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SIMULTANEOUS ZEROS OF A SYSTEM OF TWO QUADRATIC FORMS

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Nandita Sahajpal, Student

Dr. David B. Leep, Major Professor

Dr. Peter Hislop, Director of Graduate Studies

SIMULTANEOUS ZEROS OF A SYSTEM OF TWO QUADRATIC FORMS

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Nandita Sahajpal
Lexington, Kentucky

Director: Dr. David Leep, Professor of Mathematics
Lexington, Kentucky

2020

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ABSTRACT OF DISSERTATION

SIMULTANEOUS ZEROS OF A SYSTEM OF TWO QUADRATIC FORMS

In this dissertation we investigate the existence of a nontrivial solution to a system of two quadratic forms over local fields and global fields. We specifically study a system of two quadratic forms over an arbitrary number field $\mathbb{K}$. The questions that are of particular interest are:

1. How many variables are necessary to guarantee a nontrivial zero to a system of two quadratic forms over a global field or a local field? In other words, what is the u–invariant of a pair of quadratic forms over any global or local field?

2. What is the relation between u–invariants of a pair of quadratic forms over any global field and the local fields associated with it?

3. How is the u–invariant of a pair of quadratic forms over any global field related to the u–invariant of its residue field?

There are many known results that address 1, 2, and 3:

(A) In the context of $\mathfrak{p}$–adic fields, a classical result by Dem'yanov states that two homogeneous quadratic forms over a $\mathfrak{p}$–adic field have a common nontrivial $\mathfrak{p}$–adic zero, provided that the number of variables is at least 9. In 1962, Birch-Lewis-Murphy gave an alternative proof to this result by Dem'yanov.

(B) In a 1964 paper, Swinnerton-Dyer showed that a system of two quadratic forms over the field of rational numbers in 11 variables, satisfying certain number-theoretic conditions, has a nontrivial rational zero.

(C) An even more remarkable result proven by Colliot-Thélène, Sansuc, and Swinnerton-Dyer extends Dem'yanov's result to an imaginary number field and also to an arbitrary number field if certain number-theoretic conditions are satisfied.

Our work in this dissertation is motivated by the work on the results stated above.

- With respect to (A), we generalize the result as well as the proof techniques to prove an analogous result over a complete discretely valued field with characteristic not 2.
- With respect to (B), we demonstrate that this result, and the techniques used in the proof can be extended to a system of two quadratic forms in at least 11 variables over an arbitrary number field.
- With respect to (C), we give a more comprehensible and self-contained proof of this result over an arbitrary number field using primarily number-theoretic arguments.

KEYWORDS: Quadratic Form, Local Field, Global Field, Simultaneous Zeros, u–invariant

Nandita Sahajpal

August 5, 2020

SIMULTANEOUS ZEROS OF A SYSTEM OF TWO QUADRATIC FORMS

By
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Dr. David Leep

Director of Dissertation

Dr. Peter Hislop

Director of Graduate Studies

August 5, 2020

Date

DEDICATION

Dedicated to my mother, Mrs. Seema Tyagi, my father, Dr. Dinesh Sahajpal, and my beloved cat, Hilbert (who left the world too soon).

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CHAPTER 1. PRELIMINARIES AND INTRODUCTION

1.1 Partial List of Notations

Below is a list of symbols that have a constant meaning throughout the dissertation or in a substantial portion of it.

Symbol	Meaning
$\mathbb{Z}$	The ring of integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathbb{Q}_p$	The field of $p-$adic numbers
$\mathbb{F}$	An arbitrary field
$\mathbb{F}^\times$	The multiplicative group of $\mathbb{F}$
$\mathbb{F}_q$	The finite field with q elements
$\mathbb{H}$	The hyperbolic plane
$\mathbb{K}$	An algebraic number field
$\overline{\mathbb{K}}$	Algebraic closure of $\mathbb{K}$
$\mathfrak{p}$	A place associated with a number field $\mathbb{K}$
Ω	The set of all places associated with a number field $\mathbb{K}$
$\mathbb{K}_\mathfrak{p}$	$\mathfrak{p}-$adic completion of a number field $\mathbb{K}$ with respect to $\mathfrak{p}$
$\mathbb{K}_{\mathfrak{p}_i}$	A real $\mathfrak{p}-$adic completion of a number field $\mathbb{K}$ with respect to $\mathfrak{p}$
$\theta_{\mathfrak{p}_i}$	$\theta_{\mathfrak{p}_i} : \mathbb{K} \to \mathbb{R}$ represents an embedding of $\mathbb{K}$ into $\mathbb{R}$

Symbol	Meaning
$u(\mathbb{F})$	The u–invariant of the field $\mathbb{F}$
$u_{\mathbb{F}}(r)$	The r-th system u–invariant of the field $\mathbb{F}$
$[n]$	For $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$
$\vec{e}_k^{\,t}$	$(0,\ldots,0,\underset{\substack{\uparrow\\k}}{1},0,\ldots,0)$, k–th standard basis vector
$\mathbb{F}^n$	An n–dimensional vector space over the field $\mathbb{F}$
$\dim W$	Dimension of the vector space W
f, $f(X_1,\ldots,X_n)$	An (n-ary) quadratic form
$B_f(\vec{X},\vec{Y})$	Bilinear form associated with the quadratic form f
$\det(f)$	Determinant of the quadratic form f
$\mathrm{rad}(f)$	Radical of the quadratic form f
$\mathrm{sgn}(f)$	Signature of the quadratic form f
$GL_n(A)$	$n \times n$ general linear group over a ring A

1.2 Preliminary Definitions and Concepts

In this section we discuss some of the very basic definitions, facts and terminology related to quadratic forms over an arbitrary field $\mathbb{F}$. [8] and [13] are the main sources for the definitions and facts that are provided in this section.

Definition 1.2.1. An (n–ary) quadratic form over a field $\mathbb{F}$ is a polynomial f in n variables over $\mathbb{F}$ that is homgeneous of degree 2. It has the general form

$$f(\vec{X}) = f(X_1,\ldots,X_n) = \sum_{i,j=1}^{n} a_{ij}X_iX_i \in \mathbb{F}[X_1,\ldots,X_n] = \mathbb{F}[X] \tag{1.1}$$

Let $\mathbb{F}$ be a field with characteristic not 2.

1. In the above form (1.1), $a_{ij} = a_{ji}$, so we can rewrite f as

$$f(\vec{X}) = \sum_{i,j=1}^{n} \frac{1}{2}(a_{ij} + a_{ji})X_i X_j = \sum_{i,j=1}^{n} a'_{ij} X_i X_j,$$

where $a'_{ij} = \dfrac{1}{2}(a_{ij} + a_{ji})$.

2. Using the above form of f, we get a symmetric matrix (a'_{ij}) determined uniquely by the coefficients of f, which we shall denote by M_f. In matrix notation we have,

$$f(\vec{X}) = \vec{X}^t \cdot M_f \cdot \vec{X}, \quad \text{(t = tanspose)},$$

where $\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$

3. For vectors $\vec{X}, \vec{Y}$ in $\mathbb{F}^n$,

$$B_f(\vec{X}, \vec{Y}) = \frac{1}{2}\Big(f(\vec{X} + \vec{Y}) - f(\vec{X}) - f(\vec{Y})\Big) \tag{1.2}$$

is a symmetric bilinear form associated with f. Note that

$$B_f(\vec{X}, \vec{X}) = f(\vec{X}), \quad \text{for any } X \in \mathbb{F}^n \tag{1.3}$$

4. Let f and g be n–ary quadratic forms over $\mathbb{F}$. We say that f is *equivalent* to g ($f \sim g$) if there exists an invertible matrix $C \in GL_n(\mathbb{F})$ such that

$$f(X) = g(C \cdot X).$$

This means that there exists a nonsingular, homogeneous linear change of variables $X_1, \ldots, X_n$ that takes g to the form f.

Definition 1.2.2 (Isotropic and Anisotropic Quadratic Forms)**.** An (n–ary) quadratic form f over a field $\mathbb{F}$ is said to be *isotropic* if there exists a nonzero vector $\vec{X} \in \mathbb{F}^n$

such that $f(\vec{X}) = 0$. The nontrivial vector $\vec{X}$ is called an *isotropic vector* of f. If f does not have a *isotropic vector* over $\mathbb{F}$, then it is said to be *anisotropic*.

Definition 1.2.3. Let f be a quadratic form over $\mathbb{F}$, and $d \in \mathbb{F}$. We say that f *represents* d *over* $\mathbb{F}$ if there exists a nontrivial vector $\vec{X} \in \mathbb{F}^n$ such that

$$f(\vec{X}) = d.$$

Definition 1.2.4 (Universal Quadratic Forms)**.** A quadratic form is called *universal* over a field $\mathbb{F}$ if it represents all the nonzero elements of $\mathbb{F}$.

Definition 1.2.5 (Nonsingular Zero, Singular Zero)**.**

1. An isotropic vector $\mathscr{X}$ of a quadratic form $f = f(X_1, \ldots, X_n)$ is said to be a *nonsingular zero of* f if

$$\frac{\partial f}{\partial X}(\mathscr{X}) = \left(\frac{\partial f}{\partial X_1}(\mathscr{X}), \ldots, \frac{\partial f}{\partial X_n}(\mathscr{X})\right)$$

   is not the zero vector, and is said to be a *singular zero* otherwise.

2. A common isotropic vector $\mathscr{X}$ of a pair of quadratic forms f, g is said to be a *nonsingular zero of* f *and* g if the vectors

$$\frac{\partial f}{\partial X}(\mathscr{X}), \frac{\partial g}{\partial X}(\mathscr{X})$$

   are linearly independent over $\mathbb{F}$, and is said to be a *singular common zero* otherwise.

Definition 1.2.6 (u–invariant of a Field)**.** The u–invariant of a field $\mathbb{F}$, denoted by $u(\mathbb{F})$, is defined to be the largest integer such that a quadratic form f over $\mathbb{F}$ in n variables is isotropic whenever $n > u(\mathbb{F})$. If no such integer exists, then $u(\mathbb{F}) = \infty$.

Example.

- $u(\mathbb{R}) = \infty$
- For a finite field $\mathbb{F}_q$, $u(\mathbb{F}_q) = 2$
- For a $\mathfrak{p}-$adic field $\mathbb{Q}_{\mathfrak{p}}$, $u(\mathbb{Q}_{\mathfrak{p}}) = 4$.

Definition 1.2.7 (System $u-$invariant)**.** For $r \geq 1$, the system $u-$invariant, denoted by $u_{\mathbb{F}}(r)$, is defined to the largest integer such that every system of r quadratic forms over $\mathbb{F}$ in n variables has a common nontrivial zero over $\mathbb{F}$, whenever $n > u_{\mathbb{F}}(r)$. Note that $u_{\mathbb{F}}(1) = u(\mathbb{F})$.

Definition 1.2.8 (Order of a Quadratic Form)**.**

1. If f is a quadratic form and $T : \mathbb{F}[X_1, \ldots, X_n] \to \mathbb{F}[X_1, \ldots, X_n]$ is a nonsingular linear transformation over $\mathbb{F}$, then $f_T(X) := f(TX)$
2. $\gamma(f)$ denotes the number of variables appearing explicitly in f.
3. Order of $f := o(f) = \min_T\{\gamma(f_T)\}$, where the minimum is taken over all nonsingular linear transformations T, defined over $\mathbb{F}$.
4. A form f is called degenerate if $o(f) < n$.

Definition 1.2.9 (Order of a Pair of Quadratic Forms)**.**

1. $o(f,g) := \min_T[\gamma(f_T) + \gamma(g_T)]$,where the minimum is taken over all nonsingular linear transformations T, defined over $\mathbb{F}$.
2. (f,g) is degenerate pair of quadratic forms if $os(f,g) < n$.

