2021

Some Proofs Regarding Minami Estimates and Local Eigenvalue Statistics for some Random Schrödinger Operator Models

Samuel Herschenfeld
University of Kentucky, samhersche@gmail.com
Digital Object Identifier: https://doi.org/10.13023/etd.2021.409

Samuel Herschenfeld, Student

Dr. Peter Hislop, Major Professor

Dr. Benjamin Braun, Director of Graduate Studies

Some Proofs Regarding Minami Estimates and Local Eigenvalue Statistics for some Random Schrödinger Operator Models

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Samuel Herschenfeld
Lexington, Kentucky

Director: Dr. Peter Hislop, Professor of Mathematics
Lexington, Kentucky
2021

ABSTRACT OF DISSERTATION

Some Proofs Regarding Minami Estimates and Local Eigenvalue Statistics for some Random Schrödinger Operator Models

We provide three proofs on different, but related models in the field of random Schrödinger operators. All three results are motivated by the desire to extend results and techniques on eigenvalue statistics or Minami estimates (an essential ingredient Poisson eigenvalue statistics).

Chapters 2 and 4 are explorations of the only two known techniques for proving Minami estimates for continuum Minami estimates. In Chapter 2, we provide an alternative and simplified proof of Klopp that holds in $d = 1$. Chapter 4 is an application of the techniques of Dietlein and Elgart to prove a Minami estimate for finite rank lattice models, which is an improvement on known results. Chapter 3 is an improvement on a result of Dolai and Krishna, in which we show the statistics for a RSO with a decaying potential is the same as the free Laplacian for decay down to $|n|^{-\alpha}$ for $\alpha > 1$.

The first chapter is a brief, general introduction to the Anderson model and relevant concepts in the field random Schrödinger operators. Each subsequent chapter has a more specific introduction before proceeding with the results.

KEYWORDS: mathematical physics, Anderson model, Minami estimate, eigenvalue statistics, decaying potential

Samuel Herschenfeld

October 4, 2021

Some Proofs Regarding Minami Estimates and Local Eigenvalue Statistics for some Random Schrödinger Operator Models

By
Samuel Herschenfeld

Dr. Peter Hislop
Director of Dissertation

Dr. Benjamin Braun
Director of Graduate Studies

October 4, 2021
Date

Dedicated to Ben Herschenfeld, who supported and inspired me my whole life.

ACKNOWLEDGMENTS

I'd like to thank my adviser, Dr. Peter Hislop, for his support, advice and collaboration which he gave with exceeding generosity.

I'd also like to thank the rest of the math department of the University of Kentucky who contributed to my education in mathematics, faculty, staff, and students. I benefited from being part of whole analysis and PDE research group and in particular I appreciate the many discussions I had with Ben Brodie.

My research and the writing of my dissertation were interrupted by COVID-19 in 2020. There were many people who advocated for a variety of kinds of support for all the researchers and graduate students in a similar position when the university closed down and many of us were isolated. From the university, I'd like to thank David Blackwell, Lisa Cassis and the committee that provided them with a proposal for support. From the department, I'd like to again thank my adviser and Peter Hislop, and the director of graduate studies Ben Braun, who both advocated for me personally.

Finally, I'd like to thank my family. I would never have accomplished any of this without the support of my parents, grandparents, brother and extended family.

TABLE OF CONTENTS

Acknowledgments iii

Chapter 1 Introduction and Preliminaries 1
1.1 Random Schrödinger Operators 1
1.2 Localization 2
1.3 Eigenvalue Statistics 3
1.4 Spectral Averaging and Wegner Estimates 4
1.5 Minami Estimates and Applications 6

Chapter 2 A Minami Estimate in $\mathbb{R}$ by Inverse Tunneling - a Simple Proof . 9
2.1 Introduction 9
2.2 Model and Results 10
2.2.1 Model 10
2.2.2 Results 11
2.3 Proof of the Minami-type estimate Without Localization 11
2.4 Improving the Minami-Type Estimate Using Localization 17
2.5 Conclusion and Further Questions 19

Chapter 3 Eigenvalue Statistics for Lattice RSO's with Decaying Potentials . 21
3.1 Inroduction 21
3.2 Model and Main Theorem 22
3.3 Proof of Main Theorem 24
3.4 Some Spectral Properties of H_0^L 26
3.5 Conclusion and Further Questions 31

Chapter 4 A Minami Estimate for a Finite Rank Lattice Model 32
4.1 Introduction 32
4.2 Model 33
4.3 Functional Analytic Lemmas 34
4.4 Eigenvalue Spacing and Minami Estimates 36
4.5 Discrete Laplacian Boundary Conditions and Eigenvalues 44
4.6 Conclusion and Further Questions 46

Bibliography 47

Vita 50

Chapter 1 Introduction and Preliminaries

We begin by laying out the organization of this thesis. In this introductory chapter, we will give a brief introduction to the field of random Schrödinger operators (RSO) with a focus on concepts relevant to the remainder of this thesis.

The next three chapters each give a new result or proof or application on a different random Schrödinger model. The common theme between these different topics is eigenvalue statistics and the Minami estimate (an essential ingredient to proofs of Poisson eigenvalue statistics).

Chapter 2 discusses a Minami estimate for a RSO model on $\mathbb{R}$ first proven by Klopp [22] and provides a new and simpler proof.

Chapter 3 is inspired by a result of Dolai and Krishna [13] that investigates eigenvalue statistics for a model with random, decaying potential. We provide an improvement of this theorem.

Chapter 4 provides a discussion of the techniques of Dietlein and Elgart [12], the first proof of a Minami Estimate for a RSO model in $\mathbb{R}^d$, $d > 1$. We focus on the essential ingredients and give an appication to a discrete, alloy type model with single site potential uniform but high rank that improves a previously known result of Hislop and Krishna [17] and provide a framework for how the techniques may be used for more models.

1.1 Random Schrödinger Operators

The study of random Schrödinger operators came about in the physics community in order to study the properties of electrons in a random medium. Random Schrödinger operators are really collections of operators, $\{H_\omega\}_\omega$ on some Hilbert space $\mathcal{H}$, where ω varies over some probability space. There are many ways to realize the randomness in a specific model, but we will only discuss models in which we have a deterministic, self-adjoint free operator, H_0 and a random potential, V_ω so that each realization (i.e. for each realization of ω) is a self-adjoint operator,

$$H_0 + V_\omega =: H_\omega \ : \ \mathcal{H} \mapsto \mathcal{H} \tag{1.1.1}$$

The simplest RSO model is the **Anderson tight binding model** on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, in which H_0 is the centered, discrete, nearest neighbor Laplacian and V_ω consists of an independent randomly scaled projection onto each site in $\mathbb{Z}^d$,

$$\begin{aligned} H_0 f(n) &:= \sum_{|m-n|=1} f(m) \\ V_\omega &:= \sum_{n\in\mathbb{Z}^d} \omega_n \langle \delta_n, \cdot \rangle \delta_n \end{aligned} \tag{1.1.2}$$

where $\delta_n \in \ell^2(\mathbb{Z}^d)$ is the vector that is 1 at n and 0 everywhere else, and $\{\omega_n\}_{n\in\mathbb{Z}^d}$ is a collection of independent and identically distributed (iid), real valued random variables. This model is named after physicist Philip Anderson, who introduced it in [1].

The **continuum Anderson model** is a model on $\mathcal{H} = L^2(\mathbb{R}^d)$. The free operator is the positive Laplacian and the potential consists of multiplication by some real-valued function, $v : \mathbb{R}^d \mapsto \mathbb{R}$, randomly scaled and centered at each point in $\mathbb{Z}^d$,

$$\begin{aligned} H_0 &:= -\Delta \\ V_\omega(x) &:= \sum_{n\in\mathbb{Z}^d} \omega_n v(x-n) \end{aligned} \tag{1.1.3}$$

where $\{\omega_n\}_{n\in\mathbb{Z}^d}$ is a collection of iid, real valued, random variables.

For simplicity, we assume ω_n is associated to the probability space $(\mathbb{R}, \rho)$ for some probability density, ρ. Then we denote the probability space for $\omega := (\omega_n)_{n\in\mathbb{Z}^d}$ as $\Omega \sim (\text{supp}\rho)^{\mathbb{Z}^d}$, the infinite product space whose existence is guaranteed by the Kolmogorov extension theorem. We emphasize here that the independence of $\{\omega_n\}$ implies that the probability distribution on Ω, is a product measure,

$$\begin{aligned} \mathbb{P}[E] &:= \int_{(\text{supp}\rho)^{\mathbb{Z}^d}} \mathbb{1}_E(\omega) \prod_{n\in\mathbb{Z}^d} \mathrm{d}\rho(\omega_n) \\ \mathbb{E}[F(\omega)] &:= \int_{(\text{supp}\rho)^{\mathbb{Z}^d}} F(\omega) \prod_{n\in\mathbb{Z}^d} \mathrm{d}\rho(\omega_n) \end{aligned}$$

where $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$.

There are many other random operator models that are studied: on different Hilbert spaces, different free operators, H_0, different realizations of the randomness. However, the models we will discuss in the rest of this thesis will be one of the above operators with possible variations on the potential.

Finally, we describe a standard fact about the spectrum of the above operators that makes our discussion and notation simpler, though it is not essential to any of our proofs. It is clear that for each ω, H_ω may have a different spectrum, $\sigma(H_\omega)$, though it is clear we have,

$$\sigma(H_\omega) \subset \sigma(H_0) + [-\sup_\omega \|V_\omega\|, \sup_\omega \|V_\omega\|]$$

However, we may say something stronger. The above two operators are examples of ergodic families of operators. This implies [16] there exists a deterministic set, $\Sigma \subset \mathbb{R}$, such that,

$$\sigma(H_\omega) = \Sigma \text{ almost surely.}$$

1.2 Localization

One of the main reasons random Schrödinger operator models are of interest is the phenomenon of localization. It is known that both the above models exhibit localization. There are a few mathematical expressions of localization.

Spectral localization refers to the existence of an interval (or union of intervals), $\Sigma_{\text{loc}} \subset \Sigma$, the deterministic spectrum defined in Section 1.1, such that with probability one, Σ_{loc} consists entirely of pure point spectrum. This means that spectrum in Σ_{loc} consists of a countable collection of dense eigenvalues. Furthermore, the eigenfunctions associated to the eigenvalues in Σ_{loc} decay exponentially in space. This is often referred to as Anderson Localization.

The first proofs of Anderson localization were given in [14], [10] and [27]. There have been many results since (see the reviews [16] and [19]).

Specifically, for the Anderson models above, localization is known to hold in the following cases,

- $d = 1$ at all energies
- $d > 1$ at any energy for sufficiently large potential
- $d > 1$ at the band edges or the bottom of the spectrum for the lattice or the continuum models respectively.

We will only explicitly use, without proof, a form of localization for finite volume operators from [15] (Lemma 2.4.1).

1.3 Eigenvalue Statistics

Another feature exhibited by RSO models related to localization involves the limiting behavior of the rescaled eigenvalues of H_ω restricted to a finite volume box near a fixed energy.

Let Λ_L be a box of length L centered at 0 in $\mathbb{Z}^d$ or $\mathbb{R}^d$ for the discrete, or respectively, continuum Anderson model. For the purposes of this section, we consider the following cutoff operators,

$$\begin{aligned} H_\omega^L &:= \chi_{\Lambda_L} H_\omega \chi_{\Lambda_L} && \text{(discrete)} \\ H_\omega^L &:= -\Delta_{\Lambda_L}^{(D)} + \chi_{\Lambda_L} V_\omega && \text{(continuum)} \end{aligned} \tag{1.3.1}$$

In either case, the spectrum of H_ω^L consists of discrete eigenvalues.

In the rest of this thesis, we will use the notation $\chi_B(H)$ to denote the spectral projection of the operator H onto the set B. Thus, for operators with discrete spectrum like the cutoff operators above,

$$\operatorname{tr} \chi_B(H_\omega^L) = |\sigma(H_\omega^L) \cap B| = \#\text{ of eigenvalues of } H_\omega^L \text{ in } B \tag{1.3.2}$$

For $E \in \mathbb{R}$, define $\xi_{\omega,E}^L$ as the random point measure on $\mathbb{R}$ with weights at the rescaled eigenvalues of $H_\omega^L - E$,

$$\xi_{\omega,E}^L := \sum_{E_\omega^L \in \sigma(H_\omega^L)} \delta(x - |\Lambda_L|(E_\omega^L - E)) \tag{1.3.3}$$

Or, for a Borel set, $B \subset \mathbb{R}$,

$$\xi_{\omega,E}^L(B) := \operatorname{tr} \chi_{E+|\Lambda_L|^{-1}B}(H_\omega^L) \tag{1.3.4}$$

$\{\xi^L_{\omega,E}\}_\omega$ is a random point process on $\mathbb{R}$.

We note that in the definition of $\xi^L_{\omega,E}$, the eigenvalues are scaled by $|\Lambda_L|$ because $|\Lambda_L|^{-1}$ is proportional to the average eigenvalue spacing for large L. For example, this scaling and the Wegner estimate (see Section 1.4) guarantees that for any fixed $B \subset \mathbb{R}$,

$$\sup_L \mathbb{E}[\xi^L_{\omega,E}(B)] < \infty \tag{1.3.5}$$

Generally, for the Anderson models above, if there exists $\Sigma_{\text{loc,M}}$ such that both spectral localization and a Minami estimate hold, then for almost every $E \in \Sigma_{\text{loc,M}}$, $\{\xi^L_{\omega,E}\}_L$ converges weakly to a Poisson point process with intensity equal to the density of states, $n(E)$ as $L \to \infty$. For the Anderson, tight binding model, Poisson local eigenvalue statistics were originally proven by Minami in [24].

1.4 Spectral Averaging and Wegner Estimates

Here, we include two fundamental results in the field of Random Schrödinger operators: spectral averaging and the Wegner estimate.

Spectral averaging is an important technique that estimates quantities like the average of the matrix element of the spectral projection of a one-parameter family of operators, $H_0 + \omega V$ for some $V > 0$ and a real, random variable ω,

$$\mathbb{E}[\langle \psi, V^{1/2} \chi_I(H_\omega) V^{1/2} \psi \rangle] \tag{1.4.1}$$

There are many variations on spectral averaging depending on the context and what assumptions are made on the operators. Usually, the proof involves changing the average over the parameter ω into an average of a spectral parameter over the interval I.

The use of spectral averaging in the field of random Schrödinger operators goes back to Kotani [23] and Simon and Wolff [27] with many more advancements in other sources.

Below, we include a spectral averaging theorem and proof in the simplest context, in which V is a rank-one operator. For more discussions of spectral averaging and proofs for more general models, see [19] and [16].

Theorem 1.4.1 (Spectral Averaging)**.** *Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $\phi \in \mathcal{H}$, with $\|\phi\| = 1$ and let H_ω be a rank one perturbation of H_0,*

$$H_\omega - H_0 = \omega P_\phi := \omega \langle \phi, \cdot \rangle \phi \tag{1.4.2}$$

Then for any interval $I \subset \mathbb{R}$,

$$\int_{\mathbb{R}} \langle \phi, \chi_{[a,b]}(H_\omega) \phi \rangle d\omega \leq |I| \tag{1.4.3}$$

As a consequence, if ω is a random variable with bounded probability density ρ, then,

$$\mathbb{E}\langle \phi, \chi_I(H_\omega)\phi \rangle \leq \|\rho\|_\infty |I| \tag{1.4.4}$$

Proof. We follow the proof from [5]. From the second resolvent identity, and using that P_ϕ is a rank one operator, we get,

$$\begin{aligned}(H_0 - z)^{-1}\phi - (H_\omega - z)^{-1}\phi &= \omega(H_\omega - z)^{-1}P_\phi(H_0 - z)^{-1}\phi \\ &= \omega(H_\omega - z)^{-1}\langle\phi, (H_0 - z)^{-1}\phi\rangle\phi\end{aligned} \tag{1.4.5}$$

Taking inner product with ϕ and rearranging yields,

$$\langle\phi, (H_\omega - z)^{-1}\phi\rangle = \frac{1}{\langle\phi, (H_0 - z)^{-1}\phi\rangle^{-1} + \omega} \tag{1.4.6}$$

Let $\langle\phi, (H_0 - z)^{-1}\phi\rangle^{-1} = x + iy$. Then,

$$\begin{aligned}\int_{\mathbb{R}} \operatorname{Im}\langle\phi, (H_\omega - z)^{-1}\phi\rangle d\omega &= \int_{\mathbb{R}} \frac{y}{(x+\omega)^2 + y^2} d\omega \\ &= \pi\end{aligned} \tag{1.4.7}$$

To get a bound on on the spectral projection, we use Stone's formula,

$$\langle\phi, \chi_I(H_\omega)\phi\rangle \leq \lim_{\delta\to 0^+} \frac{1}{\pi}\int_a^b \langle\phi, \operatorname{Im}(H - E - i\delta)^{-1}\phi\rangle dE \tag{1.4.8}$$

So, integrating over ω yields,

$$\int_{\mathbb{R}} \langle\phi, \chi_{[a,b]}(H_\omega)\phi\rangle d\omega \leq |I| \tag{1.4.9}$$

Using that ρ is bounded, (1.4.9) implies,

$$\begin{aligned}\mathbb{E}\langle\phi, \chi_I(H_\omega)\phi\rangle &= \int_{\mathbb{R}} \rho(\omega)\langle\phi, \chi_I(H_\omega)\phi\rangle d\omega \\ &\leq \|\rho\|_\infty |I|\end{aligned} \tag{1.4.10}$$

□

The Wegner estimate is a first result on the distribution of the eigenvalues of the cutoff RSO model, H_ω^L and an essential ingredient to proofs of localization. It is also important context for understanding the Minami estimate and will be essential to proofs in chapter 2. We include here the standard proof for the tight-binding model, for which the Wegner estimate is an immediate consequence of spectral averaging result stated above. For a longer discussion and proofs for some continuum models, see [16].

Theorem 1.4.2 (Wegner Estimate)**.** *Let H_ω be the discrete Anderson model from Section 1.1. Assume $\{\omega_n\}$ have a common, bounded probability density ρ . Then, for any interval $I \subset \mathbb{R}$*

$$\mathbb{P}[\operatorname{tr}\chi_I(H_\omega^L) \geq 1] \leq \|\rho\|_\infty |\Lambda_L||I| \tag{1.4.11}$$

Proof. For this model, the Wegner estimate is an immediate consequence of spectral averaging,

$$\begin{aligned} \mathbb{P}[\operatorname{tr}\chi_I(H_\omega^L) \geq 1] &\leq \mathbb{E}[\operatorname{tr}\chi_I(H_\omega^L)] \\ &= \sum_{n\in\Lambda_L} \mathbb{E}\langle \delta_n, \chi_I(H_\omega^L)\delta_n\rangle \\ &= \sum_{n\in\Lambda_L} \mathbb{E}_{\omega_n^\perp}\mathbb{E}_{\omega_n}\langle \delta_n, \chi_I(H_\omega^L)\delta_n\rangle \\ &\leq \|\rho\|_\infty |\Lambda||I| \end{aligned} \tag{1.4.12}$$

We used the notation $\mathbb{E}_{\omega_n}$ to denote the expectation with respect to just ω_n and and $\mathbb{E}_{\omega_n^\perp}$ to denote the expectation with respect to all other variables, $\{\omega_m\}_{m\neq n}$. Specifically, we take advantage of the independence of the random variables by iterating $\mathbb{E} = \mathbb{E}_{\omega_n^\perp}\mathbb{E}_{\omega_n}$.

