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The v_1-Periodic Region in Complex Motivic Ext And a Real Motivic v_1-Selfmap

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Digital Object Identifier: https://doi.org/10.13023/etd.2022.186

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The v_1-Periodic Region in $\mathbb{C}$-Motivic Ext And an $\mathbb{R}$-Motivic v_1-Selfmap

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Ang Li
Lexington, Kentucky

Director: Dr. Bertrand Guillou, Professor of Mathematics
Lexington, Kentucky
2022

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ABSTRACT OF DISSERTATION

The v_1-Periodic Region in $\mathbb{C}$-Motivic Ext And an $\mathbb{R}$-Motivic v_1-Selfmap

My thesis work consists of two main projects with some connections. In the first project we establish a v_1 periodicity theorem in Ext over the $\mathbb{C}$-motivic Steenrod algebra. The element h_1 of Ext, which detects the homotopy class η in the motivic Adams spectral sequence, is non-nilpotent and therefore generates h_1-towers. Our result is that, apart from these h_1-towers, v_1 periodicity operators give isomorphisms in a range near the top of the Adams chart. This result generalizes well-known classical behavior.

In the second project we consider a nontrivial action of C_2 on the type 1 spectrum $\mathcal{Y}$, which is well-known for admitting a v_1 selfmap. The resultant finite C_2-equivariant spectrum can also be viewed as the complex points of a finite $\mathbb{R}$-motivic spectrum. We show that one of the v_1 selfmaps of $\mathcal{Y}$ can be lifted to a selfmap in the real motivic case. Further, the cofiber of the real motivic selfmap is a realization of the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ of the $\mathbb{R}$-motivic Steenrod algebra. The finite subalgebra $\mathcal{A}^{\mathbb{R}}(1)$, generated by Sq^1 and Sq^2, of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ can be given 128 different $\mathcal{A}^{\mathbb{R}}$-module structures. We also show that all of these $\mathcal{A}^{\mathbb{R}}$-modules can be realized as the cohomology of a 2-local finite $\mathbb{R}$-motivic spectrum. The realization results are obtained using an $\mathbb{R}$-motivic analogue of the Toda realization theorem. We notice that each realization of $\mathcal{A}^{\mathbb{R}}(1)$ can be expressed as a cofiber of an $\mathbb{R}$-motivic v_1 selfmap. The C_2-equivariant analogue of the above results then follows because of the Betti realization functor. We identify a relationship between the $\mathrm{RO}(\mathrm{C}_2)$-graded Steenrod operations on a C_2-equivariant space and the classical Steenrod operations on both its underlying space and its fixed-points. The second project is joint work with Prasit Bhattacharya and Bertrand Guillou.

KEYWORDS: Steenrod algebra, v_1 selfmap, $\mathbb{C}$ and $\mathbb{R}$ motivic homotopy theory, C_2 equivariant homotopy theory

Ang Li

April 22, 2022

The v_1-Periodic Region in $\mathbb{C}$-Motivic Ext And an $\mathbb{R}$-Motivic v_1-Selfmap

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献给生长于华夏大地，驰骋纵横于古今人世的万千英雄儿女。

ACKNOWLEDGMENTS

First I would like to devote my sincere appreciation and respect to my advisor Bertrand Guillou, for his existence, gentleness, and glaring wisdom, for tolerating the farces I created in various ways, and for the willingness to understand my often devastated situations.

I would like to express my gratitude to Gabriel Angelini-Knoll, Eva Belmont, Andrew Blumberg, Mike Hill, Hana Jia Kong, Prasit Bhattacharya, J.D. Quigley, Jon Rubin, Nat Stapleton, Ningchuan Zhang, Foling Zou (and other potential persons in the Homotopy Theory family that I forget to list). Thank you for showing your love and passion to the pure aesthetic of Topology, and illuminate my way as brilliant predecessors.

I would like to thank Nat again because he is special.

I appreciate Eric Kaper, Julianne Vega, Rafael Rojas, Michael Morrow, Jamison Wallace, and Jamin Kochman (in the partial order of the time we met), as well as Dr. David Leep and Dr. Jing Qin, to be my local math mates, have me feel supported and keep my sanity above the right level. I treasure the experience and the friendship.

I'm grateful to my grandparents for their love and encouragements. Those combined with the approval of my WoW guild members equip me the mental gears to fight. I wish to thank all the characters in my fictions, who have brought me hallucinations, power and dreams. I also would like to thank Coffea (the coffee shop closed during the pandemic), for offering me a cute place to have most of my work done.

At last, I would like to thank Adrielle Stapleton, for being a real and true friend, for the braveness as a writer, and for the music.

TABLE OF CONTENTS

Acknowledgments iii

List of Tables vi

List of Figures vii

Chapter 1 Introduction 1

Chapter 2 The Steenrod algebra and the squaring operation 6
2.1 Squaring operation 7
2.1.1 Power operation construction 8
2.1.2 The C_2-equivairant squaring operation 10
2.1.3 Comparison theorem 13
2.2 The $\mathbb{R}$-motivic Steenrod algebra and a freeness criterion 17

Chapter 3 The v_1-Periodic Region in the cohomology of the $\mathbb{C}$-motivic Ext . 21
3.1 The stable (co)module category $Stab(\Gamma)$ 22
3.2 Massey products 23
3.3 The Cartan-Eilenberg spectral sequence 27
3.4 The motivic periodicity theorem 34

Chapter 4 Topological realization of $\mathcal{A}^{\mathbb{R}}(1)$ 37
4.1 $\mathbb{R}$-motivic Toda realization theorem 37
4.1.1 Weak $\mathbb{R}$-motivic Toda realization – version (I) 38
4.1.2 Weak $\mathbb{R}$-motivic Toda realization – version (II) 39
4.1.3 Weak $\mathbb{R}$-motivic Toda realization – version (III) 39
4.2 The 128 $\mathcal{A}^{\mathbb{R}}$-module structure 40
4.3 A realization of $\mathcal{A}^{\mathbb{R}}(1)$ 43
4.3.1 The spectrum $\mathcal{A}_1^{\mathbb{R}}$ via Smith's construction 43
4.3.2 The Betti realization of $\mathcal{A}_1^{\mathbb{R}}$ 51

Chapter 5 $\mathbb{R}$-motivic selfmap 56
5.1 The spectrum $\mathcal{Y}$ 58
5.1.1 Construction of $\mathcal{Y}$ and its lifts 59
5.1.2 Selfmaps between the lifts of $\mathcal{Y}$ 59
5.2 A $v_{(1,nil)}$-selfmap on $\mathcal{Y}_{(\mathsf{h},1)}$ 60
5.3 Nonexistence of $v_{(1,0)}$-selfmap on $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ 67

Appendices 69
Appendix A: An $\mathbb{R}$-motivic analogue of the spectrum $\mathcal{Z}$ 69
A topological realization of $\mathcal{A}^{\mathbb{R}}(2)$ 69

An $\mathbb{R}$-motivic lift of $\mathcal{B}(2)$ 72
The construction of $\mathcal{Z}_{\mathbb{R}}$ 76
Appendix B: The $\mathbb{R}$-motivic Adem relations 76

Bibliography 78

Vita 82

LIST OF TABLES

5.1 The maps f_2 and f_{h} 62

LIST OF FIGURES

1.1 The 4 $\mathcal{A}$-module structures of classical $\mathcal{A}(1)$ 3
1.2 Cell-diagram of the $\mathbb{R}$-motivic question mark complex 3

3.1 $[S, F_0]^{\mathcal{A}^{\mathbb{C}\vee}}_{*,*,*}$ 24
3.2 $[S, F_{01}]^{\mathcal{A}^{\mathbb{C}\vee}}_{*,*,*}$ 24
3.3 $[S/h_0, F_0]^{\mathcal{A}^{\mathbb{C}}(1)_*}$ 28
3.4 $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$ 29
3.5 $[S/(h_0, P), \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$ 30
3.6 $[S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$ 35

4.1 The $\mathcal{A}^{\mathbb{R}}(1)$ as an $\mathcal{A}^{\mathbb{R}}(1)$-module 40
4.2 The free $\mathbb{M}_2^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}(1)$ 42
4.3 The $\mathcal{A}$-structure of $\mathrm{H}^*(\mathcal{Q}; \mathbb{F}_2)$ 44
4.4 The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}); \mathbb{F}_2)$ 45
4.5 The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}); \mathbb{F}_2)$. 46
4.6 $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ and $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{C}^{\mathbb{R}}(2))$ 48
4.7 The $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1)$ 51
4.8 The $\mathcal{A}$-module $\mathrm{H}^*(\Phi^{C_2}\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$ 53
4.9 Some underlying and fixed $\mathcal{A}$-modules of $\mathcal{A}_1^{\mathrm{C}_2}$ 54

5.1 $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ and $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$ 62

Chapter 1 Introduction

The calculation of homotopy groups of topological spaces and spectra is a major focus in algebraic topology. While the homotopy groups of spaces are hard to compute, stabilization into the realm of spectra makes homotopy calculations more accessible. The stable homotopy category, or homotopy category of spectra, is therefore a good context for homotopy calculations.

Motivic homotopy theory, also known as $\mathbb{A}^1$-homotopy theory, is a way to apply the techniques of algebraic topology, specifically homotopy, to algebraic varieties and, more generally, to schemes. The theory was formulated by Morel and Voevodsky [46]. Equivariant homotopy theory studies spaces with group actions and homotopy classes of equivariant maps between them. Let C_2 denote the cyclic group of order 2. In the stable context, there are many connections between the classical, the complex or real motivic, and the C_2-equivariant stable homotopy categories. As a consequence, many difficult questions located in the classical stable homotopy category can be reconsidered and solved in the motivic and C_2-equivariant contexts. For instance, the Kervaire invariant one problem occurs classically, but is answered via a computation in the C_2-equivariant environment by Hill, Hopkins and Ravenel [33].

One of the primary tools for computing stable homotopy groups of spheres is the Adams spectral sequence. The E_2-page of the Adams spectral sequence is given by $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathcal{A})$, which we denote by Ext, where $\mathcal{A}$ is the classical Steenrod algebra. Adams [2] showed that there is a vanishing line of slope $\frac{1}{2}$ and intercept $\frac{3}{2}$, and J. P. May showed there is a periodicity line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$, where the periodicity operation is defined by the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$.

In classical stable homotopy theory, the interest in periodic v_n-selfmaps of finite spectra lies in the fact that one can associate to each v_n-selfmap an infinite family in the chromatic layer n stable homotopy groups of spheres. Therefore, interest lies in constructing type n spectra and finding v_n-selfmaps of lowest possible periodicity on a given type n spectrum. This, in general, is a difficult problem, though progress has been made sporadically throughout the history of the subject [53, 19, 10, 9, 47, 15, 14]. With the modern development of motivic stable homotopy theory, one may ask if there are similar periodic selfmaps of finite motivic spectra.

In this thesis, we establish a v_1-periodicity region in the E_2-page of the analogous $\mathbb{C}$-motivic Adams spectral sequence. With the modern development of motivic stable homotopy theory, we study analogous periodic self-maps of finite motivic spectra. In particular, we develop a notion of type (n, k) $\mathbb{R}$-motivic spectra parallel to the classical chromatic type n spectra. We construct an $\mathbb{R}$-motivic spectrum $\mathcal{Y}$ and show that it admits a $v_{1,nil}$-selfmap. Let $\mathcal{A}^{\mathbb{R}}$ denote the $\mathbb{R}$-motivic Steenrod algebra and $\mathcal{A}^{\mathbb{R}}(1)$ be the $\mathbb{R}$-motivic Steenrod subalgebra generated by Sq^1 and Sq^2. We prove that $\mathcal{A}^{\mathbb{R}}(1)$ admits 128 $\mathcal{A}^{\mathbb{R}}$-module structures. We also analyze the 128 $\mathcal{A}^{\mathbb{R}}$-module structures, and identify one of them as the cohomology of the cofiber of the $v_{1,nil}$-selfmap of $\mathcal{Y}$.

Chapter 2 provides background on the Steenrod algebra and the stable homotopy category. We review the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic and C_2-equivariant stable homotopy

categories, as well as the restriction and the geometric fixed points functors. We construct the equivariant Steenrod operations using the equivariant extended power construction, which establishes comparisons with the classical Steenrod operations. We also review the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$, discuss the structure of its subalgebra $\mathcal{A}^{\mathbb{R}}(n)$, and give a freeness criterion.

In Chapter 3, we discuss a v_1-periodicity region in the $\mathbb{C}$-motivic Ext groups. The $\mathbb{C}$-motivic Adams spectral sequence, introduced by Morel and developed by Dugger and Isaksen in [24], is a tri-graded spectral sequence that converges to the p-completion of the $\mathbb{C}$-motivic stable homotopy groups of spheres. Voevodsky also developed an analogous $\mathbb{C}$-motivic Steenrod algebra. In the case $p = 2$, there is a vanishing line in $\mathrm{Ext}_{\mathbb{C}}$ of slope 1 and intercept 0. This result was obtained by Guillou and Isaksen in [29]. Quigley has a partial result that $\mathrm{Ext}_{\mathbb{C}}$ has a periodicity line of slope $\frac{1}{3}$ under the condition $s \leq w$ in the case $r = 2$ [49, Corollary 5.4].

Krause used a different approach to obtain the classical periodicity line in his thesis [36]. We adapted this approach for $\mathrm{Ext}_{\mathbb{C}}$. The obstruction is that classical Ext and $\mathrm{Ext}_{\mathbb{C}}$ have different vanishing lines. The vanishing line of $\mathrm{Ext}_{\mathbb{C}}$ is elevated by the h_1-towers, a collection of rays of slope 1. Collecting all the parts except those h_1-towers, i.e. the h_1-torsion part, we get a subgroup of $\mathrm{Ext}_{\mathbb{C}}$.

Let $\mathcal{A}^{\mathbb{C}}_*$ denote the dual $\mathbb{C}$-motivic Steenrod algebra. For $\mathrm{Ext}_{\mathbb{C}}$, we can work over $\mathcal{A}^{\mathbb{C}}_*$ instead of $\mathcal{A}^{\mathbb{C}}$. We first reduce our periodicity question to establishing the vanishing region of certain $\mathrm{Ext}_{\mathbb{C}}$ groups. Then we make an explicit computation for these $\mathrm{Ext}_{\mathbb{C}}$ groups over the dual Steenrod subalgebra $\mathcal{A}^{\mathbb{C}}(1)_*$ to get a starting vanishing region. We transport this vanishing region using the Cartan-Eilenberg spectral sequence along normal extensions of Hopf algebras and obtain the vanishing region of these groups over $\mathcal{A}^{\mathbb{C}}(2)_*$, which is the same as the vanishing region of these $\mathrm{Ext}_{\mathbb{C}}$ groups over $\mathcal{A}^{\mathbb{C}}_*$. The h_1-torsion part has a periodicity region that coincides with the classical case. i.e. above the line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$ we have a v_1-periodic pattern if we ignore h_1-towers.

In Chapter 4, we analyze in detail the question of realizing $\mathcal{A}^{\mathbb{R}}(1)$ as an $\mathbb{R}$-motivic spectrum. Given an $\mathcal{A}$-module M, we say M is realized by X if there is a spectrum X such that $H^*(X) \cong M$ as an $\mathcal{A}$-module. Classically, $\mathcal{A}(1)$ has 4 different $\mathcal{A}$-module structures, which are distinguished by the action of Sq^4 (as in Figure 1.1, where we depict a singly-generated free $\mathcal{A}(1)$-module, where each $\bullet$ represents a $\mathbb{F}_2$-generator. The black and blue lines represent the action of Sq^1 and Sq^2, respectively. The red boxed lines represent the action of Sq^4. Whether or not the dotted red lines exist gives the 4 different $\mathcal{A}$-module structures). And each can be realized as the cohomology of a spectrum.

The existence and uniqueness of a realization is guaranteed by the $\mathbb{R}$-motivic Toda realization theorem. The classical Toda realization theorem [53] (see also [14, Theorem 3.1]), is recast in the modern literature as a special case of Goerss-Hopkins obstruction theory [25] (when the chosen operad is trivial). This obstruction theory can be generalized to the $\mathbb{R}$-motivic setting [43], and 4.1 would then be a special case of such a generalization.

We describe all the 128 $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$. The $\mathbb{R}$-motivic Toda realization theorem indicates that all of them can be realized.

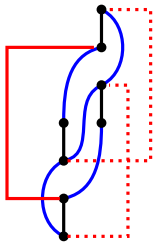


Figure 1.1: The 4 $\mathcal{A}$-module structures of classical $\mathcal{A}(1)$

We then construct one specific realization $\mathcal{A}_1^{\mathbb{R}}$ of the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ using a method of Smith(outlined in [50, Appendix C]), which constructs new finite spectra from known ones. The idea is as follows. If X is a p-local finite spectrum then the permutation group Σ_n acts on $X^{\wedge n}$. One may then use an idempotent $e \in \mathbb{Z}_{(p)}[\Sigma_n]$ to obtain a split summand of the spectrum $X^{\wedge n}$. As explained in [50, Appendix C], Young tableaux provide a rich source of such idempotents. For a judicious choice of e and X, the spectrum $e(X^{\wedge n})$ can be interesting.

We exploit the relation that $\mathsf{h} \cdot \eta_{1,1} = 0$ in $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ [45] to construct an $\mathbb{R}$-motivic analogue of the question mark complex $\mathcal{Q}_{\mathbb{R}}$. The cell-diagram of $\mathcal{Q}_{\mathbb{R}}$ is as described in the picture below. For a choice of idempotent element e in the group ring $\mathbb{Z}_{(2)}[\Sigma_3]$,

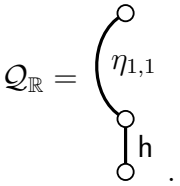


Figure 1.2: Cell-diagram of the $\mathbb{R}$-motivic question mark complex

we observe that $e(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})^{\otimes 3})$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module. This is the cohomology of an $\mathbb{R}$-motivic spectrum $\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3})$, which we call $\Sigma^{1,0}\mathcal{A}_1^{\mathbb{R}}$ (see (4.3.5) for details). The freeness criterion gives that the cohomology of $\mathcal{A}_1^{\mathbb{R}}$ is free over $\mathcal{A}^{\mathbb{R}}(1)$.

We analyze the properties of the image of $\mathcal{A}_1^{\mathbb{R}}$, denoted $\mathcal{A}_1^{\mathrm{C}_2}$, under the Betti realization functor, by applying the comparison theorem. We provide the $\mathcal{A}$-module structures of the underlying and the geometric fixed points for some selected module structures of $\mathcal{A}_1^{\mathrm{C}_2}$.

Chapter 5 describes a v_1-selfmap in the $\mathbb{R}$-motivic setting. There requires a new notion of the chromatic layers. Classically any non-contractible finite p-local spectrum admits a periodic v_n-selfmap for some $n \geq 0$. This is a consequence of the thick-subcategory theorem [34, Theorem 7], aided by a vanishing line argument [34, §4.2]. In the classical case all the thick tensor ideals of $\mathbf{Sp}_{p,\mathrm{fin}}$ (the homotopy category of finite p-local spectra) are also prime (in the sense of [4]). The thick tensor-ideals of the homotopy category of cellular motivic spectra over $\mathbb{C}$ or $\mathbb{R}$ are not completely known (but see [31, 36]). However, one can gather some knowledge about the prime thick tensor-ideals in $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}})$ (the homotopy category of 2-local cellular $\mathbb{R}$-motivic

spectra) through the Betti realization functor

$$\beta : \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$$

using the complete knowledge of prime thick subcategories of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ [5].

The prime thick tensor-ideals of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ are essentially the pull-back of the classical thick subcategories along the two functors, the geometric fixed-point functor

$$\Phi^{\mathrm{C}_2} : \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}})$$

and the forgetful functor

$$\Phi^{e} : \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}) \longrightarrow \mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}}).$$

Let $\mathcal{C}_n$ denote the thick subcategory of $\mathrm{Ho}(\mathbf{Sp}_{2,\mathrm{fin}})$ consisting of spectra of type at least n. The prime thick subcategories,

$$\mathcal{C}(e,n) = (\Phi^e)^{-1}(\mathcal{C}_n) \text{ and } \mathcal{C}(\mathrm{C}_2,n) = (\Phi^{\mathrm{C}_2})^{-1}(\mathcal{C}_n),$$

are the only prime thick subcategories of $\mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$.

Definition 1.0.3. *We say a spectrum $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ is of type (n,m) if $\Phi^e(X)$ is of type n and $\Phi^{\mathrm{C}_2}(X)$ is of type m.*

For a type (n,m) spectrum X, a self-map $f : X \to X$ is periodic if and only if at least one of $\{\Phi^e(f), \Phi^{\mathrm{C}_2}(f)\}$ are periodic (see [6, Proposition 3.17]).

Definition 1.0.4. *Let $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ be of type (n,m). We say a self-map $f : X \to X$ is*

(i) a $v_{(n,m)}$-selfmap of mixed periodicity (i,j) if $\Phi^e(f)$ is a v_n-selfmap of periodicity i and $\Phi^{\mathrm{C}_2}(f)$ is a v_m-selfmap of periodicity j,

(ii) a $v_{(n,\mathrm{nil})}$-selfmap of periodicity i if $\Phi^e(f)$ is a v_n-selfmap of periodicity i and $\Phi^{\mathrm{C}_2}(f)$ is nilpotent, and,

(iii) a $v_{(\mathrm{nil},m)}$-selfmap of periodicity j if $\Phi^e(f)$ is a nilpotent self-map and $\Phi^{\mathrm{C}_2}(f)$ is a v_m-selfmap of periodicity j.

Under this notion, we consider the classical spectrum

$$\mathcal{Y} := \mathrm{M}_2(1) \wedge \mathrm{C}(\eta)$$

that admits, up to homotopy, 8 different v_1-selfmaps of periodicity 1 [19, Section 2] (see also [15]). Here $\mathrm{M}_2(1)$ is the Moore spectrum and $\mathrm{C}(\eta)$ is the cone on η. We ask ourselves if the v_1-selfmaps are equivariant upon providing $\mathcal{Y}$ with interesting C_2-equivariant structures. There are two $\mathbb{C}_2$-equivairant lifts of classical multiplication by 2: 2 and $\mathsf{h} = 1 - \epsilon$ [20], where $\epsilon : S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}$ is the twist map. Similarly, $\eta_{1,0}$ and $\eta_{1,1}$ are the C_2-equivariant lifts of classical Hopf map η.

We will consider four C_2-equivariant lifts of the spectrum $\mathcal{Y}$,

(i) $\mathcal{Y}^{C_2}_{\text{triv}}$, where the action of C_2 is trivial,

(ii) $\mathcal{Y}^{C_2}_{(2,1)} := C^{C_2}(2) \wedge C^{C_2}(\eta_{1,1})$, with $\Phi^{C_2}(\mathcal{Y}^{C_2}_{(2,1)}) = M_2(1) \wedge M_2(1)$,

(iii) $\mathcal{Y}^{C_2}_{(\mathsf{h},0)} := C^{C_2}(\mathsf{h}) \wedge C^{C_2}(\eta_{1,0})$, with $\Phi^{C_2}(\mathcal{Y}^{C_2}_{(\mathsf{h},0)}) = \Sigma C(\eta) \vee C(\eta)$, and,

(iv) $\mathcal{Y}^{C_2}_{(\mathsf{h},1)} := C^{C_2}(\mathsf{h}) \wedge C^{C_2}(\eta_{1,1})$, with $\Phi^{C_2}(\mathcal{Y}^{C_2}_{(\mathsf{h},1)}) = \Sigma M_2(1) \vee M_2(1)$.

The C_2-spectra $\mathcal{Y}^{C_2}_{\text{triv}}$, $\mathcal{Y}^{C_2}_{(2,1)}$ and $\mathcal{Y}^{C_2}_{(\mathsf{h},1)}$ are of type $(1,1)$, and $\mathcal{Y}^{C_2}_{(\mathsf{h},0)}$ is of type $(1,0)$. There are unique $\mathbb{R}$-motivic lifts of the classes 2, h, $\eta_{1,0}$, and $\eta_{1,1}$, and therefore we have unique $\mathbb{R}$-motivic lifts of $\mathcal{Y}^{C_2}_{\text{triv}}$, $\mathcal{Y}^{C_2}_{(2,1)}$, $\mathcal{Y}^{C_2}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{C_2}_{(\mathsf{h},1)}$ which we will simply denote by $\mathcal{Y}^{\mathbb{R}}_{\text{triv}}$, $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$, $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$, respectively.

Using an Atiyah-Hirzebruch like spectral sequence, we show that every map

$$v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

is a non zero permanent cycle, and is necessarily a $v_{(1,\text{nil})}$-selfmap. We also identify its cofiber as one of the 128 $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$.

We then show the non-existence of a $v_{(1,0)}$-selfmap on $C^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$. Classically, the Moore spectrum $M_2(1)$ does admit a v_1-selfmap. But the v_1-selfmaps of $M_2(1)$ are not in the image of the underlying homomorphism

$$\Phi^e \circ \beta \colon [\Sigma^{8k,8k} C^{\mathbb{R}}(\mathsf{h}), C^{\mathbb{R}}(\mathsf{h})]^{\mathbb{R}} \longrightarrow [\Sigma^{8k} M_2(1), M_2(1)].$$

Similarly, the v_1-selfmaps of $\mathcal{Y}$ are not in the image of the underlying homomorphism

$$\Phi^e \circ \beta \colon [\Sigma^{2k,2k} \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}]^{\mathbb{R}} \longrightarrow [\Sigma^{8k} \mathcal{Y}, \mathcal{Y}].$$

However, these results do not preclude the existence of a $v_{(1,0)}$-selfmap on $C^{C_2}(\mathsf{h})$ and $\mathcal{Y}^{C_2}_{(\mathsf{h},0)}$. Forthcoming work [27] of Guillou and Isaksen shows that 8σ is in the image of $\Phi^e \colon \pi_{7,8}(\mathbb{S}_{C_2}) \longrightarrow \pi_7(\mathbb{S})$ and suggests that $C^{C_2}(\mathsf{h})$ supports a $v_{(1,0)}$-selfmap.

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Chapter 2 The Steenrod algebra and the squaring operation

The stable homotopy category $\mathrm{Ho}(\mathbf{Sp})$ is the category of spectra and homotopy classes of morphisms between them. If we use the ($\mathbb{C}$ or $\mathbb{R}$-) motivic spectra or the C_2-equivariant spectra, we obtain the ($\mathbb{C}$ or $\mathbb{R}$-) motivic stable homotopy category or the C_2-equivariant stable homotopy category (see [46] and [33]). They all have the notion of Eilenberg-Mac Lane spectra and satisfy the Brown's representability theorem.

$$\begin{array}{ccc} \mathrm{Ho}(\mathbf{Sp})^{\mathbb{R}} & \xrightarrow{\mathbb{C}\otimes_{\mathbb{R}}(-)} & \mathrm{Ho}(\mathbf{Sp})^{\mathbb{C}} \\ {\scriptstyle Be^{\mathbb{R}}}\downarrow & & \downarrow{\scriptstyle Be^{\mathbb{C}}} \\ \mathrm{Ho}(\mathbf{Sp})^{C_2} & \xrightarrow[\Phi^e]{} & \mathrm{Ho}(\mathbf{Sp}) \end{array}$$

The vertical functors are called the Betti realization functors.

For any C_2-equivariant space $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$ we can functorially assign two non-equivariant spaces – the underlying space $\Phi^e(\mathrm{X})$, which is obtained by restricting the action of C_2 to the trivial group, and the space of C_2-fixed-points $\mathrm{X}^{\mathrm{C}_2}$. For a C_2-equivariant spectrum $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$, restricting the action to the trivial subgroup results in a monoidal functor

$$\Phi^e : \mathbf{Sp}^{\mathrm{C}_2} \longrightarrow \mathbf{Sp}$$

that identifies the underlying spectrum. However, there are two different notions of fixed-point spectrum – the categorial fixed-points and the geometric fixed-points.

The categorical fixed-points functor is a lax monodial functor

$$(-)^{\mathrm{C}_2} : \mathbf{Sp}^{\mathrm{C}_2} \longrightarrow \mathbf{Sp},$$

which is defined so that $\pi_k(\mathrm{E}^{\mathrm{C}_2}) \cong \pi_k^{\mathrm{C}_2}(\mathrm{E})$, but it does not interact well with infinite suspensions. The correction term is explained by the tom Dieck splitting [37, Theorem V.11.1]:

$$(\Sigma^\infty_{\mathrm{C}_2}\mathrm{X})^{\mathrm{C}_2} \simeq \Sigma^\infty(\mathrm{X}^{\mathrm{C}_2}) \vee \Sigma^\infty(\mathrm{X}_{\mathrm{hC}_2}), \tag{2.0.1}$$

where $\mathrm{X}_{\mathrm{hC}_2}$ is the homotopy orbit space. Let $\widetilde{\mathrm{EC}_2} := \mathrm{Cof}(\mathrm{EC}_{2+} \to \mathbb{S})$. The geometric fixed-point functor

$$\Phi^{C_2} : \mathbf{Sp}^{\mathrm{C}_2} \longrightarrow \mathbf{Sp},$$

is a symmetric monoidal functor given by $\Phi^{C_2}\mathrm{E} := (\mathrm{E} \wedge \widetilde{\mathrm{EC}_2})^{\mathrm{C}_2}$. When $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$,

$$\Phi^{C_2}(\Sigma^\infty_{\mathrm{C}_2}\mathrm{E}) \simeq \Sigma^\infty\mathrm{E}^{\mathrm{C}_2} \tag{2.0.2}$$

is the first component in (2.0.1). For any $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$, there is a natural map of spectra

$$\iota_{\mathrm{E}} : \mathrm{E}^{\mathrm{C}_2} \longrightarrow \Phi^{C_2}\mathrm{E}$$

induced by the map $\mathbb{S} \to \widetilde{\mathrm{EC}_2}$.

The Eilenberg-Mac Lane spectrum $\mathrm{H}\underline{\mathbb{F}_2}$ is an $\mathbb{E}_\infty^{\mathrm{C}_2}$-ring ([37, VII]), i.e. a commutative monoid as a genuine C_2-spectrum. The restriction $\Phi^e\mathrm{H}\underline{\mathbb{F}_2} \simeq \mathrm{H}\mathbb{F}_2$, the categorical fixed-points $\mathrm{H}\underline{\mathbb{F}_2}^{\mathrm{C}_2} \simeq \mathrm{H}\mathbb{F}_2$ and the geometric fixed-points $\Phi^{C_2}\mathrm{H}\underline{\mathbb{F}_2} \simeq \mathrm{H}\mathbb{F}_2[t]$ are $\mathbb{E}_\infty$-rings. It follows from the knowledge of $\mathbb{M}_2^{\mathrm{C}_2} := \pi_\star^{\mathrm{C}_2}\mathrm{H}\underline{\mathbb{F}_2}$ that for $n \geq 0$

$$(\Sigma^{n\sigma}\mathrm{H}\underline{\mathbb{F}_2})^{\mathrm{C}_2} \simeq \bigvee_{i=0}^{n} \Sigma^i \mathrm{H}\mathbb{F}_2 \hookrightarrow \Phi^{C_2}\mathrm{H}\underline{\mathbb{F}_2} \simeq \operatorname*{colim}_{n\to\infty} (\Sigma^{n\sigma}\mathrm{H}\underline{\mathbb{F}_2})^{\mathrm{C}_2} \simeq \mathrm{H}\mathbb{F}_2[t]$$

is the inclusion of the first $(n+1)$ components. The above map clearly splits. One can endow $(\Sigma^{n\sigma}\mathrm{H}\underline{\mathbb{F}_2})^{\mathrm{C}_2}$ with an $\mathbb{E}_\infty$-structure isomorphic to the truncated polynomial algebra $\mathrm{H}\mathbb{F}_2[t]/(t^{n+1})$ so that the splitting map

$$\pi_{\mathbb{F}_2}^{(n)} : \Phi^{C_2}\mathrm{H}\underline{\mathbb{F}_2} \simeq \mathrm{H}\mathbb{F}_2[t] \twoheadrightarrow (\Sigma^{n\sigma}\mathrm{H}\underline{\mathbb{F}_2})^{\mathrm{C}_2} \simeq \mathrm{H}\mathbb{F}_2[t]/(t^{n+1})$$

is an $\mathbb{E}_\infty$-map. Alternatively, $\pi_{\mathbb{F}_2}^{(n)}$ can be obtained as an $\mathbb{E}_\infty$-map by applying the functor H to the map of commutative rings $\mathbb{F}_2[t] \twoheadrightarrow \mathbb{F}_2[t]/(t^{n+1})$. The composition

$$\mathrm{H}\underline{\mathbb{F}_2}^{\mathrm{C}_2} \xhookrightarrow{\iota_{\mathbb{F}_2}} \Phi^{C_2}\mathrm{H}\underline{\mathbb{F}_2} \xrightarrow{\pi_{\mathbb{F}_2}^{(0)}} \mathrm{H}\underline{\mathbb{F}_2}^{\mathrm{C}_2} \tag{2.0.3}$$

is the identity and exhibits $\Phi^{C_2}\mathrm{H}\underline{\mathbb{F}_2}$ as an augmented $\mathrm{H}\mathbb{F}_2$-algebra.

For any C_2-space $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$, the restriction functor induces a natural transformation

$$\Phi_*^e : \mathrm{H}_{C_2}^{i,j}(\mathrm{X}_+) \longrightarrow \mathrm{H}^i(\Phi^e(\mathrm{X})_+).$$

To compare the cohomology of $\mathrm{X}^{\mathrm{C}_2}$ with the $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology of X, we make use of the splitting (2.0.3) to define the natural ring map

$$\widehat{\Phi^{C_2}}_* : \mathrm{H}_{C_2}^{i,j}(\mathrm{X}_+) \longrightarrow \mathrm{H}^{i-j}(\mathrm{X}_+^{\mathrm{C}_2}),$$

which sends $u \in \mathrm{H}_{C_2}^{i,j}(\mathrm{X}_+)$ to the composite (as defined in [11, 2.7])

$$\Sigma^\infty \mathrm{X}^{\mathrm{C}_2} \xrightarrow{\Phi^{C_2}u} \Sigma^{i-j}\Phi^{C_2}\mathrm{H}\underline{\mathbb{F}_2} \xrightarrow{\pi_{\mathbb{F}_2}^{(0)}} \Sigma^{i-j}\mathrm{H}\mathbb{F}_2.$$

2.1 Squaring operation

For prime $p = 2$, the classical squaring operations are natural transformations of the classical cohomology groups.

$$\mathrm{Sq}^n : \mathrm{H}^*(-) \to \mathrm{H}^{*+n}(-)$$

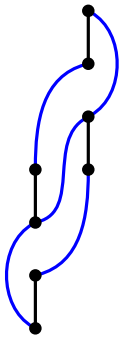
They commute with the suspension. The classical Steenrod algebra $\mathcal{A}$ is the graded Hopf algebra generated by the Steenrod squares. If we truncate on the squaring operations generation $\mathcal{A}$, we get the Steenrod subalgebra.

Example 2.1.1.

- *The Steenrod subalgebra $\mathcal{A}(0)$ is generated by Sq^1. The black line represents the action of* Sq^1.



- *The Steenrod subalgebra $\mathcal{A}(1)$ is generated by Sq^1 and Sq^2. The black and blue lines represent the action of* Sq^1 *and* Sq^2*, respectively.*



If we replace the classical cohomology groups with the $\mathbb{C}$ or $\mathbb{R}$-motivic cohomologies or the C_2-equivariant cohomology (also known as the $RO(C_2)$-graded cohomology), we obtain the notions of the $\mathbb{C}$ or $\mathbb{R}$-motivic Steenrod algebras and the C_2-equivariant Steenrod algebra. We denote them as $\mathcal{A}^{\mathbb{C}}$, $\mathcal{A}^{\mathbb{R}}$ and $\mathcal{A}^{C_2}$, respectively.

