q} H_Q \underline{g}^{\frac{4k-n}{2}} & 8k \in [2n+4, 4n-16] \\ & \quad \textit{and} \quad n \equiv 0 \pmod 2 \\ \Sigma^{k\rho_Q} H_Q \underline{g}^{\frac{4k-n-3}{2}} \oplus \phi^*_{LDR}\underline{\mathbb{F}_2}^* & 8k \in [2n+4, 4n-12] \\ & \quad \textit{and} \quad n \equiv 1 \pmod 2 \\ \Sigma^{k\rho_Q} H_Q \underline{mg}^* & 8k = 2n+2 \\ \Sigma^{k\rho_Q} H_Q \phi^*_Z \underline{B}(2,0) & 8k = 2n-2 \\ \Sigma^{k\rho_Q} H_Q \underline{B}(3,0) & 8k \in [n+3, 2n-10] \\ & \quad \textit{and} \quad n \equiv 1 \pmod 4 \\ \Sigma^{k\rho_Q} H_Q \underline{\mathbb{M}g} & 8k \in [n+1, 2n] \\ & \quad \textit{and} \quad n \equiv 3 \pmod 4. \end{cases}$$

Proof. This is a translation of Proposition 5.2.10. Alternatively, the slices above dimension $2n$ follow from Theorem 5.1.15 and Proposition 4.4.6. The slices in dimensions $2n$ and lower follow from the towers computed in Section 5.4.1. □

Proposition 5.4.8 (The $8k+4$-slices)**.** *For $n \geq 5$ and $8k+4 > n$, the $8k+4$-slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ are*

$$P_{8k+4}^{8k+4}\left(\Sigma^n H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{3+k\rho_Q} H_Q \phi^*_{LDR}\underline{\mathbb{F}_2} & 8k+4 \in [2n+4, 4n-12], \quad n \textit{ even} \\ \Sigma^{2+k\rho_Q} H_Q \phi^*_Z \underline{\mathbb{F}_2} & 8k+4 \in [n+1, 2n-4], \quad n \textit{ odd} \\ \Sigma^{1+k\rho_Q} H_Q \underline{m} & 8k+4 \in [n+2, 2n], \quad n \equiv 2 \pmod 4 \\ \Sigma^{1+k\rho_Q} H_Q \underline{mg} & 8k+4 \in [n+4, 2n-4], \quad n \equiv 0 \pmod 4 \end{cases}$$

Proof. The first case follows from Proposition 4.4.7. The remaining cases follow from Equation (5.8) and Section 5.4.1. □

Proposition 5.4.9 (The $4k+2$-slices)**.** *Let $n \geq 5$. If n is odd, then $\Sigma^n H_Q\underline{\mathbb{Z}}$ has no nontrivial $4k+2$-slices if $4k+2 > n$. If n is even and $8k+2 > n$, then the $8k+2$-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is nontrivial only if $8k+2 \in [n+1, 2n]$, in which case the slice is*

$$P_{8k+2}^{8k+2}\left(\Sigma^n H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+k\rho_Q} H_Q \underline{w} & n \equiv 0 \pmod 4 \\ \Sigma^{1+k\rho_Q} H_Q \phi^*_Z \underline{\mathbb{F}_2} & n \equiv 2 \pmod 4 \end{cases}$$

Similarly, if n is even and $8k-2 > n$, the $8k-2$-slice is nontrivial only if $8k-2 \in [n+1, 2n]$, in which case the slice is

$$P_{8k-2}^{8k-2}\left(\Sigma^n H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{-1+k\rho_Q} H_Q \phi^*_Z \underline{\mathbb{F}_2}^* & n \equiv 0 \pmod 4 \\ \Sigma^{-1+k\rho_Q} H_Q \underline{w}^* & n \equiv 2 \pmod 4. \end{cases}$$

Proof. According to Chapter 4, the K_4-spectrum $\Sigma^n H_K\underline{\mathbb{Z}}$ does not have any nontrivial slices in odd dimensions, except for the n-slice. By Theorem 5.1.15, this implies that $\Sigma^n H_Q\underline{\mathbb{Z}}$ does not have any $4k+2$-slices above dimension $2n$. The slices in dimensions below $2n$ are given by Section 5.4.1. □

5.4.3 Slice towers for $\Sigma^n H_Q\underline{\mathbb{Z}}$

By Proposition 5.4.5, $\Sigma^n H_Q\underline{\mathbb{Z}}$ is an n-slice for $n \in \{0, \ldots, 4\}$. The slice tower for $\Sigma^5 H_Q\underline{\mathbb{Z}}$ was given in Section 5.4.1. We now display a few more examples of slice towers.

Example 5.4.10. The slice tower for $\Sigma^6 H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P_{16}^{16} = \Sigma^2 H_Q\underline{g} \arrow[r] & \Sigma^6 H_Q\underline{\mathbb{Z}} \arrow[d] \\
P_{12}^{12} = \Sigma^{1+\rho} H_Q\underline{m} \arrow[r] & \Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \arrow[d] \\
& P_6^6 = \Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}.
\end{tikzcd}$$

This follows immediately from combining Example 4.4.2, Equation (5.8), and Section 5.4.1.

Example 5.4.11. The slice tower for $\Sigma^7 H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P_{24}^{24} = \Sigma^3 H_Q\underline{g} \arrow[r] & \Sigma^7 H_Q\underline{\mathbb{Z}} \arrow[d] \\
P_{16}^{16} = \Sigma^{2+\rho_Q} H_Q\underline{m} \arrow[r] & \Sigma^{3+\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \arrow[d] \\
P_8^8 = \Sigma^{\rho_Q} H_Q\underline{\mathbb{M}g} \arrow[r] & \Sigma^{3+\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
& P_7^7 = \Sigma^{\rho_Q - 1} H_{Q_8}\underline{\mathbb{Z}}^*.
\end{tikzcd}$$

This follows immediately from combining Example 4.4.3 and Section 5.4.1.

Example 5.4.12. The slices, but not the slice tower, for $\Sigma^8 H_K\underline{\mathbb{Z}}$ were determined in Section 4.4. Let us denote by F the fiber of the map $H_Q\underline{\mathbb{Z}} \longrightarrow H_Q\phi_{LDR}\underline{\mathbb{F}_2}$ induced by the map of Q_8-Mackey functors $\underline{\mathbb{Z}} \longrightarrow \phi_{LDR}\underline{\mathbb{F}_2}$ that is surjective at L, D, and R. Then the nontrivial homotopy Mackey functors of F are $\underline{\pi}_0(F) \simeq \underline{\mathbb{Z}}(2,1)$ and

$\underline{\pi}_{-1}(F) \cong \underline{g}^2$. The slice tower for $\Sigma^8 H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P_{32}^{32} = \Sigma^4 H_Q\underline{g} \arrow[r] & \Sigma^8 H_Q\underline{\mathbb{Z}} \simeq \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P_{24}^{24} = \Sigma^3 H_Q\underline{g}^2 \arrow[r] & \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \arrow[d] \\
P_{20}^{20} = \Sigma^{3+\rho_Q} H_Q\phi^*_{LDR}\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{4+\rho_K} F \arrow[d] \\
P_{12}^{12} = \Sigma^{1+\rho_Q} H_Q\underline{mg} \arrow[r] & \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P_{10}^{10} = \Sigma^{\rho_Q+1}\underline{w} \arrow[r] & \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,2) \arrow[d] \\
 & P_8^8 = \Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}^*,
\end{tikzcd}$$

where the bottom of the tower comes from Section 5.4.1.