□

1.5 Minami Estimates and Applications

Minami estimates, originally proven by Minami in [24] is a bound on the probability that H_ω^L has at least two eigenvalues in a small interval. Below, we provide a simple proof for the discrete Anderson model that is provided in [5]. For more general RSO models, this estimate, which is affected by the correlation between distinct eigenvalues, is more difficult to prove and is the topic of chapters 2 and 4.

Theorem 1.5.1 (Minami Estimate). *Let H_ω be the discrete Anderson model from Section 1.1. Assume $\{\omega_n\}$ have a common, absolutely continuous probability density ρ.*

Then, there exists a constant depending only on ρ such that for any $I \subset \mathbb{R}$,

$$\mathbb{P}[\operatorname{tr}\chi_I(H_\omega^L) \geq 2] \leq C|\Lambda_L|^2|I|^2 \tag{1.5.1}$$

Proof. Let $H_{\omega_n^\perp,\tilde{\omega}_n}$ denote the rank one perturbation of H_ω,

$$H_\omega^L - H_{\omega_n^\perp,\tilde{\omega}_n}^L = (\omega_n - \tilde{\omega}_n)\delta_n \tag{1.5.2}$$

and let $\tilde{\omega}_n$ be a random variable, independent of, and with identical distribution to, ω_n. Weyl's inequality for rank one perturbations implies that for any interval, I,

$$|\operatorname{tr}\chi_I(H_\omega^L) - \operatorname{tr}\chi_I(H_{\omega_n^\perp,\tilde{\omega}_n}^L)| \leq 1 \tag{1.5.3}$$

We then proceed in a similar manner to the proof of the Wegner estimate, and use the trick above to get a product of independent random variables,

$$\mathbb{P}[\,\mathrm{tr}\,\chi_I(H_\omega^L) \geq 2] \leq \mathbb{E}[\,\mathrm{tr}\,\chi_I(H_\omega^L)(\,\mathrm{tr}\,\chi_I(H_\omega^L) - 1)] \tag{1.5.4}$$

$$= \sum_{n\in\Lambda_L} \mathbb{E}_{\omega_n^\perp}\mathbb{E}_{\omega_n}[\langle\delta_n, \chi_I(H_\omega^L)\delta_n\rangle(\,\mathrm{tr}\,\chi_I(H_\omega^L) - 1)] \tag{1.5.5}$$

$$\leq \sum_{n\in\Lambda_L} \mathbb{E}_{\omega_n^\perp,\tilde{\omega}_n}\mathbb{E}_{\omega_n}[\langle\delta_n, \chi_I(H_\omega^L)\delta_n\rangle(\,\mathrm{tr}\,\chi_I(H^L_{\omega_n^\perp,\tilde{\omega}_n})] \tag{1.5.6}$$

$$\leq \sum_{n\in\Lambda_L} \mathbb{E}_{\omega_n^\perp,\tilde{\omega}_n}\,\mathrm{tr}\,\chi_I(H^L_{\omega_n^\perp,\tilde{\omega}_n})\|\rho\||I| \tag{1.5.7}$$

$$\leq (\|\rho\|_\infty|\Lambda||I|)^2 \tag{1.5.8}$$

To go from (1.5.5) to (1.5.6), we used (1.5.3), which importantly, is true for any $\omega_n, \tilde{\omega_n} \in \mathbb{R}$.

□

One of the main applications of a Minami estimate proving the local eigenvalue statistics, $\xi^L_{\omega,E}$ converges weakly to a Poisson point process.

Below, we outline the proof given in [6], [7] to see the role of Minami estimates, and specifically, the fact that weaker forms of the Minami estimate are still sufficient to prove Poisson statistics.

For some $\ell << L$ to be specified later, we divide Λ_L into disjoint boxes of side length ℓ,

$$\Lambda_L = \cup_{j=1}^{M_\ell}\Lambda_{\ell,j} \tag{1.5.9}$$

such that $\{H_\omega^{\Lambda_{\ell,j}}\}$ are independent operators. We define random, point measures for each of these operators,

$$\xi^{\ell,j}_{\omega,E}(I) = \,\mathrm{tr}\,\chi_{E+|\Lambda_L|^{-1}I}(H_\omega^{\Lambda_{\ell,j}}) \tag{1.5.10}$$

We let $\tilde{\xi}^L_{\omega,E} = \sum_j \xi^{\ell,j}_{\omega,E}$. Then, the first step of the proof is to show that $\tilde{\xi}^L_{\omega,E}$ and $\xi^L_{\omega,E}$ have the same limit as $L \to \infty$. For appropriately chosen ℓ, this is a consequence of spectral localization.

The second step is to show that the superposition measure, $\tilde{\xi}^L_{\omega,E}$ satisfies a set of three conditions, which, from the theory of point processes, imply weak convergence to a Poisson point process with intensity, $\nu_E > 0$ [8]. The conditions are that for any bounded, fixed interval, I, the following hold,

$$\lim_{L\to\infty}\sup_j \mathbb{P}[\xi^{\ell,j}_{\omega,E}(I) \geq 1] = 0 \tag{1.5.11}$$

$$\lim_{L\to\infty}\sum_j \mathbb{P}[\xi^{\ell,j}_{\omega,E}(I) \geq 1] = \nu_E|I| \tag{1.5.12}$$

$$\lim_{L\to\infty}\sum_j \mathbb{P}[\xi^{\ell,j}_{\omega,E}(I) \geq 2] = 0 \tag{1.5.13}$$

The Wegner estimate quickly implies (1.5.11). For the second quantity, the intensity is equal to an important quantity called the density of states, $n(E) = \nu_E$. Then, (1.5.12) is a consequence of standard results on density of states, along with (1.5.13). A Minami-type estimate is required to prove (1.5.13). Let us examine a (1.5.13) a bit further. Using (1.5.1),

$$\sum_j \mathbb{P}[\xi^{\ell,j}_{\omega,E}(I) \geq 2] \leq \frac{L^d}{\ell^d} \sup_j \mathbb{P}[\xi^{\ell,j}_{\omega,E}(I) \geq 2] \tag{1.5.14}$$

$$\leq \frac{L^d}{\ell^d} \sup_j \mathbb{P}[\operatorname{tr} \chi_{E+|\Lambda_L|^{-1}I}(H^{\Lambda_{\ell,j}}_\omega) \geq 2] \tag{1.5.15}$$

$$\lesssim \frac{L^d}{\ell^d}\, \ell^{2d}(L^{-d}|I|)^2 \tag{1.5.16}$$

$$\lesssim \ell^d L^{-d} \tag{1.5.17}$$

which goes to 0 as long as $\ell = o(L)$. For the discrete Anderson model above, the choosing $\ell = L^{1/2}$ is sufficient for the proof.

However, the Minami-type estimates proven in Chapters 2 and 4 for models with higher rank perturbations are weaker than the (1.5.1). But they are sufficient to prove (1.5.13) if we choose different values for ℓ.

Chapter 2 A Minami Estimate in $\mathbb{R}$ by Inverse Tunneling - a Simple Proof

2.1 Introduction

The proof given in the first chapter of a Minami estimate for the discrete Anderson model is heavily reliant on the fact that the single site potential is a rank-one operator.

Since Minami's original work in 1996 [24], there have been few generalizations of a Minami-type estimate and Poisson statistics to models with non-rank one perturbations, including continuum Anderson models, for which the single site potential is an infinite rank operator.

In 2014, Klopp [22] proved a Minami-type estimate for a one-dimensional, continuum Anderson model with a new approach that attempts to make rigorous common heuristics about the Minami estimate. In 2018, Deitlein and Elgart [12] proved a Minami-type estimate in $\mathbb{R}^d$ for $d \geq 1$ that holds in an interval at the bottom of the spectrum (in both cases, we say "Minami-type" because the bound acquired is weaker than in the tight-binding model, (1.5.1)). These two proofs are quite different, as is common for the difference between one-dimensional and multi-dimensional random Schrödinger operators. We explore both in Chapters 2 and 4 of this thesis.

This chapter focuses on the $d = 1$ case. We will provide an alternative proof to Klopp's. The proof follows the same general strategy while removing some technical arguments.

Let us discuss here Klopp's strategy, which is to prove that a Minami estimate on $\mathbb{R}$ is a consequence of the Wegner estimate and localization. We recall here the Wegner and Minami estimates,

$$\mathbb{P}[\operatorname{tr}\chi_I(H_\omega^L) \geq 1] \leq C|\Lambda_L||I|$$
$$\mathbb{P}[\operatorname{tr}\chi_I(H_\omega^L) \geq 2] \leq C|\Lambda_L|^2|I|^2$$

We see the bound for the Minami estimate is the square of the bound for the Wegner estimate and might conclude that the Minami estimate is a result of the Wegner estimate and some form of independence of close-lying eigenvalues. This is reflected in the proof of Theorem 1.5.1, where we use a rank-one perturbation trick to get a product of independent random variables.

This trick is not available in the continuum. However, we may still use the idea of indepedence of the eigenvalues by showing the associated eigenfunctions lie in mostly distinct regions of space, so that they depend mostly on distinct sets of random variables and thus are approximately independent. In the case that eigenfunctions are exponentially localized in space, it is more likely they are independent. We note that eigenfunctions will be at most approximately independent because of the unique continuation principle.

Klopp's proof is in two parts. First, given two eigenfunctions associated to eigenvalues of H_ω^L in a small interval, I, he proves they are approximate eigenfunctions for operators restricted to smaller, disjoint, independent domains. This allows for independent applications of the Wegner estimate.

In case the two eigenfunctions significantly overlap in space, the idea is that this is the result of tunneling. So, we find linear combinations that do lie in disjoint spaces and control them sufficiently so they are still approximate eigenfunctions. Klopp refers to this as "inverse tunneling," to which he attributes the main idea of the proof.

The result of the above arguments gives a bound on the probability with less than ideal dependence on the volume. Localization is used to apply the inverse tunneling Minami estimate to boxes of length $\ell \sim (\log L)^r$, instead of the full domain. This yields an improved estimate.

We provide below an alternative proof to the first step. In order to execute the "inverse tunneling," the linear combinations of the eigenfunctions must both

1. be in the domain of H_ω restricted to some smaller, predetermined intervals, and

2. still be approximate eigenfunctions for nearby eigenvalues

Klopp uses Prüfer variables to do this. His technique provides for some quantitative bounds from assuming the eigenfunctions "significantly overlap" that can be leveraged into control of the linear combinations.

In our proof, we provide a different approach that avoids the technical arguments using Prüfer variables and instead uses the related but qualitative Sturm's oscillation theorem. The proof given below also gives a better estimate than the Prüfer variable approach for the part of the proof without localization. This improved estimate still falls short of (1.5.1), but may lead to a proof that does not.

2.2 Model and Results

2.2.1 Model

We consider the random Schrödinger operator on $\mathbb{R}$,

$$H_\omega := H_0 + V_\omega \tag{2.2.1}$$

where, $H_0 = -\Delta$ is the positive Laplacian and

$$V_\omega(x) = \sum_{n\in\mathbb{Z}} \omega_n v(x - n + \frac{1}{2}) \tag{2.2.2}$$

for some single site potential function, v, and random variables, $\{\omega_n\}_{n\in\mathbb{Z}}$. We make the following assumptions on V_ω,

(A1) v is a compactly supported, smooth, positive function with $\operatorname{supp}(v) \subset [-R, R]$, for $R > 0$, and $\sup v =: V_+$

(A2) We have the covering condition, $v > V_- \chi_{[0,1]}$, for some $V_- > 0$

(A3) $\{\omega_n\}_{n\in\mathbb{Z}}$ is a family of independent and identically distributed (i.i.d.) bounded, random variables with absolutely continuous distribution, μ, with $\operatorname{supp}(\mu) \subset [0,1]$

We are concerned with the finite volume operators on $[0, L]$ for $L \in \mathbb{N}$

$$H_\omega^L := H_0^L + V_\omega^L \tag{2.2.3}$$

where H_0^L is the Laplacian on $[0, L]$ with Dirichlet boundary conditions and $V_\omega^L := V_\omega \chi_{[0,L]}$. Similarly, we will use H_ω^Λ to denote restrictions to other domains Λ. The results should hold with other self adjoint boundary conditions, we will restrict ourselves to Dirichlet boundary conditions for simplicity.

We note that our assumptions imply $V_\omega \geq 0$ so that,

$$\sigma(H_\omega^L) \subset [0, \infty) \tag{2.2.4}$$

2.2.2 Results

The new argument appears in the next section. There, we prove a Minami estimate with still less than ideal dependence on L. The result is the main theorem of the chapter,

Theorem 2.2.1. *Let $I \subset [0, \infty)$ be any interval. There exists $C > 0, L_0 > 0$ depending on I such that for $L > L_0$, we have,*

$$\mathbb{P}[\operatorname{tr}(\mathbb{1}_I(H_\omega^L) \geq 2] \leq CL^3|I|^2 \tag{2.2.5}$$

In section 2.4, we use localization to improve the dependence on L to get the following result. This final Minami estimate is the same result proven by Klopp [22] and the argument is the same, though we include the proof for completeness. We recall also that for one dimensional operators like this one, localization holds at all energies.

Theorem 2.2.2. *Let $I \subset [0, \infty)$ be any interval. Fix any $p > 0$, $\beta > 1$ Then, for sufficiently large L (depending on p, β, I), there exist $C > 0$ such that*

$$\mathbb{P}[\operatorname{tr}(\mathbb{1}_I(H_\omega^L) \geq 2] \leq L^{-p} + CL^2(|I| + L^{-\beta})^2 \tag{2.2.6}$$

2.3 Proof of the Minami-type estimate Without Localization

The proof of Theorem 2.2.1 begins by showing that if H_ω^L has two eigenvalues close together, then we may find close by eigenvalues for Hamiltonians on two disjoint intervals by choosing from a finite number of points on which to divide $[0, L]$.

This is our version of an inverse tunneling result. A result like this is the heart of the proof of this Minami estimate as it is in Klopp's proof as well. This proof is also where our proof is furthest from Klopp's technique. Despite this, both our technique and Klopp's share the same limitation of only being available for one dimensional operators.

Proof of Theorem 2.2.1. For any interval $J \subset \mathbb{R}$, we will use the notation $\{\lambda_k(H^J)\}_{k=1}^{\infty}$ to denote the eigenvalues of H^J in increasing order. We note that these eigenvalues are simple, because $d = 1$.

The first step is to consider a fixed configuration, ω, such that H_ω^L has at least two eigenvalues in the inteval I and show the following deterministic fact. We can find smaller, mostly disjoint domains $J_1, J_2 \subset [0, L]$ such that the restrictions $H_\omega^{J_1}, H_\omega^{J_2}$ each have an eigenvalue in I. The specific properties of J_1, J_2 will be set in the argument below.

Let u_1 and u_2 be normalized eigenfunctions of H_ω^L associated to eigenvalues $E_1, E_2 \subset I$. Suppose $E_1 = \lambda_N(H_\omega^L)$, $E_2 = \lambda_{N+1}(H_\omega^L)$. Let x_n be the n^{th} zero of u_1 and y_m the m^{th} zero of u_2, with $x_0 = y_0 = 0$.

We recall that, by Sturm's oscillation theorem [28, Lemma 5.21], u_1 has exactly $N+1$ zeros, u_2 has exactly $N+2$ zeros and the zeros of u_2 interlace the zeros of u_1. Thus, $x_{N+1} = y_{N+2} = L$ and,

$$0 = y_0 = x_0 \leq y_1 \leq x_1 \leq \ldots \leq y_N \leq x_N = y_{N+1} = L \tag{2.3.1}$$

If necessary, let us change the signs of the eigenfunctions so that u_1 and u_2 are the same sign on (x_0, y_1). Note, in particular, that the domain consists of intervals of the form (x_m, y_{m+1}) on which $u_1 u_2 > 0$ and intervals of the form (y_m, x_m) on which $u_1 u_2 < 0$.

Fix some $m \in \{1, \cdots, N\}$. Note that $u_1|_{[0,x_m]}$ is a Dirichlet eigenfunction of $H_\omega^{[0,x_m]}$ with eigenvalue E_1. Furthermore, since $u_1|_{[0,x_m]}$ has exactly $m+1$ zeros, Sturm's oscillation theorem gives that $E_1 = \lambda_m(H_\omega^{[0,x_m]})$. Repeating this observation for u_2, we have the following,

$$E_1 = \lambda_m(H_\omega^{[0,x_m]}) = \lambda_{N-m}(H_\omega^{[x_m,L]}) \tag{2.3.2}$$

and

$$E_2 = \lambda_m(H_\omega^{[0,y_m]}) = \lambda_{N-m}(H_\omega^{[y_{m+1},L]}) \tag{2.3.3}$$

We consider $p_1 \in (y_m, x_m)$ and $p_2 \in (x_m, y_{m+1})$. This allows us to control eigenvalues on $[0, p_1]$ and on $[p_2, L]$. Recall that as a consequence of the minimax theorem, we have the domain monotonicity property [25]: if $\Omega_1 \subset \Omega_2$, then $\lambda_k(H^{\Omega_2}) \leq \lambda_k(H^{\Omega_1})$. Therefore,

$$\lambda_m(H_\omega^{[0,p_1]}) \in [E_1, E_2] \tag{2.3.4}$$

and

$$\lambda_{N-m}(H_\omega^{[p_2,L]}) \in [E_1, E_2] \tag{2.3.5}$$

We wish to find some $k \in \mathbb{Z}$ such that $H_\omega^{[0,k]}$ and $H_\omega^{[k+1,L]}$ both have eigenvalues in I, close to E_1 or E_2.