Definition 1.2.10 (Rank of a Quadratic Form)**.** Let f be a quadratic form in n variables over a field $\mathbb{F}$ of characteristic not 2. Let M_f be the symmetric matrix corresponding to f with entries in $\mathbb{F}$. The rank of f, denoted by $\text{rank}(f)$, is equal to the rank of the matrix M_f. We say that f is *nondegenerate* or *nonsingular* if $\operatorname{rank}(f) = n$, otherwise we say that f is *degenerate* or *singular*.

Definition 1.2.11 (Hyperbolic Quadratic Form)**.** A binary form f in two variables X_1 and X_2 over a field $\mathbb{F}$ is called *hyperbolic* if after a nonsingular linear change of variables

$$f \sim X_1X_2 \sim X_1^2 - X_2^2.$$

If f is a hyperbolic form as defined above, that is, $f \sim X_1X_2$, then

$$\mathbb{H} = \{\vec{v} \in \mathbb{F}^2 : f(\vec{v}) = \text{nonzero constant}\}$$

is a *hyperbolic plane* corresponding to f.

We state an important lemma regarding quadratic forms and hyperbolic planes that is used implicitly in Chapters 4 and 5.

Lemma 1.2.12. *([13, Propostion 3']) Let f be a quadratic form over $\mathbb{F}$. If f represents 0 and is nondegenerate, one has that $f \sim f_2 + g$ where f_2 is hyperbolic. Moreover, f represents all elements of $\mathbb{F}$.*

Lemma 1.2.13. *[13, Corollary, page 34] If f is a nonsingular quadratic form over a field $\mathbb{F}$, then*

$$f \sim f_1 + \cdots + f_m + f_a,$$

where $f_1, \ldots, f_m$ are hyperbolic quadratic forms over $\mathbb{F}$, and f_a is an anisotropic quadratic form over $\mathbb{F}$. This decomposition is unique up to equivalence.

Definition 1.2.14 (Kernel of a Quadratic Form)**.** Let f be a nonsingular quadratic form over $\mathbb{F}$. By Lemma 1.2.13,

$$f \sim f_1 + \cdots + f_m + f_a,$$

where $f_1, \ldots, f_m$ are hyperbolic quadratic forms over $\mathbb{F}$, and f_a is an anisotropic quadratic form over $\mathbb{F}$. f_a is called the anisotropic part or the *kernel of f* over $\mathbb{F}$. The *kernel of f* is unique up to equivalence.

Definition 1.2.15 (Absolute Values on a Field)**.** Let $\mathbb{F}$ be a field and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. An *absolute value on* $\mathbb{F}$ is a function

$$| \quad | : \mathbb{F} \to \mathbb{R}_+$$

that satisfies the following properties

1. $|x| = 0$ if and only if $x = 0$;
2. $|xy| = |x||y|$ for all $x, y \in \mathbb{F}$;
3. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{F}$.

We say that an absolute value on $\mathbb{F}$ is nonarchimedean if it satisfies the additional condition

4. $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{F}$;

otherwise, we say that the absolute value if *archimedean.*

5. Two absolute values on a field $\mathbb{F}$ are equivalent whenever they induce the same metric topology on $\mathbb{F}$.

Definition 1.2.16 (Places of a field)**.** The *places of a field* $\mathbb{F}$ are defined to be the absolute values on $\mathbb{F}$ up to equivalence.

- A *finite place on* $\mathbb{F}$ is a place corresponding to an equivalence class of nonarchimedean absolute values on $\mathbb{F}$.
- A *real place on* $\mathbb{F}$ is a place corresponding to an equivalence class of archimedean absolute values on $\mathbb{F}$ such that the completion of $\mathbb{F}$ with respect to the met-

ric induced by the archimedean absolute values in that equivalence class is isomorphic to $\mathbb{R}$.

Definition 1.2.17 (Global Fields)**.** A *global field* is any field $\mathbb{K}$ that is, either a finite extension of $\mathbb{Q}$ (called a number field), or of $\mathbb{F}_q(t)$ (called a function field in one variable over a finite field $\mathbb{F}_q$).

Definition 1.2.18 (Local Fields)**.** A completion of a global $\mathbb{K}$ under any nonarchimedean absolute value is called a *local field.*

1.3 Introductory Note

A (quadratic) form over a field $\mathbb{F}$ is a homogeneous polynomial of degree 2 with coefficients in the field $\mathbb{F}$.

In this dissertation we study a system of two quadratic forms over a number field. Before studying the case of two quadratic forms, it is worth recalling what is known in the case of a single quadratic form.

Local-Global Principle: The Local-Global Principle, also known as the Hasse Principle, is an idea that the existence or non existence of solutions in $\mathbb{Q}$ (global solutions) of a diophantine equation can be detected by studying the solutions of the equation over $\mathbb{R}$ as well as in $\mathbb{Q}_p$ (local solutions modulo all powers of p) for each prime p.

Given a diophantine equation, if it has a nontrivial solution in $\mathbb{Q}$, then this also yields a nontrivial solution in $\mathbb{R}$ and as well as in $\mathbb{Q}_p$ for each prime p. However, the Hasse Principle asks when is the converse true, that is, when can you patch the solutions over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p to yield a solution over $\mathbb{Q}$, or rather, can we always detect the lack of a global solution by studying the solutions locally. This question is not limited to $\mathbb{Q}$ and has been extended to other rings and fields.

For instance, when dealing with a single quadratic form over a number field, the following result is the central pillar of the global theory of quadratic forms.

Theorem (Hasse-Minkowski Theorem)**.** *If q is a quadratic form over a global field $\mathbb{K}$, then q has a nontrivial solution over $\mathbb{K}$ if and only if q has a nontrivial solution in each completion $\mathbb{K}_{\mathfrak{p}}$ of $\mathbb{K}$ for all $\mathfrak{p} \in \Omega$.*

Consequently, for a single quadratic form over a number field $\mathbb{K}$, the problem of finding a $\mathbb{K}-$rational solution is completely solved.

Now we move on to a system of two quadratic forms over a number field. In [4, Theorem 10.1], Colliot-Thélène, Sansuc, and Swinnerton-Dyer prove that the Hasse Principle can be successfully applied to a system of two quadratic forms in at least 9 variables over a number field. Although correct, the proof of the result in [4, Theorem 10.1] requires prior knowledge of several key results that are often very geometric and/or analytic in nature. Therefore, the main aim of this dissertation is to clarify as well as provide a detailed number-theoretic proof of [4, Theorem 10.1] that avoids using prior analytic and/or geometric results.

Chapter 2 of this dissertation is devoted to providing the reader with the necessary preliminary results and techniques that are used extensively throughout the dissertation and are vital to understanding the proof of the main theorems in the chapters that follow.

In **Chapter 3** we study a system of two quadratic forms over a *c.d.v. field* of characteristic not 2. We show that the proof of [1, Theorem 1] naturally extends to an analogous result over any *c.d.v field* of characteristic different from 2, which in turn gives us a nice relationship between the $u-$invariant of a *c.d.v field* $\mathbb{F}$ and its finite residue field $\overline{\mathbb{F}}$.

Theorem 3.1.3. *Over a complete discretely valued (c.d.v.) field $\mathbb{F}$ with characteristic* *not* 2*, and* $u_{\overline{\mathbb{F}}}(1) < \infty$*,*

$$u_{\mathbb{F}}(2) = 2u_{\overline{\mathbb{F}}}(2)$$

In **Chapter 4,** we present our work that is motivated by the work in [14] for a system two quadratic forms in $n = 11$ variables over $\mathbb{Q}$. We demonstrate that the result as well as the proof technique in [14] can be generalized to a system two quadratic forms in $n \geq 11$ variables over an arbitrary number field. In our proof of the main theorem 4.1.3 we not only provide rigorous, algebraic justification for the arguments used in [14], but also provide self-contained arguments that are necessary to generalize them to an arbitrary number field. We state the main result of Chapter 4 below:

Theorem 4.1.3. *Let* $\mathbb{K}$ *be a number field with s distinct real places denoted by* $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$*. Let* f, g *be quadratic forms in at least 11 variables, defined over* $\mathbb{K}$*; Suppose that every form in the* $\mathbb{K}$*–pencil has rank at least 5 and if* $s \geq 1$*, suppose that every nonzero quadratic form* $\lambda f + \mu g$ *in the* $\mathbb{K}_{\mathfrak{p}_i}$*–pencil is indefinite for all* $1 \leq i \leq s$*. Then* f, g *have infinitely many nontrivial common zeros over* $\mathbb{K}$*.*

Our final chapter, **Chapter 5,** is where we give a more comprehensible self-contained proof of [4, Theorem 10.1]. In particular, our proof avoids using several prior key results including [4, Theorem 9.2, Thoerem 9.4, Theorem 9.5]. We state the main result of Chapter 5 below:

Theorem 5.1.1. *Let* $\mathbb{K}$ *be a number field with s distinct real places denoted by* $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$*. Let* f, g *be quadratic forms in at least 9 variables, defined over* $\mathbb{K}$*; Suppose that every form in the* $\mathbb{K}$*–pencil has rank at least 5 and if* $s \geq 1$*, suppose that every nonzero quadratic form* $\lambda f + \mu g$ *in the* $\mathbb{K}_{\mathfrak{p}_i}$*–pencil is indefinite for all* $1 \leq i \leq s$*. Then* f, g *have infinitely many nontrivial common zeros over* $\mathbb{K}$*.*

2.1 Quadratic Forms over an Arbitrary Field.

In this section we have collected some preliminary results about quadratic forms over an arbitrary field $\mathbb{F}$ that are used extensively throughout this dissertation including some of the preliminary lemmas that originated out of the necessity to fill in the details in the arguments and statements from [1], [3], [4], and [14]. We give detailed self-contained proofs of all the results in this section using primarily number-theoretic techniques.

Lemma 2.1.1. *([3, Lemma 1.8]) Let f be a quadratic form in n variables over $\mathbb{F}$. Let $W \subset \mathbb{F}^n$ be a subspace. Let $\bar{f} = f\big|_W$ represent the quadratic form given by restriction of the quadratic map $f : \mathbb{F}^n \to \mathbb{F}$ to the subspace W. Then*

$$\operatorname{rank}(\bar{f}) \geq \operatorname{rank}(f) - 2(n - \dim W). \tag{2.1}$$

Proof. Let $\dim W = n - k$, $0 \leq k \leq n$. If $k = 0$, then $W = \mathbb{F}^n$ and equation (2.1) holds. Now suppose that $k = 1$. W.L.O.G., let $e_2, \ldots, e_n$ be a basis of W over $\mathbb{F}$ and let

$$f = aX_1^2 + X_1 L_1(X_2, \ldots, X_n) + q(X_2, \ldots, X_n).$$

Note that

$$\begin{aligned}
\operatorname{rank}(f) &= \operatorname{rank}(aX_1^2 + X_1 L_1(X_2, \ldots, X_n) + q(X_2, \ldots, X_n)) \\
&\leq \operatorname{rank}(aX_1^2 + X_1 L_1(X_2, \ldots, X_n)) + \operatorname{rank}(q(X_2, \ldots, X_n)) \\
&\leq 2 + \operatorname{rank}(q(X_2, \ldots, X_n) \\
&= 2[n - (n-1)] + \operatorname{rank}(\bar{f}) \\
&= 2(n - \dim W) + \operatorname{rank}(\bar{f}).
\end{aligned}$$

This implies that

$$\text{rank}(\bar{f}) \geq \text{rank}(f) - 2(n - \dim W),$$

when $k = 1$, that is, when the dimension of the space drops down by 1, the rank of the quadratic form drops down by at most 2. Hence if the $\dim W = n - k$, $k \geq 1$, then

$$\text{rank}(\bar{f}) \geq \text{rank}(f) - 2(k)$$

$$\text{rank}(\bar{f}) \geq \text{rank}(f) - 2(n - \dim W).$$

This completes the proof of the Lemma.