5.5 Homology calculations

In Section 5.4, we described the slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$. In Section 5.6 below, we will give the corresponding slice spectral sequences. The E_2-pages of those spectral sequences are given by the homotopy Mackey functors of the slices. We describe those homotopy Mackey functors here.

5.5.1 The n-slice

We start with the n-slices in the order listed in Proposition 5.4.6. The homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}$ were calculated in Proposition 5.2.9. We use the same methods to determine the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}$.

Proposition 5.5.1. *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_i(\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j+4 \\ \underline{\mathbb{M}g} & \begin{array}{l} i \in [4j+4, 8j+3], \\ \quad i \equiv 2 \pmod 4 \end{array} \\ \underline{B}(3,0) & \begin{array}{l} i \in [4j+4, 8j+3], \\ \quad i \equiv 0 \pmod 4 \end{array} \\ \phi_Z^*\underline{\pi}_i(\Sigma^{(j+1)\rho_K} H_K\underline{\mathbb{Z}}) & i \in [j+1, 4j+3]. \end{cases}$$

See Proposition 5.2.2 or Chapter 6 for the homotopy Mackey functors of $\Sigma^{(j+1)\rho_K} H_K\underline{\mathbb{Z}}$.

We may now use Proposition 5.5.1 and the exact sequence $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$ to get the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)$.

Proposition 5.5.2. *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho_K + j\rho_Q} H_Q \underline{\mathbb{Z}}(3,2)$ are*

$$\underline{\pi}_i(\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j+4 \\ \underline{\mathbb{M}g} & i \in [4j+4, 8j+3], \\ & \quad i \equiv 2 \pmod 4 \\ \underline{B}(3,0) & i \in [4j+4, 8j+3], \\ & \quad i \equiv 0 \pmod 4 \\ \phi_Z^* \underline{\pi}_i(\Sigma^{(j+1)\rho_K} H_K \underline{\mathbb{Z}}) & i \in [j+2, 4j+3]. \end{cases}$$

The key point here is that the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)$ are the same as that of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}$, except that the $\underline{g}$ in degree $j+1$ has been removed.

In Proposition 5.2.10 we list the homotopy Mackey functors of $\Sigma^{-j\rho_Q} H_Q\underline{\mathbb{Z}}$. Anderson duality then provides us with the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}^*$.

Proposition 5.5.3. *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}^*$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}^*) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j \\ \underline{\mathbb{M}g} & i \in [4j+1, 8j-1], \\ & \quad i \equiv 2 \mod 4 \\ \underline{B}(3,0) & i \in [4j+1, 8j-1], \\ & \quad i \equiv 0 \mod 4 \\ \phi_Z^* \underline{B}(2,0) & i = 4j \\ \phi_Z^* \underline{\pi}_{i-4}(\Sigma^{(j-1)\rho_K} H_K \underline{\mathbb{Z}}) & i \in [j+3, 4j-1]. \end{cases}$$

Finally, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)$ follow from the exact sequence $\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{\mathbb{F}}_2$.

Proposition 5.5.4. *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j \\ \underline{\mathbb{M}g} & i \in [4j+1, 8j-2], \\ & \quad i \equiv 2 \pmod 4 \\ \underline{B}(3,0) & i \in [4j+1, 8j-2], \\ & \quad i \equiv 0 \pmod 4 \\ \phi_Z^* \underline{B}(2,0) & i = 4j \\ \phi_Z^* \underline{\pi}_i(\Sigma^{j\rho_K} H_K \underline{\mathbb{Z}}) & i \in [j, 4j-1]. \end{cases}$$

5.5.2 The $8k$-slices

We now move on to the $8k$-slices.

Proposition 5.5.5. *For $j = 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0)$ are*

$$\underline{\pi}_i(\Sigma^{\rho_Q} H_Q \phi_Z^* \underline{B}(2,0)) \cong \begin{cases} \underline{mg} & i = 2 \\ \underline{g} & i = 1. \end{cases}$$

For $j \geq 2$, they are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0)) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j+2, 2j-1] \\ \underline{g}^2 & i = j+1 \\ \underline{g} & i = j. \end{cases}$$

Proof. Because $\phi_Z^* \underline{B}(2,0)$ is a pullback,

$$\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0) \simeq \Sigma^{j\rho_K} H_Q \phi_Z^* \underline{B}(2,0).$$

The exact sequence of K-Mackey functors $\underline{m}^* \longrightarrow \underline{B}(2,0) \longrightarrow \underline{g}$ provides us with $\Sigma^{j\rho_K} H_K \underline{m}^* \longrightarrow \Sigma^{j\rho_K} H_K \underline{B}(2,0) \longrightarrow \Sigma^{j\rho_K} H_K \underline{g}$. The conclusion follows from [GY, Propositions 4.8 and 7.4] and the resulting long exact sequence in homotopy. □

We may again use this strategy of reducing the calculations from Q to K for determining the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$.

Proposition 5.5.6. *For $j = 1$ the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$ are*

$$\underline{\pi}_i(\Sigma^{\rho_K} H_K \underline{B}(3,0)) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4 \\ \underline{mg} & i = 2 \\ \underline{g} & i = 1. \end{cases}$$

For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j-1 \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus g^{4j-2-i} & i \in [2j+2, 4j-2] \\ \underline{g}^{2(k-2)+1} & i = 2j+1 \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{2(j-3)+1} & i = 2j \\ \underline{g}^{2(i-j-1)} & i \in [j+3, 2j-1] \\ \underline{g}^{i-j+1} & i \in [j, j+2]. \end{cases}$$

Proof. Because the underlying spectrum of $H_Q \underline{B}(3,0)$ is contractible,

$$\Sigma^{\rho_Q} H_Q \underline{B}(3,0) \simeq \Sigma^{\rho_K} H_Q \underline{B}(3,0).$$

Now, we may consider $\underline{B}(3,0)$ as a pullback $\phi_Z^* B := \underline{B}(3,0)$, thus the calculation is reduced to one of K-Mackey functors. The sequence of K-Mackey functors $\underline{\mathbb{Z}}^* \xrightarrow{2} \underline{\mathbb{Z}} \longrightarrow \underline{B}$ provides us with

$$\Sigma^{j\rho_K} H_K \underline{\mathbb{Z}}^* \longrightarrow \Sigma^{j\rho_K} H_K \underline{\mathbb{Z}} \longrightarrow \Sigma^{j\rho_K} H_K \underline{B}.$$

