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Inverse Boundary Value Problems for Polyharmonic Operators With Non-Smooth Coefficients

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DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
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ABSTRACT OF DISSERTATION

Inverse Boundary Value Problems for Polyharmonic Operators With Non-Smooth Coefficients

We consider inverse boundary problems for polyharmonic operators and in particular, the problem of recovering the coefficients of terms up to order one. The main interest of our result is that it further relaxes the regularity required to establish uniqueness. The proof relies on an averaging technique introduced by Haberman and Tataru for the study of an inverse boundary value problem for a second order operator.

KEYWORDS: Inverse Problem, Polyharmonic, Differential Equations, Analysis

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Dedicated to Katherine Gauthier, for her never ending support.

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Chapter 1 Inverse Problem

To define what an inverse problem is, let us start with a partial differential equation. Suppose

$$\begin{cases} \mathcal{L}u = 0 & \text{in}\,\Omega \\ \text{boundary conditions on } u \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ and $\mathcal{L} = \sum_{|\alpha| \leq N} c_\alpha(x) D^\alpha$ where α is a multi-index. An inverse problem is the problem of recovering the coefficients $c_\alpha(x)$ given information about solutions of the operator $\mathcal{L}$. Calderón was one of the first to pose such a question in [12].

Inverse problems do not just lie in the theoretical world, but instead, have many applications. A few examples are Electrical Impedance Tomography (EIT), Electron Beam Tomography (EBT), and Inverse Heat Conduction (IHC).

EIT is used in medical imaging, ground exploration, and more. I will go into more detail about EIT in the next section. EBT is used in computerized tomography (CT) scans to determine tissues and material inside the body using boundary measurements. Lastly, IHC is used for determining surface heat flux of an object where surface sensors are not practical.

As we see, there are practical applications to inverse problems, however, I am more interested in the mathematics of them. A few questions one may ask when studying inverse problems are: can we show that the coefficients are unique, can we recover the coefficients, is the recovery stable. My research lies in showing uniqueness. Specifically, my research lies in how general can the coefficients be to guarantee that uniqueness is satisfied for a specific class of operators.

1.1 The Calderón Problem

The Calderón problem asks if we can recover the conductivity of a domain. Let $\gamma(x)$ be the conductivity and Ω be an open, bounded, simply connected domain with smooth boundary. We will also assume that the domain has no sources or sinks of current and $\gamma(x) > \epsilon > 0$. Ohm's Law along with conservation of charge states that for a voltage potential f on $\partial\Omega$ we have that the voltage potential $u(x)$ in Ω satisfies

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

If we assume that γ has at least two derivatives, we can transform the problem into the following Schrödinger equation. Let $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$ and $v = \gamma^{1/2}u$. Then if u solves (1.1), we have

$$\begin{cases} -\Delta v + qv = 0 & \text{in } \Omega \\ v = \dfrac{f}{\gamma^{1/2}} & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The key idea to our approach for studying this type of inverse problem is to construct Complex Geometric Optics (CGO) solutions. That is, solutions to

$$-\Delta u + qu = 0 \qquad \text{in } \Omega$$

of the form

$$u(x) = e^{-x\cdot\zeta/h}(1 + r(x)).$$

Here, $\zeta \in \mathbb{C}^d$ is a specifically chosen vector such that $e^{-x\cdot\zeta}$ is harmonic, $0 < h < 1$, and $r(x)$ is thought of as a remainder. The idea is to find enough of these solutions so that the products are dense in some space What we want to happen is if we have two equations,

$$\begin{cases} -\Delta u_1 + q_1 u_1 = 0 & \text{in } \Omega \\ -\Delta u_2 + q_2 u_2 = 0 & \text{in } \Omega \\ u_1 = u_2 = f & \text{on } \partial\Omega. \end{cases}$$

and we can construct enough of these solutions where the remainder is small in terms of h, then after taking limits, we end up with

$$0 = \int_{\mathbb{R}^d} e^{-ix\cdot k}(q_1(x) - q_2(x))\,dx$$

for every $k \in \mathbb{R}^d$. This tells us that boundary measurements uniquely determine the coefficients. This technique will be discussed later in the dissertation so the details are omitted here. A difficulty that arises is what do we have to assume about q. If we assume q is bounded, taking limits is not as difficult. However, once that assumption is removed, such as $q \in L^p$, taking limits becomes more difficult. The next section gives more information and references of the study into the smoothness conditions on q (and γ) and what conditions q needs to satisfy needed to construct solutions and take limits.

1.2 Smoothness of the Coefficient

The main idea in this section is to determine what smoothness condition coefficients need to satisfy in order to recover the coefficients. To start, one may want to assume that the coefficients are smooth or bounded. We then try to see where we can relax some conditions and study how this relaxation can be done. The importance of this, is that many times in a practical setting, coefficients are non smooth. For example, when taking an x-ray or CT scan, there is not a smooth transition from bone to muscle, so a smoothness condition may not be ideal.

It will be important to know that results for γ in (1.1) with s derivative is implied by a result for Schrödinger operators with coefficients that have $s - 2$ derivatives, at least for $s \geq 1$. Calderón [12] established an early result for a linearized problem and introduced the technique of CGO solutions. The first uniqueness result came from Sylvester and Uhlmann [21] for the full non-linear problem. Their results was for γ being a bounded scalar function. Later, Brown [9] and Brown and Torres [10] proved uniqueness for γ with around $3/2$ derivatives. Paivarinta, Panchenko and Uhlmann [19] were able to construct CGO solutions for γ having 1 derivative, but were not able to establish uniqueness results. A breakthrough came from Haberman and Tataru [14] where they established the uniqueness result for γ having 1 derivative. The breakthrough idea came from looking at an average behavior instead of a fixed behavior. We saw that we will have to bound the remainder $r(x)$, and if we fix ζ, we obtain a bound. However, if we choose a variety of ζ's in a certain class and look at the average of the bounds, it turns out we obtain a better estimate.

Chapter 2 Polyharmonic Operator

2.1 Polyharmonic Operators

The main result of this dissertation is a result for polyharmonic operators with nonsmooth coefficients up to order one. For an integer $m \geq 2$, the operator we study is

$$\mathcal{L} = (-\Delta)^m + Q \cdot D + q \tag{2.1}$$

where $D = -i\nabla$ the coefficients q and $Q = (Q_1, \cdots, Q_d)$ lie in certain negative order Sobolev spaces. The main result allows for less regular coefficients than previously studied. An example of a polyharmonic boundary value problem is

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ (-\Delta)^j u = \phi_j, & \text{on } \partial\Omega, j = 0, \ldots, m-1. \end{cases} \tag{2.2}$$

We then measure $-\frac{\partial}{\partial \nu}(-\Delta)^j u$ for $j = 0, \cdots m-1$ where $\frac{\partial}{\partial \nu}$ is the outward normal derivative. We define the Cauchy data of a solution u of (2.2) to be

$$(u, \frac{\partial u}{\partial \nu}, -\Delta u, \ldots, (-\Delta)^{m-1} u, \frac{\partial}{\partial \nu}(-\Delta)^{m-1} u). \tag{2.3}$$

The collection of all Cauchy data of the operator (2.1) is defined to be the collection of all Cauchy data for all weak solutions u, see (2.3). Note that there are other boundary conditions that may be studied. The choice of Cauchy data will also differ depending on the boundary conditions. The question we ask is if the coefficients are uniquely determined by the set of Cauchy data. With the assumptions we make on q and Q, we will be able to guarantee uniqueness.

2.2 Previous Results

Inverse problems dealing with the polyharmonic operator were first studied by Krupchyk, Lassas, and Uhlmann [15]. Their assumptions on the coefficients were $Q \in W^{1,\infty}(\Omega; \mathbb{C}^d)$ and $q \in L^\infty(\Omega, \mathbb{C})$ with some restrictions on the support of the coefficients. This work was extended to unbounded potentials by Krupchyk and Uhlmann [16]. Their assumption was $q \in L^p(\Omega)$ for $p \geq 2d/m$. Assylbekov [2] and Assylbekov and Iyer [1] started the study of the polyharmonic operator with coefficients in negative order Sobolev spaces. One key component of their proof was use of a Sobolev multiplication theorem, (3.2). The final assumptions on the coefficients were $Q \in W^{-\frac{m}{2}+1,p}(\mathbb{R}^n)$ and $q \in W^{-\frac{m}{2}+\delta,r}(\mathbb{R}^n)$ where p and r are dependent on the dimension and m and $\delta > 0$. They also make some assumptions on the support of the coefficients. It also remains an open problem to extend the techniques we discuss for nonsmooth first order perturbations to second order perturbations as studied in the works of Bhattarcharyya and Ghosh [4] and Bhattacharyya, Krishnan, and Sahoo [5]. Other work includes Yan studied the biharmonic operators on transversally anisotropic manifolds [23].

The key innovation of our research is to adapt the arguments of Haberman and Tataru to polyharmonic operators. We consider operators whose principal part is the polyharmonic operator and include lower order terms involving at most one derivative as in (2.1). For these operators we are able to establish uniqueness for potentials q which lie in Sobolev spaces with smoothness index $-s > -m/2 - 1$. We believe that the cutoff in the smoothness index should be $-m$ for operators where the principal part is $(-\Delta)^m$. This is consistent with the result of Haberman and Tataru when $m = 1$. Our result allows us to get arbitrarily close to this cut off when $m = 2$. However, when $m \geq 3$ we believe that there is a large gap between our results and the optimal result. Since L^p embeds into $W^{-s,q}$ for $1/p = 1/q + s/d$ our result below in Theorem 2.4 will give uniqueness for potentials in L^p for $p > d/m$ which is a weaker result for L^p spaces than Krupchyk and Uhlmann give [16, Theorem 1.1]. This is one of several indications that our result, while an advance over earlier work, is not the last word on this subject.

2.3 Main Result

The main result establishes a condition on the smoothness of coefficients. This result is nearly 1 degree less smooth than previous known results.

Theorem 2.4. *Fix $m \geq 2$ and $d \geq 3$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let $\mathcal{L}_j = (-\Delta)^m + Q^j \cdot D + q^j$ with $q^j \in \tilde{W}^{-s,p}(\Omega)$ and each component Q^j lies in $\tilde{W}^{-s+1,p}(\Omega)$ with $s < m/2 + 1$, $1/p + (s - m)/d < 0$, and $p \geq 2$. If each of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same linear space of Cauchy data for (2.2), then $Q^1 = Q^2$ and $q^1 = q^2$.*

Chapter 3 CGO and X^λ Spaces

The goal of this section is to construct Complex Geometric Optics (CGO) solutions to the operator (2.1). To do this, we will define X^λ spaces that work well with our operator. The other important item in this section is it provides many of the estimates of operators between various function spaces. These estimates involve how the function spaces behave with our operator and the relative size of the operators in terms of a small parameter, $0 < h < 1$.

3.1 Definitions and preliminaries

The main theorem gives the assumptions of the coefficients in terms of Sobolev spaces, as they give a precise meaning on the smoothness of the coefficients. Given $s \in \mathbb{R}$ and $1 < p < \infty$, we let $W^{s,p}(\mathbb{R}^d)$ be the Sobolev space of order s. Informally, distributions in $W^{s,p}(\mathbb{R}^d)$ have s derivatives in $L^p(\mathbb{R}^d)$. The space $W^{s,p}(\mathbb{R}^d)$ has the norm $\|u\|_{W^{s,p}(\mathbb{R}^d)} = \left\|(1-\Delta)^{s/2}u\right\|_{L^p(\mathbb{R}^d)}$. Often times, we work on a domain $\Omega \subset \mathbb{R}^d$, so it will be useful to have a notion of Sobolev spaces on Ω. We let $W^{s,p}(\Omega)$ denote the Sobolev space on Ω obtained by by restricting distributions in $W^{s,p}(\mathbb{R}^d)$ to Ω. The norm of $W^{s,p}(\Omega)$

$$\|u\|_{W^{s,p}(\Omega)} = \inf_{\substack{v\in W^{s,p}(\mathbb{R}^d)\\ v=u \text{ in } \Omega}} \|u\|_{W^{s,p}(\mathbb{R}^d)}.$$

We let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^{s,p}(\Omega)$. Lastly, we define

$$\tilde{W}^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^d) \,:\, \operatorname{supp}(u) \subset \bar{\Omega}\}.$$

This space can get confusing since a distribution in $\tilde{W}^{s,p}(\Omega)$ may not be a distribution on Ω, instead is a distribution on $\mathbb{R}^d$. For us, Ω is at least Lipschitz, therefore, $C_0^\infty(\Omega)$ is dense in the space $\tilde{W}^{s,p}(\Omega)$. See McLean [17, Theorem 3.29] for the case $p = 2$ and a proof that may be generalized to $p \in (1,\infty)$. However, this does not give us that $W_0^{s,p}(\Omega)$ and $\tilde{W}^{s,p}(\Omega)$ are the same space, because the norms used are different. For many cases of s and p, the two spaces are equal, however, in other cases $W_0^{s,p}(\Omega)$ will be a larger space. The norm in $\tilde{W}^{s,p}(\Omega)$ is the norm of a distribution that is 0 outside of $\bar{\Omega}$. The norm of an element in $W_0^{s,p}(\Omega)$ is the infinum of the norm in $\mathbb{R}^d$ over all possible extensions and therefore may be smaller.

Another important function space that we will use is the Hölder space $C^\theta(\mathbb{R}^d)$. This is defined for $\theta \in (0,1]$ and is the collection of functions on $\mathbb{R}^d$ for which the norm

$$\|f\|_{C^\theta(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\theta}$$

is finite.

Since we assume Ω is at least Lipschitz, we have a characterization of the dual space of $W^{s,p}(\Omega)$. That is, for $1 < p < \infty$ and p' such that $1/p + 1/p' = 1$, we have

$$W^{s,p}(\Omega)^* = \tilde{W}^{-s,p'}(\Omega), \qquad 1 < p < \infty, s \geq 0. \tag{3.1}$$

The case $p = 2$ may be found in the monograph of McLean [17, Theorem 3.30] and his argument extends to $p \in (1, \infty)$. Using the spaces $\tilde{W}^{s,p}(\Omega)$ is useful for dealing with solutions of (2.1) with nonzero boundary data. Also, for a range of s and p, we will be able to define the product of a function in a positive order Sobolev space with coefficients taken from Sobolev spaces of negative order. The Sobolev embedding theorem and the product rule gives us that the bilinear map $(u, v) \to vD^\alpha u$ maps $W^{m,2}(\Omega) \times W^{m,2}(\Omega) \to W^{m-|\alpha|,t}(\Omega)$ provided $|\alpha| \leq m$ and where t is given by

$$\begin{cases} \frac{1}{t} = 1 - \frac{m}{d}, & \frac{m}{d} < \frac{1}{2} \\ \frac{1}{t} = \frac{1}{2}, & \frac{m}{d} > \frac{1}{2} \\ \frac{1}{t} > \frac{1}{2}, & \frac{m}{d} = \frac{1}{2}. \end{cases} \tag{3.2}$$

This is a special case of a more general result in the monograph of Runst and Sickel [20, §4.4.4, Theorems 1,2]. Thus, if $f \in \tilde{W}^{|\alpha|-m,t'}(\Omega)$ and $u \in W^{m,2}(\Omega)$, then we may define $fD^\alpha u \in \tilde{W}^{-m,2}(\Omega)$. If $v \in W^{m,2}(\Omega)$, then we define $\langle fD^\alpha u, v\rangle$ as equal to $\langle f, vD^\alpha u\rangle$ and we have the estimate

$$|\langle fD^\alpha u, v\rangle| \lesssim \|f\|_{\tilde{W}^{|\alpha|-m,t'}(\Omega)} \|u\|_{W^{m,2}(\Omega)} \|v\|_{W^{m,2}(\Omega)}. \tag{3.3}$$

Now, we are able to describe a notion of a weak solution to (2.1). Let B_0 be a bilinear form of $(-\Delta)^m$. That is, $B_0 : W^{m,2}(\Omega) \times W^{m,2}(\Omega) \to \mathbb{C}$ given by

$$B_0(u, v) = \int_\Omega \sum_{|\alpha|=|\beta|=m} c_{\alpha\,\beta} D^\alpha u\, D^\beta v.$$

This form must also have the property

$$B_0(u, v) = \int_\Omega [(-\Delta)^m u]v\,, \qquad u \in C^\infty(\Omega), v \in C_0^\infty(\Omega). \tag{3.4}$$

There are many choices possible to choose for B_0, it depends on how we integrate by parts. One particular nice choice for the Navier boundary value problem is (2.2),

$$B_0(u, v) = \begin{cases} \displaystyle\int_\Omega (-\Delta)^{m/2} u (-\Delta)^{m/2} v\, dx, & \text{if } m \text{ is even} \\ \displaystyle\int_\Omega \nabla(-\Delta)^{(m-1)/2} u \cdot \nabla(-\Delta)^{(m-1)/2} v\, dx, & \text{if } m \text{ is odd.} \end{cases} \tag{3.5}$$

Another nice choice is the form

$$B_0(u, v) = \int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} D^\alpha u D^\alpha v\, dx. \tag{3.6}$$

as it is easier to see that this form is coercive. However, the choice of forms is not as important as one may think. We note that in our uniqueness theorem based on forms, Theorem 6.8, will hold for any choice of B_0 satisfying (3.4).

