Solubility of Additive Forms over Local Fields

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Solubility of Additive Forms over Local Fields

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

Solubility of Additive Forms over Local Fields

Michael Knapp, in a previous work, conjectured that every additive sextic form over $\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{-5})$ in seven variables has a nontrivial zero. In this dissertation, I show that this conjecture is true, establishing that

$$\Gamma^*(6, \mathbb{Q}_2(\sqrt{-1})) = \Gamma^*(6, \mathbb{Q}_2(\sqrt{-5})) = 7.$$

I then determine the minimal number of variables $\Gamma^*(d, K)$ which guarantees a non-trivial solution for every additive form of degree $d = 2m$, m odd, $m \geq 3$ over the six ramified quadratic extensions of $\mathbb{Q}_2$. We prove that if

$$K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\},$$

then

$$\Gamma^*(d, K) = \frac{3}{2}d,$$

and if

$$K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\},$$

then

$$\Gamma^*(d, K) = d + 1$$

. Finally, I show that for degree $d = 4$, if

$$K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\},$$

then

$$\Gamma^*(4, K) = 11.$$

KEYWORDS: Forms in many variables, p-adic fields, ramified extensions, additive forms

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Solubility of Additive Forms over Local Fields

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Homogeneous forms of the type

$$a_1x_1^d + a_2x_2^d + \ldots + a_sx_s^d \tag{1.1}$$

where $a_1, \ldots, a_s$ belong to some field K are known as *additive forms* of degree d. A famous conjecture of Artin claimed that any homogeneous form over a local field K (i.e., a finite extension of $\mathbb{Q}_p$) of degree d in d^2+1 variables has a nontrivial zero regardless of the choice of coefficients from K. Although many well-known counterexamples to this conjecture have been discovered (see [10], [4], [1]), none of them have been additive forms. It has therefore been proposed that the conjecture holds when restricted to additive forms. We will refer to this as Artin's Additive Form Conjecture.

Let $\Gamma^*(d, K)$ represent the minimum number of variables s such that every form (1.1) is guaranteed to have a nontrivial zero, regardless of the choice of $a_i \in K$. In this language, Artin's Additive Form Conjecture posits that $\Gamma^*(d, K) \leq d^2 + 1$. Davenport and Lewis introduced the method of *contraction* in [3], and used it to establish the truth of Artin's Additive Form Conjecture for every field of p-adic numbers $\mathbb{Q}_p$. Indeed, this bound is exact when $d = p - 1$; i.e., $\Gamma^*(p-1, \mathbb{Q}_p) = (p-1)^2 + 1$. They noted, however, that when $d \neq p - 1$ very often a much lower value of $\Gamma^*(d, \mathbb{Q}_p)$ can be found. Exact values of $\Gamma^*(d, \mathbb{Q}_p)$ are known only for small values of d. Even less is known in general about Γ^* for finite extensions of $\mathbb{Q}_p$. For unramified extensions of $K/\mathbb{Q}_p$, $p > 2$, the bound $\Gamma^*(d, K) \leq d^2 + 1$ was established in [8]. We will examine the question of exact values of $\Gamma^*(d, K)$ in more detail.

In [6], Knapp, using the method of contraction, showed that for every ramified quadratic extension K of $\mathbb{Q}_2$, we have $\Gamma^*(6, K) \leq 9$, with equality holding for $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$. For the remaining two extensions, $\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{-5})$, Knapp further showed that $\Gamma^*(6, K) \geq 7$ and conjectured that $\Gamma^*(6, K) = 7$. Aside from some trivial cases, this was all that was known about exact values of Γ^* for proper extensions of $\mathbb{Q}_p$. In chapter 2 we will show that Knapp's conjecture is true, and then in chapter 3 expand these results over the same fields to any degree $d = 2m$, where $m \geq 3$ is an odd number. Next, in chapter 4 we will derive exact values of Γ^* for quartic additive forms with $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$. Finally, we will examine some related results.

1.1 Preliminaries

Notation

For this dissertation, let K denote one of the ramified quadratic extensions of $\mathbb{Q}_2$, $\mathcal{O}$ denote its ring of integers, and $e = 2$ denote its degree of ramification. Without loss of generality, assume $a_i \in \mathcal{O}\backslash\{0\}$. Let π be a uniformizer (generator of the unique

Table 1.1: Uniformizer and representation of 2

K	π	$2 \pmod{\pi^7}$
$\mathbb{Q}_2(\sqrt{2})$	$\sqrt{2}$	π^2
$\mathbb{Q}_2(\sqrt{-2})$	$\sqrt{-2}$	$\pi^2+\pi^4$
$\mathbb{Q}_2(\sqrt{10})$	$\sqrt{10}$	$\pi^2+\pi^6$
$\mathbb{Q}_2(\sqrt{-10})$	$\sqrt{-10}$	$\pi^2+\pi^4+\pi^6$
$\mathbb{Q}_2(\sqrt{-1})$	$1+\sqrt{-1}$	$\pi^2+\pi^3+\pi^5+\pi^6$
$\mathbb{Q}_2(\sqrt{-5})$	$1+\sqrt{-5}$	$\pi^2+\pi^3+\pi^4+\pi^6$

maximal ideal) of $\mathcal{O}$, so that any $c \in \mathcal{O}$ can be written $c = c_0 + c_1\pi + c_2\pi^2 + c_3\pi^3 + \ldots$, with $c_i \in \{0, 1\}$. The choices of uniformizer and corresponding representation of 2 used in this dissertation are listed in Table 1.1 (cf. Table 1 of [6]).

It is straightforward to see that the choices of π for $\mathbb{Q}_2(\sqrt{2})$, $\mathbb{Q}_2(\sqrt{-2})$, $\mathbb{Q}_2(\sqrt{10})$, and $\mathbb{Q}_2(\sqrt{-10})$ are suitable; in each case π^2 give 2 times some odd integer, and every odd integer is a unit in $\mathcal{O}$. In the case of $\mathbb{Q}_2(\sqrt{2})$, $\pi^2 = 2$ and in $\mathbb{Q}_2(\sqrt{-2})$, $\pi^4 + \pi^2 = 2$ are the exact representations. However, in the other four fields, the exact representations have an infinite number of nonzero terms. For $\pi = \sqrt{10}$, we have $16|\pi^2 + \pi^6 - 2 = 1008$, and for $\pi = \sqrt{-10}$ we have $16|\pi^2 + \pi^4 + \pi^6 - 2 = -912$. For $\pi = 1 + \sqrt{-5}$ we have $\pi^2 + \pi^3 + \pi^4 + \pi^6 - 2 = 152 + 40\sqrt{-5} = (8\pi)(\frac{22-7\sqrt{-5}}{3})$. (Note that in this case, $\frac{1}{3} \in \mathcal{O}$.) For $\pi = 1 + \sqrt{-1}$, we again have a closed form for the representation $2 = \pi^2 + \pi^3 + \frac{\pi^5}{1-\pi}$. Chapter 5 further examines the question of the choice of uniformizer and representation of p for ramified extensions of $\mathbb{Q}_p$.

Level

Factoring out the highest power of π, the coefficient c of any variable x can be written in the form $c = \pi^r(c_0 + c_1\pi + c_2\pi^2 + c_3\pi^3 + \ldots)$, $c_0 \neq 0$. Such a variable is said to be at *level* r, and we will refer to the value of c_1 as its π-*coefficient*, and more generally c_k as its π^k-*coefficient*. We will also have occasion to refer to selections of coefficients as a *coefficient class*.

The change of variables $\pi^r x^d = \pi^{r-id}(\pi^i x)^d = \pi^{r-id} y^d$ for $i \in \mathbb{Z}$ replaces any variable with one at a level that differs from the level of the original variable by a multiple of d. Intuitively, we can think of this as moving a variable up or down a multiple of d levels. Because this change of variables doesn't change whether or not the form has a nontrivial zero, we will often simply consider the level of a variable modulo d.

Normalization

Multiplying a form by π increases the level of each variable by one, and does not affect the existence of a nontrivial zero. Considering the levels of variables modulo d, applying this transformation any number of times effects a cyclic permutation of the levels. This is useful for arranging the variables by level in an order which is more convenient, a process to which we will refer as *normalization*. See Lemma 3 of [3] for a proof of the following Lemma.

Lemma 1.1.1. *Given an additive form of degree d in an arbitrary local field K, let s be the total number of variables, and s_i be the number of variables in level $i \pmod d$. By a change of variables, the form may be transformed to one with:*

$$s_0 \geq \frac{s}{d}, \qquad s_0 + s_1 \geq \frac{2s}{d}, \ldots, \qquad s_0 + \ldots + s_{d-1} = s. \tag{1.2}$$

Contraction

Consider two variables x_1, x_2 in the same level, and without loss of generality assume their coefficients a_1, a_2 are not divisible by π (by the above cyclic permutation of levels). Suppose there is an assignment $x_1 = b_1, x_2 = b_2$ with at least one of b_1, b_2 in $\mathcal{O}^\times$, such that $\pi^k | (a_1 b_1^d + a_2 b_2^d)$. Then the change of variables $x_1 = b_1 y, x_2 = b_2 y$ yields a form in which the two variables x_1, x_2 are replaced with the new variable y at least k levels higher. If this new form has a nontrivial zero, then there is a nontrivial zero of the original form. This transformation is known as a *contraction*, and it is key to all of the results that follow. Intuitively, contractions represent partial solutions, and by combining a series of contractions, we arrive at a complete solution.

When two variables are contracted to produce a new variable at a higher level, the level and coefficient of the resulting variable depend on numerous factors: the coefficients of the contracted variables, the choices of assignment to the variables, the degree d, and the field of coefficients K. Thus it is difficult to speak generally about the outcome of any given contraction without more information. For the remainder of this dissertation, K is a totally ramified extension of $\mathbb{Q}_2$, and so has a residue field of two elements. This allows us to define a sort of *trivial* contraction.

Lemma 1.1.2. *Suppose that 1.1 has two variables in level k. They may be contracted to a variable at least one level higher.*

Proof. Without loss of generality, assume the variables are in level 0. Then $a_1 \equiv a_2 \equiv 1 \bmod \pi$. Any assignment with $b_1 \equiv b_2 \equiv 1 \bmod \pi$ gives

$$a_1 b_1^d + a_2 b_2^d \equiv 2 \equiv 0 \bmod \pi.$$

□

Note that without more information we cannot determine the exact level of the resulting variable, or any other information about its coefficient. Despite this, we can still obtain the following, surprisingly useful lemma.

Lemma 1.1.3. *Suppose that (1.1) has two variables in level k, and at least one variable in levels $k+1$, $k+2$, ... $k+t-1$, then contractions can be performed to produce a variable at level at least $k+t$.*

Proof. Any two variables in the same level can be contracted to a variable at least one level higher. By repeated contractions, we obtain the desired variable. □

In chapters 2, 3, and 4, I will describe more specific contractions about whose resulting variable we can say much more.

Type

A set of constraints on a form will be referred to as a type, and any form satisfying those constraints will be said to be of that type. Because the proofs below involve inspecting numerous subcases, it is useful to introduce a compact notation to convey important information about the type of particular forms and the results of operations performed which result in forms of a different type. In particular for this dissertation, the notation $(s_0, s_1, \ldots, s_\ell)$ will describe the type of a form which has *at least* s_i variables in level i for each level i listed. Levels which are omitted in this notation may be assumed to contain as few as no variables. For example, $(4, 2, 0, 1)$ indicates a form with at least four variables at level 0, at least two variables at level 1, as few as no variables at level 2, at least one variable in level 3, and as few as no variables at any higher level.

To indicate that variables in a particular level fall into certain π-coefficient classes, the number of variables in that level are partitioned into two numbers stacked vertically. For example, $(\substack{3\\1}, 2, 0, 1)$ indicates a form as above with at least four variables in level 0, at least three of which are in one π-coefficient class, and at least one of which is in the other. Note that this notation does not give any indication of which π-coefficient class contains which number of variables.

I indicate that a contraction performed on a certain type of form results in a form of another specified type with an arrow, labeled with the contraction performed. For example, consider a form with at least two variables in a level with the same π-coefficient, and having at least two variables two levels higher with differing π-coefficients. The first two can be used to perform an *s2*-contraction (defined in Chapters 2 and 3), resulting in a variable exactly two levels higher:

$$(\substack{2\\0}, 0, \substack{1\\1}) \xrightarrow{s2} (0, 0, \substack{2\\1}).$$

Applying the change of variables which results in a cyclic permutation of the levels of the variables is indicated in a similar way. For example, in the case $d = 6$, we have the following:

$$(4, 2, 1, 0, 0, 0) \xrightarrow{c} (0, 4, 2, 1, 0, 0).$$

In some places square brackets are used instead of parentheses to succinctly indicate an *exact* number of variables in the specified levels and coefficient classes (e.g., $[\substack{3\\1}, 2, 0, 1]$).

Hensel's Lemma

Contractions can be used to demonstrate the existence of a nontrivial zero by applying the following version of Hensel's Lemma, specialized for diagonal forms (see [8], Theorem 2.3 for a proof):

Lemma 1.1.4 (Hensel's Lemma)**.** *Let* $\gamma = \begin{cases} 1 & \text{if } \tau = 0 \\ \lfloor \frac{e}{p-1} \rfloor + e\tau + 1 & \text{if } \tau \geq 1. \end{cases}$ *Let* $d = mp^\tau$*, where* $(m, p) = 1$*. Suppose that* $b, c \in \mathcal{O}^\times$ *and that the congruence* $cx^d \equiv b \bmod \pi^\nu$ *has a solution* $a \in \mathcal{O}$ *for some* $\nu \geq \gamma$*. Then, the congruence* $cs^d \equiv b \bmod \pi^{\nu+1}$ *has a solution* t *where* $t \equiv a \bmod \pi^{\nu - e\tau}$*. Consequently, the equation* $cx^d = b$ *has a solution in* $\mathcal{O}$*.*

The aim is then to show that a series of contractions can be performed which "raises" a variable γ levels. To simplify showing that such a series of contractions can be formed, one may designate a level k. Thus, by showing that contractions involving at least one variable from the designated level produce a variable at level $k + \gamma$, a nontrivial solution follows. To further simplify the counting of levels that a variable is raised, I borrow the notion of *primary variable* from [3]. A variable in the designated level is considered to be a primary variable, as is any variable formed from a contraction involving a primary variable. The presence of a primary variable in level $k + \gamma$ indicates the existence of a nontrivial solution. Further, it is often convenient to apply a cyclic permutation to the level of the variables and then designate level 0. In this way, we may show the existence of a nontrivial solution by raising a primary variable to level γ. I indicate with an asterisk a level (or π-coefficient class within a level) which contains a primary variable. For example:

$$(\substack{3\\1}, 1, 1, 0, 1, 0) \xrightarrow{t} (\substack{1\\1}, 1, 1, 0, 2*, 0).$$

Note that if a series of contractions involving a variable at level k produces a new variable at level $k + \gamma$ levels up, a nontrivial zero immediately follows by Hensel's Lemma. Thus in the proofs below it is always assumed (when possible) that a single contraction produces a variable at level at most $k + \gamma - 1$.

Chapter 2 Sextic Forms

2.1 Introduction

Knapp [6], using the method of contraction, showed that for all six ramified quadratic extensions K of $\mathbb{Q}_2$, we have $\Gamma^*(6, K) \leq 9$, with equality holding for $\mathbb{Q}_2(\sqrt{2})$, $\mathbb{Q}_2(\sqrt{-2})$, $\mathbb{Q}_2(\sqrt{10})$, and $\mathbb{Q}_2(\sqrt{-10})$. For the remaining two extensions ($\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{-5})$), Knapp further showed that $\Gamma^*(6, K) \geq 7$ and conjectured that $\Gamma^*(6, K) = 7$. Indeed, it is straightforward to see that $x_0^6 + \pi x_1^6 + \ldots + \pi^5 x_5^6 = 0$ has no nontrivial solution over the fields considered. In this chapter, we show that this conjecture is correct.

Theorem 2.1.1. *We have*

$$\Gamma^*(6, \mathbb{Q}_2(\sqrt{-1})) = \Gamma^*(6, \mathbb{Q}_2(\sqrt{-5})) = 7.$$

2.2 Preliminaries

For the remainder of this chapter, let K denote one of the ramified quadratic extensions $\mathbb{Q}_2(\sqrt{-1})$ or $\mathbb{Q}_2(\sqrt{-5})$, assume that the number of variables s in the form (1.1) is 7, and the exponent d is 6, i.e,

$$a_1 x_1^6 + a_2 x_2^6 + \ldots + a_7 x_7^6. \tag{2.1}$$

Knapp defines the four types of contraction which I use for this proof in his Lemmas 5 and 8 of [6]. I restate them verbatim here as Lemmas 2.2.1 and 2.2.2, without proof, and they will henceforth be referenced by the numbering of this paper.

Lemma 2.2.1. *Suppose that x and y are variables at level k.*

If x and y have different π-coefficients, then they can be contracted to a variable T at level $k+1$. Moreover, we can arrange so that T has whichever π-coefficient we like.

If x and y have the same π-coefficient, then they can be contracted to a variable T at level $k+2$.

Also, in this case they can be contracted to a variable T at level at least $k+3$.

We note that in the case where x and y have the same π-coefficient, we cannot control the π-coefficient of T. Moreover, if we contract to level at least $k+3$, then we cannot control the exact level of T.

For the sake of convenient reference, I name each of these contractions. I indicate a contraction of two variables with different π-coefficients with an arrow and a d above, e.g.,

$$(^3_1, 2, 1, 0, 0, 0) \xrightarrow{d} (2, 3, 1, 0, 0, 0).$$

I indicate a contraction of two variables with the same π-coefficients resulting in a variable exactly 2 levels higher with *s2* above the arrow, e.g.,

$$\left(\begin{smallmatrix}3\\1\end{smallmatrix}, 2, 1, 0, 0, 0\right) \xrightarrow{s2} \left(\begin{smallmatrix}1\\1\end{smallmatrix}, 2, 2, 0, 0, 0\right).$$

I indicate a contraction of two variables with the same π-coefficients resulting in a variable at least 3 levels higher with *s3* above the arrow, e.g.,

$$\left(\begin{smallmatrix}3\\1\end{smallmatrix}, 2, 1, 0, 0, 0\right) \xrightarrow{s3} \left(\begin{smallmatrix}1\\1\end{smallmatrix}, 2, 1, 1, 0, 0\right).$$

Note that in the case of an *s3*-contraction, the example given is just one of the possible outcomes of the contractions, since the resulting variable may appear in any level at least 3 higher than the level of the contracted variables.