Depending on the location of the zeros of u_1 and u_2, one of the following occurs:

1. there exists k, $0 < k < L-1$ such that $k \in [y_m, x_m]$ and $(k+1) \in [x_m, y_{m+1}]$ for some m
2. there exists k, $0 < k < L-1$ such that $k \in [x_m, y_{m+1}]$ and $(k+1) \in [y_{m+1}, x_{m+1}]$ for some m
3. for all $k \in \{1, \cdots, L-1\}$, we have $k \in [y_m, x_m]$ for some m
4. for all $k \in \{1, \cdots, L-1\}$, we have $k \in [x_m, y_{m+1}]$ for some m

Let us name the events in which these conditions hold for $i = 1, 2, 3, 4$:

$$\Omega_i := \{\omega \text{ such that } (i) \text{ holds}\} \tag{2.3.6}$$

Before continuing, let us emphasize here the consequences of these cases. If (1) holds then for the given k, the domain monotonicity argument gives that $H_\omega^{[0,k]}$ has an eigenvalue in I and $H_\omega^{[k+1,L]}$ has an eigenvalue in I. Similarly, if (2) holds then for the given k, we have that $H_\omega^{[k,L]}$ has an eigenvalue in I and $H_\omega^{[0,k+1]}$ has an eigenvalue in I. If (3) or (4) holds we get that for all k, $H_\omega^{[0,k]}$ or respectively, $H_\omega^{[k,L]}$ has an eigenvalue in I.

Now, we have,

$$\{\omega : \operatorname{tr}(\mathbb{1}_I(H_\omega^L)) \geq 2\} \subset \cup_{i=1}^4 \Omega_i \tag{2.3.7}$$

and,

$$\mathbb{P}[\operatorname{tr}(\mathbb{1}_I(H_\omega^L)) \geq 2] \leq \sum_{i=1}^4 \mathbb{P}[\Omega_i] \tag{2.3.8}$$

We are now ready to reduce our Minami estimate to applications of a Wegner estimate. The arguments to estimate the probability each of these four events are similar. To demonstrate the ideas, let us first estimate the most straight forward term, $\mathbb{P}[\Omega_1]$ using the domain monotonicity argument:

$$\mathbb{P}[\Omega_1] \leq \sum_{k=1}^{L-2} \mathbb{P}[\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]})) \geq 1 \text{ and } \operatorname{tr}(\mathbb{1}_I(H_\omega^{[k+1,L]})) \geq 1] \tag{2.3.9}$$

We would like to estimate these summands by a product of Wegner estimates. This is what we will do, but we should be careful since the operators $H_\omega^{[0,k]}$ and $H_\omega^{[k+1,L]}$ are not completely independent. However, they are *almost* independent since the number of random variables appearing in both operators is at most $2R$ (small relative to the number of random variables).

To resolve this issue, we need a Wegner estimate in which we are allowed to hold a fixed number of random variables fixed. We will return to this issue below in the moment it becomes technically necessary. Now, we return to bounding the summands above,

$$\mathbb{P}[\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]})) \geq 1 \text{ and } \operatorname{tr}(\mathbb{1}_I(H_\omega^{[k+1,L]})) \geq 1] \tag{2.3.10}$$

for fixed $k \in \{1, ..., L-2\}$. Let us assume $k \geq \lfloor L/2 \rfloor$ (the argument in case $k < L/2$ will be the same). Let us define the two random variables $X_{\omega,1} := \operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]}))$ and $X_{\omega,2} := \operatorname{tr}(\mathbb{1}_I(H_\omega^{[k+1,L]}))$. Our main obstacle in estimating 2.3.10 is that $X_{\omega,1}$ and $X_{\omega,2}$ may both depend on ω_j for $j = k-R, \cdots, k+R$. Thus, we freeze some of these random variables so that they are independent (in probability terms, we are taking the conditional probability of $X_{\omega,1}$ given $\{\omega_j\}_{j=k-R}^{k+R}$).

$$\mathbb{P}[X_{\omega,1} \geq 1, X_{\omega,2} \geq 1] = \int_{\Omega^L} (\mathbb{1}_{\{X_{\omega,1} \geq 1\}})(\mathbb{1}_{\{X_{\omega,2} \geq 1\}}) \prod_{j=0}^{L} d\omega_j \tag{2.3.11}$$

$$= \int \Pi_{j=k-R}^{L} d\omega_j \int \Pi_{j=0}^{k-R-1} d\omega_j (\mathbb{1}_{\{X_{\omega,1} \geq 1\}})(\mathbb{1}_{\{X_{\omega,2} \geq 1\}}) \tag{2.3.12}$$

$$= \int \Pi_{j=k-R}^{L} d\omega_j (\mathbb{1}_{\{X_{\omega,2} \geq 1\}}) \int \Pi_{j=0}^{k-R-1} d\omega_j (\mathbb{1}_{\{X_{\omega,1} \geq 1\}}) \tag{2.3.13}$$

Now in this last line, we would like to estimate the inner and outer integrals both by a Wegner estimate, but we must take care with the inner integral especially since we are considering the operator $H_\omega^{[0,k]}$ but only varying $\{\omega_j\}_{j=0}^{k-R-1}$. It is now well known that a Wegner estimate like this, in which some random variables are held fixed can be proven with a quantitative unique continuation principle (QUCP) such as the ones proven in [4] and [26]. These papers have slightly different perspectives and intended applications than we do here, so we believe it may be helpful to provide here a proof of the necessary Wegner estimate given as an application of a QUCP proven in [21] which in turn is inspired by the result in [4]. For convenience, we restate the result here,

Theorem 2.3.1 (QUCP (Corollary A.2 from [21]))**.** *Consider the Schrödinger operator $H_\Lambda := -\Delta_\Lambda^{(D)} + V$, where $\Lambda = \Lambda_L(x_0)$ and $||V||_\infty \leq K < \infty$. Let $\psi \in \mathcal{D}(\Delta_\Lambda)$. Fix δ, D such that $0 < \delta \leq D$. There exists a constant $\tilde{m} = \tilde{m}(d, \delta, D) > 0$, such that given a measurable set Θ with $diam(\Theta) \leq D$ and $x \in \Lambda$ such that $B(x,\delta) \subset \Lambda$ and $R := d(x, \Theta) \geq D$, we have,*

$$(1+K)||\psi|_{B(x,\delta)}||^2 + (29\sqrt{d})^d ||H_\Lambda \psi||^2 \geq R^{-\tilde{m}(1+K^{2/3}+\log(||\psi_\Lambda|| ||\psi_\Theta||^{-1}))R^{4/3}} ||\psi|_\Theta||^2 \tag{2.3.14}$$

We wish to prove that,

$$\int \Pi_{j=0}^{k-R-1} d\omega_j (\mathbb{1}_{\{X_{\omega,1}\} \geq 1}) \lesssim |I| k \tag{2.3.15}$$

or, written differently,

$$\mathbb{P}_{\{j<k-R\}}[\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]}) \geq 1] \lesssim |I| k \tag{2.3.16}$$

where we have used the shorthand $\mathbb{P}_{\{j<k-R\}}$ to mean we are only varying ω_j for $j < k-R$.

We will follow the usual strategy for proving Wegner estimates, beginning by expanding the trace, to localize in space. We will use the notation $\Lambda_j := [j-1, j]$.

$$\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]})) = \sum_{j=1}^{k} \operatorname{tr}(\chi_{\Lambda_j} \mathbb{1}_I(H_\omega^{[0,k]}) \chi_{\Lambda_j}) \tag{2.3.17}$$

Thus,

$$\mathbb{P}_{\{i<k-R\}}[\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]})) \geq 1] \leq \mathbb{E}_{\{i<k-R\}}[\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]}))] \tag{2.3.18}$$

$$= \sum_{j=1}^{k} \mathbb{E}_{\{i<k-R\}}[\operatorname{tr}(\chi_{\Lambda_j} \mathbb{1}_I(H_\omega^{[0,k]}) \chi_{\Lambda_j})] \tag{2.3.19}$$

By spectral averaging as in for example [16], we can estimate the terms with $j < k - R$,

$$\mathbb{E}_{\{i<k-R\}}[\operatorname{tr}(\chi_{\Lambda_j} \mathbb{1}_I(H_\omega^{[0,k]}) \chi_{\Lambda_j})] \leq \frac{1}{V_-} \mathbb{E}_{\omega_j^\perp} \mathbb{E}_{\omega_j}[\operatorname{tr}(v_j^{1/2} \mathbb{1}_I(H_\omega^{[0,k]}) v_j^{1/2})] \leq C_v |I| \tag{2.3.20}$$

For $k - R < j \leq k$, we must use the QUCP. In particular, we will show there exists $C \in (0,1)$ such that,

$$\operatorname{tr}(\chi_{\Lambda_{[k-R,k]}} \mathbb{1}_I(H_\omega^{[0,k]}) \chi_{\Lambda_{[k-R,k]}}) < C \operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]})) \tag{2.3.21}$$

In this case, 2.3.21 along with 2.3.20 implies,

$$\begin{aligned} &\mathbb{E}_{\{i<k-R\}}[\operatorname{tr}(\mathbb{1}_I(H_\omega^{[0,k]}))] \\ &\quad \leq \frac{1}{1-C} \mathbb{E}_{\{i<k-R\}}[\operatorname{tr} \chi_{\Lambda_{[0,k-R]}} \mathbb{1}_I(H_\omega^{[0,k]}) \chi_{\Lambda_{[0,k-R]}}] \\ &\quad \lesssim k|I| \end{aligned} \tag{2.3.22}$$

which is exactly the Wegner estimate desired. So, to complete this Wegner estimate, we return to proving 2.3.21 using the QUCP.

Let ϕ be a normalized eigenfunction of $H_\omega^{[0,k]}$ with associated eigenvalue $E \in I$. We will apply the QUCP to the operator $H_E := H_\omega^{[0,k]}$ and ϕ. Our exceptional set will play the role, $\Theta := [k-R, k]$, with $D = |\Theta| = R$, $x := k - 2R$, $\delta := R/2$, and the R of the QUCP equal to D which is the same as our R from our potential. Then the QUCP, gives the following inequality,

$$\begin{aligned} (1+K)||\phi|_{B(x,\delta)}||^2 &\geq D^{-\tilde{m}(1+K^{2/3}+\log(1/||\phi|_\Theta||)D^{4/3}} ||\phi|_\Theta||^2 \\ &= C_1 ||\phi|_\Theta||^{\tilde{m}D^{4/3}\log(D)+2} \end{aligned} \tag{2.3.23}$$

where $C_1 > 0$ depends on $D, \tilde{m}, K$ and $K > 0$ depends on V_+ and $\sup I$. In particular, we get,

$$||\phi|_\Theta|| \leq C_2 ||\phi|_{x,\delta}||^\rho \leq C_2(1 - ||\phi|_\Theta||)^\rho \tag{2.3.24}$$

where C_2 is a positive constant independent of k and

$$0 < \rho := 2/(2 + \tilde{m}R^{4/3}\log(R)) \leq 1 \tag{2.3.25}$$

We recall here that $R \geq 1$.

Note that the function $f(x) = C_2(1-x)^\rho - x$ is decreasing with $f(0) = C_2 > 0$, $f(1) = -1$, so that $f(C_0) = 0$ for some $C_0 \in (0,1)$ depending on C_2 and ρ. Thus, (2.3.24) implies (importantly) that $||\phi_\Theta|| \leq C_0 < 1$, where C_0 depends on ρ which in turn depends on R.

Finally, we return to proving 2.3.21, recalling that $[k-R,k] = \Theta$,

$$\mathrm{tr}\,(\chi_\Theta \mathbb{1}_I(H_\omega^{[0,k]})\chi_\Theta) = \mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[0,k]})\chi_\Theta \mathbb{1}_I(H_\omega^{[0,k]})) \tag{2.3.26}$$

$$= \sum_{\phi_E : E \in \sigma(H_\omega^{[0,k]}) \cap I} ||(\phi_E)_\Theta||^2 \leq C_0^2\,\mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[0,k]})) \tag{2.3.27}$$

which completes the proof of (2.3.21).

We can now finish bounding $\mathbb{P}[\Omega_1]$,

$$\begin{aligned}(2.3.13) &\leq Ck|I| \int \Pi_{j=k-R}^{L} d\omega_j (\mathbb{1}_{\{X_{\omega,2} \geq 1\}}) \\ &\leq (Ck|I|)\,(C'(L-k-1)|I|) \\ &\leq C'' L^2 |I|^2\end{aligned} \tag{2.3.28}$$

where the first inequality requires the Wegner estimate proven in this section. The second inequality uses a Wegner estimate in which we vary all the random variables (the QUCP argument is not used). Now, summing over k, we have,

$$\mathbb{P}[\Omega_1] \lesssim L^3 |I|^2 \tag{2.3.29}$$

When estimating $\mathbb{P}[\Omega_2]$, we are given k, we proceed in exactly the same way, but we have operators that overlap by at most one more random variable. In particular, 2.3.10 is replaced by,

$$\mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[0,k+1]})) \geq 1 \text{ and } \mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[k,L]})) \geq 1] \tag{2.3.30}$$

Thus, the same argument will give the desired estimate, but our exceptional set in the Wegner set is $[k-R-1, k+1]$ which of course will change the constants but they will remain independent of L and I.

Let us describe how to estimate $\mathbb{P}(\Omega_3)$ and $\mathbb{P}(\Omega_4)$. At first glance, it appears that in cases (3) and (4) above, we cannot use the same arguments, since we do not have eigenfunctions extending from both 0 and L. However, we are guaranteed an eigenfunction on a small domain which saves us. For example, take case (3). Recall that our domain monotonicity argument tells us that in this case, $H_\omega^{[0,k]}$ has an eigenvalue in I for every k. In particular, we take advantage of the fact that $H_\omega^{[0,L]}$ and $H_\omega^{[0,1]}$ both have eigenvalues in I. So, in the same way we estimated $\mathbb{P}[\Omega_1]$,

$$\mathbb{P}[\Omega_3] \leq \mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[0,1]}) \geq 1 \text{ and } \mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[0,L]}) \geq 1] \tag{2.3.31}$$

$$= \int \Pi_{j=1}^{R} d\omega_j (\mathbb{1}_{\{Y_{\omega,1} \geq 1\}}) \int \Pi_{j=R+1}^{L} d\omega_j (\mathbb{1}_{\{Y_{\omega,2} \geq 1\}}) \tag{2.3.32}$$

where $Y_{\omega,1} := \mathbb{1}_I(H_\omega^{[0,1]})$ and $Y_{\omega,2} := \mathbb{1}_I(H_\omega^{[0,L]})$. So, the internal integral is estimated by a Wegner estimate for $H_\omega^{[0,L]}$ but we have frozen the first R random variables. Thus, the same Wegner estimate we used to bound $\mathbb{P}[\Omega_1]$ allows us to bound this in the same way. In this case, we get,

$$\mathbb{P}[\Omega_3] \lesssim L|I|^2 \tag{2.3.33}$$

Estimating $\mathbb{P}[\Omega_4]$ is similar but we have an eigenfunction for $H_\omega^{[L-1,L]}$.

Putting together the estimates for Ω_j, $j = 1, 2, 3, 4$, we get,

$$\mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_I(H_\omega^{[0,L]}) \geq 2] \lesssim L^3 |I|^2 \tag{2.3.34}$$

□

2.4 Improving the Minami-Type Estimate Using Localization

In this section, we prove Theorem 2.2.2 by following Klopp's arguments. We recall the idea of the proof is to use the localization of eigenfunctions to reduce to smaller intervals. We consider the event that eigenfunctions in I are exponentially decaying and cover our domain $[0, L]$ with intervals of length $\ell \sim (\log L)^{1/\xi}$. We show that localized eigenfunctions are approximate eigenfunctions on the scale ℓ and so if H_ω^L has two eigenvalues in I, the restriction of H_ω^L to two length ℓ intervals have an eigenvalue near I. In the event the two intervals are far apart, so that the Hamiltonians are independent, we may again use Wegner estimates. In the event the two intervals are close together, we apply Theorem 2.2.1.

Localization can be characterized in many ways. Our focus is on proving the Minami-type estimate for our model and so we will not provide here a proof of localization. For this section, we will assume a result that we can directly apply and that directly expresses the features of localization we have described earlier, namely the exponenetial decay of eigenfunctions. In particular, we assume the following lemma from [15],

Lemma 2.4.1. *[Lemma 1.1 from [15]] For any $p > 0$ and $\xi \in (0, 1)$, for $L \geq 1$ large enough, there exists a set of configurations $\mathcal{Z}$ such that $\mathbb{P}(\mathcal{Z}) \geq 1 - L^{-p}$ and for $\omega \in \mathcal{Z}$, if*

(1) ϕ is a normalized eigenvector of H_ω^L with associated eigenvalue $E \in I$

(2) $x_m \in \Lambda_L$ is a maximum of $x \mapsto ||\phi||_x =: \|\phi|_{B(x,1)}\|$,

then, for $x \in \Lambda_L$, one has,

$$||\phi||_x \leq L^{p+d} e^{-|x-x_m|^\xi}$$

In [15], and in [22] this result is proven as a consequence of a stronger expression of localization for multiple families of random Schrödinger operators including the one discussed here. To see a complete proof of localization for a category of one dimensional continuum operators that includes ours and that holds at all energies, see [9].

As in [22, Lemma 3.6], lemma 2.4.1 can be used to prove the following lemma in which we see that our localized eigenfunctions are approximate eigenfunctions for operators on the scale ℓ. Here and in the remainder of the chapter, we use the notation $\Lambda_d(x) := [x-d, x+d] \cap [0, L]$.

Lemma 2.4.2. *Let $l = (r\log(L))^{1/\xi}$ for $r > 0$, $\xi \in (0,1)$. For L sufficiently large, $\omega \in \mathcal{Z}$, $\gamma \in \Lambda_L$, if H_ω^L has k eigenvalues in I with localization centers in $\Lambda_l(\gamma)$, then $H_\omega^{\Lambda_{3l}(\gamma)}$ has k eigenvalues in*

$$I + 2[-L^{-\alpha}, L^{-\alpha}]$$

for $\alpha = r - 1 - p$ as long as $\alpha > 1$.

We include a proof for $k = 1, 2$, the cases used in this section.

Proof of Lemma 2.4.2. Let $k = 1$ and ϕ a normalized eigenfunction of H_ω^L with eigenvalue $e \in I$. Then let χ be a smooth cutoff function that is identically 1 on $\Lambda_{2l}(\gamma)$ and has support contained in $\Lambda_{3l}(\gamma)$. Then, an application of Lemma 2.4.1 gives,

$$||(H_\omega^{\Lambda_{3l}(\gamma)} - e)(\chi\phi)|| \leq L^{p+1}e^{-\xi} = L^{-(r-1-p)} \tag{2.4.1}$$

By a similar calculation, using that $\alpha > 1$, we get that $||\chi\phi|| > 1 - L^{-(\alpha-1)} > \frac{1}{2}$ so we get the desired result.

If $k = 2$ and ϕ_1, ϕ_2 are normalized eigenfunctions with associated eigenvalues in I, then,

$$| < \chi\phi_1, \chi\phi_2 > | \leq 3L^{-(\alpha-1)}$$

This, along with the argument above gives the desired result. □

We now complete the proof Theorem 2.2.2.