Remark 2.1.2. *In contrast with the names, the $\mathbb{R}$-motivic Steenrod algebra and the C_2-equivariant Steenrod algebra are not Hopf algebras. Their duals, both have a left and a right units, are Hopf algebroids. (See [50] for detail definitions.)*

2.1.1 Power operation construction

The construction of the classical mod 2 Steenrod algebra, which is the algebra of stable cohomology operations for ordinary cohomology with $\mathbb{F}_2$-coefficients, involves the $\mathbb{E}_\infty$-structure[1] of $\mathrm{H}\mathbb{F}_2$ and the fact that the tautological line bundle γ over $\mathbb{RP}^\infty$ is $\mathrm{H}\mathbb{F}_2$-orientable. We review here how the mod 2 Steenrod operations are derived from that structure. A similar discussion can be found in [16, Section VIII.2].

Notation 2.1.3. *For any space or spectrum* X *and $n \geq 1$, we let*

$$\mathrm{D}_n(\mathrm{X}) := (\mathrm{E}\Sigma_n)_+ \wedge_{\Sigma_n} (\mathrm{X}^{\wedge n}),$$

where Σ_n acts by permuting the factors of $\mathrm{X}^{\wedge n}$*. By convention,* $\mathrm{D}_0(\mathrm{X}) = \mathbb{S}$.

An $\mathbb{E}_\infty$-ring structure on a spectrum R gives a collection of maps of the form

$$\Theta_n^{\mathrm{R}} : \mathrm{D}_n(\mathrm{R}) \longrightarrow \mathrm{R}$$

for each $n \geq 0$, which satisfy the usual coherence conditions (see [42]). By assumption, Θ_0^{R} is the unit map of R and Θ_1^{R} is the identity map.

[1] Technically, we only make use of the $\mathbb{H}_\infty$-ring structure that underlies the $\mathbb{E}_\infty$-structure of $\mathrm{H}\mathbb{F}_2$

The $\mathrm{H}\mathbb{F}_2$-orientibility of $\boldsymbol{\gamma}$ implies the existence of an $\mathrm{H}\mathbb{F}_2$-Thom class

$$\mathsf{u}_n : \mathrm{Th}(\boldsymbol{\gamma}^{\oplus n}) \simeq \mathbb{RP}^\infty_n \longrightarrow \Sigma^n \mathrm{H}\mathbb{F}_2 \tag{2.1.4}$$

for each $n \geq 0$. These are compatible as n varies, in the sense that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Th}(\boldsymbol{\gamma}^{\oplus(m+n)}) & \longrightarrow & \mathrm{Th}(\boldsymbol{\gamma}^{\oplus m}) \wedge \mathrm{Th}(\boldsymbol{\gamma}^{\oplus n}) & \xrightarrow{\mathsf{u}_m \wedge \mathsf{u}_n} & \Sigma^m \mathrm{H}\mathbb{F}_2 \wedge \Sigma^n \mathrm{H}\mathbb{F}_2 \\ & \searrow^{\mathsf{u}_{m+n}} & & & \downarrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ & & & & \mathrm{H}\mathbb{F}_2. \end{array} \tag{2.1.5}$$

For any spectra E and F, there is a natural map

$$\delta_n : \mathrm{D}_n(\mathrm{E} \wedge \mathrm{F}) \longrightarrow \mathrm{D}_n(\mathrm{E}) \wedge \mathrm{D}_n(\mathrm{F})$$

induced by the diagonal on $\mathrm{E}\Sigma_n$ and the isomorphism $(\mathrm{E} \wedge \mathrm{F})^{\wedge n} \cong \mathrm{E}^{\wedge n} \wedge \mathrm{F}^{\wedge n}$. Thus, we may define the map τ_n as the composition

$$\begin{array}{ccc} \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2) & \xrightarrow{\qquad\qquad \tau_n \qquad\qquad} & \Sigma^{2n} \mathrm{H}\mathbb{F}_2 \\ \downarrow{\scriptstyle \delta_2} & & \uparrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ \mathrm{D}_2(\mathrm{S}^n) \wedge \mathrm{D}_2(\mathrm{H}\mathbb{F}_2) & \xrightarrow{\simeq} \Sigma^n \mathbb{RP}^\infty_n \wedge \mathrm{D}_2(\mathrm{H}\mathbb{F}_2) \xrightarrow{\Sigma^n \mathsf{u}_n \wedge \Theta_2^{\mathbb{F}_2}} & \Sigma^{2n} \mathrm{H}\mathbb{F}_2 \wedge \mathrm{H}\mathbb{F}_2. \end{array} \tag{2.1.6}$$

Definition 2.1.7. *The power operation is the natural transformation*

$$\mathcal{P}_2 \colon \mathrm{H}^n(-) \longrightarrow \mathrm{H}^{2n}(\mathrm{D}_2(-))$$

which takes a class $u \in \mathrm{H}^n(\mathrm{E})$ to the composite class

$$\mathcal{P}_2(u) \colon \mathrm{D}_2(\mathrm{E}) \xrightarrow{\mathrm{D}_2(u)} \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2) \xrightarrow{\tau_n} \Sigma^n \mathrm{H}\mathbb{F}_2$$

for any $\mathrm{E} \in \mathbf{Sp}$.

From (2.1.5), we deduce the commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2 \wedge \Sigma^m \mathrm{H}\mathbb{F}_2) & \longrightarrow & \mathrm{D}_2(\Sigma^n \mathrm{H}\mathbb{F}_2) \wedge \mathrm{D}_2(\Sigma^m \mathrm{H}\mathbb{F}_2) & \xrightarrow{\tau_n \wedge \tau_m} & \Sigma^{2n} \mathrm{H}\mathbb{F}_2 \wedge \Sigma^{2m} \mathrm{H}\mathbb{F}_2 \\ \downarrow{\scriptstyle \mathrm{D}_2(\mu_{\mathbb{F}_2})} & & & & \downarrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ \mathrm{D}_2(\Sigma^{n+m} \mathrm{H}\mathbb{F}_2) & & \xrightarrow[\tau_{n+m}]{\qquad\qquad\qquad\qquad} & & \Sigma^{2n+2m} \mathrm{H}\mathbb{F}_2. \end{array} \tag{2.1.8}$$

As a result, we have

$$\delta_2^*(\mathcal{P}_2(u) \otimes \mathcal{P}_2(v)) = \mathcal{P}_2(u \otimes v)$$

which leads to the Cartan formula for the Steenrod algebra.

If $X \in \mathbf{Top}_*$ is given the trivial Σ_2-action and $X \wedge X$ the permutation action, the diagonal map $X \to X \wedge X$ is Σ_2-equivariant. Consequently, we have an induced map

$$\Delta_X : (B\Sigma_2)_+ \wedge X \simeq (E\Sigma_2)_+ \wedge_{\Sigma_2} X \longrightarrow D_2(X).$$

Since $H^*(B\Sigma_2) \cong \mathbb{F}_2[\mathsf{t}]$, we may write (using the Kunneth isomorphism)

$$\Delta_X^*(\mathcal{P}_2(u)) = \sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \mathrm{Sq}^i(u), \tag{2.1.9}$$

which defines the natural transformations $\mathrm{Sq}^i \colon H^n(-) \longrightarrow H^{n+i}(-)$.

Remark 2.1.10. *The squaring operation $\mathrm{Sq}^i(u)$ for any class $u \in H^n(X)$ is determined by $\mathrm{Sq}^i(\iota_n)$, where $\iota_n \in H^*(K(\mathbb{F}_2, n))$ is the fundamental class, because of the universal property of $K(\mathbb{F}_2, n)$. A priori, $\mathrm{Sq}^i(u)$ depends on the cohomological degree of u. However, this dependence is eradicated by the fact that the squaring operations are stable, i.e. for any $u \in H^*(X)$*

$$\mathrm{Sq}^i(\sigma_*(u)) = \sigma_*(\mathrm{Sq}^i(u)),$$

where $\sigma_* : H^*(X) \cong H^{*+1}(\Sigma X)$ is the suspension isomorphism. The $H\mathbb{F}_2$-orientibility of γ implies $\mathrm{Sq}^0(\iota) = \iota$ for the generator $\iota \in H^1(S^1)$, which, along with Cartan formula, implies stability.

2.1.2 The C_2-equivairant squaring operation

The construction of the classical squaring operations can be adapted to construct squaring operations on the $\mathrm{RO}(C_2)$-graded cohomology of a C_2-space.

Remark 2.1.11. *Our ideas are closely related to the construction of the $\mathbb{R}$-motivic squaring operations due to Voevodsky [54]. Certain parts, such as the construction of the power operation Definition 2.1.20, though different, can be compared to [56, 57], where the author studies C_2-equivariant power operations on the homology of spaces.*

Notation 2.1.12. *For any group G and a family of subgroups $\mathcal{F}$ closed under subconjugacy, there exists a space $E\mathcal{F}$ determined up to a G-weak equivalence by its universal property*

$$E\mathcal{F}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F}, \\ \emptyset & \text{otherwise.} \end{cases}$$

When $G = C_2 \times \Sigma_n$ and $\mathcal{F}_n = \{H \subset G : H \cap \Sigma_n = \mathbb{1}\}$, we denote $E\mathcal{F}_n$ by $E_{C_2}\Sigma_n$. Note that there is a natural C_2-equivariant map $E\Sigma_n \longrightarrow E_{C_2}\Sigma_n$.

Notation 2.1.13. *For a based C_2-space or a C_2-spectrum X, we let*

$$D_n^{C_2}(X) := (E_{C_2}\Sigma_n)_+ \wedge_{\Sigma_n} (X^{\wedge n})$$

the n-th equivariant extended power construction on X. *There is a natural* C_2*-equivariant map*

$$\delta_n^{\mathrm{C}_2} : \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{X} \wedge \mathrm{Y}) \longrightarrow \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{X}) \wedge \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{Y})$$

induced by the diagonal map of $\mathrm{E}_{\mathrm{C}_2}\Sigma_n$ *for any pair* X *and* Y *of* C_2 *space or spectra.*

For a C_2-equivariant space $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$, the inclusions $\mathrm{X}^{\mathrm{C}_2} \hookrightarrow \mathrm{X}$ and $\mathrm{E}\Sigma_n \longrightarrow \mathrm{E}_{\mathrm{C}_2}\Sigma_n$ together induce a natural map

$$\lambda_{\mathrm{X}} : \mathrm{D}_2(\mathrm{X}^{\mathrm{C}_2}) \longrightarrow \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{X})^{\mathrm{C}_2} \tag{2.1.14}$$

which is usually not an equivalence.

Example 2.1.15. *When* $\mathrm{X} \simeq \mathrm{S}^0$, $\lambda_{\mathrm{S}^0} : (\mathrm{B}\Sigma_2)_+ \longrightarrow (\mathrm{B}_{\mathrm{C}_2}\Sigma_2)^{\mathrm{C}_2} \simeq \mathrm{B}\Sigma_2 \wedge \mathrm{S}^0_+$ *is the inclusion of a summand.*

Likewise, when $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$, the map $\mathrm{E}^{\mathrm{C}_2} \hookrightarrow \mathrm{E}$ induces a natural map

$$\lambda_{\mathrm{E}} : \mathrm{D}_2(\mathrm{E}^{\mathrm{C}_2}) \longrightarrow \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{E})^{\mathrm{C}_2}.$$

Using the fact that $\widetilde{\mathrm{EC}_2}$ is an $\mathbb{E}_\infty$-ring C_2-spectrum we define a map $\lambda_{\mathrm{E}}^{\Phi}$ as the composition

$$\begin{array}{ccccc}
\mathrm{D}_2(\Phi^{C_2}\mathrm{E}) & & \xrightarrow{\qquad\qquad\lambda_{\mathrm{E}}^{\Phi}\qquad\qquad} & & \Phi^{C_2}\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{E}) \\
\downarrow{\scriptstyle \lambda_{\widetilde{\mathrm{EC}_2}\wedge\mathrm{E}}} & & & & \uparrow \\
(\mathrm{D}_2(\widetilde{\mathrm{EC}_2} \wedge \mathrm{E}))^{\mathrm{C}_2} & \longrightarrow & (\mathrm{D}_2(\widetilde{\mathrm{EC}_2}) \wedge \mathrm{D}_2(\mathrm{E}))^{\mathrm{C}_2} & \longrightarrow & (\widetilde{\mathrm{EC}_2} \wedge \mathrm{D}_2(\mathrm{E}))^{\mathrm{C}_2} \cong \Phi^{C_2}\mathrm{D}_2(\mathrm{E})
\end{array} \tag{2.1.16}$$

By definition, an $\mathbb{E}_\infty^{\mathrm{C}_2}$-ring structure on a spectrum R consists of a system of maps

$$\Theta_n^{\mathrm{R}} : \mathrm{D}_n^{\mathrm{C}_2}(\mathrm{R}) \longrightarrow \mathrm{R}$$

for each $n \geq 0$, which satisfy certain compatibility criteria [37, §VII.2]. The categorical fixed-point spectrum $\mathrm{R}^{\mathrm{C}_2}$ as well as the geometric-fixed point spectrum $\Phi^{C_2}\mathrm{R}$ of an $\mathbb{E}_\infty^{\mathrm{C}_2}$-ring spectrum R are $\mathbb{E}_\infty$-ring spectra with structure maps

$$\Theta_n^{\mathrm{R}^{\mathrm{C}_2}} : \mathrm{D}_2(\mathrm{R}^{\mathrm{C}_2}) \xrightarrow{\lambda_{\mathrm{R}}} \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{R})^{\mathrm{C}_2} \xrightarrow{(\Theta_n^{\mathrm{R}})^{\mathrm{C}_2}} \mathrm{R}^{\mathrm{C}_2}$$

and

$$\Theta_n^{\Phi^{C_2}\mathrm{R}} : \mathrm{D}_2(\Phi^{C_2}\mathrm{R}) \xrightarrow{\lambda_{\mathrm{R}}^{\Phi}} \Phi^{C_2}\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{R}) \xrightarrow{\Phi^{C_2}\Theta_n^{\mathrm{R}}} \Phi^{C_2}\mathrm{R},$$

respectively. Further, the natural map

$$\iota_{\mathrm{R}} : \mathrm{R}^{\mathrm{C}_2} \longrightarrow \Phi^{C_2}\mathrm{R}$$

is an $\mathbb{E}_\infty$-ring map.

Let ω denote the sign representation of Σ_2. The equivariant Eilenberg-Mac Lane spectrum $\mathrm{H}\underline{\mathbb{F}_2}$ does not distinguish between the C_2-equivariant bundles

$$\overline{\epsilon} : \mathrm{E}_{\mathrm{C}_2}\Sigma_2 \times_{\Sigma_2} (\rho) \longrightarrow \mathrm{B}_{\mathrm{C}_2}\Sigma_2$$

$$\overline{\gamma} : \mathrm{E}_{\mathrm{C}_2}\Sigma_2 \times_{\Sigma_2} (\rho \otimes \omega) \longrightarrow \mathrm{B}_{\mathrm{C}_2}\Sigma_2,$$

i.e. there exists a C_2-equivariant Thom isomorphism (see Remark 2.1.19)

$$\mathrm{Th}(\overline{\gamma}) \wedge \mathrm{H}\underline{\mathbb{F}_2} \simeq \mathrm{Th}(\overline{\epsilon}) \wedge \mathrm{H}\underline{\mathbb{F}_2} \simeq \Sigma^{\rho}(\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+ \wedge \mathrm{H}\underline{\mathbb{F}_2}. \tag{2.1.17}$$

The above Thom isomorphism results in an $\mathrm{H}\underline{\mathbb{F}_2}$-Thom class

$$\underline{\mathfrak{u}}_n : \mathrm{Th}(\overline{\gamma}^{\oplus n}) \longrightarrow \Sigma^{n\rho}\mathrm{H}\underline{\mathbb{F}_2}$$

for each $n \geq 0$, and these Thom classes can be used to define the C_2-equivariant power operations. Since

$$\mathrm{D}_2^{\mathrm{C}_2}(\mathrm{S}^{n\rho}) \simeq \mathrm{Th}(n\rho \oplus n(\rho \otimes \omega)) \simeq \Sigma^{n\rho}\mathrm{Th}(\overline{\gamma}^{\oplus n}),$$

we define the map $\underline{\tau}_n$ as the composition

$$\begin{array}{ccc} \mathrm{D}_2(\Sigma^{n\rho}\mathrm{H}\underline{\mathbb{F}_2}) & \xrightarrow{\qquad\qquad\underline{\tau}_n\qquad\qquad} & \Sigma^{2n\rho}\mathrm{H}\underline{\mathbb{F}_2} \\ \downarrow & & \uparrow{\scriptstyle \mu_{\mathbb{F}_2}} \\ \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{S}^{n\rho}) \wedge \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{H}\underline{\mathbb{F}_2}) & \xrightarrow{\simeq} \Sigma^{n\rho}\mathrm{Th}(\overline{\gamma}^{\oplus n}) \wedge \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{H}\underline{\mathbb{F}_2}) & \xrightarrow{\Sigma^n \underline{\mathfrak{u}}_n \wedge \Theta_2^{\mathbb{F}_2}} \Sigma^{2n\rho}\mathrm{H}\underline{\mathbb{F}_2} \wedge \mathrm{H}\underline{\mathbb{F}_2}. \end{array} \tag{2.1.18}$$

Remark 2.1.19. *The Thom isomorphism* (2.1.17) *does not follow immediately from the general theory of equivariant Thom isomorphisms [18] because the basespace* $\mathrm{B}_{\mathrm{C}_2}\Sigma_2$ *is not* C_2*-connected. In such cases, there is no guarantee of an* $\mathrm{H}\underline{\mathbb{F}_2}$*-Thom class; instead an* $\mathrm{H}\underline{\mathbb{F}_2}$*-orientation is encoded by a family of classes (see [41]). However, there is a map of* C_2*-equivariant* $\mathbb{R}$*-vector bundles from* $\overline{\gamma}$ *to the tautological Atiyah Real line bundle over* $\mathrm{BS}^{\sigma} = \mathrm{BU}_{\mathbb{R}}(1)$*, which is* C_2*-connected. Further, the tautological Atiyah Real line bundle admits a single* $\mathrm{H}\underline{\mathbb{F}_2}$*-Thom class as* $\mathrm{H}\underline{\mathbb{F}_2^{\times}}$ *is* C_2*-equivariantly contractible. Therefore,* $\overline{\gamma}$ *also admits a single* $\mathrm{H}\underline{\mathbb{F}_2}$*-Thom class which leads to* (2.1.17) *using a standard argument involving the Thom diagonal map.*

Definition 2.1.20. *The equivariant power operation is the natural transformation*

$$\mathcal{P}_2^{\mathrm{C}_2} : \mathrm{H}_{C_2}^{n\rho}(-) \longrightarrow \mathrm{H}_{C_2}^{2n\rho}(\mathrm{D}_2^{\mathrm{C}_2}(-))$$

which takes a class $u \in \mathrm{H}_{C_2}^{n\rho}(\mathrm{E})$ *to the composite class*

$$\mathcal{P}_2^{\mathrm{C}_2}(u) : \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{E}) \xrightarrow{\mathrm{D}_2^{\mathrm{C}_2}(u)} \mathrm{D}_2^{\mathrm{C}_2}(\Sigma^{n\rho}\mathrm{H}\underline{\mathbb{F}_2}) \xrightarrow{\underline{\tau}_n} \Sigma^{2n\rho}\mathrm{H}\underline{\mathbb{F}_2}$$

for any $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}$.

When $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$ is given the trivial Σ_2-action and $\mathrm{X} \wedge \mathrm{X}$ is given the permutation action, the diagonal map $\mathrm{X} \to \mathrm{X} \wedge \mathrm{X}$ is a $\mathrm{C}_2 \times \Sigma_2$-equivariant map. Consequently, we have a C_2-equivariant map

$$\Delta_{\mathrm{X}}^{\mathrm{C}_2} : (\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+ \wedge \mathrm{X} \simeq (\mathrm{E}_{\mathrm{C}_2}\Sigma_2)_+ \wedge_{\Sigma_2} \mathrm{X} \longrightarrow \mathrm{D}_2^{\mathrm{C}_2}(\mathrm{X}).$$

By [35, Lemma 6.27] (also see [56, Proposition 3.2]),

$$\mathrm{H}^\star_{C_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+) \cong \mathbb{M}_2^{\mathrm{C}_2}[\mathsf{y}, \mathsf{x}]/(\mathsf{y}^2 = a_\sigma \mathsf{y} + u_\sigma \mathsf{x}),$$

where $|\mathsf{y}| = (1,1)$ and $|\mathsf{x}| = (2,1)$. Since $\mathrm{H}^\star_{C_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+)$ is $\mathbb{M}_2^{\mathrm{C}_2}$-free, we also have a Kunneth isomorphism

$$\mathrm{H}^\star_{C_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+ \wedge \mathrm{X}) \cong \mathrm{H}^\star_{C_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+) \otimes_{\mathbb{M}_2^{\mathrm{C}_2}} \mathrm{H}^\star_{C_2}(\mathrm{X}).$$

Thus, for any $u \in \mathrm{H}^{n\rho}_{C_2}(\mathrm{X})$, we may write $(\Delta_{\mathrm{X}}^{\mathrm{C}_2})^*(\mathcal{P}_2^{\mathrm{C}_2}(u))$ using the formula

$$(\Delta_{\mathrm{X}}^{\mathrm{C}_2})^*(\mathcal{P}_2^{\mathrm{C}_2}(u)) = \sum_{i=0}^{n} \mathsf{x}^{n-i} \otimes \underline{\mathrm{Sq}}^{2i}(u) + \sum_{i=0}^{n} \mathsf{y}\mathsf{x}^{n-i-1} \otimes \underline{\mathrm{Sq}}^{2i+1}(u), \tag{2.1.21}$$

which defines the equivariant squaring operations $\underline{\mathrm{Sq}}^i$ for all $i \geq 0$. These can be extended to operations on the entire $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology ring as in [54, Prop 2.6]).

Remark 2.1.22. *Just like the classical case, one can easily deduce that the* $\mathrm{RO}(\mathrm{C}_2)$*-graded squaring operations defined this way are natural, stable and obey the Cartan formula. In fact, Voevodsky [54] uses a similar approach to establish these properties for the* $\mathbb{R}$*-motivic Steenrod algebra, which can be emulated in the* C_2*-equivariant case using the Betti realization functor.*

2.1.3 Comparison theorem

The purpose of this section is to compare the $\mathrm{RO}(\mathrm{C}_2)$-graded squaring operations with the classical squaring operations along the maps Φ^e_* and $\widehat{\Phi^{C_2}}_*$, which renders the following theorems.

Theorem 2.1.23. *For* $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}$ *and any class* $u \in \mathrm{H}^\star_{C_2}(\mathrm{E})$, $\Phi^e_*(\underline{\mathrm{Sq}}^n(u)) = \mathrm{Sq}^n(\Phi^e_*(u))$.

Theorem 2.1.24. *For* $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}$ *and any class* $u \in \mathrm{H}^\star_{C_2}(\mathrm{E})$,

$$\widehat{\Phi^{C_2}}_*(\underline{\mathrm{Sq}}^{2n}(u)) = \mathrm{Sq}^n(\widehat{\Phi^{C_2}}_*(u)).$$

Since the restriction functor is monoidal, it induces a ring map

$$\Phi^e_* : \mathrm{H}^\star_{C_2}(\mathrm{X}_+) \longrightarrow \mathrm{H}^*(\Phi^e(\mathrm{X})_+)$$

for any $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$.

Example 2.1.25. *When* $\mathrm{X} = *$*, the map*

$$\Phi^e_* : \pi^{\mathrm{C}_2}_\star \mathrm{H}\underline{\mathbb{F}_2} \longrightarrow \pi_* \mathrm{H}\mathbb{F}_2 \cong \mathbb{F}_2$$

sends $u_\sigma \mapsto 1$*,* $a_\sigma \mapsto 0$*, and* $\Theta \mapsto 0$*. This follows from the fact that the cofiber sequence* $\mathrm{C}_{2+} \longrightarrow S^0 \xrightarrow{a_\sigma} S^\sigma$ *shows that the kernel of* Φ^e_* *consists of precisely the* a_σ*-divisible elements.*

Proposition 2.1.26. *For any* $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$ *and a class* $u \in \mathrm{H}^{n\rho}_{C_2}(\mathrm{X})$

$$\Phi^e_*(\mathcal{P}^{\mathrm{C}_2}_2(u)) = \mathcal{P}_2(\Phi^e_*(u)).$$

Proof. Since, $\Phi^e(\overline{\gamma}) = 2\gamma$, it follows that $\Phi^e_*(\underline{\mathsf{u}}_n) = \mathsf{u}_{2n}$. This, along with the fact that $\Phi^e(\Theta^{\mathbb{F}_2}_2) = \Theta^{\mathbb{F}_2}_2$ shows $\Phi^e(\underline{\tau}_n) = \tau_{2n}$, and the result follows. □

Proof of Theorem 2.1.23. Let $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$ and $u \in \mathrm{H}^{n\rho}_{C_2}(\mathrm{X})$. Since $\Phi^e(\mathrm{B}_{\mathrm{C}_2}\Sigma_2) \simeq \mathrm{B}\Sigma_2$, $\Phi^e(\Delta^{\mathrm{C}_2}) = \Delta$, $\Phi^e_*(\mathsf{y}) = \mathsf{t}$ and $\Phi^e_*(\mathsf{x}) = \mathsf{t}^2$, it follows that

$$\Delta^*_{\Phi^e(\mathrm{X})}(\mathcal{P}_2(\Phi^e_*(u))) \;=\; \sum_{i=-n}^{n} \mathsf{t}^{n-i} \otimes \mathrm{Sq}^i(\Phi^e_*(u))$$

must equal

$$\begin{aligned}
\Phi^e_*((\Delta^{\mathrm{C}_2}_{\mathrm{X}})^*(\mathcal{P}^{\mathrm{C}_2}_2(u))) &= \Phi^e_*\Big(\sum_{i=-n}^{n} \mathsf{x}^{n-i} \otimes \underline{\mathrm{Sq}}^{2i}(u) + \sum_{i=-n}^{n} \mathsf{y}\mathsf{x}^{n-i-1} \otimes \underline{\mathrm{Sq}}^{2i+1}(u)\Big) \\
&= \sum_{i=-n}^{n} \mathsf{t}^{2n-2i} \otimes \Phi^e_*(\underline{\mathrm{Sq}}^{2i}(u)) \\
&\qquad + \sum_{i=-n}^{n} \mathsf{t}^{2n-2i-1} \otimes \Phi^e_*(\underline{\mathrm{Sq}}^{2i+1}(u)).
\end{aligned}$$

Thus, the result is true for cohomology classes $u \in \mathrm{H}^{n\rho}_{C_2}(\mathrm{X})$ for any space $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$.

Since the squaring operations are stable, the result extends to arbitrary $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology classes. Moreover, since $\Re$ commutes with suspensions, in the sense that $\Phi^e \circ \Sigma^\infty_{\mathrm{C}_2} \simeq \Sigma^\infty \circ \Phi^e$, and any $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}$ is equivalent to $\Sigma^{-n}\Sigma^\infty_{\mathrm{C}_2}\mathrm{X}$ for some n and $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$, we conclude the same for any $u \in \mathrm{H}^\star_{C_2}(\mathrm{E})$. □

Now we draw our attention towards comparing the action of the C_2-equivariant Steenrod algebra $\mathcal{A}^{\mathrm{C}_2}$ on $\mathrm{H}^\star(\mathrm{X}_+)$ to the action of the classical Steenrod algebra $\mathcal{A}$ on $\mathrm{H}^*(\mathrm{X}^{\mathrm{C}_2}_+)$, where $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$. Note that

$$\widehat{\Phi^{C_2}}_* : \mathrm{H}^\star_{C_2}(\mathrm{X}_+) \longrightarrow \mathrm{H}^*(\mathrm{X}^{\mathrm{C}_2}_+)$$

is a ring map.

Example 2.1.27. *When* $\mathrm{X} = *$*, the map*

$$\widehat{\Phi^{C_2}}_*\colon \pi_\star^{\mathrm{C}_2}\mathrm{H}\underline{\mathbb{F}_2} \cong \mathbb{F}_2[u_\sigma, a_\sigma] \oplus \Theta\{u_\sigma^{-i}a_\sigma^{-j}\} \longrightarrow \pi_*\mathrm{H}\mathbb{F}_2 \cong \mathbb{F}_2$$

sends $a_\sigma \mapsto 1$*,* $u_\sigma \mapsto 0$*, and* $\Theta \mapsto 0$*. This is essentially because smashing with*

$$\widetilde{\mathrm{EC}_2} \simeq \mathrm{colim}\{\mathrm{S}^0 \xrightarrow{a_\sigma} \mathrm{S}^\sigma \xrightarrow{a_\sigma} \mathrm{S}^{2\sigma} \longrightarrow \dots\}$$

amounts to inverting a_σ *and the projection* $\pi^{(0)}_{\mathbb{F}_2}$ *kills* u_σ*.*

Remark 2.1.28. *One can deduce from Example 2.1.15 that in cohomology, the map*

$$\lambda^*_{\mathrm{S}^0} : \mathrm{H}^*(\mathrm{B}_{\mathrm{C}_2}\Sigma_2^{\mathrm{C}_2}_+) \cong \mathbb{F}_2[t][\iota]/(\iota^2 - \iota) \twoheadrightarrow \mathrm{H}^*((\mathrm{B}\Sigma_2)_+) \cong \mathbb{F}_2[t],$$

is the quotient map sending $\iota \mapsto 0$*.*

Example 2.1.29. *The map* $\widehat{\Phi^{C_2}}_*\colon \mathrm{H}^\star_{C_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+) \to \mathrm{H}^*(\mathrm{B}_{\mathrm{C}_2}\Sigma_2^{\mathrm{C}_2}_+)$ *sends* $x \mapsto t$ *and* $y \to \iota$*,* $a_\sigma \mapsto 1$ *and* $u_\sigma \mapsto 0$*.*

Lemma 2.1.30. *The composition*

$$\mathrm{H}^\star_{C_2}(\mathrm{Th}(\overline{\gamma}^{\oplus n})) \xrightarrow{\widehat{\Phi^{C_2}}_*} \mathrm{H}^*(\mathrm{Th}(\overline{\gamma}^{\oplus n})^{\mathrm{C}_2}) \xrightarrow{\lambda_{\mathrm{S}^{\rho\otimes\omega}}} \mathrm{H}^*(\mathrm{Th}(\gamma^{\oplus n}))$$

sends $\underline{u}_n \mapsto u_n$*.*

Proof. Let $\zeta_{\mathrm{C}_2} : (\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+ \to \mathrm{Th}(\overline{\gamma}^{\oplus n})$ denote the zero-section. Under the zero section map the Thom class is mapped to the Euler class, and therefore $\zeta^*_{\mathrm{C}_2}(\underline{u}_n) = x^n$. Likewise, the zero-section for the nonequivariant bundle $\zeta : (\mathrm{B}\Sigma_2)_+ \to \mathrm{Th}(\gamma^{\oplus n})$ sends $u_n \mapsto t^n$. By naturality of $\widehat{\Phi^{C_2}}_*$ and λ, we get a commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^\star_{C_2}(\mathrm{Th}(\overline{\gamma}^{\oplus n})) & \xrightarrow{\widehat{\Phi^{C_2}}_*} & \mathrm{H}^*(\mathrm{Th}(\overline{\gamma}^{\oplus n})^{\mathrm{C}_2}) & \xrightarrow{\lambda^*_{\mathrm{S}^{\rho\otimes\omega}}} & \mathrm{H}^*(\mathrm{Th}(\gamma^{\oplus n})) \\
\downarrow{\scriptstyle \zeta^*_{\mathrm{C}_2}} & & \downarrow{\scriptstyle (\zeta^{\mathrm{C}_2}_{\mathrm{C}_2})^*} & & \downarrow{\scriptstyle \zeta^*} \\
\mathrm{H}^\star_{C_2}((\mathrm{B}_{\mathrm{C}_2}\Sigma_2)_+) & \xrightarrow[\widehat{\Phi^{C_2}}_*]{} & \mathrm{H}^*(\mathrm{B}_{\mathrm{C}_2}\Sigma_2^{\mathrm{C}_2}_+) & \xrightarrow[\lambda^*_{\mathrm{S}^0}]{} & \mathrm{H}^*((\mathrm{B}\Sigma_2)_+).
\end{array}$$

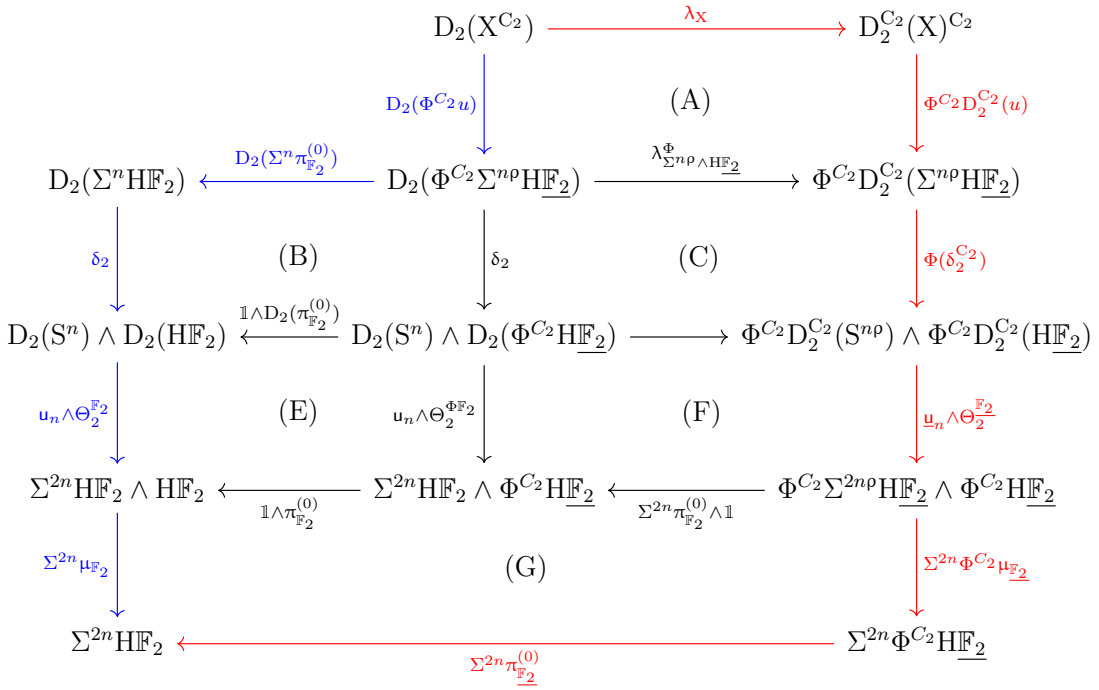
which along with Remark 2.1.28 and injectivity of ζ^* implies the result. □

Corollary 2.1.31. *For any space* $\mathrm{X} \in \mathbf{Top}_*^{\mathrm{C}_2}$ *and a class* $u \in \mathrm{H}^{n\rho}_{C_2}(\mathrm{X})$*,*

$$\mathcal{P}_2(\widehat{\Phi^{C_2}}_*(u)) = \lambda^*_{\mathrm{X}}(\widehat{\Phi^{C_2}}_*(\mathcal{P}_2^{\mathrm{C}_2}(u))). \tag{2.1.32}$$

Proof. It is enough to show that in the following diagram commutes as the blue path and the red path indicates the left-hand side and the right-hand side of (2.1.32)

respectively.