Lemma 2.2.2. *Suppose that the form contains at least 3 variables at level k which all have the same π-coefficient. Suppose also that the coefficients belong to $\mathbb{Q}_2(\sqrt{-1})$ or $\mathbb{Q}_2(\sqrt{-5})$. Then there are two variables at level k which can be contracted to a variable at level at least $k+4$.*

I indicate such a contraction with t above the arrow, e.g.,

$$\left(\begin{smallmatrix}3\\1\end{smallmatrix}, 2, 1, 0, 0, 0\right) \xrightarrow{t} \left(\begin{smallmatrix}1\\1\end{smallmatrix}, 2, 1, 0, 1, 0\right).$$

Finally, we give a statement of Hensel's Lemma specific to the needs of the proof below (see Theorem 2.1 of [8]). This is the same version of Hensel's Lemma used in [6].

Lemma 2.2.3 (Hensel's Lemma)**.** *Let x_i be a variable of (2.1) at level h. Suppose that x_i can be used in a contraction of variables (or one in a series of contractions) which produces a new variable at level at least $h+5$. Then (2.1) has a nontrivial solution.*

Note that if any contraction produces a new variable five levels up, a nontrivial zero immediately follows by Hensel's Lemma. Thus in the proofs below it is always assumed that an *s3*-contraction produces a variable exactly 3 or 4 levels up, and that a t-contraction produces a variable exactly 4 levels up.

2.3 Proof of the conjecture

Lemma 2.3.1. *Suppose that (2.1) has at least seven variables at the same level. Then (2.1) has a nontrivial zero.*

Proof. By the pigeonhole principle, there are three pairs of variables each sharing the same π-coefficient. Perform three *s2*-contractions on these three pairs to produce three new variables exactly 2 levels up. Again, by the pigeonhole principle, two of these new variables have the same π-coefficient. Perform an *s3*-contraction on this pair of variables to produce a new variable at least 3 additional levels up. A (nontrivial) zero follows from Hensel's Lemma. □

(Note that Lemma 2.3.1 is stated in greater generality than necessary, as we assume that every form considered in this proof has exactly seven variables.)

Lemma 2.3.2. *Designate level k. If after a series of contractions there are at least two variables in level $k+4$, at least one of which is primary, then (2.1) has a nontrivial zero.*

Proof. Any contraction involving the primary variable will result in a variable at level at least $k+5$. The nontrivial zero follows from Hensel's lemma. $\square$

Lemma 2.3.3. *Suppose that (2.1) has at least five variables at the same level with the same π-coefficient. Then (2.1) has a nontrivial zero.*

Proof. Without loss of generality, assume that the 5 variables appear in level 0. Then,

$$(\substack{5\\0}, 0, 0, 0, 0, 0) \xrightarrow{t} (\substack{3\\0}, 0, 0, 0, 1*, 0) \xrightarrow{t} (\substack{1\\0}, 0, 0, 0, 2*, 0).$$

The result follows from Lemma 2.3.2. $\square$

Lemma 2.3.4. *Designate level k. If after a series of contractions there are at least two variables in the same level with the same π-coefficient, at least level $k+2$, at least one of which is primary, then (2.1) has a nontrivial zero.*

Proof. Perform an $s3$-contraction on the two variables. The zero follows from Hensel's lemma. $\square$

Lemma 2.3.5. *Designate level k. If after a series of contractions there are at least three variables in the same level, at least level $k+2$, at least two of which are primary, then (2.1) has a nontrivial zero.*

Proof. This follows from Lemma 2.3.4 and the pigeonhole principle. $\square$

Lemma 2.3.6. *Suppose that (2.1) has at least six variables at the same level. Then (2.1) has a nontrivial zero.*

Proof. Without loss of generality, assume that the six variables appear in level 0. By lemma 2.3.3, assume there are at most four variables with the same π-coefficient in level 0. First suppose there are four variables in one of the π-coefficient classes in level 0. Then, after the series of contractions

$$(\substack{4\\2}, 0, 0, 0, 0, 0) \xrightarrow{s2} (\substack{2\\2}, 0, 1*, 0, 0, 0) \xrightarrow{s2} (\substack{0\\2}, 0, 2*, 0, 0, 0) \xrightarrow{s2} (\substack{0\\0}, 0, 3*, 0, 0, 0),$$

the result follows from Lemma 2.3.5. Next assume there are three variables in each π-coefficient class. After the series of contractions

$$(\substack{3\\3}, 0, 0, 0, 0, 0) \xrightarrow{t} (\substack{1\\3}, 0, 0, 0, 1*, 0) \xrightarrow{t} (\substack{1\\1}, 0, 0, 0, 2*, 0),$$

the result follows from Lemma 2.3.2. $\square$

Lemma 2.3.7. *Suppose that (2.1) has three variables with the same π-coefficient in level k and at least one variable in level $k+4$. Then (2.1) has a nontrivial zero.*

Proof. Perform a t-contraction among the three variables. The zero follows from Lemma 2.3.2. $\square$

Lemma 2.3.8. *Suppose that (2.1) has two variables with different π-coefficient in level k and at least one variable in each of levels $k+1$ and $k+2$. Then (2.1) has a nontrivial zero.*

Proof. After the series of contractions:

$$\left(\substack{1\\1}, 1, 1, 0, 0, 0\right) \xrightarrow{d} \left(0, \substack{1*\\1}, 1, 0, 0, 0\right) \xrightarrow{d} \left(0, 0, \substack{2*\\0}, 0, 0, 0\right),$$

the zero follows from Lemma 2.3.4. $\square$

Lemma 2.3.9. *Suppose that (2.1) has at least two variables with the same π-coefficient in level k and at least two variables with differing π-coefficients in level $k+1$. Then (2.1) has a nontrivial zero.*

Proof. After the series of contractions:

$$\left(\substack{2\\0}, \substack{1\\1}, 0, 0, 0, 0\right) \xrightarrow{s2} \left(0, \substack{1\\1}, 1*, 0, 0, 0\right) \xrightarrow{d} \left(0, 0, \substack{2*\\0}, 0, 0, 0\right),$$

the zero follows from Lemma 2.3.4. $\square$

Lemma 2.3.10. *Suppose that (2.1) has two variables with the same π-coefficient in level k and at least one variable in each of levels $k+2$ and $k+3$. Then (2.1) has a nontrivial solution.*

Proof. Perform an *s2*-contraction on the two variables. By Lemma 2.3.4, assume the resulting variable and the existing variable in level $k+2$ have different π-coefficients. Perform a d-contraction to create a variable in level $k+3$ with the same π-coefficient as the existing variable. The solution follows from Lemma 2.3.4. $\square$

Lemma 2.3.11. *Suppose that (2.1) has two variables with the same π-coefficient in level k and at least two variables in level $k+2$ not having the same π-coefficient. Then (2.1) has a nontrivial zero.*

Proof. Perform an *s2*-contraction on the two variables in level k. The zero following from Lemma 2.3.4. $\square$

Lemma 2.3.12. *Suppose that (2.1) has two variables with the same π-coefficient in level k and at least two variables in level $k+3$ not having the same π-coefficient. Then (2.1) has a nontrivial zero.*

Proof. Perform an *s3*-contraction on the two variables in level k. If the new variable goes to level $k+3$, a zero follows from Lemma 2.3.4. If the new variable goes to level $k+4$, perform a d-contraction on the variables in level $k+3$ to produce a variable in level $k+4$. A zero follows from any contraction on the two variables in level $k+4$. $\square$

Lemma 2.3.13. *Suppose that (2.1) has two variables with the same π-coefficient in level k and at least one variable in each of levels $k+3$ and $k+4$. Then (2.1) has a nontrivial zero.*

Proof. Perform an *s3*-contraction on the two variables. By Hensel's Lemma assume that the new variable is created at level $k+3$ or $k+4$. By Lemma 2.3.2, assume it is level $k+3$. The zero follows from Lemma 1.1.3 with $t=2$ and k replaced by $k+3$. $\square$

Lemma 2.3.14. *Suppose that (2.1) has at least four variables in level k which can be used to form two pairs each with the same π-coefficient, and at least one variable in one of levels $k+2$, $k+3$, or $k+4$. Then (2.1) has a nontrivial zero.*

Proof. If level $k+2$ contains at least one variable, perform *s2*-contractions on the two pairs. The zero follows from Lemma 2.3.4.

If level $k+3$ contains at least one variable, perform *s3*-contractions on the two pairs. By Lemma 2.3.2, assume that both of the resulting variables don't go to level $k+4$, and by Lemma 2.3.4 and the pigeonhole principle, assume that both variables don't go to level $k+3$. Then the zero follows from Lemma 1.1.3 and Hensel's Lemma.

If level $k+4$ contains at least one variable, perform *s3*-contractions on the two pairs. By Lemma 1.1.3 and Hensel's Lemma, assume both variables don't go to level $k+3$, and thus at least one variable goes to level $k+4$. The zero follows from Lemma 2.3.2. $\square$

Lemma 2.3.15. *Suppose that (2.1) has two variables in level k with different π-coefficients, and has at least one variable in both of levels $k+1$ and $k+4$. Then (2.1) has a nontrivial zero.*

Proof. Without loss of generality, assume $k=0$. Consider the series of contractions:

$$(\tbinom{1}{1}, 1, 0, 0, 1, 0) \xrightarrow{d} (0, 2*, 0, 0, 1, 0) \xrightarrow{s3} (0, 0, 0, 0, 2*, 0).$$

Here, because the s3-contraction involves a primary variable that has already been raised one level, if it creates a variable at least four levels up, then a solution follows from Hensel's Lemma. Hence, we assume that it produces a variables exactly three levels up. Perform a contraction with the other variable in level 4, and a zero follows from Hensel's Lemma. $\square$

Lemma 2.3.16. *Suppose that (2.1) has at least four variables in level k which can be used to form two pairs, one with the same π-coefficient and one with different π-coefficients, and has at least one variable in one of levels $k+1$ or $k+4$. Then (2.1) has a nontrivial solution.*

Proof. Suppose that level $k+1$ contains at least one variable. After the contraction:

$$(\tbinom{3}{1}, 1, 0, 0, 0, 0) \xrightarrow{s2} (\tbinom{1}{1}, 1, 1, 0, 0, 0)$$

the solution follows from Lemma 2.3.8.

Suppose that level $k+4$ contains at least one variable. The zero follows from Lemma 2.3.7. $\square$

Lemma 2.3.17. *Suppose that (2.1) has at least five variables at the same level. Then (2.1) has a nontrivial solution.*

Proof. Without loss of generality, assume the five variables appear in level 0. By lemma 2.3.6, assume level 0 contains exactly five variables. We will need to consider the locations of the remaining two variables.

By lemma 2.3.3, there are at most four variables with the same π-coefficient in level 0. By the pigeonhole principle, one may construct two pairs of variables, one with the same π-coefficient and one with different π-coefficients. Thus by Lemma 2.3.16, assume there are no variables in levels 1 or 4.

By the pigeonhole principle, one may also construct two pairs of variables, each with the same π-coefficient. Thus by Lemma 2.3.14, assume that there are no variables in levels 2, 3, or 4.

Therefore, assume the remaining two variables are in level 5. They can be contracted to a variable at least one level higher. By Hensel's Lemma, assume this resulting variable is not at least five levels higher, and so is formed in level 6, 7, 8, or 9. By a change of variables, the variable can be moved down six levels to level 0, 1, 2, or 3. The resulting form is covered by one of the preceding cases in this Lemma or by Lemma 2.3.6, and a zero follows. $\square$

Note that the change of variables used in the proof of the preceding lemma allows a variable in level k to be regarded as a variable in any level $\ell \equiv k \pmod 6$. For the remainder of the chapter, we will exploit this fact implicitly, referring only to the lemma which determines the chosen level ℓ.

Lemma 2.3.18. *Suppose that (2.1) has at least three variables in level k and at least two variables in level $k+5$. Suppose further that it is not the case that all of the variables in level k have the same π-coefficient and all the variables in level $k+5$ have the same π-coefficient. Then (2.1) has a nontrivial solution.*

Proof. Without loss of generality, assume $k=0$. By the pigeonhole principle, at least two variables in level 0 have the same π-coefficient.

Suppose two of the variables in level 5 have differing π-coefficients. First consider the case where not all of the variables in level 0 have the same π-coefficient. After the change of variables and series of contractions:

$$({}^{2}_{1}, 0, 0, 0, 0, {}^{1}_{1}) \xrightarrow{c} ({}^{1}_{1}, {}^{2}_{1}, 0, 0, 0, 0) \xrightarrow{d} (0, {}^{2*}_{2}, 0, 0, 0, 0) \xrightarrow{d,d} (0, 0, {}^{2*}_{0}, 0, 0, 0),$$

the solution follows from Lemma 2.3.4.

Now, consider the case where all of the variables in level 0 have the same π-coefficient. We examine the locations of the remaining two variables. First, assume level 3 is occupied. After the change of variables:

$$({}^{3}_{0}, 0, 0, 1, 0, {}^{1}_{1}) \xrightarrow{c} ({}^{1}_{1}, {}^{3}_{0}, 0, 0, 1, 0),$$

a zero follows from Lemma 2.3.15. Thus, assume level 3 is unoccupied. By Lemma 2.3.7, assume that level 4 is unoccupied, and by Lemma 2.3.8 with $k = 5$, assume level 1 is unoccupied. By Lemma 2.3.11 with $k = 0$, assume all variables in level 2 have the same π-coefficient. If level 2 contains two variables, a solution follows from Lemma 2.3.12 with $k = 2$. If level 0 contains four variables, a d-contraction can be performed on level 5 so that there are five variables in level 0 with the same π-coefficient, and a solution follows from Lemma 2.3.3. If the variables in level 5 can be used to form two pairs with differing π-coefficients, then two d-contractions can be performed, and again a solution follows from Lemma 2.3.3. If the variables in level 5 can be used to form two pairs, one with the same π-coefficients and one with differing π-coefficients, then a solution follows from Lemma 2.3.16 with $k = 5$. One case remains:

$$({}^{3}_{0}, 0, 1, 0, 0, {}^{2}_{1}) \xrightarrow{c} ({}^{2}_{1}, {}^{3}_{0}, 0, 1, 0, 0) \xrightarrow{d} (1, {}^{4}_{0}, 0, 1, 0, 0) \xrightarrow{s3} (1, {}^{2}_{0}, 0, 1, 1, 0).$$

A solution follows from Lemma 2.3.10 with $k = 1$. (Here, the *s3*-contraction is performed with a variable which has already been raised one level, and so by Hensel's Lemma, we may assume that the resulting variable is created exactly three levels up.)

Now, suppose that the two variables in level 5 have the same π-coefficient. The hypothesis implies that the three variables in level 0 do not have the same π-coefficient. After the change of variables and series of contractions:

$$({}^{2}_{1}, 0, 0, 0, 0, {}^{2}_{0}) \xrightarrow{c} ({}^{2}_{0}, {}^{2}_{1}, 0, 0, 0, 0) \xrightarrow{s2} (0, {}^{2}_{1}, 1*, 0, 0, 0) \xrightarrow{d} (0, 1, {}^{2*}_{0}, 0, 0, 0),$$

the solution follows from Lemma 2.3.4.

□

Lemma 2.3.19. *Suppose that (2.1) has at least four variables at the same level. Then (2.1) has a nontrivial solution.*

Proof. By Lemma 2.3.17, we may assume that the level with at least four variables has exactly four variables; without loss of generality, assume that this is level 0.

First, suppose at least two of the remaining three variables are in level 5. By Lemma 2.3.18, assume that all the variables in level 0 have the same π-coefficient, and that all the variables in level 5 have the same π-coefficient. Thus the four variables can be used to form two pairs, each having the same π-coefficient. Perform an *s3*-contraction on the two variables in level 5. After a change of variables, the resulting variable will be in level 2 or 3, and the solution follows from Lemma 2.3.14. Thus, assume that level 5 contains at most one variable. We will examine the locations of the remaining two variables.

By Lemma 2.3.14 if the four variables form two pairs with matching π-coefficients we may assume levels 2, 3, and 4 are unoccupied, and therefore that level 1 contains at least two variables. These may be contracted to a variable in level 2 or 3, and so the solution follows again from Lemma 2.3.14.

Thus, assume that three of the four variables in level 0 are in one π-coefficient class, and the one remaining variable is in the other class. By Lemma 2.3.16, we may assume levels 1 and 4 are unoccupied, and by Lemma 2.3.10 that one of levels 2 or 3 is unoccupied.

If two of the remaining variables lie in level 2 and have the same π-coefficient, they can be contracted to a variable in level 4 and the solution follows from Lemma 2.3.7. Suppose they have differing π-coefficients. After the contraction:

$$({}^{3}_{1}, 0, {}^{1}_{1}, 0, 0, 0) \xrightarrow{s2} ({}^{1}_{1}, 0, {}^{2*}_{1}, 0, 0, 0),$$

the solution follows from Lemma 2.3.4.

Thus we may assume both variables lie in level 3. If they have differing π-coefficients, by the pigeonhole principle there is a pair of variables in level 0 with the same π-coefficients, and a solution follows from Lemma 2.3.12. Thus assume they have the same π-coefficient. If there is a variable in level 5, by a change of variables we may consider the variables in level 0 to be in level 6, and the solution follows from Lemma 2.3.10 with $k = 3$. Thus we may assume there are three variables in level 3, all with the same π-coefficient. After the change of variables and series of contractions:

$$({}^{3}_{1}, 0, 0, {}^{3}_{0}, 0, 0) \xrightarrow{c} ({}^{3}_{0}, 0, 0, {}^{3}_{1}, 0, 0) \xrightarrow{t} (1, 0, 0, {}^{3}_{1}, 1*, 0) \xrightarrow{d} (1, 0, 0, 2, 2*, 0),$$

the solution follows from Lemma 2.3.2. $\square$

Lemma 2.3.20. *Suppose that (2.1) has at least three variables in level k not all having the same π-coefficient, and at least two variables in level $k+1$. Then (2.1) has a nontrivial solution.*

Proof. Without loss of generality, assume $k = 0$. By Lemma 2.3.19, level 0 has exactly three variables. If there are at least three variables in level 1, by multiplying by π^5 we may consider level 5 and 6 to have three variables each. By a change of variables, the variables in level 6 may be considered to be in level 0. Then a solution follows from Lemma 2.3.18. Thus assume level 1 contains exactly two variables. Suppose that the two variables in level 1 have different π-coefficients. After the series of contractions:

$$({}^{2}_{1}, {}^{1}_{1}, 0, 0, 0, 0) \xrightarrow{s2} (1, {}^{1}_{1}, 1*, 0, 0, 0) \xrightarrow{d} (1, 0, {}^{2*}_{0}, 0, 0, 0),$$

the solution follows from Lemma 2.3.4.