Proof of Theorem 2.2.2. Fix $\xi \in (0,1)$ and $p, r > 0$. Let $l = (r\log(L))^{1/\xi}$ and $\Gamma = \frac{l}{8}\mathbb{Z}$. Suppose for $\omega \in \mathcal{Z}$, H_ω^L has two eigenvalues in I with localization centers $x_1 < x_2$. Consider the two cases,

1. $|x_1 - x_2| \geq l$
2. $|x_1 - x_2| < l$.

In case of (1), there exist $\gamma_1, \gamma_2 \in \Gamma$ with $d(\gamma_i, x_i) \leq \frac{l}{8}$ for $i = 1, 2$ and $d(\Lambda_{3l/8}(\gamma_1), \Lambda_{3l/8}(\gamma_2)) \geq \frac{l}{4}$. Thus by Lemma 2.4.2, $H_\omega^{\Lambda_{3l/8}(\gamma_i)}$ has an eigenvalue in $I + [-L^{-\alpha_1}, L^{-\alpha_1}]$ where $\alpha_1 = (\frac{1}{8})^\xi r - 1 - p$. Furthermore, we have that $H_\omega^{\Lambda_{3l/8}(\gamma_1)}$ and $H_\omega^{\Lambda_{3l/8}(\gamma_2)}$ are independent for L_0 sufficiently large so that $\frac{l}{4} > 2R$.

In case of (2), there exists $\gamma \in \Gamma$ such that $x_1, x_2 \in \Lambda_{9l/16}(\gamma) \subset \Lambda_l(\gamma)$ (by taking the point closest to the midpoint of x_1 and x_2). So, by lemma 2.4.2, we get that $\Lambda_{3l}(\gamma)$ has two eigenvalues in $I + [-L^{\alpha_2}, L^{-\alpha_2}]$ where $\alpha_2 = r - 1 - p$.

We may now bound the probability of having two eigenvalues in I by the event that (1) or (2) from above occurs for some point(s) in Γ. Let $I_\alpha = I + [-L^{-\alpha}, L^{-\alpha}]$.

$$\mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_I(H^L_\omega)) \geq 2] \;\leq\; L^{-p} + \sum_{\gamma \in \Gamma} \mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_{I_{\alpha_2}}(H^{\Lambda_{3l}(\gamma)})) \geq 2]$$

$$+ \sum_{d(\Lambda_{3\ell/8}(\gamma_1), \Lambda_{3\ell/8}(\gamma_2)) \geq l/4} \mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_{I_{\alpha_1}}(H^{\Lambda_{3\ell/8}(\gamma_1)})) \geq 1 \text{ and } \mathrm{tr}\,(\mathbb{1}_{I_{\alpha_1}}(H^{\Lambda_{3\ell/8}(\gamma_2)}) \geq 1] \tag{2.4.2}$$

$$\leq L^{-p} + \frac{L}{\ell/8} C(3l)^3 |I_{\alpha_2}|^2 + (\frac{L}{\ell/8})^2 C'(3l/8)^2 |I_{\alpha_1}|^2 \tag{2.4.3}$$

$$\leq L^{-p} + C'' L^2 |I_{\alpha_1}|^2 \tag{2.4.4}$$

where we used Theorem 2.2.1 for the terms in the first sum and the Wegner estimate for the terms in the second sum. In the final inequality, we use $|I_{\alpha_2}| < |I_{\alpha_1}|$.

Since $\alpha_1 = (1/8)^\xi r - 1 - p$, and we can fix any $\xi \in (0,1)$, we can find r so that $\alpha_1 = \beta$ for any β, which gives us,

$$\mathbb{P}[\,\mathrm{tr}\,(\mathbb{1}_I(H^L_\omega)) \geq 2] \leq L^{-p} + C'' L^2 (|I| + L^{-\beta})^2 \tag{2.4.5}$$

□

2.5 Conclusion and Further Questions

This work, along with the work in Chapter 4 is motivated by a desire to improve and generalize techniques for proofs of Minami estimates for non-rank one models. It was an accomplishment in this chapter to replace the technical Prüfer variable argument Klopp uses with an argument only based on Sturm's oscillation theorem and domain monotonicity. However, Sturm's oscillation theorem is often proven using Prüfer variable techniques, and so if we wanted to extend the argument to models where Prüfer variables are unavailable such as in $\mathbb{R}^d$, for $d > 1$, then we would likely need some significant additional arguments. It is an interesting and difficult question (that we could not answer) if any proof could be given for $d > 1$ that still is generally based on "inverse tunneling" and the approximate independence of eigenfuntions in space.

It was also an accomplishment of this chapter to improve the L dependence of the Minami estimate (without localization) from the dependence acquired by Klopp. It is perhaps an easier problem than the one above to improve the estimate further to the ideal dependence, L^2, without the explicit use of localization. After all, localization holds at all energies for this one dimensional model, so perhaps using other one-dimensional techniques, like Sturm's oscillation theorem, are sufficient to prove an ideal Minami estimate. Looking at where there is room to improve the proof, one might try to find a way to identify events that are overcounted in estimates like (2.3.9).

Chapter 3 Eigenvalue Statistics for Lattice RSO's with Decaying Potentials

3.1 Introduction

In this chapter, we will prove that the random point measures of the local, rescaled eigenvalues of some random Schrödinger operators with decaying, random potential have the same limit as the free Laplacian. This is an improvement of a result of Dolai and Krishna [13] in the rate of decay allowed for the potential.

It is now well known that the limiting process for the local eigenvalue statistics of the Anderson model in the localization regime is a Poisson point process. The existence of a delocalized regime and any distinct limiting process for it are open problems.

Local eigenvalue statistics for other random operator models are also of interest, both because the models are of independent interest, and because progressing towards the demonstration of any phase transition might be illuminating for many models.

In this chapter, we study random Schrödinger operators with a decaying, random potential. The perturbation of the free Laplacian by a potential with power decay has been studied in the context of scattering theory. For example, if the potential, V, decays sufficiently fast, we expect states to propagate at infinity like free states. Indeed, [25, Theorem XIII.33] if V is a short range potential $V(x) \lesssim (1+|x|)^{-\alpha}$ for $\alpha > 1$, the absolutely continuous parts of the spectra of H_V and H_0 are unitarily equivalent.

Results like this make decaying, random potentials a tempting context to try to prove properties related to delocalization, which have been more elusive than proofs related to localization for random operators like the Anderson model.

What is known about random, decaying potentials is more complicated than the result described above. In [20], the main theorems concern discrete, random, decaying potentials, $V_\omega(n) = a_n\omega_n$ on $\ell^2(\mathbb{Z}^d)$, $d \geq 3$ with $|a_n| \lesssim |n|^{-\alpha}$, $\alpha > 1$, and certain unbounded but finite variance random variables ω_n. They prove that for these potentials, $\sigma(H_\omega) = \mathbb{R}$, $\sigma_c(H_\omega) \subset [-2d, 2d]$, $\sigma_{ac}(H_\omega) = [-2d, 2d]$ almost surely. The part of their result concerning the continuous spectrum resembles the deterministic result on short range potentials from the previous paragraph. Indeed, it would be a direct consequence of the scattering theory if the $\{\omega_n\}$ were bounded. However, the previous result does not directly apply because V_ω is almost surely unbounded. Nevertheless, it seems randomness allows for an "effective decay" that behaves like $|n|^{-\alpha}$ almost surely.

In [11], the main theorem provides an interesting picture for the one-dimensional, discrete case. In this model, $V_\omega(n) = \lambda a_n \omega_n$, with $|a_n| \sim |n|^{-\alpha}$, with some moment conditions on $\{\omega_n\}$. They prove that if $\alpha \in [0, 1/2)$, the spectrum of H_ω is pure point and if $\alpha > 1/2$, the spectrum in $[-2, 2]$ is purely continuous. If $\alpha = 1/2$, there exists transitions in both the coupling constant, λ, and in the energy from pure point to continuous spectrum. The main theorem gives results for more specific cases and provides decay rates for eigenfunctions, but we emphasize here that we see that the randomness allows for even slower decay (down to $\alpha > 1/2$) to still provide for continuous spectrum in $[-2, 2]$.

In [2], Bourgain extended part of this generalization to $d = 2$ proving $[-4, 4] \subset \sigma_{ac}(H_\omega)$ for power decay only requiring $\alpha > \frac{1}{2}$.

There are not as many results investigating the eigenvalue statistics of RSO's with random, decaying potentials. Dolai and Krishna investigate the local eigenvalue statistics for some of these models, proving they have the same limit as the free operator for power decay down to $\alpha > 2$. We give a variation on their argument, which improves the result to $\alpha > 1$.

3.2 Model and Main Theorem

We consider the random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$,

$$H_\omega = H_0 + V_\omega \tag{3.2.1}$$

where H_0 is the centered, discrete Laplacian and V_ω is a random potential. We define them for $f \in \ell^2(\mathbb{Z}^d)$,

$$\begin{aligned} H_0 f(n) &= \sum_{|m-n|=1} f(m) \\ V_\omega f(n) &= a_n \omega_n f(n) \end{aligned} \tag{3.2.2}$$

We make the following assumptions on the potential V_ω,

- $\{\omega_n\}_{\{n\in\mathbb{Z}^d\}}$ is a collection of i.i.d. random variables with bounded, absolutely continuous density.
- $\{a_n\}_{n\in\mathbb{Z}^d}$ is a sequence of real, constants satisfying a power decay bound,

$$|a_n| \leq V_+(1 + |n|)^{-\alpha} \tag{3.2.3}$$

  for some $V_+ > 0$ and $\alpha > 0$.

Let $\Lambda_L = [-L, L] \subset \mathbb{Z}^d$ and define the cutoff operators by,

1. $H_0^L = \chi_{\Lambda_L} H_0 \chi_{\Lambda_L}$,
2. $H_\omega^L = \chi_{\Lambda_L} H_\omega \chi_{\Lambda_L}$,
3. $V_\omega^L = \chi_{\Lambda_L} V_\omega \chi_{\Lambda_L}$.

We define the following point measures on $\mathbb{R}$ for $E \in \mathbb{R}$, by the locally rescaled eigenvalues of the cutoff operators,

$$\begin{aligned} \mu_{L,E}^0(x) &= \frac{1}{(2L+1)^{d-1}} \sum_{\lambda\in\sigma((2L+1)(H_0^L - E))} \delta(x - \lambda) \\ \mu_{L,E}^\omega(x) &= \frac{1}{(2L+1)^{d-1}} \sum_{\lambda\in\sigma((2L+1)(H_\omega^L - E))} \delta(x - \lambda) \end{aligned} \tag{3.2.4}$$

The main theorem of the chapter is the following,

Theorem 3.2.1. *Let $f \in C_0^\infty(\mathbb{R})$. Suppose the coupling constants have power decay, for $V_+, \alpha > 0$,*

$$|a_n| \le V_+(1+|n|)^{-\alpha} \tag{3.2.5}$$

Then, for any $\alpha > 1$,

$$\lim_{L\to\infty} \int_{\mathbb{R}} f d\mu^0_{L,E} - \int_{\mathbb{R}} f d\mu^\omega_{L,E} = 0 \tag{3.2.6}$$

for all $E \in \mathbb{R}$ and any ω.

Remark. 1. *We note here that this result is a an improvement on the of Dolai and Krishna in [13]. In particular, their result holds for $\alpha > 2$ instead of $\alpha > 1$.*

2. *In [13], the random variables are allowed to be unbounded with finite first moment, while here the random variables are assumed to be bounded. In exchange for this restriction, the result holds for all ω instead of almost surely.*

Remark. *Let us make make some comments here on the scaling in the definition of these measures. The scaling we used here is the same as in [13]. However, this scaling is different than the scaling used for the point processes for the ergodic Anderson model, H_A^L, in the localized regime,*

$$\sum_{\lambda \in \sigma((2L+1)^d (H_A^L - E))} \delta(x-\lambda) \tag{3.2.7}$$

The scaling used here is related to spectral properties of the free Laplacian. Consider the range of measures, for $0 \le r \le d$,

$$\mu^{0,r}_{L,E}(x) = \frac{1}{(2L+1)^{d-r}} \sum_{\lambda \in \sigma((2L+1)^r (H_0^L - E))} \delta(x-\lambda) \tag{3.2.8}$$

Since H_0^L may have eigenvalues with multiplicity up to L^{d-1} for infinitely many L, (see [18] for the continuum case or examine the formulas in Lemma 3.4.1 for the discrete case), we see that $\sup_L \int f \; d\mu^{0,r}_{L,E}$ may be unbounded for $r > 1$ and $f \in C_0^\infty(\mathbb{R})$.

However, eigenvalues of H_0^L have multiplicity bounded by CL^{d-1}. Also, in Lemma 3.4.1, we show the related fact that for $|I| \ge CL^{-1}$, $d \ge 3$ we have the Wegner type bound,

$$|\sigma(H_0^L) \cap I| \lesssim L^d |I| \tag{3.2.9}$$

This fact, in turn, guarantees that $\sup_L \int f \; d\mu^{0,r}_{L,E} < \infty$ for $r \le 1$. Thus, the choice $r = 1$ seems correct, being the largest value (giving the finest window on the eigenvalues) for which $\mu^{0,r}_{L,E}$ might have a limit. In [13] Dolai and Krishna prove $\{\mu^0_{L,E}\}_L$ has a limit point for $E \in (-2d, -2d+2) \cup (2d-2, 2d)$ and a formula is conjectured as a limit at all $E \in (-2d, 2d)$.

3.3 Proof of Main Theorem

We now proceed with a proof of the main theorem.

Proof. Let λ_k be the k^{th} eigenvalue of H_0^L counting multiplicity with associated orthonormal eigenfunctions $\{\psi_k\}$, and let ν_k be the k^{th} eigenvalue of H_ω^L counting multiplicity with associated orthonormal eigenfunctions $\{\phi_k\}$.

For ease of notation, let $\{\tilde{\lambda}_i\}$, $\{\tilde{\nu}_i\}$ be the eigenvalues of $(2L+1)(H_0^L - E)$ and $(2L+1)(H_\omega^L - E)$ respectively,

$$\begin{aligned}\tilde{\lambda}_i &:= (2L+1)(\lambda_i^L - E),\\ \tilde{\nu}_i &:= (2L+1)(\nu_i^L - E)\end{aligned} \tag{3.3.1}$$

Then, we evaluate the difference, expanding in the eigenfunctions,

$$\int_{\mathbb{R}} f d\mu_{L,E}^0 - \int_{\mathbb{R}} f d\mu_{L,E}^\omega = \frac{1}{(2L+1)^{d-1}} \left(\sum_i f(\tilde{\lambda}_i) - \sum_i f(\tilde{\nu}_i) \right) \tag{3.3.2}$$

$$= \frac{1}{(2L+1)^{d-1}} \left(\sum_i f(\tilde{\lambda}_i)\langle \psi_i, \psi_i\rangle - \sum_i f(\tilde{\nu}_i)\langle \phi_i, \phi_i\rangle \right) \tag{3.3.3}$$

$$= \frac{1}{(2L+1)^{d-1}} \sum_{i,j} (f(\tilde{\lambda}_i) - f(\tilde{\nu}_j))\langle \psi_i, \phi_j\rangle\langle \phi_j, \psi_i\rangle \tag{3.3.4}$$

$$= \frac{1}{(2L+1)^{d-1}} \sum_{\lambda_i \neq \nu_j} \frac{f(\tilde{\lambda}_i) - f(\tilde{\nu}_j)}{\tilde{\lambda}_i - \tilde{\nu}_j} (\tilde{\lambda}_i - \tilde{\nu}_j)\langle \psi_i, \phi_j\rangle\langle \phi_j, \psi_i\rangle \tag{3.3.5}$$

$$= \frac{1}{(2L+1)^{d-2}} \sum_{\lambda_i \neq \nu_j} \frac{f(\tilde{\lambda}_i) - f(\tilde{\nu}_j)}{\tilde{\lambda}_i - \tilde{\nu}_j} (\lambda_i - \nu_j)\langle \psi_i, \phi_j\rangle\langle \phi_j, \psi_i\rangle \tag{3.3.6}$$

$$= \frac{1}{(2L+1)^{d-2}} \sum_{\lambda_i \neq \nu_j} \frac{f(\tilde{\lambda}_i) - f(\tilde{\nu}_j)}{\tilde{\lambda}_i - \tilde{\nu}_j} \langle V_\omega \psi_i, \phi_j\rangle\langle \phi_j, \psi_i\rangle \tag{3.3.7}$$

Let $\epsilon = \frac{\alpha-1}{2} > 0$. We divide the above sum into sections determined by the size of $\tilde{\lambda}_i$, i.e. the distance between λ_i and E. We define the partition of index sets below,

$$\begin{aligned}\mathcal{I}_0 &= \{i : |\lambda_i - E| \leq L^{-1+\epsilon}\}\\ \mathcal{I}_k &= \{i : L^{-1+k\epsilon} < |\lambda_i - E| \leq L^{-1+(k+1)\epsilon}\}\end{aligned}$$

for $k = 1, \cdots, \lfloor \frac{1}{\epsilon} \rfloor$. We further note, that for sufficiently large L, $d \geq 3$, Lemma 3.4.1, implies that,

$$|\mathcal{I}_k| \lesssim L^{d-1+(k+1)\epsilon} \tag{3.3.8}$$

Starting with $\mathcal{I}_0$, we get,

$$\frac{1}{(2L+1)^{d-2}} \sum_{i \in \mathcal{I}_0} \sum_{\nu_j \neq \lambda_i} \frac{f(\tilde{\lambda_i}) - f(\tilde{\nu_j})}{\tilde{\lambda_i} - \tilde{\nu_j}} (\lambda_i - \nu_j) \langle \psi_i, \phi_j \rangle \langle \phi_j, \psi_i \rangle \tag{3.3.9}$$

$$= \frac{1}{(2L+1)^{d-2}} \sum_{i \in \mathcal{I}_0} \sum_{\nu_j \neq \lambda_i} \frac{f(\tilde{\lambda_i}) - f(\tilde{\nu_j})}{\tilde{\lambda_i} - \tilde{\nu_j}} \langle V_\omega \psi_i, \phi_j \rangle \langle \phi_j, \psi_i \rangle \tag{3.3.10}$$

$$\leq \frac{1}{(2L+1)^{d-2}} \sum_{i \in \mathcal{I}_0} \sum_{\nu_j \neq \lambda_i} \|f'\|_\infty |\langle V_\omega \psi_i, \phi_j \rangle \langle \phi_j, \psi_i \rangle| \tag{3.3.11}$$

$$\leq \frac{\|f'\|_\infty}{(2L+1)^{d-2}} \left(\sum_{i \in \mathcal{I}_0} \sum_{\nu_j \neq \lambda_i} |\langle V_\omega \psi_i, \phi_j \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in \mathcal{I}_0} \sum_{\nu_j \neq \lambda_i} |\langle \phi_j, \psi_i \rangle|^2 \right)^{\frac{1}{2}} \tag{3.3.12}$$

$$\leq \frac{\|f'\|_\infty}{(2L+1)^{d-2}} \left(\sum_{i \in \mathcal{I}_0} \|V_\omega \psi_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{i \in \mathcal{I}_0} \|\psi_i\|^2 \right)^{\frac{1}{2}} \tag{3.3.13}$$

$$\leq \frac{\|f'\|_\infty}{(2L+1)^{d-2}} \left(\sup_i \|V_\omega \psi_i\|^2 |\mathcal{I}_0| \right)^{\frac{1}{2}} |\mathcal{I}_0|^{\frac{1}{2}} \tag{3.3.14}$$

$$\leq C \frac{\|f'\|_\infty}{(2L+1)^{d-2}} L^{-\alpha} L^{d-1+\epsilon} \lesssim L^{1-\alpha+\epsilon} = L^{-\epsilon} \tag{3.3.15}$$

Here, we used that $\sup \|V_\omega \psi_i\| \lesssim L^{-\alpha}$. We can see this as a consequence of the delocalization of the eigenfunctions, ψ_i. Using lemma 3.4.2, and $|V_\omega(n)| \lesssim (1+|n|)^{-\alpha}$, we get,

$$\sup_i \|V_\omega \psi_i\|^2 \lesssim \sum_{n \in \Lambda_L} L^{-d} (1+|n|)^{-2\alpha} \lesssim L^{-2\alpha} \tag{3.3.16}$$

as desired.