The squares (A), (B) and (C) commute naturally, the squares (E) and (G) commute because $\pi_{\mathbb{F}_2}^{(0)}$ is an $\mathbb{E}_\infty$-ring map, and (F) commutes because of Lemma 2.1.30. □

Proof of Theorem 2.1.24. For any space $X \in \mathbf{Top}_*^{C_2}$ and a class $u \in H_{C_2}^{n\rho}(X)$, we have a commutative diagram

$$\begin{array}{ccc} (B\Sigma_2)_+\wedge X^{C_2} & \xrightarrow{\Delta_{X^{C_2}}} & D_2(X^{C_2})) \\ {\scriptstyle \lambda_{S^0}\wedge\mathbb{1}_{X^{C_2}}}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ (B_{C_2}\Sigma_2)_+\wedge X^{C_2} & \xrightarrow[(\Delta_X^{C_2})^{C_2}]{} & D_2^{C_2}(X)^{C_2}. \end{array}$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{n} \mathsf{t}^{n-i}\otimes \mathrm{Sq}^i(\Phi_*(u)) &= \Delta^*_{X^{C_2}}(\mathcal{P}_2(\Phi_*(u))) \\ &= \Delta^*_{X^{C_2}}(\lambda_X^*(\Phi_*(\mathcal{P}_2^{C_2}(u)))) \\ &= (\lambda_{S^0}^*\otimes\mathbb{1}^*_{X^{C_2}})(((\Delta_X^{C_2})^{C_2})^*(\Phi_*(\mathcal{P}_2^{C_2}(u)))) \\ &= (\lambda_{S^0}^*\otimes\mathbb{1}^*_{X^{C_2}})\Phi_*((\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u))) \\ &= (\lambda_{S^0}^*\otimes\mathbb{1}^*_{X^{C_2}})\Phi_*\left(\sum_{i=0}^{n}\mathsf{x}^{n-i}\otimes \mathrm{Sq}^{2i,i}(u)\right. \\ &\qquad \left. +\sum_{i=0}^{n}\mathsf{y}\mathsf{x}^{n-i-1}\otimes \underline{\mathrm{Sq}}^{2i+1}(u)\right) \end{aligned}$$

$$
\begin{aligned}
&= (\lambda^*_{\mathrm{S}^0} \otimes \mathbb{1}^*_{\mathrm{X}^{\mathrm{C}_2}}) \left(\sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \Phi_*(\underline{\mathrm{Sq}}^{2i}(u)) \right. \\
&\qquad\qquad \left. + \sum_{i=0}^{n} \iota\mathsf{t}^{n-i-1} \otimes \Phi_*(\underline{\mathrm{Sq}}^{2i+1}(u)) \right) \\
&= \sum_{i=0}^{n} \mathsf{t}^{n-i} \otimes \Phi_*(\underline{\mathrm{Sq}}^{2i}(u)),
\end{aligned}
$$

and hence, the result is true for all $u \in \mathrm{H}^{n\rho}_{C_2}(\mathrm{X})$ for any $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$.

Since the squaring operations are stable, the result extends to arbitrary $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology classes. Moreover, since the geometric fixed point functor Φ commutes with suspensions (2.0.2), and any $\mathrm{E} \in \mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}}$ is equivalent to $\Sigma^{-n}\Sigma^\infty_{\mathrm{C}_2}\mathrm{X}$ for some n and $\mathrm{X} \in \mathbf{Top}^{\mathrm{C}_2}_*$, we conclude the same for any $u \in \mathrm{H}^\star_{C_2}(\mathrm{E})$. □

2.2 The $\mathbb{R}$-motivic Steenrod algebra and a freeness criterion

In this section, we focus particularly on the $\mathbb{R}$-motivic Steenrod algebra. We give a criterion that will detect freeness for modules over certain subalgebras of $\mathcal{A}^{\mathbb{R}}$. Writing $\mathbb{M}^{\mathbb{R}}_2$ for the $\mathbb{R}$-motivic cohomology of a point, we prove:

Theorem 2.2.1. *A finitely generated $\mathcal{A}^{\mathbb{R}}(n)$-module* M *is free if and only if*

1. M *is free as an $\mathbb{M}^{\mathbb{R}}_2$-module, and*
2. $\mathbb{F}_2 \otimes_{\mathbb{M}^{\mathbb{R}}_2} \mathrm{M}$ *is a free $\mathbb{F}_2 \otimes_{\mathbb{M}^{\mathbb{R}}_2} \mathcal{A}^{\mathbb{R}}(n)$-module.*

We begin by reviewing the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ following Voevodsky [54]. The algebra $\mathcal{A}^{\mathbb{R}}$ is the ring of bigraded homotopy classes of self-maps of the $\mathbb{R}$-motivic Eilenberg-Mac Lane spectrum $\mathrm{H}\mathbb{F}^{\mathbb{R}}_2$:

$$\mathcal{A}^{\mathbb{R}} = [\mathrm{H}\mathbb{F}^{\mathbb{R}}_2, \mathrm{H}\mathbb{F}^{\mathbb{R}}_2]_{*,*}.$$

The unit map $\mathbb{S}_{\mathbb{R}} \to \mathrm{H}\mathbb{F}^{\mathbb{R}}_2$ induces a canonical projection map

$$\epsilon : \mathcal{A}^{\mathbb{R}} \longrightarrow \mathbb{M}^{\mathbb{R}}_2 := [\mathbb{S}_{\mathbb{R}}, \mathrm{H}\mathbb{F}^{\mathbb{R}}_2]_{*,*} \cong \mathbb{F}_2[\tau, \rho],$$

where $|\tau| = (0, -1)$ and $|\rho| = (-1, -1)$. Further, using the multiplication map $\mathrm{H}\mathbb{F}^{\mathbb{R}}_2 \wedge \mathrm{H}\mathbb{F}^{\mathbb{R}}_2 \to \mathrm{H}\mathbb{F}^{\mathbb{R}}_2$ one can give $\mathcal{A}^{\mathbb{R}}$ a left $\mathbb{M}^{\mathbb{R}}_2$-module structure as well as a right $\mathbb{M}^{\mathbb{R}}_2$-module structure. Voevodsky shows that $\mathcal{A}^{\mathbb{R}}$ is a free left $\mathbb{M}^{\mathbb{R}}_2$-module. There is an analogue of the classical Adem basis in the motivic setting, and Voevodsy established motivic Adem relations, thereby completely describing the multiplicative structure of $\mathcal{A}^{\mathbb{R}}$.

The motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ also admits a diagonal map, so that its left $\mathbb{M}^{\mathbb{R}}_2$-linear dual is an algebra over $\mathbb{F}_2$. Note that $\mathcal{A}^{\mathbb{R}}$ is an $\mathbb{F}_2$-algebra but not an $\mathbb{M}^{\mathbb{R}}_2$-algebra as τ is not a central element since

$$\mathrm{Sq}^1(\tau) = \rho \neq \tau\mathrm{Sq}^1. \tag{2.2.2}$$

This complication is also reflected in the fact that the pair $(\mathbb{M}_2^{\mathbb{R}}, \hom_{\mathbb{M}_2^{\mathbb{R}}}(\mathcal{A}^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}))$ is a Hopf algebroid, and not a Hopf algebra like its complex counterpart. The underlying algebra of the dual $\mathbb{R}$-motivic Steenrod algebra is given by

$$\mathcal{A}_*^{\mathbb{R}} \cong \mathbb{M}_2^{\mathbb{R}}[\xi_{i+1}, \tau_i : i \geq 0]/(\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1})$$

where ξ_i and τ_i live in bidegree $(2^{i+1} - 2, 2^i - 1)$ and $(2^{i+1} - 1, 2^i - 1)$, respectively. The complete description of the Hopf algebroid structure can be found in [54].

Ricka[2] [51] identified the quotient Hopf algebroids of $\mathcal{A}_*^{\mathbb{R}}$ (see also [32]). In particular, there are quotient Hopf algebroids

$$\mathcal{A}^{\mathbb{R}}(n)_* = \mathcal{A}_*^{\mathbb{R}}/(\xi_1^{2^n}, \dots, \xi_n^2, \xi_{n+1}, \dots, \tau_0^{2^{n+1}}, \dots, \tau_n^2, \tau_{n+1}, \dots)$$

which can be thought of as analogues of the quotient Hopf algebras

$$\mathcal{A}(n)_* = \mathcal{A}_*/(\xi_1^{2^{n+1}}, \dots, \xi_{n+1}^2, \xi_{n+2}, \dots)$$

of the classical dual Steenrod algebra $\mathcal{A}_*$. It is an algebraic fact that

$$\tau^{-1}(\mathcal{A}^{\mathbb{R}}(n)_*/(\rho)) \cong \mathbb{F}_2[\tau^{\pm 1}] \otimes \mathcal{A}(n)_* \tag{2.2.3}$$

as Hopf algebras (see [21, Corollary 2.9]). The above isomorphism sends $\tau_i \mapsto \tau^{1-2^i}\xi_{i+1}$ and $\xi_{i+1} \mapsto \tau^{1-2^{i+1}}\xi_{i+1}^2$. The quotient Hopf algebroid $\mathcal{A}^{\mathbb{R}}(n)_*$ is the $\mathbb{M}_2^{\mathbb{R}}$-linear dual of the subalgebra $\mathcal{A}^{\mathbb{R}}(n)$ of $\mathcal{A}^{\mathbb{R}}$, which is generated by the elements $\{\tau, \rho, \mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n}\}$.

Although τ is not in the center (see (2.2.2)) of $\mathcal{A}^{\mathbb{R}}$ or $\mathcal{A}^{\mathbb{R}}(n)$, the element ρ is in the center. We make use of this fact to prove the following result.

Lemma 2.2.4. *A finitely-generated $\mathcal{A}^{\mathbb{R}}(n)$-module* M *is free if and only if*

1. M *is free as an $\mathbb{F}_2[\rho]$-module, and,*
2. $\mathrm{M}/(\rho)$ *is a free $\mathcal{A}^{\mathbb{R}}(n)/(\rho)$-module.*

Proof. The 'only if' part is trivial. For the 'if' part, choose a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $\mathrm{M}/(\rho)$ and let $\tilde{b}_i \in M$ be any lift of b_i. Let F denote the free $\mathcal{A}^{\mathbb{R}}(n)$-module generated by $\mathcal{B}$ and consider the map

$$f : \mathrm{F} \to \mathrm{M}$$

which sends $b_i \mapsto \tilde{b}_i$. We show that f is an isomorphism by inductively proving that f induces an isomorphism $\mathrm{F}/(\rho^n) \cong \mathrm{M}/(\rho^n)$ for all $n \geq 1$. The case of $n = 1$ is true by assumption.

[2]Ricka actually identified the quotient Hopf algebroids of the C_2-equivariant dual Steenrod algebra. However, the difference between the $\mathbb{R}$-motivic Steenrod algebra and the C_2-equivariant Steenrod algebra lies only in the coefficient rings and results of Ricka easily identifies the quotient Hopf algebroids of the $\mathbb{R}$-motivic Steenrod algebra.

For the inductive argument, first note that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{F}/(\rho^{n-1}) & \xrightarrow{\cdot\rho} & \mathrm{F}/(\rho^{n}) & \longrightarrow & \mathrm{F}/(\rho) & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_0} & & \| \\ 0 & \longrightarrow & \mathrm{M}/(\rho^{n-1}) & \xrightarrow{\cdot\rho} & \mathrm{M}/(\rho^{n}) & \longrightarrow & \mathrm{M}/(\rho) & \longrightarrow & 0 \end{array}$$

is a diagram of $\mathcal{A}^{\mathbb{R}}(n)$-modules (since ρ is in the center) where the horizontal rows are exact. The map f_0 is an isomorphism by assumption *(2)*, and f_{n-1} is an isomorphism by the inductive hypothesis; hence, f_n is an isomorphism by the five lemma. □

Proof of 2.2.1. The result follows immediately from Lemma 2.2.4 combined with [30, Theorem B] and the fact that $\mathcal{A}^{\mathbb{C}}(n) = \mathcal{A}^{\mathbb{R}}(n)/(\rho)$. □

In order to employ 2.2.1, we use the work of Adams and Margolis [3], which provides a freeness criterion for modules over finite-dimensional subalgebras of the classical Steenrod algebra in terms of Margolis homology. Recall that, for an algebra A and an element $x \in \mathrm{A}$ such that $x^2 = 0$, the Margolis homology of M with respect to x is defined as

$$\mathcal{M}(\mathrm{M}, x) = \frac{\ker(x : \mathrm{M} \to \mathrm{M})}{\mathrm{img}(x : \mathrm{M} \to \mathrm{M})}.$$

In the classical Steenrod algebra, the element P^s_t is defined to be dual to $\xi_t^{2^s} \in \mathcal{A}_*$. In terms of the Milnor basis,

$$\mathrm{P}^s_t := \mathrm{Sq}(\underbrace{0, \ldots, 0}_{t-1}, 2^s).$$

The element P^0_t is often denoted by Q_{t-1}. One may define the $\mathbb{R}$-motivic analogues of $\mathrm{P}^s_t \in \mathcal{A}$ by setting

$$\overline{\mathrm{Q}}_t := \tau_t^* \quad \text{and} \quad \overline{\mathrm{P}}^s_t := (\xi_t^{2^{s-1}})^*$$

in $\mathcal{A}^{\mathbb{R}}(n)$ for $s \geq 1$, recalling that the motivic ξ_t plays the role of the classical ξ_t^2. It is easy to see that under the isomorphism (2.2.3), $\overline{\mathrm{Q}}_t$ corresponds to $\tau^{1-2^t}\mathrm{Q}_t$ and $\overline{\mathrm{P}}^s_t$ corresponds to $\tau^{2^s(1-2^t)}\mathrm{P}^s_t$.

In the context of 2.2.1, freeness over

$$\mathbb{F}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(n) := \mathcal{A}^{\mathbb{R}}(n)/(\rho, \tau) \cong \mathcal{A}^{\mathbb{C}}(n)/(\tau)$$

can be detected using Margolis homology calculations following [30, Theorem B(i)].

Corollary 2.2.5. *Let* M *be a finitely generated left* $\mathcal{A}^{\mathbb{R}}(n)$*-module and let*

$$\mathrm{M}/(\rho, \tau) := \mathrm{M} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2.$$

Then M *is a free* $\mathcal{A}^{\mathbb{R}}(n)$*-module if and only if*

1. M *is free over* $\mathbb{M}_2^{\mathbb{R}}$*,*

2. $\mathcal{M}(\mathrm{M}/(\rho,\tau),\overline{\mathrm{Q}}_i)=0$ *for* $0\leq i\leq n$, *and*

3. $\mathcal{M}(\mathrm{M}/(\rho,\tau),\overline{\mathrm{P}}^s_t)=0$ *for* $1\leq s\leq t\leq n$.

Remark 2.2.6. *The quotient* $\mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau)$ *fits into a short exact sequence*

$$\overline{\mathrm{E}}(n)\hookrightarrow\mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau)\twoheadrightarrow\overline{\mathrm{P}}(n) \tag{2.2.7}$$

of connected finite-dimensional Hopf algebras over $\mathbb{F}_2$, *where* $\overline{\mathrm{E}}(n):=\Lambda_{\mathbb{F}_2}(\overline{\mathrm{Q}}_0,\dots,\overline{\mathrm{Q}}_n)$ *and* $\overline{\mathrm{P}}(n):=\mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau,\overline{\mathrm{Q}}_0,\dots,\overline{\mathrm{Q}}_n)$. *The short exact sequence* (2.2.7) *splits from the right. This right splitting map confirms that* (2.2.7) *is a split exact sequence of coalgebras as all of the Hopf algebras involved in* (2.2.7) *admit a cocommutative comultiplication. However, when* (2.2.7) *is viewed as an exact sequence of algebras, it does not split because the algebras involved are not commutative. For example, when* $n=1$ *then a left splitting map in* (2.2.7) *would imply that* $\overline{\mathrm{Q}}_0$ *commutes with* Sq^2 *and contradicts the fact that* $\overline{\mathrm{Q}}_1:=[\mathrm{Sq}^2,\overline{\mathrm{Q}}_0]$. *Dually, there is a splitting*

$$\mathcal{A}^{\mathbb{R}}(n)_*/(\rho,\tau)\cong\frac{\mathbb{F}_2[\xi_1,\dots,\xi_n]}{(\xi_1^{2^n},\dots,\xi_n^2)}\otimes\Lambda(\tau_0,\dots,\tau_n)$$

as an algebra, though it does not split as a coalgebra. This is clear from the fact that

$$\Delta(\tau_k)\equiv\sum_{i=0}^{k}\xi_{k-i}^{2^i}\otimes\tau_i\not\equiv\tau_k\otimes 1+1\otimes\tau_k\mod(\rho,\tau).$$

Remark 2.2.8 (A minor correction to [30])**.** *Note that Remark 2.2.6 stands in contradiction to [30, Corollary 4.2]. However, this does not affect [30, Corollary 4.3] which claims* $(\overline{\mathrm{P}}^t_t)^2=0$. *This is because* $\overline{\mathrm{P}}(n)$ *is a sub-Hopf algebra of* $\mathcal{A}^{\mathbb{R}}(n)/(\rho,\tau)$. *We also note that the proof of [30, Theorem B(i)] remains unaffected by this change.*

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Chapter 3 The v_1-Periodic Region in the cohomology of the $\mathbb{C}$-motivic Ext

The E_2-page of the Adams spectral sequence is given by $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathcal{A})$, which we denote by Ext, where $\mathcal{A}$ is the classical Steenrod algebra. For Ext, Adams [2] showed that there is a vanishing line of slope $\frac{1}{2}$ and intercept $\frac{3}{2}$, and J. P. May showed there is a periodicity line of slope $\frac{1}{5}$ and intercept $\frac{12}{5}$, where the periodicity operation is defined by the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$. This result has not been published by May, but can be found in the thesis of Krause:

Theorem 3.0.1. *[36, Theorem 5.14] For $r \geq 2$, the Massey product operation $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$ is uniquely defined on $\mathrm{Ext}^{s,f} = H^{s,f}(\mathcal{A})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$, where s is the stem, and f is the Adams filtration.*

Furthermore, for $f > \frac{1}{5}s + \frac{12}{5}$, the operation

$$P_r \colon H^{s,f}(\mathcal{A}) \xrightarrow{\cong} H^{s+2^{r+1},f+2^r}(\mathcal{A})$$

is an isomorphism.

The purpose of this chapter is to discuss an analog of the theorem above in the $\mathbb{C}$-motivic context. Motivic homotopy theory, also known as $\mathbb{A}^1$-homotopy theory, is a way to apply the techniques of algebraic topology, specifically homotopy, to algebraic varieties and, more generally, to schemes. The theory was formulated by Morel and Voevodsky [46].

We analyze the case where the base field F is the complex numbers $\mathbb{C}$. Let $\mathbb{M}_2$ denote the bigraded motivic cohomology ring of Spec $\mathbb{C}$, with $\mathbb{F}_2 = \mathbb{Z}/2$-coefficients. Voevodsky [55] proved that $\mathbb{M}_2 \cong \mathbb{F}_2[\tau]$. Let $\mathcal{A}^{\mathbb{C}}$ be the mod 2 motivic Steenrod algebra over $\mathbb{C}$. The motivic Adams spectral sequence is a trigraded spectral sequence with

$$E_2^{*,*,*} = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2),$$

where the third grading is the motivic weight. (See Dugger and Isaksen [24]). The $\mathbb{C}$-motivic E_2-page, which we denote by $\mathrm{Ext}_{\mathbb{C}}$, has a vanishing line computed by Guillou and Isaksen [29]. Quigley has a partial result that $\mathrm{Ext}^{s,f,w}$ has a periodicity line of slope $\frac{1}{3}$ under the condition $s \leq w$ in the case $r = 2$ [49, Corollary 5.4].

The multiplication by 2 map $S^{0,0} \xrightarrow{2} S^{0,0}$ is detected by h_0, and the Hopf map $S^{1,1} \xrightarrow{\eta} S^{0,0}$ is detected by h_1 in $\mathrm{Ext}_{\mathbb{C}}$. These elements have degrees $(0,1,0)$ and $(1,1,1)$ respectively. By an infinite h_1-tower we will mean a non-zero sequence of elements of the form $h_1^k x$ in $\mathrm{Ext}_{\mathbb{C}}$ with $k \geq 0$, where x is not h_1 divisible. We will write h_1-towers for infinite h_1-towers, and refer to x as the base of the h_1-tower $h_1^k x$ ($k \geq 0$). Since all h_1-towers are τ-torsion, one might guess that the motivic $\mathrm{Ext}_{\mathbb{C}}$ groups differ from the classical Ext groups by only infinite h_1-towers. This is not true, but we may expect the h_1-torsion part of $\mathrm{Ext}_{\mathbb{C}}$ to obtain a pattern similar to Ext. Our result pertains solely to this h_1-torsion region.

Remark 3.0.2. *Let $\mathcal{A}_*^{\mathbb{C}}$ denote the dual Steenrod algebra. For $\mathrm{Ext}_{\mathbb{C}}$, we can work over $\mathcal{A}_*^{\mathbb{C}}$ instead of $\mathcal{A}^{\mathbb{C}}$. i.e.*

$$E_2^{*,*,*} \cong \mathrm{Ext}_{\mathcal{A}_*^{\mathbb{C}}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2)_*.$$

Here we view $\mathbb{M}_2$ as the homology of the motivic sphere instead of the cohomology; this is an $\mathcal{A}_*^{\mathbb{C}}$-comodule.

The goal of this chapter is the following theorem:

Theorem 3.0.3. *For $r \geq 2$, the Massey product operation $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$ is uniquely defined on $\mathrm{Ext}^{s,f,w} = H^{s,f,w}(\mathcal{A}^{\mathbb{C}})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$.*

Furthermore, for $f > \frac{1}{5}s + \frac{12}{5}$, the restriction of P_r to the h_1-torsion

$$P_r \colon [H^{s,f,w}(\mathcal{A}^{\mathbb{C}})]_{h_1-torsion} \to [H^{s+2^{r+1},f+2^r,w+2^r}(\mathcal{A}^{\mathbb{C}})]_{h_1-torsion}$$

is an isomorphism.

3.1 The stable (co)module category $Stab(\Gamma)$

In order to restrict to working with only the h_1-torsion (also h_0-torsion) part, first we would like to choose a suitable working environment: a category with some nice properties that will serve our purposes. Usually $\mathrm{Ext}_{\mathbb{C}}$ is defined in the derived category of $\mathcal{A}_*^{\mathbb{C}}$-comodules, which we denote $D(\mathcal{A}_*^{\mathbb{C}})$. However, the coefficient ring $\mathbb{M}_2$ is not compact in $D(\mathcal{A}_*^{\mathbb{C}})$, which means that $\mathbb{M}_2$ does not interact well with colimits. The stable comodule category will better serve our purposes. That is a category $\mathscr{C}$ such that:

1. If M is a $\mathcal{A}_*^{\mathbb{C}}$-comodule that is free of finite rank over $\mathbb{M}_2$ and N is a $\mathcal{A}_*^{\mathbb{C}}$-comodule, then $\mathrm{Hom}_{\mathscr{C}}(M, N) \cong \mathrm{Ext}_{\mathcal{A}_*^{\mathbb{C}}}(M, N)$.
2. If M is a $\mathcal{A}_*^{\mathbb{C}}$-comodule that is free of finite rank over $\mathbb{M}_2$, then M is compact in $\mathscr{C}$. That is to say, for any sequential colimit in $\mathscr{C}$ of $\mathcal{A}_*^{\mathbb{C}}$-comodules

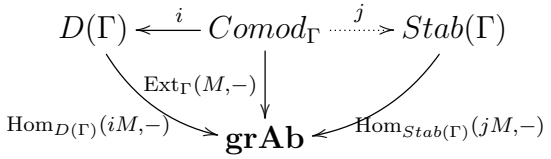
   $$\underset{i}{\mathrm{Colim}} N_i := \mathrm{Colim}(N_0 \xrightarrow{f_0} N_1 \to \cdots \to N_i \xrightarrow{f_i} \cdots),$$

   we have $\underset{i}{\mathrm{Colim}}\, \mathrm{Ext}_{\mathcal{A}_*^{\mathbb{C}}}(M, N_i) \cong \mathrm{Hom}_{\mathscr{C}}(M, \underset{i}{\mathrm{Colim}} N_i)$

The correct choice of $\mathscr{C}$ is called $Stab(\mathcal{A}_*^{\mathbb{C}})$. The category can be constructed in various ways (see [12, Sec. 2.1] for details), and has several useful properties for our case. The following proposition summarizes some of the discussion in [7, Sec. 4]:

Proposition 3.1.1. *The category $Stab(\mathcal{A}_*^{\mathbb{C}})$ satisfies conditions* (1) *and* (2) *above.*

Namely, for a Hopf algebra Γ and comodule M that is free of finite rank, we have a diagram



where i is the canonical functor and j is well-defined only for comodules that are free of finite rank over $\mathbb{M}_2$. This diagram commutes. Because the stable comodule category cooperates nicely with taking colimits in the sense that the condition (2) holds, we can compute the colimit of a sequence of $\mathrm{Ext}_\Gamma(M,N)$.

Here we introduce notation that will be used in future sections.

Notation 3.1.2. *For a motivic spectrum M such that $H_*(M)$ is free of finite rank over $\mathbb{M}_2$, let M also denote the embedded image of the homology of the spectrum M in the stable comodule category (i.e., $M = j(H_*(M))$). We use $[M,N]^\Gamma_{*,*,*}$ to denote $\mathrm{Hom}_{Stab(\Gamma)}(M,N)$, where M, $N \in Stab(\Gamma)$. For example, if $M = S$, then $H_*(S) = \mathbb{M}_2$, which we also denote by S. Thus $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{M}_2,\mathbb{M}_2) = [S,S]^{\mathcal{A}^{\mathbb{C}}_*}_{s,f,w}$. When Γ is the motivic dual Steenrod algebra, we omit the superscript Γ. This notation is consistent with [36].*

We use the grading (s,f,w), where s is the stem, f is the Adams filtration and w is the motivic weight. Notice that $t = s+f$ is the internal degree. Given a self-map θ: $\Sigma^{s_0,f_0,w_0}M \xrightarrow{\theta} M$ in $Stab(\mathcal{A}^{\mathbb{C}}_*)$, we have a cofiber sequence $\Sigma^{s_0,f_0,w_0}M \xrightarrow{\theta} M \to M/\theta$ in $Stab(\mathcal{A}^{\mathbb{C}}_*)$. The associated long exact sequence will be indexed as follows:

$$\cdots \to [M,N]_{s+s_0+1,f+f_0-1,w+w_0} \to [M/\theta,N]_{s,f,w} \to [M,N]_{s,f,w} \to [M,N]_{s+s_0,f+f_0,w+w_0} \to \cdots$$

Sometimes we omit indices when there is no risk of confusion.

3.2 Massey products

In this section, we show that the cofiber S/h_0^k admits a self-map and identify it with the Massey product in Theorem 3.0.3. Self-maps are maps of suspensions of an object to itself. For a dualizable object Y, self maps $\Sigma^n Y \to Y$ can also be described as elements of $\pi_*(Y \otimes DY)$, with DY the $\otimes$-dual of Y. In this section we mainly deal with homological self-maps in $Stab(\mathcal{A}^{\mathbb{C}}_*)$.

When considering the vanishing region and the periodicity region, we only work with the h_0-torsion part. (Of course, this is not much of a loss: as classically, the only h_0-local elements are in the 0-stem.) We next investigate the h_1-torsion part inside the h_0-torsion. For this purpose, we introduce the following notion.

Definition 3.2.1. *Let F_0 be the fiber of $S \to S[h_0^{-1}]$, where $S[h_0^{-1}] := \mathrm{Colim}(S \xrightarrow{h_0} S \xrightarrow{h_0} \cdots)$ in $Stab(\mathcal{A}^{\mathbb{C}}_*)$. Similarly, let F_{01} be the fiber of $F_0 \to F_0[h_1^{-1}]$ with $F_0[h_1^{-1}]$ defined as an analogous colimit.*

The group $[S, F_{01}]$ contains the subset of $[S, S]$ consisting of elements that are both h_0- and h_1-torsion, as well as the negative parts of those h_0 and h_1-towers in $F_0[h_1^{-1}]$. The regions we are considering are unaffected. We display the corresponding $\mathrm{Ext}_{\mathbb{C}}$ groups in Figure 3.1 and 3.2.

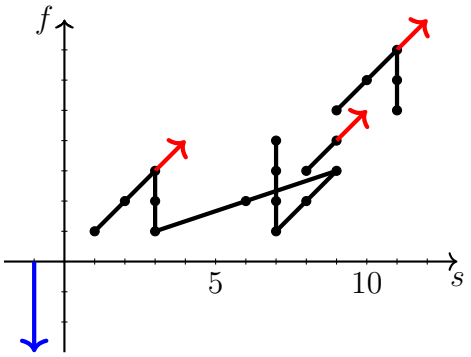


Figure 3.1: $[S, F_0]^{\mathcal{A}^{\mathbb{C}\vee}}_{*,*,*}$

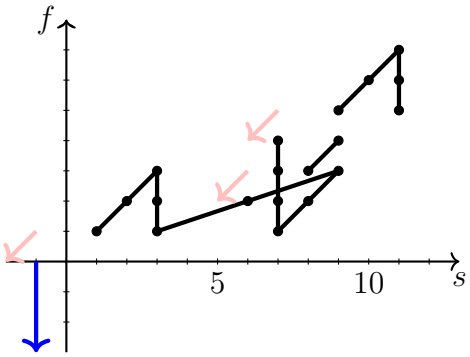


Figure 3.2: $[S, F_{01}]^{\mathcal{A}^{\mathbb{C}\vee}}_{*,*,*}$

The periodicity operator P corresponds to multiplying by the element h_{20}^4 of the May spectral sequence, meaning that for many values of x, $h_{20}^4 x \in \langle h_3, h_0^4, x\rangle$. However, h_{20}^4 does not survive to $\mathrm{Ext}_{\mathbb{C}}$. As a result, multiplying by P is not a map from $[S, S]$ to $[S, S]$. Luckily, [29, Figure 2] shows that P survives in $[S/h_0, S]$. Similarly, we have the following proposition:

Proposition 3.2.3 ([1])**.** *The element $h_{20}^{2^r}$ survives the May spectral sequence to $[S/h_0^k, S]$ for $k \leq 2^r$, and thus gives a corresponding element $P^{2^{r-2}}$ in $[S/h_0^k, S/h_0^k]$, i.e. a self-map of S/h_0^k.*

If N is an $\mathcal{A}_*^{\mathbb{C}}$-comodule in $Stab(\mathcal{A}_*^{\mathbb{C}})$, then $[S/h_0^k, S/h_0^k]$ acts on $[S/h_0^k, N]$. The corresponding element P (or some power of P) inside $[S/h_0^k, S/h_0^k]$ induces a map from $[S/h_0^k, N]$ to itself. We would like to show that for any $k \leq 2^r$ and $r \geq 2$, multiplying by $P^{2^{r-2}}$ on $[S/h_0^k, S]$ coincides with the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, -\rangle$ in a certain region. In other words, we must show that there is zero indeterminacy.

The Massey product is defined on the kernel of $h_0^{2^r}$ on $[S, S]$, which we will denote $ker(h_0^{2^r})$. It lands in the cokernel of multiplication by h_{r+1}:

$$P_r(-) : ker(h_0^{2^r}) \to [S, S]/h_{r+1}.$$

Remark 3.2.4. *Originally one would like to consider the following square and see that it commutes in a certain region*

$$\begin{array}{ccc} [S/h_0^k, S] & \xrightarrow{-\cdot P^{2^{r-2}}} & [S/h_0^k, S] \\ \downarrow & & \downarrow \\ ker(h_0^{2^r}) & \xrightarrow[P_r(-)]{} & [S, S]/h_{r+1}. \end{array}$$

The vertical maps are induced by $S \to S/h_0^k$. However, since we lost the advantage of a vanishing region of $f > \frac{1}{2}s + \frac{3}{2}$ that we need in the classical setting, the region

where the vertical maps are isomorphisms is not satisfactory. We solve this problem by restricting attention to the h_0 and h_1-torsion.

To better fit our purposes, consider the Massey product defined on $[S, F_{01}]$

$$P_r(-) : ker_{F_{01}}(h_0^{2^r}) \to [S, F_{01}]/h_{r+1.}$$

This gives the following squares, over which we have more control:

$$\begin{array}{ccc} [S/h_0^k, F_{01}] & \xrightarrow{-\cdot P^{2^{r-2}}} & [S/h_0^k, F_{01}] \\ \downarrow & & \downarrow \\ ker_{F_{01}}(h_0^{2^r}) & \xrightarrow[P_r(-)]{} & [S, F_{01}]/h_{r+1} \\ \downarrow & & \downarrow \\ ker_S(h_0^{2^r}) & \xrightarrow[P_r(-)]{} & [S, S]/h_{r+1} \end{array} \tag{3.2.5}$$

The canonical map $F_{01} \to S$ induces a map $[S, F_{01}] \to [S, S]$ given by inclusion on the h_0- and h_1-torsion elements and which sends negative towers to zero. The bottom square commutes for $s > 0$ and $f > 0$ modulo potential indeterminacy. We would like to show that the indeterminacy vanishes under some conditions.

Let $C(\eta)$ denote the cofiber of the first Hopf map

$$S^{1,1} \xrightarrow{\eta} S^{0,0}.$$

Writing C_η for the cohomology $H^{*,*}(C(\eta))$, we have the following result:

Theorem 3.2.6. *[29, Theorem 1.1] The group* $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s,f,w}(\mathbb{M}_2, C_\eta)$ *vanishes when* $s > 0$ *and* $f > \frac{1}{2}s + \frac{3}{2}$.

Theorem 3.2.6 gives us that $[S, C_\eta]_{s,f,w}$ vanishes when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$. In other words, there are only h_1-towers when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$ in $[S, S]_{s,f,w}$. Moreover, we have the following fact:

Proposition 3.2.7 (Corollary of [28, Theorem 1.1])**.** *For $r \geq 1$, h_{r+1} does not support an h_1-tower.*

Therefore the indeterminacy $(h_{r+1}[S, S])_{s,f,w}$ must vanish when $f > \frac{1}{2}s + 3 - 2^r$, under the following two conditions: that h_{r+1} has $s = 2^{r+1} - 1$, and that there are only h_1-towers in $[S, S]_{s,f,w}$ when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$, which are h_{r+1}-torsion groups.

Remark 3.2.8. *It is easy to see that the indeterminacy $(h_{r+1}[S, F_{01}])_{s,f,w}$ also vanishes when $f > \frac{1}{2}s + 3 - 2^r$.*

The first row of the top square in (3.2.5) is multiplication by some power of the element P. We next determine when the vertical maps are isomorphisms.

Lemma 3.2.9 (Motivic version of [36, Lemma 5.2])**.** *Let $M, N \in Stab(\mathcal{A}_*^{\mathbb{C}})$. Assume that $[M, N]$ vanishes when $f > as+bw+c$ for some $a, b, c \in \mathbb{R}$, let $\theta : \Sigma^{s_0,f_0,w_0}M \to M$ be a map with $f_0 > as_0 + bw_0$, and let M/θ denote the cofiber of $\Sigma^{s_0,f_0,w_0}M \xrightarrow{\theta} M$. Then*

$$[M/\theta, N] \to [M, N]$$

is an isomorphism above a vanishing plane parallel with the one in $[M, N]$ but with f-intercept given by $c - (f_0 - as_0 - bw_0)$.

Proof. The result follows from the long exact sequence associated to the cofiber sequence $\Sigma^{s_0,f_0,w_0}M \xrightarrow{\theta} M \to M/\theta$:

$$\cdots \to [M, N]_{s+s_0+1,f+f_0-1,w+w_0} \to [M/\theta, N]_{s,f,w} \to [M, N]_{s,f,w} \to [M, N]_{s+s_0,f+f_0,w+w_0} \to \cdots$$

□

Remark 3.2.10. *This approach could also apply to a vanishing region above several planes or even a surface. The vanishing condition of Lemma 3.2.9 could be rephrased as the following:*

Assume that $[M, N]_{*,*,*}$ vanishes when $f > \varphi(s, w)$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then the gradient $v(-,-) = (\frac{\partial\varphi}{\partial s}(-), \frac{\partial\varphi}{\partial w}(-))$ is a vector field. Let $d = \max_{(s_0,w_0)} |v(s_0, w_0)|$, and assume both $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0}$ are greater than d. The remaining proof would follow similarly, with the f-intercept given by $max\{c-(f_0-ds_0), c-(f_0-dw_0)\}$.