Thus, suppose that the two variables in level 1 have the same π-coefficient. By Lemma 2.3.8, assume level 2 is unoccupied. By Lemma 2.3.15, assume level 4 is unoccupied. Suppose level 5 is occupied. After the contractions and change of variables:

$$({}^{2}_{1}, {}^{2}_{0}, 0, 0, 0, 1) \xrightarrow{d} (1, {}^{3}_{0}, 0, 0, 0, 1) \xrightarrow{c} ({}^{3}_{0}, 0, 0, 0, 1, 1) \xrightarrow{t} (1, 0, 0, 0, 2*, 1),$$

the solution follows from Lemma 2.3.2. Thus, assume level 5 is unoccupied, and the remaining two variables lie in level 3. If they have differing π-coefficients, then they can be contracted to a variable in level 4, and again a solution follows from Lemma 2.3.15. If they have the same π-coefficient, they can be contracted to a variable in level 5, and a solution follows from the case described above. $\square$

Lemma 2.3.21. *Suppose that (2.1) has at least three variables at the same level, not all having the same π-coefficient. Then (2.1) has a nontrivial solution.*

Proof. By Lemma 2.3.19, we may assume that the level(s) with at least three variables have exactly three variables. Without loss of generality assume that one of the levels with three variables is level 0. By Lemma 2.3.18, we may assume there is at most one variable in level 5. We examine the locations of the remaining three variables.

By Lemma 2.3.10, we may assume that at least one of levels 2 and 3 is unoccupied, and by Lemma 2.3.13 that at least one of levels 3 and 4 is unoccupied. By Lemma 2.3.8, we may assume that at least one of levels 1 and 2 is unoccupied, and by Lemma 2.3.15, we may assume at least one of levels 1 and 4 is unoccupied.

Suppose level 1 is occupied. By Lemma 2.3.20, assume level 1 contains exactly one variable. By the above considerations, assume levels 2 and 4 are unoccupied. If there are two variables with different π-coefficients in level 3, they can be contracted to a variable in level 4 and the solution follows from Lemma 2.3.15. Thus, assume there are at least two variables in level 3, all having the same π-coefficient. By Lemma 2.3.10 with $k = 3$, assume that level 5 is unoccupied, and thus that level 3 has exactly three variables with the same π-coefficient, and then by Lemma 2.3.7 with $k = 3$, there is a nontrivial solution. Thus assume level 1 is unoccupied.

Suppose level 2 has at least two variables, and so assume levels 1 and 3 are unoccupied. By Lemma 2.3.11 with $k = 0$, assume the variables in level 2 have the same π-coefficient. By Lemma 2.3.7 with $k = 2$, assume there are at most two variables in level 2, and thus level 4 must contain at least one variable. If level 5 also contains a variable, then the solution follows from Lemma 2.3.10 with $k = 2$, and so assume level 5 is unoccupied, and thus level 4 contains two variables. If the two variables have differing π-coefficients, the solution follows from Lemma 2.3.11 with $k = 2$. If the two variables have the same π-coefficient, the solution follows from Lemma 2.3.11 with $k = 4$. Thus assume that level 2 contains at most one variable.

Suppose level 3 is occupied. By the above, assume levels 1, 2, and 4 are unoccupied, and so level 3 contains exactly three variables and level 5 contains exactly one. By the pigeonhole principle, at least two of the variables in level 3 must have the same π-coefficient, and the solution follows from Lemma 2.3.12 with $k = 3$. Thus assume that level 3 is unoccupied.

Suppose level 4 is occupied. By the preceding reasoning, assume levels 1 and 3 are unoccupied and levels 2 and 5 contain at most one variable. Thus assume level 4 contains at least two variables. If two of these variables have the same π-coefficient, then the solution follows from Lemma 2.3.11 with $k = 4$. Thus assume level 4 contains exactly two variables with differing π-coefficients, level 2 contains one variable, and level 5 contains one variable. The solution follows from Lemma 2.3.8 with $k = 4$. □

Lemma 2.3.22. *Suppose that (2.1) has at least three variables at level k and at least three variables at level $k + 1$. Then (2.1) has a nontrivial solution.*

Proof. Without loss of generality, assume $k = 0$. By Lemma 2.3.19 and Lemma 2.3.21, assume that levels 0 and 1 each have exactly three variables with the same π-coefficient. By Lemma 2.3.7, assume that levels 4 and 5 are unoccupied. Thus, assume the remaining variable is in level 2 or 3. First, suppose it is in level 2. After the contraction

$$\left(\genfrac{}{}{0pt}{}{3}{0},\genfrac{}{}{0pt}{}{3}{0},1,0,0,0\right)\xrightarrow{s2}\left(\genfrac{}{}{0pt}{}{3}{0},1,1,1,0,0\right)$$

the solution follows from Lemma 2.3.10. Now, suppose the variable is in level 3. After an initial *s2*-contraction, there are two possible cases:
$\left(\genfrac{}{}{0pt}{}{3}{0},\genfrac{}{}{0pt}{}{3}{0},0,1,0,0\right)\xrightarrow{s2}$

- $\left(\genfrac{}{}{0pt}{}{3}{0},1,0,\genfrac{}{}{0pt}{}{1}{1},0,0\right)\xrightarrow{d}\left(\genfrac{}{}{0pt}{}{3}{0},1,0,0,1,0\right)$, and the solution follows from Lemma 2.3.7.
- $\left(\genfrac{}{}{0pt}{}{3}{0},1,0,\genfrac{}{}{0pt}{}{2*}{0},0,0\right)$, and the solution follows from Lemma 2.3.4 with $k=1$.

$\square$

Lemma 2.3.23. *Suppose that (2.1) has at least three variables at the same level. Then (2.1) has a nontrivial solution.*

Proof. By Lemma 2.3.19, we may assume that any level with at least three variables has exactly three variables. By Lemma 2.3.21, we may assume that the variables in any level with three variables all have the same π-coefficient. Without loss of generality assume that one of the levels with three variables is level 0.

By Lemma 2.3.10, we may assume that at least one of levels 2 and 3 is unoccupied. By Lemma 2.3.7, we may assume that level 4 is unoccupied. By Lemma 2.3.22, assume that levels 5 and 1 each contain at most two variables.

By Lemma 2.3.9, we may assume that all variables in level 1 have the same π-coefficient. By Lemma 2.3.11, assume that all variables in level 2 have the same π-coefficient. If there are at least two variables in level 2, they may be contracted to a variable in level 4, and a solution follows from Lemma 2.3.7. Thus assume that level 2 contains at most one variable. If there are two variables in level 3 having differing π-coefficients, they may be contracted to a variable in level 4 and a solution follows from Lemma 2.3.7, and so assume all variables in level 3 have the same π-coefficient. By Lemma 2.3.18, we may assume that all variables in level 5 have the same π-coefficient.

First suppose that level 5 contains no variables, and so there are four variables that occupy levels 1, 2, and 3. By the assumptions made, assume level 1 contains at most two, level 2 contains at most one, and thus level 3 contains a variable. Therefore, assume that level 2 is unoccupied, and so level 1 is occupied, and level 3 has at least two variables. A solution follows from Lemma 2.3.13 with $k=3$. Thus we may assume that level 5 contains either one or two variables, and so (by Lemma 2.3.10 with $k=3$ and the assumptions above) that levels 2 and 3 together contain at most one variable, and thus that level 1 contains either one or two variables.

Now suppose that level 5 contains exactly one variable. By the above assumptions, level 1 contains exactly two variables, and levels 2 and 3 together contain exactly one variable. Perform an *s2*-contraction on the two variables in level 1 to create a new variable in level 3. If level 2 contains a variable, then a solution follows from Lemma

2.3.10 with $k = 0$. Thus suppose that level 2 is unoccupied and level 3 contains exactly one variable. If this variable has the same π-coefficient as the variable resulting from the contraction, the solution follows from Lemma 2.3.4. However, if the π-coefficients of the two variables differ, they can be contracted to a variable in level 4 and the solution follows from Lemma 2.3.7.

Thus, we may assume that level 5 contains exactly two variables having the same π-coefficient and level 1 contains at least one variable. Perform an *s2*-contraction on a pair of variables from level 0 to produce a variable in level 2. A solution follows from Lemma 2.3.10 with $k = 5$. □

Lemma 2.3.24. *Suppose that (2.1) has a least two variables at the same level. Then (2.1) has a nontrivial solution.*

Proof. By Lemma 2.3.23, we may assume that any level with at least two variables has exactly two variables. It is possible by applying an appropriate cyclic permutation to have a form with at least $n + 1$ variables in the first n levels for $1 \leq n \leq 6$ (cf. Lemma 3 of [6]). This process is known as normalization. It follows that we may assume that level 0 has two variables, level 1 is occupied, and level 5 has at most one variable.

First suppose that the two variables in level 0 have differing π-coefficients. By Lemma 2.3.8, assume level 2 is unoccupied. Thus levels 0 through 2 contain at most four variables, and so by normalization assume that level 3 is occupied. Perform a d-contraction on the pair in level 0 so that there are two variables in level 1 with the same π-coefficient, and use this pair to perform an *s3*-contraction from level 1. By Hensel's Lemma, assume this results in a new variable in level 4. By Lemma 2.3.2, assume level 4 is unoccupied, and therefore by normalization that level 3 contains two variables. If the two variables in level 3 have differing π-coefficients, a solution follows from Lemma 2.3.11 with $k = 1$. If these two variables have the same π-coefficient, omit the initial d-contraction, and a solution follows from Lemma 2.3.10 with $k = 3$.

Thus, we may assume that the two variables in level 0 have the same π-coefficient. Suppose that level 1 contains just one variable. By normalization, assume level 2 is occupied, and so by Lemma 2.3.10 that level 3 is unoccupied, and again by normalization, that level 2 contains two variables, and by Lemma 2.3.11 that they have the same π-coefficient. By Lemma 2.3.13 with $k = 2$, assume that level 5 is unoccupied, and so level 4 contains the remaining two variables. By Lemma 2.3.11 with $k = 2$, assume that they have the same π-coefficient. A solution follows from Lemma 2.3.13 with $k = 4$.

Thus we may assume level 1 contains two variables, and by Lemma 2.3.9 that they have the same π-coefficient. If level 2 is occupied, perform an *s2*-contraction on the two variables in level 1, and a solution follows from Lemma 2.3.10. Therefore, assume that level 2 is unoccupied and by normalization that level 3 is occupied. By Lemma 2.3.13, assume level 4 is unoccupied, and so by normalization that level 3 contains two variables, and by Lemma 2.3.11 with $k = 1$ that they have the same π-coefficient. The remaining variable is in level 5, and a solution follows from Lemma 2.3.10 with $k = 3$. □

The forms we consider have seven variables and six levels, and thus there is some level that contains at least two variables. By Lemma 2.3.24, this completes the proof of the theorem.

Chapter 3 Forms of Twice Odd Degree

3.1 Introduction

Here, techniques similar to those used in the previous chapter will be used to extend those results to forms of degree $d = 2m$, where m is an odd integer, at least 3.

Theorem 3.1.1. *Let $d = 2m$, where m is an odd integer at least 3.*

- *If $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$, then $\Gamma^*(d, K) = \frac{3}{2}d$.*
- *If $K \in \{\mathbb{Q}_2(\sqrt{-1}),\ \mathbb{Q}_2(\sqrt{-5})\}$, then $\Gamma^*(d, K) = d + 1$.*

3.2 Preliminaries

For the remainder of this chapter, let K denote one of the six ramified quadratic extensions of $\mathbb{Q}_2$, and the exponent $d = 2m$, $m \geq 3$ odd, i.e,

$$a_1x_1^{2m} + a_2x_2^{2m} + \ldots + a_sx_s^{2m}. \tag{3.1}$$

The following lemma establishes the existence of the types of contractions for $d = 2m$, m odd, at least 3, which will be used in this proof. Each type of contraction is named for convenient reference.

Lemma 3.2.1.

1. *Two variables in the same level with differing π-coefficients can be contracted to a variable one level up having a π-coefficient of one's choosing. (d-contraction)*
2. *Two variables in the same level with the same π-coefficient can be contracted to a variable exactly two levels up. (s2-contraction)*
3. *Two variables in the same level with the same π-coefficient can be contracted to a variable at least three levels up. (s3-contraction)*
4. *Suppose $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$. Among three variables in the same level with the same π-coefficient, two can be contracted to a variable at least four levels up. (t-contraction)*
5. *Suppose $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$. Among three variables in the same level with the same π-coefficient, two can be contracted to a variable exactly two levels up having the same π-coefficient. (st-contraction)*

Proof. Using the fact that $2 \equiv \pi^2 \pmod{\pi^3}$,

$(1 + a\pi + b\pi^2 + c\pi^3)^{2m} \equiv (1 + 2a\pi + (2b + a^2)\pi^2 + (2c + 2ab)\pi^3)^m \equiv$
$(1 + a^2\pi^2 + a\pi^3)^m \equiv 1 + ma^2\pi^2 + ma\pi^3 \equiv$
$1 + a\pi^2 + a\pi^3 \pmod{\pi^4}$.

Therefore, the d^{th} powers modulo π^4 are 0 and $1 + c\pi^2 + c\pi^3$ for $c \in \{0, 1\}$.

Using $2 \equiv \pi^2 + j\pi^3 \pmod{\pi^4}, j \in \{0, 1\}$,
$(1 + a_1\pi + a_2\pi^2 + a_3\pi^3) + (1 + b_1\pi + b_2\pi^2 + b_3\pi^3)(1 + c\pi^2 + c\pi^3) \equiv$
$(1 + a_1\pi + a_2\pi^2 + a_3\pi^3) + [1 + b_1\pi + (b_2 + c)\pi^2 + (b_3 + b_1c + c)\pi^3] \equiv$
$2 + (a_1 + b_1)\pi + (a_2 + b_2 + c)\pi^2 + (a_3 + b_3 + b_1c + c)\pi^3 \equiv$
$(a_1 + b_1)\pi + (1 + a_2 + b_2 + c)\pi^2 + (j + a_3 + b_3 + b_1c + c)\pi^3 \pmod{\pi^4}$.

First suppose that the two variables have differing π-coefficients. We have $a_1+b_1 = 1$, and so the resulting variable is one level higher. Choose c so that $1 + a_2 + b_2 + c$ (the π-coefficient of the resulting variable) is the desired value $\pmod 2$. This proves (1).

Suppose now that $a_1 = b_1$. Then we have
$(a_1 + b_1)\pi + (1 + a_2 + b_2 + c)\pi^2 + (j + a_3 + b_3 + b_1c + c)\pi^3 \equiv$
$2a_1\pi + (1 + a_2 + b_2 + c)\pi^2 + (j + a_3 + b_3 + a_1c + c)\pi^3 \equiv$
$(1 + a_2 + b_2 + c)\pi^2 + (a_1 + j + a_3 + b_3 + a_1c + c)\pi^3 \pmod{\pi^4}$.

Choosing c so that $1 + a_2 + b_2 + c \equiv 1 \pmod 2$ gives a variable raised exactly two levels. This proves (2). Choosing c so that $1 + a_2 + b_2 + c \equiv 0 \pmod 2$ gives a variable raised at least three levels. This proves (3).

If $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, then $j = 1$. Choose c so that $c \equiv 1 + a_2 + b_2 \pmod 2$. It follows that
$(1 + a_2 + b_2 + c)\pi^2 + (a_1 + j + a_3 + b_3 + a_1c + c)\pi^3 \equiv$
$(a_3 + b_3 + (1 + a_1)(a_2 + b_2))\pi^3 \pmod{\pi^4}$.

If $a_1 = 1$, the expression becomes $a_3 + b_3$, and by the pigeonhole principle the two variables can be chosen so that their π^3-coefficients are the same. If $a_1 = 0$, the expression becomes $a_2 + b_2 + a_3 + b_3$. If there is a pair in the same π^2, π^3-coefficient class, then choose that pair. If no such pair exists, then by the pigeonhole principle there exists a pair with both differing π^2- and π^3-coefficients. It follows that $a_2 + b_2 + a_3 + b_3 \equiv 0 \pmod 2$. This proves (4).

If $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$, then $j = 0$. Choose c so that $c \equiv a_2 + b_2 \pmod 2$. It follows that
$(1 + a_2 + b_2 + c)\pi^2 + (a_1 + j + a_3 + b_3 + a_1c + c)\pi^3 \equiv$
$\pi^2 + (a_1 + a_3 + b_3 + (1 + a_1)(a_2 + b_2))\pi^3 \pmod{\pi^4}$.

As above, the two variables can be chosen so that $(a_3 + b_3 + (1 + a_1)(a_2 + b_2)) \equiv 0 \pmod 2$. This proves (5). □

Finally, we give a statement of Hensel's Lemma specific to the needs of the proof below (see Theorem 2.1 of [8]). This is the same version of Hensel's Lemma used in [6].

Lemma 3.2.2 (Hensel's Lemma)**.** *Let $d = 2m$, m odd, and x_i be a variable of (3.1) at level h. Suppose that x_i can be used in a contraction of variables (or one in a series of contractions) which produces a new variable at level at least $h+5$. Then (3.1) has a nontrivial zero.*

By Hensel's Lemma, for the rest of this chapter we may assume that an *s3*-contraction produces a variable either 3 or 4 levels higher, and a t-contraction produces a variable exactly 4 levels higher. To draw the reader's attention to a level from which a variable satisfying this statement of Hensel's Lemma will originate, we append an asterisk. For example:

$$\left(\begin{smallmatrix}2\\0\end{smallmatrix}, 0, \begin{smallmatrix}1\\1\end{smallmatrix}\right) \xrightarrow{s2} \left(0, 0, \begin{smallmatrix}2*\\1\end{smallmatrix}\right).$$

3.3 General Lemmas

Lemma 3.3.1. *Suppose that after a series of contractions, (3.1) has two variables in the same level with the same π-coefficient, one of which has been raised at least two levels, or it has two variables in the same level, one of which has been raised at least four levels. Then (3.1) has a nontrivial zero.*

Proof. If one of the variables has been raised at least two levels, performing an *s3*-contraction with the two variables results in a variable that has been raised at least an additional three levels, for a total of at least five levels. If one of the variables has been raised at least four levels, performing any contraction results in a variable that has been raised at least one additional level, for a total of at least five levels. A zero follows from Hensel's Lemma. □

Lemma 3.3.2. *Suppose that (3.1) has at least four variables in level k which can be used to form a pair with the same π-coefficient and a pair with differing π-coefficients, and at least one variable in level $k+1$. Then (3.1) has a nontrivial zero.*

Proof. Without loss of generality, assume $k = 0$. The hypotheses imply that (3.1) is of type $\left(\begin{smallmatrix}3\\1\end{smallmatrix}\right)$. After the series of contractions

$$\left(\begin{smallmatrix}3\\1\end{smallmatrix}, 1\right) \xrightarrow{s2} \left(\begin{smallmatrix}1\\1\end{smallmatrix}, 1, 1\right) \xrightarrow{d} \left(0, \begin{smallmatrix}1*\\1\end{smallmatrix}, 1\right) \xrightarrow{d} \left(0, 0, \begin{smallmatrix}2*\\0\end{smallmatrix}\right),$$

a zero follows from Lemma 3.3.1. □

Lemma 3.3.3. *Suppose $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, that (3.1) has at least three variables in level k with the same π-coefficient, and at least one variable in level $k+4$. Then (3.1) has a nontrivial zero.*

Proof. Without loss of generality, assume $k = 0$. The hypotheses imply that (3.1) is of type $\left(\begin{smallmatrix}3\\0\end{smallmatrix}, 0, 0, 0, 1\right)$. After the contraction

$$\left(\begin{smallmatrix}3\\0\end{smallmatrix}, 0, 0, 0, 1\right) \xrightarrow{t} (1, 0, 0, 0, 2*),$$

a zero follows from Lemma 3.3.1. □

Lemma 3.3.4. *Suppose that (3.1) has at least four variables in level k which can be used to form two pairs with the same π-coefficient, and a variable in at least one of levels $k+2$, $k+3$, or $k+4$. Then (3.1) has a nontrivial zero.*

Proof. The hypotheses imply that (3.1) is of the type $\binom{2}{2}$ or $\binom{4}{0}$. In either case, we are able to perform a pair of *s2*- or *s3*-contractions. Assume that it is of type $\binom{2}{2}$. (The same argument applies if $\binom{2}{2}$ is replaced with $\binom{4}{0}$.) Without loss of generality, assume $k=0$. If there is a variable in level $k+2$, then after the contractions

$$(\genfrac{}{}{0pt}{}{2}{2}, 0, 1) \xrightarrow{s2,s2} (0, 0, 3*),$$

a zero follows from Lemma 3.3.1.