Except for $i = 4j-2$, the result follows from the associated long exact sequence in homotopy. In degree $4j-2$ we have an extension

$$\underline{mg} \longrightarrow \underline{\pi}_{4j-2}(\Sigma^{j\rho_K} H \underline{B}) \longrightarrow \underline{g}.$$

We need to show this is not the split extension. This follows from the exact sequence $\underline{B}(2,0) \longrightarrow \underline{B} \longrightarrow \underline{\mathbb{F}_2}$ of K-Mackey functors. □

Proposition 5.5.7. *For $j = 1$ and $j = 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}$ are*

$$\underline{\pi}_i(\Sigma^{\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4 \\ \phi_Z^* \underline{B}(2,0) & i = 2. \end{cases}$$

and

$$\underline{\pi}_i(\Sigma^{2\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 8 \\ \underline{mg} & i = 7 \\ \phi_{LDR} \underline{\mathbb{F}_2} & i = 6 \\ \underline{g} & i = 5 \\ \underline{mg} & i = 4 \\ \underline{g} & i = 3. \end{cases}$$

For $j \geq 3$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j - 1 \\ \phi_{LDR} \underline{\mathbb{F}_2} \oplus \underline{g}^{4j-i-2} & i \in [2j+2, 4j-2] \\ \underline{g}^{2j-3} & i = 2j+1 \\ \underline{g}^{2j-5} \oplus \phi_{LDR} \underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^{2(i-j)-2} & i \in [j+2, 2j-1] \\ \underline{g} & i = j+1 \end{cases}$$

Proof. We first deal with the case $j = 1$. The short exact sequence of Mackey functors

$$\underline{w}^* \hookrightarrow \underline{\mathbb{M}g} \twoheadrightarrow \underline{mg}^*$$

combines with Proposition 5.5.16 and Proposition 5.5.8 to show that the only non-trivial Mackey functors are $\phi_Z^* \underline{\mathbb{F}_2}$ in degree 4 and an extension of $\underline{m}$ by $\underline{g}$ in degree 2. It remains to see that this extension is $\phi_Z^* \underline{B}(2,0)$. According to Proposition 5.2.10, the Postnikov tower for $\Sigma^{-\rho_Q} H_Q \underline{\mathbb{Z}}$ is

$$\begin{tikzcd} \Sigma^{-5} H_Q \phi_Z^* \underline{B}(2,0) \arrow[r] & \Sigma^{-\rho_Q} H_Q \underline{\mathbb{Z}} \arrow[d] \\ \Sigma^{-7} H_Q \underline{\mathbb{M}g} \arrow[r] & X \arrow[d] \\ & \Sigma^{-8} H_Q \underline{\mathbb{Z}}^*. \end{tikzcd}$$

Desuspending this diagram once by ρ_Q gives a tower for computing the homotopy Mackey functors of $\Sigma^{-2\rho_Q} H_Q \underline{\mathbb{Z}}$. The homotopy Mackey functors for $\Sigma^{-8-\rho_Q} H_Q \underline{\mathbb{Z}}^*$ and $\Sigma^{-5\rho_Q} H_Q \Psi^* \underline{B}(2,0)$ follow, using Anderson duality, from Proposition 5.2.9 and Proposition 5.5.5. Long exact sequences in homotopy then imply that

$$\underline{\pi}_{-9}(\Sigma^{-7-\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \phi_Z^* \underline{B}(2,0).$$

Dualizing gives that $\underline{\pi}_2(\Sigma^{\rho_Q} H_Q \underline{\mathbb{M}g})$ is $\phi_Z^* \underline{B}(2,0)$.

We now have a fiber sequence

$$\Sigma^4 H_Q \phi_Z^* \underline{\mathbb{F}_2} \longrightarrow \Sigma^{\rho_Q} H_Q \underline{\mathbb{M}g} \longrightarrow \Sigma^2 H_Q \phi_Z^* \underline{B}(2,0). \tag{5.9}$$

Suspending this sequence by ρ_Q immediately gives the homotopy Mackey functors of $\Sigma^{2\rho_Q} H_Q \underline{\mathbb{M}g}$. The same is true in the case $j = 3$, except that we have an extension

$$\underline{g} \hookrightarrow \underline{\pi}_6 \Sigma^{3\rho_Q} H_Q \underline{\mathbb{M}g} \twoheadrightarrow \phi_{LDR} \underline{\mathbb{F}_2}.$$

We claim that, more generally, any extension of $\underline{\mathbb{Z}}$-modules

$$\underline{g}^m \hookrightarrow \underline{E} \twoheadrightarrow \phi_{LDR} \underline{\mathbb{F}_2}$$

is necessarily the split extension. To see this, first note that $\phi_{LDR}\underline{\mathbb{F}_2}$ is, by definition, the direct sum $\phi_L^* \underline{\mathbb{F}_2} \oplus \phi_D^* \underline{\mathbb{F}_2} \oplus \phi_R^* \underline{\mathbb{F}_2}$. It therefore suffices to show that the only $\underline{\mathbb{Z}}$-module extension of $\phi_L^* \underline{\mathbb{F}_2}$ by $\underline{g}^m$ is the split extension. Since any such extension will vanish at the subgroups D and R, the $\underline{\mathbb{Z}}$-module structure forces the value at Q to be 2-torsion and therefore equal to $\underline{\mathbb{F}_2}^{m+1}$. Since there is a nontrivial restriction to the subgroup L, the $\underline{\mathbb{Z}}$-module structure forces the transfer from L to vanish. Thus the extension must be the split extension.

The suspension by $(j-1)\rho_Q$ of Equation (5.9) gives the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}$ in degrees $2j+1$ and higher. Now we argue by induction that the Mackey functors for $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}$ are as claimed, for $j \geq 3$. For instance, since the bottom Mackey functor is

$$\underline{\pi}_j(\Sigma^{(j-1)\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \underline{g},$$

we see by decomposing $\Sigma^{(j-1)\rho_Q} H_Q \underline{\mathbb{M}g}$ using the Postnikov tower that

$$\underline{\pi}_{j+1}(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \underline{g}.$$

The values of the Mackey functors $\underline{\pi}_i$, for $i \leq 2j-2$, follow in a similar way. The values

$$\underline{\pi}_{2j-2}(\Sigma^{(j-1)\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \underline{g}^{2j-7} \oplus \phi_{LDR} \underline{\mathbb{F}_2},$$

and

$$\underline{\pi}_{2j-1}(\Sigma^{(j-1)\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \underline{g}^{2j-5}$$

give that

$$\underline{\pi}_{2j-1}(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{M}g}) \cong \underline{g}^{2j-4}$$

and that we have an extension of $\underline{\mathbb{Z}}$-modules

$$\underline{g}^{2j-5} \hookrightarrow \underline{\pi}_{2j}(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}) \twoheadrightarrow \phi_{LDR} \underline{\mathbb{F}_2}.$$

By the argument given above, this must be the split extension. □

The homotopy Mackey functors for the remaining $8k$-slices follow from Proposition 4.5.5 and Proposition 4.5.8.