Given our form B_0, we will define solutions to the equation $\mathcal{L}u = 0$. We first require

$$Q \in \tilde{W}^{1-m,t'}(\Omega) \quad \text{and} \quad q \in \tilde{W}^{-m,t'}(\Omega), \quad \text{with } t \text{ as in (3.2).} \tag{3.7}$$

We say that u is a *weak solution* of

$$\mathcal{L}u = (-\Delta)^m u + Q \cdot Du + qu = 0$$

if $u \in W^{m,2}(\Omega)$ and

$$B_0(u, v) + \langle Q \cdot Du, v\rangle + \langle qu, v\rangle = 0, \qquad v \in W_0^{m,2}(\Omega).$$

Because of the estimate (3.3) and by our assumption (3.7), we have that the bilinear form is continuous on $W^{m,2}(\Omega) \times W^{m,2}(\Omega)$. Notice that the conditions in (3.7) are weaker than the conditions we assume the coefficients satisfy in the main theorem. This is because we can define weak solutions for a more broad class of coefficients, but if we want to show uniqueness of the coefficients, we have to have stronger assumptions. The goal for this this research is to find the minimal amount of regularity needed on the coefficients where we can still guarantee uniqueness.

We will adopt the convention of using $a \lesssim b$ to mean that there is a constant C so that $a \leq Cb$. The constants C are allowed to depend on m, the L^p-index, the order of a Sobolev space, the dimension, the domain and the coefficients Q and q appearing in the operator. However, it is an important point that C will always be independent of h for h small as this is needed to evaluate limits as h tends to zero. We use $a \approx b$ to mean $a \lesssim b$ and $b \lesssim a$.

3.2 X^λ Spaces

We let $\mathcal{V}$ denote the collection of vectors

$$\mathcal{V} = \{\zeta \in \mathbb{C}^d : \zeta = \mu_1 + i\mu_2,\ \mu_j \in \mathbb{R}^d,\ \mu_i \cdot \mu_j = \delta_{ij}\}. \tag{3.8}$$

We want to find solutions of (2.1) of the form

$$u(x) = e^{x\cdot\zeta/h}(a(x) + \psi(x)) \text{ in } \Omega \tag{3.9}$$

where $\zeta \in \mathcal{V}$. We have $h > 0$ is a small limiting parameter, $a(x)$ is an amplitude which will solve certain transport equations. We will make the assumption that $a(x)$ is smooth, which will be discussed soon. Lastly, ψ is a remainder term that will have a certain X^λ norm bound depending on a power of h.

If we plug (3.9) into (2.1) and left multiply by $h^{2m}e^{-x\cdot\zeta/h}$, we obtain

$$L_\zeta \psi = -L_\zeta a. \tag{3.10}$$

We introduce the operator $P_\zeta(hD)$ defined by

$$P_\zeta(hD) = P(hD) = e^{-x\cdot\zeta/h}(-h^2\Delta)e^{x\cdot\zeta/h},$$

where we will view this as a semi-classical operator with symbol $P_\zeta(\xi) = P(\xi) = |\xi|^2 - 2i\zeta\cdot\xi$. With this notation we may write

$$L_\zeta = P(hD)^m + h^{2m}Q\cdot(\frac{\zeta}{ih} + D) + h^{2m}q. \tag{3.11}$$

We will give estimates for solutions of the equation $L_\zeta u = f$, at least for h small. We note that $P(hD)$ vanishes on a set of codimension two and is therefore not elliptic.

To study the equation (3.10), it is useful to use function spaces that work well with $P(hD)$. Similar spaces were used by Bourgain [7, 8] in his study of evolution equations. The spaces described below were introduced in the study of the inverse conductivity problem by Haberman and Tataru [14]. For $\lambda \in \mathbb{R}$, $\zeta \in \mathcal{V}$, and $h > 0$, we introduce a space $X^\lambda_{h\zeta}$ by

$$X^\lambda_{h\zeta} = X^\lambda = \{u \in \mathcal{S}'(\mathbb{R}^d) : \hat{u} \text{ is a function and } \|u\|_{X^\lambda} < \infty\}$$

where the norm in X^λ is given by

$$\|u\|^2_{X^\lambda} = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2(h + |P_\zeta(h\xi)|)^{2\lambda}\, d\xi.$$

Notice that the operator $P_\zeta(hD)$ and the spaces $X^\lambda_{h\zeta}$ depend on h and ζ but often times, this dependency is left out to simplify notation. There are estimates that depend on h, but not ζ, so we infer the h is coming from the X^λ norms without statement. We will make it clear when the dependency matters. Some properties of the X^λ spaces is that the dual is $X^{-\lambda}$ and that the Schwartz space is dense since the symbol $(h + |P(h\xi)|)^\lambda$ is bounded above and below on compact sets.

What we want to do is construct a right inverse of $P(hD)$ between X^λ spaces. Let $\phi \in C^\infty_0(\mathbb{R}^d)$ such that $\phi = 1$ in a neighborhood of $\bar{\Omega}$. Let $J_\phi = \phi P(hD)^{-1/2}$ and

$$I_\phi = J_\phi \frac{|P(hD)|}{P(hD)} J^*_\phi = \phi P(hD)^{-1}\phi.$$

We will show that I_ϕ is a right inverse and is a bounded operator between $X^\lambda \to X^{\lambda+1}$ for any λ. To show I_ϕ is a bounded operator, we use a lemma from Haberman and Tataru which they used to get an estimate for $u \in L^2_{\text{loc}}$ in terms of $P(hD)^{1/2}u$. We refer to [14, Lemma 2.1] for a proof.

Lemma 3.12 ([14, Lemma 2.1])**.** *Let v and w be nonnegative weights on $\mathbb{R}^d$. If $\phi \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\|\phi * f\|_{L^2_w} \lesssim_\phi \min\left(\sup_\xi \int J(\xi,\eta)\, d\eta, \sup_\eta \int J(\xi,\eta)\, d\xi\right)^{1/2} \|f\|_{L^2_v},$$

where

$$J(\xi,\eta) = |\phi(\xi-\eta)|\frac{w(\xi)}{v(\eta)}.$$

In order to use 3.12, we will need the following two lemmas. The lemmas were proved in [11].

Lemma 3.13. *Let $h > 0$ and $0 \le s \le 1$. We have*

$$\int_{B(x,r)} \frac{|\xi|^s}{|P(h\xi)|}\, d\xi \lesssim \frac{r^{d+1}}{h^{1+s}}, \qquad x \in \mathbb{R}^d,\ r > 0. \tag{3.14}$$

If $\phi \in \mathcal{S}(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} \frac{|\phi(x-\xi)||\xi|^s}{|P(h\xi)|}\, d\xi \lesssim \frac{1}{h^{1+s}}. \tag{3.15}$$

Proof. We begin with the proof of (3.14). By applying the change of variables, $\xi \to \xi/h$ we may reduce to the case where $h = 1$. To show (3.14) with $h = 1$, we will consider three cases: a) $|x| < 8$, $r \le 1$, b) $|x| \ge 8$, $r \le 1$, c) $r > 1$. We let $\Sigma = \{\xi : P(\xi) = 0\}$ and since $P(\xi) = |\xi + \operatorname{Im}\zeta \cdot \xi|^2 - 1 - 2i \operatorname{Re}\zeta \cdot \xi$, we have that Σ is a sphere of radius 1 and codimension two centered at $-\operatorname{Im}\zeta$. Furthermore, one can check that $\nabla P(\xi) \neq 0$ on Σ and thus it follows that $P(\xi) \approx \operatorname{dist}(\xi, \Sigma)$ on $B(0, 9)$. To establish (3.14) in case a) we need to estimate integrals of $|\xi|^s/|P(\xi)| \lesssim 1/\operatorname{dist}(\xi, \Sigma)$. We can obtain the necessary estimate by using a change of variables to straighten out a portion of the sphere to reduce to integrating the distance to a plane of codimension two in $\mathbb{R}^d$.

To establish (3.14) in case b), we use that $|\xi|^s/|P(\xi)| \approx |\xi|^{s-2} \lesssim 1$ for $|\xi| \ge 7$. From this, we have

$$\int_{B(x,r)} \frac{|\xi|^s}{|P(\xi)|}\, d\xi \lesssim r^d \le r^{d-1}$$

where the last inequality uses that $r \le 1$. Finally, in case c) we use that $|P(\xi)| \approx |\xi|^2$ for $|\xi|$ large to obtain

$$\int_{B(x,r)} \frac{|\xi|^s}{|P(\xi)|}\, d\xi \le \int_{\{|\xi| \le 8\}} \frac{|\xi|^s}{|P(\xi)|}\, d\xi + \int_{\{|\xi|>8\} \cap B(x,r)} |\xi|^{s-2}\, d\xi.$$

We use case a) to bound the first term and obtain that the previous display is at most $1 + r^{d-2+s} \lesssim r^{d-1}$. The last inequality follows since $r \ge 1$ and $s \le 1$. We note that if $d = 2$ and $s = 0$, the intermediate estimate involves a logarithm, but the conclusion still holds.

We turn to the proof of (3.15) and give the proof for $h > 0$. We write $\mathbb{R}^d = \cup_{k \ge 0} A_k$ where $A_0 = B(x, 1)$ and for $k \ge 1$ $A_k = B(x, 2^k) \setminus B(x, 2^{k-1})$. If $\phi \in \mathcal{S}(\mathbb{R}^d)$, then it is rapidly decreasing and, in particular, we have $|\phi(x - \xi)| \lesssim 2^{-kd}$ if $\xi \in A_k$. We break the integral in (3.15) into integrals over the sets A_k, and use (3.14) to obtain

$$\int_{\mathbb{R}^d} \frac{|\phi(x-\xi)||\xi|^s}{|P(h\xi)|}\, d\xi \lesssim \frac{1}{h^{1+s}} \sum_{k=1}^{\infty} 2^{k(d-1)} \cdot 2^{-kd} \lesssim \frac{1}{h^{1+s}}.$$

□

Lemma 3.16. *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. For any $\eta \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\big(\big||P(h\eta)| - |P(h\xi)|\big|\big)\, d\xi \lesssim 1$$

Proof. Using the triangle inequality and the definition of $P(h\xi)$, we have

$$\big||P(h\eta)| - |P(h\xi)|\big| \lesssim h|\xi-\eta|(h|\xi+\eta|+1). \tag{3.17}$$

Using (3.17), we obtain the following inequality.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\varphi(\xi-\eta)|}{|P(h\xi)|}\big(\big||P(h\eta)| - |P(h\xi)|\big|\big)\, d\xi \\
&\quad \lesssim \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h|\xi-\eta|\, d\xi + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h^2|\xi-\eta|^2\, d\xi \\
&\qquad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h^2|\xi-\eta||\xi|\, d\xi.
\end{aligned}$$

Using Lemma 3.13 and recalling that $0 < h \leq 1$ completes the proof. □

Theorem 3.18. *The operator $J_\phi : X^\lambda \to X^{\lambda+1/2}$ is a bounded operator for all $\lambda \in \mathbb{R}$.*

Proof. We have that

$$(J_\phi u)\hat{} = \hat{\phi} * (\hat{u}/|P(h\cdot)|^{1/2}).$$

Since $\hat{\phi}$ is again a Schwartz function, we replace $\hat{\phi}$ by ϕ, and we will show that for a Schwartz function, ϕ, we have

$$\sup_\eta \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\xi)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \lesssim 1, \qquad 0 \leq h \leq 1. \tag{3.19}$$

Then an application of Lemma 3.12 will give the result of the Theorem. To establish (3.19), we begin by adding and subtracting the quantity $(h+|P(h\eta)|)^{2\lambda+1}$ and breaking the integral into two parts.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\xi)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&= \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \Big| \frac{(h+|P(h\xi)|)^{2\lambda+1} - (h+|P(h\eta)|)^{2\lambda+1} + (h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}} \Big|\, d\xi \\
&\leq \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \Big| \frac{(h+|P(h\xi)|)^{2\lambda+1} - (h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}} \Big|\, d\xi
\end{aligned}$$

The first integral simplifies nicely and will be broken up into two more parts, leaving us with three integrals.

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}(h+|P(h\eta)|)\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\, d\xi \\
&= \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h\, d\xi + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}|P(h\eta)|\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\, d\xi
\end{aligned}$$

The first integral is as simplified as we need. The second integral will require adding and subtracting $|P(h\xi)|$ and broken into two more integrals. The last integral will remain the same.

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h\, d\xi + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|(|P(h\eta)|-|P(h\xi)|+|P(h\xi)|)\Big|\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\, d\xi \\
&= \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h\, d\xi + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|(|P(h\eta)|-|P(h\xi)|)\Big|\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}|P(h\xi)|\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\, d\xi
\end{aligned}$$

The third integral will simplify nicely and we will be left with four integrals. We will want to show that all are uniformly bounded independent of h to show (3.19).