If there is a variable in level $k+3$, then after an initial pair of *s2*-contractions, there are two possible cases.

$(\genfrac{}{}{0pt}{}{2}{2}, 0, 0, 1) \xrightarrow{s2,s2}$

- $(0, 0, \genfrac{}{}{0pt}{}{2*}{0}, 1)$
- $(0, 0, \genfrac{}{}{0pt}{}{1*}{1*}, 1) \xrightarrow{d} (0, 0, 0, \genfrac{}{}{0pt}{}{2*}{0})$

A zero follows in both cases from Lemma 3.3.1.

If there is a variable in level $k+4$, assume by Lemma 3.3.1 that the variables resulting from the following two *s3*-contractions are not produced in the same level:

$$(\genfrac{}{}{0pt}{}{2}{2}, 0, 0, 0, 1) \xrightarrow{s3,s3} (0, 0, 0, 1*, 2*).$$

A zero follows from Lemma 3.3.1.

□

3.4 Lemmas Regarding Large Numbers of Variables

Lemma 3.4.1. *Suppose (3.1) has at least six variables in the same level which can be used to form three pairs with the same π-coefficient. Then (3.1) has a nontrivial zero.*

In particular, if (3.1) has seven variables in the same level, then (3.1) has a nontrivial zero.

Proof. If there are seven variables in the same level, by the pigeonhole principle, three pairs of variables may be formed having the same π-coefficient. After performing an *s2*-contraction with one of these pairs, a zero follows from Lemma 3.3.4. □

Corollary 3.4.2. *Artin's Additive Form Conjecture holds for all quadratic ramified extensions of $\mathbb{Q}_2$ and degrees $d = 2m$, m odd.*

Proof. The case $m = 1$ is treated separately; see [7], Chapter VI. In the cases considered, $d \geq 6$, and so $d^2 + 1 \geq 6d + 1$, and by the pigeonhole principle, the form has a level with at least seven variables. A zero follows from Lemma 3.4.1. □

Lemma 3.4.3. *Suppose $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$ and (3.1) has at least five variables in the same level with the same π-coefficient, or at least six variables with at least three variables in each π-coefficient class. Then (3.1) has a nontrivial zero.*

Proof. Without loss of generality, assume the level indicated is 0. The hypotheses imply that (3.1) is of type $\binom{5}{0}$ or $\binom{3}{3}$. Consider the contractions given in each case:

$$\binom{5}{0} \xrightarrow{t,t} (1, 0, 0, 0, 2*),$$

$$\binom{3}{3} \xrightarrow{t,t} (\tbinom{1}{1}, 0, 0, 0, 2*).$$

A zero follows from Lemma 3.3.1. □

Lemma 3.4.4. *Suppose that $s \geq \frac{7}{5}d$ and after normalization (3.1) has at least six variables in level 0. Then (3.1) has a nontrivial zero.*

If $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, then the condition on s is unnecessary.

Proof. By Lemma 3.4.1, assume that level 0 contains exactly six variables. By Lemma 3.3.4, assume levels 2, 3, and 4 are unoccupied. By normalization, $s \geq \frac{7}{5}d$ implies that levels 0 through 4 together contain at least seven variables, and so level 1 contains at least one variable. By Lemma 3.3.2, assume all of the variables in level 0 have the same π-coefficient. A zero follows from Lemma 3.4.1.

Now, suppose that $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$. By Lemma 3.4.3, assume that four of the variables in level 0 are in one π-coefficient class, and two are in the other. A zero follows from Lemma 3.4.1. □

Lemma 3.4.5. *Suppose that $s \geq \frac{7}{5}d$ and after normalization (3.1) has at least five variables in level 0. Then (3.1) has a nontrivial zero.*

If $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, then $s \geq d + 1$ is sufficient.

Proof. By Lemma 3.4.4, assume level 0 contains exactly five variables. By Lemma 3.3.4, assume levels 2, 3, and 4 are unoccupied.

By normalization, $s \geq \frac{7}{5}d$ implies that levels 0 through 4 together contain at least seven variables, and so level 1 contains at least two variables. These variables can be contracted to a variable in level 2 or 3 using a *d*- or *s2*-contraction, and a zero follows from Lemma 3.3.4.

Now, suppose that $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$. By normalization, $s \geq d + 1$ implies that levels 0 through 4 together contain at least six variables, and so level 1 contains at least one variable. By Lemma 3.3.2, assume all the variables in level 0 have the same π-coefficient. A zero follows from Lemma 3.4.3. □

Lemma 3.4.6. *Suppose that $s \geq \frac{7}{5}d$ and after normalization (3.1) has at least four variables in level 0 which can be used to form two pairs with the same π-coefficient. Then (3.1) has a nontrivial zero.*

If $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, then $s \geq d + 1$ is sufficient.

Proof. By Lemma 3.4.5, assume level 0 contains exactly four variables. By Lemma 3.3.4, assume levels 2, 3, and 4 are unoccupied. By normalization, $s \geq d+1$ implies that levels 0 through 4 together contain at least six variables, and so level 1 contains at least two variables. Any two of the variables in level 1 can be contracted to a variable in level 2 or 3, and a zero follows from Lemma 3.3.4. $\square$

3.5 Lemmas Regarding Small Numbers of Variables

Lemma 3.5.1. *Suppose that (3.1) has two variables in level k with the same π-coefficient, and two variables in at least one of levels $k+1$, $k+2$, or $k+3$ with differing π-coefficients. Then (3.1) has a nontrivial zero.*

Proof. Without loss of generality, the hypotheses imply that (3.1) is of type $(\genfrac{}{}{0pt}{}{2}{0}, \genfrac{}{}{0pt}{}{1}{1})$, $(\genfrac{}{}{0pt}{}{2}{0}, 0, \genfrac{}{}{0pt}{}{1}{1})$, or $(\genfrac{}{}{0pt}{}{2}{0}, 0, 0, \genfrac{}{}{0pt}{}{1}{1})$. In the first two cases, consider the following series of contractions:

$$(\genfrac{}{}{0pt}{}{2}{0}, \genfrac{}{}{0pt}{}{1}{1}) \xrightarrow{s2} (0, \genfrac{}{}{0pt}{}{1}{1}, 1*) \xrightarrow{d} (0, 0, \genfrac{}{}{0pt}{}{2*}{0}),$$

$$(\genfrac{}{}{0pt}{}{2}{0}, 0, \genfrac{}{}{0pt}{}{1}{1}) \xrightarrow{s2} (0, 0, \genfrac{}{}{0pt}{}{2*}{1}).$$

In the final case, consider the two following subcases:

$(\genfrac{}{}{0pt}{}{2}{0}, 0, 0, \genfrac{}{}{0pt}{}{1}{1}) \xrightarrow{s3}$

- $(0, 0, 0, \genfrac{}{}{0pt}{}{2*}{1})$
- $(0, 0, 0, \genfrac{}{}{0pt}{}{1}{1}, 1*) \xrightarrow{d} (0, 0, 0, 0, \genfrac{}{}{0pt}{}{2*}{0})$

In every case, a zero follows from Lemma 3.3.1. $\square$

Lemma 3.5.2. *Suppose that (3.1) has two variables in level k with the same π-coefficient, and a variable in both of levels $k+2$ and $k+3$, or both of levels $k+3$ and $k+4$. Then (3.1) has a nontrivial zero.*

Proof. If there is a variable in each of levels $k+2$ and $k+3$, then after an initial *s2*-contraction, there are two possible cases:

$(\genfrac{}{}{0pt}{}{2}{0}, 0, 1, 1) \xrightarrow{s2}$

- $(0, 0, \genfrac{}{}{0pt}{}{2*}{0}, 1)$
- $(0, 0, \genfrac{}{}{0pt}{}{1*}{1}, 1) \xrightarrow{d} (0, 0, 0, \genfrac{}{}{0pt}{}{2*}{0})$

A zero follows from Lemma 3.3.1.

If there is a variable in each of levels $k+3$ and $k+4$, then after an initial s3-contraction, there are two possible cases:

$(\genfrac{}{}{0pt}{}{2}{0}, 0, 0, 1, 1) \xrightarrow{s3}$

- $(0, 0, 0, 2*, 1)$
- $(0, 0, 0, 1, 2*)$

A zero follows from Lemma 1.1.3 and Hensel's Lemma. □

Lemma 3.5.3. *Suppose that (3.1) has two variables in level k with the same π-coefficient, at least two variables in level $k+1$, and a variable in at least one of level $k+2$, $k+3$, or $k+4$. Then (3.1) has a nontrivial zero.*

Proof. By Lemma 3.5.1, assume the variables in level $k+1$ have the same π-coefficient. They may be contracted to a variable in level $k+3$, and so if there is a variable in level $k+2$ or $k+4$, a zero follows from Lemma 3.5.2. If there is a variable in level $k+3$, by Lemma 3.5.1, assume it has the same π-coefficient as the resulting variable. A zero follows from Lemma 3.3.1. □

Lemma 3.5.4. *Suppose that (3.1) has two variables in level k with differing π-coefficients, a variable in level $k+1$, and a variable in at least one of levels $k+2$ or $k+4$. Then (3.1) has a nontrivial zero.*

Proof. Consider the following series of contractions for each case:

$$(^1_1, 1, 1) \xrightarrow{d} (0, ^{1*}_1, 1) \xrightarrow{d} (0, 0, ^{2*}_0),$$

$$(^1_1, 1, 0, 0, 1) \xrightarrow{d} (0, ^2_0, 0, 0, 1) \xrightarrow{s3} (0, 0, 0, 0, 2*).$$

In both cases, a zero follows from Lemma 3.3.1. □

Lemma 3.5.5. *Suppose that $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$, $s \geq \frac{3}{2}d$, and after normalization, for some k with $0 \leq k \leq d-4$, (3.1) has two variables in level k with the same π-coefficient, at least four variables in level $k+2$, and at most $\frac{3}{2}(k+3)$ variables in levels 0 through $k+1$ together. Then (3.1) has a nontrivial zero.*

Proof. By Lemma 3.5.1, assume all of the variables in level $k+2$ have the same π-coefficient. By Lemma 3.5.2, assume level $k+3$ is unoccupied. By normalization, level 0 is occupied, and so if any of levels $k+4$, $k+5$, and $k+6$ is congruent to 0 modulo d, then a zero follows from Lemma 3.3.4. Thus, assume $k+6 < d$. By Lemma 3.3.4, assume levels $k+4$, $k+5$, and $k+6$ are unoccupied. By normalization, $s \geq \frac{3}{2}d$ implies that levels 0 through $k+6$ contain at least $\frac{3}{2}(k+7) = \frac{3}{2}(k+3)+6$ variables, and so level $k+2$ contains at least six variables. A zero follows from Lemma **??**. □

Lemma 3.5.6. *Suppose that $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$, $s \geq \frac{3}{2}d$. Then (3.1) has a nontrivial zero.*

Proof. Assume that the form is normalized. By Lemma 3.4.5, assume that level 0 has at most four variables.

First, suppose that level 0 contains four variables. By Lemma 3.4.6, assume that three of the variables fall into one π-coefficient class, and one into the other. By Lemma 3.3.2, assume level 1 is unoccupied. Therefore in this case, the form begins $[^3_1, 0]$.

Now, suppose level 0 contains exactly three variables, not all having the same π-coefficient. Suppose that level 1 is occupied. By Lemma 3.5.4, assume level 2 is unoccupied. By normalization, level 1 contains at least two variables. By Lemma 3.5.3, assume level 3 is unoccupied. By normalization, level 1 contains at least three variables, and by Lemma 3.5.1 assume they have the same π-coefficient. After the series of contractions

$$(\genfrac{}{}{0pt}{}{2}{1}, \genfrac{}{}{0pt}{}{3}{0}) \xrightarrow{d} (1, \genfrac{}{}{0pt}{}{4*}{0}) \xrightarrow{s2,s2} (1, 0, 0, 2*),$$

a zero follows from Lemma 3.3.1. Thus assume level 1 is unoccupied. Therefore in this case, the form begins $[\genfrac{}{}{0pt}{}{2}{1}, 0]$.

Next, suppose that level 0 contains exactly three variables, all having the same π-coefficient. First, suppose that level 1 contains at least two variables. By Lemma 3.5.3, assume levels 2, 3, and 4 are unoccupied. By normalization, level 1 contains at least five variables, a pair of which can be contracted up exactly two levels, and thus a zero follows from Lemma 3.5.3. Thus, assume level 1 contains at most one variable. Therefore in this case, the form begins $[\genfrac{}{}{0pt}{}{3}{0}, 1]$ or $[\genfrac{}{}{0pt}{}{3}{0}, 0]$.

Finally, suppose that level 0 contains exactly two variables. By normalization level 1 is occupied. Suppose the variables in level 0 have differing π-coefficients. By Lemma 3.5.4, assume level 2 is unoccupied, and so by normalization, level 1 contains at least three variables. The two variables in level 0 can be contracted to a variable in level 1 so that its resulting four variables can be used to form two pairs with the same π-coefficient. By Lemma 3.3.4 then, assume levels 3, 4, and 5 are unoccupied, and so by normalization level 1 contains at least seven variables. A zero follows from Lemma 3.4.1. Thus, assume the variables in level 0 have the same π-coefficient. Suppose level 1 contains at least two variables. By Lemma 3.5.1, assume they have the same π-coefficient. By Lemma 3.5.3, assume levels 2, 3, and 4 are unoccupied. and so by normalization level 1 contains at least six variables with the same π-coefficient, and a zero follows from Lemma 3.4.1. Thus assume level 1 contains exactly one variable. Therefore in this case, the form begins $[\genfrac{}{}{0pt}{}{2}{0}, 1]$.

It follows from the above that the form begins with one of the blocks $[\genfrac{}{}{0pt}{}{2}{1}, 0]$, $[\genfrac{}{}{0pt}{}{3}{0}, 0]$, $[\genfrac{}{}{0pt}{}{2}{0}, 1]$, $[\genfrac{}{}{0pt}{}{3}{1}, 0]$, or $[\genfrac{}{}{0pt}{}{3}{0}, 1]$. In each case there are either 3 or 4 variables in the first two levels. Now suppose that there are $\frac{3}{2}\ell$ or $\frac{3}{2}\ell + 1$ variables in the first ℓ levels specified, for even ℓ. We will show by induction that the subsequent pairs of levels are composed of the blocks $[\genfrac{}{}{0pt}{}{2}{0}, 0]$ and $[\genfrac{}{}{0pt}{}{3}{0}, 0]$. By normalization, level ℓ is occupied, by Lemma 3.5.2 assume level $\ell + 1$ is unoccupied, and so by normalization and Lemmas 3.5.5 and 3.5.1, assume level ℓ contains two or three variables having the same π-coefficient. If $\frac{3}{2}\ell$ variables are contained in levels 0 through $\ell - 1$, then by normalization level ℓ contains exactly three. By induction, it follows that level $d - 2$ contains at least two variables with the same π-coefficient, and so if levels 0 and 1 are both occupied, a zero follows from Lemma 3.5.2 with $k = d - 2$, and if level 0 contains two variables with differing π-coefficients, then a zero follows from Lemma 3.5.1 with $k = d - 2$.

The only case remaining is forms that begin $[\genfrac{}{}{0pt}{}{3}{0}, 0, \genfrac{}{}{0pt}{}{3}{0}, 0, \genfrac{}{}{0pt}{}{3}{0}, 0, \ldots]$. If for some even k the variables in *both* of levels k and $k + 2$ are all in the same π-coefficient class, then perform an st-contraction from level k to level $k + 2$ and a zero follows from Lemma 3.3.1. Thus assume the π-coefficient classes of the variables in the even numbered levels alternate. Using a slight modification of our notation, this might be written

$[{3 \atop 0}, 0, {0 \atop 3}, 0, {3 \atop 0}, 0, {0 \atop 3}, \ldots]$. However, because $d = 2m$, where m is odd, the variables in levels $d-2$ and 0 all have the same π-coefficient, and a zero follows.

□

Lemma 3.5.7. *Suppose that $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, $s \geq d+1$ and after normalization (3.1) has at least three variables in level 0 and at least two variables in level 1. Then (3.1) has a nontrivial zero.*

Proof. By Lemmas 3.3.2 and 3.4.6, assume level 0 contains exactly three variables. By Lemma 3.5.1, assume the variables in level 1 have the same π-coefficient. By Lemma 3.5.3, assume levels 2, 3, and 4 are unoccupied. By normalization, levels 0 through 4 together contain at least six variables, and so level 1 contains at least three variables having the same π-coefficient. Thus by Lemma 3.3.3 with $k = 1$ assume level 5 is unoccupied. By normalization, levels 0 through 5 together contain at least seven variables, and so level 1 contains at least four variables with the same π-coefficient. Contract both pairs to level 3, and by Lemma 3.5.1, assume the resulting variables have the same π-coefficients. A zero follows from Lemma 3.3.1. □

Lemma 3.5.8. *Suppose that $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, $s \geq d+1$ and after normalization, for some $i \in \{2, 3\}$ and k with $0 \leq k+i \leq d-1$, (3.1) has two variables in level k with the same π-coefficient, at least four variables in level $k+i$, no variables in level $k+i+1$, and at most $k+i+1$ variables in levels 0 through $k+i-1$ together. Then (3.1) has a nontrivial zero.*

Proof. By Lemma 3.5.1, assume all of the variables in level $k+i$ have the same π-coefficient. If any of $k+i+2$, $k+i+3$, or $k+i+4$ is congruent to 0 modulo d, then a zero follows from Lemma 3.3.4. Otherwise, assume these levels are unoccupied, and so by normalization, $s \geq d+1$ implies that levels 0 through $k+i+4$ contain at least $k+i+6$ variables, and so level $k+i$ contains at least five variables. A zero follows from Lemma 3.4.3. □

Lemma 3.5.9. *Suppose that $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$ and $s \geq d+1$. Then (3.1) has a nontrivial zero.*

Proof. Assume that the form is normalized. By Lemma 3.4.5, assume that level 0 has at most four variables.