We have this as a corollary:

Corollary 3.2.11 (Motivic version of [36, Lemma 5.9])**.** *Let $k \geq 1$. For $f > \frac{1}{2}s+\frac{3}{2}-k$, the natural map $[S/h_0^k, F_{01}]_{s,f,w} \to [S, F_{01}]_{s,f,w}$ is an isomorphism.*

Proof. To determine this, we need to confirm that $[S, F_{01}]_{s,f,w}$ admits a vanishing region of $f > \frac{1}{2}s + \frac{3}{2}$. The fiber sequence $F_{01} \to F_0 \hookrightarrow F_0[h_1^{-1}]$ gives us an exact sequence:

$$\cdots \to [S, F_{01}]_{s,f,w} \to [S, F_0]_{s,f,w} \xrightarrow{h_1^{-1}} [S, F_0[h_1^{-1}]]_{s,f,w} \to [S, \Sigma^{1,-1,0}F_{01}]_{s,f,w} \to \cdots$$

Since $[S, F_0]$ differs from $[S, S]$ only in the 0-stem, there are only h_1-towers when $f > \frac{1}{2}s + \frac{3}{2}$. And by Theorem 3.2.6 again, $[S, C_\eta]_{s,f,w}$ vanishes when $s > 0$ and $f > \frac{1}{2}s + \frac{3}{2}$. In other words, above the plane $f = \frac{1}{2}s + \frac{3}{2}$, multiplying by h_1, which detects η, is an isomorphism from $[S, F_0]_{s,f,w}$ to $[S, F_0]_{s+1,f+1,w+1}$.

As a result, inverting h_1 would be an isomorphism from $[S, F_0]_{s,f,w}$ to $[S, F_0[h_1^{-1}]]_{s,f,w}$ when $f > \frac{1}{2}s+\frac{3}{2}$. Therefore, $[S, F_{01}]_{s,f,w}$ vanishes when $f > \frac{1}{2}s+\frac{3}{2}$. Applying Lemma 3.2.9 gives the corollary. □

The results in 3.2.3 and 3.2.8 locate the region where both squares commute, thus obtaining the first part of Theorem 3.0.3.

Theorem 3.2.12 (Motivic version of [36, Proposition 5.12])**.** *For $k \leq 2^r$ and $r \geq 2$, the cofiber S/h_0^k admits a self-map $P^{2^{r-2}}$ of degree $(2^{r+1}, 2^r, 2^r)$. Thus, for any $N \in Stab(\mathcal{A}_*^{\mathbb{C}})$, composition with $P^{2^{r-2}}$ defines a self-map on $[S/h_0^k, N]$.*

When $f > \frac{1}{2}s + 3 - k$, in the case $N = F_{01}$, the induced map coincides with the Massey product $P_r(-) := \langle h_{r+1}, h_0^{2^r}, - \rangle$ with zero indeterminacy.

3.3 The Cartan-Eilenberg spectral sequence

We will obtain a vanishing region for $[S/(h_0, P), F_{01}]_{*,*,*}$ in this section. Consider the colimit

$$F_0/h_1^\infty := \operatorname*{Colim}_i(\Sigma^{-1,-1,-1}F_0/h_1 \xrightarrow{h_1} \cdots \xrightarrow{h_1} \Sigma^{-i,-i,-i}F_0/h_1^i \xrightarrow{h_1} \cdots)$$

in $Stab(\mathcal{A}_*^{\mathbb{C}})$. As we show in the following result, it differs from F_{01} by a suspension in the region we are considering.

Proposition 3.3.1. *When $f > \frac{1}{2}s + \frac{3}{2}$,*

$$[S, \Sigma^{-1,1,0}F_0/h_1^\infty]_{s,f,w} \cong [S, F_{01}]_{s,f,w}$$

Proof. To see this, note that the colimit F_0/h_1^∞ is a union of all the h_1-torsion in F_0, while the fiber F_{01} detects the h_1-torsion together with those negative h_1-towers. □

Note that F_0 coincides with

$$\Sigma^{-1,1,0}S/h_0^\infty := \Sigma^{-1,1,0}\operatorname*{Colim}_i(\Sigma^{0,1,0}S/h_0 \xrightarrow{h_0} \cdots \xrightarrow{h_0} \Sigma^{0,i,0}S/h_0^i \xrightarrow{h_0} \cdots),$$

if we ignore the negative h_0-tower. That is, we have $[S, \Sigma^{-1,1,0}S/h_0^\infty]_{s,f,w} \cong [S, F_0]_{s,f,w}$ when $f > 0$.

Remark 3.3.2. *We have shown that the map*

$$[S/h_0^k, F_0/h_1^\infty]_{s,f,w} \to [S, F_0/h_1^\infty]_{s,f,w}$$

is an isomorphism when $f > \frac{1}{2}s + 3 - k$. We consider this colimit because it is better for computational purposes (the fiber F_{01} is harder to deal with than the colimit F_0/h_1^∞).

Let θ be a self-map of S/h_0^k, and consider the cofiber sequence $S/h_0^k \xrightarrow{\theta} S/h_0^k \to S/(h_0^k, \theta)$. The vanishing region for $[S/(h_0^k, \theta), F_0/h_1^\infty]_{*,*,*}$ is the region where

$$[S/h_0^k, F_0/h_1^\infty]_{s,f,w} \xrightarrow{\theta} [S/h_0^k, F_0/h_1^\infty]_{s+s_0,f+f_0,w+w_0}$$

is an isomorphism. The goal of this section is to obtain a vanishing region for $[S/(h_0^k, \theta), F_0/h_1^\infty]_{*,*,*}$ in the case $k = 1$ and $\theta = P$.

The dual Steenrod algebra is too large to work with, so we would like to start with a smaller one, namely $\mathcal{A}^{\mathbb{C}}(1)_* \cong \mathbb{M}_2[\tau_0, \tau_1, \xi_1]/(\tau_0^2 = \tau\xi_1, \tau_1^2, \xi_1^2)$. Then for $\mathcal{A}^{\mathbb{C}}_*$-comodules M and N (thus also $\mathcal{A}^{\mathbb{C}}(1)_*$-comodules), we can recover $[M,N]^{\mathcal{A}^{\mathbb{C}}_*}$ from $[M,N]^{\mathcal{A}^{\mathbb{C}}(1)_*}$ via infinitely many Cartan-Eilenberg spectral sequences along normal extensions of Hopf algebras, as we will explain later.

Let $N = F_0/h_1^\infty$. We will compute $[S/h_0, F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}$ as an intermediate step before reaching our goal of $[S/(h_0, P), F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}$. As a starting point, we can compute $[S/h_0, F_0]$ over $\mathcal{A}^{\mathbb{C}}(1)_*$, via the cofiber sequence $S \xrightarrow{h_0} S \to S/h_0$.

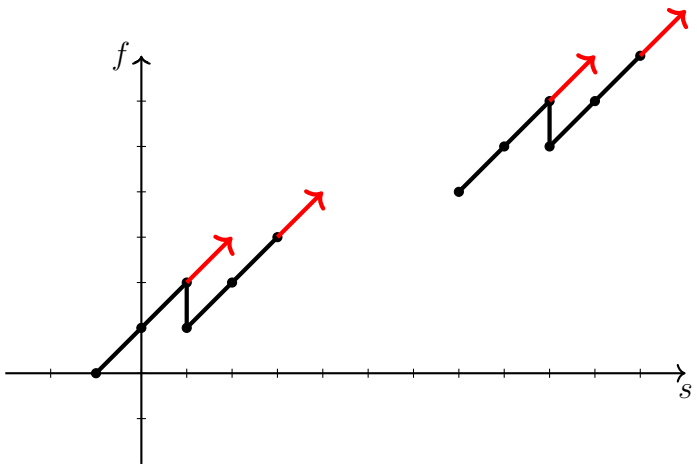


Figure 3.3: $[S/h_0, F_0]^{\mathcal{A}^{\mathbb{C}}(1)_*}$

This is periodic,where the periodicity shifts degree by $(8,4,4)$. Since $[S/h_0, F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}$ is a colimit, it is essential to know the maps over which we are taking the colimit. First let us take a look at the maps induced by multiplying by h_1 (we abbreviate $\Sigma^{-i,-i,-i}$ to Σ^{-i} in this diagram):

$$\begin{array}{ccccccc}
\xrightarrow{h_1} & [S/h_0, \Sigma^{-1}F_0] & \longrightarrow & [S/h_0, \Sigma^{-1}F_0/h_1] & \longrightarrow & \Sigma^{2,0,1}[S/h_0, \Sigma^{-1}F_0] & \xrightarrow{h_1} \\
 & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow{\scriptstyle id} & \\
\xrightarrow{h_1^2} & [S/h_0, \Sigma^{-2}F_0] & \longrightarrow & [S/h_0, \Sigma^{-2}F_0/h_1^2] & \longrightarrow & \Sigma^{3,1,2}[S/h_0, \Sigma^{-2}F_0] & \xrightarrow{h_1^2} \\
 & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow & \\
\end{array} \tag{3.3.4}$$

All rows are exact. From this we yield a more illuminating diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & coker(h_1) & \longrightarrow & [S/h_0, \Sigma^{-1}F_0/h_1] & \longrightarrow & ker(h_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & coker(h_1^2) & \longrightarrow & [S/h_0, \Sigma^{-2}F_0/h_1^2] & \longrightarrow & ker(h_1^2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_1\circ\Sigma^{-1}} & & \downarrow & & \downarrow & &
\end{array}$$

The maps i on the right column are canonical inclusions, and passing to colimits gives

$$\operatorname*{Colim}_k(coker(h_1^k)) \to [S/h_0, F_0/h_1^\infty] \to \operatorname*{Colim}_k(ker(h_1^k)).$$

Working over the dual subalgebra $\mathcal{A}^{\mathbb{C}}(1)_*$ we calculate $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$ directly. Furthermore we have:

Proposition 3.3.5. *For any $k \in \mathbb{Z}$, $k \geq 1$, the maps $[S/h_0, \Sigma^{-k}F_0/h_1^k]^{\mathcal{A}^{\mathbb{C}}(1)_*} \to [S/h_0, \Sigma^{-k-1}F_0/h_1^{k+1}]^{\mathcal{A}^{\mathbb{C}}(1)_*}$ are injective.*

The result of the calculation is shown in Figure 3.4. The shift in the figure appears as result of Proposition 3.3.1.

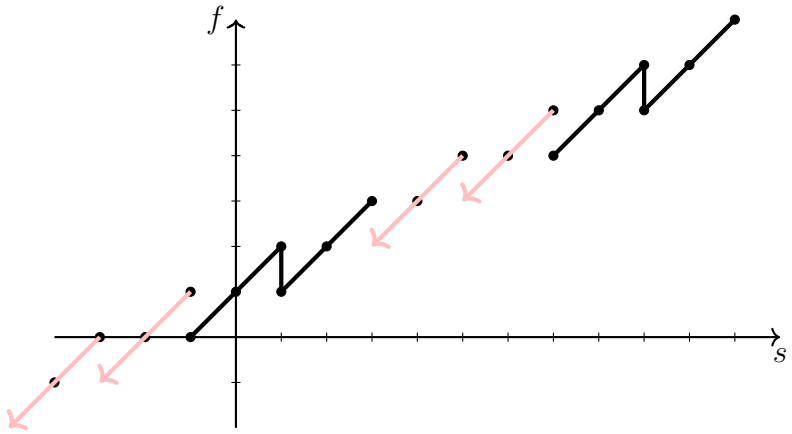


Figure 3.4: $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$

This is periodic, with a periodicity degree shift of $(8, 4, 4)$, just as with $[S/h_0, F_0]^{\mathcal{A}^{\mathbb{C}}(1)_*}$. Note that $[S/h_0, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$ differs from the classical $[S/h_0, S]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*}$ with two extra negative h_1-towers associated to each "lighting flash". The element in degree $(-1, 0, -1)$ in the first pattern is generated by τ with a shift.

Recall the self-map P on S/h_0 acts injectively as can be seen in Figure 3.4. Combining this with the long exact sequence:

$$\cdots \longrightarrow [S/(h_0, P), F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{s,f,w} \longrightarrow [S/h_0, F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{s,f,w} \xrightarrow{P}$$

$$\xrightarrow{P} [S/h_0, F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{s+8,f+4,w+4} \longrightarrow [S/(h_0, P), F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{s-1,f+1,w} \longrightarrow \cdots$$

gives $[S/(h_0,P),\Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$ as in Figure 3.5.

Remark 3.3.5. *Analogously to Proposition 3.3.5, for any $k\in\mathbb{Z}$, $k\geq 1$, the following maps are also injective:*

$$[S/(h_0,P),\Sigma^{-k}F_0/h_1^k]^{\mathcal{A}^{\mathbb{C}}(1)_*}\to[S/(h_0,P),\Sigma^{-k-1}F_0/h_1^{k+1}]^{\mathcal{A}^{\mathbb{C}}(1)_*}.$$

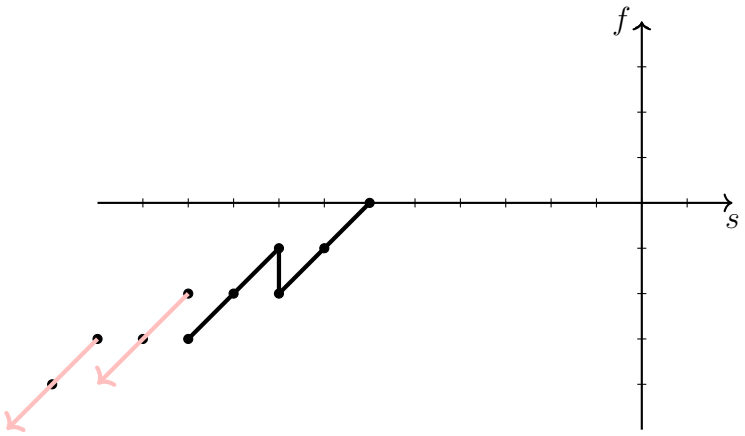


Figure 3.5: $[S/(h_0,P),\Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$

Next we will use the Cartan-Eilenberg spectral sequence to bootstrap our result from $\mathcal{A}^{\mathbb{C}}(1)_*$-homology to $\mathcal{A}^{\mathbb{C}}_*$-homology. A brief introduction to the Cartan-Eilenberg spectral sequence (see [17, Ch.XV] for details) is relevant at this point. Given an extension of Hopf algebras over $\mathbb{M}_2$

$$E\to\Gamma\to C$$

(so in particular $E\cong\Gamma\square_C\mathbb{M}_2$), the Cartan-Eilenberg spectral sequence computes $Cotor_\Gamma(M,N)$ for a Γ-comodule M and an E-comodule N. The spectral sequence arises from the double complex (Γ-resolution of $M)\square_\Gamma(E$-resolution of $N)$, and we have $Cotor_\Gamma(M,N)\cong\mathrm{Ext}_\Gamma(M,N)$ when M and N are τ-free.

The Cartan-Eilenberg spectral sequence has the form

$$E_1^{s,t,*,*}=Cotor_C^{t,*}(M,\bar{E}^{\otimes s}\otimes N)\Rightarrow Cotor_\Gamma^{s+t,*}(M,N).$$

If E has trivial C-coaction, then we have $E_1^{s,t,*,*}\cong Cotor_C^{t,*}(M,N)\otimes\bar{E}^{\otimes s}$. Taking the cohomology we obtain the E_2-page:

$$E_2^{s,t,*,*}=Cotor_E^{s,*}(\mathbb{M}_2,Cotor_C^{t,*}(M,N))\cong\mathrm{Ext}_E^{s,*}(\mathbb{M}_2,\mathbb{M}_2)\otimes\mathrm{Ext}_C^{t,*}(M,N).$$

The Cartan-Eilenberg spectral sequence converges when the input is a bounded-below $\mathcal{A}^{\mathbb{C}}_*$-comodule. We will obtain a vanishing region for each finite stage $[S/(h_0,P),\Sigma^{-k}F_0/h_1^k]^{\mathcal{A}^{\mathbb{C}}_*}$ and then deduce the vanishing region for $[S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}_*}$ by passing to the colimit.

$$\begin{array}{ccccccc}
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(1)_*} & \longrightarrow & [S/(h_0,P),\Sigma^{-2}F_0/h_1^2]^{\mathcal{A}^{\mathbb{C}}(1)_*} & \longrightarrow\cdots\longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*} \\
\downarrow & & \overset{CESS}{\downarrow} & & \\
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}_*} & \longrightarrow & [S/(h_0,P),\Sigma^{-2}F_0/h_1^2]^{\mathcal{A}^{\mathbb{C}}_*} & \longrightarrow\cdots\longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}_*}
\end{array}$$

We first calculate $[S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*}$, where

$$\mathcal{A}^{\mathbb{C}}(2)_* = \mathbb{M}_2[\tau_0,\tau_1,\tau_2,\xi_1,\xi_2]/(\tau_0^2=\tau\xi_1,\tau_1^2=\tau\xi_2,\tau_2^2,\xi_1^4,\xi_2^2).$$

To do this, we will use a sequence of normal maps of Hopf algebras:

$$\mathcal{A}^{\mathbb{C}}(2)_* \to \mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2 \to \mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2) \to \mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2,\tau_2) = \mathcal{A}^{\mathbb{C}}(1)_*.$$

First we consider the Cartan-Eilenberg spectral sequence corresponding to the extension

$$E(\tau_2) \to \mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2) \to \mathcal{A}^{\mathbb{C}}(1)_*.$$

The element τ_2, which has degree $(6,1,3)$, corresponds to h_{30} in the May spectral sequence. The $\mathcal{A}^{\mathbb{C}}(1)_*$-coaction on $E(\tau_2)$ is trivial for degree reasons. So we start with the $E_1=E_2$-page, and deduce a vanishing region on $[S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)}$.

$$\begin{array}{ccc}
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(1)_*}\otimes\mathbb{M}_2[h_{30}] & \longrightarrow\cdots\longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}\otimes\mathbb{M}_2[h_{30}] \\
\Big\Downarrow & & \\
[S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)} & \longrightarrow\cdots\longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)}
\end{array}$$

For the normal extension $E(\beta)\to\Gamma\to C$ of Hopf algebras we state a motivic version of [36, Lemma 4.10], which gives a relationship between the vanishing region for $[M,N]^\Gamma$ and the vanishing condition of $[M,N]^C$ together with the two "slopes" associated to β. Note that if β has degree (s_0,f_0,w_0), then $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0}$ are the slopes of the projections of (s_0,f_0,w_0) onto the plane $w=0$ and the plane $s=0$.

Theorem 3.3.6. *Let $E(\alpha)\to\Gamma\xrightarrow{q}C$ be a normal extension of Hopf algebras and M,$N\in$ Stab(Γ). Suppose β is an element in $[S,S]^E$ of degree (s_0,f_0,w_0) with s_0,f_0,w_0 all positive. Its image in $[S,S]^\Gamma$ (which we also call β) acts on $[M,N]^\Gamma$. Suppose for some $a,b,c,m,c_0\in\mathbb{R}$ with $a,b>0$ and $m\geq\frac{f_0}{s_0}>0$, the group $[q_*(M),q_*(N)]^C$ vanishes when $f>as+bw+c$ and also vanishes when $f>ms+c_0$. Then*

1. *if $f_0\leq as_0+bw_0$, or β acts nilpotently on $[M,N]^\Gamma$, then $[M,N]^\Gamma$ has a parallel vanishing region. In other words, it vanishes when $f>as+bw+c'$ for some constant c' and also vanishes when $f>ms+c_0$.*
2. *otherwise, $[M,N]^\Gamma$ vanishes when $f>\frac{mbw_0-f_0(m-a)}{bw_0-s_0(m-a)}s+\frac{bf_0-mbs_0}{bw_0-s_0(m-a)}w+c'$ and vanishes when $f>ms+c_0$.*

Remark 3.3.7. *The additional vanishing plane $f>ms+c_0$ generalizes the bounded below condition. In the classical setting, we have that $[M,N]^\Gamma$ vanishes when $s<c_0$, but due to the negative h_1-towers we do not have a vertical vanishing plane. So we adjust the "∞-slope" plane to be $f=ms+c_0$ to fulfill our purpose. This bound does not affect the periodicity region we study here, so we omit it henceforth.*

Proof of Theorem 3.3.6. If β has $f_0 \leq as_0+bw_0$, then β multiples of classes in $[M,N]^C$ will lie under the existing vanishing planes.

If $f_0 > as_0 + bw_0$, then every infinite β tower will contain classes lying outside of the rigion $f > as + bw + c$. If β acts nilpotently, there exists an integer k such that $\beta^k x$ is zero for all $x \in [M,N]^\Gamma$. Then there is a maximum length for all β-towers, and so we can still get a parallel vanishing plane $f > as+bw+c'$ on $[M,N]^\Gamma$ by adjusting the f-intercept.

Now we turn to case (2). If $f_0 > as_0 + bw_0$ and β acts non-nilpotently, then there must exist an element $x \in [M,N]^\Gamma$ for which the classes $\beta^k x$ are not zero on the E_∞ page of the Cartan-Eilenberg spectral sequence for every k. Thus no matter how we move up the existing vanishing plane $f > as + bw + c$, some β multiples of x will lie above the plane. Instead, we will find a new vanishing plane $f > a's + b'w + c'$ for $a', b', c' \in \mathbb{R}$. The new vanishing region $f > a's + b'w + c'$ must satisfy the condition $f_0 \leq a's_0 + b'w_0 + c'$. This plane is spanned by the direction of β and the intersecting line of the two existing vanishing planes. Hence we can solve to obtain $a' = \frac{mbw_0 - f_0(m-a)}{bw_0 - s_0(m-a)}$ and $b' = \frac{bf_0 - mbs_0}{bw_0 - s_0(m-a)}$. □

Remark 3.3.8. *In the relevant cases, the starting vanishing regions will have $b = 0$. In this case, the 3-dimensional conditions in Theorem 3.3.6 simplify to the following 2-dimensional conditions.*

Suppose for some $a, c, m, c_0 \in \mathbb{R}$ with $a > 0$ and $m \geq \frac{f_0}{s_0} > 0$, the group $[q_*(M), q_*(N)]^C$ vanishes when $f > as + c$ and also vanishes when $f > ms + c_0$. Then:

1. *if $f_0 \leq as_0$, or β acts nilpotently on $[M,N]^\Gamma$, then $[M,N]^\Gamma$ has a parallel vanishing region. That is to say, it vanishes when $f > as + c'$ for some constant c', and also vanishes when $f > ms + c_0$,*

2. *if otherwise, then $[M,N]^\Gamma$ vanishes when $f > \frac{f_0}{s_0}s + c'$ for some constant c', and vanishes when $f > ms + c_0$.*

Remark 3.3.9. *Similarly, we could generalize to the statement that the group $[q_*(M), q_*(N)]^C$ vanishes when $f > \varphi(s,w)$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then the gradient $v(-,-) = (\frac{\partial \varphi}{\partial s}(-), \frac{\partial \varphi}{\partial w}(-))$ is a vector field. Now we would like to consider $g = \underset{(s_0,w_0)}{Min} |v(s_0,w_0)|$ and compare g with $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0}$. The conditions can be rewritten as follows:*

1. *if $\frac{f_0}{s_0} \leq g$ or $\frac{f_0}{w_0} \leq g$, or β acts nilpotently, then $[M,N]^\Gamma$ has the same vanishing region translated vertically.*

2. *if both $\frac{f_0}{s_0}$ and $\frac{f_0}{w_0} > g$, and β acts non-nilpotently, then we must modify the vanishing region of $[M,N]^\Gamma$. However, it takes some work to write down a precise modification, so we omit it here.*

Remark 3.3.10. *From the cofiber sequence $S \xrightarrow{h_0^k} S \to S/h_0^k$ we can take tensor duals to derive the fiber sequence $D(S/h_0^k) \to S \to S$. Since $D(S/h_0^k) \simeq \Sigma^{-1,1-k,0} S/h_0^k$, we*

have

$$[S/h_0^k, S]_{s,f,w} = [S, D(S/h_0^k)]_{s,f,w} = [S, S/h_0^k]_{s+1,f+k-1,w.}$$

Because S/h_0^k *is compact in* $Stab(\mathcal{A}_*^{\mathbb{C}})$*, smashing with some* $N \in Stab(\mathcal{A}_*^{\mathbb{C}})$*, we get*

$$[S/h_0^k, N]_{s,f,w} \cong [S, D(S/h_0^k) \wedge N]_{s,f,w} \cong [S, S/h_0^k \wedge N]_{s+1,f+k-1,w.}$$

As a result $\beta \in [S,S]^\Gamma$ *acts on* $[M,N]^\Gamma$ *for compact* $M \in Stab(\mathcal{A}_*^{\mathbb{C}})$*, since* β *acts on* $[S, DM \wedge N]^\Gamma$*.*

The group $[S/(h_0,P), \Sigma^{-1,1,0}F_0/h_1^\infty]_{*,*,*}^{\mathcal{A}^{\mathbb{C}}(1)_*}$ has a single "lighting flash" pattern along with two negative h_1-towers (see Figure 3.5), so the vanishing region to start off with is $f > c$. (We obtain the same vanishing region of

$$[S/(h_0,P), \Sigma^{-1,1,0}(\Sigma^{-k}F_0/h_1^k)]_{*,*,*}^{\mathcal{A}^{\mathbb{C}}(1)_*}$$

for each k, since the maps we are taking colimit over are injections by Remark 3.3.5.) In our case, $[M,N]^\Gamma = [S/(h_0,P), \Sigma^{-1,1,0}F_0/h_1^\infty]_{*,*,*}^{\mathcal{A}^{\mathbb{C}}(1)_*}$, and we will apply Theorem 3.3.6 in the following three cases: (i) β is τ_2 of degree $(6,1,3)$; (ii) β is ξ_2 of degree $(5,1,3)$; (iii) β is ξ_1^2 of degree $(3,1,2)$.

Recall that we are working with the Cartan-Eilenberg spectral sequence

$$[S/(h_0,P), \Sigma^{-1,1,0}(\Sigma^{-k}F_0/h_1^k)]^{\mathcal{A}^{\mathbb{C}}(1)_*} \otimes \mathbb{M}_2[h_{30}] \Rightarrow [S/(h_0,P), \Sigma^{-1,1,0}(\Sigma^{-k}F_0/h_1^k)]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)}$$

There cannot be any differentials for degree reasons. By Theorem 3.3.6 the element h_{30} will bring us a vanishing region $f > \frac{1}{6}s + c_1$ for each k, where c_1 is some constant (we obtain the same constant for all k). Passing to the colimit, we conclude that $[S/(h_0,P), \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)}$ shares the same vanishing region $f > \frac{1}{6}s + c_1$.

The second step is to consider the normal extension in which we add ξ_2, corresponding to the class h_{21}:

$$E(\xi_2) \to \mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2 \to \mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2, \xi_2).$$

The $\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)$-coaction on $E(\xi_2)$ is trivial. We have E_2-pages as the first row:

$$\begin{array}{ccccc}
[S/(h_0,P), \Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)} \otimes \mathbb{M}_2[h_{21}] & \longrightarrow \cdots \longrightarrow & [S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)} \otimes \mathbb{M}_2[h_{21}] \\
\Downarrow & & \\
[S/(h_0,P), \Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2} & \longrightarrow \cdots \longrightarrow & [S/(h_0,P), F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2}
\end{array}$$

The spectral sequence collapses at the E_2-page. This is because in the May spectral sequence over $\mathcal{A}^{\mathbb{C}}(2)$ or $\mathcal{A}^{\mathbb{C}}$, there is a differential $d_1(h_{30}) = h_1h_{21} + h_2h_{20}$, but in the group $[S/(h_0,P), \Sigma^{-1,1,0}(\Sigma^{-k}F_0/h_1^k)]^{\mathcal{A}^{\mathbb{C}}(2)_*/(\xi_1^2,\xi_2)}$, h_0 and h_2 are zero. As a result, h_{21} is also non-nilpotent. For some constant c_2, the vanishing region of $[S/(h_0,P), \Sigma^{-1,1,0}(\Sigma^{-k}F_0/h_1^k)]^{\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2}$ is $f > \frac{1}{5}s + c_2$ for each k according to Theorem 3.3.6, and the same is true for the colimit $[S/(h_0,P), \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2}$.

Next we consider the Cartan-Eilenberg spectral sequence corresponding to the extension:

$$E(\xi_1^2) \to \mathcal{A}^{\mathbb{C}}(2)_* \to \mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2.$$

Here the class ξ_1^2 corresponds to the class h_2 in the May spectral sequence. The $\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2$-coaction on $E(\xi_1^2)$ is trivial as well. We have E_2-pages as in the first row:

$$\begin{array}{ccccc} [S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2}\otimes\mathbb{M}_2[h_2] & \longrightarrow & \cdots & \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*/\xi_1^2}\otimes\mathbb{M}_2[h_2] \\ \Downarrow & & & & \\ [S/(h_0,P),\Sigma^{-1}F_0/h_1]^{\mathcal{A}^{\mathbb{C}}(2)_*} & \longrightarrow & \cdots & \longrightarrow & [S/(h_0,P),F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(2)_*} \end{array}$$

We do have some non-zero differentials appear. In the previous steps, by introducing $[\tau_2]=(6,1,3)$ and $[\xi_2]=(5,1,3)$, which give rise to non-nilpotent elements in $\mathrm{Ext}_{\mathbb{C}}$, we arrived a vanishing region of $f>\frac{1}{5}s+c_3$, where c_3 is a constant. However $[\xi_1^2]=(3,1,2)$ is nilpotent since $h_2^4=0$ in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}(2)_*}$ and $\mathrm{Ext}_{\mathbb{C}}$.

Moving from $\mathcal{A}^{\mathbb{C}}(2)_*$ to $\mathcal{A}^{\mathbb{C}}{}_*$, we have many more elements to introduce. However those elements won't satisfy $\frac{f}{s}>\frac{1}{5}$. By Theorem 3.3.6 (or Remark 3.3.8), for each k, $[S/(h_0,P),\Sigma^{-1,1,0}(\Sigma^{-k}F_0/h_1^k)]^{\mathcal{A}^{\mathbb{C}}}$ vanishes if $f=\frac{1}{5}s+c_3$. Since the vanishing plane passes through the point $(-6,0,-1)+3\cdot(3,1,2)=(3,3,5)$, the constant c_3 is $\frac{12}{5}$ and the region $f>\frac{1}{5}s+\frac{12}{5}$ would be carried through to $\mathcal{A}^{\mathbb{C}}{}_*$. We conclude that

Proposition 3.3.11. *The group $[S/(h_0,P),\Sigma^{-1,1,0}F_0/h_1^\infty]_{s,f,w}$ has a vanishing region of $f>\frac{1}{5}s+\frac{12}{5}$.*

Note that it is possible for many reasons that the vanishing region we have found is not optimal. First, we could consider the "slope" of the motivic weight side $\frac{f}{w}$ instead of $\frac{f}{s}$ under certain bounded below conditions. Second, if other elements were included, more differentials would occur, allowing for a larger vanishing region. More calculation is required to clarify these cases.

3.4 The motivic periodicity theorem

Let F_0 and F_{01} still be the same as in Definition 3.2.1, so that

$$[S,\Sigma^{-1,1,0}F_0/h_1^\infty]_{s,f,w}\cong[S,F_{01}]_{s,f,w}$$

when $f>\frac{1}{2}s+3$. Given a self-map θ on S/h_0^k let us recall the diagram where the first row is exact:

$$\begin{array}{ccccccc} [S/(h_0^k,\theta),\Sigma^{-1,1,0}F_0/h_1^\infty] & \longrightarrow & [S/h_0^k,\Sigma^{-1,1,0}F_0/h_1^\infty] & \xrightarrow{\theta} & [S/h_0^k,\Sigma^{-1,1,0}F_0/h_1^\infty] & \longrightarrow & \Sigma^{-1,1,0}[S/(h_0^k,\theta),\Sigma^{-1,1,0}F_0/h_1^\infty] \\ & & \downarrow & & \downarrow & & \\ & & [S,\Sigma^{-1,1,0}F_0/h_1^\infty] & \xrightarrow{P_r(-)} & [S,\Sigma^{-1,1,0}F_0/h_1^\infty] & & \end{array}$$

The vertical maps are isomorphisms whenever $f>\frac{1}{2}s+\frac{3}{2}-k$ due to Corollary 3.2.11. We would like to further restrict the condition to $f>\frac{1}{2}s+3-k$ in order to eliminate the indeterminacy. The vanishing condition on $[S/(h_0^k,\theta),\Sigma^{-1,1,0}F_0/h_1^\infty]$, which is the same as the vanishing condition on $[S/(h_0^k,\theta),F_{01}]_{s,f,w}$, tells us whether θ is an isomorphism.

In the previous section, we established the case when $k = 1$, given in Proposition 3.3.11. We show in Figure 3.6 the $(2^{r+1}, 2^r, 2^r)$-periodic pattern for $[S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}$, where $k \leq 2^r$. By an analogous computation, one can see that for a general positive integer $k \leq 2^r$, the groups $[S/(h_0^k, P^{2^{r-2}}), F_{01}]_{s,f,w}$ admit a parallel vanishing region as in the $k = 1$ case.

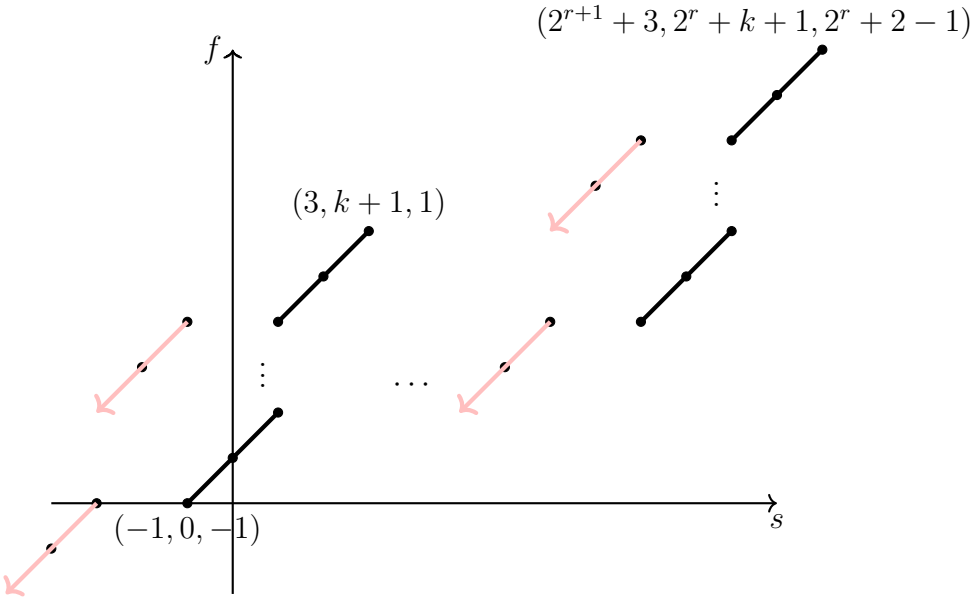


Figure 3.6: $[S/h_0^k, \Sigma^{-1,1,0}F_0/h_1^\infty]^{\mathcal{A}^{\mathbb{C}}(1)_*}_{*,*,*}$

We have the following lemma for the f-intercept:

Lemma 3.4.7 (Corollary of [36, Lemma 5.4])**.** *Let $M, N \in Stable(\mathcal{A}_*^{\mathbb{C}})$ with M compact. Let $M_1 = M/\theta_1$ be the cofiber of the self-map $\Sigma^{s_1,f_1,w_1}M \xrightarrow{\theta_1} M$, and let $M_2 = M/(\theta_1, \theta_2)$ be the cofiber of the self-map $\Sigma^{s_2,f_2,w_2}M/\theta_1 \xrightarrow{\theta_2} M/\theta_1$. Define M_1' and M_2' with respect to the self-maps $\Sigma^{s_1',f_1',w_1'}M \xrightarrow{\theta_1'} M$ and $\Sigma^{s_2',f_2',w_2'}M/\theta_1' \xrightarrow{\theta_2'} M/\theta_1'$ in the same way. Suppose θ_i and θ_i' are parallel, i.e. $(s_i, f_i, w_i) = \lambda_i(s_i', f_i', w_i')$ where λ_i are non-zero real numbers and $i = 1, 2$.*

Further let $a, b \in \mathbb{R}$ and suppose $f_i > as_i + bw_i$ and $f_i' > as_i' + bw_i'$ for $i = 1, 2$. We make the convention that the f-intercept is ∞ if there is no such vanishing plane. Then the minimal f-intercepts of the vanishing planes parallel to $f = as + bw$ on $[M_2, N]$ and $[M_2', N]$ agree.