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} h\, d\xi + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|(|P(h\eta)|-|P(h\xi)|)\Big|\, d\xi + \int_{\mathbb{R}^d} |\phi(\xi-\eta)|\, d\xi \\
&\quad + \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\, d\xi \\
&= I + II + III + IV
\end{aligned}$$

Using (3.15) of Lemma 3.13 we see that I is bounded by a constant which is independent of η and h, Lemma 3.16 gives the same estimate for II, and the estimate for III is easy. This leaves us to bound IV. We turn to an application of the mean value theorem to show the bound for IV. We set $f(t) = t^{2\lambda+1}$ for $t \geq 0$. For $\eta, \xi \in \mathbb{R}^d$, let $a = \min(\, h + |P(h\xi)|,\, h + |P(h\eta)|\,)$ and $b = ||P(h\xi)| - |P(h\eta)||$. Then, we get that $a + b = \max(\, h + |P(h\xi)|,\, h + |P(h\eta)|\,)$. The mean value theorem states that there exists θ between a and $a + b$ such that

$$f(a+b) - f(a) = (2\lambda+1)\theta^{2\lambda} b.$$

Also, note that we have previously shown that $b \lesssim h|\xi-\eta|(h|\xi+\eta|+1)$.
Let $A_0 = B(\eta,1)$ and $A_k = B(\eta,2^k) \setminus B(\eta,2^{k-1})$. This gives us that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\,d\xi \\
&= \sum_{k=0}^{\infty}\int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\Big|\frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\Big|\,d\xi \\
&= \sum_{k=0}^{\infty}\int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\frac{(2\lambda+1)\theta^{2\lambda}h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\,d\xi \\
&\lesssim \sum_{k=0}^{\infty}\int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\frac{\theta^{2\lambda}h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\,d\xi
\end{aligned}$$

Now, we will start by breaking this case up for $\lambda \geq 0$ and $\lambda < 0$.
Supposing that $\lambda \geq 0$, we have that $|\theta| \leq h + |P(h\xi)| + |P(h\eta)|$. We now will look at the following cases.

1. $|h\eta| < 8$ and $2^k \leq 32/h$
2. $|h\eta| < 8$ and $2^k \geq 32/h$
3. $|h\eta| \geq 8$ and $2^k \leq |\eta|/8$
4. $|h\eta| \geq 8$ and $2^k \geq |\eta|/8$

In the first case, since $|h\eta| < 8$ and $2^k \leq 32/h$, we have

$$|\xi+\eta| \leq 2|\eta| + |\xi-\eta| \leq \frac{16}{h} + 2^k \leq \frac{48}{h}.$$

And,

$$\begin{aligned}
|P(h\xi)| &\leq |P(h\eta)| + |P(h\xi) - P(h\eta)| \\
&\lesssim |P(h\eta)| + h|\xi-\eta|(h|\xi+\eta|+1) \\
&\lesssim |P(h\eta)| + h|\xi-\eta|49 \\
&\lesssim |P(h\eta)| + h|\xi-\eta| \\
&\lesssim |P(h\eta)| + h2^k
\end{aligned}$$

Then,

$$\begin{aligned}
\theta &\lesssim (h + |P(h\eta)| + h2^k + |P(h\eta)|)^{2\lambda} \\
&\lesssim (h2^k + |P(h\eta)|)^{2\lambda}
\end{aligned}$$

Using the fact that $h|\xi+\eta|+1 \lesssim 1$ we have

$$\int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\frac{\theta^{2\lambda}h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\,d\xi \lesssim \int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|}\frac{\theta^{2\lambda}h2^k}{(h+|P(h\eta)|)^{2\lambda}}\,d\xi.$$

We now use the bound for θ to obtain

$$\leq \int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h2^k + |P(h\eta)|)^{2\lambda} h2^k}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi.$$

We can now combine terms that are raised to the 2λ and separate the numerator. Also, for each fixed k, we use the bound $|\phi(\xi-\eta)| \lesssim 2^{-Nk}$ for N that will be determined later.

$$\begin{aligned}
&\leq 2^k \int_{A_k} \frac{h|\phi(\xi-\eta)|}{|P(h\xi)|} \Big(\frac{h2^k + |P(h\eta)|}{h+|P(h\eta)|}\Big)^{2\lambda} d\xi \\
&\leq 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|} \Big(\frac{h2^k + |P(h\eta)|}{h+|P(h\eta)|}\Big)^{2\lambda} d\xi \\
&= 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|} \Big(\frac{h2^k}{h+|P(h\eta)|} + \frac{|P(h\eta)|}{h+|P(h\eta)|}\Big)^{2\lambda} d\xi.
\end{aligned}$$

We now use the fact that $h \leq 1$ to obtain $h2^k \lesssim 1$ and that $\frac{|P(h\eta)|}{h+|P(h\eta)|} \lesssim 1 \lesssim 2^k$.

$$\begin{aligned}
&\leq 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|} \Big(2^k + 1\Big)^{2\lambda} d\xi \\
&\lesssim 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|} \Big(2^k\Big)^{2\lambda} d\xi.
\end{aligned}$$

Lastly, to bound the remaining integral, we use 3.13.

$$\begin{aligned}
&= 2^k 2^{-Nk} 2^{2k\lambda} \int_{A_k} \frac{h}{|P(h\xi)|}\, d\xi \\
&\lesssim 2^k 2^{-Nk} 2^{2k\lambda} 2^{k(d+1)} \\
&= 2^{k(-N+2+2\lambda+d)}.
\end{aligned}$$

We now turn to the second case. Since $|h\eta| < 8$ and $32/h \leq 2^k$, we have $16/h \leq 2^{k-1}$. Therefore $16/h \leq |\xi-\eta|$ since $2^{k-1} \leq |\xi-\eta|$. Therefore, we have

$$2^{k-2} \leq 8/h = 16/h - 8/h \leq |\xi-\eta| - |\eta| \leq |\xi|$$

which tells us that $|P(h\xi)| \approx |h\xi|^2$. Therefore, for $\xi \in A_k$ we have $|P(h\xi)| \approx |h\xi|^2$. We will also use the fact that

$$|\xi| \leq |\eta| + |\xi-\eta| \leq 8/h + 2^k \lesssim 2^k.$$

Then,

$$h|\xi+\eta| \leq h(2|\xi| + |\xi-\eta||) \lesssim 8 + h2^k.$$

To bound our integral, we start by using our bound on θ, the bound on $|\xi-\eta|$, and the bound for $h|\xi+\eta|$.

$$\begin{aligned}
&\int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\lesssim \int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\xi)|+|P(h\eta)|)^{2\lambda}\, h\, 2^k\, (h2^k+8+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi.
\end{aligned}$$

We now use the fact that $|P(h\xi)| \approx |h\xi|^2$ along with factoring an h out of $h2^k + 9$.

$$\lesssim \int_{A_k} \frac{|\phi(\xi-\eta)|}{|h\xi|^2} \frac{(h+|h\xi|^2+|P(h\eta)|)^{2\lambda} h^2 2^k (2^k + \frac{9}{h})}{(h+|P(h\eta)|)^{2\lambda}} \, d\xi.$$

We now use the bounds that $|\xi| \lesssim 2^k$, $|\phi(\xi-\eta)| \lesssim 2^{-Nk}$, and $h \leq 1$.

$$\lesssim 2^{-Nk} \frac{h^2 2^k}{(h2^{k-2})^2} \int_{A_k} \frac{(h+h^2(2^k)^2+|P(h\eta)|)^{2\lambda}(2^k+\frac{9}{h})}{(h+|P(h\eta)|)^{2\lambda}} \, d\xi.$$

Since the integrand is independent of ξ we can bound the integral with the volume of the ball with radius 2^k.

$$\lesssim 2^{-Nk} \frac{2^k}{(2^{k-2})^2} 2^{dk} \frac{(h+h^2(2^k)^2+|P(h\eta)|)^{2\lambda}(2^k+\frac{9}{h})}{h^{2\lambda}}.$$

We now use the definition $P(h\eta)$ to simplify more along with the bound $9/h \lesssim 2^k$.

$$\begin{aligned}
&\lesssim 2^{-Nk}2^k2^{dk}\frac{(h+h^22^{2k}+|e\cdot h\eta|+|h\eta|^2)^{2\lambda}(2^k+2^k)}{h^{2\lambda}} \\
&\lesssim 2^{-Nk}2^k2^{dk}\frac{(h+h^22^{2k}+|e\cdot h\eta|+|h\eta|^2)^{2\lambda}}{h^{2\lambda}}.
\end{aligned}$$

Since $h \leq 1$, $|e\cdot h\eta| \leq |h\eta|$, and $|h\eta| \leq 8$, we can bound the rest using $8/h \lesssim 2^k$.

$$\begin{aligned}
&\lesssim 2^{-Nk}2^k2^{dk}(1+h2^{2k}+|h\eta|+h|\eta|^2)^{2\lambda} \\
&\lesssim 2^{-Nk}2^k2^{dk}(2^{2k}+2^{2k}+8+\frac{8^2}{h})^{2\lambda} \\
&\lesssim 2^{-Nk}2^k2^{dk}(2^{2k}+2^{2k}+2^{2k}+2^{2k})^{2\lambda} \\
&\lesssim 2^{-Nk}2^k2^{dk}2^{4k\lambda} \\
&= 2^{k(-N+d+1+4\lambda)}.
\end{aligned}$$

Turning to the third case, since $|h\eta| \geq 8$ and $2^k \leq |\eta|/8$, we have

$$\begin{aligned}
A_k &= \{\xi : 2^{k-1} \leq |\xi-\eta| \leq 2^k\} \\
&\subseteq \{\xi : |\xi-\eta| \leq |\eta|/8\} \\
&\subseteq \{\xi : 7|\eta|/8 \leq |\xi| \leq 9|\eta|/8\}
\end{aligned}$$

Therefore, we have that $|\xi| \sim |\eta|$ and since $|\eta| > 8/h$ we have that $|P(h\eta)| \approx |h\eta|^2$ and $|P(h\xi)| \approx |h\xi|^2 \approx |h\eta|^2$. We also have that $h|\xi+\eta| \approx h|\xi|$. Note also that

$$1 = \frac{h|\eta|}{h|\eta|} \lesssim \frac{1}{8}h|\xi|.$$

Using $1 + h|\xi + \eta| \approx h|\xi|$ and our bound on θ, we can start to bound the integral.

$$\int_{A_k} \frac{\phi(\xi - \eta)}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi - \eta|(h|\xi + \eta| + 1)}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi$$
$$\lesssim \int_{A_k} \frac{\phi(\xi - \eta)}{|P(h\xi)|} \frac{(h + |P(h\xi)| + |P(h\eta)|)^{2\lambda} h 2^k h|\xi|}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi.$$

We now use the fact that $|P(h\xi)| \approx h|\xi| \, h|\eta|$.

$$\lesssim \int_{A_k} \frac{\phi(\xi - \eta)}{h|\xi|h|\eta|} \frac{(h + |P(h\xi)| + |P(h\eta)|)^{2\lambda} h 2^k h|\xi|}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi.$$

Since $|\eta|$ is constant when integrating with respect to ξ we can pull it out front. We also cancel out one $|\xi|$ in the numerator and denominator. Using the bound $|\phi(\xi - \eta)| \lesssim 2^{-Nk}$, we simplify further.

$$\lesssim 2^{-Nk} 2^k \frac{1}{|\eta|} \int_{A_k} \frac{(h + |P(h\xi)| + |P(h\eta)|)^{2\lambda}}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi.$$

Now we use the fact that $|P(h\eta)| \approx |P(h\xi)|$. And the fact that we can bound the integral with the volume of a ball with radius 2^k.

$$\approx 2^{-Nk} 2^k \frac{1}{|\eta|} \int_{A_k} \frac{(h + |P(h\eta)|)^{2\lambda}}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi$$
$$\lesssim 2^{-Nk} 2^k \frac{1}{|\eta|} 2^{dk}.$$

We now use the fact that $h \leq 1$ and $1/|\eta| \lesssim 1$.

$$\lesssim 2^{-Nk} 2^k h 2^{dk}$$
$$\lesssim 2^{-Nk} 2^k 2^{dk}$$
$$= 2^{k(-N+1+d)}.$$

We now turn to the last case given that $\lambda \geq 0$. Since $|h\eta| \geq 8$ and $2^k \geq |\eta|/8$, we have

$$|\xi| \leq |\eta| + |\xi - \eta| \lesssim 2^k$$

which gives us

$$|P(h\xi)| \leq |e \cdot h\xi| + |h\xi|^2 \lesssim h2^k + h^2 2^{2k}$$

We also have

$$|\xi + \eta| \leq |\xi| + |\eta| \lesssim 2^k$$

which gives $|P(h\xi)| \lesssim h2^k + h^2 2^{2k}$. Using the bound on θ, $|\xi|$, $|\eta|$, $|P(h\xi)|$, $|P(h\eta)|$, and $|\phi(\xi-\eta)|$ we simplify the integral.

$$
\begin{aligned}
&= \int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\lesssim \int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\xi)|+|P(h\eta)|)^{2\lambda} h2^k(h2^k+2^k)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\lesssim 2^{-Nk}2^k2^k \int_{A_k} \frac{h}{|P(h\xi)|} \frac{(h+h2^k+h^2 2^{2k}+h2^k+h^2 2^{2k})^{2\lambda}}{h^{2\lambda}}\, d\xi.
\end{aligned}
$$

We now use the fact that $h \le 1$ and use 3.13 to simplify the rest,

$$
\begin{aligned}
&\lesssim 2^{-Nk}2^k2^k \int_{A_k} \frac{h}{|P(h\xi)|}(1+2^k+h2^{2k}+2^k+h2^{2k})^{2\lambda}\, d\xi \\
&\lesssim 2^{-Nk}2^k2^k \int_{A_k} \frac{h}{|P(h\xi)|}(2^{2k}+2^{2k}+2^{2k}+2^{2k}+2^{2k})^{2\lambda}\, d\xi \\
&\lesssim 2^{-Nk}2^k2^k2^{4k\lambda} \int_{A_k} \frac{h}{|P(h\xi)|}\, d\xi \\
&\lesssim 2^{-Nk}2^k2^k2^{4k\lambda}2^{k(d+1)} \\
&= 2^{k(-N+3+4\lambda+d)}
\end{aligned}
$$

Therefore, if we choose N such that all the integrals from cases one through four all all bounded by 2^{-k}. Specifically, we obtain

$$
\int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \lesssim 2^{-k}.
$$

Therefore, looking at the IV integral, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\lesssim \sum_{k=0}^{\infty} \int_{A_k} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\lesssim \sum_{k=0}^{\infty} 2^{-k} \\
&< \infty.
\end{aligned}
$$

Therefore we satisfy (3.19). Now we turn to the case when $\lambda < 0$. The big difference in this case is how we bound θ. We will again break into four cases, and θ will be bounded dependent on the case. Since $\lambda < 0$ we have different inequalities to work with. We now want to choose θ small to make $\theta^{2\lambda}$ large. We note that $h \le \theta$, but this only helps in a few cases. We will sometimes need better estimates on θ. We will again look at the following cases.