First, suppose level 0 has four variables. By Lemma 3.4.6, assume three of them fall into one π-coefficient class, and one into the other. By Lemmas 3.3.2 and 3.3.3, assume levels 1 and 4 are unoccupied. By normalization, levels 2 and 3 together contain at least two variables, and by Lemma 3.5.2, assume at most one is occupied. By Lemma 3.5.1, assume they have the same π-coefficient. If level 2 contains at least two variables, they can be contracted to a variable in level 4 and a zero follows from Lemma 3.3.3. Thus assume level 2 is unoccupied, level 3 contains at least two variables, and by Lemma 3.5.8 at most three variables. Therefore in this case, the form begins with $[{3 \atop 1}, 0, 0]$ followed by $[{2 \atop 0}, 0]$ or $[{3 \atop 0}, 0]$.

Next, suppose there are exactly three variables in level 0, not all having the same π-coefficient. By Lemma 3.5.7, assume level 1 has at most one variable. Suppose level 1 has exactly one variable. Then by Lemma 3.5.4, assume levels 2 and 4 are unoccupied. By normalization, level 3 contains at least two variables, and by Lemma 3.5.8 at most three. By Lemma 3.5.1 assume they have the same π-coefficient. Thus assume level 3 contains two or three variables having the same π-coefficient, and level 4 contains zero. Now, suppose level 1 is unoccupied. By normalization level 2 is occupied, and by Lemma 3.5.2, assume level 3 is unoccupied. By normalization, level 2 contains at least two variables, and by Lemma 3.5.8 at most three. By Lemma 3.5.1 assume they have the same π-coefficient. Thus assume level 2 contains two or three variables and level 3 contains zero. Therefore in this case, the form begins with $[^2_1, 1, 0]$ or $[^2_1, 0]$ followed by $[^2_0, 0]$ or $[^3_0, 0]$.

Now suppose level 0 contains exactly three variables having the same π-coefficient. By Lemma 3.5.7, assume level 1 has at most one variable. Suppose level 1 is unoccupied. By normalization level 2 contains at least one variable, and so by Lemma 3.5.2, assume level 3 is unoccupied. By normalization, level 2 contains at least two variables. By Lemma 3.5.1, assume they have the same π-coefficient and thus can be contracted to a variable in level 4. A zero follows from Lemma 3.3.3. Thus assume level 1 contains exactly one variable. By Lemma 3.3.3, assume level 4 is unoccupied. By normalization, levels 2 and 3 together contain at least two variables, and by Lemma 3.5.2, assume at most one is occupied, and by Lemma 3.5.1 that they have the same π-coefficient. If the variables are in level 2, they can be contracted to a variable in level 4, and a zero follows from Lemma 3.3.3. Thus, by Lemma 3.5.8 assume level 3 contains two or three variables with the same π-coefficient and level 4 contains zero. Therefore in this case, the form begins $[^3_0, 1, 0]$ followed by $[^2_0, 0]$ or $[^3_0, 0]$.

Next, suppose level 0 contains exactly two variables with differing π-coefficients. By normalization, level 1 contains at least one variable, and by Lemma 3.5.4, assume levels 2 and 4 are unoccupied, and so by normalization level 1 contains at least two variables. First, suppose level 1 contains at least three variables. Because a d-contraction could be performed from level 0 so that the resulting variables in level 1 would form two pairs with the same π-coefficients, by Lemma 3.3.4 with $k = 1$, assume levels 3, 4, and 5 are also unoccupied. By normalization level 1 contains at least five variables. A d-contraction can be performed from level 0 such that the resulting six variables in level 1 can be used to form three pairs each having the same π-coefficient; a zero follows from Lemma 3.4.1. Thus, assume level 1 contains exactly two variables, and so by normalization applied to levels 0 through 4, level 3 contains at least two variables. After a d-contraction from level 0, assume by Lemma 3.5.1 with $k = 1$ that the variables in level 3 have the same π-coefficient. Suppose level 3 contains at least four variables. By Lemma 3.3.4, if $d = 6$ a zero follows; otherwise assume levels 5, 6, and 7 are unoccupied. By normalization level 3 contains at least five variables, and a zero follows from Lemma 3.4.3. Thus assume level 3 contains at most three variables. Therefore in this case, the form begins with $[^1_1, 2, 0]$ and is followed by $[^2_0, 0]$ or $[^3_0, 0]$.

Finally, suppose level 0 contains exactly two variables with the same π-coefficient. By normalization, level 1 is occupied. Suppose level 1 contains at least two variables; by Lemma 3.5.1, assume they have the same π-coefficient. By Lemma 3.5.3, assume levels 2, 3, and 4 are unoccupied. By normalization, level 1 contains at least four variables. By Lemma 3.3.3 with $k = 1$, assume level 5 is unoccupied, and so by normalization level 1 contains at least five variables. By Lemma 3.5.1, assume they have the same π-coefficient; a zero follows from Lemma 3.4.3. Thus assume level 1 contains exactly one variable, and by normalization level 2 is occupied. By Lemma 3.5.2, assume level 3 is unoccupied, and so by normalization level 2 contains at least two variables, and by Lemma 3.5.8, at most three. By Lemma 3.5.1, assume they have the same π-coefficient. Therefore in this case, the form begins with $[{2 \atop 0}, 1]$ followed by $[{2 \atop 0}, 0]$ or $[{3 \atop 0}, 0]$.

It follows from the above that the form begins with $[{3 \atop 1}, 0, 0]$, $[{2 \atop 1}, 1, 0]$, $[{2 \atop 1}, 0]$, $[{3 \atop 0}, 1, 0]$, $[{1 \atop 1}, 2, 0]$, or $[{2 \atop 0}, 1]$ followed by $[{2 \atop 0}, 0]$ or $[{3 \atop 0}, 0]$. In each case, the form begins with a block of $\ell = 2$ or $\ell = 3$ levels, level ℓ is assumed to have either two or three variables having the same π-coefficient, level $\ell + 1$ is assumed to be unoccupied, and levels 0 through $\ell - 1$ contain exactly $\ell + 1$ variables. If $\ell = d - 2$ or $\ell = d - 3$, a zero follows from the argument below. Otherwise, we proceed by induction as follows.

First suppose that level ℓ contains exactly two variables. By normalization, level $\ell + 2$ is occupied, and by Lemma 3.5.2 with $k = \ell$, assume level $\ell + 3$ is unoccupied. By normalization, level $\ell + 2$ contains at least two variables, by Lemma 3.5.8 with $k = \ell$ and $i = 2$ at most three, and by Lemma 3.5.1 with $k = \ell$, they have the same π-coefficient.

Now, suppose that level ℓ contains three variables. By Lemma 3.3.3, assume level $\ell + 4$ is unoccupied, and by Lemma 3.5.1, assume the variables in each of levels $\ell + 2$ and $\ell + 3$ have the same π-coefficient. By normalization, levels $\ell + 2$ and $\ell + 3$ together contain at least two variables, and by Lemma 3.5.2 only one is occupied. If the variables are in level $\ell + 2$, they can be contracted to a variable in level $\ell + 4$ and a zero follows from Lemma 3.3.3. Thus assume level $\ell + 2$ is unoccupied and by Lemma 3.5.8, assume level $\ell + 3$ contains two or three variables.

In both cases, the initial block of ℓ levels is followed by $[{2 \atop 0}, 0]$ or $[{3 \atop 0}, 0, 0]$. Further, the next two levels are subject to the same constraints as were levels ℓ and $\ell + 1$, and so by induction it follows that at least one of levels $d - 3$ or $d - 2$ contains two variables with the same π-coefficient, or level $d - 4$ contains three variables with the same π-coefficient. In the former case, a zero follows from Lemmas 3.5.2 and 3.5.1 with $k = d - 3$ or $k = d - 2$. In the latter case, a zero follows from Lemma 3.3.3 with $k = d - 4$.

$\square$

If $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$, from Lemma 3.5.6 we have $\Gamma(d, K) \leq \frac{3}{2}d$. To demonstrate we have the equality from the first part of Theorem 3.1.1, we present a form in $s = \frac{3}{2}d - 1$ variables that has no nontrivial zero. Let

$$f = \sum_{i=0}^{\frac{d-2}{4}-1} \pi^{4i}[(x_{6i}^d + x_{6i+1}^d + x_{6i+2}^d) + (\pi^2 + \pi^3)(x_{6i+3}^d + x_{6i+4}^d + x_{6i+5}^d)] + \pi y^d + \pi^{d-2} z^d.$$

Recall that for these four fields $2 \equiv \pi^2 \pmod{\pi^4}$ and that the d^{th} powers modulo π^4 are 0, 1, and $1+\pi^2+\pi^3$.

Suppose that f has a nontrivial zero $(a_0, \ldots, a_{s-3}, b, c)$ in $\mathcal{O}$. We have $a_0^d + a_1^d + a_2^d + \pi b^d \equiv 0 \pmod{\pi^2}$. Since the d^{th} powers modulo π^2 are 0 and 1, it follows that $\pi | b$. Note that

$$(1+\pi)(x^d + y^d + z^d) + \pi^2(u^d + v^d + w^d) \equiv$$
$$(1+\pi)[(x^d + y^d + z^d) + (\pi^2 + \pi^3)(u^d + v^d + w^d)] \pmod{\pi^4}.$$

Suppose π divides all of a_i for $i < 3j$. Then π^2 divides $a_{3j}^d + a_{3j+1}^d + a_{3j+2}^d$. If π does not divide all of a_{3j}, a_{3j+1}, and a_{3j+2}, then modulo π^4, $a_{3j}^d + a_{3j+1}^d + a_{3j+2}^d \in \{\pi^2, \pi^3\}$, and so $a_{3j}^d + a_{3j+1}^d + a_{3j+2}^d + (\pi^2 + \pi^3)(a_{3j+3}^d + a_{3j+4}^d + a_{3j+5}^d) \not\equiv 0 \pmod{\pi^4}$. Thus assume all of a_{3j}, a_{3j+1}, and a_{3j+2} are divisible by π. By induction, π divides each of $a_0, \ldots, a_{s-3}$. So, we must have $\pi | c$, a contradiction. Thus f has no primitive zero modulo π^d, and therefore f has no nontrivial zero over K.

If $K \in \{\mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{-5})\}$, from Lemma 3.5.9 we have $\Gamma(d, K) \leq d + 1$. To demonstrate we have the equality from the second part of Theorem 3.1.1, we present a form in $s = d$ variables that has no nontrivial zero over K. Let

$$g = \sum_{i=0}^{d-1} \pi^i x_i^d.$$

Chapter 4 Quartic Forms

4.1 Introduction

Here, we extend the method of contraction used in chapters 2 and 3 to additive forms of degree 4 over four ramified quadratic extensions of $\mathbb{Q}_2$, demonstrating the following result.

Theorem 4.1.1. *If* $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$*, then*

$$\Gamma^*(4, K) = 11$$

Further, I conjecture that the following holds for the remaining two quadratic ramified extensions.

Conjecture 4.1.2. *Artin's Additive Form Conjecture holds for all quartic forms over ramified quadratic extensions of* $\mathbb{Q}_2$*. In particular:*

$$\Gamma^*(4, \mathbb{Q}_2(\sqrt{-1})) = 11$$

$$\Gamma^*(4, \mathbb{Q}_2(\sqrt{-5})) = 9$$

The following anisotropic forms:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + (\pi + \pi^3)(x_4^4 + x_5^4 + x_6^4 + x_7^4) + (\pi^2 + \pi^3)x_8^4 + (\pi^3 + \pi^4 + \pi^5)x_9^4$$

and

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + \pi(x_4^4 + x_5^4 + x_6^4 + x_7^4)$$

were obtained by direct computation in $\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{-5})$, respectively, where π is as in Table 1.1.

4.2 Preliminaries

For the remainder of this chapter, let K denote one of the ramified quadratic extensions $\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})$, assume that the number of variables s in the form (1.1) is 11, and the exponent d is 4, i.e,

$$a_1x_1^4 + a_2x_2^4 + \ldots + a_{11}x_{11}^4. \tag{4.1}$$

By a change of variables, for $d = 4$ it suffices to consider only forms which have all of their variables in levels 0 through 3. Thus, throughout this chapter we will consider types of forms with only four levels listed, e.g., forms of type $(5, 0, 1, 1)$. We extend the type notation in chapter 1 by considering all cyclic permutations of the type notation to denote the same type. For example, a form of type $(1, 1, 5, 0)$ will also be said to be of type $(5, 0, 1, 1)$.

In order to see how contractions will behave in the four chosen fields, we compute the fourth powers modulo π^7. Let $2 = \pi^2 u$, where π is as in Table 1.1 and $u \in \mathcal{O}^\times$, $a = a_0 + a_1\pi + a_2\pi^2 + \ldots \in \mathcal{O}$ with $a_i \in \{0, 1\}$. Note that for the four fields considered, $2 \equiv \pi^2 \pmod{\pi^4}$, i.e., $u \equiv 1 \pmod{\pi^2}$. Then

$$\begin{aligned}
(1 + a\pi)^4 &= 1 + 4a\pi + 6a^2\pi^2 + 4a^3\pi^3 + a^4\pi^4 \\
&= 1 + u^2 a\pi^5 + (ua^2\pi^4 + u^2a^2\pi^6) + u^2a^3\pi^7 + a^4\pi^4 \\
&\equiv 1 + (ua^2 + a^4)\pi^4 + u^2a\pi^5 + u^2a^2\pi^6 \\
&\equiv 1 + (ua^2 + a^4)\pi^4 + a\pi^5 + a\pi^6 \\
&\equiv 1 + (ua_0 + ua_1\pi^2 + a_0)\pi^4 + (a_0 + a_1\pi)\pi^5 + a_0\pi^6 \\
&\equiv 1 + (u + 1)a_0\pi^4 + a_0\pi^5 + a_0\pi^6 \pmod{\pi^7}.
\end{aligned}$$

Then the unit fourth powers modulo π^7 are $\{1, 1+\pi^5\}$ if $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10})\}$ because $u \equiv 1 \pmod{\pi^4}$, and $\{1, 1+\pi^5+\pi^6\}$ if $K \in \{\mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$ because $u \equiv 1 + \pi^2 \pmod{\pi^4}$. In both cases, there exists an $\alpha \in \mathcal{O}$ such that $\alpha^4 \equiv 1 + \pi^5 \pmod{\pi^6}$. Now, suppose

$$a_0\beta_0^4 + a_1\beta_1^4 = b_0 + b_1\pi + b_2\pi^2 + \ldots + b_5\pi^5 + \ldots.$$

with $a_i, \beta_i \in \mathcal{O}^\times$. Then

$$a_0\beta_0^4 + a_1(\beta_1\alpha)^4 = b_0 + b_1\pi + b_2\pi^2 + \ldots + (1 + b_5)\pi^5 + \ldots.$$

If $b_5 = 1$, then $(1 + b_5) = 2 \equiv \pi^2 \pmod{\pi^3}$. Thus the coefficient of the π^5 term gets switched from 0 to 1, or vice-versa, possibly changing the values of the higher order terms, while leaving the lower order terms unchanged. The consequence of this is that for a variable formed by the corresponding contraction, we may choose the coefficient of the π^5 term. (Here the level of the term π^5 is considered relative to the level of the original variables x_0 and x_1, not the resulting variable.) We will say that a variable is *free* at some level if the coefficient corresponding to that level can be chosen in this way. Further, any variable resulting from a contraction or series of contractions involving a variable *free at level k* will retain this very useful property. Because of this, we will (for the sake of convenience) refer to the originating variables as free at this level also, even though they are not the result of a contraction.

As an example, consider a pair of variables in level 0 which could be contracted to a variable in level 3, and thence contracted with a variable there to one in level 5, i.e., one with a π^5-coefficient of 1 (relative to level 0). By choosing to form the contraction in the way described so the π^5-coefficient is 0, we instead contract to a variable which has level at least 6. (One may thus think of getting level 5 "for free"). Similarly, if after the same series of contractions we would create a variable in a level greater than 5, we may instead arrange for the new variable to be precisely in level 5.

We now add to this notion of free variable the basic contractions which can be performed, depending only on the π- and π^2-coefficients of a pair of variables in the same level.

The results of the following two lemmas will be used frequently and without reference.

Lemma 4.2.1.

1. *Two variables in the same level with differing π-coefficients can be contracted to a variable exactly one level up.*

2. *Two variables in the same level with the same π- and π^2-coefficients can be contracted to a variable exactly two levels up.*

3. *Two variables in the same level with the same π-coefficient and differing π^2-coefficients can be contracted to a variable at least three levels up.*

4. *If a pair of variables in level k are contracted, the resulting variable is free at level $k+5$.*

Proof. $(1 + a_1\pi + a_2\pi^2) + (1 + b_1\pi + b_2\pi^2) \equiv 2 + (a_1 + b_1)\pi + (a_2 + b_2)\pi^2 \equiv (a_1 + b_1)\pi + (1 + a_2 + b_2)\pi^2 \pmod{\pi^3}$.

(1) If $a_1 \neq b_1$, then $(a_1 + b_1)\pi + (1 + a_2 + b_2)\pi^2 \equiv \pi \pmod{\pi^2}$.

(2) If $a_1 = b_1$ and $a_2 = b_2$, then $(a_1 + b_1)\pi + (1 + a_2 + b_2)\pi^2 \equiv 2\pi + 3\pi^2 \equiv \pi^2 \pmod{\pi^3}$.

(3) If $a_1 = b_1$ and $a_2 \neq b_2$, then $(a_1 + b_1)\pi + (1 + a_2 + b_2)\pi^2 \equiv 2\pi + 2\pi^2 \equiv 0 \pmod{\pi^3}$.

Statement (4) follows from the discussion above. □

Lemma 4.2.2. *If there are at least three variables in the same level, a pair may be contracted at least two levels. If there are at least five variables in the same level, a pair may be contracted exactly two levels.*

Proof. By the pigeonhole principle, among three variables there are two with the same π-coefficient, and among five variables, there are two with the same π, π^2-coefficients. The contractions follow from Lemma 4.2.1. □

The following is a more complicated contraction involving four variables.