Proof of Lemma 3.4.7. We construct the iterated cofiber $L_1 = M/(\theta_1, \theta_1')$ and $L_2 = M/(\theta_1, \theta_2, \theta_1', \theta_2')$. Since $f_i > as_i + bw_i$ and $f_i' > as_i' + bw_i'$ for $i = 1, 2$, the minimal f-intercepts for the vanishing planes parellel to $f = as + bw$ agree on $[M_i, N]$, $[M_i', N]$ and $[L_i, N]$ by inductively applying Lemma 3.2.9.

Note that the notation for L_1 and L_2 is ambiguous. The notation does not indicate that M/θ_1 should admit a θ_1' self-map or vice versa. Because of the uniqueness of (homological) self-maps that Krause has shown in [36, Sec. 4], there is a self-map θ_1'' compatible with both θ_1 and θ_1', which acts on M by a power of θ_1, and by a power

of θ_1'. We will take L_1 to be the cofiber of the self-map θ_1''. Similarly, there exists a self-map θ_2'' on L_1 that acts on M_1 by a power of θ_2, and on M_1' by a power of θ_2'. So we can set L_2 as the cofiber of the self-map θ_2''. □

Remark 3.4.8. *Krause's proof of the uniqueness of self-maps is in the classical setting, yet for the $\mathbb{C}$-motivic case the proof is analogous.*

Remark 3.4.9. *The cofiber sequences arising from the Verdier's axiom and the 3×3 lemma offer an alternative way to view the vanishing condition of $[S/(h_0^k, P^{2^{r-2}}), F_{01}]_{s,f,w}$. Let $m, n, l, l' \in \mathbb{N}$ be positive with $m \leq 4l$ and $m+n \leq 4(l+l')$. We have the following cofiber sequences:*

$$S/h_0^m \to S/h_0^{m+n} \to S/h_0^n$$
$$S/(h_0^m, P^{l+l'}) \to S/(h_0^{m+n}, P^{l+l'}) \to S/(h_0^n, P^{l+l'})$$
$$S/(h_0^m, P^{l}) \to S/(h_0^{m}, P^{l+l'}) \to S/(h_0^m, P^{l'}).$$

Passing to the induced long exact sequences in homology, we conclude that for $k \leq 2^r$, the groups $[S/(h_0^k, P^{2^{r-2}}), F_{01}]_{s,f,w}$ admit the same vanishing condition as $[S/(h_0, P), F_{01}]_{s,f,w}$.

It follows that for any $k \leq 2^r$ and any self-map $\theta = P^{2^{r-2}}$ of S/h_0^k, the corresponding groups $[S/(h_0^k, \theta), F_{01}]$ have a vanishing region of $f > \frac{1}{5}s + \frac{12}{5}$. Combining with Theorem 3.2.12, we arrive at the motivic version of Theorem 3.0.1:

Theorem 3.4.10 (Another way of stating Theorem 3.0.3)**.** *For $r \geq 2$, the Massey product operation $P_r(-) := \langle h_{r+1}, h_0^{2^r}, -\rangle$ is uniquely defined on $\mathrm{Ext}^{s,f,w} = H^{s,f,w}(\mathcal{A}^{\mathbb{C}})$ when $s > 0$ and $f > \frac{1}{2}s + 3 - 2^r$.*

Furthermore, for $f > \frac{1}{5}s + \frac{12}{5}$,

$$P_r \colon [S, F_{01}]_{s,f,w} \xrightarrow{P_r(-)} [S, F_{01}]_{s+2^{r+1}, f+2^r, w+2^r}.$$

is an isomorphism when restricted to the subgroup consisting of elements that are torsion with respect to both h_0 and h_1.

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Chapter 4 Topological realization of $\mathcal{A}^{\mathbb{R}}(1)$

Given an $\mathcal{A}$-module M, we say M is realized by X if there is a spectrum X such that $H^*(X) \cong M$ as an $\mathcal{A}$-module. Classically, $\mathcal{A}(1)$ has 4 different $\mathcal{A}$-module structures, which are distinguished by the action of Sq^4 (as in Figure 1.1, where we depict a singly-generated free $\mathcal{A}(1)$-module, where each $\bullet$ represents a $\mathbb{F}_2$-generator. The black and blue lines represent the action of Sq^1 and Sq^2, respectively. The red boxed lines represent the action of Sq^4. Whether or not the dotted red lines exist gives the 4 different $\mathcal{A}$-module structures). And each can be realized as the cohomology of a spectrum.

The existence and uniqueness of a realization is guaranteed by the $\mathbb{R}$-motivic Toda realization theorem. The classical Toda realization theorem [53] (see also [14, Theorem 3.1]), is recast in the modern literature as a special case of Goerss-Hopkins obstruction theory [25] (when the chosen operad is trivial). This obstruction theory can be generalized to the $\mathbb{R}$-motivic setting [43], and 4.1 would then be a special case of such a generalization.

We describe all the 128 $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$. The $\mathbb{R}$-motivic Toda realization theorem indicates that all of them can be realized. We then construct one specific realization $\mathcal{A}_1^{\mathbb{R}}$ of the subalgebra $\mathcal{A}^{\mathbb{R}}(1)$ using a method of Smith(outlined in [50, Appendix C]), which constructs new finite spectra from known ones.

4.1 $\mathbb{R}$-motivic Toda realization theorem

The classical Toda realization theorem [53] (see also [14, Theorem 3.1]), is recast in the modern literature as a special case of Goerss-Hopkins obstruction theory [25] (when the chosen operad is trivial). This obstruction theory can be generalized to the $\mathbb{R}$-motivic setting [43], and Section 4.1 would then be a special case of such a generalization.

More recent work of [48] conceptualizes Goerss-Hopkins obstruction theory in the general setup of stable ∞-categories with t-structures. If we set $\mathcal{C} = \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$, $\mathrm{A} = \mathbb{S}_{\mathrm{H}_{\mathbb{R}}\mathbb{F}_2}$, and let K to be a finite $\mathcal{A}^{\mathbb{R}}_*$-comodule in [48, Corollary 4.10], then we get a sequence of obstruction classes

$$\theta_n \in \mathrm{Ext}^{-2,n+2,0}_{\mathcal{A}^{\mathbb{R}}_*}(\mathrm{K},\mathrm{K}) \tag{4.1.1}$$

for each $n \geq 0$, the vanishing of which guarantees the existence of an $\mathbb{S}_{\mathrm{H}_{\mathbb{R}}\mathbb{F}_2}$-module whose homology is isomorphic to K as an $\mathcal{A}^{\mathbb{R}}_*$-comodule. Since the t-structure in $\mathbf{Sp}^{\mathbb{R}}$ does not change the motivic weight, the obstruction classes in (4.1.1) lie in the Ext-groups of motivic weight 0.

If M is a finite $\mathbb{M}_2^{\mathbb{R}}$-free $\mathcal{A}^{\mathbb{R}}$-module then $\mathrm{K} := \hom_{\mathbb{M}_2^{\mathbb{R}}}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}})$ is a finite $\mathcal{A}^{\mathbb{R}}_*$-comodule,

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}_*}(\mathrm{K},\mathrm{K}) \cong \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{M},\mathrm{M}),$$

and therefore, Section 4.1 follows. Alternatively, one can prove Section 4.1 simply by emulating the classical proof (as exposed in [14, §3]).

The purpose of this section is to deduce, from the $\mathbb{R}$-motivic Toda realization theorem (Section 4.1), various weaker forms which are perhaps more convenient for application purposes. Explicit calculation of $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{*,*,*}(\mathrm{M},\mathrm{M})$ can often be difficult, and one can use a sequence of spectral sequences to approximate these ext groups. Each such approximation leads to a corresponding weaker form.

4.1.1 Weak $\mathbb{R}$-motivic Toda realization – version (I)

Let M be an $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathbb{M}_2^{\mathbb{R}}$-module is free and finitely generated. Let $\mathcal{B}_{\mathrm{M}}$ denote its $\mathbb{M}_2^{\mathbb{R}}$-basis and $\mathcal{D}_{\mathrm{M}}$ denote the collection of bidegrees in which there is an element in $\mathcal{B}_{\mathrm{M}}$. For any element $x \in \mathrm{M}^{s,w}$, we let $\mathtt{t}(x) = s + w$ and define

$$\mathrm{M}_{\geq n} := \mathbb{M}_2^{\mathbb{R}} \cdot \{b \in \mathcal{B}_{\mathrm{M}} : \mathtt{t}(b) \geq n\}$$

as the free sub $\mathbb{M}_2^{\mathbb{R}}$-module of M generated by $\{b \in \mathcal{B}_{\mathrm{M}} : \mathtt{t}(b) \geq n\}$.

Note that the $\mathcal{A}^{\mathbb{R}}$-module structure of M is determined by the action of $\mathcal{A}^{\mathbb{R}}$ on the elements of $\mathcal{B}_{\mathrm{M}}$ and the Cartan formula. This, along with the fact that $\mathtt{t}(a) \geq 0$ for all $a \in \mathcal{A}^{\mathbb{R}}$, implies that $\mathrm{M}_{\geq n}$ are also a sub $\mathcal{A}^{\mathbb{R}}$-module of M. Therefore, we get an $\mathcal{A}^{\mathbb{R}}$-module filtration of M

$$\mathrm{M} = \mathrm{M}_{\geq k} \supset \mathrm{M}_{\geq k+1} \supset \cdots \supset \mathrm{M}_{\geq k+l} = \mathbf{0}$$

such that we for each i there is a short exact sequences

$$0 \longrightarrow \mathrm{M}_{\geq i+1} \longrightarrow \mathrm{M}_{\geq i} \longrightarrow \bigoplus_{\{b \in \mathcal{B}_{\mathrm{M}} : \mathtt{t}(b) = i\}} \Sigma^{|b|}\mathbb{M}_2^{\mathbb{R}} \longrightarrow 0 \tag{4.1.2}$$

of $\mathcal{A}^{\mathbb{R}}$-modules.

A short exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules gives a long exact sequence in Ext. By splicing the long exact sequences induced by (4.1.2), we get an "algebraic" Atiyah-Hirzebruch spectral sequence

$$\mathrm{E}_2^{s',w',s,f,w} := \mathcal{B}_{\mathrm{M}}^{s',w'} \otimes \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) \Rightarrow \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-s',f,w-w'}(\mathrm{M},\mathrm{M}) \tag{4.1.3}$$

and a corresponding weak version of Section 4.1, along with a uniqueness criterion.

Theorem 4.1.4. *Let* M *denote an* $\mathcal{A}^{\mathbb{R}}$*-module whose underlying* $\mathbb{M}_2^{\mathbb{R}}$*-module is free and finite. Suppose*

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-2,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) = 0$$

for $f \geq 3$ *whenever* $(s,w) \in \mathcal{D}_{\mathrm{M}}$*. Then there exists an* $\mathrm{X} \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ *such that* $\mathrm{H}_{\mathbb{R}}^{*,*}(\mathrm{X}) \cong \mathrm{M}$ *as an* $\mathcal{A}^{\mathbb{R}}$*-module. Further, such a realization is unique if*

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-1,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) = 0$$

for all $f \geq 2$ *and* $(s,w) \in \mathcal{D}_{\mathrm{M}}$*.*

4.1.2 Weak $\mathbb{R}$-motivic Toda realization – version (II)

For any $\mathcal{A}^{\mathbb{R}}$-module M which is $\mathbb{M}_2^{\mathbb{R}}$-free, the quotient $\mathrm{M}/(\rho)$ is an $\mathcal{A}^{\mathbb{C}}$-module. In particular,

$$\mathcal{A}^{\mathbb{R}}/(\rho) \cong \mathcal{A}^{\mathbb{C}}$$

as a graded Hopf-algebra. Therefore, we have a spectral sequence

$${}^{\rho}\mathrm{E}_2^{s,f,w,i} := \bigoplus_{i\geq 0} \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s+i,f,w+i}(\mathrm{M}/(\rho), \mathbb{M}_2^{\mathbb{C}}) \Longrightarrow \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(\mathrm{M}, \mathbb{M}_2^{\mathbb{R}}) \tag{4.1.5}$$

which is often called the *(algebraic) ρ-Bockstein spectral sequence*. Thus we get the following version of the $\mathbb{R}$-motivic Toda realization and uniqueness theorem which is weaker than Theorem 4.1.4.

Theorem 4.1.6. *Let* M *denote an $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathbb{M}_2^{\mathbb{R}}$-module is free and finite. Suppose*

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s-2+i,f,w+i}(\mathrm{M}/(\rho), \mathbb{M}_2^{\mathbb{C}}) = 0$$

for $f \geq 3$ and all $i \geq 0$ whenever $(s,w) \in \mathcal{D}_{\mathrm{M}}$, then there exists an $\mathrm{X} \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ such that $\mathrm{H}_{\mathbb{R}}^{*,*}(\mathrm{X}) \cong \mathrm{M}$ as an $\mathcal{A}^{\mathbb{R}}$-module. Further, such a realization is unique if

$$\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s-1+i,f,w+i}(\mathrm{M}/(\rho), \mathbb{M}_2^{\mathbb{C}}) = 0$$

for all $f \geq 2$, $i \geq 0$ and $(s,w) \in \mathcal{D}_{\mathrm{M}}$.

4.1.3 Weak $\mathbb{R}$-motivic Toda realization – version (III)

Similarly to the classical case, the $\mathbb{C}$-motivic Steenrod algebra enjoys an increasing filtration called the May filtration (see [20]), which is easier to express on its dual. On $\mathcal{A}_*^{\mathbb{C}}$, the May filtration is induced by assigning the May weights

$$\mathsf{m}(\tau_{i-1}) = \mathsf{m}(\xi_i^{2^j}) = 2i-1$$

and extending it multiplicatively. The associated graded is an exterior algebra

$$\mathrm{gr}(\mathcal{A}^{\mathbb{C}}) \cong \Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{i,j} : i \geq 1, j \geq 0), \tag{4.1.7}$$

where $\xi_{i,0}$ represents $(\tau_{i-1})_*$ and $(\xi_{i,j+1})_*$ represents $(\xi_i^{2^j})_*$ in the associated graded. When $\mathrm{M} = \mathbb{M}_2^{\mathbb{R}}$ in (4.1.10), then

$${}^{\mathrm{May}}\mathrm{E}_{1,\mathbb{M}_2^{\mathbb{C}}}^{*,*,*,*} \cong \mathbb{M}_2^{\mathbb{C}}[\mathsf{h}_{i,j} : i \geq 1, j \geq 0], \tag{4.1.8}$$

where $\mathsf{h}_{i,j}$ represents the class $\xi_{i,j}$. The (s,f,w,m)-degrees of these generators are given by

$$|\mathsf{h}_{i,j}| = \begin{cases} (2^i-2, 1, 2^{i-1}-1, 2i-1) & \text{if } j=0, \text{ and,} \\ (2^j(2^i-1)-1, 1, 2^{j-1}(2^i-1), 2i-1) & \text{otherwise.} \end{cases}$$

Remark 4.1.9. *After reindexing the May filtration of* (4.1.8) *by setting the May weight of* $h_{i,j}$ *equal to* i*, it is consistent with the indexing used in [20].*

When M is a cyclic $\mathcal{A}^{\mathbb{R}}$-module, $M/(\rho)$ is also cyclic as an $\mathcal{A}^{\mathbb{C}}$-module, thus the May filtration induces a filtration on $M/(\rho)$. Thus, we get a corresponding May spectral sequence

$$ {}^{\text{May}}E_{1,M/(\rho)}^{s,f,w,\mathsf{m}} := \text{Ext}_{\text{gr}(\mathcal{A}^{\mathbb{C}})}^{s,f,w,m}(\text{gr}(M/(\rho)), \mathbb{M}_2^{\mathbb{C}}) \Rightarrow \text{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s,f,w}(M/(\rho), \mathbb{M}_2^{\mathbb{C}}) \tag{4.1.10} $$

computing the input of the ρ-Bockstein spectral sequence (4.1.5). Thus we can formulate a version of $\mathbb{R}$-motivic Toda realization theorem which is even weaker than Theorem 4.1.6.

Theorem 4.1.11. *Let* M *denote an cyclic* $\mathcal{A}^{\mathbb{R}}$*-module whose underlying* $\mathbb{M}_2^{\mathbb{R}}$*-module is free and finite. Suppose*

$$ {}^{\text{May}}E_{1,M/(\rho)}^{s-2+i,f,w+i,*} = 0. $$

for $f \geq 3$ *and all* $i \geq 0$ *whenever* $(s,w) \in \mathcal{D}_{\text{M}}$*. Then there exists an* $X \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$ *such that* $H_{\mathbb{R}}^{*,*}(X) \cong M$ *as an* $\mathcal{A}^{\mathbb{R}}$*-module. Further, such a realization is unique if*

$$ {}^{\text{May}}E_{1,M/(\rho)}^{s-1+i,f,w+i,*} = 0 $$

for $f \geq 2$*,* $i \geq 0$ *and* $(s,w) \in \mathcal{D}_{\text{M}}$*.*

4.2 The 128 $\mathcal{A}^{\mathbb{R}}$-module structure

Our result concerns realizations of $\mathcal{A}^{\mathbb{R}}(1)$.

Theorem 4.2.1. *There exists* 128 *different* $\mathcal{A}^{\mathbb{R}}$*-modules whose underlying* $\mathcal{A}^{\mathbb{R}}(1)$*-module structures are free on one generator, all of which can be realized as* $H_{\mathbb{R}}^{*,*}(X)$ *for some* $X \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$*.*

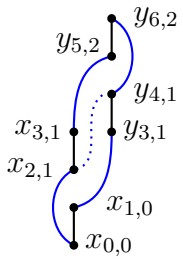


Figure 4.1: The $\mathcal{A}^{\mathbb{R}}(1)$ as an $\mathcal{A}^{\mathbb{R}}(1)$-module

Notation 4.2.2. *We fix an* $\mathbb{M}_2^{\mathbb{R}}$*-basis*

$$ \{x_{0,0}, x_{1,0}, x_{2,1}, x_{3,1}, y_{3,1}, y_{4,1}, y_{5,2}, y_{6,2}\} $$

of $\mathcal{A}^{\mathbb{R}}(1)$ *as in Figure 4.7 (where we depict a singly-generated free* $\mathcal{A}^{\mathbb{R}}(1)$*-module, where each* $\bullet$ *represents a* $\mathbb{M}_2^{\mathbb{R}}$*-generator. The black and blue lines represent the action of motivic* Sq^1 *and* Sq^2*, respectively. A dotted line represents that the action hits the* τ*-multiple of the given* $\mathbb{M}_2^{\mathbb{R}}$*-generator), so that*

- $\mathrm{Sq}^1(x_{0,0}) = x_{1,0}$
- $\mathrm{Sq}^1(y_{5,2}) = y_{6,2}$
- $\mathrm{Sq}^2(x_{2,1}) = \tau y_{4,1}$
- $\mathrm{Sq}^1(x_{2,1}) = x_{3,1}$
- $\mathrm{Sq}^2(x_{0,0}) = x_{2,1}$
- $\mathrm{Sq}^2(x_{3,1}) = y_{5,2}$
- $\mathrm{Sq}^1(y_{3,1}) = y_{4,1}$
- $\mathrm{Sq}^2(x_{1,0}) = y_{3,1}$
- $\mathrm{Sq}^2(y_{4,1}) = y_{6,2}$.

We now record all 128 $\mathcal{A}^{\mathbb{R}}$-modules of Theorem 4.2.1 using the basis above.

Theorem 4.2.3. *For every vector* $(\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V} = \mathbb{F}_2^7$ *and*

$$\jmath_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25} + \beta_{26}),$$

there exists a unique isomorphism class of $\mathcal{A}^{\mathbb{R}}$*-module structures on* $\mathcal{A}^{\mathbb{R}}(1)$ *determined by the formulas*

(i) $\mathrm{Sq}^4(x_{0,0}) = \beta_{03}(\rho \cdot y_{3,1}) + (1 + \beta_{03} + \beta_{14})(\tau \cdot y_{4,1}) + \alpha_{03}(\rho \cdot x_{3,1})$

(ii) $\mathrm{Sq}^4(x_{1,0}) = y_{5,2} + \beta_{14}(\rho \cdot y_{4,1})$

(iii) $\mathrm{Sq}^4(x_{2,1}) = \beta_{26}(\tau \cdot y_{6,2}) + \beta_{25}(\rho \cdot y_{5,2}) + \jmath_{24}(\rho^2 \cdot y_{4,1})$

(iv) $\mathrm{Sq}^4(x_{3,1}) = (\beta_{25} + \beta_{26})(\rho \cdot y_{6,2})$

(v) $\mathrm{Sq}^4(y_{3,1}) = \gamma_{36}(\rho \cdot y_{6,2})$

(vi) $\mathrm{Sq}^8(x_{0,0}) = \beta_{06}(\rho^2 \cdot y_{6,2})$.

Further, any $\mathcal{A}^{\mathbb{R}}$*-module whose underlying* $\mathcal{A}^{\mathbb{R}}(1)$*-module is free on one generator is isomorphic to one listed above.*

Notation 4.2.4. *For any vector* $\overline{\mathrm{v}} \in \mathcal{V}$*, we denote the corresponding* $\mathcal{A}^{\mathbb{R}}$*-module in Theorem* 4.2.3 *by* $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$*. By* $\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}]$*, we denote an object of* $\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$*, whose cohomology is isomorphic to* $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$ *as an* $\mathcal{A}^{\mathbb{R}}$*-module. We let*

$$\mathcal{A}_1^{\mathbb{R}} := \{\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}] : \overline{\mathrm{v}} \in \mathcal{V}\}/(\textit{weak equivalence})$$

denote the set of equivalence classes of finite $\mathbb{R}$*-motivic spectra whose cohomology are free of rank 1 over* $\mathcal{A}^{\mathbb{R}}(1)$*.*

We begin by proving Theorem 4.2.3, which identifies all possible $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$ up to isomorphism.

Proof of Theorem 4.2.3. Note that the Cartan formula of $\mathcal{A}^{\mathbb{R}}$ and finiteness of $\mathcal{A}^{\mathbb{R}}(1)$ imply that the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ is determined once the action of Sq^4 and Sq^8 are specified on its $\mathbb{M}_2^{\mathbb{R}}$-generators. The following are possible Sq^4 and Sq^8-actions on the $\mathbb{M}_2^{\mathbb{R}}$-module generators. As can be seen in Figure 4.2, there is no room

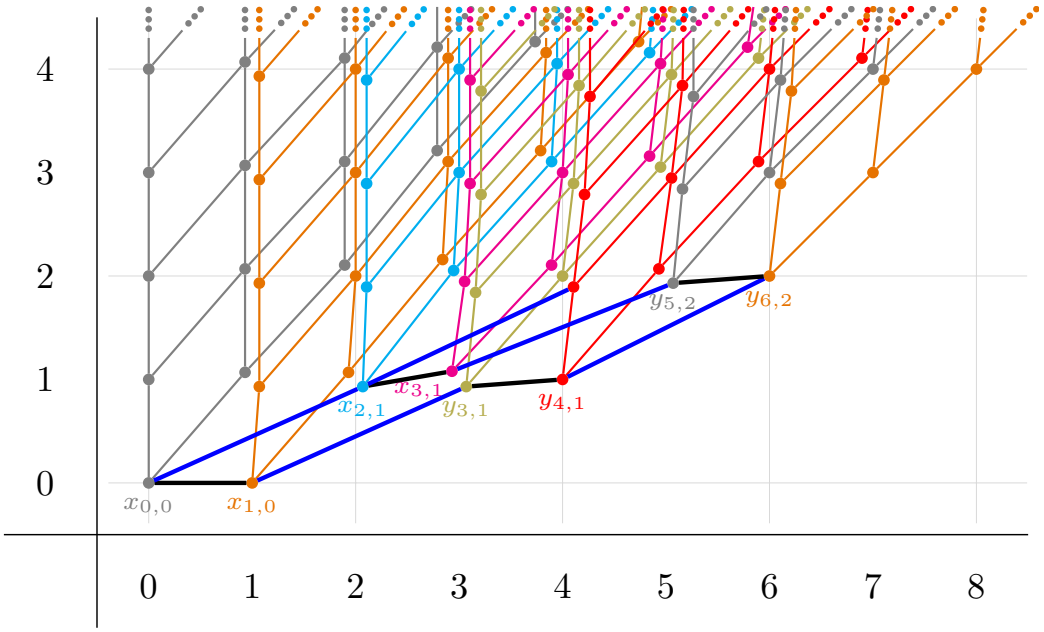


Figure 4.2: The free $\mathbb{M}_2^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}(1)$

for other possible actions.

$$\begin{aligned}
\mathrm{Sq}^4(x_{0,0}) &= \beta_{03}(\rho \cdot y_{3,1}) + \beta_{04}(\tau \cdot y_{4,1}) + \alpha_{03}(\rho \cdot x_{3,1}) \\
\mathrm{Sq}^4(x_{1,0}) &= \beta_{14}(\rho \cdot y_{4,1}) + \beta_{15}(y_{5,2}) \\
\mathrm{Sq}^4(x_{2,1}) &= \jmath_{24}(\rho^2 \cdot y_{4,1}) + \beta_{25}(\rho \cdot y_{5,2}) + \beta_{26}(\tau \cdot y_{6,2}) \\
\mathrm{Sq}^4(x_{3,1}) &= \beta_{36}(\rho \cdot y_{6,2}) \\
\mathrm{Sq}^4(y_{3,1}) &= \gamma_{36}(\rho \cdot y_{6,2}) \\
\mathrm{Sq}^8(x_{0,0}) &= \beta_{06}(\rho^2 \cdot y_{6,2})
\end{aligned}$$

The Adem relation $\mathrm{Sq}^2\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1 + \rho\mathrm{Sq}^3\mathrm{Sq}^1$ (see Proposition B.1), when applied to $x_{0,0}$ and $x_{2,1}$, yields $\beta_{15} = 1$, $\beta_{03} + \beta_{04} + \beta_{14} = 1$ and $\beta_{25} + \beta_{26} = \beta_{36}$. The equation

$$\jmath_{24} = \beta_{03}\gamma_{36} + \alpha_{03}\beta_{36},$$

is forced by the Adem relation $\mathrm{Sq}^4\mathrm{Sq}^4 = \mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2 + \tau\mathrm{Sq}^3\mathrm{Sq}^4\mathrm{Sq}^1$ when applied to $x_{0,0}$. This exhausts all constraints imposed by Adem relations in these dimensions. □

In Theorem 4.2.3, there are exactly seven free variables taking values in $\mathbb{F}_2$, and therefore, there are exactly 128 different $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$. Thus, in order to complete the proof of Theorem 4.2.1, we realize these $\mathcal{A}^{\mathbb{R}}$-modules as spectra using Theorem 4.1.11, which is a weak form of the $\mathbb{R}$-motivic Toda realization theorem.

Proof of Theorem 4.2.1. Firstly, note that $\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$ is a cyclic $\mathcal{A}^{\mathbb{R}}$-module for all $\overline{\mathrm{v}} \in \mathcal{V}$, therefore $\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1) := \mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)/(\rho)$ admits a May filtration. Secondly, note that

$$\mathrm{gr}(\mathcal{A}^{\mathbb{C}}_{\overline{\mathrm{v}}}(1)) \cong \Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{1,0}, \xi_{1,1}, \xi_{2,0})$$

as an $\mathrm{gr}(\mathcal{A}^{\mathbb{C}})$-module (see (4.1.7) for notation). Consequently,

$$ {}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{A}^{\mathbb{C}}_{\bar{\mathrm{v}}}(1)} \cong {}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathbb{M}^{\mathbb{C}}_2}/(\mathsf{h}_{1,0},\mathsf{h}_{1,1},\mathsf{h}_{2,0}) \cong \frac{\mathbb{M}^{\mathbb{C}}_2[\mathsf{h}_{i,j} : i \geq 1, j \geq 0]}{(\mathsf{h}_{1,0},\mathsf{h}_{1,1},\mathsf{h}_{2,0})}. \tag{4.2.3} $$

In the notation of Subsection 4.1.3

$$ \mathcal{D}_{\mathcal{A}^{\mathbb{R}}(1)} = \{(0,0),(1,0),(2,1),(3,1),(4,1),(5,2),(6,2)\} $$

By directly inspecting the (s,f,w)-degree of ${}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{A}^{\mathbb{C}}_{\bar{\mathrm{v}}}(1)}$, we see that the condition necessary for existence in Theorem 4.1.11 is satisfied. Hence, the result. □

Remark 4.2.4. *The vanishing region of* ${}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathcal{A}^{\mathbb{C}}_{\bar{\mathrm{v}}}(1)}$ *does not preclude the possibility of having a nonzero element in* $\mathrm{Ext}^{-1,2,0}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{M},\mathrm{M})$. *We suspect (even after running the differentials in* (4.1.3) *and* (4.1.5)*), that the above group is nonzero for a given* $\mathcal{A}^{\mathbb{R}}$*-module structure on* $\mathcal{A}^{\mathbb{R}}(1)$*, and that there are, up to homotopy, multiple realizations as* $\mathbb{R}$*-motivic spectra.*

4.3 A realization of $\mathcal{A}^{\mathbb{R}}(1)$

4.3.1 The spectrum $\mathcal{A}^{\mathbb{R}}_1$ via Smith's construction

Consider the $\mathbb{R}$-motivic question mark complex $\mathcal{Q}_{\mathbb{R}}$. Let Σ_n act on $\mathcal{Q}^{\wedge n}_{\mathbb{R}}$ by permutation. Any element $e \in \mathbb{Z}_{(2)}[\Sigma_n]$ produces a canonical map

$$ \tilde{e} : \mathcal{Q}^{\wedge n}_{\mathbb{R}} \longrightarrow \mathcal{Q}^{\wedge n}_{\mathbb{R}}. $$

Now let e be the idempotent

$$ e = \tfrac{1+(1\ 2)-(1\ 3)-(1\ 3\ 2)}{3} $$

in $\mathbb{Z}_{(2)}[\Sigma_3]$, and denote by $\bar{e}$ the resulting idempotent of $\mathbb{F}_2[\Sigma_3]$. For an $\mathbb{R}$-motivic spectrum X with action of Σ_n, we then define

$$ \tilde{e}(X) = \operatorname*{hocolim}_{\longrightarrow}(X \xrightarrow{\tilde{e}} X \xrightarrow{\tilde{e}} \dots), $$

and we employ the same notation in the C_2-equivariant or classical contexts. We will use that for a spectrum X with action of Σ_n, we have an isomorphism

$$ \mathrm{H}^*(\tilde{e}X;\mathbb{F}_2) \cong \bar{e}\mathrm{H}^*(X;\mathbb{F}_2). \tag{4.3.1} $$

We record the following important property of $\bar{e}$ which is a special case of [50, Theorem C.1.5].

Lemma 4.3.2. *If* V *is a finite-dimensional* $\mathbb{F}_2$*-vector space, then* $\bar{e}(V^{\otimes 3}) = 0$ *if and only if* $\dim V \leq 1$.

The following result, which gives the values of $\overline{e}$ on induced representations, is also straightforward to verify:

Lemma 4.3.3. *Suppose that $W = \mathrm{Ind}_{C_2}^{\Sigma_3}\mathbb{F}_2$ is induced up from the trivial representation of a cyclic 2-subgroup. Then $\overline{e}(W) \cong \mathbb{F}_2$. Moreover, for the regular representation $\mathbb{F}_2[\Sigma_3] = \mathrm{Ind}_{e}^{\Sigma_3}\mathbb{F}_2$, we have $\dim \overline{e}(\mathbb{F}_2[\Sigma_3]) = 2$.*

We also record the fact that when $\dim_{\mathbb{F}_2} V = 2$ and $\dim_{\mathbb{F}_2} W = 3$ then

$$\dim_{\mathbb{F}_2} \overline{e}(V^{\otimes 3}) = 2 \quad \text{and} \quad \dim_{\mathbb{F}_2} \overline{e}(W^{\otimes 3}) = 8, \tag{4.3.4}$$

as we will often use this.

The bottom cell of $\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3})$ is in degree $(1,0)$, and we define

$$\mathcal{A}_1^{\mathbb{R}} := \Sigma^{-1,0}\tilde{e}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3}) = \Sigma^{-1,0}\mathrm{hocolim}_{\longrightarrow}(\mathcal{Q}_{\mathbb{R}}^{\wedge 3} \xrightarrow{\tilde{e}} \mathcal{Q}_{\mathbb{R}}^{\wedge 3} \xrightarrow{\tilde{e}} \dots). \tag{4.3.5}$$

The purpose of this section is to prove the following theorem.

Theorem 4.3.6. *The spectrum $\mathcal{A}_1^{\mathbb{R}}$ is a type $(2,1)$ complex whose bi-graded cohomology $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module on one generator.*

Let $\mathcal{A}_1^{\mathrm{C}_2} := \beta(\mathcal{A}_1^{\mathbb{R}})$ and $\mathcal{Q}_{\mathrm{C}_2} := \beta(\mathcal{Q}_{\mathbb{R}})$. Note that we have a C_2-equivariant splitting

$$\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3} \simeq \tilde{e}(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3}) \vee (1-\tilde{e})(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3})$$

which splits the underlying spectra as well as the geometric fixed-points, as both Φ^e and Φ^{C_2} are additive functors.

We will identify the underlying spectrum $\Phi^e(\mathcal{A}_1^{\mathrm{C}_2})$ by studying the $\mathcal{A}$-module structure of its cohomology with $\mathbb{F}_2$-coefficients. Firstly, note that

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}) \simeq \Sigma^{-1}\tilde{e}(\Phi^e(\mathcal{Q}_{\mathrm{C}_2}^{\wedge 3})) \simeq \Sigma^{-1}\tilde{e}(\mathcal{Q}^{\wedge 3}),$$

where $\mathcal{Q}$ is the classical question mark complex, whose $\mathrm{H}\mathbb{F}_2$-cohomology as an $\mathcal{A}$-module is well understood. It consists of three $\mathbb{F}_2$-generators a, b, and c in internal degrees 0, 1, and 3, such that $\mathrm{Sq}^1(a) = b$ and $\mathrm{Sq}^2(b) = c$ are the only nontrivial relations, as displayed in Figure 4.3.

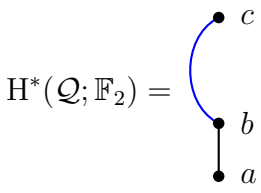


Figure 4.3: The $\mathcal{A}$-structure of $\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)$

Because of the Kunneth isomorphism and the fact that the Steenrod algebra is cocommutative, we have an isomorphism of $\mathcal{A}$-modules

$$\mathrm{H}^{*+1}(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2) \cong \mathrm{H}^*(\tilde{e}(\mathcal{Q}^{\wedge 3});\mathbb{F}_2) \cong \overline{e}(\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3}),$$

where the second isomorphism is (4.3.1).

Lemma 4.3.4. *The underlying $\mathcal{A}(1)$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$ is free on a single generator.*

Proof. Let us denote the $\mathcal{A}$-module $\mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)$ by V. Since $\dim \mathcal{M}(\mathrm{V}, Q_i) = 1$ for $i \in \{0,1\}$, it follows from the Kunnneth isomorphism of Q_i-Margolis homology groups, cocommutativity of the Steenrod algebra, and Lemma 4.3.2 that

$$\mathcal{M}(\overline{e}(\mathrm{V}^{\otimes 3}), Q_i) = \overline{e}(\mathcal{M}(\mathrm{V}, Q_i)^{\otimes 3}) = 0$$

for $i \in \{0,1\}$. It follows from [3, Theorem 3.1] that $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathbb{R}});\mathbb{F}_2)$ is free as an $\mathcal{A}(1)$-module. It is singly generated because of (4.3.4). □

We explicitly identify the image of $\overline{e} : \mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3} \longrightarrow \mathrm{H}^*(\mathcal{Q};\mathbb{F}_2)^{\otimes 3}$ in Figure 4.4.

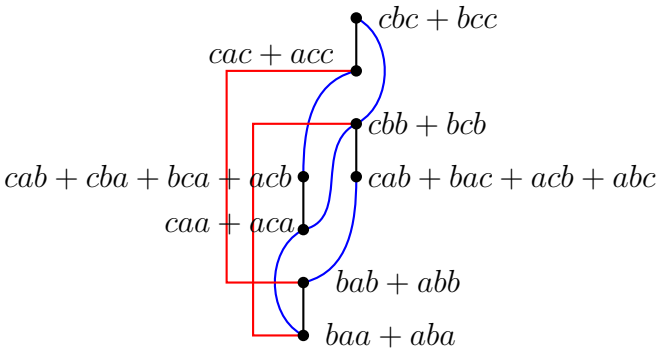


Figure 4.4: The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2)$

Remark 4.3.5. *Using the Cartan formula, we can identify the action of Sq^4 on $\Phi^e(\mathcal{A}_1^{\mathrm{C}_2})$. We notice that its $\mathcal{A}$-module structure is isomorphic to $A_1[10]$ of [15]. Since such an $\mathcal{A}$-module is realized by a unique 2-local finite spectrum, we conclude*

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}) \simeq A_1[10]$$

and is of type 2.