1. $|h\eta| < 8$ and $2^k \leq |\eta|/8$
2. $|h\eta| < 8$ and $2^k \geq |\eta|/8$
3. $|h\eta| \geq 8$ and $2^k \leq |\eta|/8$
4. $|h\eta| \geq 8$ and $2^k \geq |\eta|/8$

In the first cases, since $|h\eta| < 8$ and $2^k \leq |\eta|/8$, we have

$$\begin{aligned} A_k &= \{\xi : 2^{k-1} \leq |\xi - \eta| \leq 2^k\} \\ &\subseteq \{\xi : |\xi - \eta| \leq |\eta|/8\} \\ &\subseteq \{\xi : 7|\eta|/8 \leq |\xi| \leq 9|\eta|/8\} \end{aligned}$$

which gives us that $|\xi| \approx |\eta$ so $|P(h\eta)| \approx |P(h\xi)|$. Then, we see that $\theta \approx h + |P(h\eta)|$. Lastly, we see that

$$\begin{aligned} h|\xi + \eta| + 1 &\leq h(2|\eta| + |\xi - \eta|) + 1 \\ &\lesssim 16 + h2^k + 1 \\ &\lesssim 2^k \end{aligned}$$

Therefore

$$\begin{aligned} \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi - \eta|(h|\xi + \eta| + 1)}{(h + |P(h\eta)|)^{2\lambda}}\, d\xi &\lesssim \int_{A_k} \frac{\phi(\xi - \eta)}{|P(h\xi)|} \frac{\theta^{2\lambda} h 2^k 2^k}{(h + |P(h\eta)|)^{2\lambda}}\, d\xi \\ &\lesssim 2^k 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|} \frac{(h + |P(h\eta)|)^{2\lambda}}{(h + |P(h\eta)|)^{2\lambda}}\, d\xi \\ &= 2^k 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|}\, d\xi \\ &\lesssim 2^k 2^k 2^{-Nk} 2^{k(d+1)} \\ &= 2^{k(-N+d+3)}. \end{aligned}$$

In the second case, we have $|h\eta| < 8$ and $2^k \geq |\eta|/8$. Here, we use the fact that $h \leq \theta$. We see that

$$\begin{aligned} \Big(\frac{\theta}{h + |P(h\eta)|}\Big)^{2\lambda} &\lesssim \Big(\frac{h}{h + h|\eta| + h^2|\eta|^2}\Big)^{2\lambda} \\ &\leq 1 \end{aligned}$$

And,

$$|\xi + \eta| \leq 2|\eta| + |\xi - \eta| \lesssim 2^k + 2^k \lesssim 2^k.$$

Therefore, we have

$$\begin{aligned} \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi - \eta|(h|\xi + \eta| + 1)}{(h + |P(h\eta)|)^{2\lambda}}\, d\xi &\lesssim \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} h2^k(h2^k + 1)\, d\xi \\ &\lesssim 2^{-Nk} 2^{2k} \int_{A_k} \frac{h}{|P(h\xi)|}\, d\xi \\ &\lesssim 2^{-Nk} 2^{2k} 2^{k(d+1)} \\ &= 2^{k(-N+3+d)}. \end{aligned}$$

Turning to the third case, since $|h\eta| \geq 8$ and $2^k \leq |\eta|/8$, we have

$$\begin{aligned} A_k &= \{\xi : 2^{k-1} \leq |\xi - \eta| \leq 2^k\} \\ &\subseteq \{\xi : |\xi - \eta| \leq |\eta|/8\} \\ &\subseteq \{\xi : 7|\eta|/8 \leq |\xi| \leq 9|\eta|/8\} \end{aligned}$$

Which gives us that $|\eta| \approx |\xi|$ so $\theta \approx h + |P(h\eta)|$. Therefore, we have that $|\xi| \approx |\eta|$ and since $|\eta| > 8/h$ we have that $|P(h\eta)| \approx |h\eta|^2$ and $|P(h\xi)| \approx |h\xi|^2 \approx |h\eta|^2$. We also have that $h|\xi + \eta| \approx h|\xi|$. Note also that

$$1 = \frac{h|\eta|}{h|\eta|} \lesssim \frac{1}{8} h|\xi|.$$

So, $1 + h|\xi + \eta| \approx h|\xi|$. Therefore, we have

$$\begin{aligned} \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi - \eta|(h|\xi + \eta| + 1)}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi &\lesssim \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} h 2^k h|\xi| \, d\xi \\ &\lesssim 2^{-Nk} 2^k \int_{A_k} \frac{1}{|\xi|} \, d\xi \\ &\lesssim 2^{-Nk} 2^k \frac{1}{|\eta|} \int_{A_k} \, d\xi \\ &\lesssim 2^{-Nk} 2^k 2^{dk} \\ &= 2^{k(-N+1+d)} \end{aligned}$$

Lastly, in case four, we have $|h\eta| \geq 8$ and $2^k \geq |\eta|/8$. Here, again, we use the fact that $h \leq \theta$. And, we note that

$$|\xi| \leq |\eta| + |\eta - \xi| \lesssim 2^k$$

and the fact that

$$|P(h\eta)| \approx |h\eta|^2.$$

Thus,

$$\begin{aligned} \Big(\frac{\theta}{h + |P(h\eta)|}\Big)^{2\lambda} &\lesssim \Big(\frac{h}{h + |h\eta|^2}\Big)^{2\lambda} \\ &\leq 1 \end{aligned}$$

And, as we saw before,

$$h|\xi - \eta|(h|\xi + \eta| + 1) \lesssim h 2^k 2^k.$$

Therefore,

$$\begin{aligned} \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi - \eta|(h|\xi + \eta| + 1)}{(h + |P(h\eta)|)^{2\lambda}} \, d\xi &\lesssim 2^{-4k\lambda} \int_{A_k} \frac{|\phi(\xi - \eta)|}{|P(h\xi)|} h 2^k 2^k \, d\xi \\ &\lesssim 2^{-4k\lambda} 2^k 2^k 2^{-Nk} \int_{A_k} \frac{h}{|P(h\xi)|} \, d\xi \\ &\lesssim 2^{-4k\lambda} 2^k 2^k 2^{-Nk} 2^{k(d+1)} \\ &= 2^{k(-N-4\lambda+3+d)} \end{aligned}$$

Now, to finish this case of IV, we choose N such that

$$\int_{A_k} \frac{\phi(\xi-\eta)}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \lesssim 2^{-k}$$

which gives us

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\phi(\xi-\eta)}{|P(h\xi)|} \Big| \frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}} \Big| \, d\xi \\
&\lesssim \sum_{k=0}^{\infty} \int_{A_k} \frac{\phi(\xi-\eta)}{|P(h\xi)|} \frac{\theta^{2\lambda} h|\xi-\eta|(h|\xi+\eta|+1)}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi \\
&\lesssim \sum_{k=0}^{\infty} 2^{-k} \\
&< \infty
\end{aligned}$$

So,

$$\int_{\mathbb{R}^d} \frac{\phi(\xi-\eta)}{|P(h\xi)|} \Big| \frac{(h+|P(h\xi)|)^{2\lambda+1}-(h+|P(h\eta)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}} \Big| \, d\xi \le C$$

for any λ. Therefore, we have shown that

$$\sup_{\eta} \int_{\mathbb{R}^d} \frac{|\phi(\xi-\eta)|}{|P(h\xi)|} \frac{(h+|P(h\xi)|)^{2\lambda+1}}{(h+|P(h\eta)|)^{2\lambda}}\, d\xi < \infty$$

which completes the proof. □

Corollary 3.20. *For any $\lambda \in \mathbb{R}$, we have that $I_\phi : X^\lambda \to X^{\lambda+1}$ is a bounded right inverse of $P(hD)$ inside Ω.*

Proof. We can write $I_\phi = J_\phi \frac{|P(hD)|}{P(hD)} J_\phi^*$ we get that $I_\phi : X^\lambda \to X^{\lambda+1}$ is a bounded operator. To show that I_ϕ is a right inverse to $P(hD)$ we need to first show that $[D, \phi] = 0$ in Ω. Note that $D\phi = 0$ in Ω since $\phi = 1$ in a neighborhood of $\bar{\Omega}$. Therefore,

$$[D,\phi]u = D(\phi u) - \phi D u = u D\phi + \phi D u - \phi D_u = 0 \qquad \text{in } \Omega.$$

Now, to show I_ϕ is a right inverse, we have

$$\begin{aligned}
P(hD) I_\phi u &= P(hD)\phi P(hD)^{-1} \phi u \\
&= \phi P(hD) P(hD)^{-1} \phi u \\
&= \phi^2 u \\
&= u, \qquad \text{in } \Omega
\end{aligned}$$

since $\phi^2 = 1$ in Ω. This completes the proof. □

We are trying to solve $L_\zeta \psi = f$ where L_ζ is as in (3.11). It is sufficient to solve the following integral equation for ψ due to m applications of 3.20,

$$\psi + h^{2m} I_\phi^m (Q \cdot (\frac{\zeta}{ih} + D)\phi + q\psi) = I_\phi^m f. \tag{3.21}$$

To show this integral equation has a solution, I will show that the map

$$\psi \to h^{2m} I_\phi^m (Q \cdot (\frac{\zeta}{ih} + D)\psi) + h^{2m} I_\phi^m (q\psi) \tag{3.22}$$

is a contraction on $X^{m/2}$ for suitable conditions on q, Q, and h.

3.3 Estimates

A natural space for the polyharmonic operator is the Sobolev space $W^{m,2}(\mathbb{R}^d)$. In contrast, a family spaces for the operator $P(hD)$ are the X^λ spaces. We would like to give estimates of different operators between X^λ spaces, as well as different Sobolev spaces. This will pave the way to showing that a solution to the integral equation (3.21) is also a weak solution to (2.1). These estimates will also help in showing uniqueness of the coefficients.

To start, we will prove a useful estimate that relates smoothness in the X^λ spaces with smoothness in L^2 Sobolev spaces. Suppose that $0 < h < 1$ and $0 \le s \le 2\lambda$, then we will want to show that

$$\sup_{\xi \in \mathbb{R}^d} \frac{\langle \xi \rangle^s}{(h + |P(h\xi)|)^\lambda} \lesssim h^{-\lambda - s}. \tag{3.23}$$

To prove this estimate, we break into two cases. The first case is when $|h\xi| < 8$. In this case, $P(h\xi)$ may be zero at times. We use the fact that $|\langle \xi \rangle| \le h^{-1}$ and $\frac{1}{h+|P(h\xi)|} \le h$ to give give us

$$\frac{\langle \xi \rangle^s}{(h + |P(h\xi)|)^\lambda} \lesssim h^{-s-\lambda}.$$

The second case we look at is when $|h\xi| \ge 8$. What happens here is that $|P(h\xi)| \sim |h\xi|^2$. This gives us

$$\frac{\langle \xi \rangle^s}{(h + |P(h\xi)|)^\lambda} \lesssim \langle \xi \rangle^{s-2\lambda} h^{-2\lambda} \le h^{-s} \langle \xi \rangle^{s-2\lambda} h^{s-2\lambda} \lesssim h^{-s}.$$

This is a better estimate since $0 < h < 1$. Combining both cases proves (3.23). This estimate will allow us to show relationships between a variety of function spaces.

Proposition 3.24. *If $|\alpha| \le 2(\lambda_2 - \lambda_1)$, then*

$$\|D^\alpha u\|_{X^{\lambda_1}} \lesssim h^{-|\alpha| + \lambda_1 - \lambda_2} \|u\|_{X^{\lambda_2}}. \tag{3.25}$$

and if $0 \le s \le 2\lambda$, then

$$\|u\|_{W^{s,2}(\mathbb{R}^d)} \lesssim h^{-s-\lambda} \|u\|_{X^\lambda} \tag{3.26}$$

$$\|u\|_{X^{-\lambda}} \lesssim h^{-s-\lambda} \|u\|_{W^{-s,2}(\mathbb{R}^d)}. \tag{3.27}$$

Proof. We will first prove (3.25) which will help prove (3.26). We have

$$\begin{aligned}
\|D^\alpha u\|^2_{X^{\lambda_1}} &= \int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha}}{(h+|P(h\xi)|)^{2\lambda_1}} |\hat{u}|^2 \, d\xi \\
&= \int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha}}{(h+|P(h\xi)|)^{2\lambda_1-2\lambda_2}} \frac{1}{(h+|P(h\xi)|)^{2\lambda_2}} |\hat{u}|^2 \, d\xi \\
&\le \sup_\xi \Big(\frac{|\xi|^{2\alpha}}{(h+|P(h\xi)|)^{2\lambda_1-2\lambda_2}} \Big) \int_{\mathbb{R}^d} \frac{1}{(h+|P(h\xi)|)^{2\lambda_2}} |\hat{u}|^2 \, d\xi \\
&\lesssim h^{-2|\alpha|-2\lambda_1+2\lambda_2} \, \|u\|^2_{X^{\lambda_2}} .
\end{aligned}$$

This proves our first estimate. Now, to show (3.26), we have

$$\begin{aligned}
\|u\|^2_{W^{s,2}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\hat{u}|^2 \, d\xi \\
&= \int_{\mathbb{R}^d} \frac{\langle\xi\rangle^{2s}}{(h+|P(h\xi)|)^{2\lambda}} (h+|P(h\xi)|)^{2\lambda} |\hat{u}|^2 \, d\xi \\
&\lesssim \sup_\xi \Big(\frac{\langle\xi\rangle^{2s}}{(h+|P(h\xi)|)^{2\lambda}} \Big) \int_{\mathbb{R}^d} (h+|P(h\xi)|)^{2\lambda} |\hat{u}|^2 \, d\xi \\
&\lesssim h^{-2s-2\lambda} \, \|u\|^2_{X^{\lambda}} .
\end{aligned}$$

Lastly, to prove (3.27), we notice that the embedding of $W^{-s,2}(\mathbb{R}^d)$ into $X^{-\lambda}$ is the adjoint of the embedding $X^\lambda \subset W^{s,2}$, the estimate (3.27) follows from (3.26) which completes the proof. □

Proposition 3.28. *Suppose $f \in L^\infty(\mathbb{R}^d)$, ζ, ζ_1 and ζ_2 are in $\mathcal{V}$ (see (3.8)) and $0 < h \le 1$. Then, when $0 \le |\alpha| + |\beta| \le 2\lambda$, we have the trilinear estimate*

$$|\langle D^\alpha f D^\beta u, v\rangle| \lesssim h^{-2\lambda-|\alpha|-|\beta|} \|f\|_\infty \|u\|_{X^\lambda_{h\zeta_1}} \|v\|_{X^\lambda_{h\zeta_2}} . \tag{3.29}$$

Let $Tu = (D^\alpha f)(D + \frac{\zeta}{ih})^\beta u$ with $f \in L^\infty$ and assume $|\alpha| + |\beta| \le 2\lambda$, then

$$\|Tu\|_{X^{-\lambda}_{h\zeta_2}} \lesssim \|f\|_\infty h^{-2\lambda-|\alpha|-|\beta|} \|u\|_{X^\lambda_{h\zeta_1}} . \tag{3.30}$$

If in addition, we have $f \in C^\theta(\mathbb{R}^d)$ and $|\alpha| \ge 1$, then we have

$$\|Tu\|_{X^{-\lambda}_{h\zeta_2}} \lesssim \|f\|_{C^\theta(\mathbb{R}^d)} h^{-2\lambda-|\alpha|-|\beta|+\theta} \|u\|_{X^\lambda_{h\zeta_1}} . \tag{3.31}$$

Proof. To start, let us assume that u and v are in the Schwartz class. It suffices to consider functions in the Schwartz class since the Schwartz class is dune in any X^λ

space. We have that

$$
\begin{aligned}
|\langle D^\alpha f D^\beta u, v\rangle| &= |(-1)^{|\alpha|} \int_{\mathbb{R}^d} f D^\alpha \Big((D^\beta u) v \Big)\, dx| \\
&\leq \int_{\mathbb{R}^d} |f| \sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!} |D^{\alpha_1+\beta} u| |D^{\alpha_2} v|\, dx \\
&\lesssim \|f\|_\infty \sum_{\alpha_1+\alpha_2=\alpha} \left\| D^{\alpha_1+\beta} u \right\|_{L^2(\mathbb{R}^d)} \left\| D_2^\alpha v \right\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \|f\|_\infty \sum_{\alpha_1+\alpha_2=\alpha} h^{-\lambda-|\alpha_1|-|\beta|} \|u\|_{X^\lambda_{h\zeta_1}} h^{-\lambda-|\alpha_2|} \|v\|_{X^\lambda_{h\zeta_2}} \\
&\lesssim \|f\|_\infty h^{-2\lambda-|\alpha|-|\beta|} \|u\|_{X^\lambda_{h\zeta_1}} \|v\|_{X^\lambda_{h\zeta_2}} .
\end{aligned}
$$

The first line is by definition, the next three lines use product rule, Cauchy Schwarz, and inclusion. The last few lines use (3.26). It is important to note that we allow different values of ζ for the spaces $X^\lambda_{h\,\zeta_j}$. To finish proving (3.29), we take the supremum over all choices of v with norm 1 to obtain the desired estimate.