Proposition 5.5.8 (Proposition 4.5.5, [GY, Proposition 4.8])**.** *We have the equivalence* $\Sigma^{\rho_Q} H_Q \underline{mg}^* \simeq \Sigma^2 H_Q \underline{m}$*. For* $j \geq 2$*, the homotopy Mackey functors of* $\Sigma^{j\rho_Q} H_Q \underline{mg}^*$ *are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{mg}^*) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j+2, 2j-1] \\ \underline{g} & i = j+1. \end{cases}$$

Proposition 5.5.9 (Proposition 4.5.8)**.** *We have equivalences*

$$\Sigma^{j\rho_Q} H\phi^*_{LDR}\underline{\mathbb{F}_2}^* \simeq \begin{cases} \Sigma^2 H\phi^*_{LDR}\underline{f} & j = 1 \\ \Sigma^4 H\phi^*_{LDR}\underline{\mathbb{F}_2} & j = 2. \end{cases}$$

Then for $j \geq 3$*, the nontrivial homotopy Mackey functors of* $\Sigma^{j\rho_Q} H\phi^*_{LDR}\underline{\mathbb{F}_2}^*$ *are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\phi^*_{LDR}\underline{\mathbb{F}_2}^*) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j+2, 2j-1]. \end{cases}$$

5.5.3 The $8k+4$-slices

Similarly, the homotopy Mackey functors of the $(8k+4)$-slices follow from Proposition 4.5.8 and [GY, Corollary 7.2, Propositions 7.3, 7.4].

Proposition 5.5.10 ([GY, Proposition 3.6])**.** *For* $j \geq 1$*, the homotopy Mackey functors of* $\Sigma^{j\rho_Q} H_Q\phi^*_{LDR}\underline{\mathbb{F}_2}$ *are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\phi^*_{LDR}\underline{\mathbb{F}_2}) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j, 2j-1]. \end{cases}$$

Proposition 5.5.11 ([GY, Corollary 7.2])**.** *For* $j \geq 1$*, the homotopy Mackey functors of* $\Sigma^{j\rho_Q} H_Q\phi^*_Z\underline{\mathbb{F}_2}$ *are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\phi^*_Z\underline{\mathbb{F}_2}) \cong \begin{cases} \phi^*_Z\underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j-1 \\ \phi^*_{LDR}\underline{\mathbb{F}_2} \oplus g^{4j-2-i} & i \in [2j, 4j-2] \\ \underline{g}^{2(i-j)+1} & i \in [j, 2j-1]. \end{cases}$$

Proposition 5.5.12 ([GY, Proposition 7.3])**.** *For* $j \geq 1$*, the homotopy Mackey functors of* $\Sigma^{j\rho_Q} H_Q\underline{m}$ *are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{m}) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j+1, 2j-1] \\ \underline{g} & i = j. \end{cases}$$

Proposition 5.5.13 ([GY, Proposition 7.4])**.** *For* $j \geq 1$*, the homotopy Mackey functors of* $\Sigma^{j\rho_Q} H_Q\underline{mg}$ *are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{mg}) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j+1, 2j-1]. \\ \underline{g}^2 & i = j. \end{cases}$$

5.5.4 The $4k+2$-slices

The homotopy Mackey functors of the $(4k+2)$-slice $\Sigma^{1+k\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}$ are given in Proposition 5.5.11. The homotopy Mackey functors of the remaining $(4k+2)$-slices are as follows.

Proposition 5.5.14 ([GY, Proposition 4.8, Corollary 7.2])**.** *We have the equivalence $\Sigma^{\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}^* \simeq \Sigma^4 H_Q \phi_Z^* \underline{\mathbb{F}_2}$. For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}^*$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j-1 \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus g^{4j-2-i} & i \in [2j+2, 4j-2] \\ \underline{g}^{2(i-j)-5} & i \in [j+3, 2j+1]. \end{cases}$$

Finally, we have the homotopy of $\Sigma^{j\rho_Q} H_Q \underline{w}$ and $\Sigma^{j\rho_Q} H_Q \underline{w}^*$.

Proposition 5.5.15. *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{w}$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j-1 \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus g^{4j-2-i} & i \in [2j, 4j-2] \\ \underline{g}^{2(i-j)+1} & i \in [j+1, 2j-1]. \end{cases}$$

Proof. The underlying spectrum of $\Sigma^{j\rho_Q} H_Q \underline{w}$ is contractible; thus,

$$\Sigma^{j\rho_Q} H_Q \underline{w} \simeq \Sigma^{j\rho_K} H_Q \underline{w}.$$

Then, because $\underline{w}$ is a pullback over Z, the calculation is essentially K-equivariant. Consider the short exact sequence of K-Mackey functors $\underline{w} \longrightarrow \underline{\mathbb{F}_2} \longrightarrow \underline{g}$ and the corresponding cofiber sequence $\Sigma^{j\rho_K} H_K \underline{w} \longrightarrow \Sigma^{j\rho_K} H_K \underline{\mathbb{F}_2} \longrightarrow \Sigma^{j\rho_K} H_K \underline{g}$. The statement follows immediately from the resulting long exact sequence in homotopy. □

Proposition 5.5.16. *For $j = 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{w}^*$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4 \\ \underline{g} & i = 2. \end{cases}$$

For $j \geq 2$, they are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j-1 \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus g^{4j-2-i} & i \in [2j+2, 4j-2] \\ \underline{g}^{2(i-j)-5} & i \in [j+3, 2j+1] \\ \underline{g} & i = j+1. \end{cases}$$

Proof. The proof is the same as that in Proposition 5.5.15, except that we start with the exact sequence of K-Mackey functors $\underline{g} \longrightarrow \underline{\mathbb{F}_2}^* \longrightarrow \underline{w}^*$. □

5.6 Slice spectral sequences

Here we include the slice spectral sequences for $\Sigma^n H_Q\underline{\mathbb{Z}}$ for several values of n between 5 and 15. In some cases, we use the restriction to the C_4-subgroups to determine some of the slice differentials.

The grading is the same as that in [HHR1, Section 4.4.2]. The Mackey functor $\underline{E}_2^{t-n,t}$ is $\underline{\pi}_n P_t^t(X)$. We also follow the Adams convention, where $\underline{\pi}_n P_t^t(X)$ has coordinates $(n, t-n)$ and the differential

$$d_r : \underline{E}_r^{s,t} \longrightarrow \underline{E}_r^{s+r,t+r-1}$$

points left one and up r.

The Q-Mackey functors that appear in these spectral sequences are listed in Table 5.2. We also display some companion C_4-slice spectral sequences, and the C_4-Mackey functors that appear are listed in Table 5.3.