Our next goal is to understand the homotopy type of the geometric fixed-point spectrum $\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2})$. First observe that the geometric fixed-points of the C_2-equivariant question mark complex $\mathcal{Q}_{\mathrm{C}_2}$ is the *exclamation mark* complex

$$\mathcal{E} := \quad \simeq \quad \mathbb{S}^0 \vee \Sigma\mathbb{M}_2(1)!$$



This is because $\Phi^{\mathrm{C}_2}(\mathsf{h}) = 0$ and $\Phi^{\mathrm{C}_2}(\eta_{1,1}) = 2$. Secondly,

$$\mathrm{H}^{*+1}(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2});\mathbb{F}_2) \cong \mathrm{H}^*(\tilde{e}(\mathcal{E}^{\wedge 3});\mathbb{F}_2) \cong \overline{e}(\mathrm{H}^*(\mathcal{E};\mathbb{F}_2)^{\otimes 3})$$

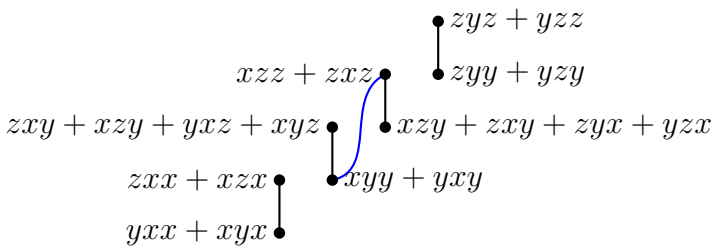


Figure 4.5: The $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}); \mathbb{F}_2)$.

is an isomorphism of $\mathcal{A}$-modules, where again the second isomorphism is (4.3.1). We explicitly calculate the $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}); \mathbb{F}_2)$ from the above isomorphism and record it in Figure 4.5 as a subcomplex of $\mathrm{H}^*(\mathcal{E}; \mathbb{F}_2)^{\otimes 3}$, with the convention that x, y and z are generators in $\mathrm{H}^*(\mathcal{E}; \mathbb{F}_2)$ in degree 0, 1 and 2 respectively.

Lemma 4.3.6. *There is an equivalence*

$$\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}) \simeq \mathrm{M}_2(1) \vee \Sigma\Big(\mathrm{M}_2(1) \wedge \mathrm{M}_2(1)\Big) \vee \Sigma^3 \mathrm{M}_2(1).$$

In particular, $\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2})$ is a type 1 spectrum.

Proof. From Figure 4.5, it is clear that we have an isomorphism of $\mathcal{A}$-modules

$$\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}); \mathbb{F}_2) \cong \mathrm{H}^*\Big(\mathrm{M}_2(1) \vee \Sigma\big(\mathrm{M}_2(1) \wedge \mathrm{M}_2(1)\big) \vee \Sigma^3 \mathrm{M}_2(1); \mathbb{F}_2\Big).$$

It is possible that the $\mathcal{A}$-module $\mathrm{H}^*(\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2}); \mathbb{F}_2)$ may not realize to a unique finite spectrum (up to weak equivalence). However, other possibilities can be eliminated from the fact that $\mathcal{E}^{\wedge 3}$ splits Σ_3-equivariantly into four components:

$$\mathcal{E}^{\wedge 3} \simeq \mathbb{S} \vee \left(\bigvee_{i=1}^{3} \Sigma \mathrm{M}_2(1)\right) \vee \left(\bigvee_{i=1}^{3} \Sigma^2 \mathrm{M}_2(1)^{\wedge 2}\right) \vee \Sigma^3 \mathrm{M}_2(1)^{\wedge 3}.$$

The idempotent $\tilde{e}$ annihilates $\mathbb{S} \simeq \mathbb{S}^{\wedge 3}$, and Lemma 4.3.3 implies that

$$\tilde{e}\left(\bigvee_{i=1}^{3} \Sigma \mathrm{M}_2(1)\right) \simeq \Sigma \mathrm{M}_2(1) \qquad \text{and}$$

$$\tilde{e}\left(\bigvee_{i=1}^{3} \Sigma^2 \mathrm{M}_2(1) \wedge \mathrm{M}_2(1)\right) \simeq \Sigma^2 \mathrm{M}_2(1) \wedge \mathrm{M}_2(1).$$

Similarly, we see using (4.3.4) that

$$\mathrm{H}^*\left(\tilde{e}\left(\mathrm{M}_2(1)^{\wedge 3}\right)\right) \cong \overline{e}\left(\mathrm{H}^*\left(\mathrm{M}_2(1)\right)^{\otimes 3}\right) \cong \mathrm{H}^*(\Sigma \mathrm{M}_2(1)).$$

Therefore, as an $\mathcal{A}$-module

$$\mathrm{H}^*\left(\tilde{e}\left(\Sigma^3 \mathrm{M}_2(1)^{\wedge 3}\right)\right) \cong \mathrm{H}^*(\Sigma^4 \mathrm{M}_2(1)).$$

Since, the $\mathcal{A}$-module $\mathrm{H}^*(\mathrm{M}_2(1))$ has a unique lift as a finite spectrum up to homotopy (also see Remark 4.3.7), we conclude $\tilde{e}\,(\Sigma^3\mathrm{M}_2(1)^{\wedge 3}) \simeq \Sigma^4\mathrm{M}_2(1)$.

As $\Phi^{\mathrm{C}_2}(\mathcal{A}_1^{\mathrm{C}_2})$ is the desuspension of $\tilde{e}(\mathcal{E}^{\wedge 3})$, the result follows. □

Remark 4.3.7. *It is well-known that if* $\mathrm{H}^*(X) \cong \mathcal{A}(0) \cong \mathrm{H}^*(\mathrm{M}_2(1))$ *as an* $\mathcal{A}$*-module and* X *is a* 2*-local finite spectrum, then* $X \simeq \mathrm{M}_2(1)$*. Firstly note that the group* $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathcal{A}(0), \mathcal{A}(0))$ *vanishes in stem equal to* -1 *and cohomological degree at least* 2*. It follows that the identity map* $\mathcal{A}(0) \to \mathcal{A}(0)$*, which is a nonzero element in degree* $(0,0)$ *in the* E_2*-page of the Adams spectral sequence*

$$\mathrm{E}_2^{s,t} := \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathrm{H}^*(\mathrm{M}_2(1)), \mathrm{H}^*(X)) \Rightarrow [X, \mathrm{M}_2(1)]_{t-s},$$

survives to produce a map from X *to* $\mathrm{M}_2(1)$*. This map, by construction, induces an isomorphism in homology. Therefore, by Whitehead's theorem it is an equivalence (also see [14,* § *5]).*

Next, we analyze the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$. We begin by recalling some general properties of the cohomology of motivic spectra.

If $X, Y \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$ such that $\mathrm{H}^{*,*}(X)$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$-module, then we have a Kunneth isomorphism [22, Proposition 7.7]

$$\mathrm{H}^{*,*}(X \wedge Y) \cong \mathrm{H}^{*,*}(X) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{H}^{*,*}(Y) \tag{4.3.8}$$

as the relevant Kunneth spectral sequence collapses. Further, if $\mathrm{H}^{*,*}(Y)$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$-module, then so is $\mathrm{H}^{*,*}(X \wedge Y)$. The $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(X \wedge Y)$ can then be computed using the Cartan formula. The comultiplication map of $\mathcal{A}^{\mathbb{R}}$ is left $\mathbb{M}_2^{\mathbb{R}}$-linear, coassociative and cocommutative [54, Lemma 11.9], which is also reflected in the fact that its $\mathbb{M}_2^{\mathbb{R}}$-linear dual is a commutative and associative algebra. Thus, when $\mathrm{H}^{*,*}(X)$ is a free left $\mathbb{M}_2^{\mathbb{R}}$-module, the elements of $\mathbb{F}_2[\Sigma_n]$ act on

$$\mathrm{H}^{*,*}(X^{\wedge n}) \cong \mathrm{H}^{*,*}(X) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \cdots \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{H}^{*,*}(X)$$

via permutation and commute with the action of $\mathcal{A}^{\mathbb{R}}$. This also implies that $\mathbb{F}_2[\Sigma_n]$ also acts on

$$\mathrm{H}^{*,*}(X^{\wedge n})/(\rho,\tau) \cong \mathrm{H}^{*,*}(X)/(\rho,\tau) \otimes \cdots \otimes \mathrm{H}^{*,*}(X)/(\rho,\tau)$$

and commutes with the action of $\mathcal{A}^{\mathbb{R}} /\!\!/ \mathbb{M}_2^{\mathbb{R}}$. From the above discussion we may conclude that

$$\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}}) \cong \Sigma^{-1}\overline{e}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})^{\otimes 3}) \tag{4.3.9}$$

is an isomorphism of $\mathcal{A}^{\mathbb{R}}$-modules.

We will also rely upon the following important property of the action of the motivic Steenrod algebra on the cohomology of a motivic space (as opposed to a motivic spectrum):

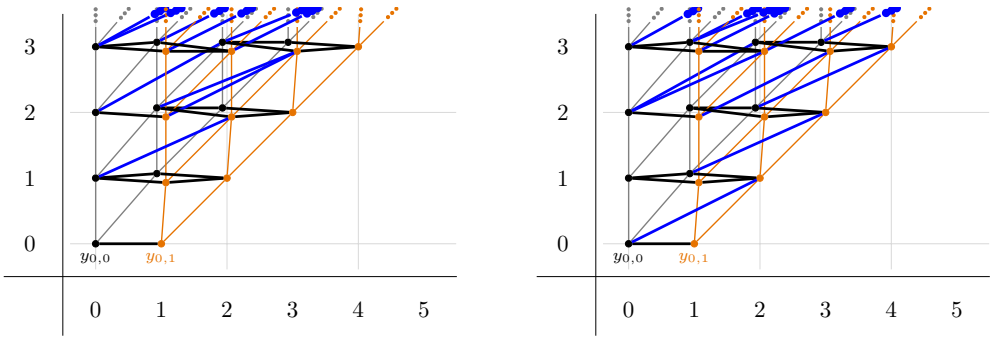
Remark 4.3.10 (Instability condition for $\mathbb{R}$-motivic cohomology)**.** *If X is an $\mathbb{R}$-motivic space then $\mathrm{H}^{*,*}(X)$ admits a ring structure, and, for any $u \in \mathrm{H}^{n,i}(X)$, the $\mathbb{R}$-motivic squaring operations obey the rule*

$$\mathrm{Sq}^{2i}(u) = \begin{cases} 0 & \text{if } n < 2i \\ u^2 & \text{if } n = 2i. \end{cases}$$

This is often referred to as the instability condition.

To understand the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$, we first make the following observation regarding $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ (as $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ is a sub-complex of $\mathcal{Q}_{\mathbb{R}}$) using an argument very similar to [20, Lemma 7.4].

Proposition 4.3.11. *There are two extensions of $\mathcal{A}^{\mathbb{R}}(0)$ to an $\mathcal{A}^{\mathbb{R}}$-module, and these $\mathcal{A}^{\mathbb{R}}$-modules are realized as the cohomology of $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathrm{C}^{\mathbb{R}}(2)$.*



$\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{C}^{\mathbb{R}}(2))$

Figure 4.6: $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ and $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathrm{C}^{\mathbb{R}}(2))$

Proof. For degree reasons, the only choice in extending $\mathcal{A}^{\mathbb{R}}(0)$ to an $\mathcal{A}^{\mathbb{R}}$-module is the action of Sq^2 on the generator in bidegree $(0,0)$. We write $y_{0,0}$ for the generator in degree $(0,\ 0)$ and $y_{1,0}$ for $\mathrm{Sq}^1(y_{0,0})$ in (cohomological) bidegree $(1,0)$. The two possible choices are

- $\mathrm{Sq}^2(y_{0,0}) = 0$ and
- $\mathrm{Sq}^2(y_{0,0}) = \rho \cdot y_{1,0}$.

We can realize the degree 2 map as an unstable map $\mathrm{S}^{1,0} \longrightarrow \mathrm{S}^{1,0}$, and we will write $\mathrm{C}^{\mathbb{R}}(2)^u$ for the cofiber. We deduce information about the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$ by analyzing the cohomology ring of $\mathrm{S}^{1,1} \wedge \mathrm{C}^{\mathbb{R}}(2)^u$ using the instability condition of Remark 4.3.10. First, note that in

$$\mathrm{H}^{*,*}(\mathrm{S}^{1,1}) \cong \mathbb{M}_2^{\mathbb{R}} \cdot \iota_{1,1}$$

we have the relation $\iota_{1,1}^2 = \rho \cdot \iota_{1,1}$ [54, Lemma 6.8]. Also note that

$$\mathrm{H}^{*,*}((\mathrm{C}^{\mathbb{R}}(2)^u)_+) \cong \mathbb{M}_2^{\mathbb{R}}[x]/(x^3)$$

where x is in cohomological degrees $(1,0)$. Therefore, in

$$\mathrm{H}^{*,*}(\mathrm{S}^{1,1} \wedge \mathrm{C}^{\mathbb{R}}(2)^u) = \mathbb{M}_2^{\mathbb{R}} \cdot \iota_{1,1} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{\mathbb{R}}\{x, x^2\}$$

the instability condition implies

$$\mathrm{Sq}^2(\iota_{1,1} \otimes x) = \iota_{1,1}^2 \otimes x^2 = \rho \cdot \iota_{1,1} \otimes x^2.$$

Here the space-level cohomology class x^2 corresponds to the spectrum-level class $y_{1,0}$. Therefore, $\mathrm{Sq}^2(y_{0,0}) = \rho \cdot y_{1,0}$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$. This is also reflected in the fact that multiplication by 2 is detected by $h_0 + \rho h_1$ in the $\mathbb{R}$-motivic Adams spectral sequence [20, §8].

On the other hand h is the 'zeroth $\mathbb{R}$-motivic Hopf map' detected by the element h_0 in the motivic Adams spectral sequence. It follows that $\mathrm{Sq}^2(y_{0,0}) = 0$. □

In order to express the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^{*,*}(X)$ for a finite spectrum X, it is enough to specify the action of $\mathcal{A}^{\mathbb{R}}$ on its left $\mathbb{M}_2^{\mathbb{R}}$-generators as the action of τ and ρ multiples are determined by the Cartan formula.

Example 4.3.7. *Let $\{y_{0,0}, y_{1,0}\} \subset \mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ denote a left $\mathbb{M}_2^{\mathbb{R}}$-basis of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$. The data that*

- $\mathrm{Sq}^1(y_{0,0}) = y_{1,0}$
- $\mathrm{Sq}^2(y_{0,0}) = 0$

completely determines the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$.

Proposition 4.3.8. *$\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ is a free $\mathbb{M}_2^{\mathbb{R}}$-module generated by a, b and c in cohomological bidegrees $(0,0), (1,0)$ and $(3,1)$, and the relations*

1. $\mathrm{Sq}^1(a) = b$,
2. $\mathrm{Sq}^2(b) = c$,
3. $\mathrm{Sq}^4(a) = 0$.

completely determine the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$.

Proof. $\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ is a free $\mathbb{M}_2^{\mathbb{R}}$-module because the attaching maps of $\mathcal{Q}_{\mathbb{R}}$ induce trivial maps in $\mathrm{H}^{*,*}(-)$. The first two relations can be deduced from the obvious maps

1. $\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \to \mathcal{Q}_{\mathbb{R}}$
2. $\mathcal{Q}_{\mathbb{R}} \to \Sigma^{1,0}\,\mathrm{C}^{\mathbb{R}}(\eta_{1,1})$

which are respectively surjective and injective in cohomology.

Let $\mathsf{h}^u : \mathrm{S}^{3,2} \to \mathrm{S}^{3,2}$ and $\eta_{1,1}^u : \mathrm{S}^{3,2} \to \mathrm{S}^{2,1}$ denote the unstable maps that stabilize to h and $\eta_{1,1}$, respectively. The unstable $\mathbb{R}$-motivic space $\mathcal{Q}_{\mathbb{R}}^u$ (which stabilizes to $\mathcal{Q}_{\mathbb{R}}$) can be constructed using the fact that the composite of the unstable maps

$$\mathrm{S}^{4,3} \xrightarrow{\Sigma^{1,1}\eta_{1,1}^u} \mathrm{S}^{3,2} \xrightarrow{\mathsf{h}^u} \mathrm{S}^{3,2}$$

is null. Thus $\mathrm{H}^{*,*}(\mathcal{Q}^u_{\mathbb{R}})$ consists of three generators a_u, b_u and c_u in bidegrees $(3,2)$, $(4,2)$ and $(6,3)$. It follows from the instability condition that $\mathrm{Sq}^4(a_u) = 0$. □

Proof of Theorem 4.3.6. From Remark 4.3.5 and Lemma 4.3.6, we deduce that $\mathcal{A}^{\mathbb{R}}_1$ is a type $(2,1)$ complex. To show that the bi-graded $\mathbb{R}$-motivic cohomology of $\mathcal{A}^{\mathbb{R}}_1$ is free as an $\mathcal{A}^{\mathbb{R}}(1)$-module, we make use of Corollary 2.2.5.

Since $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is a summand of a free $\mathbb{M}^{\mathbb{R}}_2$-module, it is projective as an $\mathbb{M}^{\mathbb{R}}_2$-module. In fact, $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is free, as projective modules over (graded) local rings are free. Also note that the elements

$$\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}^1_1, \overline{\mathrm{Q}}_1 \in \mathcal{A}^{\mathbb{R}}(1)/(\rho,\tau)$$

are primitive. Hence we have a Kunneth isomorphism in the respective Margolis homologies, in particular we have,

$$\mathcal{M}(\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)/(\rho,\tau), x) = \overline{e}(\mathcal{M}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})/(\rho,\tau), x)^{\otimes 3})$$

for $x \in \{\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}^1_1, \overline{\mathrm{Q}}_1\}$. Since $\dim_{\mathbb{F}_2} \mathcal{M}(\mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})/(\rho,\tau), x) = 1$ for all $x \in \{\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}^1_1, \overline{\mathrm{Q}}_1\}$, by Lemma 4.3.2

$$\mathcal{M}(\mathcal{A}^{\mathbb{R}}_1/(\rho,\tau), x) = 0$$

for $x \in \{\overline{\mathrm{Q}}_0, \overline{\mathrm{P}}^1_1, \overline{\mathrm{Q}}_1\}$. Thus, by Corollary 2.2.5 we conclude that $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module. A direct computation shows that

$$\dim_{\mathbb{F}_2} \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)/(\rho,\tau) = 8,$$

hence $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ is $\mathcal{A}^{\mathbb{R}}(1)$-free of rank one. □

Using the description (4.3.9) and Cartan formula we make a complete calculation of the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$. Let $a, b, c \in \mathrm{H}^{*,*}(\mathcal{Q}_{\mathbb{R}})$ as in Proposition 4.3.8. In Figure 4.7 we provide a pictorial representation with the names of the generators that are in the image of the idempotent $\overline{e}$. For convenience we relabel the generators in Figure 4.7, where the indexing on a new label records the cohomological bidegrees of the corresponding generator. The following result is straightforward, and we leave it to the reader to verify.

Lemma 4.3.9. *In $\mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$, the underlying $\mathcal{A}^{\mathbb{R}}(1)$-module structure, along with the relations*

1. $\mathrm{Sq}^4(v_{0,0}) = \tau \cdot w_{4,1}$,
2. $\mathrm{Sq}^4(v_{1,0}) = w_{5,2}$,
3. $\mathrm{Sq}^4(v_{2,1}) = 0$,
4. $\mathrm{Sq}^4(v_{3,1}) = 0 = \mathrm{Sq}^4(w_{3,1})$,
5. $\mathrm{Sq}^8(v_{0,0}) = 0$,

completely determine the $\mathcal{A}^{\mathbb{R}}$-module structure.

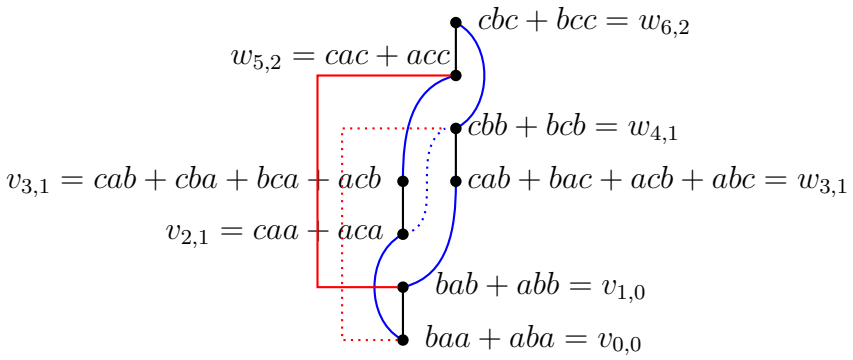


Figure 4.7: The $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{H}^{*,*}(\mathcal{A}_1)$

4.3.2 The Betti realization of $\mathcal{A}_1^{\mathbb{R}}$

Under the Betti-realization map

$$\beta_* : \pi_{*,*}\mathrm{H}_{\mathbb{R}}\mathbb{F}_2 \cong \mathbb{F}_2[\rho,\tau] \longrightarrow \pi_\star \mathrm{H}\underline{\mathbb{F}_2} \tag{4.3.8}$$

$\rho \mapsto a_\sigma$ and $\tau \mapsto u_\sigma$. Since the functor β is symmetric monoidal and $\beta(\mathrm{H}_{\mathbb{R}}\mathbb{F}_2) = \mathrm{H}\underline{\mathbb{F}_2}$, the i-th $\mathbb{R}$-motivic squaring operations maps to the i-th $\mathrm{RO}(\mathrm{C}_2)$-graded squaring operations under the map

$$\beta_* : \mathcal{A}^{\mathbb{R}} \longrightarrow \mathcal{A}^{\mathrm{C}_2}.$$

Hence, $\mathrm{H}^\star_{C_2}(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$ is $\mathbb{M}_2^{\mathrm{C}_2}$-free (as $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}])$ is $\mathbb{M}_2^{\mathbb{R}}$-free) and its $\mathcal{A}^{\mathrm{C}_2}$-module structure is essentially given by Theorem 4.2.3 (after replacing Sq^i with $\underline{\mathrm{Sq}}^i$ and $\mathbb{M}_2^{\mathbb{R}}$-basis elements by its image under β_*).

Corollary 4.3.9. *There exists* 128 *different $\mathcal{A}^{\mathrm{C}_2}$-modules whose underlying $\mathcal{A}^{\mathrm{C}_2}(1)$-module structures are free on one generator, all of which can be realized as the $\mathrm{RO}(\mathrm{C}_2)$-graded cohomology of a 2-local finite C_2-spectrum.*

Remark 4.3.10. *The map β_* of* (4.3.8) *is only an injection with cokernel the summand $\Theta\{u_\sigma^{-i}a_\sigma^{-j} : i,j \geq 0\}$ of $\mathbb{M}_2^{\mathrm{C}_2}$. In general, for an $\mathcal{A}^{\mathbb{R}}(1)$-module $\mathrm{M}_{\mathbb{R}}$, the number of $\mathcal{A}^{\mathrm{C}_2}$-module structures on $\beta(\mathrm{M}_{\mathbb{R}})$ can be strictly larger than the number of $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathrm{M}_{\mathbb{R}}$. But this is not the case when $\mathrm{M}_{\mathbb{R}} = \mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)$ simply for degree reasons, therefore Corollary 4.3.9 holds.*

As discussed in Example 2.1.25, the restriction map

$$\Phi^e_* : \mathbb{M}_2^{\mathrm{C}_2} \longrightarrow \mathbb{F}_2$$

sends $a_\sigma \mapsto 0$, $u_\sigma \mapsto 1$, and $\Theta \mapsto 0$. Thus, when $\mathrm{H}^\star_{C_2}(\mathrm{E})$ is $\mathbb{M}_2^{\mathrm{C}_2}$-free, Φ^e_* is simply "setting $a_\sigma = 0$, $u_\sigma = 1$, and $\Theta = 0$". This observation, along with Theorem 2.1.23, allows us to completely deduce the $\mathcal{A}$-module structure of $\mathrm{H}^*(\Phi^e(\mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1)))$ from Theorem 4.2.3. Together with the fact that the $\mathcal{A}$-module structures on $\mathcal{A}(1)$ are uniquely-realized, our observations yield the following theorem, where the notation $\mathrm{A}_1[i,j]$ is adopted from [15].

Theorem 4.3.11. *For* $\overline{\mathrm{v}} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$ *(as in Theorem 4.2.3),*

$$\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]) \simeq \mathrm{A}_1[1 + \beta_{03} + \beta_{14}, \beta_{26}].$$

Now we shift our attention towards understand the geometric fixed-points of $\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$. As discussed in Example 2.1.27, the modified geometric fixed-points functor

$$\widehat{\Phi^{C_2}}_*\colon \mathbb{M}_2^{\mathrm{C}_2} \longrightarrow \mathbb{F}_2$$

sends $a_\sigma \mapsto 1$, $u_\sigma \mapsto 0$, and $\Theta \mapsto 0$. Thus, when $\mathrm{H}^\star_{C_2}(\mathrm{E})$ is $\mathbb{M}_2^{\mathrm{C}_2}$-free, $\widehat{\Phi^{C_2}}_*$ is simply "setting $a_\sigma = 1$, $u_\sigma = 0$, and $\Theta = 0$". This, along with Theorem 4.2.3 and Theorem 2.1.24, gives the following.

Notation 4.3.12. *Because* $\mathrm{H}^\star_{C_2}(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$ *is* $\mathbb{M}_2^{\mathrm{C}_2}$*-free, the* $\mathrm{H}\mathbb{F}_2$*-cohomology of* $\Phi^{C_2}\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$ *consists of eight* $\mathbb{F}_2$*-generators, all of which are in the image of* $\widehat{\Phi^{C_2}}_*$*. We let*

$$s_0 := \widehat{\Phi^{C_2}}_*(x_{0,0}),\ s_{1a} := \widehat{\Phi^{C_2}}_*(x_{2,1}),\ s_{1b} := \widehat{\Phi^{C_2}}_*(x_{1,0}),\ s_2 := \widehat{\Phi^{C_2}}_*(y_{3,1})$$

$$t_2 := \widehat{\Phi^{C_2}}_*(x_{3,1}),\ t_{3a} := \widehat{\Phi^{C_2}}_*(y_{5,2}),\ t_{3b} := \widehat{\Phi^{C_2}}_*(y_{4,1}),\ t_4 := \widehat{\Phi^{C_2}}_*(y_{6,2}).$$

Note that $|s_{i(-)}| = |t_{i(-)}| = i$.

Theorem 4.3.13. *Let* $\overline{\mathrm{v}} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$*, and let*

$$\jmath_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25} + \beta_{26})$$

as in Theorem 4.2.3. The $\mathcal{A}$*-module structure on* $\mathrm{H}^*(\Phi^{C_2}\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$ *is determined by the following relations, as depicted in Figure 4.8:*

- $\mathrm{Sq}^1(s_0) = s_{1a}$
- $\mathrm{Sq}^1(s_{1b}) = s_2$
- $\mathrm{Sq}^1(t_2) = t_{3a}$
- $\mathrm{Sq}^1(t_{3b}) = t_4$
- $\mathrm{Sq}^2(s_0) = \beta_{03}s_2 + \alpha_{03}t_2$
- $\mathrm{Sq}^2(s_{1a}) = \beta_{25}t_{3a} + \jmath_{24}t_{3b}$
- $\mathrm{Sq}^2(s_{1b}) = t_{3a} + \beta_{14}t_{3b}$
- $\mathrm{Sq}^2(s_2) = \gamma_{36}t_4$
- $\mathrm{Sq}^2(t_2) = (\beta_{25} + \beta_{26})t_4$
- $\mathrm{Sq}^4(s_0) = \beta_{06}t_4$.

We find Theorem 2.1.23 and Theorem 2.1.24 very handy for computational purposes. These results can be applied to understand the $\mathrm{RO}(\mathrm{C}_2)$-graded squaring operations on the cohomology of a wide variety of C_2-spectra whose underlying and geometric fixed-point spectra are known. Alternatively, one can identify the action of the classical Steenrod algebra on the cohomology of the underlying as well as the geometric fixed-points of a C_2-spectrum from the knowledge of $\mathrm{RO}(\mathrm{C}_2)$-graded Steenrod operations. We apply Theorem 2.1.23 and Theorem 2.1.24 to identify the $\mathcal{A}$-module structure of the underlying and the geometric fixed-points of $\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$ (see Theorem 4.3.11 and Theorem 4.3.13).

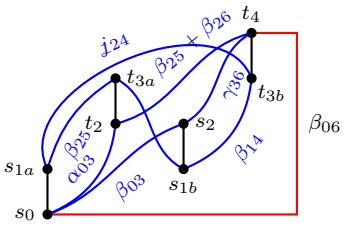
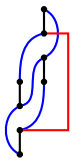
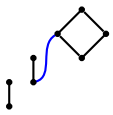
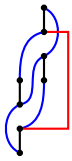
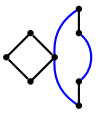
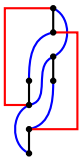
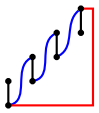
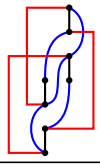
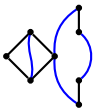
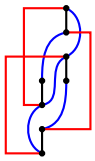
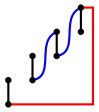
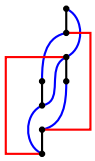
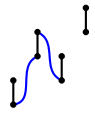
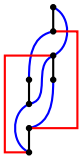
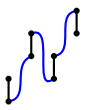
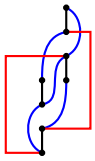
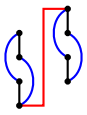


Figure 4.8: The $\mathcal{A}$-module $\mathrm{H}^*(\Phi^{C_2}\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$

In Figure 4.9, we provide the $\mathcal{A}$-module structure of the underlying and the geometric fixed points of $\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$ for selected values of $\overline{\mathrm{v}} \in \mathcal{V}$. We express Sq^1, Sq^2 and Sq^4 using black, blue, and red lines respectively.

Figure 4.9: Some underlying and fixed $\mathcal{A}$-modules of $\mathcal{A}_1^{\mathrm{C}_2}$

$\overline{\mathrm{v}} \in \mathcal{V}$	$\mathrm{H}^*(\Phi^e(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]))$	$\mathrm{H}^*(\Phi^{C_2}\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$	Cofiber of
$(0,0,1,0,0,0,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(\mathsf{h},1)} \to \mathcal{Y}_{(\mathsf{h},1)}$
$(1,1,0,0,0,0,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(\mathsf{h},1)}$
$(0,1,0,1,0,1,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(2,1)}$
$(1,0,0,0,0,1,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(\mathsf{h},1)} \to \mathcal{Y}_{(2,1)}$
$(0,0,0,1,0,1,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(\mathsf{h},1)}$
$(1,0,0,0,0,0,0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(\mathsf{h},1)} \to \mathcal{Y}_{(2,1)}$
$(1,0,0,0,0,0,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(2,1)}$
$(1,1,1,1,1,0,1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \to \mathcal{Y}_{(2,1)}$

Remark 4.3.10 (Appearance of the Joker)**.** *We note that the* $\mathcal{A}(1)$*-module*



often called the Joker, is a subcomplex of the geometric fixed point of $\mathcal{A}_1^{\mathbb{R}}[\overline{\mathrm{v}}]$ *if and only if* $\mathfrak{j}_{24} = 1$*. Further, when* $\mathfrak{j}_{24} = 1$ *then in* (5.1.2)*,* ϵ *and* δ *cannot both equal* h*. This can easily be derived from Theorem 4.2.3 and Theorem 2.1.24.*

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Chapter 5 $\mathbb{R}$-motivic selfmap

Classically any non-contractible finite p-local spectrum admits a periodic v_n-selfmap for some $n \geq 0$. This is a consequence of the thick-subcategory theorem [34, Theorem 7], aided by a vanishing line argument [34, §4.2]. In the classical case all the thick tensor ideals of $\mathbf{Sp}_{p,\text{fin}}$ (the homotopy category of finite p-local spectra) are also prime (in the sense of [4]). The thick tensor-ideals of the homotopy category of cellular motivic spectra over $\mathbb{C}$ or $\mathbb{R}$ are not completely known (but see [31, 36]). However, one can gather some knowledge about the prime thick tensor-ideals in $\text{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\text{fin}})$ (the homotopy category of 2-local cellular $\mathbb{R}$-motivic spectra) through the Betti realization functor

$$\beta : \text{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\text{fin}}) \longrightarrow \text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}})$$

using the complete knowledge of prime thick subcategories of $\text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}})$ [5].

The prime thick tensor-ideals of $\text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}})$ are essentially the pull-back of the classical thick subcategories along the two functors, the geometric fixed-point functor

$$\Phi^{\text{C}_2} : \text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}}) \longrightarrow \text{Ho}(\mathbf{Sp}_{2,\text{fin}})$$

and the forgetful functor

$$\Phi^{e} : \text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}}) \longrightarrow \text{Ho}(\mathbf{Sp}_{2,\text{fin}}).$$

Let $\mathcal{C}_n$ denote the thick subcategory of $\text{Ho}(\mathbf{Sp}_{2,\text{fin}})$ consisting of spectra of type at least n. The prime thick subcategories,

$$\mathcal{C}(e,n) = (\Phi^e)^{-1}(\mathcal{C}_n) \text{ and } \mathcal{C}(\text{C}_2,n) = (\Phi^{\text{C}_2})^{-1}(\mathcal{C}_n),$$

are the only prime thick subcategories of $\text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}})$.

Definition 5.0.1. *We say a spectrum $X \in \text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}})$ is of type (n,m) if $\Phi^e(X)$ is of type n and $\Phi^{\text{C}_2}(X)$ is of type m.*

For a type (n,m) spectrum X, a self-map $f : X \to X$ is periodic if and only if at least one of $\{\Phi^e(f), \Phi^{\text{C}_2}(f)\}$ are periodic (see [6, Proposition 3.17]).

Definition 5.0.2. *Let $X \in \text{Ho}(\mathbf{Sp}^{\text{C}_2}_{2,\text{fin}})$ be of type (n,m). We say a self-map $f : X \to X$ is*

(i) a $v_{(n,m)}$-selfmap of mixed periodicity (i,j) if $\Phi^e(f)$ is a v_n-selfmap of periodicity i and $\Phi^{\text{C}_2}(f)$ is a v_m-selfmap of periodicity j,

(ii) a $v_{(n,\text{nil})}$-selfmap of periodicity i if $\Phi^e(f)$ is a v_n-selfmap of periodicity i and $\Phi^{\text{C}_2}(f)$ is nilpotent, and,

(iii) a $v_{(\mathrm{nil},m)}$-selfmap of periodicity j if $\Phi^e(f)$ is a nilpotent self-map and $\Phi^{\mathrm{C}_2}(f)$ is a v_m-selfmap of periodicity j.

Example 5.0.3. *The sphere spectrum $\mathbb{S}_{\mathrm{C}_2}$ is of type $(0,0)$. The degree 2 map is a $v_{(0,0)}$-selfmap. In general, if we consider the v_n-selfmap of a type n spectrum with trivial action of C_2, then the resultant equivariant self-map is a $v_{(n,n)}$-selfmap.*

Example 5.0.4. *Let $\mathrm{S}^{1,1}_{\mathrm{C}_2}$ denote the C_2-equivariant sphere which is the one-point compactification of the real sign representation. The unstable twist-map*

$$\epsilon^u : \mathrm{S}^{1,1}_{\mathrm{C}_2} \wedge \mathrm{S}^{1,1}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{1,1}_{\mathrm{C}_2} \wedge \mathrm{S}^{1,1}_{\mathrm{C}_2}$$

stabilizes to a nonzero element $\epsilon \in \pi_{0,0}(\mathbb{S}_{\mathrm{C}_2})$. Let $\mathsf{h} = 1 - \epsilon \in \pi_{0,0}(\mathbb{S}_{\mathrm{C}_2})$ be the stabilization of the map

$$\mathsf{h}^u = 1 - \epsilon^u : \mathrm{S}^{3,2}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{3,2}_{\mathrm{C}_2}.$$

Note that on the underlying space ϵ is of degree -1, while on the fixed points it is the identity. Therefore $\Phi^e(\mathsf{h})$ is multiplication by 2, whereas $\Phi^{\mathrm{C}_2}(\mathsf{h})$ is trivial. Thus h is a $v_{(0,\mathrm{nil})}$-selfmap, and the cofiber $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ is of type $(1,0)$.