□

Definition 2.1.2 (Polar Hyperplane to a Quadratic Form at a vector in $\mathbb{F}^n$.)**.** The *Polar Hyperplane* to a quadratic form $f(X_1, \ldots, X_n)$ over a field $\mathbb{K}$ at a nontrivial vector $\vec{a} = (a_1, \ldots, a_n)^t$ is set of all zeros of the linear form

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\vec{a}) X_i. \tag{2.2}$$

We say that $\vec{v} = (v_1, \ldots, v_n)$ lies on the polar hyperplane to f at $\vec{a}$ if

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\vec{a}) v_i = 0$$

Notation. $\mathbb{H}_f^{\vec{a}}$ denotes the polar hyperplane to the quadratic form f at $\vec{a}$.

Definition 2.1.3 (Tangent Hyperplane to a Quadratic Form at a vector in $\mathbb{F}^n$.)**.** If $f(\vec{a}) = 0$, then the polar hyperplane corresponding to f at $\vec{a}$ is called the Tangent Hyperplane to the quadratic form f at the vector $\vec{a}$.

Notation. $\mathbb{T}_f^{\vec{a}}$ denotes the tangent hyperplane to the quadratic form f at $\vec{a}$ to the quadratic form f.

Definition 2.1.4 (Radical of a Bilinear Form)**.** Let $B(X,Y): \mathbb{F}^n \times F^n \to \mathbb{F}$ denote a bilinear form over $\mathbb{F}$. Then the *radical of B over* $\mathbb{F}$, is the subspace

$$\text{rad}_{\mathbb{F}}(B) = \{\vec{v} \in \mathbb{F}^n : B(\vec{v}, \mathbb{F}^n) = 0\}.$$

Definition 2.1.5 (Radical of a Quadratic Form)**.** Let f be a quadratic form in n variables over a field $\mathbb{F}$, and $B_f(\vec{u},\vec{v}) := f(\vec{u}+\vec{v}) - f(\vec{u}) - f(\vec{v})$ denote the bilinear form associated with f. Then the *radical of f* over $\mathbb{F}$ is the subspace

$$\text{rad}_{\mathbb{F}}(f) = \{\vec{v} \in \text{rad}_{\mathbb{F}}(B_f) : B(\vec{v}, \mathbb{F}^n) = 0\}.$$

Lemma 2.1.6. *([3, Lemma 1.16] Let f be a quadratic form in $n \geq 3$ variables over $\mathbb{F}$. Let $W \subset \mathbb{F}^n$ be an $(n-1)$-dimensional subspace. Let $\bar{f} = f\big|_W$ represent the quadratic form given by restriction of the quadratic map $f: \mathbb{F}^n \to \mathbb{F}$ to the subspace W. Assume that $r_f = \text{rank}(f) \geq 3$. Let Q be the quadric (curve or surface) defined by f in $\mathbb{F}^n$, and let $\mathbb{H}$ be the hyperplane defined by W. Let $r_{\bar{f}} = \text{rank}(\bar{f})$. Then*

1. *Assume that $r_f = n$. Then*
   a) *$r_{\bar{f}} = n-2$, if and only if $\mathbb{H}$ is tangent to Q.*
   b) *$r_{\bar{f}} = n-1$, if and only if $\mathbb{H}$ is not tangent to Q.*
2. *Assume $r_f < n$, and let $\text{rad}(f) := \text{rad}_{\mathbb{F}}(f)$. The dimension of $\text{rad}(f)$ is $n - r_f$.*
   a) *$r_{\bar{f}} = r_f$ if and only if $\mathbb{H}$ does not contain $\text{rad}(f)$.*
   b) *$r_{\bar{f}} = r_f - 2$ if and only if $\text{rad}(f) \subset \mathbb{H}$ and $\mathbb{H}$ is tangent to Q at a nonsingular point.*
   c) *$r_{\bar{f}} = r_f - 1$ if and only if $\text{rad}(f) \subset \mathbb{H}$ and $\mathbb{H}$ is not tangent to Q at a nonsingular point.*

Proof. (1a) Suppose that $\mathbb{H}$ is tangent to Q. This means that there exists a nonsingular point P on Q such that the tangent hyperplane to f at P is $\mathbb{H}: \mathbb{T}_f^P$.

So, after a nonsingular linear transformation over $\mathbb{F}$, we may assume that $P = (1, 0, \ldots, 0) \in \mathbb{F}^n$, and

$$f = X_1 L(X_2, \ldots, X_n) + q(X_2, \ldots, X_n),$$

where $L = \alpha_2 X_2 + \cdots + \alpha_n X_n$, with $\alpha_i \in \mathbb{F}$, and $\mathbb{T}_f^P = \ker(L)$ over $\mathbb{F}$. Since $r_f = n$, $L \not\equiv 0$. W.L.O.G., let $\alpha_2 \neq 0$. After another nonsingular linear transformation over $\mathbb{F}$, we get that $L = X_2 = 0$, and f can be rewritten as

$$f = X_1 X_2 + q'(X_2, \ldots, X_n).$$

Then

$$r_{\bar{f}} = \operatorname{rank}(f|_{X_2=0}) = \operatorname{rank}(q'(0, X_3 \ldots, X_n)) \leq n - 2.$$

By Lemma 2.1.1, we get that $r_{\bar{f}} = n - 2$.

Conversely, suppose that $r_{\bar{f}} = n - 2$. W.L.O.G., by a nonsingular linear change of variables over the field $\mathbb{F}$, we may assume that $\mathbb{H} : X_1$, and

$$f = \alpha_1 X_1^2 + X_1(\alpha_2 X_2 + \cdots + \alpha_n X_n) + q(X_2, \ldots, X_n).$$

Then

$$\bar{f} = f|_{X_1=0} = q(X_2, \ldots, X_n)$$

has rank $n - 2$.

After another nonsingular linear transformation over $\mathbb{F}$ that involves only $X_2, \ldots, X_n$, we can rewrite f as

$$f = \alpha_1 X_1^2 + X_1(\alpha_2' X_2 + \cdots + \alpha_n' X_n) + q'(X_3, \ldots, X_n),$$

where $\operatorname{rank}(q'(X_3, \ldots, X_n)) = n - 2$.

Since $r_f = n$, X_2 must appear in f, and hence $\alpha_2' \neq 0$. Therefore, $P = (0, 1, 0 \ldots, 0)$ is a nonsingular zero of f.

We will now show that $X_1 = 0$ is the tangent hyperplane to Q at P. Note that

$$\frac{\partial f}{\partial X_1} = 2\alpha_1 X_1 + \alpha_2' X_2 + \cdots, \alpha_n' X_n; \frac{\partial f}{\partial X_1}(P) = \alpha_2' \neq 0$$
$$\frac{\partial f}{\partial X_i}(P) = 0; 2 \leq i \leq n.$$

This implies that tangent hyperplane to Q at P is X_1.

(1b) By Lemma 2.1.1, $r_{\bar{f}} \in \{n, n-1, n-2\}$.

Since $r_f = n$, $r_{\bar{f}} \leq \dim W < n$.

By part (1a), we know that $r_{\bar{f}} = n-2$ if and only if $\mathbb{H}$ is tangent to Q.

Hence, $r_{\bar{f}} = n-1$ if and only if $\mathbb{H}$ is not tangent to Q.

(2a) Let $f = f(X_1, \ldots, X_{r_f})$, and $\mathbb{H} : a_1 X_1 + \cdots + a_n X_n = 0$, $a_i \in \mathbb{F}$, $i \in [n]$. Note that $\{e_{r_f+1}, \ldots, e_n\}$ is a basis for L over $\mathbb{F}$, and hence, $\mathbb{H}$ contains L if and only if $a_{r_f+1} = \cdots = a_n = 0$.

If $\mathbb{H}$ does not contain L, then a_i is nonzero for some i, $r+1 \leq i \leq n$.

W.L.O.G., let $a_n \neq 0$. We define a nonsingular linear change of variables over $\mathbb{F}$ as follows:

$$X_i \mapsto X_i; \qquad 1 \leq i \leq n-1$$
$$X_n \mapsto X_n - \frac{1}{a_n}(a_1 X_1 + \cdots + a_{n-1} X_{n-1}).$$

Under this change of variables f (and hence Q) stays fixed and $\mathbb{H} : a_n X_n$. Now note the

$$\bar{f} = f\big|_{a_n X_n = 0} = f,$$

and hence $r_{\bar{f}} = r_f$.

(2b) By a nonsingular linear change of variables,

$$f = a_1 X_1^2 + X_1(a_2 X_2 + \ldots + a_{r_f} X_{r_f}) + q(X_2, \ldots, X_{r_f}),$$

and

$$\mathbb{H} : c_1 X_1 + \cdots + c_n X_n.$$

This implies that $\mathrm{rad}(f) = \langle e_{r_f+1}, \ldots, e_n \rangle$. Note that $\{e_{r_f+1}, \ldots, e_n\}$ is a basis for $\mathrm{rad}(f)$ over $\mathbb{F}$, and hence $\mathbb{H}$ contains $\mathrm{rad}(f)$ if and only if $c_{r_f+1} = \cdots = c_n = 0$. Therefore, $\mathbb{H} : c_1 X_1 + \cdots + c_{r_f} X_{r_f}$, where at least one of the c_i's $\in \mathbb{F}$ is nonzero. W.L.O.G., let $c_1 \neq 0$. By a nonsingular linear change of variables over $\mathbb{F}$ involving only $X_1, \ldots, X_{r_f}$, we can rewrite

$$\mathbb{H} : X_1, \tag{2.3}$$

and

$$f = a'_1 X_1^2 + X_1(a'_2 X_2 + \ldots + a'_{r_f} X_{r_f}) + q'(X_2, \ldots, X_{r_f}), \tag{2.4}$$

Then,

$$\begin{aligned} r_{\bar{f}} &= \mathrm{rank}(f\big|_{X_1=0}) \\ &= \mathrm{rank}(q'(X_2, \ldots, X_{r_f})) \\ &\leq r_f - 1 < r_f \end{aligned}$$

By (2a), we can conclude that $\mathrm{rad}(f) \subset \mathbb{H}$ if and only if $r_{\bar{f}} < r_f$.

Now suppose that $r_{\bar{f}} = r_f - 2$. Using equation 2.4,

$$r_{\bar{f}} = \mathrm{rank}(f\big|_{X_1=0}) = \mathrm{rank}(q'(X_2, \ldots, X_{r_f})) = r_f - 2$$

So after another nonsingular linear transformation over $\mathbb{F}$, involving only $X_2, \ldots, X_{r_f}$, we may assume that

$$f = a_1' X_1^2 + X_1 L_1(X_2, \dots, X_{r_f}) + q''(X_3, \dots, X_{r_f}),$$

where $L_1 = a_2'' X_2 + \dots + a_{r_f}'' X_{r_f}$. Note that $L_1 \not\equiv 0$ as $\mathrm{rank}(f) = r_f$. Hence at least one of the a_i'' 's is nonzero. W.L.O.G., let $a_2'' \neq 0$. Then $e_2 = (0, 1, 0, \dots, 0)$ is a zero of f such that

$$\frac{\partial f}{\partial X_1}(e_2) = a_2'' \neq 0;$$
$$\frac{\partial f}{\partial X_i}(e_2) = 0; 2 \leq i \leq n.$$

This implies that e_2 is a nonsingular zero of f and $e_2 \in \mathbb{H}$ *i.e,* $\mathbb{H}$ is tangent to Q at a nonsingular point.