Lemma 4.2.3. *Suppose there are four variables in the same level k having the same π-coefficient, and which can be used to form two pairs, the variables of each pair being in the same π^2, π^3-coefficient class. Then those four variables can be contracted to a variable at least four levels up which is free at level $k+5$.*

Proof. Let $a, b, c, d \in \mathcal{O}^\times$, with $a = 1 + a_1\pi + a_2\pi^2 + \ldots$, and the other units following the analogous convention, and with $(a, b), (c, d)$ forming the pairs of the hypothesis. Then,

$$\begin{aligned}a+b+c+d &= 4+4a_1\pi+2(a_2+c_2)\pi^2+2(a_3+c_3)\pi^3+(a_4+b_4+c_4+d_4)\pi^4+\dots\\ &= u^2\pi^4+u^2a_1\pi^5+u(a_2+c_2)\pi^4+u(a_3+c_3)\pi^5+(a_4+b_4+c_4+d_4)\pi^4+\dots\\ &\equiv 0 \pmod{\pi^4}\end{aligned}$$

The variable formed from this contraction is free at level $k+5$ by the same calculation used to prove Lemma 4.2.1 (4). □

Finally, we give a statement without proof of Hensel's Lemma specific to the needs of this chapter. (For a proof, see [8, Theorem 2.1].)

Lemma 4.2.4 (Hensel's Lemma)**.** *Let $d = 4$, and x_i be a variable of (4.1) at level h. Suppose that x_i can be used in a contraction of variables (or one in a series of contractions) which produces a new variable at level at least $h + 7$. Then (4.1) has a nontrivial zero.*

Lemma 4.2.5. *Suppose that (4.1) has two variables in level k, and for each of levels $k+1, k+2, \dots, k+t-1$, either one of the two chosen variables in level k is free at that level, or there is a distinct variable at that level.*

Then contractions can be performed to produce a variable at level at least $k + t$. Alternatively, contractions can be performed to produce a variable exactly at any level at which any of the initial variables was free.

Proof. Any two variables in the same level can be contracted to a variable at least one level higher. By repeated contractions, bypassing (or stopping at) levels on which the variables are free, we obtain the desired variable. □

4.3 Archetypal Forms

Lemma 4.3.1. *If (4.1) is of type $(3, 0, 2, 1)$, then it has a nontrivial zero.*

Proof. Contract a pair from level 0 to a variable at least two levels higher, say level k. If $k \geq 7$, then a zero follows from Hensel's Lemma. Otherwise, we will use Lemma 4.2.5 to produce a variable in level at least 7.

There is at least one variable remaining in level $0 \equiv 4 \pmod{d}$. Treat this as a variable in level 4. The variable resulting from the contraction is free at level 5, and so we may assume that $k \neq 5$. The two variables in level 2 can be treated as one variable in level 2 and one in level 6. Thus level k contains two variables (including the one produced by the contraction) and levels k through 6 either contain a variable or are free with respect to the variable formed by the initial contraction. A zero follows from Lemma 4.2.5 and Hensel's Lemma. □

Corollary 4.3.2. *If (4.1) is of type $(5, 0, 1, 1)$, then it has a nontrivial zero.*

Proof. By Lemma 4.2.2, contract a pair from level 0 to level 2. The resulting form is of type $(3, 0, 2, 1)$ and a zero follows from Lemma 4.3.1. □

Corollary 4.3.3. *If (4.1) is of type* $(7, 0, 0, 1)$*, then it has a nontrivial zero.*

Proof. Contract a pair from level 0 to level 2. A zero follows from Corollary 4.3.2. □

Lemma 4.3.4. *If (4.1) is of type* $(6, 0, 1, 0)$*, then it has a nontrivial zero.*

Proof. Suppose two pairs contract from level 0 to level 2. The two resulting variables are free at level 5. By the pigeonhole principle, two of the three variables now in level 2 have the same π-coefficient, and so a pair can be contracted to a variable at least in level 4 which is also free at level 5. Assume this level k is either 4 or 6. The remaining variable in level 2 can be treated as being in level 6. Thus level k contains two variables and levels k through 6 contain a variable or are free with respect the variable in level k resulting from the previous contractions. A zero follows from Lemma 4.2.5 and Hensel's Lemma.

Thus assume there is at most one pair in level 0 having both variables in the same π, π^2-coefficient class. It follows that there are three variables in some π, π^2-coefficient class, and one in each of the other three classes. Thus there are two pairs which can be contracted at least three levels up. If any pair contracts at least four levels, a zero follows as above from Lemma 4.2.5 (with $k = 4$ or $k = 6$) and Hensel's Lemma. Thus assume both pairs contract to level 3. A zero again follows from Lemma 4.2.5 (with $k = 3$) and Hensel's Lemma. □

Corollary 4.3.5. *If (4.1) is of type* $(5, 0, 3, 0)$*, then it has a nontrivial zero.*

Proof. By Corollary 4.3.2, assume all variables in level 2 have the same π-coefficient, for otherwise a pair could be contracted to level 3, creating a form of type $(5, 0, 1, 1)$. By the pigeonhole principle, there are two in the same π, π^2-coefficient class. Contract them to level 0. A zero follows from Lemma 4.3.4. □

Corollary 4.3.6. *If (4.1) is of type* $(8, 0, 0, 0)$*, then it has a nontrivial zero.*

Proof. Contract a pair from level 0 to level 2. A zero follows from Lemma 4.3.4. □

4.4 Remaining Cases

Lemma 4.4.1. *Suppose that* $s \geq 11$ *and after normalization (4.1) has at least seven variables in level 0. Then (4.1) has a nontrivial zero.*

Proof. By Corollary 4.3.6, assume level 0 contains exactly seven variables. By Corollary 4.3.3, assume level 3 is empty, and by Lemma 4.3.4, assume level 2 is empty. Then level 1 has at least four variables. By Lemma 4.3.4, assume they have the same π-coefficient. By the pigeonhole principle, there is a pair which can be contracted to level 3. A zero follows from Corollary 4.3.3. □

Lemma 4.4.2. *Suppose that* $s \geq 11$ *and after normalization (4.1) has at least six variables in level 0. Then (4.1) has a nontrivial zero.*

Proof. By Lemma 4.4.1, assume level 0 contains exactly six variables. By Lemma 4.3.4, assume level 2 is empty. By normalization, level 1 contains at least three variables, and so by Lemma 4.3.1, assume level 3 contains at most one variable. Thus assume (4.1) is of type $(6, 4, 0, 1)$ or $(6, 5, 0, 0)$. By Lemma 4.3.4 assume the variables in level 1 all have the same π-coefficient.

Suppose first that (4.1) is of type $(6, 5, 0, 0)$. Note that any variable formed from a contraction from level 1 will be free at level 6. Supposing a pair in level 1 contracts to level 4, by the pigeonhole principle there is a pair among the three remaining variables which contracts to level 3, and a zero follows from Corollary 4.3.3. If a pair in level 1 contracts to at least level 5, then the resulting variable is free at level 6, and another variable from level 1 may be treated as being in level 5. Thus by Lemma 4.2.5, a variable can be formed at level 6, and a zero follows from Lemma 4.3.4. Thus assume every pair of variables in level 1 can only contract to level 3, and so they all belong to the same π, π^2-class. By the pigeonhole principle and Lemma 4.2.3, four of these variables contract to a variable in at least level 5. Because this variable is free at level 6, by Lemma 4.2.5, a variable can be formed at level 6 as before, and a zero follows from Lemma 4.3.4.

Suppose now that (4.1) is of type $(6, 4, 0, 1)$. By Corollary 4.3.2, assume the variables in level 0 have the same π-coefficient. If a pair from level 0 contracts to level 3, a zero follows from Lemma 4.3.1. If a pair from level 0 contracts to level 4 or higher, by Lemma 4.2.5 a variable can be formed in level 5, and a zero follows from Corollary 4.3.2. Thus assume all variables in level 0 are in the same π, π^2-class. By the pigeonhole principle and Lemma 4.2.3, four variables contract to a variable in level at least 4. By Lemma 4.2.5, a variable can be formed in level 5, and a zero follows from Corollary 4.3.2. □

Lemma 4.4.3. *Suppose that $s \geq 11$ and after normalization (4.1) has at least five variables in level 0. Then (4.1) has a nontrivial zero.*

Proof. By Lemma 4.4.2, assume level 0 contains exactly five variables. By Corollary 4.3.5, assume level 2 contains at most two variables, and so by normalization level 1 contains at least two variables.

If level 2 contains one or two variables, then by Corollary 4.3.2, assume level 3 is empty, and so level 1 contains five or four variables, respectively. If level 1 contains five variables, by the pigeonhole principle there is a pair that can be contracted to level 3, and a zero follows from Corollary 4.3.2. If, on the other hand, level 2 contains two variables, contract a pair from level 0 to level 2, and a zero follows from Lemma 4.3.1.

Thus, assume level 2 is empty. By normalization, level 1 contains at least four variables. By Lemma 4.3.1, assume level 3 contains at most one variable, and by normalization level 1 contains at least five variables. If it contains exactly five, then level 3 contains one, and a zero follows from Corollary 4.3.2.

Thus, assume (4.1) is of type $(5, 6, 0, 0)$. We first describe two possibilities for forming variables using those in level 0. By Corollary 4.3.3, assume all variables in level 0 have the same π-coefficient. By Lemma 4.3.4, assume no pair from level 0 contracts to level 3. If a pair from level 0 contracts at least to level 4, then by Lemma 4.2.5 it can be used to form a variable in level 5, and a zero follows from Corollary 4.3.3. Thus assume all variables in level 0 are in the same π, π^2-class and so by the pigeonhole principle there are two pairs in the same π^2, π^3-class. Therefore there are four variables that can be contracted to a variable in at least level 4. If they contract to at least level 5, it can be contracted to exactly level 5 and a zero follows from Corollary 4.3.3. Thus, assume a variable y which is free at level 5 could be formed at level 4 by consuming four variables in level 0. Further by Lemma 4.2.5 the variable y and the remaining variable from level 0 could be consumed to form a variable y' at level 5.

Now, considering the original form of type $(5, 6, 0, 0)$, a pair can be contracted from level 1 to level 3, and so by Corollary 4.3.2, assume all variables in level 1 have the same π-coefficient. If a pair from level 1 contracts to at least level 6, it can be contracted exactly to level 6 (and regarded as a variable in level 2), another pair can be contracted to level 3, and a zero follows from Corollary 4.3.2. If a pair from level 1 contracts to level 5, then the resulting variable is free at level 6, and thus can be contracted with y' to a variable at level at least 7, and a zero follows from Hensel's Lemma. If a pair from level 1 contracts to level 4, then the resulting variable is free at level 6, may be contracted with y which is free at level 5, resulting in a variable at level at least 7, and a zero follows from Hensel's Lemma. Thus assume all pairs in level 1 contract to level 3, and so are all in the same π, π^2-class. Therefore, by the pigeonhole principle and Lemma 4.2.3 four variables can be contracted to a variable at level at least 5. If they contract to level at least 6, they can be contracted exactly to level 6, the remaining pair in level 1 can be contracted to level 3, and a zero follows from Corollary 4.3.2. Thus, the four variables contract to a variable in level 5 which can then be contracted with y', resulting in a variable at level at least 7, and a zero follows from Hensel's Lemma. □

Lemma 4.4.4. *Suppose that $s \geq 11$ and after normalization (4.1) has at least four variables in level 0. Then (4.1) has a nontrivial zero.*

Proof. By Lemma 4.4.3, assume level 0 has exactly four variables. By normalization, level 1 has at least two variables. By Lemma 4.3.1, assume level 2 has at most two variables, and so by normalization level 1 has at least three variables. By Lemma 4.3.1, assume level 3 has at most one variable, and further by the same lemma that either level 2 has at most one variable or level 3 is empty. In either case, levels 2 and 3 together have at most two variables, and so by normalization level 1 has at least five variables. By Corollary 4.3.2, assume level 3 is empty, and by Corollary 4.3.3 that level 1 has at most six variables. Thus assume (4.1) is of type $(4, 6, 1, 0)$ or $(4, 5, 2, 0)$. If it is of type $(4, 5, 2, 0)$, then a pair from level 1 can be contracted to level 3, and a zero follows from Lemma 4.3.1.

Thus, assume (4.1) is of type $(4, 6, 1, 0)$. By Corollary 4.3.3, assume all variables in level 0 have the same π-coefficient. If a pair from level 0 contracts to level 3, a zero follows from Lemma 4.3.4, and if a pair contracts to level at least 4, then a zero follows from Lemma 4.2.5 (with $k = 4$) and Hensel's Lemma. Thus assume that two pairs in level 0 contract to level 2. If any two of the three resulting variables in level 2 have different π-coefficients, then a zero follows from Lemma 4.3.4. Thus, contract just one pair to level 2, and the resulting pair contracts at least to level 4. We may assume that it contracts to level 4 or 5 (because it is free at level 5). By Lemma 4.2.5, if the variable was formed at level 4, it can be contracted to create a variable in level 5, which by a change of variables can be moved to level 1. A zero follows from Corollary 4.3.3. $\square$

Lemma 4.4.5. *Suppose that $s \geq 11$ and after normalization (4.1) has at least three variables in level 0. Then (4.1) has a nontrivial zero.*

Proof. By Lemma 4.4.4, assume level 0 contains exactly three variables. Suppose level 3 contains one variable. Then by Lemma 4.3.1, assume level 2 contains at most one variable, and so level 1 contains at least six variables, and a zero follows from Lemma 4.3.4.

Thus assume level 3 is empty. By Corollary 4.3.5, assume level 2 has at most four variables, and so level 1 has at least four variables. By Lemma 4.3.1, assume level 2 has at most two variables, and so level 1 has at least six variables. By Corollary 4.3.3, assume level 1 has exactly six variables, and thus level 2 has exactly two. Contract a pair from level 1 to level 3. A zero follows from Lemma 4.3.1. $\square$

Because $s \geq 11$, level 0 will contain at least three variables after normalization, and so by Lemma 4.4.5 we have that $\Gamma^*(4, K) \leq 11$. In order to establish equality we will demonstrate an anisotropic form in ten variables.

Theorem 4.4.6. *Let $G = (x_0^4 + x_1^4 + x_2^4 + x_3^4)$ and $H = (x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4 + x_9^4)$.*

For $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10})\}$, $F = G + \pi H$ has no nontrivial zero.

For $K \in \{\mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$, $F = G + (\pi + \pi^2)H$ has no nontrivial zero.

Proof. Let v_π be the valuation function giving the greatest power of π dividing $\alpha \in K$. For F to have a nontrivial solution, we must have that $v_\pi(G) = v_\pi(H) + 1 \neq \infty$ for some substitution of the variables, and so one of $v_\pi(G)$ and $v_\pi(H)$ is odd and the other is even. G represents no values with odd valuation and so $v_\pi(G)$ must be even and $v_\pi(H)$ odd. The only such value modulo π^7 represented by H is $\pi^5 + \pi^6$ if $K \in \{\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{10})\}$, and π^5 if $K \in \{\mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-10})\}$. In the former case, G does not represent $\pi^6 + \pi^7$ modulo π^8, and in the latter case G does not represent π^6 modulo π^8, and so F has no nontrivial zero. $\square$

This completes the proof.

Let $\mathbb{Z}[[x]]$ denote the ring of formal power series with integer coefficients, and $\mathbb{Z}_p$ denote the ring of p-adic integers.

5.1 Isomorphisms

Theorem 5.1.1. *Let f be a power series over $\mathbb{Z}$ with constant coefficient a prime power $q = p^n$. Then $\mathbb{Z}[[x]]/f\mathbb{Z}[[x]] \cong \mathbb{Z}_p[[x]]/f\mathbb{Z}_p[[x]]$.*

Proof. Given some coset $a + f\mathbb{Z}_p[[x]]$ with $a(x) = a_0 + a_1x + a_2x^2 + \ldots \in \mathbb{Z}_p[[x]]$, write a_0 as $a_0 = \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \ldots$ with $\alpha_i \in \{0, 1, \ldots, q-1\}$. Then

$$a_0 = \alpha_0 + q(\alpha_1 + \alpha_2 q + \ldots) = \alpha_0 + (q - f)(\alpha_1 + \alpha_2 q + \ldots) + f(\alpha_1 + \alpha_2 q + \ldots).$$

$q - f$ is an element of $x\mathbb{Z}_p[[x]]$, and so collecting terms we have $a(x) = \alpha_0 + b_1x + b_2x^2 + \ldots + \beta f$, $b_i \in \mathbb{Z}_p$, and $\beta \in \mathbb{Z}_p[[x]]$. Continuing in this way, we obtain a representative $\gamma(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \ldots$ of the coset $a + f\mathbb{Z}_p[[x]] = \gamma + f\mathbb{Z}_p[[x]]$ with $\gamma_i \in \{0, 1, \ldots, q-1\}$. Any other representative γ' of this coset gives $(\gamma - \gamma') \in f\mathbb{Z}_p[[x]]$. Thus the coefficient of the smallest power of x in $\gamma - \gamma'$ with a nonzero coefficient is a multiple of q, and so γ' differs from γ in some coefficient by a multiple of q. Thus, γ is the unique representative satisfying this property.

A fortiori, we can perform this same rewriting on an element $b(x) \in \mathbb{Z}[[x]]$ to obtain a unique canonical representative with coefficients from 0 to $q - 1$. Because in both cases we get the same set of canonical representatives, this yields a bijection between $\mathbb{Z}[[x]]/f\mathbb{Z}[[x]]$ and $\mathbb{Z}_p[[x]]/f\mathbb{Z}_p[[x]]$ which is evidently a homomorphism. □

Comment: This does not hold when the constant coefficient is merely a multiple of p. The proof breaks down because the sets of canonical representatives in the two rings will not be the same.

Theorem 5.1.2. *Let $f = f_e x^e - f_{e-1}x^{e-1} - \ldots - f_0$ be a polynomial over $\mathbb{Z}_p$ with $p|f_i$ for $i \neq e$, $p \nmid f_e$, and $e \geq 1$. Then $\mathbb{Z}_p[[x]]/f\mathbb{Z}_p[[x]] \cong \mathbb{Z}_p[x]/f\mathbb{Z}_p[x]$.*

Proof. Without loss of generality, assume $f_e = 1$. Let $f = x^e - pg$ where $g \in \mathbb{Z}_p[x]$, $\deg(g) \leq e - 1$. Let $h \in \mathbb{Z}_p[[x]]$, $h = a_0 + x^e b_0$ where $a_0 \in \mathbb{Z}_p[x]$, $\deg(a_0) \leq e - 1$, $b_0 \in \mathbb{Z}_p[[x]]$. $h = a_0 + (x^e - f)b_0 + fb_0 = a_0 + pgb_0 + fb_0 = a_0 + pc_0 + fb_0$, where $c_0 = gb_0 \in \mathbb{Z}_p[[x]]$. $h = a_0 + p(a_1 + pc_1 + fb_1) + fb_0 = (a_0 + pa_1) + p^2(c_1) + f(b_0 + pb_1) = \ldots = (a_0 + \ldots + p^i a_i) + p^{i+1}(c_i) + f(b_0 + \ldots + p^i b_i) = \ldots = \hat{a} + f\hat{b}$, where $\hat{a} \in \mathbb{Z}_p[x]$, $\deg(\hat{a}) \leq e - 1$, and $\hat{b} \in \mathbb{Z}_p[[x]]$. Therefore, the coset $h + f\mathbb{Z}_p[[x]] = \hat{a} + f\mathbb{Z}_p[[x]]$.