Example 5.0.5. *The equivariant Hopf map $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathrm{C}_2})$ is the Betti realization of the $\mathbb{R}$-motivic Hopf map η [45, 23]. Up to a unit, it is the stabilization of the projection map*

$$\eta^u_{1,1} := \pi : \mathrm{S}^{3,2}_{\mathrm{C}_2} \simeq \mathbb{C}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{CP}^1 \cong \mathrm{S}^{2,1}_{\mathrm{C}_2},$$

where the domain and the codomain are given the C_2-structure using complex conjugation. On fixed-points, the map π is the projection map

$$\pi : \mathbb{R}^2 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{RP}^1,$$

which is a degree 2 map. From this we learn that while $\Phi^e(\eta_{1,1})$ is nilpotent, $\Phi^{\mathrm{C}_2}(\eta_{1,1})$ is the periodic v_0-selfmap. Hence, $\eta_{1,1}$ is a $v_{(\mathrm{nil},0)}$-selfmap and the cofiber $\mathrm{C}(\eta_{1,1})$ is of type $(0,1)$.

Remark 5.0.6. *In the C_2-equivariant stable homotopy groups, the usual Hopf map (sometimes referred to as the 'topological Hopf map') is different from $\eta_{1,1}$ of Example 5.0.5. The 'topological Hopf map' $\eta_{1,0} \in \pi_{1,0}(\mathbb{S}_{\mathrm{C}_2})$ should be thought of as the stabilization of the unstable Hopf map*

$$\eta^u_{1,0} \colon \mathrm{S}^{3,0}_{\mathrm{C}_2} \longrightarrow \mathrm{S}^{2,0}_{\mathrm{C}_2}$$

where both domain and codomain are given the trivial C_2-action.

Definition 5.0.7. *We say a spectrum $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}})$ is of type (n,m) if $\beta(X)$ is of type (n,m). We call an $\mathbb{R}$-motivic self-map*

$$f : X \to X$$

a $v_{(n,m)}$-selfmap, where m and n are in $\mathbb{N} \cup \{\mathrm{nil}\}$ (but not both nil*), if $\beta(f)$ is a C_2-equivariant $v_{(n,m)}$-selfmap.*

Remark 5.0.8. *The maps 'multiplication by 2' (of Example 5.0.3), h (of Example 5.0.4), and $\eta_{1,1}$ (of Example 5.0.5) admit $\mathbb{R}$-motivic lifts along β and provide us with examples of a $v_{(0,0)}$-selfmap, $v_{(0,\mathrm{nil})}$-selfmap and $v_{(\mathrm{nil},0)}$-selfmap of the $\mathbb{R}$-motivic sphere spectrum $\mathbb{S}_{\mathbb{R}}$, respectively.*

A theorem of Balmer and Sanders [5] asserts that $\mathcal{C}(e,n) \subset \mathcal{C}(\mathrm{C}_2, m)$ if and only if $n \geq m+1$. In particular, $\mathcal{C}(e,n)$ is contained in $\mathcal{C}(\mathrm{C}_2, n-1)$. Consequently, there are no type (n,m) (C_2-equivariant or $\mathbb{R}$-motivic) spectra if $n \geq m+2$. Their result also implies the following:

Proposition 5.0.9. *Let $X \in \mathrm{Ho}(\mathbf{Sp}^{\mathrm{C}_2}_{2,\mathrm{fin}})$ be of type $(n+1,n)$ for some n. Then X cannot support a $v_{(n+1,\mathrm{nil})}$-selfmap.*

The proposition holds since the cofiber of such a self-map would be of type $(n+2,n)$, contradicting the results of Balmer-Sanders. In particular, neither the C_2-equivariant cofiber $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ nor the $\mathbb{R}$-motivic cofiber $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ supports a $v_{(1,\mathrm{nil})}$-selfmap. However, it is possible that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ as well as $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ can admit a $v_{(1,0)}$-selfmap or a $v_{(\mathrm{nil},0)}$-selfmap. In fact, $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathbb{R}})$ and $\eta_{1,1} \in \pi_{1,1}(\mathbb{S}_{\mathrm{C}_2})$ induce $v_{(\mathrm{nil},0)}$-selfmaps of $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ respectively. In Section 5.3, we show that:

Theorem 5.0.10. *The spectrum $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ does not admit a $v_{(1,0)}$-selfmap.*

However, it is possible that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ admits a $v_{(1,0)}$-selfmap (see Remark 5.3.7 for details). In contrast to the classical case, there is no guarantee that a finite C_2-equivariant or $\mathbb{R}$-motivic spectrum will admit *any* periodic self-map, or at least nothing concrete is known yet.

5.1 The spectrum $\mathcal{Y}$

This section we study the $\mathbb{R}$-motivic lifts of the classical spectrum

$$\mathcal{Y} := \Sigma^{-3}\mathbb{C}\mathbb{P}^2 \wedge \mathbb{R}\mathbb{P}^2,$$

and selfmaps between them.

From the chromatic point of view, the spectrum $\mathcal{Y}$ is extremely useful because it supports a v_1-self-map of lowest possible periodicity, that is, one. Famously, Mark Mahowald used the spectrum $\mathcal{Y}$ and the low periodicity of its v_1-self-map to prove the height 1 telescope conjecture at the prime 2 [38, 39]. However, 1-periodic v_1-self-maps of $\mathcal{Y}$ are not unique. In fact, up to homotopy, there are eight different v_1-self-maps supported by $\mathcal{Y}$, all of whose cofibers are realizations of $\mathcal{A}(1)$ (see [19]). Up to weak

equivalence, there are four different finite spectra realizing $\mathcal{A}(1)$, and all of them can be obtained as the cofiber of some v_1-self-map of $\mathcal{Y}$. These four different realizations can be distinguished by their $\mathcal{A}$-module structures. Therefore, it is natural to ask if all of the v_1-self-maps of $\mathcal{Y}$ can be lifted to $\mathbb{R}$-motivic analogues, and whether all of the $\mathbb{R}$-motivic realizations of $\mathcal{A}^{\mathbb{R}}(1)$ can be obtained as the cofiber of such a lift.

The answer to the above question is complicated by the fact that there are multiple $\mathbb{R}$-motivic lifts of the spectrum $\mathcal{Y}$. Even if we insist on those lifts which can potentially realize $\mathcal{A}^{\mathbb{R}}(1)$ as a cofiber of a periodic self-map, we are left with two choices; $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$. We state our first result towards answering these questions after establishing some notations. Further, we shall see that some realizations of $\mathcal{A}^{\mathbb{R}}(1)$ must be given as the cofiber of a map between $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ rather than as the cofiber of a self-map of either.

5.1.1 Construction of $\mathcal{Y}$ and its lifts

We consider the classical spectrum

$$\mathcal{Y} := \mathrm{M}_2(1) \wedge \mathrm{C}(\eta)$$

that admits, up to homotopy, 8 different v_1-selfmaps of periodicity 1 [19, Section 2] (see also [15]). We ask ourselves if the v_1-selfmaps are equivariant upon providing $\mathcal{Y}$ with interesting C_2-equivariant structures. We will consider four C_2-equivariant lifts of the spectrum $\mathcal{Y}$,

(i) $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, where the action of C_2 is trivial,

(ii) $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)} := \mathrm{C}^{\mathrm{C}_2}(2) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,1})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}) = \mathrm{M}_2(1) \wedge \mathrm{M}_2(1)$,

(iii) $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)} := \mathrm{C}^{\mathrm{C}_2}(\mathsf{h}) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,0})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}) = \Sigma\mathrm{C}(\eta) \vee \mathrm{C}(\eta)$, and,

(iv) $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)} := \mathrm{C}^{\mathrm{C}_2}(\mathsf{h}) \wedge \mathrm{C}^{\mathrm{C}_2}(\eta_{1,1})$, with $\Phi^{\mathrm{C}_2}(\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}) = \Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)$.

The C_2-spectra $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$ and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ are of type $(1,1)$, and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$ is of type $(1,0)$. There are unique $\mathbb{R}$-motivic lifts of the classes 2, h, $\eta_{1,0}$, and $\eta_{1,1}$, and therefore we have unique $\mathbb{R}$-motivic lifts of $\mathcal{Y}^{\mathrm{C}_2}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$, $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ which we will simply denote by $\mathcal{Y}^{\mathbb{R}}_{\mathrm{triv}}$, $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$, $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$, and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$, respectively.

5.1.2 Selfmaps between the lifts of $\mathcal{Y}$

Let $\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$ and $\mathcal{B}^{\mathbb{R}}_{2}(1)$ denote the $\mathcal{A}^{\mathbb{R}}$-modules $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ and $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$, respectively. As shown in Lemma 5.2.6, these differ in that the bottom cell of $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ supports a Sq^4, whereas the bottom cell of $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ does not. In Subsection 4.3.1, we used a method due to Smith ([50, Appendix C]) to produce the $\mathcal{A}^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}_{\mathbf{0}}(1)$. Then we observed that $\mathcal{A}^{\mathbb{R}}_{\mathbf{0}}(1)$ fits into a short exact sequence

$$\Sigma^{3,1}\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1) \hookrightarrow \mathcal{A}^{\mathbb{R}}_{\mathbf{0}}(1) \twoheadrightarrow \mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$$

that can be realized as a cofiber sequence of finite spectra. We extend the above result to prove the following.

Theorem 5.1.1. *Given* $\overline{\mathrm{v}} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$*, define*

$$\epsilon = \begin{cases} \mathsf{h} & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 0 \\ 2 & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 1, \end{cases} \qquad \text{and} \qquad \delta = \begin{cases} \mathsf{h} & \text{if } \alpha_{03} + \beta_{03} = 0 \\ 2 & \text{if } \alpha_{03} + \beta_{03} = 1. \end{cases}$$

Then there exists a short exact sequence

$$\Sigma^{3,1}\mathcal{B}^{\mathbb{R}}_{\epsilon}(1) \hookrightarrow \mathcal{A}^{\mathbb{R}}_{\overline{\mathrm{v}}}(1) \twoheadrightarrow \mathcal{B}^{\mathbb{R}}_{\delta}(1) \tag{5.1.2}$$

of $\mathcal{A}^{\mathbb{R}}$*-modules. Moreover, this exact sequence can be realized as the cohomology of a cofiber sequence*

$$\mathcal{Y}^{\mathbb{R}}_{(\delta,1)} \longrightarrow \mathcal{A}^{\mathbb{R}}_{1}[\overline{\mathrm{v}}] \longrightarrow \Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\epsilon,1)} \tag{5.1.3}$$

in the category $\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$*.*

The map of spectra that underlies the connecting map

$$v\colon \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\epsilon,1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\delta,1)} \tag{5.1.4}$$

of (5.1.3) is a v_1-self-map of $\mathcal{Y}$ of periodicity 1.

The algebraic part of Theorem 5.1.1 is a straightforward consequence of Theorem 4.2.3 once we identify the $\mathcal{A}^{\mathbb{R}}$-modules $\mathcal{B}^{\mathbb{R}}_{\mathsf{h}}(1)$ and $\mathcal{B}^{\mathbb{R}}_{2}(1)$. However, the topological assertions in Theorem 5.1.1, as well as in Theorem 4.2.1, require a technical tool, which we refer to as the $\mathbb{R}$-*motivic Toda realization theorem.*

5.2 A $v_{(1,nil)}$-selfmap on $\mathcal{Y}_{(\mathsf{h},1)}$

With the construction of $\mathcal{A}^{\mathbb{R}}_1$, one might hope that any one of $\mathcal{Y}^{\mathbb{R}}_{(i,j)}$ fits into an exact triangle

$$\Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(i,j)} \xrightarrow{v} \mathcal{Y}^{\mathbb{R}}_{(i,j)} \longrightarrow \mathcal{A}^{\mathbb{R}}_{1} \longrightarrow \Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(i,j)} \xrightarrow{\Sigma v} \dots \tag{5.2.1}$$

in $\mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}})$. The motivic weights prohibit $\mathcal{A}^{\mathbb{R}}_1$ from being the cofiber of a self-map on $\mathcal{Y}_{\mathrm{triv}}$ or $\mathcal{Y}_{(\mathsf{h},0)}$, as the 2-cell in these complexes appears in weight 0, whereas in $\mathcal{A}^{\mathbb{R}}_1$ the 2-cell is in weight 1. We will also see that the spectrum $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ cannot be a part of (5.2.1) because of its $\mathcal{A}^{\mathbb{R}}$-module structure (see Lemma 5.2.6). If $\mathcal{Y}_{(i,j)} = \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ in (5.2.1), then the map v will necessarily be a $v_{(1,\mathrm{nil})}$-selfmap because $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ is of type $(1,1)$ and $\mathcal{A}^{\mathbb{R}}_1$ is of type $(2,1)$. The main purpose of this section is to prove Theorem 5.2.14 and Theorem 5.2.17 by showing that $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ does fit into an exact triangle very similar to (5.2.1)

$$\Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(i,j)} \xrightarrow{v} \mathcal{Y}^{\mathbb{R}}_{(i,j)} \longrightarrow \mathrm{C}^{\mathbb{R}}(v) \longrightarrow \Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(i,j)} \xrightarrow{\Sigma v} \dots$$

where $\mathrm{C}^{\mathbb{R}}(v)$ is of type $(2,1)$ and $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v)) \cong \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$ as $\mathcal{A}^{\mathbb{R}}$-modules.

Remark 5.2.2. *The fact that* $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v))$ *is isomorphic to* $\mathrm{H}^{*,*}(\mathcal{A}_1^{\mathbb{R}})$ *as* $\mathcal{A}^{\mathbb{R}}$*-modules does not imply that* $\mathrm{C}^{\mathbb{R}}(v)$ *and* $\mathcal{A}_1^{\mathbb{R}}$ *are equivalent as* $\mathbb{R}$*-motivic spectra. There are a plethora of examples of Steenrod modules that are realized by spectra of different homotopy types.*

We begin by discussing the $\mathcal{A}^{\mathbb{R}}$-module structures of $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$. Using Adem relations, one can show that the element

$$\overline{\mathrm{Q}}_1 := \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1 \in \mathcal{A}^{\mathbb{R}}(1)$$

squares to zero. Let $\Lambda(\overline{\mathrm{Q}}_1)$ denote the exterior subalgebra $\mathbb{M}_2^{\mathbb{R}}[\overline{\mathrm{Q}}_1]/(\overline{\mathrm{Q}}_1^2)$ of $\mathcal{A}^{\mathbb{R}}(1)$. Let $\mathcal{B}^{\mathbb{R}}(1)$ denote the $\mathcal{A}^{\mathbb{R}}(1)$-module

$$\mathcal{B}^{\mathbb{R}}(1) := \mathcal{A}^{\mathbb{R}}(1) \otimes_{\Lambda(\overline{\mathrm{Q}}_1)} \mathbb{M}_2^{\mathbb{R}}.$$

Both $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ are realizations of $\mathcal{B}^{\mathbb{R}}(1)$. In other words:

Proposition 5.2.3. *There is an isomorphism of* $\mathcal{A}^{\mathbb{R}}(1)$*-modules*

$$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)}) \cong \mathcal{B}^{\mathbb{R}}(1)$$

for $(i,j) \in \{(2,1),(\mathsf{h},1)\}$.

Proof. By direct inspection, $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)})$ is cyclic as an $\mathcal{A}^{\mathbb{R}}(1)$-module for $(i,j) \in \{(2,1),(\mathsf{h},1)\}$. Thus we have an $\mathcal{A}^{\mathbb{R}}(1)$-module map

$$f_i : \mathcal{A}^{\mathbb{R}}(1) \to \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)}). \tag{5.2.4}$$

The result follows from the fact that $\overline{\mathrm{Q}}_1$ acts trivially on $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(i,j)})$ and a dimension counting argument. □

Remark 5.2.5. *Let* $\{y_{0,0}, y_{1,0}\}$ *be the* $\mathbb{M}_2^{\mathbb{R}}$*-basis of* $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ *or* $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$*, so that* $\mathrm{Sq}^1(y_{0,0}) = y_{1,0}$*, and let* $\{x_{0,0}, x_{2,1}\}$ *a basis of* $\mathrm{C}^{\mathbb{R}}(\eta_{1,1})$*, so that* $\mathrm{Sq}^2(x_{0,0}) = x_{2,1}$*. If we consider the* $\mathbb{M}_2^{\mathbb{R}}$*-basis* $\{v_{0,0}, v_{1,0}, v_{2,1}, v_{3,1}, w_{3,1}, w_{3,2}, w_{4,2}, w_{5,3}, w_{6,3}\}$ *of* $\mathcal{A}^{\mathbb{R}}(1)$ *from Figure 4.7, then the maps* f_i *of* (5.2.4) *are given as in Table 5.1.*

Lemma 5.2.6. *The* $\mathcal{A}^{\mathbb{R}}$*-module structures on* $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$ *and* $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ *are given as in Figure 5.1.*

Proof. The result is an easy consequence of a calculation using the Cartan formula,

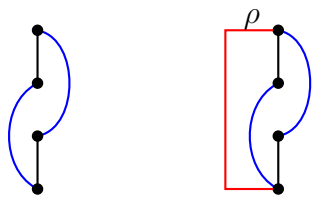
$$\mathrm{Sq}^4(xy) = \mathrm{Sq}^4(x)y + \tau\mathrm{Sq}^3(x)\mathrm{Sq}^1(y) + \mathrm{Sq}^2(x)\mathrm{Sq}^2(y) + \tau\mathrm{Sq}^1(x)\mathrm{Sq}^3(y) + x\mathrm{Sq}^4(y),$$

and the fact that $\mathrm{Sq}^2(y_{0,0}) = \rho y_{1,0}$ in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(2))$, whereas $\mathrm{Sq}^2(y_{0,0})$ vanishes in $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(\mathsf{h}))$ (see Proposition 4.3.11). □

Remark 5.2.2. *Comparing Lemma 5.2.6 and Lemma 4.3.9, we see that the* $\mathcal{A}^{\mathbb{R}}(1)$*-module map* f_2*, as in Remark 5.2.5, cannot be extended to a map of* $\mathcal{A}^{\mathbb{R}}$*-modules.*

Table 5.1: The maps f_2 and f_{h}

x	$f_2(x)$	$f_{\mathsf{h}}(x)$
$v_{0,0}$	$y_{0,0}x_{0,0}$	$y_{0,0}x_{0,0}$
$v_{1,0}$	$y_{1,0}x_{0,0}$	$y_{1,0}x_{0,0}$
$v_{2,1}$	$y_{0,0}x_{2,0} + \rho \cdot y_{1,0}\ x_{0,0}$	$y_{0,0}x_{2,0}$
$v_{3,1}$	$y_{1,0}x_{2,0}$	$y_{1,0}x_{2,0}$
$w_{3,1}$	$y_{1,0}x_{2,0}$	$y_{1,0}x_{2,0}$
$w_{4,2}$	0	0
$w_{5,3}$	0	0
$w_{6,3}$	0	0



$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$

Figure 5.1: $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ and $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(2,1)})$

Corollary 5.2.3. *There is an exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules*

$$0 \longrightarrow \mathrm{H}^{*,*}(\Sigma^{3,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \xrightarrow{\pi^*} \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1) \xrightarrow{\iota^*} \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \longrightarrow 0. \tag{5.2.4}$$

Proof. From the description of the map f_{h} in Remark 5.2.5, along with Lemma 4.3.9 and Lemma 5.2.6, it is easy to check that f_{h} extends to an $\mathcal{A}^{\mathbb{R}}$-module map and that

$$\ker f_{\mathsf{h}} \cong \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$$

as $\mathcal{A}^{\mathbb{R}}$-modules. □

The exact sequence (5.2.4) corresponds to a nonzero element in the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence (also see Remark 5.2.7 and Remark 5.2.9)

$$\overline{v} \in \mathrm{Ext}^{2,1,1}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2) \Rightarrow [\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}]_{2,1}, \tag{5.2.5}$$

where $D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} := \mathrm{F}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}, \mathbb{S}_{\mathbb{R}})$ is the Spanier-Whitehead dual of $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$. If

Notation 5.2.6. *Note that we follow [20, 13] in grading* $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ *as* $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}$*, where s is the stem, f is the Adams filtration, and w is the weight. We will also follow [26] in referring to the difference $s - w$ as the coweight.*

Remark 5.2.7. *Since* $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ *is* $\mathbb{M}^{\mathbb{R}}_2$*-free, an appropriate universal-coefficient spectral sequence collapses and we get* $\mathrm{H}^{*,*}(D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \hom_{\mathbb{M}^{\mathbb{R}}_2}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2)$*. Further, the Kunneth isomorphism of* (4.3.8) *gives us*

$$\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \otimes_{\mathbb{M}^{\mathbb{R}}_2} \mathrm{H}^{*,*}(D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}),$$

and therefore,

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \wedge D\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \cong \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})).$$

Theorem 5.2.14 follows immediately if we show that the element $\overline{v}$ is a nonzero permanent cycle. The following result implies that a d_r-differential (for $r \geq 2$) supported by $\overline{v}$ has no potential nonzero target.

Proposition 5.2.8. *For* $f \geq 3$, $\mathrm{Ext}^{1,f,1}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) = 0$.

Proof. In order to calculate $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$, we filter the spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ via the evident maps

$$\begin{array}{ccccccc} Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3. \\ \| & & \| & & \| & & \| \\ \mathbb{S}_{\mathbb{R}} & & \mathrm{C}^{\mathbb{R}}(\mathsf{h}) & & \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \cup_{\mathbb{S}_{\mathbb{R}}} \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) & & \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \end{array}$$

Note that $\mathrm{H}^{*,*}(Y_j)$ are free $\mathbb{M}^{\mathbb{R}}_2$-modules. The above filtration results in cofiber sequences

$$Y_0 \longrightarrow Y_1 \longrightarrow \Sigma^{1,0}\mathbb{S}_{\mathbb{R}},$$

$$Y_1 \longrightarrow Y_2 \longrightarrow \Sigma^{2,1}\mathbb{S}_{\mathbb{R}}, \qquad \text{and}$$

$$Y_2 \longrightarrow Y_3 \longrightarrow \Sigma^{3,1}\mathbb{S}_{\mathbb{R}},$$

which induce short exact sequences of $\mathcal{A}^{\mathbb{R}}$-modules as the connecting map

$$\mathrm{C}^{\mathbb{R}}(Y_j \to Y_{j+1}) \longrightarrow \Sigma Y_j$$

induces the zero map in $\mathrm{H}^{*,*}(-)$. Thus, applying the functor $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), -)$ to these short-exact sequences, we get long exact sequences, which can be spliced together to obtain an Atiyah-Hirzebruch like spectral sequence

$$\begin{array}{c} \mathrm{E}_1^{*,*,*,*} = \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2)\{g_{0,0}, g_{1,0}, g_{2,1}, g_{3,1}\} \\ \Downarrow \\ \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})). \end{array}$$

An element $x \cdot g_{i,j}$ in the E_2-page contributes to the degree $|x| - (i, 0, j)$ of the abutment. Thus, Proposition 5.2.8 is a straightforward consequence of Proposition 5.2.10. □

Remark 5.2.9. *Because,* $\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ *is* $\mathbb{M}^{\mathbb{R}}_2$*-free and finite, we have*

$$\mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}) \cong \hom_{\mathbb{M}^{\mathbb{R}}_2}(\mathrm{H}^{*,*}(\mathcal{Y}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2),$$

and therefore, $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{H}^{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}), \mathbb{M}^{\mathbb{R}}_2) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$.

Proposition 5.2.10. *For* $f \geq 3$ *and* $(i,j) \in \{(0,0),(1,0),(2,1),(3,1)\}$*, we have that*

$$\mathrm{Ext}^{1+i,f,1+j}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) = 0.$$

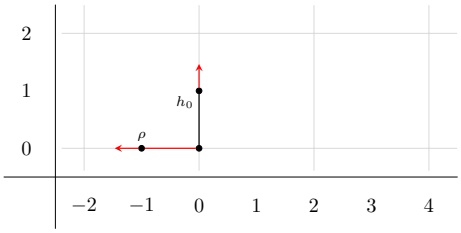
Proof. Our desired vanishing concerns only the groups $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ in coweights 0, 1 and 2. These groups can be easily calculated starting from the computations of $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathbb{M}^{\mathbb{R}}_2)$ in [20] and [13] and using the short exact sequences in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}$ arising from the cofiber sequences

$$\Sigma^{1,1} S_{\mathbb{R}} \xrightarrow{\eta_{1,1}} S_{\mathbb{R}} \longrightarrow \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \quad \text{and}$$

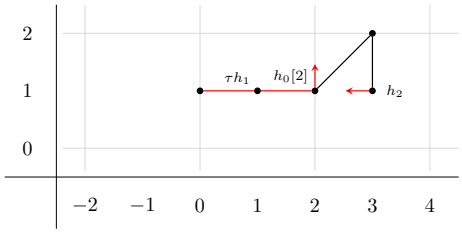
$$\mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \xrightarrow{\mathsf{h}} \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) \longrightarrow \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \wedge \mathrm{C}^{\mathbb{R}}(\eta_{1,1}) = \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}.$$

We display $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1})))$ in coweights 0, 1 and 2 in the charts below. Here horizontal, vertical, or diagonal lines denote multiplication by ρ, h_0, and h_1, respectively.

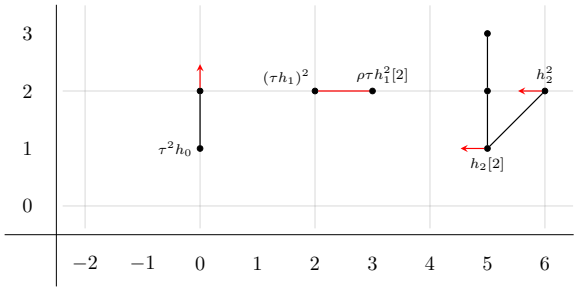
$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1}))$ in coweight zero



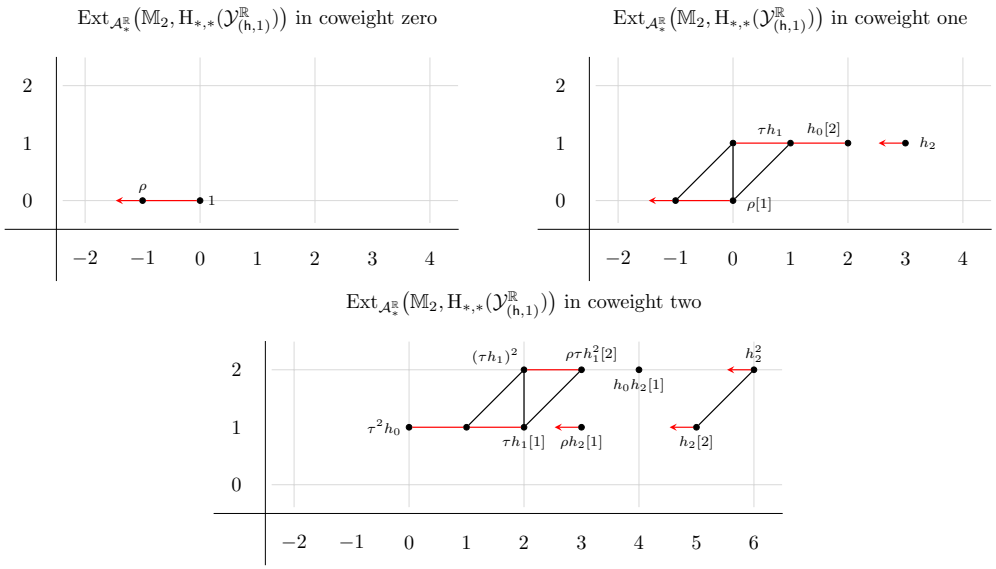
$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1}))$ in coweight one



$\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}_2, \mathrm{H}_{*,*}(\mathrm{C}^{\mathbb{R}}(\eta_{1,1}))$ in coweight two



We find that $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}))$ is, in coweights zero, one, and two, also given by the charts below.



The result follows from the above charts. □

Remark 5.2.11. *One can also resolve Proposition 5.2.10 directly using the ρ-Bockstein spectral sequence*

$$\begin{aligned} \mathrm{E}_1 &:= \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{F}_2[\tau], \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{C}}_{(\mathsf{h},1)})) \otimes \mathbb{F}_2[\rho] \\ &\Downarrow \\ &\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}_*}(\mathbb{M}^{\mathbb{R}}_2, \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})) \end{aligned} \tag{5.2.12}$$

and identifying a vanishing region for $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^{\mathbb{C}}_*}(\mathbb{F}_2[\tau], \mathrm{H}_{*,*}(\mathcal{Y}^{\mathbb{C}}_{(\mathsf{h},1)}))$. Even a rough estimate of the vanishing region using the E_1-page of the $\mathbb{C}$-motivic May spectral sequence leads to Proposition 5.2.10. Such an approach would avoid explicit calculations of $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$ as in [20] and [13].

Proof of Theorem 5.2.14. By Proposition 5.2.8 every map

$$v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

detected by $\overline{v}$ of (5.2.5) is a nonzero permanent cycle. In order to finish the proof of Theorem 5.2.14 we must show that v is necessarily a $v_{(1,\mathrm{nil})}$-selfmap of periodicity 1. It is easy to see that the underlying map

$$\Phi^e(\beta(v)) : \Sigma^2\mathcal{Y} \longrightarrow \mathcal{Y}$$

is a v_1-selfmap of periodicity 1 as

$$\mathrm{C}(\Phi^e(\beta(v))) \simeq \Phi^e(\beta(\mathrm{C}^{\mathbb{R}}(v))) \simeq \mathcal{A}_1[10]$$

is of type 1 (see Remark 4.3.5). On the other hand,

$$\Phi^{\mathrm{C}_2}(\beta(v)) : \Sigma^2(\Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)) \longrightarrow \Sigma\mathrm{M}_2(1) \vee \mathrm{M}_2(1)$$

is necessarily a nilpotent map because of [34, Theorem 3(ii)] and the fact that a v_1-selfmap of $\mathrm{M}_2(1)$ has periodicity at least 4 (see [19] for details) which lives in $[\mathrm{M}_2(1), \mathrm{M}_2(1)]_{8k}$ for $k \geq 1$. □

Proof of Theorem 5.2.17. Since v is a $v_{(1,\mathrm{nil})}$-selfmap and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ is of type $(1,1)$, it follows that $\mathrm{C}^{\mathbb{R}}(v)$ is of type $(2,1)$. Moreover,

$$\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v)) \cong \mathrm{H}^{*,*}(\mathcal{A}^{\mathbb{R}}_1)$$

as v is detected by $\overline{v}$ of (5.2.5) in the E_2-page of the Adams spectral sequence. Thus, $\mathrm{H}^{*,*}(\mathrm{C}^{\mathbb{R}}(v))$ is a free $\mathcal{A}^{\mathbb{R}}(1)$-module on single generator. □

Remark 5.2.13. *It is likely that realizing a different $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ as a spectrum may lead to a 1-periodic v_1-selfmap on $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ as well as on $\mathcal{Y}^{\mathrm{C}_2}_{(2,1)}$. We explore such possibilities in upcoming work.*

we prove:

Theorem 5.2.14. *The $\mathbb{R}$-motivic spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$ admits a $v_{(1,\mathrm{nil})}$-selfmap*

$$v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$$

of periodicity 1.

By applying the Betti realization functor we get:

Corollary 5.2.15. *The C_2-equivariant spectrum $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$ admits a 1-periodic $v_{(1,\mathrm{nil})}$-selfmap*

$$\beta(v) : \Sigma^{2,1}\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)} \longrightarrow \mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}.$$

Corollary 5.2.16. *The geometric fixed-point spectrum of the telescope*

$$\beta(v)^{-1}\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},1)}$$

is contractible.

Classically, the cofiber of a v_1-selfmap on $\mathcal{Y}$ is a realization of the finite subalgebra $\mathcal{A}(1)$ of the Steenrod algebra $\mathcal{A}$. We see a very similar phenomenon in the $\mathbb{R}$-motivic as well as in the C_2-equivariant cases. The C_2-equivariant Steenrod algebra $\mathcal{A}^{\mathrm{C}_2}$ as well as the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ admit subalgebras analogous to $\mathcal{A}(1)$ (generated by Sq^1 and Sq^2) [32, 51], which we denote by $\mathcal{A}^{\mathrm{C}_2}(1)$ and $\mathcal{A}^{\mathbb{R}}(1)$, respectively. We observe that:

Theorem 5.2.17. *The spectrum $\mathrm{C}^{\mathbb{R}}(v) := \mathrm{Cof}(v : \Sigma^{2,1}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)} \to \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)})$ is a type $(2,1)$ complex whose bigraded cohomology is a free $\mathcal{A}^{\mathbb{R}}(1)$-module on one generator.*

Corollary 5.2.18. *The bigraded cohomology of the C_2-equivariant spectrum*

$$\mathrm{C}^{\mathrm{C}_2}(\beta(v)) \simeq \beta(\mathrm{C}^{\mathbb{R}}(v))$$

is a free $\mathcal{A}^{\mathrm{C}_2}(1)$-module on one generator.

5.3 Nonexistence of $v_{(1,0)}$-selfmap on $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$

Our last main result is the following.

Theorem 5.3.1. *The spectrum $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ does not admit a $v_{(1,0)}$-selfmap.*

Let X be a finite $\mathbb{R}$-motivic spectrum and let $f\colon \Sigma^{i,j}X \to X$ be a map such that

$$\Phi^{\mathrm{C}_2}(\beta(f))\colon \Sigma^{i-j}\Phi^{\mathrm{C}_2}(\beta(X)) \longrightarrow \Phi^{\mathrm{C}_2}(\beta(X))$$

is a v_0-selfmap. Then it must be the case that $i = j$, as v_0-selfmaps preserve dimension. Note that both $\mathrm{C}^{\mathbb{R}}(\mathsf{h})$ and $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ are of type $(1, 0)$.

Proposition 5.3.2. *The v_1-selfmaps of $\mathrm{M}_2(1)$ are not in the image of the underlying homomorphism*

$$\Phi^e \circ \beta\colon [\Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}), \mathrm{C}^{\mathbb{R}}(\mathsf{h})]^{\mathbb{R}} \longrightarrow [\Sigma^{8k}\mathrm{M}_2(1), \mathrm{M}_2(1)].$$

Proof. The minimal periodicity of a v_1-selfmap of $\mathrm{M}_2(1)$ is 4. Let $v : \Sigma^{8k}\mathrm{M}_2(1) \to \mathrm{M}_2(1)$ be a $4k$-periodic v_1-selfmap. It is well-known that the composite

$$\Sigma^{8k}\mathbb{S} \longhookrightarrow \Sigma^{8k}\mathrm{M}_2(1) \xrightarrow{v} \mathrm{M}_2(1) \longrightarrow \Sigma^{1}\mathbb{S} \tag{5.3.3}$$

is not null (and equals $P^{k-1}(8\sigma)$ where P is a periodic operator given by the Toda bracket $\langle \sigma, 16, - \rangle$).