Conversely, suppose $\mathbb{H}$ is tangent to Q at a nonsingular point. W.L.O.G., after a nonsingular linear transformation over $\mathbb{F}$, let $\vec{e}_1$ be that nonsingular point,

$$\mathbb{H} : \alpha_2 X_2 + \cdots + \alpha_{r_f} X_{r_f},$$

and

$$f = X_1(\alpha_2 X_2 + \cdots + \alpha_{r_f} X_{r_f}) + q(X_2, \dots, X_{r_f}),$$

where at least one of α_is is nonzero because $\mathrm{rank}(f) = r_f$. W.L.O.G., let $\alpha_2 \neq 0$. Then after a nonsingular linear transformation over $\mathbb{F}$ involving only $X_2, \dots, X_{r_f}$, we can write

$$f = X_1 X_2 + q'(X_2, \dots, X_{r_f}),$$

and

$$\mathbb{H} : X_2,$$

which implies that

$$r_{\bar{f}} = \text{rank}(f\big|_{X_2=0}) = \text{rank}(q'(0, X_3 \ldots, X_{r_f})) \leq r_f - 2 \tag{2.5}$$

By Lemma 2.1.1 and equation (2.5),

$$r_f - 2 \leq r_{\bar{f}} \leq r_f - 2.$$

Therefore, $r_{\bar{f}} = r_f - 2$

(2c) In the proof of (2b,) we proved that $\text{rad}(f) \subset \mathbb{H}$ if and only if $r_{\bar{f}} < r_f$. Suppose that $\text{rad}(f) \subset H$, then $r_{\bar{f}} = r_f - 1$ or $r_f - 2$. If $\mathbb{H}$ is not tangent to Q at a nonsingular point, then by (2b) $r_{\bar{f}} \neq r_f - 2$. Hence, $r_{\bar{f}} = r_f - 1$. Conversely, if $\mathbb{H}$ is tangent to Q at a nonsingular point, then by (2b) $r_{\bar{f}} = r_f - 2$.

□

Lemmas 2.1.7 and 2.1.8 are well-know results from quadratic form theory.

Lemma 2.1.7. *[1, Lemma1] If a quadratic form f over $\mathbb{F}$ in $n \geq 2$ variables has nonsingular zeros in $\mathbb{F}^n$, then the set of nonsingular zeros does not lie in a proper linear subspace of $\mathbb{F}^n$.*

Proof. Let $f := f(X_1, X_2, \ldots, X_n)$ be a quadratic form over $\mathbb{F}$. We are given that f has a nonsingular zero in $\mathbb{F}^n$. After a linear transformation, we may assume that $\vec{e}_1 = (1, 0, \ldots, 0)$ is that zero, and f can we rewritten in the form

$$f = X_1 \left(\sum_{i=2}^{n} b_i X_i \right) + q(X_2, \ldots, X_n).$$

Note that,

$$\frac{\partial f}{\partial X_i}(\vec{e}_1) = \begin{cases} 0, & \text{if } i = 1 \\ b_i, & \text{if } i \geq 2 \end{cases}$$

Since $\vec{e}_1$ is a nonsingular zero, at least one of the b_i's is nonzero. W.L.O.G., let $b_2 \neq 0$. Using another linear transformation, we can assume that

$$f = X_1X_2 + q'(X_2,\dots,X_n). \tag{2.6}$$

Let W be a linear subspace of $\mathbb{F}^n$ such that $\dim(W) = n-1$. Then

$$W = \{(X_1,\dots,X_n) \in \mathbb{F}^n \,|\, L(X_1,\dots,X_n) = 0\}\},$$

for some linear form over $\mathbb{F}$ denoted by $L(X_1,\dots,X_n) = \sum_{i=1}^{n} c_iX_i$, not all c_i's are zero. If $\vec{e}_1 \notin W$, then we are done. Suppose that $\vec{e}_1 \in W$. We will show that there exists a nonsingular zero of f that does not lie in W. Since $\vec{e}_1 \in W$, $c_1 = 0$. Take $X_2 = 1$ and choose $X_i = a_i \in \mathbb{F}$, $i \geq 3$, such that

$$c_2 + \sum_{i=3}^{n} c_ia_i \neq 0.$$

(1) If $c_2 = 0$, W.L.O.G., we may assume that $c_3 \neq 0$. We may choose $a_3 = 1$, and $a_i = 0$ for all i, $4 \leq i \leq n$. Then,

$$\sum_{i=3}^{n} c_ia_i = c_3 \neq 0.$$

(2) If $c_2 \neq 0$, we may choose $a_i = 0$ for all i, $3 \leq i \leq n$.

Using (2.6), take $X_1 = -q'(1,a_3,\dots,a_n)$. Then $\alpha = (-q'(1,a_3,\dots,a_n),1,a_3,\dots,a_n)$ is a zero of f in $\mathbb{F}^n$ such that

$$\frac{\partial f}{\partial X_1}(\alpha) = 1 \neq 0.$$

i.e, α is a nonsingular zero of f. Note that $\alpha \notin W$ by construction.

□

Lemma 2.1.8. *[1, Lemma1] A quadratic form is degenerate if and only if it has a singular zero*

Proof. Suppose that f is degenerate *i.e*, $o(f) < n$. After a nonsingular linear transformation we may assume that $\gamma(f(X)) = o(f)$. Suppose that X_k does not appear in $f(X)$, then e_k is a singular zero of f because

$$\frac{\partial f_T}{\partial X_i}(e_k) = 0,$$

for all $i = 1, \ldots, n$. Conversely, suppose f has a singular zero. After applying a linear transformation T to f, we may assume that $\vec{e_1}$ is a singular zero of $f_T(X)$ *i.e*,

$$\frac{\partial f}{\partial X_i}(\vec{e_1}) = 0,$$

for all $i = 1, \ldots, n$.
This implies that X_1 does not appear in $f(X)$ *i.e*, $\gamma(f(X)) < n$. Hence, we get that f is a degenerate quadratic form.

□

Lemma 2.1.9. *[1, Lemma 2] If (f, g) is a pair of nondegenerate quadratic forms over $\mathbb{F}$ which have a common zero but no nonsingular common zero, then there is a form in the pencil $\mu f - \lambda g$ which has only singular zeros.*

Proof. After a nonsingular linear transformation, we may assume that $\vec{e_1}$ is a common zero of the pair (f, g). This implies that the vectors x_1^2 does not appear in f, g and $\frac{\partial f}{\partial x}(\vec{e_1})$ and $\frac{\partial g}{\partial x}(\vec{e_1})$ are proportional *i.e*, we can find $\mu, \lambda \in \mathbb{F}$ such that $(\mu, \lambda) \neq (0, 0)$ and $\mu\left(\frac{\partial f}{\partial x}(\vec{e_1})\right) = \lambda\left(\frac{\partial g}{\partial x}(\vec{e_1})\right)$.

W.L.O.G., we may assume that $\lambda \neq 0$ and let $h = \mu f - \lambda g$. Since,

$$\frac{\partial h}{\partial x}(\vec{e_1}) = \mu\left(\frac{\partial f}{\partial x}(\vec{e_1})\right) - \lambda\left(\frac{\partial g}{\partial x}(\vec{e_1})\right) = 0,$$

we get that $\vec{e_1}$ is a singular zero of h. At this point, Lemma 2.1.8 implies that h is a degenerate quadratic form. Since $h(\vec{e_1}) = 0$ and $\frac{\partial h}{\partial x}(\vec{e_1}) = 0$, we get that x_1 does not appear in h. Since (f, g) is a nondegenerate pair of quadratic forms and $\lambda \neq 0$,

x_1 must appear in f. So after a nonsingular linear transformation, we may assume that

$$f = x_1x_2 + q(x_2, \dots, x_n)$$

and

$$h = h(x_2, \dots, x_n)$$

Next, note that $\mathscr{X}$ is a common zero of f, g if and only if $\mathscr{X}$ is a common zero of f, h. If h has a nonsingular zero in $\mathbb{F}^n$, then by using Lemma 2.1.7, we know that there exists a nonsingular zero $\mathscr{X} = (a_1, a_2, \dots, a_n)$ of h such that $a_2 \neq 0$. If we choose $a_1 = \frac{-q(a_2,\dots,a_n)}{a_2}$, then $\mathscr{X} = (a_1, a_2, \dots, a_n)$ is a common nonsingular zero of f, h and hence it is a common nonsingular zero of f, g as well. Since f, g do not have any nonsingular common zeros, we get a contradiction to the assumption that h has a nonsingular zero. Therefore, h has only singular zeros. □

In the next proposition we give a detailed proof of an elementary fact that is stated in [14].

Proposition 2.1.10. *We assume that f is a quadratic form in n variables defined over $\mathbb{F}$ with rank at least 2 and that f has nonsingular zeros over $\mathbb{F}$. Let $\vec{a}$ be a nonsingular zero of f over $\mathbb{F}$, then we can find another nonsingular zero $\vec{b}$ of f over $\mathbb{F}$ such that $\vec{b}$ does not lie on the tangent hyperplane to $f = 0$ at $\vec{a}$. As a consequence, $\vec{a}$ does not lie on the tangent hyperplane to $f = 0$ at $\vec{b}$.*

Proof.

1. By Lemma 2.1.7, we know that all the nonsingular zeros of f over $\mathbb{F}$ do not lie in a hyperplane. Hence, we can find $\vec{b}$ such that it is a nonsingular zero of f over $\mathbb{F}$ and it does not lie on the hyperplane to $f = 0$ at $\vec{a}$.

2. W.L.O.G., we may assume that $\vec{a} = (1, 0 \dots, 0)$ and $\vec{b} = (0, 1, 0, \dots, 0)$.
   Then there exists linear forms $L_1 = a_2X_2 + \cdots + \alpha_nX_n$, and $L_2 = b_3X_3 + \cdots + b_nX_n$

such that

$$f = X_1L_1(X_2,\dots,X_n) + X_2L_2(X_3,\dots,X_n) + f'(X_3,\dots X_n).$$

where $L_1 = 0$ is the tangent hyperplane to $f = 0$ corresponding to $\vec{a}$, and $a_2X_1 + L_2 = 0$ is the tangent hyperplane to $f = 0$ at $\vec{b}$.

Since $\vec{b}$ does not lie on $L_1 = 0$, we get that $a_2 \neq 0$. Then it is clear that $\vec{a} = (1,0,\dots,0)$ also does not lie on $a_2X_1 + L_2 = 0$.

□

Lemma 2.1.11. *Let f be quadratic form in n variables over any field $\mathbb{F}$ such that $o(f) \geq u(\mathbb{F}) + 1$, then f has a nonsingular zero in $\mathbb{F}$.*

Proof. By a nonsingular linear transformation over $\mathbb{F}$, if necessary, we express f in terms of the minimum number of variables. So, $f = f(X_1,\dots,X_m)$, where $m = o(f)$. Since $m \geq u(\mathbb{F}) + 1$, f must have a nontrivial zero in $\mathbb{F}^n$. If all the zeros of f are singular, then by Lemma 2.1.8, f must be degenerate. This implies that there exists a nonsingular linear transformation T such that $\gamma(f_T) < m$, which is a contradiction as $o(f) = m$. □

Lemma 2.1.12. *Let f be a nonsingular quadratic form over $\mathbb{F}$ in n variables such that* char$(\mathbb{F}) \neq 2$, *and let Q_f denote the quadric generated by $f = 0$. Let $\mathbb{H}$ denote any hyerplane. Then $\mathbb{H}$ is polar to Q_f at a unique point in $\mathbb{F}^n$.*

Proof. Let $\mathbb{H}$ be given by the kernel of the linear form

$$L = c_1X_1 + \cdots + c_nX_n = \vec{c}^{\,t}X,$$

where $\vec{c} = (c_1,\dots,c_n)^t \in \mathbb{F}^n$ and $X = (X_1,\dots,X_n)^t$. Let $P = M_f^{-1}\vec{c} \in \mathbb{F}^n$. Then the polar hyerplane to Q_f at P is given by the kernel of the linear form P^tM_fX. Note

that

$$\begin{aligned} P^t M_f X &= \vec{c}^{\,t} M_f^{-1} M_f X \\ &= \vec{c}^{\,t} X \\ &= c_1 X_1 + \cdots + c_n X_n \\ &= L \end{aligned}$$

This implies that $\mathbb{H}$ is polar to Q_f at P.