Transitioning to (3.30), we see this operator in (3.11) where $\beta = 1$ or 0. To prove this estimate, we use (3.29). We have

$$
\begin{aligned}
|\langle Tu, v\rangle| &= \left| (-1)^{|\alpha|} \int_{\mathbb{R}^d} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} f(x) \,|\frac{\zeta}{ih}|^{\beta_2} D^\alpha (v\, D^{\beta_1} u)\, dx \right| \\
&\lesssim \|f\|_\infty \sum_{\beta_1+\beta_2=\beta} h^{-|\beta_2|} \int_{\mathbb{R}^d} |D^\alpha (v\, D^{\beta_1} u)|\, dx
\end{aligned}
$$

We now apply the same techniques as the first estimate to obtain

$$
\begin{aligned}
\|f\|_\infty \sum_{\beta_1+\beta_2=\beta} h^{-|\beta_2|} \int_{\mathbb{R}^d} |D^\alpha (v\, D^{\beta_1} u)|\, dx &\lesssim \|f\|_\infty \sum_{\beta_1+\beta_2=\beta} h^{-|\beta_2|} h^{-|\alpha|-|\beta_1|-2\lambda} \\
&\lesssim \|f\|_\infty h^{-|\alpha|-|\beta|-2\lambda}.
\end{aligned}
$$

This proves (3.30). The question we now ask is, what if f was a little more smooth here compared to just bounded. We can use the extra smoothness to increase the power of h by the order of the smoothness. Let $f \in C^\theta(\mathbb{R}^d)$ and $\alpha > 1$. Then, we can decompose f into an infinitely smooth part and a part with small L^∞ norm. We call this decomposition $f = f_h + f^h$ where f_h is a mollification of f. We have that $\|D^\alpha f_h\|_\infty \lesssim h^{\theta-|\alpha|}$ and $\left\| f^h \right\|_\infty \lesssim h^\theta$. Since $\alpha > 1$, we decompose $\alpha = \alpha_1 + \alpha_2$, where $|\alpha_1| = 1$. We are now able to integrate by parts with α_1 and focusing on f_h to obtain

$$
\begin{aligned}
&\left| (-1)^{|\alpha|} \int_{\mathbb{R}^d} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} f_h(x) \,|\frac{\zeta}{ih}|^{\beta_2} D^\alpha (v\, D^{\beta_1} u)\, dx \right| \\
&\quad = \left| (-1)^{\alpha+\alpha_1} \int_{\mathbb{R}^d} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} D^{\alpha_1} f_h(x) \,|\frac{\zeta}{ih}|^{\beta_2} D^{\alpha_2} (v\, D^{\beta_1} u)\, dx \right| \\
&\quad \lesssim h^{\theta-|\alpha_1} h^{-|\alpha_2|-|\beta|-2\lambda} \\
&\quad = h^{\theta-|\alpha|-|\beta|-2\lambda}.
\end{aligned}
$$

Now, running through the argument of (3.30) but using the bound $\left\|f^h\right\|_\infty \lesssim h^\theta$, we obtain

$$\begin{aligned}\Big|(-1)^{|\alpha|}\int_{\mathbb{R}^d}\sum_{\beta_1+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}\, f^h(x)\,|\frac{\zeta}{ih}|^{|\beta_2|}\, D^\alpha(v\, D^{\beta_1}u)\, dx\Big| \\ = \left\|f^h\right\|_\infty h^{-|\alpha|-|\beta|-2\lambda} \\ \lesssim h^{\theta-|\alpha|-|\beta|-2\lambda}.\end{aligned}$$

This completes the estimate and the proof. □

The last estimate in this section shows that multiplying by a smooth function preserves the norm in X^λ spaces, *i.e.* $\|uv\|_{X^\lambda} \lesssim \|u\|_{X^\lambda}$ for any smooth v with one added caveat.

Proposition 3.32. *Let $v \in C^\infty(\mathbb{R}^d)$ and assume that v and all derivatives of v are bounded on $\mathbb{R}^d$. Then for $\lambda \in \mathbb{R}$, we have*

$$\|uv\|_{X^\lambda} \lesssim \|u\|_{X^\lambda}. \tag{3.33}$$

The implied constant in (3.33) *depends on λ and $\|D^\alpha v\|_\infty$ for $|\alpha| \le c(|\lambda|)$.*

Proof. We begin by giving a proof by induction for (3.33) when $\lambda \ge 0$ is an integer. Since $X^0 = L^2$, the base case $\lambda = 0$ is elementary. Next, we assume (3.33) holds for $\lambda \ge 0$ and establish the inequality for $\lambda + 1$. Using Plancherel's theorem we have $\|uv\|_{X^{\lambda+1}} \le h\|uv\|_{X^\lambda} + \|P(hD)(uv)\|_{X^\lambda}$. Then using the product rule, we have

$$\begin{aligned}\|uv\|_{X^{\lambda+1}} &\lesssim h\|uv\|_{X^\lambda} + \|vP(hD)u\|_{X^\lambda} \\ &\quad + h\|uDv\|_{X^\lambda} + h^2\|u\Delta v\|_{X^\lambda} + h^2\|Du\cdot Dv\|_{X^\lambda} \\ &\lesssim (h+h^2)\|u\|_{X^\lambda} + \|P(hD)u\|_{X^\lambda} + h^2\|Du\|_{X^\lambda} \\ &\lesssim \|u\|_{X^{\lambda+1}}.\end{aligned}$$

The second inequality uses our induction hypothesis, the third inequality follows from (3.25) and uses that $0 < h \le 1$.

Next, we may use complex interpolation to establish that the map $u \to vu$ is bounded on X^λ for all $\lambda \ge 0$. Finally, the adjoint of this map will also be bounded on $X^{-\lambda}$ which gives (3.33) for $\lambda < 0$. □

3.4 Existence of CGO Solutions

In this section, we will show that (3.21) has a solution $\psi \in X^{m/2}$. Using the estimate (3.26) implies that ψ is in $W^{m,2}(\mathbb{R}^d)$. This gives us that ψ is also a weak solution to (2.1). We will start with assumptions on q and Q which follow from Proposition 8.3 and Theorem 2.4.

Theorem 3.34. *Suppose that*

$$Q_j = \sum_{|\beta|\le m-1} D^\beta Q_{j\beta}, \qquad q = \sum_{|\beta|\le m} D^\beta q_\beta \tag{3.35}$$

with $Q_{j\beta}, q_\beta$ in $C^\theta(\mathbb{R}^d)$ for some $\theta \in (0,1]$ and each function is supported in $\bar{\Omega}$. Then there is a value h_0 depending on θ and the Hölder norms of the functions $Q_{j\beta}, q_\beta$ so that if $0 < h \le h_0$, we may find a function $\psi \in X^{m/2}$ which satisfies $L_\zeta \psi = f$ in Ω. The solution will satisfy the estimate

$$\|\psi\|_{X^{m/2}} \lesssim \|f\|_{X^{-m/2}}.$$

Proof. We will show this is true for one $D^\beta Q_\beta$ and $D^\alpha q_\alpha$. This is enough since Q and q are decomposed into finite sums, where the number of terms is dependent on m. We are left with showing

$$\psi \to h^{2m} I_\phi^m((D^\beta Q_\beta)\cdot(\frac{\zeta}{i\,h} + D)\psi) + h^{2m} I_\phi^m((D^\alpha q_\alpha)\psi)$$

is a contraction map. We will look at the bounds of two terms. The first term uses Proposition 3.28 and we obtain the following bound.

$$\begin{aligned}\left\| h^{2m} I_\phi^m((D^\beta Q_\beta)\cdot(\frac{\zeta}{i\,h} + D)\psi)\right\|_{X^{m/2}} &\lesssim h^{2m} \left\|(D^\beta Q_\beta)\cdot(\frac{\zeta}{i\,h} + D)\psi\right\|_{X^{-m/2}} \\ &\lesssim h^{2m} h^{-m-|\beta|-1+\theta} \|\psi\|_{X^{m/2}} \\ &= h^{m-1-|\beta|+\theta} \|\psi\|_{X^{m/2}}\end{aligned}$$

where θ is from the decomposition of Q since Q_β will be Hölder continuous. Since $|\beta| \le m-1$, we have that we can pick an h_0 sufficiently small, dependent on the Hölder norms of $Q_{\beta\, j}$ and θ to guarantee this bound is less than $1/4$. Similarly, the second term is bounded by

$$\begin{aligned}\left\| h^{2m} I_\phi^m((D^\alpha q_\alpha)\psi)\right\|_{X^{m/2}} &\lesssim h^{2m} \|(D^\alpha q_\alpha)\psi\|_{X^{-m/2}} \\ &\lesssim h^{2m} h^{-m-|\alpha|+\theta} \|\psi\|_{X^{m/2}} \\ &= h^{m-|\alpha|+\theta} \|\psi\|_{X^{m/2}}\end{aligned}$$

where this θ comes from the decomposition of q since q_α will be Hölder continuous. Since $|\alpha| \le m$, we have that we can pick an h_0 sufficiently small, dependent on the Hölder norms of q_α and θ to guarantee this bound is less than $1/4$. Therefore, adding both terms together, we have that

$$\left\| h^{2m} I_\phi^m((D^\beta Q_\beta)\cdot(\frac{\zeta}{i\,h} + D)\psi) + h^{2m} I_\phi^m((D^\alpha q_\alpha)\psi)\right\|_{X^{m/2}} \le \frac{1}{4} \|\psi\|_{X^{m/2}}.$$

This gives us that (3.21) has a solution since

$$I + I_\phi^m((D^\beta Q_\beta)\cdot(\frac{\zeta}{i\,h} + D) + h^{2m} I_\phi^m((D^\alpha q_\alpha)$$

is invertible on $X^{m/2}$. This solution will thus satisfy

$$\|\psi\|_{X^{m/2}} \lesssim \|f\|_{X^{-m/2}}$$

after m applications of Corollary 3.20. To show that ψ is a weak solution of the differential equation $L_\varsigma \psi = f$ we need to show that it is a solution in the sense of distributions, which follows from Corollary 3.20 and that ψ is in $W^{m,2}(\Omega)$. We use (3.26) to show that $X^{m/2} \subset W^{m,2}(\mathbb{R}^d)$. □

Chapter 4 Averaging Estimate

This chapter will cover an important estimate for $\|\psi\|_{X^{m/2}}$ involving $\|f\|_{X^{-\lambda}}$ for $\lambda > 0$. In the previous chapter, we proved many such estimates. Using strictly those estimates, we are able to obtain similar results to Assylbekov and Iyer [1]. One factor that all the previous esimates use is that they are using a fixed ζ. The estimate (3.27) of Proposition 3.24 gives an estimate for the norm of a distribution in $X^{-\lambda}$ in terms of the $W^{-s,2}$ norm. However, as observed by Haberman and Tataru [14, Lemma 3.1], there is an improvement available if we look at averages of $X^{\lambda}_{\tau\zeta}$ norms over certain families of vectors ζ and for τ in an interval $[h, 2h]$. The improvement is that the averages grow less rapidly in h. This improved behavior is used in the main theorem to recover q and Q by studying limits as h tends to zero of expressions defined using our solutions.

4.1 Average

In this section, we describe the averages we are interested in and give an extension of Haberman and Tataru's estimate.

Fix $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ and let $\mu_1, \mu_2 \in \mathbb{R}^d$ such that $\xi_0 \cdot \mu_1 = \xi_0 \cdot \mu_2 = 0$ and $\mu_i \cdot \mu_j = \delta_{ij}$. We set

$$\mu_1(\theta) = \mathrm{Re}(e^{i\theta}(\mu_1 + i\mu_2)) = \mu_1 \cos(\theta) - \mu_2 \sin(\theta)$$
$$\mu_2(\theta) = \mathrm{Im}(e^{i\theta}(\mu_1 + i\mu_2)) = \mu_1 \sin(\theta) + \mu_2 \cos(\theta)$$

and then for τ small we put

$$\begin{aligned} \zeta^1(\tau,\theta) &= \mu_1(\theta) + i\sqrt{1 - \frac{\tau^2|\xi_0|^2}{4}}\mu_2(\theta) - i\frac{\tau\xi_0}{2}, \\ \zeta^2(\tau,\theta) &= -\mu_1(\theta) - i\sqrt{1 - \frac{\tau^2|\xi_0|^2}{4}}\mu_2(\theta) - i\frac{\tau\xi_0}{2}. \end{aligned} \tag{4.1}$$

Throughout this section, we will let $P(\tau\xi) = |\tau\xi|^2 - 2i\zeta^k(\tau,\theta)\cdot\tau\xi$ for $k = 1$ or 2. We will choose $h > 0$ such that $h \le 1/(4|\xi_0|)$ and we will have that $h \le \tau \le 2h$. For $\xi \in \mathbb{R}^d$, we define $\xi^{\perp}$ to be the projection of ξ onto the plane spanned by $\{\mu_1, \mu_2\}$. We will now prove an estimate for the average of the norms

$$\|f\|_{X^{-\lambda}_{\tau\zeta^k(\tau,\theta)}}, \qquad h \le \tau \le 2h,\ \theta \in [0, 2\pi] \tag{4.2}$$

We begin with a technical lemma which gives an estimate for the Jacobian of a change of variables that will be used in studying the average of the norms in (4.2).

Lemma 4.3. *Let $\xi, \xi_0 \in \mathbb{R}^d \setminus \{0\}$, and $0 < \tau \le \min(1, 1/(2|\xi_0|))$ and let $P(\tau\xi)$ be as defined after (4.1). We define a change of variables $z(\tau,\theta) = (z_1(\tau,\theta), z_2(\tau,\theta))$ by*

$$z_1(\tau,\theta) = \frac{Re\ P(\tau\xi)}{\tau^2} \quad and \quad z_2(\tau,\theta) = \frac{Im\ P(\tau\xi)}{\tau^2}. \tag{4.4}$$

Then the Jacobian J for this change of variables has the lower bound

$$\frac{2|\xi^\perp|^2}{\tau^3} \leq J.$$

Proof. We give the proof for ζ^1. The case of ζ^2 is similar and we omit the details. A calculation shows that

$$J = \left|\det\begin{pmatrix} \frac{-2\mu_2(\theta)\cdot\xi}{\tau^2}\Big(\sqrt{1-\frac{\tau^2|\xi_0|^2}{4}} + \frac{\tau^2|\xi_0|^2}{4\sqrt{1-\frac{\tau^2|\xi_0|^2}{4}}}\Big) & \frac{2\mu_1(\theta)\cdot\xi}{\tau}\sqrt{1-\frac{\tau^2|\xi_0|^2}{4}} \\ 2\mu_1(\theta)\cdot\xi/\tau^2 & 2\mu_2(\theta)\cdot\xi/\tau \end{pmatrix}\right|$$

$$= \Big| -\frac{|\xi_0|^2}{\sqrt{1-\frac{\tau^2|\xi_0|^2}{4}}}\frac{|\mu_2(\theta)\cdot\xi|^2}{\tau} - 4\frac{\sqrt{1-\frac{\tau^2|\xi_0|^2}{4}}}{\tau^3}|\mu_2(\theta)\cdot\xi|^2 - 4\frac{\sqrt{1-\frac{\tau^2|\xi_0|^2}{4}}}{\tau^3}|\mu_1(\theta)\cdot\xi|^2\Big|$$

$$= \frac{4|\xi^\perp|^2}{\tau^3}\Big(|\xi_0|^2\tau^2\frac{1}{4\sqrt{1-\frac{|\tau\xi_0|^2}{4}}}\frac{|\mu_2(\theta)\cdot\xi|^2}{|\xi^\perp|^2} + \sqrt{1-\frac{\tau^2|\xi_0|^2}{4}}\Big).$$

The condition that $\tau \leq 1/(2|\xi_0|)$, gives at least the following bound

$$\frac{1}{2} \leq \sqrt{1-\frac{\tau^2|\xi_0|^2}{4}} \leq 1.$$

Therefore we can conclude

$$2\frac{|\xi^\perp|^2}{\tau^3} \leq J.$$

□

Proposition 4.5. *Let $P(\tau\xi)$ depend τ and θ as defined in the beginning of this section. If $0 < \epsilon \leq 1$ and $h\langle\xi_0\rangle^2 \lesssim 1$, we have*

$$\frac{1}{h}\int_0^{2\pi}\int_h^{2h}\frac{1}{(\tau+|P(\tau\xi)|)^{2-2\epsilon}}\,d\tau\,d\theta \lesssim \frac{1}{\tau^{4-4\epsilon}\langle\xi\rangle^{4-4\epsilon}}.$$

Proof. We consider three cases depending on the value of $|\xi|$. Case 1: $|\xi| \leq \max(16|\xi_0|, 1)$ Case 2a: $|\xi| > \max(16|\xi_0|, 1)$ and $|\xi|^2 \leq 16|\xi^\perp|/h$, Case 2b: $|\xi| > \max(16|\xi_0|, 1)$ and $|\xi|^2 > 16|\xi^\perp|/h$.