Suppose there exists $f : \Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \to \mathrm{C}^{\mathbb{R}}(\mathsf{h})$ such that $\Phi^e \circ \beta(f) = v$. Then (5.3.3) implies that the composition

$$\Sigma^{8k,8k}\mathbb{S}_{\mathbb{R}} \longhookrightarrow \Sigma^{8k,8k}\mathrm{C}^{\mathbb{R}}(\mathsf{h}) \xrightarrow{v} \mathrm{C}^{\mathbb{R}}(\mathsf{h}) \longrightarrow \Sigma^{1,0}\mathbb{S} \tag{5.3.4}$$

is nonzero as the functor $\Phi^e \circ \beta$ is additive. The composite of the maps in (5.3.4) is a nonzero element of $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ in negative coweight. This contradicts the fact that $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ is trivial in negative coweights [20]. □

Proposition 5.3.5. *The v_1-selfmaps of $\mathcal{Y}$ are not in the image of the underlying homomorphism*

$$\Phi^e \circ \beta\colon [\Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}, \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}]^{\mathbb{R}} \longrightarrow [\Sigma^{8k}\mathcal{Y}, \mathcal{Y}].$$

Proof. Let $v : \Sigma^{2k}\mathcal{Y} \to \mathcal{Y}$ denote a v_1-selfmap of periodicity k. Notice that the composite

$$\mathrm{S}^{2k} \longhookrightarrow \Sigma^{2k}\mathcal{Y} \xrightarrow{v} \mathcal{Y} \longrightarrow \mathcal{Y}_{\geq 1} \tag{5.3.6}$$

where $\mathcal{Y}_{\geq 1}$ is the first coskeleton, must be nonzero. If not, then v factors through the bottom cell resulting in a map $\mathrm{S}^{2k} \to \Sigma^{2k}\mathcal{Y} \to \mathbb{S}$ which induces an isomorphism in K(1)-homology, contradicting the fact that $\mathbb{S}$ is of type 0.

If $f : \Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} \to \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)}$ were a map such that $\Phi^e \circ \beta(f) = v$, then (5.3.6) would force one among the hypothetical composites (A), (B) or (C) in the diagram

$$\begin{array}{ccccccc}
\Sigma^{2k,2k}\mathbb{S}_{\mathbb{R}} & \hookrightarrow & \Sigma^{2k,2k}\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} & \longrightarrow & \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},0)} & \xrightarrow{p_3} & \Sigma^{3,0}\mathbb{S}_{\mathbb{R}} \qquad (A) \\
 & \searrow & & & \uparrow & & \\
 & & & & \mathrm{Fib}(p_3) & \xrightarrow{p_2} & \Sigma^{2,0}\mathbb{S}_{\mathbb{R}} \qquad (B) \\
 & & & \searrow & \uparrow & & \\
 & & & & \mathrm{Fib}(p_2) & \xrightarrow{p_1} & \Sigma^{1,0}\mathbb{S}_{\mathbb{R}} \qquad (C)
\end{array}$$

to exist as a nonzero map, thereby contradicting the fact that $\pi_{*,*}(\mathbb{S}_{\mathbb{R}})$ is trivial in negative coweights. □

Remark 5.3.7. *The above results do not preclude the existence of a $v_{(1,0)}$-selfmap on $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ and $\mathcal{Y}^{\mathrm{C}_2}_{(\mathsf{h},0)}$. Forthcoming work [27] of the second author and Isaksen shows that 8σ is in the image of $\Phi^e\colon \pi_{7,8}(\mathbb{S}_{\mathrm{C}_2}) \longrightarrow \pi_7(\mathbb{S})$ and suggests that $\mathrm{C}^{\mathrm{C}_2}(\mathsf{h})$ supports a $v_{(1,0)}$-selfmap.*

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Appendices

Appendix A: An $\mathbb{R}$-motivic analogue of the spectrum $\mathcal{Z}$

Recently in [14], the authors introduced a new type 2 spectrum $\mathcal{Z}$ which is notable for admitting a v_2-self-map of lowest possible periodicity, that is 1. The low periodicity of the v_2-self-map makes the spectrum $\mathcal{Z}$ suitable for the analysis of the telescope conjecture which, if true, would imply that the natural map from the telescope of $\mathcal{Z}$ to the K(2)-localization of $\mathcal{Z}$ is a weak equivalence. While the telescope conjecture is true for finite spectra of type 1 [38, 39, 44], it is expected to be false for finite spectra of type ≥ 2 (see [40]). In fact, in [8], the authors study the prime 2, height 2 telescope conjecture using the spectrum $\mathcal{Z}$ and lay down several conjectures (see [8, § 9]), whose validity would lead to a disproof of the telescope conjecture. In this work, we also construct an $\mathbb{R}$-motivic analogue of $\mathcal{Z}$ which is likely to shed light on some of these conjectures.

Theorem A.1. *There exists $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that the underlying $\mathcal{A}^{\mathbb{R}}(2)$-module structure of its cohomology is isomorphic to*

$$\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Z}_{\mathbb{R}}) \cong_{\mathcal{A}^{\mathbb{R}}(2)} \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(\tilde{\mathrm{Q}}^{\mathbb{R}}_2)} \mathbb{M}^{\mathbb{R}}_2$$

where $\tilde{\mathrm{Q}}^{\mathbb{R}}_2 := [\mathrm{Sq}^4, \mathrm{Q}^{\mathbb{R}}_1]$.

The type 2 spectrum $\mathcal{Z} \in \mathbf{Sp}_{2,\mathrm{fin}}$, is defined by the property that its cohomology as an $\mathcal{A}(2)$-module is

$$\mathcal{B}(2) := \mathcal{A}(2) \otimes_{\Lambda(\mathrm{Q}_2)} \mathbb{F}_2,$$

where $\mathrm{Q}_2 = [\mathrm{Sq}^4, \mathrm{Q}_1]$ is dual to the Milnor generator ξ_3 of the dual Steenrod algebra. They first show that an $\mathcal{A}$-module structure on $\mathcal{A}(2)$ satisfying the criteria in [14, Lemma 2.7] leads to an $\mathcal{A}$-module structure on $\mathcal{B}(2)$. In [14], the authors show that among the 1600 possible $\mathcal{A}$-module structures on $\mathcal{A}(2)$ [52], there are some $\mathcal{A}$-modules that satisfy [14, Lemma 2.7]. Then they use the classical Toda realization theorem to show that any $\mathcal{A}$-module whose underlying $\mathcal{A}(2)$-module structure is $\mathcal{B}(2)$ can be realized as a 2-local finite spectrum, which they call $\mathcal{Z}$.

We construct $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ by emulating the construction of the classical $\mathcal{Z}$ (as in [14]) in the $\mathbb{R}$-motivic context. Since there is no a priori $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(2)$, we produce one in the following subsection. In fact, we construct an $\mathbb{R}$-motivic spectrum whose cohomology is the desired $\mathcal{A}^{\mathbb{R}}$-module.

A topological realization of $\mathcal{A}^{\mathbb{R}}(2)$

Let $\mathcal{A}^{\mathbb{R}}(2)$ denote the sub-$\mathbb{M}^{\mathbb{R}}_2$-algebra of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$, and Sq^4. We will use a method of Smith (exposed in [50, Appendix C]) to construct an $\mathbb{R}$-motivic spectrum $\mathcal{A}^{\mathbb{R}}_2 \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that its cohomology as an $\mathcal{A}^{\mathbb{R}}(2)$-module is free on one generator.

Let $\mathsf{h}, \eta_{1,1}$ and $\nu_{3,2}$ denote the first three $\mathbb{R}$-motivic Hopf-elements (these are denoted ω, η, and ν in [20, Section 8]).

Lemma A.2. *The $\mathbb{R}$-motivic Toda-bracket $\langle \mathsf{h}, \eta_{1,1}, \nu_{3,2}\rangle$ contains 0.*

Proof. In this argument, it will be convenient to refer to the "coweight", by which we mean the difference $s - w$, as in [26].

Since h and $\eta_{1,1}$ have coweight 0 while $\nu_{3,2}$ has coweight 1, it follows that the bracket $\langle \mathsf{h}, \eta_{1,1}, \nu_{3,2}\rangle$ is comprised of elements in stem 5 with coweight 2. The only element in stem 5 with coweight 1 is $\rho\cdot\nu_{3,2}^2$ [13]. Since this element is a $\nu_{3,2}$ multiple, it lies in the indeterminacy, which means that the $\mathbb{R}$-motivic Toda-bracket does contain zero. □

Lemma A.2 implies that we can construct a 4-cell complex $\mathcal{K}$ whose cohomology as an $\mathcal{A}^{\mathbb{R}}$-module has the structure described in Corollary A.3 and displayed in Figure A1.

Corollary A.3. *There exists $\mathcal{K} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{K})$ is $\mathbb{M}^{\mathbb{R}}_2$-free on four generators x_0, x_1, x_3 and x_7, such that $\mathrm{Sq}^{i+1}(x_i) = x_{2i+1}$ for $i \in \{0, 1, 3\}$.*

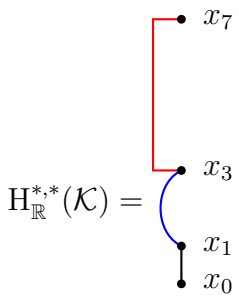


Figure A1: The $\mathcal{A}^{\mathbb{R}}$-structure of $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{K})$

Let $e \in \mathbb{Z}_{(2)}[\Sigma_6]$ denote the idempotent corresponding to the Young tableaux

1	2	3
4	5	
6		

which is constructed as follows. Let $\Sigma_{\mathrm{Row}} \subset \Sigma_6$ denote the subgroup comprised of permutations that preserve each row. Likewise, let Σ_{Col} denote the subgroup comprised of column-preserving permutations. Let

$$\mathsf{R} = \sum_{r\in\Sigma_{\mathrm{Row}}} r \qquad \text{and} \qquad \mathsf{C} = \sum_{c\in\Sigma_{\mathrm{Col}}} (-1)^{\mathrm{sign}(c)} c \tag{A.4}$$

and define

$$e = \frac{1}{\mu}\mathsf{R}\cdot\mathsf{C},$$

where μ is an odd integer defined in [50, Theorem C.1.3]. We let $\overline{e}$ denote the resulting idempotent in $\mathbb{F}_2[\Sigma_6]$.

Proposition A.5. *The idempotent $\overline{e} \in \mathbb{F}_2[\Sigma_6]$ has the property that $\overline{e}(\mathrm{V}^{\otimes 6}) = 0$ if $\dim_{\mathbb{F}_2} \mathrm{V} < 3$ and*

$$\dim_{\mathbb{F}_2} \overline{e}(\mathrm{V}^{\otimes 6}) = \begin{cases} 8 & \text{if } \dim_{\mathbb{F}_2} \mathrm{V} = 3, \\ 64 & \text{if } \dim_{\mathbb{F}_2} \mathrm{V} = 4. \end{cases}$$

Proof. Let $\overline{\mathsf{R}}$ and $\overline{\mathsf{C}}$ denote the images of R and C in $\mathbb{F}_2[\Sigma_6]$, respectively. Then $\overline{e} = \overline{\mathsf{R}} \cdot \overline{\mathsf{C}}$. It is straightforward that $\overline{\mathsf{C}}$ vanishes on $\mathrm{V}^{\otimes 6}$ if $\dim \mathrm{V} \leq 2$.

Now suppose that V has basis $\{a, b, c\}$. Then a basis for $\overline{e}(\mathrm{V}^{\otimes 6})$ is given by

$$\left\{ \overline{e}\left(\begin{array}{|c|c|c|}\hline a&b&c\\\hline b&c\\\cline{1-2} c\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline b&a&c\\\hline a&c\\\cline{1-2} c\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&b\\\hline c&b\\\cline{1-2} b\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline c&a&b\\\hline a&b\\\cline{1-2} b\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline c&b&a\\\hline b&a\\\cline{1-2} a\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline b&c&a\\\hline c&a\\\cline{1-2} a\\\cline{1-1}\end{array}\right), \right.$$

$$\left. \overline{e}\left(\begin{array}{|c|c|c|}\hline a&b&c\\\hline b&a\\\cline{1-2} c\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&b\\\hline c&a\\\cline{1-2} b\\\cline{1-1}\end{array}\right) \right\}.$$

Finally, suppose that $\dim \mathrm{V} = 4$ with basis $\{a, b, c, d\}$. For any subspace $\mathrm{W} \subset \mathrm{V}$ spanned by three of these basis elements, the space $\overline{e}(\mathrm{W}^{\otimes 6})$ has dimension 8, as we have just seen. There are 4 choices of W, which together yield a 32-dimensional subspace of $\mathrm{V}^{\otimes 6}$. Now consider Young tableaux in which all four basis elements appear and only one is repeated. In the case that d is repeated, we generate only two independent elements:

$$\overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&d\\\hline b&d\\\cline{1-2} d\\\cline{1-1}\end{array}\right) \quad \text{and} \quad \overline{e}\left(\begin{array}{|c|c|c|}\hline c&a&d\\\hline b&d\\\cline{1-2} d\\\cline{1-1}\end{array}\right).$$

Allowing any basis element to be the repeating one, this gives an 8-dimensional subspace. Finally, we consider Young tableaux in which all four basis elements appear and two are repeated. In the case that c and d are repeated, we have the four elements

$$\overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&d\\\hline b&d\\\cline{1-2} c\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline a&c&d\\\hline c&b\\\cline{1-2} d\\\cline{1-1}\end{array}\right), \overline{e}\left(\begin{array}{|c|c|c|}\hline b&d&c\\\hline b&c\\\cline{1-2} d\\\cline{1-1}\end{array}\right), \quad \text{and} \quad \overline{e}\left(\begin{array}{|c|c|c|}\hline a&d&c\\\hline d&b\\\cline{1-2} c\\\cline{1-1}\end{array}\right).$$

As there are $\binom{4}{2} = 6$ such choices, this contributes another subspace of dimension $4 \cdot 6 = 24$. □

We define

$$\mathcal{A}_2^{\mathbb{R}} := \Sigma^{-5,-1} e(\mathcal{K}^{\wedge 6}) = \Sigma^{-5,-1}(\mathrm{hocolim}\, \{\mathcal{K}^{\wedge 6} \xrightarrow{e} \mathcal{K}^{\wedge 6} \xrightarrow{e} \dots\}),$$

which is a split summand of $\Sigma^{-5,-1}\mathcal{K}^{\wedge 6}$ as e is an idempotent. We shift the grading by $(-5,-1)$ to make sure that the $\mathcal{A}^{\mathbb{R}}(2)$-module generator of $\mathrm{H}_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$ is in $(0,0)$ (see Remark A.12).

Theorem A.6. $\mathrm{H}_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}}) \cong \mathcal{A}^{\mathbb{R}}(2)$ *as an $\mathcal{A}^{\mathbb{R}}(2)$-module.*

Proof. By Corollary 2.2.5, $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ is a free $\mathcal{A}^{\mathbb{R}}(2)$-module if and only if $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ is free as an $\mathbb{M}^{\mathbb{R}}_2$-module and the Margolis homology $\mathcal{M}(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)\otimes_{\mathbb{M}^{\mathbb{R}}_2}\mathbb{F}_2, x)$ vanishes for $x \in \{\mathrm{Q}^{\mathbb{R}}_0, \mathrm{Q}^{\mathbb{R}}_1, \overline{\mathrm{P}}^1_1, \mathrm{Q}^{\mathbb{R}}_2, \overline{\mathrm{P}}^1_2\}$, where $\overline{\mathrm{P}}^1_1$ and $\overline{\mathrm{P}}^1_2$ are the elements in $\mathcal{A}^{\mathbb{R}}$ dual to ξ_1 and ξ_2, respectively.

Let $\mathrm{K}_{\mathbb{R}} := \mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{K})$. The $\mathcal{A}^{\mathbb{R}}$-module $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ is $\mathbb{M}^{\mathbb{R}}_2$-projective as it is a summand of

$$\mathrm{H}^{*,*}_{\mathbb{R}}(\Sigma^{-5}\mathcal{K}^{\wedge 6}) \cong \Sigma^{-5}\mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}^{\mathbb{R}}_2}6},$$

which is $\mathbb{M}^{\mathbb{R}}_2$-free. However, $\mathbb{M}^{\mathbb{R}}_2$ is a graded local ring, and over a local ring, being projective is equivalent to being free. Hence, $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ is $\mathbb{M}^{\mathbb{R}}_2$-free. Since $\mathrm{Q}^{\mathbb{R}}_0, \mathrm{Q}^{\mathbb{R}}_1, \mathrm{Q}^{\mathbb{R}}_2, \overline{\mathrm{P}}^1_1$, and $\overline{\mathrm{P}}^1_2$ are primitive modulo (ρ, τ), and for $\mathrm{K} := \mathrm{K}_{\mathbb{R}}\otimes_{\mathbb{M}^{\mathbb{R}}_2}\mathbb{F}_2$, $i \in \{0,1,2\}$ and $t \in \{1,2\}$

$$\dim_{\mathbb{F}_2}\mathcal{M}(\mathrm{K}, \mathrm{Q}^{\mathbb{R}}_i) = 2 = \dim_{\mathbb{F}_2}\mathcal{M}(\mathrm{K}, \overline{\mathrm{P}}^1_t),$$

it follows from Proposition A.5 that

$$\mathcal{M}(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)\otimes_{\mathbb{M}^{\mathbb{R}}_2}\mathbb{F}_2, x) \cong \mathcal{M}(\overline{e}(\mathrm{K}^{\otimes 6}), x) \cong \overline{e}(\mathcal{M}(\mathrm{K}, x)^{\otimes 6}) = 0$$

for $x \in \{\mathrm{Q}^{\mathbb{R}}_0, \mathrm{Q}^{\mathbb{R}}_1, \overline{\mathrm{P}}^1_1, \mathrm{Q}^{\mathbb{R}}_2, \overline{\mathrm{P}}^1_2\}$. Thus, $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ is free over $\mathcal{A}^{\mathbb{R}}(2)$. Proposition A.5 also implies that the $\mathbb{M}^{\mathbb{R}}_2$-rank of $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ is 64, and therefore $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$ has rank 1 over $\mathcal{A}^{\mathbb{R}}(2)$. □

An $\mathbb{R}$-motivic lift of $\mathcal{B}(2)$

Let $\tilde{\mathrm{Q}}^{\mathbb{R}}_2 = [\mathrm{Sq}^4, \mathrm{Q}^{\mathbb{R}}_1]$. Unlike the classical Steenrod algebra, $\mathrm{Q}^{\mathbb{R}}_2$ does not agree with $\tilde{\mathrm{Q}}^{\mathbb{R}}_2$. Instead, as in [54, Example 13.7], these are related by the formula

$$\mathrm{Q}^{\mathbb{R}}_2 = [\mathrm{Sq}^4, \mathrm{Q}^{\mathbb{R}}_1] + \rho\mathrm{Sq}^5\mathrm{Sq}^1.$$

However, one can check that both $\mathrm{Q}^{\mathbb{R}}_2$ and $\tilde{\mathrm{Q}}^{\mathbb{R}}_2$ square to zero, hence generate exterior algebras. We define (left) $\mathcal{A}^{\mathbb{R}}(2)$-modules

$$\mathcal{B}^{\mathbb{R}}(2) := \mathcal{A}^{\mathbb{R}}(2)\otimes_{\Lambda(\mathrm{Q}^{\mathbb{R}}_2)}\mathbb{M}^{\mathbb{R}}_2$$

and

$$\tilde{\mathcal{B}}^{\mathbb{R}}(2) := \mathcal{A}^{\mathbb{R}}(2)\otimes_{\Lambda(\tilde{\mathrm{Q}}^{\mathbb{R}}_2)}\mathbb{M}^{\mathbb{R}}_2. \tag{A.7}$$

Let $\mathrm{A}^{\mathbb{R}}_2$ denote $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{A}^{\mathbb{R}}_2)$. It is easy to check that the left ideal generated by $\mathrm{Q}^{\mathbb{R}}_2$ (likewise $\tilde{\mathrm{Q}}^{\mathbb{R}}_2$) in $\mathcal{A}^{\mathbb{R}}(2)$ is isomorphic to $\Sigma^{7,3}\mathcal{B}^{\mathbb{R}}(2)$ (likewise $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$). It follows that there is an exact sequence of $\mathcal{A}^{\mathbb{R}}(2)$-modules

$$0 \longrightarrow \Sigma^{7,3}\mathrm{B}_{\mathbb{R}} \xhookrightarrow{\iota} \mathrm{A}^{\mathbb{R}}_2 \xrightarrow{\pi\!\!\!\!\rightarrow} \mathrm{B}_{\mathbb{R}} \longrightarrow 0, \tag{A.8}$$

where $\mathrm{B}_{\mathbb{R}}$ is either $\mathcal{B}^{\mathbb{R}}(2)$ or $\tilde{\mathcal{B}}^{\mathbb{R}}(2)$. The main purpose of this subsection is to show that:

Lemma A.9. *There exists an exact sequence of $\mathcal{A}^{\mathbb{R}}$-modules whose underlying $\mathcal{A}^{\mathbb{R}}(2)$-module exact sequence is isomorphic to* (A.8) *with* $\mathrm{B}_{\mathbb{R}} \cong \tilde{\mathcal{B}}^{\mathbb{R}}(2)$.

Remark A.10. *In the case of* $\mathrm{B}_{\mathbb{R}} = \mathcal{B}^{\mathbb{R}}(2)$, *the image of* $\Sigma^{7,1}\mathcal{B}^{\mathbb{R}}(2) \longrightarrow \mathrm{A}_2^{\mathbb{R}}$ *is a sub-$\mathcal{A}^{\mathbb{R}}(2)$-module, but not a sub-$\mathcal{A}^{\mathbb{R}}$-module. See Remark A.14 for more details.*

Lemma A.9 and Remark A.10 are direct consequences of the $\mathcal{A}^{\mathbb{R}}$-module structure of $\mathrm{A}_2^{\mathbb{R}}$ which can be deduced from the injection

$$\Sigma^{5,1}\mathrm{A}_2^{\mathbb{R}} \hookrightarrow \mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}}6},$$

where $\mathrm{K}_{\mathbb{R}} = \mathrm{H}_{\mathbb{R}}^{*,*}(\mathcal{K}_{\mathbb{R}})$. We do not want to entirely leave this calculation to the reader because, without a few tricks, this calculation is likely to require computer assistance as e has 144 elements in its expression (in terms of the standard $\mathbb{F}_2$-generators of $\mathbb{F}_2[\Sigma_3]$) and $\mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}}6}$ has 2^{12} elements in its $\mathbb{M}_2^{\mathbb{R}}$-basis. We begin after setting the following notation.

Notation A.11. *Let x_i denote the $\mathbb{M}_2^{\mathbb{R}}$-generators of $\mathrm{K}_{\mathbb{R}}$ in degree i as in Corollary A.3. We use the numbered Young diagram (abbrev. NYD)*

$$\begin{array}{|c|c|c|}\hline i_1 & i_2 & i_3 \\ \hline i_4 & i_5 \\ \cline{1-2} i_6 \\ \cline{1-1}\end{array}$$

to denote the $\mathbb{M}_2^{\mathbb{R}}$-basis element $x_{i_1} \otimes \cdots \otimes x_{i_6} \in \mathrm{K}_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}}6}$, where $i_j \in \{0,1,3,7\}$.

As in Proposition A.5, let $\overline{\mathsf{R}}$ and $\overline{\mathsf{C}}$ denote the images of R and C (see (A.4)) in $\mathbb{F}_2[\Sigma_6]$, respectively. Since $\overline{e} = \overline{\mathsf{R}} \cdot \overline{\mathsf{C}}$, we record a few properties of $\overline{\mathsf{R}}$ and $\overline{\mathsf{C}}$. Note that $\overline{\mathsf{R}}$ annihilates an NYD if it has repeating digits in a row. Likewise, $\overline{\mathsf{C}}$ annihilates an NYD if there are repeating digits in a column. For instance,

$$\overline{\mathsf{R}}\left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 3 & 7 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right) = 0 = \overline{\mathsf{C}}\left(\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 3 & 7 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\right).$$

Remark A.12. *The lowest degree NYD which is not annihilated by $\overline{e}$ is*

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

which lives in degree $(5,1)$. *Of course, there are multiple NYD's in bidegree* $(5,1)$ *not annihilated by $\overline{e}$ but their images are the same. Likewise, the NYD of the highest degree not annihilated by $\overline{e}$ is*

$$\begin{array}{|c|c|c|}\hline 7 & 7 & 7 \\ \hline 3 & 3 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

which lives in bidegree $(28,11)$.

The lowest degree element $\iota := \overline{e}($  $)$, which serves as the $\mathcal{A}^{\mathbb{R}}$-module generator of $\Sigma^{5,1}\mathrm{A}_2^{\mathbb{R}}$, can also be expressed as

$$\iota = \overline{\mathsf{R}}($$ $$)$$

because the other NYDs present in the expression $\overline{\mathsf{C}}($  $)$ are annihilated by $\overline{\mathsf{R}}$.

Since the $\mathbb{R}$-motivic Steenrod algebra is cocommutative we get

$$\overline{\mathsf{R}}(\overline{\mathsf{C}}(\mathrm{Sq}^i(-))) = \overline{\mathsf{R}}(\mathrm{Sq}^i(\overline{\mathsf{C}}(-))) = \mathrm{Sq}^i(\overline{\mathsf{R}}(\overline{\mathsf{C}}(-))).$$

This, along with the Cartan formula, allows us to calculate $a \cdot \iota$ for any $a \in \mathcal{A}^{\mathbb{R}}$, fairly easily. For example,

$\mathrm{Sq}^1 \cdot \iota$ = $\overline{\mathsf{R}}(\mathrm{Sq}^1$  $)$

= $\overline{\mathsf{R}}($  $+$  $+$  $)$

= $\overline{\mathsf{R}}($  $),$

$\mathrm{Sq}^2 \cdot \iota$ = $\overline{\mathsf{R}}(\mathrm{Sq}^2$  $)$

= $\overline{\mathsf{R}}($  $+$  $+\tau($  $+$  $+$  $))$

= $\overline{\mathsf{R}}($  $),$

$\mathrm{Sq}^4 \cdot \iota$ = $\overline{\mathsf{R}}(\mathrm{Sq}^4($  $))$

= $\overline{\mathsf{R}}($  $+$  $+\tau($  $+$  $+$  $)$

$+\tau($  $+$  $+$  $))$

= $\overline{\mathsf{R}}($  $+\tau$  $).$

In this way, we calculate

$\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota$ = $\overline{\mathsf{R}}($  $+$  $+$  $)$

$\mathrm{Q}_2^{\mathbb{R}} \cdot \iota$ = $\overline{\mathsf{R}}($  $+$  $+$  $+\rho$  $),$

where the details are left to the reader.

Remark A.13. *We record (see Figure A2, in which black dots correspond to generators of $\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ and orange dots to $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$), in the notation introduced in Subsection 4.1.1, that*

$$\begin{aligned}\mathcal{D}_{\tilde{\mathcal{B}}^{\mathbb{R}}(2)} = \ & \{(0,0),(1,0),(2,1),(3,1),(4,1),(4,2),(5,2),(6,2),(6,3),(7,3),(8,3),(8,4),\\ & (9,4),(10,4),(10,5),(11,5),(12,5),(12,6),(13,6),(14,6),(15,7),(16,7)\}\end{aligned}$$

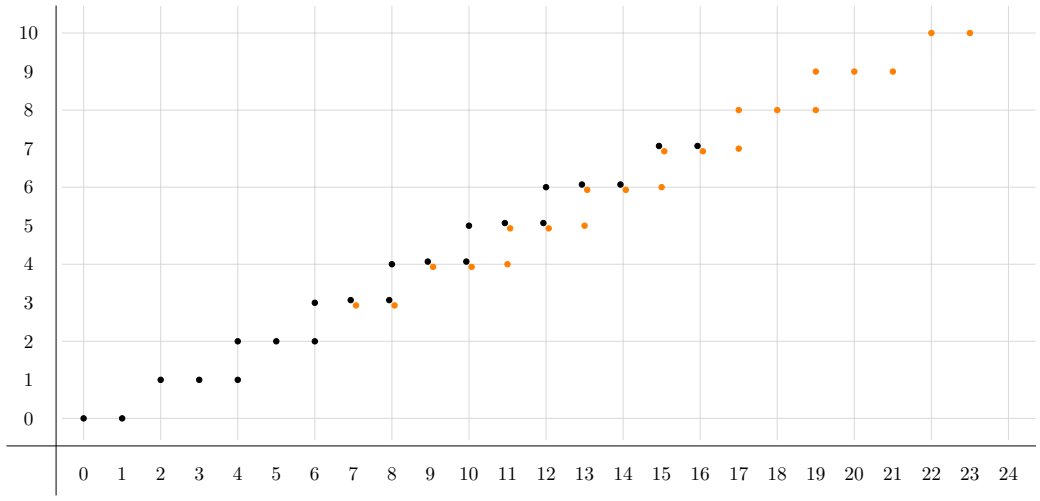


Figure A2: $\mathbb{M}_2^{\mathbb{R}}$-module generators of $\mathcal{A}^{\mathbb{R}}(2)$

and $\mathcal{D}_{\mathrm{A}_2^{\mathbb{R}}} = \{(i+7\epsilon, j+3\epsilon) : (i,j) \in \mathcal{D}_{\tilde{\mathcal{B}}^{\mathbb{R}}(2)} \text{ and } \epsilon \in \{0,1\}\}$.

Proof of Lemma A.9. Recall that the image of $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ in (A.8) is the (left) $\mathcal{A}^{\mathbb{R}}(2)$-submodule of $\mathrm{A}_2^{\mathbb{R}}$ generated by $\tilde{\mathrm{Q}}_2^{\mathbb{R}}$. We must check that this is closed under the action of $\mathcal{A}^{\mathbb{R}}$. Since $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4$ are in $\mathcal{A}^{\mathbb{R}}(2)$, it remains to check that for all $i \geq 3$ and $a \in \mathcal{A}^{\mathbb{R}}(2)$

$$\mathrm{Sq}^{2^i} \cdot (a\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) = b\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota$$

for some $b \in \mathcal{A}^{\mathbb{R}}(2)$. For degree reasons (see Remark A.13), we only need to consider the case when $i = 3$ and $a \in \{1, \mathrm{Sq}^1, \mathrm{Sq}^2\}$. We check

$$\begin{aligned}
\mathrm{Sq}^8 \cdot (\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) &= (\mathrm{Sq}^4\mathrm{Sq}^4 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^2)\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota \\
\mathrm{Sq}^8 \cdot (\mathrm{Sq}^1\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) &= (\mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^7)\mathrm{Sq}^1\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota \\
\mathrm{Sq}^8 \cdot (\mathrm{Sq}^2\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota) &= (\mathrm{Sq}^4\mathrm{Sq}^4\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^4 + \tau\mathrm{Sq}^5\mathrm{Sq}^4\mathrm{Sq}^1)\mathrm{Sq}^2\tilde{\mathrm{Q}}_2^{\mathbb{R}} \cdot \iota
\end{aligned}$$

and thus the result holds. □

Remark A.14. *We notice that*

$$\mathrm{Sq}^8 \cdot (\mathrm{Q}_2^{\mathbb{R}} \cdot \iota) = \overline{R}(\quad + \quad + \tau \quad + \rho(\quad + \quad + \quad))$$

cannot be equal to $b\mathrm{Q}_2^{\mathbb{R}} \cdot \iota$ *for any* $b \in \mathcal{A}^{\mathbb{R}}(2)$. *This is an easy but tedious calculation. For the convenience of the reader, we note that an* $\mathbb{F}_2$*-basis for the elements in degree* $|\mathrm{Sq}^8| = (8,4)$ *of* $\mathcal{A}^{\mathbb{R}}(2)$ *is given by*

$$\{\mathrm{Sq}^6\mathrm{Sq}^2, \tau\mathrm{Sq}^7\mathrm{Sq}^1, \tau\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1, \rho\mathrm{Sq}^7, \rho\mathrm{Sq}^6\mathrm{Sq}^1, \rho\mathrm{Sq}^5\mathrm{Sq}^2, \rho\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1, \rho^2\mathrm{Sq}^5\mathrm{Sq}^1\}.$$

The construction of $\mathcal{Z}_{\mathbb{R}}$

Recall the $\mathcal{A}^{\mathbb{R}}$-module $\tilde{\mathrm{B}}_2^{\mathbb{R}}$, as given in (A.7), and let

$$\mathrm{B}_2^{\mathbb{C}} := \tilde{\mathrm{B}}_2^{\mathbb{R}}/(\rho).$$

Proof of Theorem A.1. Since $\mathrm{B}_2^{\mathbb{C}}$ is cyclic as an $\mathcal{A}^{\mathbb{C}}$-module, it admits a May filtration, whose associated graded is isomorphic to

$$\mathrm{gr}(\mathrm{B}_2^{\mathbb{C}}) \cong \Lambda(\xi_{1,0}, \xi_{1,1}, \xi_{1,2}, \xi_{2,0}, \xi_{2,1})$$

and whose E_2-page of the corresponding May spectral sequence is isomorphic to

$${}^{\mathrm{May}}\mathrm{E}^{*,*,*,*}_{1,\mathrm{B}_2^{\mathbb{C}}} \cong \frac{\mathbb{M}_2^{\mathbb{C}}[\mathsf{h}_{i,j} : i \geq 1, j \geq 0]}{(\mathsf{h}_{1,0}, \mathsf{h}_{1,1}, \mathsf{h}_{1,2}, \mathsf{h}_{2,0}, \mathsf{h}_{2,1})}. \tag{A.15}$$

From this and Remark A.13, one easily checks that the condition for Theorem 4.1.11 is satisfied. Thus, there exists $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}^{\mathbb{R}}_{2,\mathrm{fin}}$ such that $\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Z}_{\mathbb{R}}) \cong \tilde{\mathrm{B}}_2^{\mathbb{R}}$. □

Remark A.16. *Since, as an $\mathcal{A}(2)$-module*

$$\mathrm{H}^*(\Phi^e(\beta(\mathcal{Z}_{\mathbb{R}}))) \cong \Phi^e_*(\beta_*(\mathrm{H}^{*,*}_{\mathbb{R}}(\mathcal{Z}_{\mathbb{R}}))) \cong \mathcal{B}(2),$$

the underlying spectrum of $\beta(\mathcal{Z}_{\mathbb{R}})$ is indeed one of the spectra $\mathcal{Z}$ considered in [14], and therefore of type 2.

Appendix B: The $\mathbb{R}$-motivic Adem relations

Voevodsky established the motivic version of the Adem relations [54, Section 10]. However, his formulas contain some typos, so for the convenience of the reader, we here present the Adem relations, in the $\mathbb{R}$-motivic case.

Proposition B.1. *In the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$, the product $\mathrm{Sq}^a\mathrm{Sq}^b$ is equal to*

1. *(a and b both even)*

$$\sum_{j=0}^{a/2} \tau^{j \bmod 2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j$$

2. *(a odd and b even)*

$$\sum_{j=0}^{(a-1)/2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j + \rho\binom{b-j}{a-2j} \mathrm{Sq}^{a+b-j-1}\mathrm{Sq}^j.$$

3. *(a even and b odd)*

$$\sum_{j=0}^{a/2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j + \rho\binom{b-1-j}{a+1-2j} \mathrm{Sq}^{a+b-j-1}\mathrm{Sq}^j.$$

4. (a and b both odd)

$$\sum_{j=0}^{(a-1)/2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^{j}$$

Remark B.2. *Given that* $\mathrm{Sq}^a = \mathrm{Sq}^1 \mathrm{Sq}^{a-1}$ *if* a *is odd and also that* $\mathrm{Sq}^1(\tau) = \rho$*, cases (2) and (4) follow from (1) and (3), respectively. Note also that (1) is the classical formula, but with* τ *thrown in whenever needed to balance the weights. In formula (2), the left term appears only when* j *is even, while the second appears only when* j *is odd. In formula (3), the second term appears only when* j *is odd.*

Example B.3. *Some examples of the* $\mathbb{R}$*-motivic Adem relation in low degrees are*

$$\mathrm{Sq}^2 \mathrm{Sq}^2 = \tau \mathrm{Sq}^3 \mathrm{Sq}^1, \qquad \mathrm{Sq}^3 \mathrm{Sq}^2 = \rho \mathrm{Sq}^3 \mathrm{Sq}^1,$$

and

$$\mathrm{Sq}^2 \mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1 + \rho \mathrm{Sq}^3 \mathrm{Sq}^1.$$

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- An $\mathbb{R}$-motivic v_1-self-map of periodicity 1
  Joint work with Prasit Bhattacharya and Bertrand Guillou
  To appear in Homology, Homotopy, and Applications. Available on the ArXiv: `https://arxiv.org/abs/2008.05547`

- The v_1-Periodic Region in the cohomology of the $\mathbb{C}$-motivic Steenrod algebra

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