Suppose that $\mathbb{H}$ is polar to Q_f at another point $P' \in \mathbb{F}^n$. Then

$$\mathbb{H} = \vec{c}^{\,t} X = P'^{\,t} M_f X.$$

Therefore, $\vec{c}^{\,t} = P'^{\,t} M_f$, which implies that $P'^{\,t} = \vec{c}^{\,t} M_f^{-1} = P^t$. Therefore, $\mathbb{H}$ is polar to Q_f at a unique point $P = M_f^{-1} \vec{c}$ in $\mathbb{F}^n$. □

Lemma 2.1.13 ([3], Lemma 1.15)**.** *Let f, g be two quadratic forms in n variables over $\mathbb{F}$. Assume that the homogeneous polynomial $P(\lambda, \mu) = \det(\lambda f + \mu g)$ of degree n does not vanish identically on $\overline{\mathbb{F}}$. If (λ_0, μ_0) is a zero of P of multiplicity m and $r (< n)$ is the rank of the quadratic form $\lambda_0 f + \mu_0 g$, then*

$$m \geq n - r.$$

Proof. Since the homogeneous polynomial $P(\lambda, \mu) = \det(\lambda f + \mu g)$ of degree n does not vanish on $\overline{\mathbb{F}}$, we get that it has only finitely many linearly independent zeros over $\overline{\mathbb{F}}$. Let (λ_0, μ_0) be a nontrivial root of $P(\lambda, \mu)$. W.L.O.G, we may assume that $\mu_0 \neq 0$. After an invertible linear change of variables, we can diagonalize and rewrite $\lambda_0 f + \mu_0 g$ as

$$\lambda_0 f + \mu_0 g = b_1 X_1^2 + \cdots + b_r X_r^2,$$

where $\text{rank}(\lambda_0 f + \mu_0 g) = r < n$.

Then

$$\begin{aligned}\lambda f + \mu g &= \lambda f + \mu \frac{\lambda_0 f + \mu_0 g - \lambda_0 f}{\mu_0} \\ &= (\lambda - \frac{\lambda_0}{\mu_0}\mu) f + \frac{\mu}{\mu_0}(b_1 X_1^2 + \cdots + b_r X_r^2) \\ &= (\lambda - \frac{\lambda_0}{\mu_0}\mu) f + \frac{\mu}{\mu_0} b_1 X_1^2 + \cdots + \frac{\mu}{\mu_0} b_r X_r^2\end{aligned}$$

Let M represent the symmetric matrix corresponding to the quadratic form $\lambda f + \mu g$. Then

$$P(\lambda, \mu) = \det(M),$$

where the matrix M is as shown below:

$$\begin{array}{c} \\ r \\ \\ \end{array}\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{r} & \multicolumn{3}{c}{n-r} \\ (\lambda - \frac{\lambda_0 \mu}{\mu_0}) a_{11} + \frac{\mu b_1}{\mu_0} & & * & & & \\ & \ddots & & & * & \\ * & & (\lambda - \frac{\lambda_0 \mu}{\mu_0}) a_{rr} + \frac{\mu b_r}{\mu_0} & & & \\ \hline & & & (\lambda - \frac{\lambda_0 \mu}{\mu_0}) a_{r+1 r+1} & & \\ & * & & & \ddots & \\ & & & & & (\lambda - \frac{\lambda_0 \mu}{\mu_0}) a_{nn} \end{array}\right)$$

We can factor out $n-r$ copies of $(\lambda - \frac{\lambda_0}{\mu_0}\mu)$ from the last $n-r$ rows of $\det(M)$. This implies that $(\lambda - \frac{\lambda_0}{\mu_0}\mu)^{n-r}$ divides $P(\lambda, \mu)$ *i.e*, the linear factor $(\mu_0 \lambda - \lambda_0 \mu)$ appears at least $n-r$ times in the linear factor decomposition

$$P(\lambda, \mu) = \prod_{i=1}^{n} (a_i \lambda - b_i \mu),$$

over $\overline{\mathbb{F}}$.

Therefore,

$$m_{(\lambda_0, \mu_0)} \geq n - r.$$

□

Lemma 2.1.14. *Let f, g be quadratic forms over $\mathbb{F}$ in n variables such that the determinant polynomial $P(\lambda,\mu) = \det(\lambda f + \mu g)$ over $\mathbb{F}$ is not identically zero. Let $r \leq \frac{n+1}{2}$ be a positive integer. Then every form in the $\mathbb{F}-$pencil generated by f, g has rank at least r if and only if every form in the $\overline{\mathbb{F}}-$ pencil has rank at least r.*

Proof. Since $\mathbb{F} \subset \overline{\mathbb{F}}$, if every form in the $\overline{\mathbb{F}}-$ pencil has rank at least r, then every form in the $\mathbb{F}-$ pencil also has rank at least r.

Conversely, suppose that every form in the $\mathbb{F}$ pencil has rank at least r, and suppose that there exists a form $\alpha f + \beta g$ in the $\left(\overline{\mathbb{F}}\backslash\mathbb{F}\right)-$pencil such that the

$$\operatorname{rank}(\alpha f + \beta g) \leq r - 1.$$

This implies that at least one of α and β is not in $\mathbb{F}$, and the pair (α, β) is a root of the determinant polynomial

$$P(\lambda,\mu) = \det(\lambda f + \mu g).$$

By Lemma 2.1.13,

$$\begin{aligned} m_{(\alpha,\beta)} &\geq n - (r-1) \\ &\geq n - \frac{n-1}{2} \\ &= \frac{n+1}{2}. \end{aligned}$$

This implies that $(\alpha,\beta) \in \mathbb{F}^2$, because otherwise the conjugate(s) of $(\alpha,\underline{)}$ will also be the root(s) of $P(\lambda,\mu)$ of multiplicity at least $\frac{n+1}{2}$, which is a contradiction as the degree of $P(\lambda,\mu)$ is n. Hence every form in the $\overline{\mathbb{F}}-$ pencil also has rank r. □

Corollary 2.1.15. *Let f, g be quadratic form over $\mathbb{F}$ in at least 9 variables such that the determinant polynomial $P(\lambda,\mu) = \det(\lambda f + \mu g)$ over $\mathbb{F}$ is not identically zero. Then every form in the $\mathbb{F}-$pencil generated by f, g has rank at least 5 if and only if every form in the $\overline{\mathbb{F}}-$ pencil has rank at least 5.*

Proof. The result follows directly from Lemma 2.1.14 by taking $r = 5$.

□

Corollary 2.1.16. *Let f, g be quadratic forms over $\mathbb{F}$ in n variables such that the determinant polynomial $P(\lambda, \mu) = \det(\lambda f + \mu g)$ over $\mathbb{F}$ is not identically zero. Let $\mathbb{L}$ be any extension of $\mathbb{F}$. Let $r \leq \dfrac{n+1}{2}$ be a positive integer. Then every form in the $\mathbb{F}-$pencil generated by f, g has rank at least r if and only if every form in the $\mathbb{L}-$ pencil has rank at least r.*

Proof. Since $\mathbb{F} \subset \mathbb{L}$, if every form in the $\mathbb{L}-$ pencil has rank at least r, then every form in the $\mathbb{F}-$pencil also has rank at least r.

Conversely, suppose that every form in the $\mathbb{F}-$pencil has rank at least r, and that there exists a form $\alpha f + \beta g$ in the $(\mathbb{L}\backslash\mathbb{F})-$pencil such that the

$$\operatorname{rank}(\alpha f + \beta g) \leq r - 1.$$

Since rank of f and g is at least r, we get that α and β are both nonzero. This implies that

- at least one of α and β is not in $\mathbb{F}$,
- the pair (α, β) is a root of the determinant polynomial over $\mathbb{F}$,

  $$P(\lambda, \mu) = \det(\lambda f + \mu g),$$

  and therefore,
- $\frac{\alpha}{\beta}$ is an algebraic element over $\mathbb{F}$, and hence belongs to $\overline{\mathbb{F}}$.

As a result, we get that $\alpha f + \beta g$ lies in the $\left(\overline{\mathbb{F}}\backslash\mathbb{F}\right)-$pencil, which is a contradiction because by Lemma 2.1.14 we know that every form in the $\overline{\mathbb{F}}-$pencil must also have rank at least r. Hence every form in the $\mathbb{L}-$ pencil also has rank at least r. □

2.2 Quadratic Forms over $\mathbb{R}$.

In this section we give detailed proofs of some facts about quadratic forms, and pairs of quadratic forms over the field of real numbers. We begin by introducing some definitions and terminology that is specific to quadratic forms over $\mathbb{R}$. Let f be a quadratic form in n variables over $\mathbb{R}$.

Definition 2.2.1 (Definite Quadratic Form)**.** We say that f *is definite quadratic form* over $\mathbb{R}$ if $f(\vec{v})$ always has the same sign for every $\vec{v} \in \mathbb{R}^n - \{\vec{0}\}$. According to that sign, the quadratic form f is called *positive-definite* or *negative-definite.*

Definition 2.2.2 (Semi-Definite Quadratic Form)**.** We say that f *is definite quadratic form* over $\mathbb{R}$ if $f(\vec{v})$ is always non-negative or always non-positive for every $\vec{v} \in \mathbb{R}^n - \{\vec{0}\}$. If $f(\vec{v})$ is always non-negative for every $\vec{v} \in \mathbb{R}^n - \{\vec{0}\}$, then f is called *positive-semi-definite*. If $f(\vec{v})$ is always non-positive for every $\vec{v} \in \mathbb{R}^n - \{\vec{0}\}$, then f is called *negative-semi-definite*.

Definition 2.2.3 (Indefinite Definite Quadratic Form)**.** We say that f *is an indefinite quadratic form* over $\mathbb{R}$ if it takes both positive and negative values when evaluated at vectors in $\mathbb{R}^n - \{\vec{0}\}$.

Example. $f(X_1, X_2, X_3) = \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_{12} X_1 X_2$ is a binary quadratic form over $\mathbb{R}$.

- f is positive definite ($f > 0$) if $\alpha_1 > 0$ and $\alpha_1\alpha_2 - \alpha_{12}^2 > 0$, and f is negative definite ($f < 0$) if $\alpha_1 < 0$ and $\alpha_1\alpha_2 - \alpha_{12}^2 > 0$.
- f is positive-semi-definite ($f \geq 0$) if $\alpha_1 > 0$ and $\alpha_1\alpha_2 - \alpha_{12}^2 = 0$, and f is negative-semi-definite ($f \leq 0$) if $\alpha_1 < 0$ and $\alpha_1\alpha_2 - \alpha_{12}^2 = 0$.
- f is indefinite if $\alpha_1\alpha_2 - \alpha_{12}^2 < 0$.

Definition 2.2.4 (Rank and Signature of a Quadratic Form over $\mathbb{R}$)**.** Let f be a quadratic form in n variables over a $\mathbb{R}$. Then f is equivalent to a diagonal form $d_1X_1^2 + \cdots + d_nX_n^2$ under an invertible linear change of variables over $\mathbb{R}$. The rank of f, denoted by rank(f) is the number of elements in the set $\{d_i; d_i \neq 0, 1 \leq i \leq n\}$. The signature of f, denoted by sgn(f), is given by

$$\operatorname{sgn}(f) = r_p - r_n,$$

where r_p is number of elements in the set $\{d_i; d_i > 0, 1 \leq i \leq n\}$, and r_n is number of elements in the set $\{d_i; d_i < 0, 1 \leq i \leq n\}$.