For $k > 0$, the estimate is similar, but we use that f is compactly supported. In particular, for some $R > 0$, if $|\nu_j - E| > RL^{-1}$, then $f(\tilde{\nu_j}) = 0$. Similarly, for all $i \in \mathcal{I}_k$, we have $f(\tilde{\lambda_i}) = 0$. So,

$$\frac{1}{(2L+1)^{d-2}}\sum_{i\in\mathcal{I}_k}\sum_{\nu_j\neq\lambda_i}\frac{f(\tilde{\lambda}_i)-f(\tilde{\nu_j})}{\tilde{\lambda}_i-\tilde{\nu_j}}(\lambda_i-\nu_j)\langle\psi_i,\phi_j\rangle\langle\phi_j,\psi_i\rangle \tag{3.3.17}$$

$$=\frac{1}{(2L+1)^{d-1}}\sum_{\substack{i\in\mathcal{I}_k\\|\nu_j-E|\leq RL^{-1}}}\frac{-f(\tilde{\nu_j})}{\lambda_i-\nu_j}\langle V_\omega\psi_i,\phi_j\rangle\langle\phi_j,\psi_i\rangle \tag{3.3.18}$$

$$\leq\frac{1}{(2L+1)^{d-1}}\sum_{\substack{i\in\mathcal{I}_k\\|\nu_j-E|\leq RL^{-1}}}\frac{\|f\|_\infty}{|\lambda_i-\nu_j|}|\langle V_\omega\psi_i,\phi_j\rangle\langle\phi_j,\psi_i\rangle| \tag{3.3.19}$$

$$\leq\frac{\|f\|_\infty L^{1-k\epsilon}}{(2L+1)^{d-1}}\left(\sum_{\substack{i\in\mathcal{I}_k\\|\nu_j-E|\leq RL^{-1}}}|\langle V_\omega\psi_i,\phi_j\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{\substack{i\in\mathcal{I}_k\\|\nu_j-E|\leq RL^{-1}}}|\langle\phi_j,\psi_i\rangle|^2\right)^{\frac{1}{2}} \tag{3.3.20}$$

$$\lesssim\frac{L^{1-k\epsilon}}{L^{d-1}}(|\mathcal{I}_k|\sup_i\|V_\omega\psi_i\|)^{\frac{1}{2}}(|\mathcal{I}_k|)^{\frac{1}{2}} \tag{3.3.21}$$

$$\lesssim\frac{L^{1-k\epsilon}}{L^{d-1}}L^{-\alpha}L^{d-1+(k+1)\epsilon}=L^{1-\alpha+\epsilon}=L^{-\epsilon} \tag{3.3.22}$$

Putting together the above estimates, we get,

$$\left|\int_{\mathbb{R}}fd\mu^0_{L,E}-\int_{\mathbb{R}}fd\mu^\omega_{L,E}\right|\lesssim(1+\frac{1}{\epsilon})L^{-\epsilon}\to 0 \tag{3.3.23}$$

Finally, we note that if $d=2$, if we perform the same calculations using lemma 3.4.1, (2), the result still holds.

□

3.4 Some Spectral Properties of H_0^L

In this section, we prove two lemmas on the structure of the eigenvalues and eigenfunctions of the cutoff, free Laplacian that are used in the previous section. First, let us list the eigenvalues and associated orthogonal, normalized eigenfunctions of H_0^L.

$$E^L_{j_1,\cdots,j_d}=2\sum_{\ell=1}^{d}\cos\left(\frac{j_\ell\pi}{2(L+1)}\right) \tag{3.4.1}$$

$$\Psi^L_{j_1,\cdots,j_d}(n)=\frac{1}{L^{\frac{d}{2}}}\prod_{\ell=1}^{d}\sin\left(\frac{j_\ell\pi n_\ell}{2(L+1)}+\frac{j_\ell\pi}{2}\right) \tag{3.4.2}$$

for $j_1,\cdots,j_d\in\{1,\cdots,2L+1\}$.

We begin with a lemma on counting the eigenvalues of H_0^L in an interval. After, we prove the normalization constant for the eigenfunctions above is correct.

Lemma 3.4.1. *Let H_0^L be as defined in the previous section and $d \geq 3$. We denote the eigenvalues of H_0^L counting multiplicity in nondecreasing order as $\{\lambda_i\}_{i=1}^{2L+1}$. Let $I \subset \mathbb{R}$ be an interval. Then, there exist constants $C_1, C_2, L_0 > 0$, such that the following holds for $|I| \geq C_1 L^{-1}$,*

1. *If $d \geq 3$,*

$$|\{i \ : \ \lambda_i \in I\}| \leq C_2 |I| L^d \tag{3.4.3}$$

2. *If $d = 2$,*

$$|\{i \ : \ \lambda_i \in I\}| \leq C_2 |I| L^d \ |\log |I|| \tag{3.4.4}$$

Remark. 1. *The bound here is similar to the bound in a Wegner estimate, telling us the eigenvalues are approximately uniformly distributed at the appropriate scale. By the appropriate scale, we mean the minimum size of I.*

This condition on the minimum size of I is strictly necessary. To see this explicitly, we can find large L for which there are eigenvalues with multiplicity up to L^{d-1}, so that .

2. *The version of this lemma in the continuum, in which we have,*

$$E^L_{j_1,\cdots,j_d} = \frac{\pi^2 |j|^2}{L^2} \tag{3.4.5}$$

is simpler to prove. In the proof below, we prove that $|m^{-1}(I)| \lesssim |I|$ for $m(\theta) = 2\sum \cos(\theta_j)$. In the continuum, we would use the function $m(k) = |k|^2$, for which this technical result is almost immediate.

Lemma 3.4.2. *Let $\{\Psi^L_{j_1,\cdots,j_d}\}$ be the eigenfunctions of H_0^L as in the beginning of this section. Then,*

$$\|\Psi^L_{j_1,\cdots,j_d}\| = 1 \tag{3.4.6}$$

In particular, the eigenfunctions are uniformly delocalized,

$$\sup_{j_1,\cdots,j_d,n} |\Psi^L_{j_1,\cdots,j_d}(n)| \leq L^{-d/2} \tag{3.4.7}$$

Now, we begin the proofs of the above two lemmas.

Proof of Lemma 3.4.1. Let $\mathcal{Z}_L = \{\frac{1}{2L+2}\pi, \cdots, \frac{2L+1}{2L+2}\pi\}^d$. Let $m(\cdot) : [0,\pi]^d \to \mathbb{R}$, $m(\theta) = 2\sum_{\ell=1}^d \cos(\theta_\ell)$ so that for an interval $I \subset \mathbb{R}$, for sufficiently large L, we have

$$|\{i : \lambda_i \in I\}| = |m^{-1}(I) \cap \mathcal{Z}_L| \tag{3.4.8}$$

It appears as though this count is simply proportional to the volume of the preimage $m^{-1}(I)$ scaled by $(\frac{1}{2L+2})^d$, but we must be a little careful. If I is too small, this is may not be the case since $m^{-1}(I)$ may contain a level surface with many eigenvalues (an energy with high multiplicity) but not many distinct energies. So, we continue our estimate rigorously below to avoid this issue. In particular, we'll see we can proceed with the intuitive estimates if $|I| \geq C\frac{1}{L}$.

Let $\mathcal{B}_L = \{\Lambda = \Lambda_{\frac{\pi}{2L+2}}(x) : \Lambda \text{ has vertices in } \mathcal{Z}_L\}$, a partition of $[0,\pi]^d$ into boxes of side length $\frac{\pi}{2L+2}$. Then,

$$|\{i : \lambda_i \in I\}| = |m^{-1}(I) \cap \mathcal{Z}_L| \tag{3.4.9}$$

$$\leq (2L+2)^d \; |\cup \{\Lambda \in \mathcal{B}_L : \Lambda \cap m^{-1}(I) \neq \emptyset\}| \tag{3.4.10}$$

The union of small cubes in (3.4.10) is contained in some neighborhood of $m^{-1}(I)$. In particular, consider $\Lambda \in \mathcal{B}$, a cube of side length $\frac{\pi}{2L+2}$, such that $\Lambda \cap m^{-1}(I) \neq \emptyset$ and $\theta \in \Lambda$. Then, we must have $|\theta - m^{-1}(I)| \leq d^{1/2}\frac{\pi}{2L+2}$. Thus,

$$(3.4.10) \leq (2L+2)^d \; |\{\theta \in [0,\pi]^d : |\theta - m^{-1}(I)| < \frac{d^{1/2}\pi}{2L+2}\}| \tag{3.4.11}$$

Above, we bounded a union of boxes by a neighborhood of $m^{-1}(I)$. We now use that this neighborhood is contained in the preimage of an expanded interval, $\tilde{I}$, $I \subset \tilde{I}$, with the same center as I and,

$$\tilde{I} := I + \left[-\sup|\nabla m|\frac{\pi d^{1/2}}{2L+2}, \sup|\nabla m|\frac{\pi d^{1/2}}{2L+2}\right] \tag{3.4.12}$$

Indeed, we check here the claimed containment. If there exists $\theta_I \in m^{-1}(I)$, and $|\theta - \theta_I| < \frac{\pi d^{1/2}}{2L+2}$, then $|m(\theta) - m(\theta_I)| \leq \sup|\nabla m|\frac{\pi d^{1/2}}{2L+2}$. So, $|m(\theta) - I| < \sup|\nabla m|\frac{\pi d^{1/2}}{2L+2}$ and $\theta \in m^{-1}(\tilde{I})$. We therefore, get,

$$(3.4.11) \leq (2L+2)^d \; |m^{-1}(\tilde{I})| \tag{3.4.13}$$

where, importantly,

$$|\tilde{I}| = |I| + \sup|\nabla m|\frac{\pi d^{1/2}}{2L+2} = |I| + \frac{2\pi d^{3/2}}{2L+2} \tag{3.4.14}$$

so that, if $|\tilde{I}| \geq C/L$, as we have assumed, then $|\tilde{I}| \lesssim |I|$. Therefore, it is sufficient to show that $|m^{-1}(\tilde{I})| \lesssim |\tilde{I}|$ (for $d = 3$). We now begin showing this technical result.

$$|m^{-1}(\tilde{I})| = \int_{m^{-1}(\tilde{I})} d\theta \tag{3.4.15}$$

Using the change of variables, $y_\ell = \cos(\theta_\ell)$, we see,

$$m^{-1}(\tilde{I}) = \int_{\{2\sum y_\ell \in \tilde{I}\} \cap [-1,1]^d} \prod_\ell \frac{1}{\sqrt{1-y_\ell^2}} dy_\ell \tag{3.4.16}$$

Fix a single octant, which we will label by the corner of the associated unit cube, $\delta = (\delta_1, \cdots, \delta_d) \in \{-1, 1\}^d$. Then, we change variables again, $z_j := 1 - |y_j| = 1 - \delta_j y_j$ and after restricting to one octant and substituting, we get the following,

$$\int_{D_\delta} \prod \frac{dz_\ell}{\sqrt{2z_\ell - z_\ell^2}} \qquad D_\delta := [0,1]^d \cap \{2 \sum \delta_\ell z_\ell \in b - \tilde{I}\} \tag{3.4.17}$$

where, $b = 2\sum_\ell \delta_\ell \in \{-d, \cdots, d\}$ depends on the octant chosen. Now, since $z_\ell \leq 1$, we have $2z_\ell - z_\ell^2 \geq z_\ell$, so we are left to bound,

$$\int_{D_\delta} \prod \frac{dz_\ell}{\sqrt{2z_\ell - z_\ell^2}} \leq \int_{D_\delta} \prod \frac{dz_\ell}{\sqrt{z_\ell}} \tag{3.4.18}$$

First, let us consider the case, $d = 2$, $\delta = (1, 1)$, the first quadrant in the $\{y_\ell\}$ coordinates. We note that in this case, $b = 2$ and that the estimates for $\delta = (-1, -1)$, the third quadrant are identical. Let $b - \tilde{I} =: [E_1, E_2]$. We may get the desired bound using polar coordinates,

$$\int_{D_\delta} \prod \frac{dz_\ell}{\sqrt{z_\ell}} \leq \int_0^{\pi/2} \int_{E_1}^{E_2} \frac{1}{\sqrt{r^2 \sin(\theta)\cos(\theta)}} r \; dr d\theta \tag{3.4.19}$$

$$\leq (E_2 - E_1) \int_0^{\pi/2} \frac{1}{\sqrt{\frac{2}{\pi}\theta(1 - \frac{2}{\pi}\theta)}} d\theta \lesssim E_2 - E_1 = |\tilde{I}|, \tag{3.4.20}$$

which would complete the desired estimate in this case. We note that we have an inequality when switching to polar coordinates, because we may have expanded the region of integration outside $[0, 1]^2$.

Now, for $d \geq 3$, for any $\delta \in \{-1, 1\}^d$, there are at least two coordinates with the same sign. So, fix δ and without loss of generality, let us permute coordinates and change signs so that $\delta_1 = \delta_2 = 1$. Then, we estimate,

$$\int_{D_\delta} \frac{dz_\ell}{\sqrt{z_\ell}} = \int_{[0,1]^{d-2}} \left(\int_{2(z_1+z_2) \in b - \tilde{I} - \sum_{d \geq 3} z_\ell} \frac{dz_1 dz_2}{\sqrt{z_1 z_2}} \right) \prod_{\ell=3}^{d} \frac{dz_\ell}{\sqrt{z_\ell}} \lesssim |\tilde{I}| \tag{3.4.21}$$

Above, the interior integral is bounded by $|\tilde{I}|$ in the same way as the $d = 2$ case and the exterior integral is equal to $2(d - 2)$. Adding up the above estimate for each octant completes the proof for $d \geq 3$.

Finally, we return to part (2) of the lemma, the $d = 2$ case. We must estimate the above integral for the second and fourth quadrant, which are the same up to changing coordinate labels. Assume $(b - \tilde{I})/2 =: [E_1, E_2]$ and we must integrate,

$$\int_{\{z_2 - z_1 \in [E_1, E_2]\} \cap [0,1]^2} \frac{dz_1 dz_2}{\sqrt{z_1 z_2}} \tag{3.4.22}$$

Let us assume that $E_1 \geq 0$. Otherwise, we can divide $[E_1, E_2]$ into $[E_1, 0] \cup [0, E_2]$. By symmetry, the integrals for $[0, E]$ and $[-E, 0]$ are equal, so it is sufficient to bound each of the two integrals by an integral over $z_2 - z_1 \in [0, max\{|E_1|, E_2\}]$. We simplify further, by noting that if $E_1 > 0$ and $E_1 - \epsilon > 0$, that the integral for $z_2 - z_1 \in [E_1, E_2]$ is bounded above by the integral for $[E_1 - \epsilon, E_2 - \epsilon]$ and so we may assume that $E_1 = 0$. So, letting $[E_1, E_2] = [0, E]$, we evaluate the integral,

$$\int_{\{z_2 - z_1 \in [0,E]\} \cap [0,1]^2} \frac{dz_1 dz_2}{\sqrt{z_1 z_2}} \leq \int_0^1 \int_{z_1}^{z_1+E} \frac{1}{\sqrt{z_1 z_2}} dz_2 dz_1 \tag{3.4.23}$$

$$= 2\int_0^1 (\frac{\sqrt{z_1 + E}}{\sqrt{z_1}} - 1) dz_1 = 2(\sqrt{z_1(z_1 + E)} + E \sinh^{-1}(\sqrt{z_1/E}) - z_1)|_0^1 \tag{3.4.24}$$

$$= 2(\sqrt{1-E} - 1) + 2E \sinh^{-1}(\sqrt{1/E}) = 2(\sqrt{1-E} - 1) + 2E \log(\sqrt{1/E} + \sqrt{1/E + 1}) \tag{3.4.25}$$

$$\leq 2E \log(2\sqrt{1/E}) \leq 2E \log(1/E) \tag{3.4.26}$$

which completes part (2) of the lemma.

□

Next, we prove the lemma on the delocalization of the eigenfunctions on H_0^L.

Proof of Lemma 3.4.2. Since we already have the formula for $\Psi^L_{j_1,\cdots,j_d}(n)$, this is an elementary calculation. In particular, it is sufficient to show,

$$\sum_{n=-L}^{L} \sin^2\left(\frac{j\pi n}{2(L+1)} + \frac{j\pi}{2}\right) = L \tag{3.4.27}$$

for any $j \in \{1, \cdots, 2L+1\}$ and any L. To simplify notation, we show the shifted, but equivalent (for odd L),

$$\sum_{n=1}^{L} \sin^2\left(\frac{j\pi n}{L+1}\right) = \frac{L+1}{2} \tag{3.4.28}$$

We proceed with this computation,

$$\begin{aligned} \sum_{n=1}^{L} \sin^2\left(\frac{j\pi n}{L+1}\right) &= \sum_{n=1}^{L} \frac{1 - \cos\left(\frac{2j\pi n}{L+1}\right)}{2} \\ &= \frac{L}{2} - \frac{1}{2}\sum_{n=1}^{L} Re(e^{\frac{2j\pi n}{L+1} i}) \\ &= \frac{L}{2} - \frac{1}{2} Re\left(\frac{e^{\frac{2j\pi L}{L+1} i} - 1}{e^{\frac{2j\pi}{L+1} i} - 1} e^{\frac{2j\pi}{L+1} i}\right) \\ &= \frac{L+1}{2} \end{aligned} \tag{3.4.29}$$

□

3.5 Conclusion and Further Questions

The result in this chapter proves the expected analogue for local eigenvalue statistics to the scattering theory of short range potentials (potentials with power decay down to $\alpha > 1$). An interesting problem that motivated this project, along with to the result of Dolai and Krishna, is to get a similar result for decay down to $\alpha > \frac{1}{2}$, the analogue of the interesting result of Bourgain, [2]. This result would surely need to use the randomness in the model in a more essential way.