In Case 1, we use that $(\tau + |P(\tau\xi)|)^{2\epsilon-2} \leq h^{2\epsilon-2}$ and $\langle\xi\rangle \lesssim \langle\xi_0\rangle$ to obtain that

$$\frac{1}{h}\int_0^{2\pi}\int_h^{2h}\frac{1}{(\tau+|P(\tau\xi)|)^{2-2\epsilon}}\,d\tau\,d\theta \lesssim \frac{1}{h^{2-2\epsilon}} \lesssim \frac{\langle\xi_0\rangle^{4-4\epsilon}h^{2-2\epsilon}}{h^{4-4\epsilon}\langle\xi\rangle^{4-4\epsilon}}$$

which gives the desired result if we require that $h^{2-2\epsilon}\langle\xi_0\rangle^{4-4\epsilon} \lesssim 1$.

In Case 2a), we will use the change of variable from Lemma 4.3. The conditions $|\xi|^2 \leq 16|\xi^{\perp}|/h$ and $|\xi| \geq 16|\xi_0|$ imply that $|z| = |z_1 + iz_2| \leq 21|\xi^{\perp}|/h$. Using this observation and the estimate for the Jacobian from Lemma 4.3 gives

$$\frac{1}{h}\int_0^{2\pi}\int_h^{2h} \frac{1}{(\tau + |P(\tau\xi)|)^{2-2\epsilon}}\, d\tau\, d\theta \lesssim \frac{h^2}{|\xi^{\perp}|^2}\int_{|\tau|\leq 21|\xi^{\perp}|/h} \frac{1}{(h^2|z|)^{2-2\epsilon}}\, dz$$
$$\lesssim \frac{1}{h^{4-4\epsilon}}\left(\frac{h}{|\xi^{\perp}|}\right)^{2-2\epsilon} \lesssim \frac{1}{h^{4-4\epsilon}\langle\xi\rangle^{4-4\epsilon}}.$$

The last step uses that $|\xi| \geq 1$ and $|\xi|^2 \leq 16|\xi^{\perp}|/h$.

Finally, in Case 2b), our conditions on ξ imply that $|P(\tau\xi)| \geq h^2\langle\xi\rangle^2$ which quickly gives the conclusion of the Proposition. □

4.2 Application

We will use Proposition 4.5 to establish an estimate relating the average of the $X^{-\lambda}$-norm of a distribution to the norm in an L^2-Sobolev space.

Theorem 4.6. *For any $0 < \epsilon < 1$ and $2 - 2\epsilon \leq s \leq 2\lambda$ we have*

$$\frac{1}{h}\int_0^{2\pi}\int_h^{2h} \|f\|^2_{X^{-\lambda}_{\zeta(\tau,\theta)}}\, d\tau\, d\theta \lesssim h^{-2(s+\lambda-1+\epsilon)}\,\|f\|^2_{W^{-s,2}(\mathbb{R}^d)}\,.$$

Proof. We have

$$\begin{aligned}
&\frac{1}{h}\int_0^{2\pi}\int_h^{2h} \|f\|^2_{X^{-\lambda}_{\zeta(\tau,\theta)}}\, d\tau\, d\theta \\
&\quad= \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \frac{1}{h}\int_0^{2\pi}\int_h^{2h} \frac{1}{(\tau + |P_\zeta(\tau\xi)|)^{2\lambda}}\, d\tau\, d\theta\, d\xi \\
&\quad\lesssim \sup_\xi \frac{\langle\xi\rangle^{2s-4+4\epsilon}}{(h + |P_\zeta(h\xi)|)^{2\lambda-2+2\epsilon}} \\
&\qquad\quad \times \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\langle\xi\rangle^{-2s+4-4\epsilon}\frac{1}{h}\int_0^{2\pi}\int_h^{2h} \frac{1}{(\tau + |P_\zeta(\tau\xi)|)^{2-2\epsilon}}\, d\tau\, d\theta\, d\xi \\
&\quad\lesssim h^{-2(s+\lambda-1+\epsilon)}\|f\|^2_{\tilde{W}^{-s,2}(\mathbb{R}^d)}.
\end{aligned}$$

The last inequality depends on Proposition 4.5 and (3.23). The conditions on s are needed to apply (3.23). □

Chapter 5 Cauchy data, bilinear forms and a Dirichlet to Neumann map

In this chapter, we show that the Cauchy data for a weak solution exists. This is important as the Cauchy data appears in the assumption of Theorem 2.4. We also define a Dirichlet to Neumann map. What we would like to do is have a theorem about bilinear forms and obtain a uniqueness result from the bilinear form. In this chapter, we will show that the bilinear form can be determined from Cauchy data or a Dirichlet to Neumann map. Using this and the fact that our solution to the inverse boundary value problem gives an injective map from the coefficients q and Q to a bilinear form. Therefore, when we prove a uniqueness result using bilinear forms, we will also prove a uniqueness result from more familiar Cauchy data or Dirichlet to Neumann maps.

One issue we run into is that a solution u will lie in $W^{m,2}(\Omega)$ so defining Cauchy data of u with higher or derivatives is tricky since if $u \in W^{m,2}$ and $|\alpha| \geq m$, then the trace of $D^\alpha u$ on $\partial\Omega$ cannot be defined. To start, for u having at least $2m$ derivatives, we define boundary operators

$$\delta_{2j} u = (-\Delta)^j u|_{\partial\Omega}, \quad \delta_{2j+1} = \frac{\partial}{\partial\nu}(-\Delta)^j u, \qquad j = 0, \dots, m-1.$$

Then, we can apply Green's identity to B_0 for $u, v \in C^\infty(\bar{\Omega})$ to obtain

$$B_0(u,v) - \int_\Omega [(-\Delta)^m u] v = \sum_{j=0}^{m-1} \int_{\partial\Omega} (-1)^j \delta_j v \, \delta_{2m-1-j} u \, d\sigma \tag{5.1}$$

If u is a solution (having at least $2m$ derivatives) of an equation with principal part $(-\Delta)^m$, the vector of distributions on the boundary $(\delta_0 u, \dots, \delta_{2m-1} u)$ are the *Cauchy data* of the solution u. (Or at least one possible choice for the Cauchy data.) Here and throughout this section, we are using the form B_0 given in (3.5). Although we cannot define all the $\delta_j u$ for $j \geq m$ using a trace theorem, the next Proposition shows that we can define Cauchy data for a weak solution of $(-\Delta)^m u = F$. One more assumption we need to make is that Ω has a smooth boundary, and not just Lipschitz.

Proposition 5.2. *We assume $\partial\Omega$ is smooth. Suppose $u \in W^{m,2}(\Omega)$ and is a weak solution of $(-\Delta)^m u = F$ in Ω with $F \in \tilde{W}^{-m,2}(\Omega)$. Then we may define the Cauchy data for u, $\delta_j u \in W^{m-\frac{1}{2}-j,2}(\partial\Omega)$ for $j = 0, \dots, 2m-1$.*

Proof. The key idea for this proof is to use duality between Sobolev spaces. For $j = 0, \dots, m-1$ the trace theorem gives us $\delta_j u$ lies in $W^{m-\frac{1}{2}-j,2}(\partial\Omega)$. Next, for $\phi_j \in W^{m-\frac{1}{2}-j,2}(\partial\Omega)$ for $j = 0, \dots, m-1$ we can solve the Dirichlet problem

$$\begin{cases} (-\Delta)^m v = 0 & \text{in } \Omega \\ \delta_j v = \phi_j & \text{on } \partial\Omega. \end{cases}$$

See, for example, the monograph of Gazzola *et. al.* [13, Theorem 2.14]. We use the identity below to define $\delta_j u$ for $j = m, \ldots, 2m-1$:

$$B_0(u,v) - \langle F, v\rangle = \sum_{j=0}^{m-1} (-1)^j \langle \delta_{2m-1-j}u, \delta_j v\rangle_{\partial\Omega}.$$

Here we are using $\langle \cdot, \cdot \rangle_{\partial\Omega}$ to denote the pairing of duality for Sobolev spaces on the boundary. Note that the left hand side of this identity is defined due to our choice of the form B_0 and our assumption that F lies in the dual space to $W^{m,2}(\Omega)$, $\tilde{W}^{-m,2}(\Omega)$. □

Remark. 1. We note that the identity (5.1) allows us to show that for a smooth solution, the weak definition of the Cauchy data in Proposition 5.2 agrees with the classical definition of the operators $\delta_j u$.

2. Our discussion of the Cauchy data requires us to assume that the boundary is smooth. For example, even to define Sobolev spaces of order k on the boundary, we would normally need the boundary to be C^k. In addition, the existence result for the Dirichlet problem for the polyharmonic operator that we quote also assumes the boundary is smooth.

3. It is possible that the work of Verchota on boundary value problems for polyharmonic operators on Lipschitz domains [22] can be used to provide a formulation of the Cauchy data for solutions on Lipschitz domains. However, this would take us far afield from the themes we are studying here and we have not pursued this direction.

For a weak solution of $Lu = 0$ with L as defined in (2.1) and coefficients satisfying (3.7), we have that qu and $Q \cdot Du$ lie in $\tilde{W}^{-m,2}(\Omega)$ (see (3.3)), thus Proposition 5.2 implies the existence of the Cauchy data for u, which is one of the key takeaways from this chapter.

The other key takeaway is how the Cauchy data, Dirichlet to Neumann maps, and bilinear forms interact. We will now consider two operators $\mathcal{L}_j = (-\Delta)^m + Q^j \cdot D + q^j$ for $j = 1, 2$ with Q^j and q^j satisfying the conditions (3.7) and let $B_j(u,v) = B_0(u,v) + \langle Q^j \cdot Du, v\rangle + \langle q^j u, v\rangle$ be the form used to define weak solutions of the operator $\mathcal{L}_j$. We define $B_1 = B_2$ if for each $u_1 \in W^{m,2}(\Omega)$ a solution of $\mathcal{L}_1 u_1 = 0$, there is a solution u_2 of $\mathcal{L}_2 u_2 = 0$ with $u_1 - u_2 \in W_0^{m,2}(\Omega)$ and $B_1(u_1, v) = B_2(u_2, v)$ for all $v \in W^{m,2}(\Omega)$. We also assume that we have the same result with the roles of $\mathcal{L}_1$ and $\mathcal{L}_2$ reversed.

Now, suppose that $B_1 = B_2$ and u_1 is a solution to $\mathcal{L}_1 u_1 = 0$ and v_2 a solution of $\mathcal{L}_2^t v_2 = 0$. If there is a solution of $\mathcal{L}_2 u_2 = 0$ with $u_1 - u_2 \in W_0^{m,2}(\Omega)$ then

$$B_1(u_1, v_2) = B_2(u_2, v_2)$$

and since v_2 is a solution of $\mathcal{L}_2^t v_2 = 0$, we have

$$B_2(u_1 - u_2, v_2) = 0.$$

Thus, we can conclude

$$B_1(u_1, v_2) = B_2(u_1, v_2). \tag{5.3}$$

Next we say that two operators L_1 and L_2 have the same Cauchy data if whenever $u_1 \in W^{m,2}(\Omega)$ is a weak solution of $L_1 u_1 = 0$, there exists $u_2 \in W^{m,2}(\Omega)$ so that $L_2 u_2 = 0$ and $\delta_j u_1 = \delta_j u_2$, $j = 0, \ldots, 2m-1$. We observe that the condition $\delta_j u_1 = \delta_j u_2$, $j = 0, \ldots, m-1$ implies $u_1 - u_2 \in W_0^{m,2}(\Omega)$ and then the definition of the Cauchy data using the form implies

$$B_0(u_1, v) + \langle (Q^1 \cdot D + q^1) u_1, v \rangle = B_0(u_2, v) + \langle (Q^2 \cdot D + q^2) u_2, v \rangle$$

and thus we have that if the Cauchy data for the two operators agree, then the forms are equal. This argument may be reversed so that we have that the equality of the forms is equivalent to the equality of the Cauchy data. This statement here tells us that if we can prove a uniqueness result using bilinear forms, the same result is true using Cauchy data.

Lastly, in the case that we have unique solvability of a boundary value problem, we may view the linear space of Cauchy data as the graph of a Dirichlet to Neumann type map. For example, if we have unique solvability for the Navier boundary value problem (2.2) with data $(\phi_0, \phi_2, \ldots, \phi_{2m-2}) \in \oplus_{j=0}^{m-1} W^{m-1/2-2j,2}(\partial\Omega)$, then we may define $\Lambda : \oplus_{j=0}^{m-1} W^{m-1/2-2j,2}(\partial\Omega) \to \oplus_{j=0}^{m-1} W^{m-3/2-2j,2}(\partial\Omega)$ by

$$\Lambda(\phi_0, \ldots, \phi_{2m-2}) = (\frac{\partial}{\partial\nu} u, \frac{\partial}{\partial\nu}(-\Delta u), \ldots, \frac{\partial}{\partial\nu}(-\Delta)^{m-1} u)$$

where u is the solution to (2.2) with data $(\phi_0, \ldots, \phi_{2m-2})$. It is clear that under the assumption of the unique solvability of the Navier boundary value problem, the equality of the Dirichlet to Neumann maps for two different operators implies the two operators have the same Cauchy data. It should be mentioned that in this section we used a specific bilinear form along with a choice of Cauchy data and a Dirchlet to Neumann map. However, we may define other such maps and bilinear form and obtain a similar connection.

The last detail in this chapter will be extending the bilinear forms from Ω to a larger domain. There will be a time where we will want to work on a simply connected domain in order to recover Q. To do so, let B be a ball such that $\Omega \subset\subset B$. Let $\mathcal{L}_j = (-\Delta)^m + Q^j \cdot D + q^j$ in Ω and an operator $\tilde{\mathcal{L}}_j$ in B. We define the forms of the operators as before and we have $\tilde{B}_j = B_j + B'$ where B' is the form B_0, with domain $W^{m,2}(B \setminus \bar{\Omega}) \times W^{m,2}(B \setminus \bar{\Omega})$.