Let $\hat{z}$ be any other representative of the coset $h + f\mathbb{Z}_p[[x]]$ with degree less than e. Then $\hat{a} - \hat{z} = f\alpha$, $g \in \mathbb{Z}_p[[x]]$. Write $\alpha = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots$. Then, $\alpha_i f_e + \alpha_{i+1} f_{e-1} + \ldots + \alpha_{i+e} f_0 = 0$ for all i, and because $p \mid f_0, \ldots, f_{e-1}$ and $p \nmid f_e$, we have $p \mid \alpha_i$ for all i, i.e., $p \mid \alpha$. Now, it follows from this that $p^2 \mid \alpha_{i+1} f_{e-1} + \ldots + \alpha_{i+e} f_0 = -\alpha_i f_e$, and so $p^2 \mid \alpha$. Continuing in this way, we get $p^j \mid \alpha$ for all j, and therefore $\alpha = 0$. Thus $\hat{a}$ is the unique representative of degree less than e.

A fortiori, this rewriting procedure works for $h \in \mathbb{Z}_p[x]$. Using the convention that $\deg(0) = -\infty$, if $h \in \mathbb{Z}_p[x]$, then each $\deg(b_i) \leq \deg(h) - e$. We thus find a unique canonical representative of the coset $h + \mathbb{Z}_p[x]$, yielding the same set of canonical representatives and thus a bijection between $\mathbb{Z}_p[[x]]/f\mathbb{Z}_p[[x]]$ and $\mathbb{Z}_p[x]/f\mathbb{Z}_p[x]$ which is evidently a homomorphism. □

Theorem 5.1.3. *Let f be a polynomial over $\mathbb{Z}$ with all coefficients except the leading coefficient divisible by p and constant coefficient a power of p. Then $\mathbb{Z}_p[[x]]/f\mathbb{Z}_p[[x]] \cong \mathbb{Z}_p[x]/f\mathbb{Z}_p[x]$.*

In particular, f is irreducible as a power series over $\mathbb{Z}$ if and only if it is irreducible as a polynomial over $\mathbb{Z}_p$.

Proof. This is an immediate consequence of the previous two isomorphisms. □

Corollary 5.1.4. *For $n \geq 1$, $\mathbb{Z}[[x]]/(p^n - x) \cong \mathbb{Z}_p$.*

Proof. This is also an immediate consequence of Theorem 5.1.3. □

A different but intimately related proof of the special case $n = 1$ can also be found in [9], page 114.

The following statement and proof of Hensel's Lemma were influenced by corresponding presentation in [9], page 129, adapted for use on power series.

Theorem 5.1.5 (Hensel's Lemma)**.** *Let $\mathcal{O}$ be a complete local ring, π be a generator of the unique maximal ideal, and $\kappa = \mathcal{O}/(\pi)$ be the residue field. If a primitive power series $f(x) \in \mathcal{O}[[x]]$ admits modulo π a factorization*

$$f(x) \equiv \bar{g}(x)\bar{h}(x) \bmod \pi$$

into relatively prime power series $\bar{g}, \bar{h} \in \kappa[[x]]$, then $f(x)$ admits a factorization

$$f(x) = g(x)h(x)$$

into power series $g, h \in \mathcal{O}[[x]]$ such that

$$g(x) \equiv \bar{g}(x) \bmod \pi,\ h(x) \equiv \bar{h}(x) \bmod \pi,$$

and $\deg(g) = \deg(\bar{g})$ (using the convention that a power series which is not a polynomial has infinite degree).

Proof. Let $g_0, h_0 \in \mathcal{O}[[x]]$ be power series such that $g_0 \equiv \bar{g} \bmod \pi$, $h_0 \equiv \bar{h} \bmod \pi$ and $\deg(g_0) = \deg(\bar{g})$, $\deg(h_0) \leq \deg(f) - \deg(\bar{g})$. Since $(\bar{g}, \bar{h}) = 1$, there exist power series $a, b \in \mathcal{O}[[x]]$ satisfying $ag_0 + bh_0 \equiv 1 \bmod \pi$. Among the coefficients of the two power series $f - g_0h_0$, $ag_0 + bh_0 - 1 \in \pi\mathcal{O}[[x]]$, pick one with minimum value and call it ρ.

Suppose now that we have

$$f \equiv (g_0 + g_1\rho + g_2\rho^2 + \ldots + g_{n-1}\rho^{n-1})(h_0 + h_1\rho + h_2\rho^2 + \ldots + h_{n-1}\rho^{n-1}) \bmod \rho^n$$

for some $n \geq 1$. Let $f_n = \rho^{-n}(f - (g_0 + \ldots + g_{n-1}\rho^{n-1})(h_0 + \ldots + h_{n-1}\rho^{n-1}))$. If $\deg(g_0) < \infty$, then it is a polynomial with a unit leading coefficient. In this case, we may write $bf_n = qg_0 + g_n$, with $\deg(g_n) < \deg(g_0)$ and $q \in \mathcal{O}[[x]]$. Otherwise, simply let $q = 1$, $g_n = bf_n$. Omit all coefficient from $af_n + h_0q$ divisible by ρ and let this be h_n, so that $\deg(h_n) \leq \deg(f) - \deg(g_0)$. Thus,

$$g_0h_n + h_0g_n = g_0(af_n + h_0q) + h_0g_n = f_n(ag_0 + bh_0) \equiv f_n \bmod \rho,$$

and so,

$$(g_0 + \ldots + g_n\rho^n)(h_0 + \ldots + h_n\rho^n) \equiv f \bmod \rho^{n+1}.$$

Continuing in this way,

$$\lim_{n\to\infty}(g_0 + \ldots + g_n\rho^n) = g, \ \lim_{n\to\infty}(h_0 + \ldots + h_n\rho^n) = h,$$

and $f = gh$.

□

The following comes from [9], page 116, exercise 9.

Theorem 5.1.6 (p-adic Weierstrass Preparation Theorem)**.** *Every nonzero power series $a(x) = a_0 + a_1x + a_2x^2 + \ldots \in \mathbb{Z}_p[[x]]$ admits a unique representation $a(x) = p^\mu a'(x)u(x)$ where p^μ is the highest power of p dividing a, $u(x)$ is a unit in $\mathbb{Z}_p[[x]]$, and $a'(x) \in \mathbb{Z}_p[x]$ is a monic polynomial of degree e, with every coefficient except the leading coefficient divisible by p and e being the least value such that $p^{\mu+1} \nmid a_e$.*

Proof. Let $b(x) = p^{-\mu}a(x)$. $b \equiv x^e(\alpha + \ldots) \bmod p$, with α a unit in $\mathbb{Z}_p$. By Hensel's Lemma, $b = a'u$ with $a' \equiv x^e \bmod p$, $\deg(a') = e$, the leading term of a' a unit, and $u \equiv \alpha + \ldots \bmod p$, i.e., u is a unit in $\mathbb{Z}_p[[x]]$. Without loss of generality, assume the leading coefficient of a' is 1. Suppose $b = b'v$ with b' and v satisfying the conditions above. Then $a' = b'(vu^{-1})$, and because $\deg(a') = \deg(b') = e$, $\deg(vu^{-1}) = 1$. Because the leading terms of a' and b' are both 1, $u = v$ and so $a' = b'$. □

Theorem 5.1.7. *A primitive power series $a(x)$ over $\mathbb{Z}$ with constant coefficient a power of prime p is irreducible as a power series if and only if the associated polynomial $a'(x)$ of Theorem 5.1.6 is irreducible as a polynomial over $\mathbb{Z}_p$.*

Proof. $\mathbb{Z}[[x]]/a\mathbb{Z}[[x]] \cong \mathbb{Z}_p[[x]]/a\mathbb{Z}_p[[x]] \cong \mathbb{Z}_p[[x]]/a'\mathbb{Z}_p[[x]] \cong \mathbb{Z}_p[x]/a'\mathbb{Z}_p[x]$. □

Corollary 5.1.8. *Let $a = a_0 + a_1x + \ldots \in \mathbb{Z}[[x]]$. Suppose a_0 is a positive power of prime p and $p \nmid a_1$. Then a is irreducible.*

Proof. The associated polynomial a' from Theorem 5.1.6 is linear. Irreducibility follows from Theorem 5.1.7. □

A completely different proof appears in [2].

5.2 Quotients of $\mathbb{Z}[[x]]$

Next I'll expand on the discussion of the relationship between $\mathbb{Z}[[x]]$ and $\mathbb{Z}_p$ hinted at in Corollary 5.1.4. I will show that certain quotients of $\mathbb{Z}[[x]]$ are isomorphic to certain extensions of $\mathbb{Z}_p$.

Theorem 5.2.1. *Let $a(x) = a_0 + a_1x + a_2x^2 + \ldots \in \mathbb{Z}[[x]]$, with a_0 a positive power of the prime p and $p \nmid a$. Let e be the least value such that a_e is not divisible by p. Then $a(x)$ factors as $a(x) = r(x)s(x)$, $r, s \in \mathbb{Z}_p[[x]]$, $deg(r) = e$, $r_0, \ldots, r_{e-1}$ divisible by p, and r_e and s_0 units in $\mathbb{Z}_p$.*

Proof. $f(x) \equiv x^e g(x) \pmod p$, and by hypothesis $g(x)$ is not divisible by x modulo p. Thus by Hensel's Lemma we get a factorization $f(x) = r(x)s(x)$ with $deg(r) = e$. Because $r(x) \equiv x^e \pmod p$, we must have that $r_0, \ldots, r_{e-1}$ are divisible by p and r_e not divisible by p. Further, because $f_0 = r_0 s_0 = p$, p^2 does not divide r_0 and s_0 is a unit. □

Theorem 5.2.2. *Let $f(x) \in \mathbb{Z}[x]$ be $f(x) = p + x^e g(x)$ with g_0 not divisible by p. Then $\mathbb{Z}[[x]]/(f)$ is isomorphic to a totally ramified extension of $\mathbb{Z}_p$ of degree e.*

Proof. By Theorem 5.2.1, $f(x) = r(x)s(x)$ with $deg(r) = e$ and s_0 a unit in $\mathbb{Z}_p$. Thus $s(x)$ is a unit in $\mathbb{Z}_p[[x]]$. Therefore, by Theorems 5.1.1 and 5.1.2, $\mathbb{Z}[[x]]/f\mathbb{Z}[[x]] \cong \mathbb{Z}_p[[x]]/f\mathbb{Z}_p[[x]] \cong \mathbb{Z}_p[[x]]/r\mathbb{Z}_p[[x]] \cong \mathbb{Z}_p[x]/r\mathbb{Z}_p[x]$.

Let v_p be the p-adic valuation. If α is a root of $r(x)$, then because $p \mid r_0, \ldots, r_{e-1}$, $v_p(\alpha^e) \geq 1$, and so $v_p(r_i\alpha^i) > 1$ for $0 < i < e$. Because $p^2 \nmid r_0$, $v_p(r_0) = 1 = v_p(\alpha^e)$, and so $v_p(\alpha) = 1/e$.

□

Chapter 6 Group of Units $(\mathcal{O}/(\pi^i))^\times$

6.1 Introduction

Let K be a finite extension of $\mathbb{Q}_p$ of degree $n = ef$, where f is the inertial degree (i.e., $\mathcal{O}/(\pi) \cong \mathbb{F}_q$ with $q = p^f$) and e is the ramification index. $\mathcal{O}$ is the valuation ring of K (also known as the ring of integers), π is a uniformizer of $\mathcal{O}$ (a generator of its maximal ideal), and $(\pi^e) = (p)$. $\mathcal{O}^\times$, the group of units of $\mathcal{O}$, is precisely the elements of $\mathcal{O}$ that are not divisible by π. The group $(\mathcal{O}/(\pi^i))^\times$ has order $(q-1)q^{i-1}$.

The following proposition is well-known.

Proposition 6.1.1. *When $e = f = 1$, $(\mathcal{O}/(\pi^i))^\times \cong (\mathbb{Z}_p/p^i\mathbb{Z}_p)^\times \cong (\mathbb{Z}/p^i\mathbb{Z})^\times$. This is cyclic when $p > 2$ or when $p = 2$ and $i \in \{1, 2\}$. When $p = 2$ and $i \geq 3$, then $(\mathbb{Z}/2^i\mathbb{Z})^\times \cong \mathbb{Z}/2^{i-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

For a proof of the last two statements see, for example, [5, Chapter 4, page 43].

Theorem 6.1.2.

1. *If $f > 1$, then $(\mathcal{O}/(\pi^i))^\times$ is cyclic if and only if $i = 1$.*
2. *If $f = 1$, $e > 1$, and $p > 2$, then $(\mathcal{O}/(\pi^i))^\times$ is cyclic if and only if $i = 1$ or $i = 2$.*
3. *If $f = 1$, $e > 1$, and $p = 2$, then $(\mathcal{O}/(\pi^i))^\times$ is cyclic if and only if $i = 1$, $i = 2$, or $i = 3$.*

The proof of this will be broken down into many subcases below.

Lemma 6.1.3. *$\mathcal{O}$ contains at least one element whose image in $(\mathcal{O}/(\pi^2))^\times$ has order $(q-1)p$.*

Proof. It is well-known that the group $\mathbb{F}_q^\times$ is cyclic; let h be an element of $\mathcal{O}$ such that its image in $\mathcal{O}/(\pi)$ is a generator of $\mathbb{F}_q^\times$.

Thus, we have that $h^{q-1} \equiv 1 \pmod{\pi}$, and so $h^{q-1} = 1 + a\pi$, where a is some element in $\mathcal{O}$. If a is a unit of $\mathcal{O}$, then h has order greater than $q-1$, i.e., at least order $(q-1)p$.

If a is not a unit in $\mathcal{O}$, then $a = b\pi$, and so $h^{q-1} = 1 + b\pi^2 \equiv 1 \pmod{\pi^2}$ and thus h has order exactly $q - 1$. However,

$$(h+\pi)^{q-1} \equiv h^{q-1} + (q-1)h^{q-2}\pi \equiv 1 - h^{q-2}\pi \not\equiv 1 \pmod{\pi^2},$$

and so $h+\pi$ has order at least $(q-1)p$. Thus $(\mathcal{O}/(\pi^2))^\times$ contains at least one element of order $(q-1)p$. $\square$

Corollary 6.1.4. *If $f = 1$ and $i \in \{1, 2\}$, then $(\mathcal{O}/(\pi^i))^\times$ is cyclic.*

Proof. Let c denote an element as in Lemma 6.1.3. In the case $f = 1$, the order of $(\mathcal{O}/(\pi^2))^\times$ is $(p-1)p = (q-1)p$, and so the image of c is a generator, and thus $(\mathcal{O}/(\pi^2))^\times$ is cyclic. $\square$

Theorem 6.1.5. *If $f > 1$, then $(\mathcal{O}/(\pi^2))^\times$ is not cyclic.*

Proof. Suppose that $g \in \mathcal{O}$ is such that its image is a generator of $(\mathcal{O}/(\pi^2))^\times$. Then $g^{q-1} = 1 + a\pi$, where a is a unit of $\mathcal{O}$.

$$g^{(q-1)p} = (1+a\pi)^p = 1 + pa\pi + \binom{p}{2}a^2\pi^2 + \ldots + a^p\pi^p \equiv 1 \pmod{\pi^i}, \qquad (6.1)$$

where $i \leq \min\{1+e, p\}$. Since $\min\{1+e, p\} \geq 2$ and $f > 1$, the order of the image of g in $(\mathcal{O}/(\pi^2))^\times$ is

$$(q-1)p < (q-1)q = |(\mathcal{O}/(\pi^2))^\times|,$$

a contradiction, and so $(\mathcal{O}/(\pi^2))^\times$ is not cyclic for $f > 1$. $\square$

Corollary 6.1.6. *If $f > 1$ then $(\mathcal{O}/(\pi^i))^\times$ is cyclic if and only if $i = 1$.*

Proof. It follows immediately from Theorem 6.1.5 that $(\mathcal{O}/(\pi^i))^\times$ is not cyclic for $i > 1$ and $f > 1$. $\square$

Theorem 6.1.7. *If $f = 1$, $e > 1$, and $p > 2$ then $(\mathcal{O}/(\pi^3))^\times$ is not cyclic.*

Proof. Suppose that $f = 1$ and that $g \in \mathcal{O}$ is such that its image is a generator of $(\mathcal{O}/(\pi^2))^\times$ as in Corollary 6.1.4. If $p > 2$ and $e \geq 2$, then $\min\{1+e, p\} \geq 3$, and so

$$g^{(p-1)p} \equiv 1 \pmod{\pi^3}$$

by Equation 6.1. However, the order of $(\mathcal{O}/(\pi^3))^\times$ is $(p-1)p^2$, and so the image of g in $(\mathcal{O}/(\pi^3))^\times$ cannot be a generator of $(\mathcal{O}/(\pi^3))^\times$. Since every element g whose image is a generator of $(\mathcal{O}/(\pi^3))^\times$ must also have an image that generates $(\mathcal{O}/(\pi^2))^\times$, it follows that no such g can exist. Thus $(\mathcal{O}/(\pi^3))^\times$ is not cyclic. $\square$

Theorem 6.1.8. *If $f = 1$, $e > 1$, and $p = 2$ then*

1. *$(\mathcal{O}/(\pi^3))^\times$ is cyclic.*
2. *$(\mathcal{O}/(\pi^4))^\times$ is not cyclic.*

Proof. Again, suppose that $g \in \mathcal{O}$ is such that its image is a generator of $(\mathcal{O}/(\pi^2))^\times$ as in Corollary 6.1.4. Then $g = 1 + a\pi$, $\pi \nmid a$. If $p = 2$, then we have

$$g^{(p-1)p} = (1+a\pi)^2 = 1 + 2a\pi + a^2\pi^2 \equiv 1 + a^2\pi^2 \not\equiv 1 \pmod{\pi^3}$$

since $(2) = (\pi^e)$, and so the image of g in $(\mathcal{O}/(\pi^3))^\times$ must have order $(p-1)p^2$. Thus $(\mathcal{O}/(\pi^3))^\times$ is cyclic.