Chapter 4 A Minami Estimate for a Finite Rank Lattice Model

4.1 Introduction

In this chapter, we prove a Minami-type estimate at the band edges of the spectrum for a polymer model with single site potential of finite, uniform rank in $\mathbb{Z}^d$ ($d \geq 1$). We apply the technique of Dietlein and Elgart [12]. In [12], they prove a Minami-type estimate for a continuum model in $\mathbb{R}^d$. Our result is an improvement on the previously proven results for the finite-rank polymer model described below. We also have the goal of illuminating the essential parts of the technique of Dietlein and Elgart, so it may be applied to a wide range of other non rank-one models.

Models on $\mathbb{Z}^d$ with uniform, rank $m < \infty$ single site potential were studied by Hislop and Krishna in [17]. They proved a generalized Minami estimate of the following form,

$$\mathbb{P}[\operatorname{tr} \chi_I(H_\omega^L) > m] \leq C_{M,m} L^{2d} |I|^2 \tag{4.1.1}$$

Additionally, they used this estimate to prove that the local eigenvalue statistics converge to a compound Poisson point process. We prove below a Minami-type estimate of the form,

$$\mathbb{P}[\operatorname{tr} \chi_I(H_\omega^L) > 1] \leq C L^{4d} |I| |\log |I||^{-p} \tag{4.1.2}$$

for arbitrary $p > 0$. We note that the bound on the probability is worse, but importantly, we bound the probability of having at least two eigenvalues in I instead of $m+1$. This is sufficient to prove that the local eigenvalue statistics converge to a Poisson point process instead of a compound Poisson point process. However, the general Minami estimate in [17] holds at all energies, while our result holds only in an interval at the edge of the spectrum.

Let us now summarize the strategy of the proof, which closely follows the proof in [12]. Most of the proof works towards proving an estimate on the minimum eigenvalue spacing on some energy interval $\Sigma_M \subset \Sigma$,

$$\operatorname{spac}_{\Sigma_M}(H_\omega^L) := \min_{E_j^\omega, E_{j+1}^\omega \in \sigma(H_\omega^L) \cap \Sigma_M} |E_j^\omega - E_{j+1}^\omega| \tag{4.1.3}$$

In particular, we acquire a bound on $\mathbb{P}[\operatorname{spac}_{\Sigma_M}(H_\omega^L) < \delta]$. The level spacing estimate itself is a type of Minami estimate, since we are bounding the probability of the event of having two eigenvalues near each other. However, the level spacing itself is too weak in its dependence on δ to be able to prove Poisson eigenvalue statistics. We follow [12] and make an improvement to get the Minami estimate.

To get a bound on $\mathbb{P}[\operatorname{spac}_{\Sigma_M}(H_\omega^L) < \delta]$, the strategy again is in two parts: an initial spacing estimate, and a Cartan-type lemma similar to one used by Bourgain in [3].

The initial spacing estimate is a perturbation theory result. It is also the key estimate (both in regards to being able to apply the technique to any specific model, and the energy interval on which the Minami estimate holds). For some small cube of configurations, Q_ϵ, we must find a fixed configuration $\omega_0 \in Q_\epsilon$, such that $\mathrm{spac}_{\Sigma_M}(H^L_{\omega_0}) \geq \delta_0$.

The initial spacing estimate is necessary to use a Cartan-type lemma on $\mathrm{spac}_{\Sigma_M}(H^L_\omega)$. The Cartan-type lemma is a bound on the variation of an analytic function (of several variables). Specifically, in [12], the lemma is applied to the discriminant of H^L_ω in an energy restricted interval, which is an analytic function of ω. The discriminant bounds the level spacing,

$$F(\omega) := \mathrm{spac}_{\Sigma_M}(H^L_\omega) \tag{4.1.4}$$

The result is a bound on the size of the set where F is small,

$$|\omega \in Q_\epsilon \ : \ \mathrm{spac}_{\Sigma_M}(H^L_\omega) < \delta| \tag{4.1.5}$$

depending on δ_0 (see Lemma 4.3.2), which will imply the level spacing estimate.

Finally, let us make some comments on the necessary properties for the proof of the initial spacing estimate. Consider $H_0^{\Lambda_m,*}$ where $*$ can mean both Dirichlet and Neumann boundary conditions and Λ_m is the support of some single site potential. The proof of the inital spacing estimate uses Dirichlet-Neumann bracketing in order to take advantage of the spectral properties of $H_0^{\Lambda_m,*}$. In particular, it is essential that an edge of the spectrum of $H_0^{\Lambda_m,*}$ consists of a simple eigenvalue and a gap between this extreme eigenvalue and the rest of the spectrum. This gap is directly tied to Σ_M, the set on which the Minami estimate holds.

We also note that Dietlein and Elgart also prove versions of the initial spacing estimate and the level spacing estimate for energies in the localization regime with a better probability bound, but we do not include one here.

4.2 Model

We consider a random Schrödinger alloy type operator on $\ell^2(\mathbb{Z}^d)$. Let H_0 be the positive, discrete Laplacian,

$$H_0 f(n) = 2d f(n) - \sum_{|n'-n|=1} f(n'). \tag{4.2.1}$$

So, $\sigma(H_0) = [0, 4d]$.

Let $\Lambda_r(n) = \prod_{j=1}^d \{n_j, \cdots, n_j + r - 1\}$ be a cube with side length $r \in \mathbb{N}$ and vertex at $n \in \mathbb{Z}^d$. Let χ_Λ be the characteristic function on Λ, or equivalently in this model, $\chi_\Lambda = P_\Lambda$, the projection onto the sites in Λ with rank $|\Lambda| = r^d$. Note that r^d is equal to the m from the introduction.

Let,

$$H_\omega = H_0 + V_\omega \tag{4.2.2}$$

where V_ω is a random potential that we define below.

- $V_\omega := \sum_{k\in r\mathbb{Z}^d} \omega_k \chi_{\Lambda_r(k)}$
- $\{\omega_n\}_{n\in r\mathbb{Z}^d}$ is a collection of i.i.d. bounded, random variables with continuous density ρ and $\mathrm{supp}\rho = [0,1]$. We also assume ρ is Lipschitz-continuous and bounded below,

$$|\rho(x) - \rho(y)| < \mathcal{K}|x-y| \text{ and } \rho_- < \rho(x) < \rho_+ \tag{4.2.3}$$

for some $\mathcal{K}, \rho_-, \rho_+ \in (0,\infty)$ and all $x, y \in [0,1]$.

For $L \in r\mathbb{Z}$, we choose Λ_L to be centered such that for an index set $\Lambda_L^* \subset \mathbb{Z}^d$ of size $(\frac{L}{r})^d$,

$$\Lambda_L = \cup_{k\in\Lambda_L^*} \Lambda_r(k)$$
$$V_\omega^L = \sum_{k\in\Lambda_L^*} \omega_k \chi_{\Lambda_r(k)} = \chi_{\Lambda_L} V_\omega$$

We also note that our assumptions imply that the deterministic spectrum for this this model is,

$$\Sigma = \sigma(H_0) + \mathrm{supp}(\rho) = [0, 4d+1] \tag{4.2.4}$$

by which we mean $\sigma(H_\omega) = \Sigma$ almost surely.

4.3 Functional Analytic Lemmas

In this section, we repeat two essential lemmas from [12]. The proofs, along with intermediate lemmas used in their proofs can be found in [12, Section 3].

These lemmas are independent of the specific model we are interested in and rely only on functional analysis. We begin with the context and notation required for the lemmas.

Let A be self-adjoint operator on a separable Hilbert space, $\mathcal{H}$ and let $I \subset \mathbb{R}$ be an interval with $|I| \le \frac{1}{2}$. Fix $\epsilon \in (0, \frac{1}{12})$. We assume A has n eigenvalues in I and there exists a gap between I and the rest of the spectrum of A,

$$\begin{aligned} n &:= \operatorname{tr} \chi_I(A) < \infty, \\ \mathrm{d}(I, \sigma(A)\backslash I) &\ge 6\epsilon \end{aligned} \tag{4.3.1}$$

where $d(A,B) =$ the distance between two sets A and B in $\mathbb{R}$.

Let B be a bounded, self-adjoint operator with $\|B\| \le 1$ and consider the one-parameter family of operators,

$$A_s := A + sB \tag{4.3.2}$$

for $s \in (-\epsilon, \epsilon)$.

Let $I_\epsilon = I + (-\epsilon, \epsilon)$ and let $\{E_i^s\}_{i=1}^n$ be the eigenvalues of A_s in I_ϵ. We also denote the average of these eigenvalues, $\overline{E^s} = \frac{1}{n}\sum_i E_i^s$.

Lemma 4.3.1 (Lemma 3.1 from [12])**.** *Let* $0 < \delta < \epsilon < \frac{1}{12}$*. If the eigenvalues satisfy*

$$\sup_{s\in[-\epsilon,\epsilon]} \sup_{i=1,\ldots,n} |E_i^s - \overline{E}^s| \le \delta, \tag{4.3.3}$$

then

$$\|P_s(B - \partial_s \overline{E}^s)P_s\| \le 9\sqrt{\frac{\delta}{\epsilon}}. \tag{4.3.4}$$

Let $N \in \mathbb{N}$ and $0 \le B_k \le 1$ be self-adjoint operators for $k \in \{1, \cdots, N\}$ such that $\sum_k B_k \le 1$. Consider the N-parameter family of operators,

$$(s_1, \cdots, s_N) \mapsto A_s := A + \sum_k s_k B_k \tag{4.3.5}$$

for $(s_1, \cdots, s_N) \in (-\epsilon, \epsilon)^N$.

The following is the Cartan-type lemma we refer to in the introduction and in the proofs below.

Lemma 4.3.2 (Lemma 3.4 from [12])**.** *Suppose there exists* $\delta_0 \in (0, \epsilon)$ *and* $s_0 \in (-\epsilon, \epsilon)^N$ *such that,*

$$\mathrm{spac}_{I_\epsilon}(A_{s_0}) > \delta_0 \tag{4.3.6}$$

Then, there exist constants C_1, C_2 *such that,*

$$|\{s \in (-\epsilon, \epsilon)^N \ : \ \mathrm{spac}_{I_\epsilon}(A_s) < \delta\}| \le C_1 N (2\epsilon)^N \exp\left(-\frac{C_2}{n^2} \frac{|\log \delta|}{|\log \delta_0|}\right) \tag{4.3.7}$$

for all $\delta \in (0, 1)$.

4.4 Eigenvalue Spacing and Minami Estimates

Recall that $r \in \mathbb{N}$ is fixed by the projections $\chi_{\Lambda_r(k)}$, which are rank r^d and we assume $L \in r\mathbb{N}$.

We begin with the key level spacing estimate, proving that there is an energy interval on which we can find a configuration where two eigenvalues in an interval, I cannot be too close together. Theorem 4.4.1 is the full initial spacing estimate and is a result of induction on this lemma. We recall the notation introduced in the previous section, $I_\epsilon = I + [-\epsilon, \epsilon]$.

Lemma 4.4.1 (Initial Spacing Estimate)**.** *Let $\epsilon \in (0, \frac{1}{12})$. We suppose there is a configuration ω_0 and an interval $I \subset \mathbb{R}$ with $|I| \leq \frac{1}{2}$ for which the following holds,*

1. *The local Hamiltonian $H^L_{\omega_0}$ has n eigenvalues $\{E_i^{\omega_0}\}_{i=1}^n$ in I, counting multiplicity, and the interval I is well-separated from $\sigma(H^L_{\omega_0})\backslash I$,*

$$d(I, \sigma(H_{\omega_0})\backslash I) \geq 8\epsilon \tag{4.4.1}$$

2. *The interval I is located in a region at an edge of the spectrum,*

$$I \subset [0, \gamma_{L,n,r}] \cup [4d+1-\gamma_{L,n,r}, 4d+1] \tag{4.4.2}$$

where,

$$\gamma_{L,n,r} := 2(1-\cos(\pi/r))(1-\frac{1}{n}-\frac{9\sqrt{2}}{\sqrt{nLr^d}}) \tag{4.4.3}$$

We note that this immediately implies the average of the eigenvalues is also located at an edge of the spectrum: $\overline{E}^{\omega_0} := \frac{1}{n}\sum_{i=1}^n E_i^{\omega_0} \in [0, \gamma_{L,n,r}] \cup [4d+1-\gamma_{L,n,r}, 4d+1]$

Then, there exists $\widehat{\omega} \in \widehat{Q}_\epsilon := \omega_0 + [-\epsilon(1-L^{-(2d+1)}), \epsilon(1-L^{-(2d+1)})]^{\Lambda_L}$, and an integer $1 \leq k \leq n-1$ so that

$$|E_{k+1}^{\widehat{\omega}} - E_k^{\widehat{\omega}}| > 8\epsilon L^{-(2d+1)}. \tag{4.4.4}$$

Proof. We proceed by proving the contrapositive: suppose under the condition (1), all the eigenvalues in I_ϵ lie close together for all $\omega \in \widehat{Q}_\epsilon$,

$$\sup_{\omega\in\widehat{Q}_\epsilon} \max_k |E_{k+1}^\omega - E_k^\omega| \leq 8\epsilon L^{-(2d+1)} \tag{4.4.5}$$

then we must have,

$$\overline{E}^\omega \in (\gamma_{L,n,r}, 4d+1-\gamma_{L,n,r}) \tag{4.4.6}$$

Note that (4.4.5) implies,

$$\sup_{\omega\in\widehat{Q}_\epsilon} \max_k |E_k^\omega - \overline{E}^\omega| \leq 8n\epsilon L^{-(2d+1)} \tag{4.4.7}$$

We recall that $L \in r\mathbb{N}$, so that Λ_L consists of $(L/r)^d$ subcubes, each containing r^d points. For $k \in r\mathbb{Z}^d$, we let $\Delta_k^N = \Delta^N_{\Lambda_r(k)}$ and $\Delta_k^D = \Delta^D_{\Lambda_r(k)}$ be the Neumann and

Dirichlet Laplacians for the subcube, $\Lambda_r(k)$ respectively. Recall the notation for the set of points, $k \in r\mathbb{Z}^d$ that index our subcubes, $\Lambda_L^* := r\mathbb{Z}^d \cap \Lambda_L$.

We will use Dirichlet-Neumann bracketing (see Section 4.5 for definitions of Dirichlet and Neumann boundary conditions in the lattice and [19, Section 5.2] for a proof of Dirichlet-Neumann bracketing in the lattice):

$$\bigoplus_{k\in\Lambda_L^*} \Delta_k^N \le \Delta_L \le \bigoplus_{k\in\Lambda_L^*} \Delta_k^D, \tag{4.4.8}$$

For ease of notation, we define the spectral projection $P_\omega := \chi_{I_\epsilon}(H_\omega^L)$, which in this case, is simply the projection onto the span of the eigenspaces of H_ω^L associated to eigenvalues in I_ϵ.

The idea is to get bounds on $\overline{E}^\omega = \frac{1}{n}\operatorname{tr} P_\omega H_\omega^L P_\omega$ using Dirichlet-Neumann bracketing,

$$\sum_{k\in\Lambda_L^*} \operatorname{tr} P_\omega \Delta_k^N \le \operatorname{tr} P_\omega H_\omega^L \le \sum_{k\in\Lambda_L^*} \operatorname{tr} P_\omega \Delta_k^D + \|V_\omega\| \operatorname{tr} P_\omega \tag{4.4.9}$$

using that $V_\omega \ge 0$ for the left most inequality.

Let $R_{k,0}^N$ be the projection onto the eigenspace corresponding to the smallest eigenvalue of Δ_k^N and $R_{k,m}^D$ be the projection onto the eigenspace corresponding to the largest eigenvalue of Δ_k^D.

Examining (4.5.7) confirms that $R_{k,0}^N$ and $R_{k,0}^D$ are both rank one projectors. Furthermore, $\gamma_r := 2(1-\cos(\pi/r))$ is equal to the second smallest eigenvalue of Δ_k^N and $4d-\gamma_r$ is equal to the second largest eigenvalue of Δ_k^D. We therefore, have,

$$\begin{aligned}\Delta_k^N &\ge 0\cdot R_{k,0}^N + \gamma_r (R_{k,0}^N)^\perp \\ &= \gamma_r \chi_{\Lambda_r(k)} - \gamma_r R_{k,0}^N, \\ \Delta_k^D &\le 4d R_{k,m}^D + (4d-\gamma_r)(R_{k,m}^D)^\perp \\ &= (4d-\gamma_r)\chi_{\Lambda_r(k)} + \gamma_r R_{k,m}^D\end{aligned} \tag{4.4.10}$$

where $(R_{k,0}^N)^\perp$ and $(R_{k,m}^D)^\perp$ are understood to be the complements in $\ell^2(\Lambda_r(k))$.