Proposition 2.2.5. *Let q be a nonsingular indefinite form in n variables. Let $q\big|_W$ denote the restriction of q to an $(n-1)$–dimensional subspace W. Then*

$$\operatorname{sgn}\left(q\big|_W\right) = \begin{cases} \operatorname{sgn}(q), & \textit{if} \operatorname{rank}\left(q\big|_W\right) = \operatorname{rank}(q) - 2 \\ \operatorname{sgn}(q) \pm 1, & \textit{if} \, rank\left(q\big|_W\right) = \operatorname{rank}(q) - 1, \end{cases}$$

where sgn(q) *denotes the signature of q.*

Proof. Let W be a subspace of dimension $n-1$ of an n–dimensional space V. Suppose that $rank(q\big|_W) = n-1$. Choose a basis $\{w_1, \ldots, w_{n-1}\}$ of $q\big|_W$ such that $q\big|_W$ can be written as a diagonal form,

$$q\big|_W = \langle d_1, \ldots, d_{n-1} \rangle,$$

where none of the $d_i's$ are zero. Then we can extend this to a basis of the whole space given by

$$\mathscr{B} = \{w_1, \ldots, w_{n-1}, v_n\}.$$

Then after a few row and column operations, the symmetric matrix of q looks like

$$\begin{array}{c} \\ n-1 \\ \\ \\ 1 \end{array}\overset{\qquad n-1 \qquad\qquad\quad 1}{\left(\begin{array}{ccc|c} d_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & d_{n-1} & \\ \hline & 0 & & b \end{array}\right)} \tag{2.7}$$

Since q is nonsingular, $b \neq 0$. This implies that

$$\operatorname{sgn}(q) = \operatorname{sgn}(q\big|_W) \pm 1$$

Now, we suppose that $rank(q\big|_W) = n-2$. We again choose a basis $\{w_1, \ldots, w_{n-1}\}$ of $q\big|_W$ such that $q\big|_W$ can be written as a diagonal form.

$$q\big|_W = \langle d_1, \ldots, d_{n-2}, 0 \rangle,$$

where none of the $d_i's$ are zero. We can extend this to a basis of the whole space given by $\mathscr{B} = \{w_1, \ldots, w_{n-1}, v_n\}$. Then after a few row and column operations, the symmetric matrix of q looks like

$$\begin{array}{c} \\ n-1 \\ \\ 2 \\ \\ \end{array}\overset{\qquad n-1 \qquad\qquad\quad 2}{\left(\begin{array}{ccc|cc} d_1 & & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & d_{n-2} & 0 & 0 \\ \hline 0 & \ldots & 0 & 0 & a \\ 0 & \ldots & 0 & a & b \end{array}\right)}$$

where a, b are nonzero.

This implies that

$$\operatorname{sgn}(q) = \operatorname{sgn}(q\big|_W) + \operatorname{sgn}\begin{pmatrix} 0 & a \\ a & b \end{pmatrix} = \operatorname{sgn}(q\big|_W)$$

because the signature of $\begin{pmatrix} 0 & a \\ a & b \end{pmatrix}$ is zero as it represents a hyperbolic form. This finishes the proof of Proposition 2.2.5. □

The next two Propositions give a proof of the result in [14, Lemma 1]. We have given detailed proofs for the nontrivial intermediate steps and statements that were used in [14, Lemma 1].

Notation. Let f be a real quadratic form. In the next two lemmas we use

$f = 0$ to denote the set $\{\vec{x} | x \in \mathbb{R}^n, f(\vec{x}) = 0\}$

$f > 0$ to denote the set $\{\vec{x} | x \in \mathbb{R}^n, f(\vec{x}) > 0\}$

$f < 0$ to denote the set $\{\vec{x} | x \in \mathbb{R}^n, f(\vec{x}) < 0\}$

Proposition 2.2.6. *Let h be any quadratic form in n variables such that the rank of h is at least 3, then*

(1) the set $h = 0$ has a nontrivial real point if and only if h is not definite; and in this case the set $h = 0$ is path-connected in $P^{n-1}(\mathbb{R})$.

(2) the sets $h > 0$ and $h < 0$ separate the projective space $P^{n-1}(\mathbb{R})$ into non-empty disjoint parts if and only if h is indefinite.

(3) If h is indefinite, then the sets $h > 0$ and $h < 0$ are path-connected.

Proof. 1. If there exists a nontrivial point $\vec{v} \in \mathbb{R}^n$ such that $h(\vec{v}) = 0$, then clearly h is not definite. Conversely, if h is not definite, then it is either semi-definite or indefinite. In either case, there exists a nontrivial $\vec{v} \in \mathbb{R}^n$ such that $h(\vec{v}) = 0$.

Suppose that the set $h = 0$ has nontrivial real points. We will show that $h = 0$ is path-connected as a subset of $P^{n-1}(\mathbb{R})$ under the euclidean topology. In the rest of the proof we will use $Z(h)$ to denote the set of all nontrivial real zeros of h in $P^{n-1}(\mathbb{R})$.

a) Suppose that h is a positive semi-definite form. We can diagonalize h to write it in the form

$$h = X_1^2 + \cdots + X_r^2,$$

where $r < n$. Since r is at least 3, this implies that $n \geq 4$. Note that any nontrivial real zero of h must have zeros in the first r coordinates.

If $\vec{a}, \vec{b}$ are any two distinct elements of $Z(h)$ *i.e*, $\vec{a} \neq c\vec{b}$, for any nonzero $c \in \mathbb{R}$. We define the map

$$\gamma : [0,1] \rightarrow Z(h)$$
$$t \mapsto t(\vec{a}) + (1-t)\vec{b}$$

Note that

- $\gamma(0) = \vec{b}$ and $\gamma(1) = \vec{a}$.
- for any $t \in [0,1]$, the vector $t(\vec{a}) + (1-t)\vec{b}$ will also have zeros in the first r coordinates and hence will be a zero of h.
- if there exists $t \in (0,1)$ such that $t(a_i) + (1-t)b_i = 0$ for $r+1 \leq i \leq n$, then this would imply that $a_i = \frac{t-1}{t} b_i$ for $r+1 \leq i \leq n$, which further implies that $\vec{a} = \frac{t-1}{t}\vec{b}$, which is a contradiction.

Therefore, we see that $\vec{a}$ and $\vec{b}$ are path-connected in $Z(h)$.

b) Suppose that h is an indefinite form of rank $r \geq 3$

We can diagonalize h over the reals to write it in the form

$$h = X_1^2 + \cdots + X_k^2 - X_{k+1}^2 - \cdots - X_r^2,$$

Note that if $\vec{v} = (v_1, \ldots, v_n)$ is a nontrivial real zero of h, then

$$v_1^2 + v_2^2 + \cdots + v_k^2 = v_{k+1}^2 + \cdots + v_r^2.$$

We will proceed by assuming that h is nonsingular *i.e*, $r = n$.

- We first look at the case when $k = n-1$. Let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ be two distinct zeros of h in $Z(h)$. Note that a_n, b_n are nonzero real numbers and we may replace $\vec{a}$ and $\vec{b}$ by $\frac{1}{a_n}\vec{a}$ and $\frac{1}{b_n}\vec{b}$, respectively to ensure that $a_n = b_n = 1$. Since $\vec{a}$ and $\vec{b}$ are zeros of h, we get that

$$a_1^2 + \cdots + a_{n-1}^2 = 1$$
$$b_1^2 + \cdots + b_{n-1}^2 = 1$$

  Now we define the following continuous map

$$\gamma : \partial(\mathscr{B}^{n-1}(0,1)) \to Z(h)$$
$$\vec{u} \mapsto (\vec{u}, 1)$$

  Note that γ is a well-defined continuous map, $\vec{a} = \gamma(a_1, \ldots, a_{n-1})$, and $\vec{b} = \gamma(b_1, \ldots, b_{n-1})$. This implies that $\vec{a}$ and $\vec{b}$ are path-connected in $Z(h)$, when $n \geq 3$.

- Suppose that $k \geq 2$ and $n - k \geq 2$ *i.e*, there are at least two positive and two negative monomials in h.
  Let $\vec{a}$ and $\vec{b}$ be two distinct zeros of h. By multiplying by a scalar if necessary, we may assume that

$$a_1^2 + \cdots + a_k^2 = a_{k+1}^2 + \cdots + a_n^2 = 1$$
$$b_1^2 + \cdots + b_k^2 = b_{k+1}^2 + \cdots + b_n^2 = 1$$

  Now we define the following continuous maps

$$\gamma_{n-k} : \partial(\mathscr{B}^{n-k}(0,1)) \to Z(h)$$
$$(u_{k+1}, \ldots, u_n) \mapsto (a_1, \ldots, a_k, u_{k+1}, \ldots, u_n),$$

and

$$\gamma_k' : \partial(\mathscr{B}^k(0,1)) \to Z(h)$$
$$(u_1, \ldots, u_k) \mapsto (u_1, \ldots, u_k, b_{k+1}, \ldots, b_n),$$

Note that γ_{n-k} and γ_k' are both well defined continuous maps. We make the following observations:

– $\vec{a} = \gamma_{n-k}(a_{k+1}, \ldots, a_n)$ is path-connected to $\gamma_{n-k}(b_{k+1}, \ldots, b_n)$ in $Z(h)$.

– $\gamma_{n-k}(b_{k+1}, \ldots, b_n) = \gamma_k'(a_1, \ldots, a_k)$.

– $\gamma_k'(a_1, \ldots, a_k)$ is path-connected to $\vec{b} = \gamma_k'(b_1, \ldots, b_k)$ in $Z(h)$.

Therefore, we get that $\vec{a}$ and $\vec{b}$ are path-connected in $Z(h)$. A similar argument can be used to prove the case when $r < n$.

2. if h is indefinite then $h > 0$ and $h < 0$ are disjoint non-empty subsets of $P^{n-1}(\mathbb{R})$. Conversely, if there exist nontrivial real points in $h > 0$ and $h < 0$, then clearly h is an indefinite quadratic form.

3. We will prove that if h is indefinite, then $h > 0$ is connected. An analogous argument will work to show that $h < 0$ is also connected.

   Let $\vec{a}$ and $\vec{b}$ be two distinct vectors in $h > 0$. We will show that $\vec{a}$ is path-connected to $\vec{b}$ or $-\vec{b}$.

   After a nonsingular linear transformation if necessary, we may assume that

$$h = \sum_{i=1}^{p} X_i^2 - \sum_{i=p+1}^{p+s} X_i^2,$$

   where $p \geq 1$ and $s \geq 1$ We define a continuous map $\gamma : [0,1] \to h > 0$, such that $\gamma(t) = t\vec{a} + (1-t)\vec{b}$.

   Then for any $t \in [0,1]$,

$$
\begin{aligned}
h(t\vec{a}+(1-t)\vec{b}) &= \sum_{i=1}^{p}(ta_i+(1-t)b_i)^2 - \sum_{i=p+1}^{p+s}(ta_i+(1-t)b_i)^2 \\
&= t^2\left[\sum_{i=1}^{p}(a_i)^2 - \sum_{i=p+1}^{p+s}(a_i)^2\right] + (1-t)^2\left[\sum_{i=1}^{p}(b_i)^2 - \sum_{i=p+1}^{p+s}(b_i)^2\right] \\
&+ 2t(1-t)\left[\sum_{i=1}^{p}(a_ib_i) - \sum_{i=p+1}^{p+s}(a_ib_i)\right] \\
&= t^2h(\vec{a}) + (1-t)^2(\vec{b}) + 2t(1-t)\left[\sum_{i=1}^{p}(a_ib_i) - \sum_{i=p+1}^{p+s}(a_ib_i)\right]
\end{aligned}
$$

If

$$2t(1-t)\left[\sum_{i=1}^{p}(a_ib_i) - \sum_{i=p+1}^{p+s}(a_ib_i)\right] \geq 0$$

then $h(t\vec{a}+(1-t)\vec{b}) > 0$.

If

$$\left[\sum_{i=1}^{p}(a_ib_i) - \sum_{i=p+1}^{p+s}(a_ib_i)\right] < 0,$$

then

$$\left[\sum_{i=1}^{p}(a_i(-b_i)) - \sum_{i=p+1}^{p+s}(a_i(-b_i))\right] > 0,$$

and hence

$h(t\vec{a}+(1-t)(-\vec{b})) > 0$.