Lemma 5.4. *If $B_1 = B_2$ then $\tilde{B}_1 = \tilde{B}_2$.*

Proof. Let $\tilde{u}_1 \in W^{m,2}(B)$ such that $\mathcal{L}_1 \tilde{u}_1 = 0$ and let $u_1 = \tilde{u}_1|_\Omega$. Since $B_1 = B_2$ there exists $u_2 \in W^{m,2}(\Omega)$ such that $u_1 - u_2 \in W_0^{m,2}(\Omega)$ and $\mathcal{L}_2 u_2 = 0$. Then, for any $v \in W_0^{m.2}(B)$

$$\tilde{B}_1(\tilde{u}_1, v) = B_1(u_1, v) + B'(\tilde{u}_1, v) = B_2(u_2, v) + B'(\tilde{u}_1, v) = \tilde{B}_2(\tilde{u}_2, v)$$

where

$$\tilde{u}_2 = \begin{cases} \tilde{u}_1 & \text{in } B \setminus \bar{\Omega} \\ u_2 & \text{in } \bar{\Omega}. \end{cases}$$

It follows that if $\tilde{\mathcal{L}}_1\tilde{u}_1 = 0$ then $\tilde{\mathcal{L}}_2\tilde{u}_2 = 0$. We have shown that if $\tilde{u}_1 \in W^{m,2}(B)$ is a solution to $\tilde{\mathcal{L}}_1\tilde{u}_1 = 0$ then there exists $\tilde{u}_2 \in W^{m,2}(B)$ with $\tilde{\mathcal{L}}_2\tilde{u}_2 = 0$ and $\tilde{u}_1 - \tilde{u}_2 \in W_0^{m,2}(B)$ since $\tilde{u}_1 - \tilde{u}_2$ is compactly supported in B. Reversing the roles of 1 and 2, it follows that $\tilde{B}_1 = \tilde{B}_2$. $\square$

Chapter 6 Main Result

In this section, we give a construction of CGO solutions. We begin with a simple result that may be viewed as an analog of a theorem of Päivärinta *et. al.* [19, Theorem 1.1] in the scale of the X^λ spaces. Our next step is to give an improvement of this result that relies on the averaging estimate from the averaging section. We then prove our main result on the uniqueness of the coefficients.

Proposition 6.1. *Let $\zeta \in \mathcal{V}$, suppose $q \in \tilde{W}^{-s,p}(\Omega)$ and $Q \in \tilde{W}^{1-s,p}(\Omega)$ with s and p satisfying $1/p + (s-m)/d < 0$ and $s < m$. Suppose $a \in C^\infty(\mathbb{R}^d)$ and $P(hD)^m a = 0$. For h small, we may find a solution ψ so that $u = e^{x\cdot\zeta/h}(a+\psi)$ is a CGO solution in Ω and with $X^\lambda = X^\lambda_{h\zeta}$ we have $\psi \in X^{m/2}$ and the estimate*

$$\|\psi\|_{X^{m/2}} \lesssim h^{2m}(\|Q\cdot Da\|_{X^{-m/2}} + h^{-1}\|Qa\|_{X^{-m/2}} + \|aq\|_{X^{-m/2}}). \qquad (6.2)$$

If in addition we have $p \geq 2$, then $\|\psi\|_{X^{m/2}} \lesssim h^{3m/2-s}$.

Proof. By Sobolev embedding [3, Theorem 6.5.1], we have $\tilde{W}^{-s,p}(\Omega) \subset \tilde{W}^{-s_1,p_1}(\Omega)$ provided $s_1 > s$, $p, p_1 \in (1,\infty)$ and

$$\frac{1}{p} + \frac{s}{d} \leq \frac{1}{p_1} + \frac{s_1}{d}.$$

Thus we may choose $p_1 \in (1,\infty)$ and $r \in (0,1)$ so that $q \in \tilde{W}^{r-m,p_1}(\Omega)$ and $Q \in \tilde{W}^{1+r-m,p_1}(\Omega)$ and $1/p_1 - r/d < 0$. By Proposition 8.3 the coefficients q and Q have a representation as in (3.35), the hypothesis of Theorem 3.34. This follows because Morrey's Lemma implies functions in $W^{r,p_1}(\mathbb{R}^d)$ are Hölder continuous. Thus we may use Theorem 3.34 to solve $L_\zeta\psi = -L_\zeta a$ and the solution will satisfy $\|\psi\|_{X^{m/2}} \lesssim \|L_\zeta a\|_{X^{-m/2}}$. The first estimate (6.2) follows from this. To obtain the estimate in terms of h, we use (3.27) and Proposition 3.32 to estimate the $X^{-m/2}$ norm in terms of the Sobolev norm. □

The next Lemma is rather technical but important. It provides a sequence of CGO solutions that are instrumental in the proof of the main theorem.

Lemma 6.3. *Suppose $1/p + (s-m)/d < 0$, $p \geq 2$, and $s < \frac{m}{2} + 1$. Assume we have potentials q^k, Q^k for $k = 1,2$ which satisfy $q^k \in \tilde{W}^{-s,p}(\Omega)$ and $Q^k \in \tilde{W}^{-s+1,p}(\Omega)$. Assume $a^k \in C^\infty(\mathbb{R}^d)$ and $P(hD)^m a^k = 0$. Given $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ and unit vectors μ_1 and μ_2 so that $\{\xi_0, \mu_1, \mu_2\}$ are mutually orthogonal, there exist sequences $\{h_j\}$, $\{\theta_j\}$ with $\lim_{j\to\infty} h_j = 0$ and $\lim_{j\to\infty}\theta_j = \theta_0$ with the following properties. If $\zeta^k_j = \zeta^k(h_j,\theta_j)$ is as in (4.1) and $X^\lambda_{k,j} = X^\lambda_{h_j\zeta^k_j}$, then we may find sequences $\{\psi^k_j\} \subset X^{m/2}_{k,j}$,*

$k = 1, 2$ *so that* $e^{x\cdot\zeta_j^k/h_j}(a^k + \psi_j^k)$ *is a CGO solution to* $L_k u = 0$ *and*

$$\|q^\ell\|_{X_{k,j}^{-m/2}} \lesssim h_j^{-s-m/2+1-\epsilon} \tag{6.4}$$

$$\|Q^\ell\|_{X_{k,j}^{-m/2}} \lesssim h_j^{-s+1-m/2+1-\epsilon} \tag{6.5}$$

$$\|Q^\ell\|_{X_{k,j}^{\frac{-m+1}{2}}} \lesssim h_j^{-s+1-m/2+1/2+1-\epsilon} \tag{6.6}$$

for $k, \ell = 1, 2$ *and*

$$\|\psi_j^k\|_{X_{k,j}^{m/2}} \lesssim h_j^{\frac{3m}{2}-s+1-\epsilon}. \tag{6.7}$$

Proof. We may use Proposition 6.1 to find solutions ψ^k for each h and θ which lie in the space $X_k^{m/2}$ where we use $X_k^\lambda = X_{h\zeta^k(h,\theta)}^\lambda$. These solutions ψ^k satisfy the estimate (6.2). Using Proposition 3.32, we can bound the right hand side of (6.2) by $h^{2m}(\|q^\ell\|_{X_k^{-m/2}} + h^{-1}\|Q^\ell\|_{X_k^{-m/2}})$. If we sum the estimates of Theorem 4.6 for $k = 1, 2$, may find sequences $\{h_j\}$ and $\{\theta_j\}$ so that for $k, \ell = 1, 2$ we have the estimates in the $X_{k,j}^{-m/2}$ norm in (6.4), (6.5), and (6.6). The estimate for the remaining term then follows from (3.25). After perhaps passing to a sub-sequence, we have that the sequence θ_j will be convergent and we let θ_0 be the limit of this sequence. The estimate (6.7) follows from (6.4), (6.5), and (6.2). □

We are now ready to give the proof of our result establishing that the bilinear form for the operator L uniquely determines the coefficients q and Q.

Theorem 6.8. *Suppose that we have two operators* L_k, $k = 1, 2$ *as in* (2.1) *and we have* $s < m/2 + 1$ *and* $p \geq 2$ *which satisfy* $1/p + (s - m)/d < 0$, $p \geq 2$. *Suppose the coefficients of the operators* L_k *satisfy* $Q^k \in \tilde{W}^{-s+1,p}(\Omega)$ *and* $q^k \in \tilde{W}^{-s,p}(\Omega)$. *If the bilinear forms for the operators satisfy* $B_1 = B_2$, *then* $Q^1 = Q^2$ *and* $q^1 = q^2$.

Proof. We begin by applying Lemma 5.4 to assume Ω is a ball. Since the Sobolev spaces $\tilde{W}^{-s,2}(\Omega)$ are increasing as $-s$ decreases, we may assume that $s > m/2$ and choose ϵ so that $s = \frac{m}{2} + 1 - \frac{3}{2}\epsilon$ for $0 < \epsilon < \frac{1}{3}$. Let $\xi_0 \in \mathbb{R}^d$ and find unit vectors $\mu_1, \mu_2 \in \mathbb{R}^d$ such that $\{\xi_0, \mu_1, \mu_2\}$ are mutually orthogonal. Given amplitudes a^k which satisfy $P(hD)_{\zeta_k}^m a^k = 0$. We apply Lemma 6.3 to the operators L_1 and L_2^t to find a sequence of solutions $\{\psi_j^k\}$, a sequence of spaces $\{X_{k,j}^{m/2}\}$ and numerical sequences $\{h_j\}$ and $\{\theta_j\}$ so that $u_k = e^{\zeta_j^k\cdot x/h_j}(a^k + \psi_j^k)$ are CGO solutions of $L_1 u_1 = 0$ and $L_2^t u_2 = 0$ which satisfy the estimates (6.4), (6.5), (6.2), and (6.7). It is clear from the definitions of ζ^k in (3.8) that $\zeta_j^1 + \zeta_j^2 = -ih_j\xi_0$. Since we have that the forms for

L_1 and L_2 are equal (see (5.3)), we have

$$
\begin{aligned}
0 &= ih_j[B_1(u_1,u_2) - B_2(u_1,u_2)] \\
&= \zeta_j^1 \cdot \langle (Q^1 - Q^2)a^1, a^2 e^{-ix\cdot\xi_0}\rangle + \zeta_j^1 \cdot \langle (Q^1 - Q^2)\psi_j^1, a^2 e^{-ix\cdot\xi_0}\rangle \\
&\quad + \zeta_j^1 \cdot \langle (Q^1 - Q^2)a^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle + \zeta_j^1 \cdot \langle (Q^1 - Q^2)\psi_j^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle \\
&\quad + ih_j\langle (Q^1 - Q^2)\cdot Da^1, a^2 e^{-ix\cdot\xi_0}\rangle + ih_j\langle (Q^1 - Q^2)\cdot Da^1, \psi_j^2 e^{-ix\cdot\xi_0}\rangle \\
&\quad + ih_j\langle (Q^1 - Q^2)\cdot D\psi_j^1, a^2 e^{-ix\cdot\xi_0}\rangle + ih_j\langle (Q^1 - Q^2)\cdot D\psi_j^1, e^{-ix\cdot\xi_0}\psi_2\rangle \\
&\quad + ih_j\langle (q^1 - q^2)(a^1 + \psi_j^1), e^{-ix\cdot\xi_0}(a^2 + \psi_j^2)\rangle \\
&= I + II + III + IV + V + VI + VII + VIII + IX.
\end{aligned}
$$

Since the only dependence on h in I is through the formula for ζ_j^1 (see (3.8)), we have

$$
\begin{aligned}
\lim_{j\to\infty} I &= \lim_{j\to\infty} \zeta_j^1 \cdot \langle (Q^1 - Q^2)a^1, a^2 e^{-ix\cdot\xi_0}\rangle \\
&= -\, i e^{i\theta_0}\langle (\mu_1 + i\mu_2)\cdot (Q^1 - Q^2), a^1 a^2 e^{-ix\cdot\xi_0}\rangle
\end{aligned}
$$

The others will all be bounded by some power of h. An important bound that will be used is

$$
\left\| D\psi_j^k \right\|_{X_{k,j}^{\frac{m-1}{2}}} \lesssim h_j^{3m/2-s-1/2-\epsilon} \tag{6.9}
$$

for $k = 1, 2$. To get (6.9) we use (3.25) along with (6.7). We are now able to prove that the remaining terms II through IX are bounded by positive powers of h_j. For II, we use the duality of $X_{1,j}^{m/2}$ and $X_{1,j}^{-m/2}$, the estimates (3.33), (6.5) and (6.7) to obtain

$$
\begin{aligned}
|II| &= |\zeta_j^1 \cdot \langle (Q^1 - Q^2)\psi_j^1, a^2 e^{-ix\cdot\xi_0}\rangle| \\
&\lesssim \left\| (Q^1 - Q^2) \right\|_{X_{1,j}^{-m/2}} \left\| \psi_j^1 \right\|_{X_{1,j}^{m/2}} \\
&\lesssim h_j^{-s+1-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\
&= h_j^{m-2s+3-2\epsilon} \\
&= h_j^{1+\epsilon}
\end{aligned}
$$

where the last equality uses $s = m/2 + 1 - 3/2\epsilon$. For III, we use the same argument with $X_{1,j}^{\lambda}$ replaced by $X_{2,j}^{\lambda}$

$$
\begin{aligned}
|III| &= |\zeta_j^1 \cdot \langle (Q^1 - Q^2)a^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\
&\lesssim \left\| (Q^1 - Q^2) \right\|_{X_{2,j}^{-m/2}} \left\| \psi_j^2 \right\|_{X_{2,j}^{m/2}} \\
&\lesssim h_j^{-s+1-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\
&= h_j^{m-2s+3-2\epsilon} \\
&= h_j^{1+\epsilon}
\end{aligned}
$$

For IV, we write Q^k in the form (3.35), use the estimate (3.29) and the estimates (6.7) for ψ_j^k to conclude

$$\begin{aligned}|IV| &= |\zeta_j^1 \cdot \langle (Q^1 - Q^2)\psi_j^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\ &\lesssim h_j^{1-2m} \left\|\psi_j^1\right\|_{X_{1,j}^{m/2}} \left\|\psi_j^2\right\|_{X_{2,j}^{m/2}} \\ &\lesssim h_j^{m+1-2s+2-2\epsilon} \\ &\lesssim h_j^{1+\epsilon}\end{aligned}$$

Since the amplitudes are smooth and there is no dependence on ψ_k^j, we have

$$\begin{aligned}|V| &= |ih_j\langle (Q^1 - Q^2)\cdot Da^1, a^2 e^{-ix\cdot\xi_0}\rangle| \\ &\lesssim h_j\end{aligned}$$

For VI, we use the same argument as in III to obtain

$$\begin{aligned}|VI| &= |ih_j\langle (Q^1 - Q^2)\cdot Da^1, \psi_j^2 e^{-ix\cdot\xi_0}\rangle| \\ &\lesssim h_j \left\|Q^1 - Q^2\right\|_{X_{2,j}^{-m/2}} \left\|\psi_j^2\right\|_{X_{2,j}^{m/2}} \\ &\lesssim h_j h_j^{-s+1-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\ &= h_j^{m-2s+4-2\epsilon} \\ &= h_j^{2+\epsilon}\end{aligned}$$