First combining the upper bounds from (4.4.10) and (4.4.9), we get,

$$\begin{aligned} n\overline{E^\omega} = \operatorname{tr} P_\omega H_\omega^L &\le n\|V_\omega\| + \sum_{k\in\Lambda_L^*} (4d-\gamma_r)\operatorname{tr} P_\omega \chi_{\Lambda_r(k)} P_\omega + \gamma_r \operatorname{tr} P_\omega R_{k,m}^D \\ &\le n + \sum_{k\in\Lambda_L^*} (4d-\gamma_r)\operatorname{tr} P_\omega \chi_{\Lambda_r(k)} P_\omega + \gamma_r \|P_\omega \chi_{\Lambda_r(k)}\| \operatorname{tr} R_{k,m}^D \\ &= n + \sum_{k\in\Lambda_L^*} (4d-\gamma_r)\operatorname{tr} P_\omega \chi_{\Lambda_r(k)} P_\omega + \gamma_r \|P_\omega \chi_{\Lambda_r(k)} P_\omega\| \\ &= n + (4d-\gamma_r)\operatorname{tr} P_\omega + \sum_{k\in\Lambda_L^*} \gamma_r \|P_\omega \chi_{\Lambda_r(k)} P_\omega\| \end{aligned} \tag{4.4.11}$$

We perform a similar calculation to get a lower bound,

$$\begin{aligned} n\overline{E^\omega} = \operatorname{tr} P_\omega H_\omega^L &\geq \sum_{k\in\Lambda_L^*} \gamma_r \operatorname{tr} P_\omega \chi_{\Lambda_r(k)} P_\omega - \gamma_r \operatorname{tr} P_\omega R_k^N \\ &\geq \sum_{k\in\Lambda_L^*} \gamma_r \operatorname{tr} P_\omega \chi_{\Lambda_r(k)} P_\omega - \gamma_r \| P_\omega \chi_{\Lambda_r(k)} P_\omega \| \operatorname{tr} R_k^N \\ &= \gamma_r n - \sum_{k\in\Lambda_L^*} \gamma_r \| P_\omega \chi_{\Lambda_r(k)} P_\omega \| \end{aligned} \tag{4.4.12}$$

Dividing by n and combining the lower and upper bounds gives,

$$\begin{aligned} &\gamma_r - \frac{\gamma_r}{n} \sum_{k\in\Lambda_L^*} \| P_\omega \chi_{\Lambda_r(k)} P_\omega \| \leq \overline{E^\omega} \\ &\leq 1 + 4d - \gamma_r + \frac{\gamma_r}{n} \sum_{k\in\Lambda_L^*} \| P_\omega \chi_{\Lambda_r(k)} P_\omega \| \end{aligned} \tag{4.4.13}$$

For the lower bound, we have also used that $V_\omega \geq 0$

Let $\alpha_k^\omega := \frac{1}{n} \operatorname{tr} P_\omega \chi_{\Lambda_r(k)} P_\omega$. An application of Lemma 4.3.1 with $\delta = 8n\epsilon L^{-2d+1}$ and $\epsilon' = \epsilon(1 - L^{-(2d+1)})$ guarantees that,

$$|\ \alpha_k^\omega - \| P_\omega \chi_{\Lambda_r(k)} P_\omega \|\ | < 9\sqrt{2n} L^{-d-\frac{1}{2}} \tag{4.4.14}$$

Summing over k, and using that $\sum_k \alpha_k^\omega = 1$, results in the bounds,

$$1 - 9\sqrt{2n} L^{-d-\frac{1}{2}} (\frac{L}{r})^d < \sum_{k\in\Lambda_L^*} \| P_\omega \chi_{\Lambda_r(k)} P_\omega \| < 1 + 9\sqrt{2n} L^{-d-\frac{1}{2}} (\frac{L}{r})^d \tag{4.4.15}$$

Substituting into (4.4.13) gives,

$$\gamma_r (1 - \frac{1}{n} - \frac{9\sqrt{2}}{\sqrt{nL} r^d}) < \overline{E}^\omega < 1 + 4d - \gamma_r (1 - \frac{1}{n} - \frac{9\sqrt{2}}{\sqrt{nL} r^d}) \tag{4.4.16}$$

or,

$$\overline{E}^\omega \in (\gamma_{L,n,r}, 1 + 4d - \gamma_{L,n,r}) \tag{4.4.17}$$

as desired.

□

The above lemma is the key result for determining if these techniques can prove a Minami estimate for a specific model. Thus, if we wish to apply the same machinery to a different model, or if we want to get an improvement in the energy at which the result holds, we should look to this lemma first.

Next, we prove the full initial spacing estimate by induction, using the previous lemma.

Theorem 4.4.1 (Initial Spacing Estimate, Inductive Step)**.** *Let $\epsilon \in (0, \frac{1}{12})$. We suppose there is a configuration ω_0 and an interval $I \subset \mathbb{R}$ with $|I| \leq \frac{1}{2}$ for which the following holds,*

1. *The local Hamiltonian $H^L_{\omega_0}$ has n eigenvalues $E_i^{\omega_0}$ in I, counting multiplicity, and the interval I is well-separated from $\sigma(H^L_{\omega_0})\backslash I$,*

$$d(I, \sigma(H_{\omega_0})\backslash I) \geq 8\epsilon \tag{4.4.18}$$

2. *The interval I is located in a region at an edge of the spectrum,*

$$I \subset [0, \gamma_{L,2,r} - \epsilon] \cup [4d + 1 - \gamma_{L,2,r} + \epsilon, 4d + 1] \tag{4.4.19}$$

Then, there exists $\widehat{\omega} \in \widehat{Q}_\epsilon := \omega_0 + [-\epsilon, \epsilon]^{\Lambda_L}$, so that

$$\min_{1 \leq k \leq n-1} |E^{\widehat{\omega}}_{k+1} - E^{\widehat{\omega}}_k| > 8\epsilon L^{-(n-1)(2d+1)}. \tag{4.4.20}$$

Proof. This theorem is a result of iterating Lemma 4.4.1.

1) We first apply the lemma with $\epsilon_1 = \epsilon$ and find ω_1, $|\omega_1 - \omega_0| \leq \epsilon_1(1 - L^{-(2d+1)})$, so that for some $1 \leq k_1 \leq n - 1$,

$$E^{\omega_1}_{k_1+1} - E^{\omega_1}_{k_1} > 8\epsilon_1 L^{-(2d+1)}. \tag{4.4.21}$$

2) Next, let $\epsilon_2 := \epsilon_1 L^{-(2d+1)}$. Define the groups of eigenvalues from the previous step, $C_1^{\omega_1} = \{E_1^{\omega_1}, \cdots, E_{k_1}^{\omega_1}\}$ and $C_2^{\omega_1} = \{E^{\omega_1}_{k_1+1}, \cdots, E_n^{\omega_1}\}$. If $k_1 \geq 2$, we apply the lemma to $C_1^{\omega_1}$ with $\epsilon = \epsilon_2$. If $k_1 = 1$, we apply the lemma to $C_2^{\omega_1}$.

We must check that the gaps between groups of eigenvalues are sufficiently large to apply the lemma. We have $d(C_1^{\omega_1}, C_2^{\omega_1}) \geq 8\epsilon_2$ as a result of the first iteration of the lemma. Furthermore, for $i = 1, 2$

$$d(C_i^{\omega_1}, \sigma(H^L_{\omega_1})\backslash(C_1^{\omega_1} \cup C_2^{\omega_1})) \geq 8\epsilon_1 - 2|\omega_1 - \omega_0| > 6\epsilon_1 > 8\epsilon_2 \tag{4.4.22}$$

Thus, we have ω_2, with $|\omega_2 - \omega_1| \leq \epsilon_2(1 - L^{-(2d+1)})$ and three groups of eigenvalues, $C_i^{\omega_2}$, for $i = 1, 2, 3$. By the same calculations as in the previous step, for $i = 1, 2, 3$,

$$d(C_i^{\omega_2}, \sigma(H^L_{\omega_2})\backslash C_i^{\omega_2}) \geq 8\epsilon_2 L^{-(2d+1)} =: 8\epsilon_3 \tag{4.4.23}$$

3) We continue iterating, letting $\epsilon_{j+1} := \epsilon_j L^{-(2d+1)}$. After $n-1$ iterations, we find ω_{n-1}, with $|\omega_{n-1} - \omega_{n-2}| \leq \epsilon_{n-1}(1 - L^{-(2d+1)})$ for which each eigenvalue is isolated,

$$d(E_i^{\omega_{n-1}}, \sigma(H^L_{\omega_{n-1}})\backslash\{E_i^{\omega_{n-1}}\}) > 8\epsilon_{n-1}L^{-(2d+1)} = 8\epsilon L^{-(n-1)(2d+1)} \tag{4.4.24}$$

Thus, (4.4.20) is satisfied for $\widehat{\omega} = \omega_{n-1}$. Finally, we must check that $|\omega_{n-1} - \omega_0| \leq \epsilon$,

$$\begin{aligned}|\omega_{n-1} - \omega_0| &\leq \sum_{i=0}^{n-2} |\omega_{i+1} - \omega_i| \leq \sum_{i=0}^{n-2} \epsilon_{i+1}(1 - L^{-(2d+1)}) \\ &= \epsilon(1 - L^{-(2d+1)}) \sum_{i=0}^{n-2} L^{-i(2d+1)} = \epsilon(1 - L^{-(2d+1)}) \frac{1 - L^{-(n-1)(2d+1)}}{1 - L^{-(2d+1)}} < \epsilon\end{aligned} \tag{4.4.25}$$

We note that at each step of the iteration, each group of eigenvalues, $C_i^{\omega_j}$, is contained in $[0, \gamma_{L,2,r}] \cup [4d+1-\gamma_{L,2,r}, 4d+1]$ and has size between 2 and n or is a singleton and does not need splitting. Therefore, the conditions of Lemma 4.4.1 are always satisfied when it is applied. □

Next, we prove the bound on the probability that the minimum eigenvalue spacing is small by using the initial spacing estimate and the Cartan type lemma.

Theorem 4.4.2 (Eigenvalue Level Spacing Estimate). *Let* $I_{sp} = [0, \gamma_{\infty,r}] \cup [4d+1-\gamma_{\infty,r}, 4d+1]$ *where,*

$$\gamma_{\infty,r} := (1 - \cos(\pi/r)) \tag{4.4.26}$$

Fix any $0 < E < \gamma_{\infty,r}$ *and* $p > 0$ *and let,*

$$I_E = [0, E] \cup [4d+1-E, 4d+1] \tag{4.4.27}$$

Then, there exists $\mathcal{L}_{sp} = \mathcal{L}_{sp}(E,p)$, $C_{sp} = C_{sp}(E,p)$ *such that,*

$$\mathbb{P}(\operatorname{spac}_{I_E}(H_\omega^L) < \delta) \le C_{sp} L^{2d} |\log \delta|^{-p} \tag{4.4.28}$$

for $L \ge \mathcal{L}_{sp}$ *and* $\delta \le \exp(-(\log L)^5)$.

Proof. Decompose I_{sp} into overlapping intervals $\{K_i\}_{\mathcal{I}}$ with length $|K_i| = \kappa$ and $|K_{i+1} \cap K_i| \ge \kappa/2$ so that $|\mathcal{I}| \le \lceil |I_{sp}|(\frac{2}{\kappa}) \rceil$.

Let $K_{i,8\epsilon} := K_i + [-8\epsilon, 8\epsilon]$ for $\epsilon \in (0, 1/12)$. Define the event $\Omega_{i,\epsilon}$,

$$\Omega_{i,\epsilon} := \{\omega \ : \ \operatorname{tr} \chi_{K_i}(H_\omega^L) \le r^d \text{ and } \operatorname{tr} \chi_{K_{i,8\epsilon} \setminus K_i}(H_\omega^L) = 0\} \tag{4.4.29}$$

The probability of $\Omega_{i,\epsilon}^c$ can be bounded using a Wegner estimate and a generalized Minami estimate [17],

$$\begin{aligned} \mathbb{P}(\Omega_{i,\epsilon}^c) &\le \mathbb{P}(\operatorname{tr} \chi_{K_{i,8\epsilon} \setminus K_i}(H_\omega^L) \ge 1) + \mathbb{P}(\operatorname{tr} \chi_{K_i}(H_\omega^L) > r^d) \\ &\le C_W L^d (16\epsilon) + C_{M,r^d} L^{2d} \kappa^2 \end{aligned} \tag{4.4.30}$$

For $0 < \delta < \kappa/2$,

$$\mathbb{P}(\operatorname{spac}_{I_E}(H_\omega^L) < \delta) \le 16 C_W L^d \epsilon |\mathcal{I}| + C_{M,r^d} L^{2d} \kappa^2 |\mathcal{I}| + \sum_{i \in \mathcal{I}} \mathbb{P}(\{\operatorname{spac}_{K_i}(H_\omega^L) < \delta\} \cap \Omega_{i,\epsilon}) \tag{4.4.31}$$

Next, partition the configuration space, $[0,1]^{\Lambda_L^*}$ into cubes Q_j, $j \in \mathcal{J}$ with side length 2ϵ so that $|\mathcal{J}| \le (\lceil \frac{1}{2\epsilon} \rceil)^{(L/r)^d}$. In case $(2\epsilon)^{-1}$ is not an integer, we may choose the Q_j to be almost disjoint except for cubes at within 2ϵ of the boundary of $[0,1]^{(L/r)^d}$. Therefore, adding the probability that is overcounted by this overlapping, we have,

$$\begin{aligned} \sum_{j \in \mathcal{J}} \mathbb{P}(Q_j) &\le \sum_{k \in \Lambda_L^*} \int_{\min(1-\omega_k, \omega_k) \le \epsilon} \prod_{n \in \Lambda_L^*} \rho(\omega_n) d\omega_n \\ &\le 1 + 4\epsilon (L/r)^d \rho_+ \end{aligned}$$

Fix i,j so that $Q_j \cap \Omega_{i,\epsilon} \neq \emptyset$ and let $\omega_{i,j} \in Q_j \cap \Omega_{i,\epsilon}$. Then, the following holds for $H^L_{\omega_{i,j}}$,

$$\begin{aligned} n_{i,j} &:= \operatorname{tr} \chi_{K_i}(H^L_{\omega_{i,j}}) \le r^d, \\ d(K_i, \sigma(H^L_{\omega_{i,j}}) \backslash K_i) &\ge 8\epsilon \end{aligned} \tag{4.4.32}$$

So, an application of Theorem 4.4.1 yields $\widehat{\omega}_{i,j} \in Q_j$ such that,

$$\operatorname{spac}_{K_{i,\epsilon}}(H^L_{\widehat{\omega}_{i,j}}) \ge 8\epsilon L^{-(n_{i,j}-1)(2d+1)} \ge 8\epsilon L^{-(r^d-1)(2d+1)} \tag{4.4.33}$$

We will see below that ϵ is small (as a function of L). So, for any fixed E, we can choose $\mathcal{L}$ large enough so that $I_E \subset [0, \gamma_{L,2,r} - \epsilon] \cup [4d+1-\gamma_{L,2,r}+\epsilon, 4d+1]$ and Theorem 4.4.1 may be applied.

Now, we apply Lemma 4.3.2, the Cartan-type lemma, with $\delta_0 = 8\epsilon L^{-(r^d-1)(2d+1)}$ to get the following bound,

$$\mathbb{P}(Q_j \cap \{\operatorname{spac}_{K_i}(H^L_\omega) < \delta\} \cap \Omega_{i,\epsilon}) \le (\prod_{k\in\Lambda^*_L} \inf_{(Q_j)_k} \rho)|\{\operatorname{spac}_{K_i}(H^L_\omega) < \delta\} \cap Q_j| \tag{4.4.34}$$

where $(Q_j)_k$ denotes the interval given by projection of Q_j on the the k^{th} coordinate. We continue,

$$\begin{aligned} (4.4.34) &\le C_1(1+2\epsilon C_\rho)^{(L/r)^d}\mathbb{P}(Q_j)(L/r)^d \exp\left(\frac{-C_2}{r^{2d}}\frac{|\log\delta|}{|\log\delta_0|}\right) \\ &\le C_1(1+2\epsilon C_\rho)^{(L/r)^d}\mathbb{P}(Q_j)\left(\frac{L}{r}\right)^d \\ &\exp\left(\frac{-C_2}{r^{2d}}\frac{|\log\delta|}{|\log(8\epsilon)|+(r^d-1)(2d+1)|\log L|}\right) \end{aligned} \tag{4.4.35}$$

For $0<\delta\le\exp(-(\log L)^5)$, choose,

$$\begin{aligned} \kappa &:= |\log\delta|^{-\alpha} \\ \epsilon &:= \exp(-|\log\delta|^{1/4}) \end{aligned} \tag{4.4.36}$$

We get,

$$\begin{aligned} \mathbb{P}(\operatorname{spac}_{I_E}(H^L_\omega) < \delta) &\le 16C_W(2|I_{sp}|)L^d\epsilon/\kappa + C_{M,r^d}(2|I_{sp}|)L^{2d}\kappa^{2-1} \\ &+ \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i}\mathbb{P}(\{\operatorname{spac}_{K_i}(H^L_\omega)<\delta\}\cap\Omega_{i,\epsilon}\cap Q_j) \\ &\le 16C_W(2|I_{sp}|)L^d\epsilon/\kappa + C_{M,r^d}(2|I_{sp}|)L^{2d}\kappa \\ &+ \kappa^{-1}C_1(1+2\epsilon C_\rho)^{(L/r)^d}(L/r)^d\exp\left(-C_{3,r}\frac{|\log\delta|}{|\log\epsilon|+|\log L|}\right) \\ &\le C'L^d\exp(-|\log\delta|^{1/4})|\log\delta|^\alpha + C''L^{2d}|\log\delta|^{-\alpha} \\ &+ C'''L^d|\log\delta|^\alpha\exp\left(-C_{3,r}|\log\delta|^{1/2}\right) \end{aligned} \tag{4.4.37}$$

Using that $\delta \le \exp(-(\log L)^5)$, for any $p > 0$, we can choose α and $\mathcal{L}_{sp}$ so that,

$$\mathbb{P}(\mathrm{spac}_{I_E}(H_\omega^L) < \delta) \le C_{sp} L^{2d} |\log \delta|^{-p} \tag{4.4.38}$$

□

Finally, we prove the Minami-type estimate as a consequence of the eigenvalue level spacing estimate. Theorem 4.4.2 already estimates the probability that there are two eigenvalues I_{sp} whose distance from each other is less than some small δ. However, we would like the probability bound for our Minami estimate to be at least $o(\delta)$ for applications. The key to get the improvement is that $|I_{sp}|$ is order one and $\delta << 1$. So, broadly, we prove the probability that two eigenvalues are within δ of each other is approximately evenly divided over a larger energy interval in I_{sp}.

Theorem 4.4.3 (Minami Estimate)**.** *Let H_ω^L be the same as in Theorem* 4.4.2*. Fix*

$$E \in [0, \gamma_{\infty,r}) \cup (4d + 1 - \gamma_{\infty,r}, 4d + 1) \tag{4.4.39}$$

and let $I = E + [-\delta/2, \delta/2]$. Recall $\gamma_{\infty,r} = 1 - \cos(\pi/r)$ Fix any $p > 0$. Then, there exists $C > 0$, $\mathcal{L}_{\mathrm{sp}} > 0$ such that,

$$\mathbb{P}(\operatorname{tr} \chi_I(H_\omega^L) \ge 2) \le C L^{4d} \delta |\log \delta|^{-p} \tag{4.4.40}$$

for $L \ge \mathcal{L}_{\mathrm{sp}}$, $\delta \le \exp(-(\log L)^5)$.