Table 5.2: Symbols for Q-Mackey functors

$\square = \underline{\mathbb{Z}}$	$\blacklozenge = \phi_Z^*\underline{\mathbb{F}_2}$	$\blacksquare = \phi_{LDR}^*\underline{\mathbb{F}_2}$
$\blacksquare = \underline{\mathbb{M}g}$	$\circ = \underline{B}(3,0)$	$\circ = \phi_Z^*\underline{B}(2,0)$
$\blacktriangle = \underline{mg}$	$n = \underline{g}^n$	

Table 5.3: Symbols for C_4-Mackey functors

$\square = \underline{\mathbb{Z}}$	$\blacklozenge = \phi_{C_2}^*\underline{\mathbb{F}_2}$	$\circ = \underline{B}(2,0)$	$\bullet = \underline{g}$

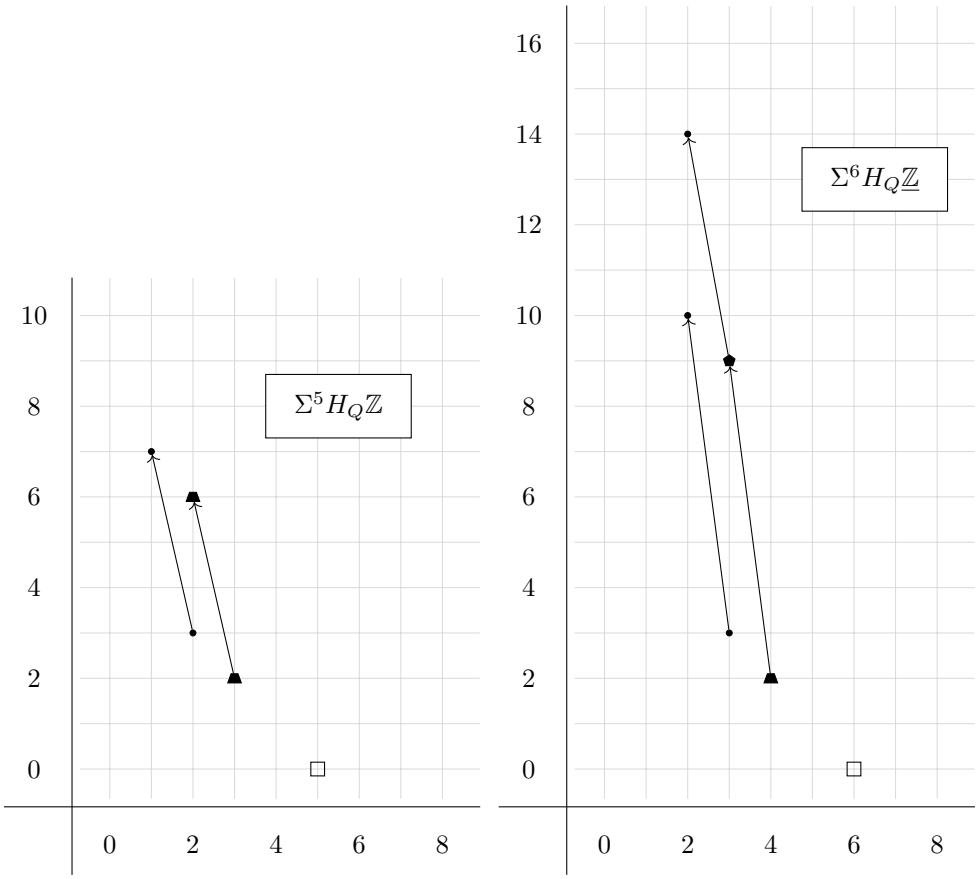
Example 5.6.1. In the spectral sequences for $\Sigma^5 H_Q\underline{\mathbb{Z}}$, $\Sigma^6 H_Q\underline{\mathbb{Z}}$, and $\Sigma^7 H_Q\underline{\mathbb{Z}}$, because we must be left with

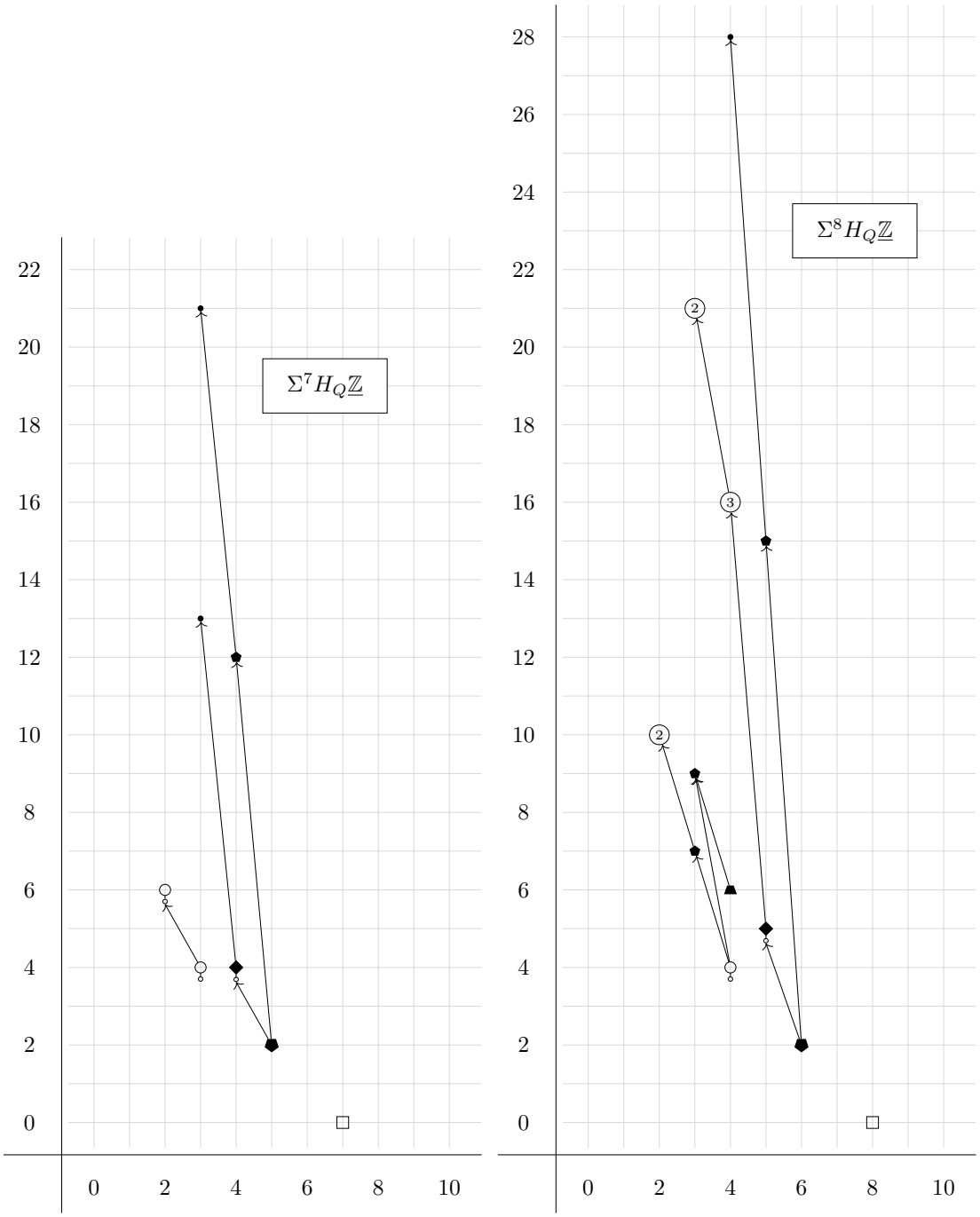
$$\underline{\pi}_n(P_n^n \Sigma^n H_Q\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}},$$

all differentials are forced.