We also have that

$$g^{(p-1)p^2} = (1+a\pi)^4 \equiv 1 + 4a\pi + 6a^2\pi^2 + 4a^3\pi^3 + a^4\pi^4 \equiv 1 \pmod{\pi^4},$$

and so by the same argument as in the proof of Theorem 6.1.7, $(\mathcal{O}/(\pi^4))^\times$ is not cyclic. $\square$

Chapter 7 Vinogradov Style Proof of Hensel's Lemma

7.1 Introduction

I begin this chapter with a relatively elementary proof of Hensel's Lemma for additive forms over $\mathbb{Q}_p$ which I took from Vinogradov [11], Chapter 2, Lemma 8. (I first found reference to Vinogradov's proof in [3].) Vinogradov's own presentation of this proof leaves out some details, which I attempt to fill in below.

In this form, Hensel's Lemma gives conditions when a solution to an additive form over $\mathbb{Q}_p$ modulo some power of p can be raised to a solution in $\mathbb{Q}_p$. This proof can be extended in a straightforward way to extensions of $\mathbb{Q}_p$ given some data on the extension.

However, there are some surprising details which arise in this extension of the proof. The first is that the proof is substantially different from that given in [8], which is the only proof known to me in the literature. Second, the power of the uniformizer π depends on different information about the extension than that used in [8]. Finally, and most surprisingly, in some cases the power of π needed to raise a solution is lower than that obtained in [8].

7.2 Additive Forms over $\mathbb{Q}_p$

Theorem 7.2.1. *Let $p > 2$, $d = mp^\tau$ with $p \nmid m$, and $\gamma = \tau + 1$. Suppose $a_1x_1^d + \ldots + a_sx_s^d \equiv a \pmod{p^\gamma}$ with $a, a_i, x_i \in \mathbb{Z}_p$, $a_1, x_1 \in \mathbb{Z}_p^\times$. Then there exists $y_1 \in \mathbb{Z}_p^\times$ such that $a_1y_1^d + a_2x_2^d + \ldots + a_sx_s^d \equiv a \pmod{p^\delta}$ for any $\delta \geq \gamma$.*

Proof. Let $x = x_1$, $b = a - a_2x_2^d - \ldots - a_sx_s^d$. Thus, it suffices to show that $x^d \equiv b \pmod{p^\gamma}$ implies that there exists y such that $y^d \equiv b \pmod{p^\delta}$ for $\delta \geq \gamma$ for any $b \in \mathbb{Z}_p^\times$. Let g be a generator for the cyclic group $(\mathbb{Z}/p^\delta\mathbb{Z})^\times$. Then we have $x^dg^k \equiv b \pmod{p^\delta}$ for some exponent k. The proof will be complete if we can show that $d \mid k$.

First, note that $g^k \equiv 1 \pmod{p^\gamma}$. Because g is a generator of $(\mathbb{Z}/p^\gamma\mathbb{Z})^\times$ (here considering g to be its equivalence class when reducing modulo p^γ), it follows that $(p-1)p^{(\gamma-1)} \mid k$, or more precisely, $p^\tau \mid k$ and $(p-1) \mid k$. We now need only show that $m \mid k$.

Next, note that because the order of $(\mathbb{Z}/p^\delta\mathbb{Z})^\times$ is $(p-1)p^{(\delta-1)}$, we can replace k with $k' = k + n(p-1)p^{(\delta-1)}$ for any integer n and all of the above statements describing k are equally true of k'. Factoring out $(p-1)$ from both terms, we get $k' = (p-1)(i + np^j)$ for some i and j. Using $p \nmid m$, solving for n in $i + np^j \equiv 0 \pmod{m}$ gives n such that $d \mid k'$. □

Notice that while showing that $m \mid k$ involved no information about the groups in question, showing that $p^\tau \mid k$ required knowing the orders of the generators of $(\mathbb{Z}/p^i\mathbb{Z})^\times$ for any i. Specifically, we needed to know a value of i for which the orders are divisible by p^τ. This is easy to obtain because for $p > 2$, $(\mathbb{Z}/p^i\mathbb{Z})^\times$ is cyclic.

For $p = 2$, the proof is a bit more subtle. Although $(\mathbb{Z}/2\mathbb{Z})^\times$ and $(\mathbb{Z}/4\mathbb{Z})^\times$ are cyclic, modulo higher powers of 2 the groups of units are not generated by a single element. However, this is not difficult to overcome.

Theorem 7.2.2. *Let $d = m2^\tau$ with $2 \nmid m$, and*

$$\gamma = \begin{cases} \tau + 2 & \text{if } \tau \geq 1 \\ \tau + 1 & \text{if } \tau = 0. \end{cases}$$

Suppose $a_1x_1^d + \ldots + a_sx_s^d \equiv a \pmod{2^\gamma}$ with $a, a_i, x_i \in \mathbb{Z}_2$, $a_1, x_1 \in \mathbb{Z}_2^\times$. Then there exists $y_1 \in \mathbb{Z}_2^\times$ such that $a_1y_1^d + a_2x_2^d + \ldots + a_sx_s^d \equiv a \pmod{2^\delta}$ for any $\delta \geq \gamma$.

Proof. The case $\tau = 0$ is straightforward, as every element is a d^{th} power. Thus assume $\tau \geq 1$.

As in Theorem 7.2.1, it suffices to show that $x^d \equiv b \pmod{2^\gamma}$ implies that there exists y such that $y^d \equiv b \pmod{2^\delta}$ for $\delta \geq \gamma$ for any $b \in \mathbb{Z}_2^\times$. $(\mathbb{Z}/2^\delta\mathbb{Z})^\times$ is generated by (the congruence classes of) -1 and 5, so let $x^d(-1)^j5^k \equiv b \pmod{2^\delta}$, and so we have $(-1)^j5^k \equiv 1 \pmod{2^\gamma}$. Because $5^k \equiv 1 \pmod 4$, it follows that we may take $j = 0$. If $\tau \geq 1$, the order of 5 in $(\mathbb{Z}/2^\gamma\mathbb{Z})^\times$ is $2^{\gamma-2} = 2^\tau$. Thus in all cases we have $2^\tau \mid j, k$. The argument for $m \mid j, k$ proceeds exactly as in Theorem 7.2.1. $\square$

7.3 Additive Forms over Extensions of $\mathbb{Q}_p$

Now we move on to finite extensions of $\mathbb{Q}_p$. Let K be a finite extension of $\mathbb{Q}_p$ of degree $n = ef$, where f is the inertial degree and e is the ramification index. $\mathcal{O}$ is the valuation ring of K (also known as the ring of integers), π is a uniformizer of $\mathcal{O}$ (a generator of its maximal ideal), and $(\pi^e) = (p)$. $\mathcal{O}^\times$, the group of units of $\mathcal{O}$, is precisely the elements of $\mathcal{O}$ that are not divisible by π.

The following is proven in [8].

Theorem 7.3.1. *Let K be a finite extension of $\mathbb{Q}_p$ with degree of ramification e, π be a uniformizer for K, and $d = mp^\tau$ and $\gamma = \left\lfloor \frac{e}{p-1} \right\rfloor + e\tau + 1$. A solution modulo π^γ lifts to a solution in K.*

The proof given in [8] involves a lot of quite complicated algebraic manipulations that don't give me much insight into the result. However, if one can compute generators and their orders for the group of units modulo a power of π, then one can proceed with the "Vinogradov-style" arguments as in Theorems 7.2.1 and 7.2.2, finding a power γ of π so that the orders of the generators in $(\mathcal{O}/(\pi^\gamma))^\times$ are divisible by p^τ.

I have directly computed generators and their orders for the six totally ramified quadratic extensions of $\mathbb{Q}_2$. For five of the extensions, they give the Hensel's lemma exponent γ which agrees with Lemma 7.3.1. However, this is not the case for $\mathbb{Q}_2(\sqrt{-5})$. The following is given without proof. It is based on direct computation, and may be proven in a way analogous to the proofs above.

Theorem 7.3.2. *Let* $K = \mathbb{Q}_2(\sqrt{-5})$, π *a uniformizer of* K, $d = m2^\tau$ *with* $2 \nmid m$, *and*

$$\gamma = \begin{cases} 1 & \text{if } \tau = 0 \\ 5 & \text{if } \tau = 1 \\ 2\tau + 2 & \text{if } \tau \geq 2. \end{cases}$$

Suppose $a_1x_1^d + \ldots + a_sx_s^d \equiv a \pmod{\pi^\gamma}$ *with* $a_i, a \in \mathcal{O}$ *and* $a_1, x_1 \in \mathcal{O}^\times$. *Then there exists* $y_1 \in \mathcal{O}^\times$ *such that* $a_1y_1^d + a_2x_2^d + \ldots + a_sx_s^d \equiv a \pmod{\pi^\delta}$ *for any* $\delta \geq \gamma$.

Appendix: Anisotropic Forms in Large Numbers of Variables

In this appendix, I give a number of anistropic forms of various degrees for the six ramified quadratic extensions of $\mathbb{Q}_2$ considered in this dissertation. They were obtained via direct computation, and are given without proof.

d=4m

d=4

$\mathbb{Q}_2(\sqrt{-1})$:

$$x_0^4+x_1^4+x_2^4+x_3^4+(\pi+\pi^3)(x_4^4+x_5^4+x_6^4+x_7^4)+(\pi^2+\pi^3)x_8^4+(\pi^3+\pi^4+\pi^5)x_9^4$$

$\mathbb{Q}_2(\sqrt{-5})$:

$$x_0^4+x_1^4+x_2^4+x_3^4+\pi(x_4^4+x_5^4+x_6^4+x_7^4)$$

$\mathbb{Q}_2(\sqrt{2})$ and $\mathbb{Q}_2(\sqrt{10})$:

$$x_0^4+x_1^4+x_2^4+x_3^4+\pi(x_4^4+x_5^4+x_6^4+x_7^4+x_8^4+x_9^4)$$

$\mathbb{Q}_2(\sqrt{-2})$ and $\mathbb{Q}_2(\sqrt{-10})$:

$$x_0^4+x_1^4+x_2^4+x_3^4+(\pi+\pi^2)(x_4^4+x_5^4+x_6^4+x_7^4+x_8^4+x_9^4)$$

d=12

$\mathbb{Q}_2(\sqrt{-1})$:

$$(x_0^{12}+\ldots+x_{14}^{12})+\pi^2x_{15}^{12}+\pi^5(x_{16}^{12}+\ldots+x_{30}^{12})+(\pi^8+\pi^9)(x_{31}^{12}+\ldots+x_{37}^{12})$$

$$(x_0^{12}+\ldots+x_{14}^{12})+\pi^2x_{15}^{12}+\pi^5(x_{16}^{12}+\ldots+x_{22}^{12})+\pi^7(x_{23}^{12}+\ldots+x_{37}^{12})$$

$\mathbb{Q}_2(\sqrt{-5})$:

$$(x_0^{12}+\ldots+x_6^{12})+\pi(x_7^{12}+\ldots+x_{13}^{12})+\pi^6(x_{14}^{12}+\ldots+x_{20}^{12})+\pi^7(x_{21}^{12}+\ldots+x_{27}^{12})$$

$\mathbb{Q}_2(\sqrt{2})$:

$$x_0^{12}+\ldots+x_{14}^{12}+\pi(x_{15}^{12}+x_{16}^{12}+x_{17}^{12})+\pi^5(x_{18}^{12}+\ldots+x_{32}^{12}) +\pi^8(x_{33}^{12}+x_{34}^{12}+x_{35}^{12})$$

$\mathbb{Q}_2(\sqrt{10})$:

$$(x_0^{12}+\ldots+x_{14}^{12})+\pi(x_{15}^{12}+x_{16}^{12}+x_{17}^{12})+\pi^5(x_{18}^{12}+\ldots+x_{32}^{12})+\pi^8(x_{33}^{12}+x_{34}^{12}+x_{35}^{12})$$

$\mathbb{Q}_2(\sqrt{-2})$:

$$(x_0^{12}+\ldots+x_{14}^{12})+\pi(x_{15}^{12}+x_{16}^{12}+x_{17}^{12})+(\pi^5+\pi^6)(x_{18}^{12}+\ldots+x_{32}^{12})+\pi^8(x_{33}^{12}+x_{34}^{12}+x_{35}^{12})$$

d=20

$\mathbb{Q}_2(\sqrt{-1})$:

$$\begin{aligned}(x_0^{20}+\ldots+x_{14}^{20})+\pi^3(x_{15}^{20}+\ldots+x_{29}^{20})+\pi^8(x_{30}^{20}+\ldots+x_{44}^{20})\\ +\pi^{10}x_{45}^{20}+\pi^{13}(x_{46}^{20}+\ldots+x_{60}^{20})+(\pi^{16}+\pi^{17})(x_{61}^{20}+\ldots+x_{67}^{20})\end{aligned}$$

$\mathbb{Q}_2(\sqrt{2})$:

$$\begin{aligned}x_0^{20}+\ldots+x_{14}^{20}+\pi(x_{15}^{20}+x_{16}^{20}+x_{17}^{20})+\pi^5(x_{18}^{20}+\ldots+x_{32}^{20})\\ +\pi^8(x_{33}^{20}+\ldots+x_{47}^{20})+\pi^{13}(x_{48}^{20}+\ldots+x_{62}^{20})\\ +\pi^{16}(x_{63}^{20}+x_{64}^{20}+x_{65}^{20})\end{aligned}$$

d=28

$\mathbb{Q}_2(\sqrt{2})$:

$$\begin{aligned}x_0^{28}+\ldots+x_{14}^{28}+\pi(x_{15}^{28}+x_{16}^{28}+x_{17}^{28})+\pi^5(x_{18}^{28}+\ldots+x_{32}^{28})\\ +\pi^8(x_{33}^{28}+\ldots+x_{47}^{28})+\pi^{13}(x_{48}^{28}+\ldots+x_{62}^{28})\\ +\pi^{16}(x_{63}^{28}+\ldots+x_{77}^{28})+\pi^{21}(x_{78}^{28}+\ldots+x_{92}^{28})\\ +\pi^{24}(x_{93}^{28}+x_{94}^{28}+x_{95}^{28})\end{aligned}$$

d=36

$\mathbb{Q}_2(\sqrt{2})$:

$$\begin{aligned}x_0^{36}+\ldots+x_{14}^{36}+\pi(x_{15}^{36}+x_{16}^{36}+x_{17}^{36})+\pi^5(x_{18}^{36}+\ldots+x_{32}^{36})\\ +\pi^8(x_{33}^{36}+\ldots+x_{47}^{36})+\pi^{13}(x_{48}^{36}+\ldots+x_{62}^{36})\\ +\pi^{16}(x_{63}^{36}+\ldots+x_{77}^{36})+\pi^{21}(x_{78}^{36}+\ldots+x_{92}^{36})\\ +\pi^{24}(x_{93}^{36}+\ldots+x_{107}^{36})+\pi^{29}(x_{108}^{36}+\ldots+x_{122}^{36})\\ +\pi^{32}(x_{123}^{36}+x_{124}^{36}+x_{125}^{36})\end{aligned}$$

d=8m

d=8

$\mathbb{Q}_2(\sqrt{-1})$:

$$x_0^8 + \ldots + x_{15}^8 + \pi(x_{16}^8 + \ldots + x_{31}^8)$$

$\mathbb{Q}_2(\sqrt{-5})$, $\mathbb{Q}_2(\sqrt{-2})$, and $\mathbb{Q}_2(\sqrt{-10})$:

$$x_0^8 + \ldots + x_{14}^8 + \pi^3(x_{16}^8 + \ldots + x_{29}^8)$$

$\mathbb{Q}_2(\sqrt{2})$ and $\mathbb{Q}_2(\sqrt{10})$:

$$x_0^8 + \ldots + x_{15}^8 + \pi^3(x_{16}^8 + \ldots + x_{31}^8)$$

d=24

$\mathbb{Q}_2(\sqrt{-1})$:

$$(x_0^{24} + \ldots + x_{30}^{24}) + \pi^3(x_{31}^{24} + \ldots + x_{61}^{24}) + \pi^{10}(x_{62}^{24} + \ldots + x_{68}^{24}) \\ + \pi^{12}(x_{69}^{24} + \ldots + x_{99}^{24}) + \pi^{17}(x_{100}^{24} + \ldots + x_{130}^{24}) + \pi^{22}x_{131}^{24}$$

$\mathbb{Q}_2(\sqrt{2})$:

$$x_0^{24} + \ldots + x_{30}^{24} + \pi(x_{31}^{24} + x_{32}^{24} + x_{33}^{24}) \\ + \pi^5(x_{34}^{24} + \ldots + x_{64}^{24}) + \pi^{10}(x_{65}^{24} + \ldots + x_{95}^{24}) + \pi^{15}(x_{96}^{24} + \ldots + x_{126}^{24}) \\ + \pi^{20}(x_{127}^{24} + x_{128}^{24} + x_{129}^{12})$$

d=16

$\mathbb{Q}_2(\sqrt{-1})$:

$$x_0^{16} + \ldots + x_{62}^{16} + \pi(x_{63}^{16} + \ldots + x_{125}^{16}) + \pi^{12}(x_{126}^{16} + x_{127}^{16} + x_{128}^{16}) + \pi^{13}(x_{129}^{16} + x_{130}^{16} + x_{131}^{16})$$

$$x_0^{16} + \ldots + x_{62}^{16} + \pi^5(x_{63}^{16} + \ldots + x_{125}^{16}) + \pi^{11}x_{126}^{16} + \pi^{14}(x_{127}^{16} + \ldots + x_{133}^{16})$$

$$(x_0^{16} + \ldots + x_{62}^{16}) + \pi^5(x_{63}^{16} + \ldots + x_{125}^{16}) + (\pi^{12} + \pi^{13})(x_{126}^{16} + \ldots + x_{140}^{16})$$

$\mathbb{Q}_2(\sqrt{2})$:

$$(x_0^{16} + \ldots + x_{62}^{16}) + \pi(x_{63}^{16} + x_{64}^{16} + x_{65}^{16}) + \pi^5(x_{66}^{16} + \ldots + x_{128}^{16}) + \pi^{12}(x_{129}^{16} + x_{130}^{16} + x_{131}^{16})$$

d=32

$\mathbb{Q}_2(\sqrt{-1})$:

$$x_0^{32}+\ldots+x_{126}^{32}+\pi(x_{127}^{32}+\ldots+x_{253}^{32})+\pi^{14}(x_{254}^{32}+\ldots+x_{380}^{32})+\pi^{15}(x_{381}^{32}+\ldots+x_{507}^{32})$$
$$+\pi^{28}(x_{508}^{32}+x_{509}^{32}+x_{510}^{32})+\pi^{29}(x_{511}^{32}+x_{512}^{32}+x_{513}^{32})$$

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