For VII, we use duality in $X_{1,j}^{(m-1)/2}$, (3.25), (6.6), and (6.9) to obtain

$$\begin{aligned}|VII| &= |ih_j\langle (Q^1 - Q^2)\cdot D\psi_j^1, a^2 e^{-ix\cdot\xi_0}\rangle \\ &\lesssim h_j \left\|Q^1 - Q^2\right\|_{X_{1,j}^{(1-m)/2}} \left\|D\psi_j^1\right\|_{X_{1,j}^{(m-1)/2}} \\ &\lesssim h_j\, h_j^{-s+1-m/2+1/2+1-\epsilon}\, h_j^{3m/2-s-1/2-\epsilon} \\ &\lesssim h_j^{m-2s+3-2\epsilon} \\ &= h_j^{1+\epsilon}.\end{aligned}$$

For $VIII$, we use the representation as in (3.35) and (3.29) to obtain

$$\begin{aligned}|VIII| &= |ih_j\langle (Q^1 - Q^2)\cdot D\psi_j^1, e^{-ix\cdot\xi_0}\psi_2\rangle| \\ &\lesssim h_j h_j^{-2m} \left\|\psi_j^1\right\|_{X_{1,j}^{m/2}} \left\|\psi_j^2\right\|_{X_{2,j}^{m/2}} \\ &\lesssim h_j^{1-2m} h_j^{3m/2-1+1-\epsilon} h_j^{3m/2-1+1-\epsilon} \\ &= h_j^{m-2s+3-2\epsilon} \\ &= h_j^{1+\epsilon}.\end{aligned}$$

For IX, we expand the product $(a^1 + \psi_j^1)(a^2 + \psi_j^2)$ to obtain

$$\begin{aligned}|IX| &= |ih_j\langle (q^1 - q^2)(a^1 + \psi_j^1), e^{-ix\cdot\xi_0}(a^2 + \psi_j^2)\rangle| \\ &\lesssim |ih_j\langle (q^1 - q^2)a^1, e^{-ix\cdot\xi_0}a^2\rangle| + |ih_j\langle (q^1 - q^2)a^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\ &+ |ih_j\langle (q^1 - q^2)\psi_j^1, e^{-ix\cdot\xi_0}a^2\rangle| + |ih_j\langle (q^1 - q^2)\psi_j^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\ &= |IX_I| + |IX_{II}| + |IX_{III}| + |IX_{IV}|.\end{aligned}$$

We now bound each term and show they are each bounded by a positive power of h. For IX_I we obtain

$$\begin{aligned}|IX_I| &= |ih_j\langle (q^1-q^2)a^1, e^{-ix\cdot\xi_0}a^2\rangle| \\ &\lesssim h_j\end{aligned}$$

using the fact that a^1 and a^2 are smooth. Next, we obtain

$$\begin{aligned}|IX_{II}| &= |ih_j\langle (q^1-q^2)a^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\ &\lesssim h_j \left\|q^1-q^2\right\|_{X_{2,j}^{-m/2}} \left\|\psi_j^2\right\|_{X_{2,j}^{m/2}} \\ &\lesssim h_j h_j^{-s-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\ &= h_j^{m-2s+3-2\epsilon} \\ &= h_j^{1+\epsilon}\end{aligned}$$

where we have used (6.4) and (6.7). For IX_{III}, we obtain

$$\begin{aligned}|IX_{III}| &= |ih_j\langle (q^1-q^2)\psi_j^1, e^{-ix\cdot\xi_0}a^2\rangle| \\ &\lesssim h_j \left\|q^1-q^2\right\|_{X_{1,j}^{-m/2}} \left\|\psi_j^1\right\|_{X_{1,j}^{m/2}} \\ &\lesssim h_j h_j^{-s-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\ &= h_j^{m-2s+3-2\epsilon} \\ &= h_j^{1+\epsilon}\end{aligned}$$

where we have used (6.4) and (6.7). Now, lastly, for IX_{IV}, we obtain

$$\begin{aligned}|IX_{IV}| &= |ih_j\langle (q^1-q^2)\psi_j^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\ &\lesssim h_j h^{-2m} \left\|\psi_j^1\right\|_{X_{1,j}^{m/2}} \left\|\psi_j^2\right\|_{X_{2,j}^{m/2}} \\ &\lesssim h_j h_j^{-2m} h_j^{3m/2-s+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\ &= h_j^{1+3m-2s+2-2\epsilon} \\ &= h_j^{1+\epsilon}\end{aligned}$$

where we use q^k in the form (3.35) and use (6.7). Thus we can conclude

$$\begin{aligned}0 &= \lim_{j\to\infty} ie^{-i\theta_0}h_j(B(u_1,u_2) - B_2(u_1,u_2) \\ &= \langle (\mu_1+i\mu_2)\cdot(Q^1-Q^2), a^1a^2e^{-ix\cdot\xi_0}\rangle\end{aligned} \tag{6.10}$$

and if we set $a^1 = a^2 = 1$, then we conclude

$$(\mu_1+i\mu_2)\cdot(\hat{Q}^1(\xi_0) - \hat{Q}^2(\xi_0)) = 0.$$

If we repeat this argument with $-\mu_2$ replacing μ_2 we obtain

$$(\mu_1-i\mu_2)\cdot(\hat{Q}^1(\xi_0) - \hat{Q}^2(\xi_0)) = 0.$$

Adding the last two displays we have $\mu \cdot (\hat{Q}^1(\xi_0) - \hat{Q}^2(\xi_0)) = 0$ if μ is perpendicular to ξ_0. Choosing μ parallel to $e_k \xi_{0,j} - e_j \xi_{0,k}$ we conclude that

$$\frac{\partial}{\partial x_k}(Q_j^1 - Q_j^2) - \frac{\partial}{\partial x_j}(Q_k^1 - Q_k^2) = 0.$$

This gives us that curl $(Q^1 - Q^2) = 0$. Since we assume Ω is a ball, we may use Proposition 8.4 to find g which is supported in $\bar{\Omega}$ and so that $Q^1 - Q^2 = Dg$. Now, we use (6.10) with $a^1 = 1$ and $a^2 = (\mu_1 - i\mu_2) \cdot x/2$. This choice for a^2 gives that $(\mu_1 + i\mu_2) \cdot Da^2 = -i$. Using that $(Q^1 - Q^2) = Dg$, we send $h_j \to 0$ and integrate by parts to obtain that

$$0 = (\mu_1 + i\mu_2) \cdot \langle Dg, e^{-ix\cdot\xi_0} a_2 \rangle = i\langle g, e^{-ix\cdot\xi_0} \rangle$$

where we used that $(\mu_1 + i\mu_2) \cdot \xi_0 = 0$. This gives us that $g = 0$ and we have that $Q^1 = Q^2$ as desired. Now that we have $Q^1 = Q^2$, we turn to the proof that $q^1 = q^2$. We use our CGO solutions with $a^1 = a^2 = 1$ to obtain

$$\begin{aligned}
0 &= B_1(u_1, u_2) - B_2(u_1, u_2) \\
&= \langle (q^1 - q^2)(1 + \psi_j^1), e^{-ix\cdot\xi_0}(1 + \psi_j^2) \rangle \\
&= \langle (q^1 - q^2), e^{-ix\cdot\xi_0} \rangle + \langle (q^1 - q^2)\psi_j^1, e^{-ix\cdot\xi_0} \rangle \\
&+ \langle (q^1 - q^2), e^{-ix\cdot\xi_0}\psi_j^2 \rangle + \langle (q^1 - q^2)\psi_j^1, e^{-ix\cdot\xi_0}\psi_j^2 \rangle \\
&= I + II + III + IV.
\end{aligned}$$

We see that I is independent of h and we will show that the remaining terms $II - IV$ will be bounded by positive powers of h. For II, we will use (6.7) and (6.4) to obtain

$$\begin{aligned}
|II| &= |\langle (q^1 - q^2)\psi_j^1, e^{-ix\cdot\xi_0} \rangle| \\
&\lesssim \left\| q^1 - q^2 \right\|_{X_{1,j}^{-m/2}} \left\| \psi_j^1 \right\|_{X_{1,j}^{m/2}} \\
&\lesssim h_j^{-s-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\
&= h^{m-2s+2-2\epsilon} \\
&= h_j^{\epsilon}
\end{aligned}$$

where we once again use $s = m/2 + 1 - 3\epsilon/2$. For III, we obtain the same bound,

$$\begin{aligned}
|II| &= |\langle (q^1 - q^2), e^{-ix\cdot\xi_0}\psi_j^2 \rangle| \\
&\lesssim \left\| q^1 - q^2 \right\|_{X_{2,j}^{-m/2}} \left\| \psi_j^2 \right\|_{X_{2,j}^{m/2}} \\
&\lesssim h_j^{-s-m/2+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\
&= h^{m-2s+2-2\epsilon} \\
&= h_j^{\epsilon}.
\end{aligned}$$

And for term IV, we write q as in (3.35) to obtain

$$\begin{aligned} |III| &= |\langle (q^1 - q^2)\psi_j^1, e^{-ix\cdot\xi_0}\psi_j^2\rangle| \\ &\lesssim h_j^{-2m} \left\|\psi_j^1\right\|_{X_{1,j}^{m/2}} \left\|\psi_j^2\right\|_{X_{2,j}^{m/2}} \\ &\lesssim h_j^{-2m} h_j^{3m/2-s+1-\epsilon} h_j^{3m/2-s+1-\epsilon} \\ &= h^{m-2s+2-2\epsilon} \\ &= h_j^{\epsilon} \end{aligned}$$

Therefore, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} \langle (q^1 - q^2)(1+\psi_j^1), e^{-ix\cdot\xi_0}(1+\psi_j^2)\rangle \\ &= \langle (q^1 - q^2), e^{-ix\cdot\xi_0}\rangle \\ &= \hat{q}^1(\xi_0) - \hat{q}^2(\xi_0) \end{aligned}$$

Since $\hat{q}^1 - \hat{q}^2 = 0$, we have $q^1 = q^2$. This completes the proof. □

Proof of Main Theorem, Theorem 2.4. In section 5, we show that if the Dirichlet to Neumann to maps for two operators are equal, then the two forms are equal. Thus this result follows immediately from Theorem 6.8. □

Chapter 7 Future Work

There are quite a few routes to take in future work regarding the polyharmonic operator. The first is to see if there is a way to further reduce the regularity. The current conjecture is that least smooth Sobolev space the zeroth order coefficient can lie in is $W^{-m,p}$. The reason this is the conjecture is that this condition is met for $m = 1$ and we get arbitrarily close to it for $m = 2$. One issue with extending q to have smoothness $< -m$ is defining weak solutions and from the Sobolev multiplication theorem, we need $-m$.

The second route is to have more coefficients. Some work has been done with second order perturbations, such as

$$\mathcal{L} = (-\Delta)^m + \sum_{j,k=1}^{d} Q_{jk} D_j D_k + Q \cdot D + q.$$

One issue that arises when trying to show uniqueness with non smooth coefficients in the second degree perturbations is that the techniques currently used require a positive degree of smoothness. The appendix in this dissertation does discuss the decomposition of negative order Sobolev spaces, and this might be enough to show uniqueness for the second order perturbations.

We may want higher degree perturbations such as

$$\mathcal{L} = (-\Delta)^m + \sum_{|\alpha| \leq m} Q_\alpha D^\alpha.$$

We restrict to having at most m order perturbations due to the challenge of finding weak solutions. Once again, there could be a work around this requirement as well.

Another, and the last of this dissertation, route could be dealing with partial data. There has been work done answering this question, however, the assumptions on the coefficients are a lot stronger than that of Theorem 2.4.

Chapter 8 Appendices

8.1 Sobolev spaces

We establish several representation theorems for elements of Sobolev spaces that are used in our arguments. Our results will depend on techniques developed by D. Mitrea, M. Mitrea, and S. Monniaux [18] who give a version of the Poincaré Lemma in their study of the Poisson problem for the exterior derivative. They construct a homotopy for the de Rham complex in a Lipschitz domain that is star-shaped with respect to each point in a small ball. The homotopy operators have the useful property that they preserve the class of forms that are supported in the domain. We briefly summarize what we need from their work and refer the reader to their article for precise definitions and complete statements of their results. They work with differential forms and initially assume the coefficients lie in $C_0^\infty(\Omega)$. Thus, let Λ denote the exterior algebra on $\mathbb{R}^d$ and let Λ^j be the elements of degree j. We will use $C_0^\infty(\Omega; \Lambda^j)$ to denote the differential forms of degree j on Ω with coefficients in $C_0^\infty(\Omega)$. In [18, Theorem 4.1], Mitrea *et. al.* find a family of operators $J_j : C_0^\infty(\Omega; \Lambda^j) \to C_0^\infty(\Omega; \Lambda^{j-1})$, for $j = 1, \ldots, d$ with the property that if $\omega \in C_0^\infty(\Omega; \Lambda^j)$

$$\omega = \begin{cases} J_1(d\omega), & j = 0 \\ J_{j+1}(d\omega) + dJ_j(\omega), & j = 1, \ldots, d-1 \\ dJ_d(\omega) + \langle \omega, \theta V_d \rangle \theta V_d, & j = d \end{cases} \tag{8.1}$$

Here, we are temporarily using d to denote both the dimension and the exterior derivative on $\mathbb{R}^d$. In the third line, $\theta \in C_0^\infty(\Omega)$ is determined in the construction of the operator J_d, $V_d = dx_1 \wedge \cdots \wedge dx_d$ is the standard volume form on $\mathbb{R}^d$ and the pairing of duality $\langle \cdot, \cdot \rangle$ is defined in section 2 of Mitrea *et. al.* [18]. In addition, the operators J_j map

$$J_j : \tilde{W}^{s,p}(\Omega; \Lambda^j) \to \tilde{W}^{s+1,p}(\Omega; \Lambda^{j-1}). \tag{8.2}$$

To see this, consider J_j acting on a larger domain, and use the estimate of Theorem 3.8 in Mitrea *et. al.* and the property that J_j preserves the property of being supported in Ω.

Proposition 8.3. *Assume that Ω is a Lipschitz domain. Suppose that $f \in \tilde{W}^{-s,p}(\Omega)$ with $s = k - r$ where $0 \le r < 1$ and $k \ge 1$ is an integer. Then we may write*

$$f = \sum_{|\alpha|=k} D^\alpha F_\alpha + F_0$$

with $F_\alpha \in \tilde{W}^{r,p}(\Omega)$ and $F_0 \in C_c^\infty(\Omega)$

Proof. Using a partition of unity, we may reduce to the case where the domain Ω is star-shaped with respect to each point in a ball. We fix $f \in \tilde{W}^{s,p}(\Omega)$ and apply the case $j = d$ of (8.1) to the form $\omega = fV_d$. If we suppress the differentials, this can be rewritten as

$$f = D \cdot F + \langle u, \theta \rangle \theta$$

with $F = (F_1, \ldots, F_d)$ and $F_j \in \tilde{W}^{s+1,p}(\Omega)$. If $k = 1$, we are done and for $k \geq 2$, we repeat this argument to represent each F_j as a smooth term and derivatives of functions in $\tilde{W}^{s+2,p}(\Omega)$. After k steps, we obtain the desired representation. □

We remark that the specific case of Mitrea's results used in the proof of Proposition 8.3 may be found in earlier work of Bogovskii [6].

Proposition 8.4. *Suppose that Ω is a ball in $\mathbb{R}^d$ and $Q = (Q_1, \ldots, Q_d)$ lies $\tilde{W}^{s,p}(\mathbb{R}^d)$ with curl $Q = 0$, then we may find $g \in \tilde{W}^{s+1,p}(\Omega)$ so that $Q = Dg$.*

Proof. We form a differential form $\omega = \sum_{k=1}^{d} Q_k dx_k$. Our hypothesis curl $Q = 0$ gives immediately that $d\omega = 0$ and then the Proposition follows from the case $j = 1$ of (8.1). □

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