This shows that $\vec{a}$ is path-connected to $\vec{b}$ or $-\vec{b}$. Since $\vec{b}$ and $-\vec{b}$ represent the same vector in $P^{n-1}(\mathbb{R})$, we get that $\vec{a}$ and $\vec{b}$ are path-connected in $h > 0$ and hence $h > 0$ is a path-connected set in $P^{n-1}(\mathbb{R})$.

□

Proposition 2.2.7. *[14, Lemma 1] Let f, g be real quadratic forms in n variables with $n \geq 3$. Then*

a. *The set $f = g = 0$ contains nontrivial real points if and only if $\lambda f + \mu g$ is never definite for any real λ, μ, not both zero.*

b. *If f is indefinite, then there exist real points on $f = 0$ that give either sign to g if and only if $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$.*

Proof. **a**. " $\Rightarrow$ "

If $f = g = 0$ contains nontrivial real points, then $\lambda f + \mu g$ is never definite for any real λ, μ, not both zero.

" $\Leftarrow$ "

Suppose that $f = g = 0$ does not contain any nontrivial real points and $\lambda f + \mu g$ is never definite for any real λ, μ, not both zero.

We get the following two cases:

1. Suppose there exists a positive semi-definite form in the pencil. W.L.O.G., we may assume that f is a semi-definite quadratic form. After a nonsingular linear transformation, we may assume that

$$f(X_1, \ldots, X_n) = X_1^2 + \cdots + X_r^2,$$

where $r < n$ is the rank of f, and

$$g = \sum_{i,j=1}^{n} a_{ij} X_i X_j.$$

Note that if we set the first r variables equal to zero, then

$$g|_{\{X_i = 0, 1 \leq i \leq r\}} = \sum_{i,j=r+1}^{n} a_{ij} X_i X_j,$$

does not vanish since $f = g = 0$ does not have any nontrivial real points.

Now using a nonsingular linear transformation involving only $X_{r+1}, \ldots, X_n$, we can assume that

$$f = f(X_1, \ldots, X_n) = X_1^2 + \cdots + X_r^2,$$

and

$$g = \sum_{i=1}^{r} \sum_{j=1}^{n} \alpha_{ij} X_i X_j + \sum_{i=r+1}^{n} \beta_i X_i^2,$$

Note that

1. if $\beta_i = 0$ for some $r+1 \le i \le n$, then $\vec{e_i}$ will be a nontrivial common rational zero of f and g.
2. if $\beta_i > 0$, $\beta_j < 0$ for some $r+1 \le i, j \le n$, then $\sqrt{|\beta_j|}\vec{e_i} + \sqrt{\beta_i}\vec{e_j}$ will be a nontrivial common real zero of f and g.

Since the set $f = g = 0$ does not have any nontrivial real points, all the β_is are nonzero real numbers that have the same sign. W.L.O.G., we may assume that all β_is are positive real numbers.

Now for any $\lambda \in \mathbb{R}$, consider the symmetric matrix corresponding to $\lambda f + g$:

$$\begin{array}{c} \\ r \\ \\ n-r \end{array}
\begin{array}{c} \quad r \qquad\qquad\qquad n-r \\
\left(\begin{array}{ccc|ccc}
\lambda + \alpha_{11} & & * & & & \\
 & \ddots & & & * & \\
* & & \lambda + \alpha_{rr} & & & \\
\hline
 & & & \beta_{r+1} & & \\
 & * & & & \ddots & \\
 & & & & & \beta_n
\end{array}\right)
\end{array}$$

Note that the first $n-r$ leading principal minors starting from the lower right corner of the above matrix are all positive and since λ appears only on the diagonal entries, we can choose λ_0 large enough so that all the leading principal minors starting from the lower right corner are all positive. Hence, using Sylvester's Criterion for a symmetric matrix we can conclude that $\lambda_0 f + g$ is a positive definite quadratic form, which is a contradiction.

2. Now we suppose that every form in the $\mathbb{R}$–pencil is indefinite. Since the set $f = 0, g = 0$ does not contain any nontrivial real points and $\lambda f + \mu g$ is indefinite for all real $\lambda, \mu \in \mathbb{R}$, not both zero, the set $\lambda f + \mu g = 0$ with $\mu > 0$ does not meet $f = 0$ in nontrivial real points. It therefore lies entirely in $f > 0$ or $f < 0$. Define

$$\mathscr{C} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda^2 + \mu^2 = 1, \mu > 0\},$$

$$M_1 = \{(\lambda, \mu) \in \mathscr{C} : \lambda f + \mu g = 0 \text{ lies in } f > 0\},$$

and

$$M_2 = \{(\lambda, \mu) \in \mathscr{C} : \lambda f + \mu g = 0 \text{ lies in } f < 0\}.$$

M_1 and M_2 are disjoint and $\mathscr{C} = M_1 \cup M_2$ by definition.

Since f is indefinite, there exist nontrivial vectors $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^n$ such that $f(\vec{u}_1) > 0$ and $f(\vec{u}_2) < 0$.

For $i = 1, 2$, we define $(\lambda_i, \mu_i) \in \mathscr{C}$ such that

$$\lambda_i = (-1)^i \frac{g(\vec{u}_i)}{\sqrt{g(\vec{u}_i)^2 + f(\vec{u}_i)^2}},$$

and

$$\mu_i = (-1)^{i+1} \frac{f(\vec{u}_i)}{\sqrt{g(\vec{u}_i)^2 + f(\vec{u}_i)^2}}.$$

Note that

$$\lambda_i f(\vec{u}_i) + \mu_i g(\vec{u}_i) = 0$$

for $i = 1, 2$.

Since $f(\vec{u}_1) > 0$ and $f(\vec{u}_2) < 0$, we get that $(\lambda_1, \mu_1) \in M_1$ and $(\lambda_2, \mu_2) \in M_2$. This shows that M_1 and M_2 are non-empty subsets of $\mathscr{C}$.

Now we will show that M_1 and M_2 as defined above are closed subsets of $\mathscr{C}$. This will give us a contradiction as $\mathscr{C}$ is connected. It is sufficient to show that every sequence in M_1 that converges to a point in $\mathscr{C}$, actually converges to a point in M_1. A similar argument will work for M_2.

Let $\{(\lambda_i, \mu_i)\}$ be a sequence in M_1 that converges to (λ, μ) in $\mathscr{C}$.

For each (λ_i, μ_i), there exists a v_i in $\mathbb{R}^n - \{0\}$ such that $(\lambda_i f + \mu_i g)(v_i) = 0$ and $f(v_i) > 0$, since $(\lambda_i, \mu_i) \in M_1$.

W.L.O.G., we may assume that $|v_i| = 1$. Let $S = \{v \in \mathbb{R}^n : |v| = 1\}$. Note that S is a compact set and $\{v_i\} \subset S$. Thus $\{v_i\}$ has a convergent subsequence in S. By restricting to this subsequence, we may assume W.L.O.G. that $\{v_i\}$ converges to v in S. Since $f(v_i) > 0$ and f is continuous, we have that

$$f(v) = f(\lim_{i \to \infty} \{v_i\}) = \lim_{i \to \infty} f(v_i) \geq 0.$$

To complete the proof we claim that $(\lambda f + \mu g)(v) = 0$.
Suppose this has been done. Then $f(v) \neq 0$ because $\mu > 0$ and the set $f = 0, g = 0$ has no real points. Since $f(v) \geq 0$, we get that $f(v) > 0$. This implies that $(\lambda, \mu) \in M_1$, as desired.

Claim 1. $(\lambda f + \mu g)(v) = 0$

We have

$$|(\lambda f + \mu g)(v)| = |(\lambda f + \mu g)(v) - (\lambda_i f + \mu_i g)(v_i)|$$

$$(\lambda f + \mu g)(v)| \leq |(\lambda f + \mu g)(v) - (\lambda f + \mu g)(v_i)| + |(\lambda f + \mu g)(v_i) - (\lambda_i f + \mu_i g)(v_i)| \tag{2.8}$$

Let $\varepsilon > 0$ be given, since $\lambda f + \mu g$ is continuous, there exists $N \in \mathbb{N}$ such that for all $i \geq N$, we have that

$$|(\lambda f + \mu g)(\vec{v}) - (\lambda f + \mu g)(\vec{v}_i)| < \frac{\varepsilon}{2}$$

We now claim that

Claim 2. *There exist $N' \in \mathbb{N}$ such that for all for any $\vec{w} \in \mathbb{R}^n$ with $|\vec{w}| = 1$, if $i \geq N'$, then*

$$|(\lambda f + \mu g)(\vec{w}) - (\lambda_i f + \mu_i g)(\vec{w})| < \frac{\varepsilon}{2}$$

Suppose that the claim is true, then for all $i \geq \max\{N, N'\}$, inequality (2.8) implies that

$$|(\lambda f + \mu g)(v)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $(\lambda f + \mu g)(\vec{v}) = 0$ as desired.

To prove claim 2, let $\vec{w} \in \mathbb{R}^n$ such that $|\vec{w}| = 1$. Then $\vec{w} = (c_1, \dots, c_n)$ such that $|c_j| \leq 1$ for all $1 \leq j \leq n$. Then

$$|(\lambda f + \mu g)(\vec{w}) - (\lambda_i f + \mu_i g)(\vec{w})| = |(\lambda - \lambda_i) f(\vec{w}) + (\mu - \mu_i) g(\vec{w})|$$

Choose $N' \in \mathbb{N}$ such that if $i \geq N'$, then

$$|\lambda - \lambda_i| < \frac{\varepsilon}{2mn(n+1)}$$

and

$$|\mu - \mu_i| < \frac{\varepsilon}{2mn(n+1)},$$

where $m = \max\left\{|a_{ij}|, |b_{ij}| \;\middle|\; 1 \leq i, j \leq n\right\}$ and a_{ij}, b_{ij} represent the coefficients of f, g, respectively. Since f, g each have at most $\frac{n(n+1)}{2}$ monomials, it follows

that

$$
\begin{aligned}
|(\lambda-\lambda_i)f(\vec{w})+(\mu-\mu_i)g(\vec{w})| &< \frac{\varepsilon}{2mn(n+1)}\cdot m\frac{n(n+1)}{2}+\frac{\varepsilon}{2mn(n+1)}\cdot m\frac{n(n+1)}{2} \\
&= \frac{\varepsilon}{4}+\frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{2}
\end{aligned}
$$

So we conclude that if $f = g = 0$ does not contain any nontrivial real points, then there exists a definite form in the pencil. □

Proof. **b**. " $\Rightarrow$ "

If there exist points on $f = 0$ which make g positive as well as negative, then $\lambda f + g$ is indefinite for all $\lambda \in \mathbb{R}$.

" $\Leftarrow$ "

Assume that $g \geq 0$ whenever $f = 0$ and that $\lambda f + g$ is indefinite for all real l. We will arrive at a contradiction. By Proposition 1(3), we know that for any $\lambda \in \mathbb{R}$, the real set $\lambda f + g < 0$ is a non-empty, open and connected set. Note that $\lambda f + g < 0$ does not meet $f = 0$ for any real λ as $f = 0$ lies entirely in $g \geq 0$. Hence, $\lambda f + g < 0$ lies entirely in $f > 0$ or $f < 0$. Define

$$\Lambda_1 := \Big\{\lambda \in \mathbb{R} \;\Big|\; \lambda f + g < 0 \text{ lies in } f > 0\Big\},$$

and

$$\Lambda_2 := \Big\{\lambda \in \mathbb{R} \;\Big|\; \lambda f + g < 0 \text{ lies in } f < 0\Big\}.$$

Note that Λ_1 and Λ_2 are disjoint and $\Lambda_1 \cup \Lambda_2 = \mathbb{R}$.