Example 5.6.2. For $\Sigma^8 H_Q\underline{\mathbb{Z}}$, the pattern of differentials emanating from the Mackey functor $\underline{\pi}_6(P_8^8\Sigma^8 H_Q\underline{\mathbb{Z}})$ is forced; no other pattern of differentials wipes out all classes in this region. The shorter differentials clearing out the smaller region are then similarly forced.

Example 5.6.3. In the cases of $\Sigma^n H_Q\underline{\mathbb{Z}}$ for $n = 10$, 12, and 15, we also display the corresponding slice spectral sequence for $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$, where we use C_4 to indiscriminately refer to any of the subgroups $L, D, R \leq Q$. The slice differentials in the C_4-case force many of the slice differentials for the Q-equivariant spectra.


$\Sigma^5 H_Q\mathbb{Z}$
$\Sigma^6 H_Q\underline{\mathbb{Z}}$
0
2
4
6
8
10
12
14
16


$\Sigma^7 H_Q\underline{\mathbb{Z}}$
$\Sigma^8 H_Q\underline{\mathbb{Z}}$
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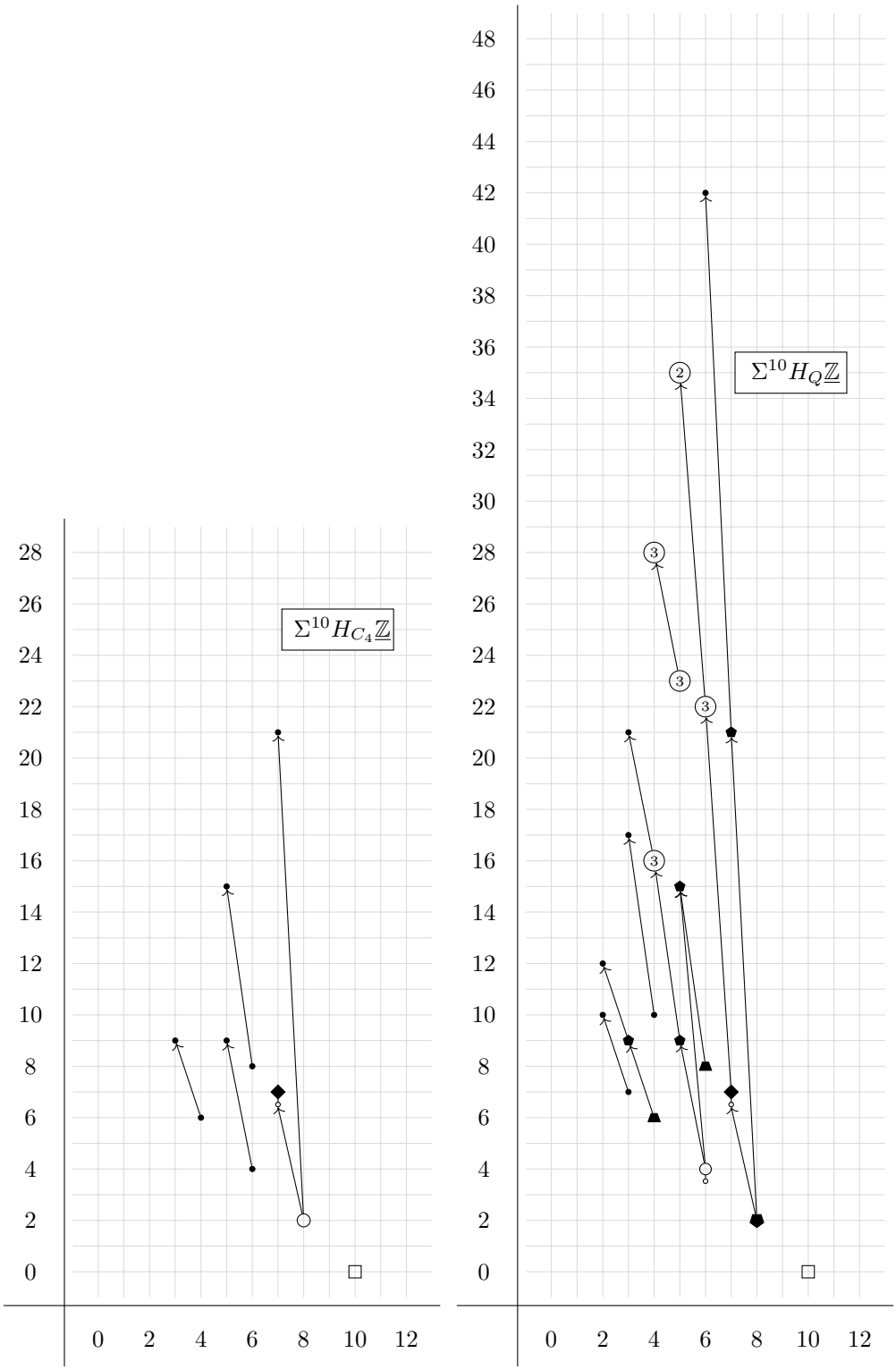
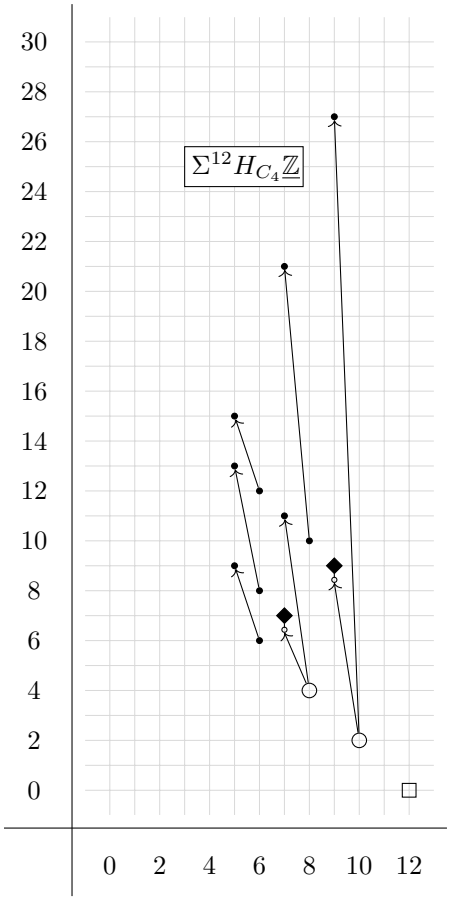
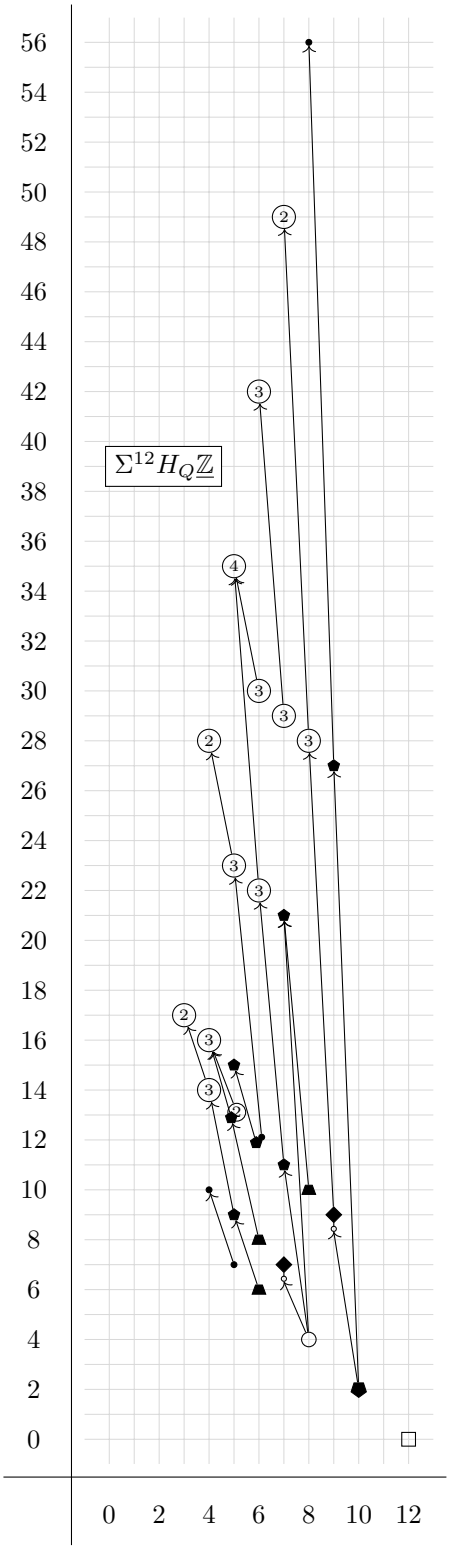
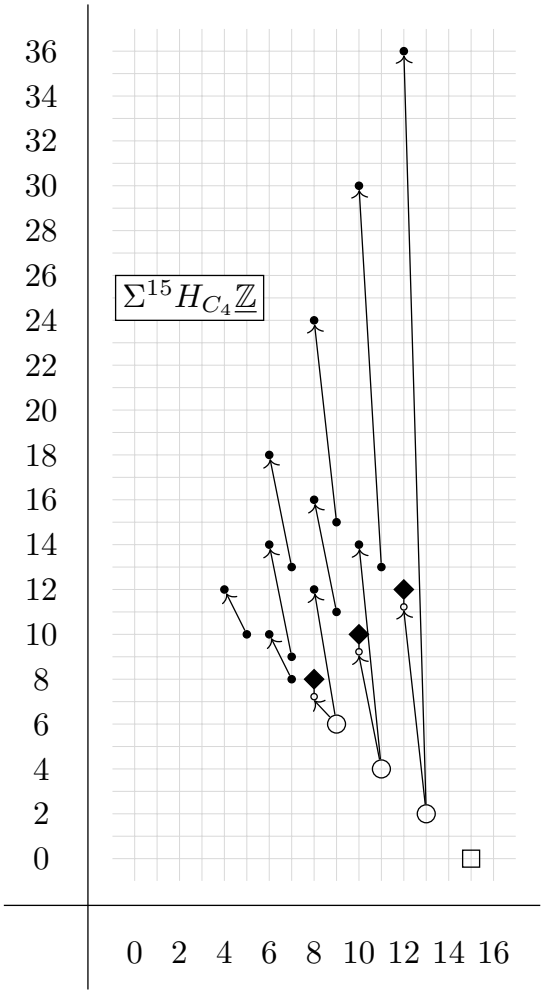
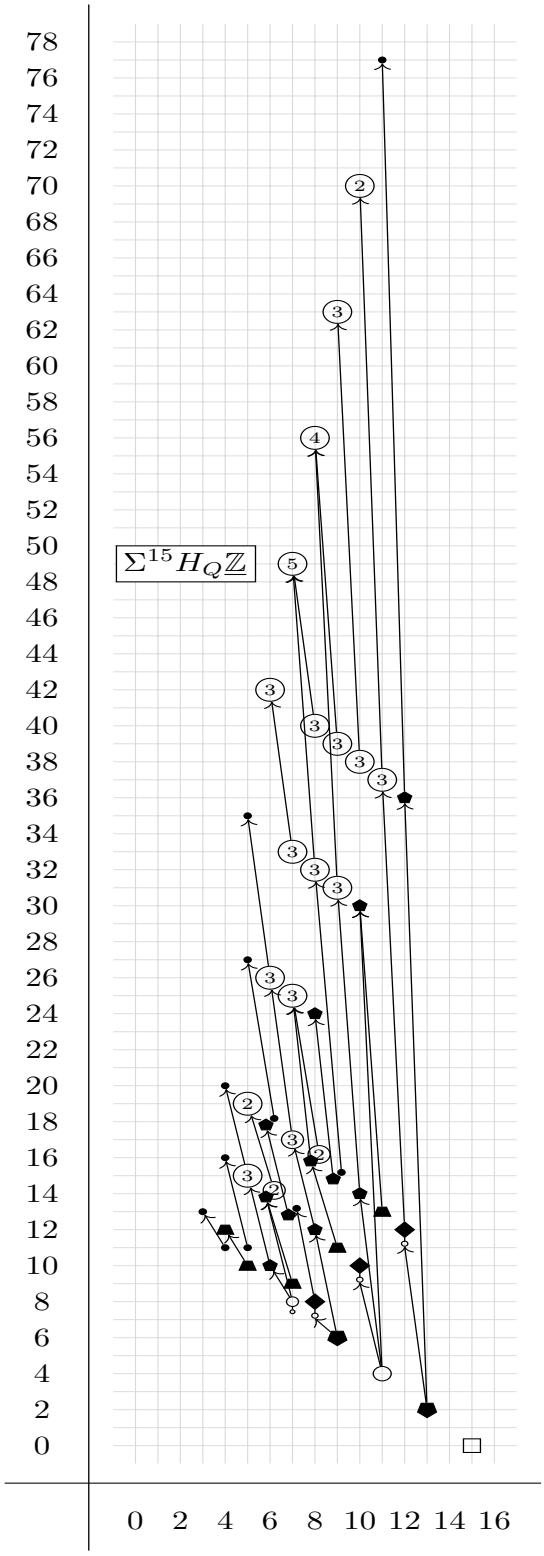
20
22
24
26
28