Remark. *Before beginning the proof of the Minami estimate, we need to be able to apply the level spacing estimate to a slightly different Hamiltonian,*

$$H_{\omega,\kappa}^L := H_\omega^L - (1 - \kappa) H_0^L = \kappa \Delta^L + V_\omega^L \tag{4.4.41}$$

for $\kappa \in \mathbb{R}$ close to 1*. For the continuum model, Dietlein and Elgart need to conjugate H_0^L by a bounded function in addition to scaling it, and so call $H_{\omega,\kappa}^L$ a deformed Schrödinger operator.*

The proofs of the previous lemma and theorems are the same when proven for $H_{\omega,\kappa}^L$ except in the initial spacing estimate, the eigenvalues of $\kappa \Delta^*_{\Lambda_r(k)}$ are of course scaled by κ, which affects the energy regime for which the estimates are valid. Specifically, the respective intervals in the hypotheses of Lemma 4.4.1 and Theorem 4.4.1 should be replaced with the following,

$$[0, \kappa \gamma_{L,n,r}] \cup [1 + \kappa(4d - \gamma_{L,n,r}), 4d + 1] \tag{4.4.42}$$

$$[0, \kappa \gamma_{L,n,r} - \epsilon] \cup [1 + \kappa(4d - \gamma_{L,n,r}) + \epsilon, 4d + 1] \tag{4.4.43}$$

Recall that the level spacing estimate, Theorem 4.4.2, holds on

$$I_E = [0, E] \cup [1 + 4d - E, 4d + 1]$$

for arbitrary $E < \gamma_{\infty,r} = \sup_L \gamma_{L,2,r}$. We can see that for any E as above, there exists $\kappa_0 \in (0, 1)$ sufficiently close to 1 so that Theorem 4.4.2 holds for $H_{\omega,\kappa}^L$ on I_E for any $\kappa \in [\kappa_0, 1]$.

Below, we will take $\kappa < 1$ and $\kappa \to 1$ as $L \to \infty$. Thus, for sufficiently large L, we will be able to apply Theorem 4.4.2 to the necessary intervals.

Proof of Theorem 4.4.3. First, we make sure to choose $\mathcal{L}_{sp}$ large enough so that $\exp(-(\log L)^5) \leq L^{-d}/4 \leq |I_{\mathrm{sp}}| = 2\gamma_{\infty,r}$.

Let $\mathcal{I}_1 = [E - \delta/2, E - \delta/2 + \frac{1}{2L^d}]$ and $\mathcal{I}_2 = [E - \delta/2, E - \delta/2 + \frac{1}{L^d}]$ so that $|\mathcal{I}_1| = \frac{1}{2L^d}$, $|\mathcal{I}_2| = \frac{1}{L^d}$. We choose $\mathcal{L}_{sp}$ large enough to ensure $I \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset I_{\mathrm{sp}}$.

Cover $\mathcal{I}_2$ with almost disjoint intervals $\{I_i\}_{i=1}^N$ of length δ (possibly shorter for the intervals intersecting boundaries of $\mathcal{I}_2$) so that $I = I_{i_0}$ for some $i_0 \leq N$ and $\frac{1}{L^d\delta} \leq N \leq \frac{1}{L^d\delta} + 2$.

We compute,

$$\mathbb{P}(\operatorname{tr}\chi_I(H^L_\omega) \geq 2) \leq \sum_{j=1}^{L^d} \mathbb{P}(\mathrm{spac}_{\mathcal{I}_1}(H^L_\omega) \leq \delta \text{ and } E^\omega_j \in I) \tag{4.4.44}$$

Define the following events,

$$\Omega^J_{i,j} := \{\mathrm{spac}_J(H^L_\omega) < \delta \text{ and } E^\omega_j \in I_i\} \tag{4.4.45}$$

The following key estimate can be thought of as uniformity of $\mathbb{P}(\Omega^{\mathcal{I}_1}_{i,j})$ over i. First, we make some definitions. Let $\kappa := \frac{1}{1+L^{-d}}$ and recall the definition of $H^L_{\omega,\kappa}$ from the remark above,

$$H^L_{\omega,\kappa} := H^L_\omega - (1-\kappa)H^L_0 = \kappa\Delta^L + V^L_\omega \tag{4.4.46}$$

We will also use the notation $E^\kappa_{\omega,j}$ to denote the eigenvalues of $H^L_{\omega,\kappa}$ in ascending order.

We may enlarge $\mathcal{L}_{sp}$ so that Theorem 4.4.2 can be applied to $H^L_{\omega,\kappa}$ on the interval $\kappa\mathcal{I}_2$. We claim that for some $C_\rho > 0$, and any $1 \leq i \leq N$,

$$\mathbb{P}(\Omega^{\mathcal{I}_1}_{i_0,j}) \leq C_\rho \mathbb{P}[\mathrm{spac}_{\kappa\mathcal{I}_2}(H^L_{\omega,\kappa}) < \delta\ ,\ E^\kappa_{\omega,j} \in \kappa I_i] \tag{4.4.47}$$

recalling that i_0 is fixed so that $I = I_{i_0}$.

Assuming (4.4.47), summing over i yields,

$$N\mathbb{P}(\Omega^{\mathcal{I}_1}_{i_0,j}) \leq \mathbb{P}(\mathrm{spac}_{\kappa\mathcal{I}_2}(H^L_{\omega,\kappa}) < \delta) \tag{4.4.48}$$

So, applying Theorem 4.4.2 and summing over j quickly yields the desired estimate,

$$\begin{aligned}\mathbb{P}(\chi_I(H^L_\omega) \geq 2) &\leq \sum_{j=1}^{L^d} \mathbb{P}(\Omega^{\mathcal{I}_1}_{i_0,j}) \\ &\leq \sum_{j=1}^{L^d} \frac{1}{N}(CL^{2d}|\log\delta|^{-p}) \\ &\leq CL^{4d}\delta|\log\delta|^{-p}\end{aligned} \tag{4.4.49}$$

To finish the proof, we must prove (4.4.47). This is an application of a standard trick, using that a shift in energy is equivalent to a shift in the potential. In particular, we are in the simplest case that $\sum_{n\in\Lambda^*_L} V_k = \chi_{\Lambda_L} = \mathrm{Id}_{\ell^2(\Lambda_L)}$, so that,

$$H^L_{\omega+\tau} = H^L_\omega + \tau \tag{4.4.50}$$

We proceed by working with integrals over the configuration space, starting with the change of variables, $\omega_k \mapsto \omega_k + \eta_i$, where $\eta_i := d(I, I_i) + \delta$ is the distance from the center of I to the center of I_i,

$$\begin{aligned}\mathbb{P}(\Omega^{\mathcal{I}_1}_{i_0,j}) &= \int_{[0,1]^{\Lambda^*_L}} \mathbb{1}_{\Omega^{\mathcal{I}_1}_{i_0,j}}(\omega) \prod_{k\in\Lambda^*_L} \rho(\omega_k)\, \mathrm{d}\omega_k \\ &\leq \int_{[\eta_i,1+\eta_i]^{\Lambda^*_L}} \mathbb{1}_{\Omega^{\mathcal{I}_2}_{i,j}}(\omega) \prod_{k\in\Lambda^*_L} \rho(\omega_k - \eta_i)\, \mathrm{d}\omega_k\end{aligned} \tag{4.4.51}$$

Next, we make another change of variables $\omega_k \mapsto \kappa\omega$ with the purpose of returning the region of integration to one contained in $[0,1]^{\Lambda^*_L}$,

$$\begin{aligned}(4.4.51) &\leq \kappa^{-(\frac{L}{r})^d} \int_{[\kappa\eta_i,\kappa(1+\eta_i)]^{\Lambda^*_L}} \mathbb{1}_{\Omega^{\mathcal{I}_2}_{i,j}}(\kappa^{-1}\omega) \prod_{k\in\Lambda^*_L} \rho(\kappa^{-1}\omega_k - \eta_i)\, \mathrm{d}\omega_k \\ &\leq \kappa^{-(\frac{L}{r})^d} \int_{[0,1]^{\Lambda^*_L}} \mathbb{1}_{\Omega^{\mathcal{I}_2}_{i,j}}(\kappa^{-1}\omega) \prod_{k\in\Lambda^*_L} \rho(\kappa^{-1}\omega_k - \eta_i)\, \mathrm{d}\omega_k\end{aligned} \tag{4.4.52}$$

Now, we note that,

$$\frac{1}{\kappa}\omega \in \Omega^{\mathcal{I}_2}_{i,j} \iff \mathrm{spac}_{\kappa\mathcal{I}_2}(H^L_{\omega,\kappa}) < \kappa\delta \text{ and } E^{\kappa}_{\omega,j} \in \kappa I_i \tag{4.4.53}$$

The Lipshitz continuity of ρ and the fact that $\rho \geq \rho_-$ implies we also have,

$$\rho(\kappa^{-1}\omega_k - \eta_i) \leq \rho(\omega_k) + 2\mathcal{K}L^{-d} \leq \rho(\omega_k)(1 + 2\mathcal{K}L^{-d}\rho_-^{-1}) \tag{4.4.54}$$

So, we get the desired estimate,

$$(4.4.52) \leq C_\rho \mathbb{P}[\mathrm{spac}_{\kappa\mathcal{I}_2}(H^L_{\omega,\kappa}) < \delta\ ,\ E^{\kappa}_{\omega,j} \in \kappa I_i] \tag{4.4.55}$$

where we used that $(\kappa(1 + 2\mathcal{K}L^{-d}\rho_-^{-1}))^{-(\frac{L}{r})^d} \leq C_\rho$ for some C_ρ independent of L. □

4.5 Discrete Laplacian Boundary Conditions and Eigenvalues

Consider the positive Laplacian on $\mathbb{Z}^d$,

$$H_0 f(n) = 2d f(n) - \sum_{|n'-n|=1} f(n') \tag{4.5.1}$$

When we restrict to finite set, $\Lambda \subset \mathbb{Z}^d$, we usually do so in the most natural way, simply cutting off the full space operator. This is known as simple boundary conditions (the definition is given below).

We need different definitions for Neumann and Dirichlet boundary conditions in order to use Dirichlet-Neumann bracketing. Our definition is equivalent the ones given in [19, Section 5.2] but we give a different expression. To help define these operators, we define an auxiliary, diagonal operator, m_Λ,

$$m_\Lambda(n,n) := \#\{n' \ :\ |n - n'| = 1, n' \notin \Lambda\}, \tag{4.5.2}$$

counting the neighbors of n that are not in Λ.

Definition 4.5.1.

1. *We denote the cutoff Laplacian with* ***simple boundary conditions*** *as* H_0^Λ,

$$H_0^\Lambda := \chi_\Lambda H_0 \chi_\Lambda \tag{4.5.3}$$

2. *We denote the cutoff Laplacian with* ***Dirichlet boundary conditions*** *as* $\Delta^{\Lambda,D}$,

$$\Delta^{\Lambda,D} := H_0^\Lambda + m_\Lambda \tag{4.5.4}$$

3. *We denote the cutoff Laplacian with* ***Neumann boundary conditions*** *as* $\Delta^{\Lambda,D}$,

$$\Delta^{\Lambda,N} := H_0^\Lambda - m_\Lambda \tag{4.5.5}$$

Remark. *Note that in this section we are using the uncentered (positive) Laplacian so* $\sigma(H_0) = [0, 4d]$*. In addition, the spectrum of each of the above cutoff operators is contained in* $[0, 4d]$*.*

Below, we enumerate the eigenvalues and eigenfunctions of the Dirichlet and Neumann Laplacians on cubes.

Let $\Lambda_L \subset \mathbb{Z}^d$ be a cube consisting of L^d sites. Define $\tilde{\Lambda}_L \subset \mathbb{R}^d$ as the union of all cubes of side length 1 centered at sites in Λ_L. For example, if $\Lambda_L = \{1, \cdots, L\}^d$, then $\tilde{\Lambda}_L = [1/2, L + 1/2]^d$.

It can be checked that eigenfunctions of the discrete Laplacian on Λ_L with Dirichlet or Neumann boundary conditions are the restriction of eigenfunctions of the continuum Laplacian on $\tilde{\Lambda}_L$ with corresponding boundary condition.

We can, therefore, simply enumerate the eigenfunctions and eigenvalues of $\Delta^{\Lambda_L,D}$ and $\Delta^{\Lambda_L,N}$:

$$\begin{aligned} \psi^{\Lambda_L,D}_{n_1,\cdots,n_d}(k) &= \prod_{i=1}^{d} \sin\left(\frac{\pi n_i}{L}(k_i - 1/2)\right) \\ \psi^{\Lambda_L,N}_{m_1,\cdots,m_d}(k) &= \prod_{i=1}^{d} \cos\left(\frac{\pi m_i}{L}(k_i - 1/2)\right) \end{aligned} \tag{4.5.6}$$

$$\begin{aligned} E_{n_1,\cdots,n_d}(\Delta^{\Lambda_L,D}) &:= 2d - 2\sum_{i=1}^{d} \cos(\pi n_i/L) \\ E_{m_1,\cdots,m_d}(\Delta^{\Lambda_L,N}) &:= 2d - 2\sum_{i=1}^{d} \cos(\pi m_i/L), \end{aligned} \tag{4.5.7}$$

where $n_i \in \{1, \cdots, L\}$ and $m_i \in \{0, \cdots, L-1\}$.

4.6 Conclusion and Further Questions

One clear area of further study is to try to apply the techniques to more models and to try to remove hypotheses. An example that is of current interest to myself and my adviser, Peter Hislop, is the random Landau model.

Another area that is of current interest is to come up with an alternative proof of the Minami estimate as a consequence of the level spacing estimate, or specifically, an alternative to the proof of (4.4.47). This would allow a relaxation of the covering condition for the continuum model and an extension to more types of potential for the lattice model.

Finally, another interesting question is how to extend the Minami estimate to more energies in the spectrum and to improve the probability estimate. Let us consider just the problem of the valid energies. Perhaps the initial spacing estimate can be extended to more energies or perhaps an entirely new proof is necessary. However, we can see that the result proven here is not ideal. If it were, we would expect that at least formally when $r = 1$, we would recover the result for the rank one Anderson model. We know that a Minami estimate holds at all energies for that model, and while parts of the initial spacing estimate break down for the $r = 1$ case, the Minami-type estimate proven here formally only applies on intervals of length 2 at the edge of the spectrum, independent of dimension.

Bibliography

[1] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492–1505, Mar 1958.

[2] J. Bourgain. On random Schrödinger operators on $\mathbb{Z}^2$. *Discrete & Continuous Dynamical Systems-A*, 8(1):1, 2002.

[3] J. Bourgain. An approach to Wegner's estimate using subharmonicity. *Journal of Statistical Physics*, 134(5):969–978, 2009.

[4] J. Bourgain and C. E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Inventiones mathematicae*, 161(2):389–426, 2005.

[5] J.-M. Combes, F. Germinet, and A. Klein. Generalized eigenvalue-counting estimates for the Anderson model. *Journal of Statistical Physics*, 135(2):201–216, Apr 2009.

[6] J.-M. Combes, F. Germinet, and A. Klein. Poisson statistics for eigenvalues of continuum random Schrödinger operators. *Analysis & PDE*, 3(1):49–80, 2010.

[7] J.-M. Combes, F. Germinet, and A. Klein. Erratum to Poisson statistics for eigenvalues of continuum random Schrödinger operators [mr2663411]. *Anal. PDE*, 7(5):1235–1236, 2014.

[8] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes.* Probability and its applications. Springer, New York, 2nd ed. edition, 2003.

[9] D. Damanik and G. Stolz. A continuum version of the Kunz-Souillard approach to localization in one dimension. *J. Reine Angew. Math.*, 660:99–130, 2011.

[10] F. Delyon, Y. Lévy, and B. Souillard. Anderson localization for multi-dimensional systems at large disorder or large energy. *Communications in mathematical physics*, 100(4):463–470, 1985.

[11] F. Delyon, B. Simon, and B. Souillard. From power pure point to continuous spectrum in disordered systems. In *Annales de l'IHP Physique théorique*, volume 42, pages 283–309, 1985.

[12] A. Dietlein and A. Elgart. Level spacing and Poisson statistics for continuum random Schrödinger operators. *Journal of the European Mathematical Society*, 23(4):1257–1293, 2020.

[13] D. R. Dolai and M. Krishna. Level repulsion for a class of decaying random potentials. *Markov Process. Related Fields*, 21(3, part 1):449–462, 2015.

[14] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer. Constructive proof of localization in the anderson tight binding model. *Communications in Mathematical Physics*, 101(1):21–46, 1985.

[15] F. Germinet and F. Klopp. Spectral statistics for random Schrödinger operators in the localized regime. *Journal of the European Mathematical Society*, 16(9):1967–2031, 2014.

[16] P. Hislop. Lectures on random Schrödinger operators. *Contemporary Mathematics*, 476:41–131, 2008.

[17] P. D. Hislop and M. Krishna. Eigenvalue statistics for random Schrödinger operators with non rank one perturbations. *Communications in Mathematical Physics*, 340(1):125–143, 2015.

[18] W. Kirsch. Small perturbations and the eigenvalues of the Laplacian on large bounded domains. *Proceedings of the American Mathematical Society*, 101(3):509–512, 1987.

[19] W. Kirsch. An invitation to random Schrödinger operators. In *Random Schrödinger operators*, volume 25 of *Panor. Synthèses*, pages 1–119. Soc. Math. France, Paris, 2008. With an appendix by Frédéric Klopp.

[20] W. Kirsch, M. Krishna, and J. Obermeit. Anderson model with decaying randomness: Mobility edge. *Mathematische Zeitschrift*, 235(3):421–433, 2000.

[21] A. Klein and F. Germinet. A comprehensive proof of localization for continuous Anderson models with singular random potentials. *Journal of the European Mathematical Society*, 15(1):53–143, 2012.

[22] F. Klopp. Inverse tunneling estimates and applications to the study of spectral statistics of random operators on the real line. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(690):79–113, 2014.

[23] S. Kotani. Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators. In *North-Holland Mathematical Library*, volume 32, pages 225–247. Elsevier, 1984.

[24] N. Minami. Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys.*, 177(3):709–725, 1996.

[25] M. Reed and B. Simon. *IV: Analysis of Operators*, volume 4. Elsevier, 1978.

[26] C. Rojas-Molina and I. Veselić. Scale-free unique continuation estimates and applications to random Schrödinger operators. *Communications in mathematical physics*, 320(1):245–274, 2013.

[27] B. Simon and T. Wolff. Singular continuous spectrum under rank one perturbations and localization for random hamiltonians. *Communications on pure and applied mathematics*, 39(1):75–90, 1986.

[28] G. Teschl. *Ordinary differential equations and dynamical systems*, volume 140. American Mathematical Soc., 2012.

Vita

Samuel Herschenfeld

Education:

- Oberlin College, Oberlin, OH,
  B.A. in Mathematics, May 2013

Professional Positions:

- Denver Math Fellow, DCIS Montbello, Denver, CO
  Fall 2013-Spring 2014
- Graduate Teaching Assistant, University of Kentucky,
  Fall 2014-Fall 2020