$\Sigma^{10} H_{C_4}\underline{\mathbb{Z}}$
$\Sigma^{10} H_Q\underline{\mathbb{Z}}$


$\Sigma^{12} H_{C_4}\underline{\mathbb{Z}}$


$\Sigma^{12} H_Q\underline{\mathbb{Z}}$





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Chapter 6 Appendices

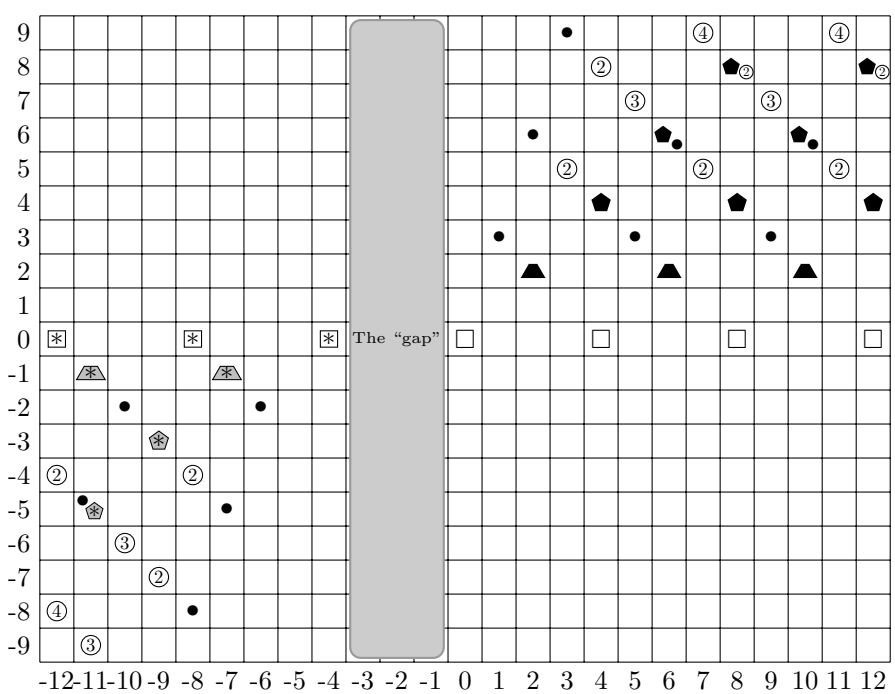
Appendix A: Homotopy Charts

The $RO(C_4)$-graded homotopy Mackey functors of $H_{C_4}\underline{\mathbb{Z}}$ are given in [HHR2]. More specifically, the homotopy Mackey functors of $\Sigma^{k\rho_{C_4}} H_{C_4}\underline{\mathbb{Z}}$, $\Sigma^{k\lambda} H_{C_4}\underline{\mathbb{Z}}$, and $\Sigma^{k\sigma} H_{C_4}\underline{\mathbb{Z}}$ are given in Figures 3 and 6 of [HHR2].

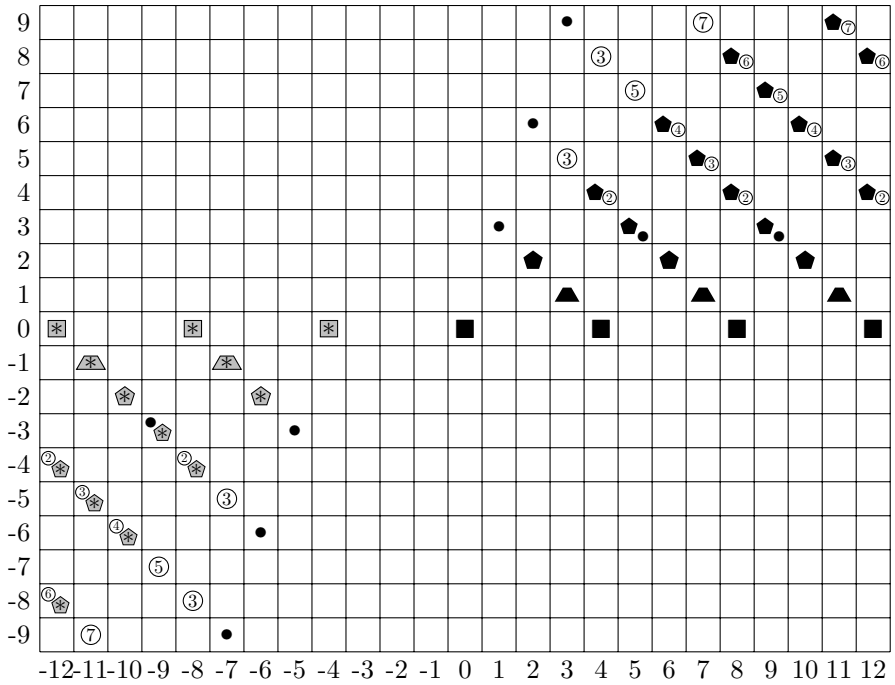
The homotopy Mackey functors of $\Sigma^{n\rho} H_K\underline{\mathbb{Z}}$ were computed in Section 4.5. The homotopy Mackey functors of $\Sigma^{n\rho} H_K\underline{\mathbb{F}_2}$ were computed in [GY, Section 7]. This homotopy is displayed in the charts below. The symbols are listed in Table 1.

Table 1: Symbols for K-Mackey functors

$\square = \underline{\mathbb{Z}}$	$\bullet = \phi^*_{LDR}\underline{\mathbb{F}_2}$	$\blacktriangle = \underline{mg}$
$\boxed{*} = \underline{\mathbb{Z}}^*$	$\circledast = \phi^*_{LDR}\underline{\mathbb{F}_2}^*$	$\boxed{*}\!\!\blacktriangle = \underline{mg}^*$
$\square = \underline{\mathbb{F}_2}$	$\boxed{*} = \underline{\mathbb{F}_2}^*$	$n = \underline{g}^n$



The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_{K_4}\underline{\mathbb{Z}}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_{K_4}\underline{\mathbb{Z}}$ appears in position $(k, 4n-k)$.



The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_{K_4}\underline{\mathbb{F}_2}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_{K_4}\underline{\mathbb{F}_2}$ appears in position $(k, 4n-k)$.

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- (with B. Guillou) The slices of quaternionic Eilenberg-Mac Lane spectra, available at arXiv:2204.03127.