Claim 3. *Λ_1 and Λ_2 are non-empty subsets of $\mathbb{R}$.*

Since f is indefinite, there exists $\vec{v}, \vec{u} \in \mathbb{R}^n - \{0\}$ such that $f(\vec{v}) > 0$ and $f(\vec{u}) < 0$. We can find a sufficiently large negative number $\lambda_1 \in \mathbb{R}$ such that

$$(\lambda_1 f + g)(\vec{v}) < 0,$$

and a sufficiently large positive number $\lambda_2 \in \mathbb{R}$ such that

$$(\lambda_2 f + g)(\vec{u}) < 0.$$

This implies that $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$.

Claim 4. *Λ_1 and Λ_2 are open sets in $\mathbb{R}$.*

Let $\lambda \in \Lambda_1$. Then there exists a nonzero $\vec{v} \in \mathbb{R}^n$ such that $\lambda f(\vec{v}) + g(\vec{v}) < 0$ and $f(\vec{v}) > 0$. This implies that $\lambda < \frac{-g(\vec{v})}{f(\vec{v})}$. Let $\varepsilon = \frac{-g(\vec{v})}{f(\vec{v})} - \lambda$. Then for $\lambda' < \lambda + \varepsilon$,

$$\begin{aligned}
\lambda' f(\vec{v}) &< \lambda f(\vec{v}) + \varepsilon f(\vec{v}), \\
\lambda' f(\vec{v}) + g(\vec{v}) &< \lambda f(\vec{v}) + g(\vec{v}) + \varepsilon f(\vec{v}) \\
&= \lambda f(\vec{v}) + g(\vec{v}) + (\frac{-g(\vec{v})}{f(\vec{v})} - \lambda) f(\vec{v}) \\
&= \lambda f(\vec{v}) + g(\vec{v}) - g(\vec{v}) - \lambda f(\vec{v}) \\
&= 0
\end{aligned}$$

Hence, $(-\infty, \lambda + \varepsilon) \subset \Lambda_1$

Similarly, for $\lambda \in \Lambda_2$ there exists a nonzero $\vec{u} \in \mathbb{R}^n$ such that $\lambda f(\vec{u}) + g(\vec{u}) < 0$ and $f(\vec{u}) < 0$. This implies that $\lambda > \frac{-g(\vec{u})}{f(\vec{u})}$.

Let $\varepsilon = \lambda + \frac{g(\vec{u})}{f(\vec{u})}$ Then for any $\lambda' > \lambda - \varepsilon$

$$\begin{aligned}
\lambda' f(\vec{u}) &< \lambda f(\vec{u}) - \varepsilon f(\vec{u}), \\
\lambda' f(\vec{u}) + g(\vec{u}) &< \lambda f(\vec{u}) - \varepsilon f(\vec{u}) + g(\vec{u}) \\
&= \lambda f(\vec{u}) + g(\vec{u}) - (\lambda + \frac{g(\vec{u})}{f(\vec{u})}) f(\vec{u}) \\
&= \lambda f(\vec{u}) + g(\vec{u}) - \lambda f(\vec{v}) - g(\vec{u}) \\
&= 0
\end{aligned}$$

Hence, $(\lambda - \varepsilon, \infty) \subset \Lambda_2$.

This proves the claim.

The previous two claims show that $\mathbb{R}$ can be written as disjoint union of two non-empty open sets, which is a contradiction.

$\square$

Remark 1. Note that in [14] in the proof of Proposition 2.2.7b the case when f is positive semi-definite (*i.e,* when $M_1 = \mathscr{C}$ and $M_2 = \emptyset$), was not considered. Hence in the proof of Proposition 2.2.7b given above, we consider the case when f is semi-definite separately.

Now let $\mathscr{C} = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda^2 + \mu^2 = 1\}$. Let f, g be two nonsingular quadratic forms over $\mathbb{R}$ in n variables. Let M_f, M_g represent the symmetric matrices corresponding to f, g, respectively. Assume that the determinant polynomial

$$\det(\lambda f + \mu g) = \det(\lambda M_f + \mu M_g)$$

is a nonzero as a polynomial over $\mathbb{R}$ in the variables λ, μ. As (λ, μ) moves on $\mathscr{C}$, $\lambda f + \mu g$ varies in the pencil.

Lemma 2.2.8. *At most $2n$ of the forms obtained by varying (λ, μ) on $\mathscr{C}$ are singular, where we consider (λ, μ) and $(-\lambda, -\mu)$ as giving two distinct forms i.e, if $S = \{(\lambda, \mu) \in \mathscr{C} : \lambda f + \mu g$ is a singular quadratic form$\}$, then $|S| \leq 2n$.*

Proof. By definition, a form in the pencil generated by f, g is singular if and only if the rank of the corresponding symmetric matrix is less that n. Since $\det(\lambda f + \mu g) = 0$ has at most $2n$ distinct zeros, there are at most $2n$ distinct singular forms in the pencil. This implies that $|S| \leq 2n$. $\square$

Next, we define

$$\begin{aligned} \operatorname{sgn} : \mathscr{C} &\to \mathbb{Z} \\ (\lambda, \mu) &\mapsto \operatorname{sgn}(\lambda f + \mu g) \end{aligned}$$

Proposition 2.2.9. sgn *is constant on each connected component of* $\mathscr{C} - S$*. This implies that* sgn *is continuous at all but finitely many points on* $\mathscr{C}$*.*

Proof. For $1 \le k \le n$, let $M^k_{\lambda f+\mu g}$:= upper $k \times k$ submatrix of $M_{\lambda f+\mu g}$, and d_k := determinant of $M^k_{\lambda f+\mu g}$. We know that any nonsingular quadratic form $\lambda f + \mu g$ can be arranged such that $d_k \neq 0$ for any k.

$$\lambda f + \mu g \cong_{\mathbb{R}} \langle d_1, \frac{d_2}{d_1}, \ldots, \frac{d_n}{d_{n-1}} \rangle.$$

Let $(\lambda_0, \mu_0) \in \mathscr{C} - S$. Because of the continuity of the determinant function, for every $\varepsilon > 0$, there exists $\delta_k > 0$ such that if

$$\|M^k_{\lambda_0 f+\mu_0 g} - M^k_{\lambda f+\mu g}\| < \delta_k,$$

then

$$|\det(M^k_{\lambda_0 f+\mu_0 g}) - \det(M^k_{\lambda f+\mu g})| < \varepsilon.$$

This holds true for all k, $1 \le k \le n$.

Let $\delta = \min\{\delta_k | 1 \le k \le n\}$. Now we can choose $\varepsilon > 0$ small enough such that $\det(M^k_{\lambda_0 f+\mu_0 g})$ and $\det(M^k_{\lambda f+\mu g})$ have the same sign in an open neighborhood U_δ around $(\lambda_0, \mu_0) \in \mathscr{C} - S$. As a result,

$$\operatorname{sgn}(\lambda_0, \mu_0) = \operatorname{sgn}(\lambda, \mu),$$

for all $(\lambda, \mu) \in U_\delta$. This shows that sgn is a locally constant function from $\mathscr{C} - S$ to $\mathbb{Z}$, where $\mathbb{Z}$ has the discrete topology on it. Hence sgn is a continuous function.

Let $\mathscr{C}_i$ be any connected component of $\mathscr{C} - S$, then $\operatorname{sgn}(\mathscr{C}_i)$ is also connected. Since the only connected sets in $\mathbb{Z}$ are singleton sets, we get that $\operatorname{sgn}(\mathscr{C}_i)$ is a constant. □

Proposition 2.2.10. *For* $(\lambda, \mu) \in \mathscr{C}$*, the signature of the quadratic form* $\lambda f + \mu g$ *changes only as we pass through a singular point on* $\mathscr{C}$ *and it changes by at most twice the nullity of the form.*

Proof. Note that the proof of the first part of this Proposition follows from the previous Proposition.

We will show that as we pass through a singularity (λ_0, μ_0) on $\mathscr{C}$, the signature changes by at most twice the nullity of the form $\lambda_0 f + \mu_0 g$.

Let $\operatorname{rank}(\lambda_0 f + \mu_0 g) = r < n$. W.L.O.G., we may assume that $\lambda_0 f + \mu_0 g$ is a form in the r variables $X_1, \ldots, X_r$.

Let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_s$ denote all the connected components of $\mathscr{C} - S$. Proposition 2.2.9 implies that sgn is constant on each $\mathscr{C}_i$. Let $\mathscr{C}_1, \mathscr{C}_2$ be the two consecutive components such that (λ_0, μ_0) is the point of singularity that disconnects $\mathscr{C}_1$ and $\mathscr{C}_2$ in $\mathscr{C}$.

Note that the form $\lambda f + \mu g$ is nonsingular for all $(\lambda, \mu) \in \mathscr{C}_1 \cup \mathscr{C}_2$

Set $X_{r+1} = \ldots = X_n = 0$ in $\lambda f + \mu g$ for all $(\lambda, \mu) \in \{\mathscr{C}_1 \cup \mathscr{C}_2 \cup (\lambda_0, \mu_0)\}$. Then $\lambda_0 f + \mu_0 g$ and $\left(\lambda f + \mu g\big|_{X_{r+1} = \cdots = X_n = 0}\right)$ are quadratic forms in r variables and in this case $\lambda_0 f + \mu_0 g$ is nonsingular when considered as a form in r variables. We define the following map which is the restriction of sgn defined in Proposition 2.2.9.

$$\begin{aligned} \operatorname{sgn}_1 : \mathscr{C}_1 \cup \mathscr{C}_2 \cup \{(\lambda_0, \mu_0)\} &\to \mathbb{Z} \\ (\lambda, \mu) \quad &\mapsto sgn\left(\lambda f + \mu g\big|_{X_{r+1} = \cdots = X_n = 0}\right) \end{aligned}$$

From Proposition 2.2.9, we know that sgn_1 is a locally constant map at a nonsingular point in $\mathscr{C}$. Since the form $\lambda_0 f + \mu_0 g$ corresponding to the point (λ_0, μ_0) is a nonsingular form in r variables, we can find $\varepsilon > 0$ such that

$$\operatorname{sgn}_1(\lambda, \mu) = \operatorname{sgn}_1(\lambda_0, \mu_0) = \operatorname{sgn}(\lambda_0, \mu_0)$$

for all $(\lambda, \mu) \in \mathscr{B}_\varepsilon(\lambda_0, \mu_0)$ in $\mathscr{C}$. Choose $(\lambda, \mu) \in \mathscr{B}_\varepsilon$ different from (λ_0, μ_0). After a few row and column operations, the symmetric matrix $M_{\lambda f + \mu g}$, can be written in the following form

$$
\begin{array}{cc}
 & \begin{array}{cc} r & n-r \end{array} \\
\begin{array}{c} r \\ \\ n-r \end{array} & \left(\begin{array}{ccc|c}
c_1 & & 0 & \\
 & \ddots & & 0 \\
0 & & c_r & \\
\hline
 & 0 & & B
\end{array}\right)
\end{array}
$$

As observed from the above matrix,

$$
\begin{aligned}
\operatorname{sgn}(\lambda f + \mu g) &= \operatorname{sgn}_1(\lambda f + \mu g) + \operatorname{sgn}(B) \\
&= \operatorname{sgn}(\lambda_0 f + \mu_0 g) + \operatorname{sgn}(B)
\end{aligned}
$$

Hence we get the following inequality,

$$
\operatorname{sgn}(\lambda_0 f + \mu_0 g) - (n-r) \leq \operatorname{sgn}(\lambda f + \mu g) \leq \operatorname{sgn}(\lambda_0 f + \mu_0 g) + (n-r)
$$

Choose $(\lambda_1, \mu_1) \in \mathscr{C}_1$ and $(\lambda_2, \mu_2) \in \mathscr{C}_2$ such that (λ_1, μ_1) and (λ_2, μ_2) lie in $\mathscr{B}_\varepsilon$. (Note: We can make this choice W.L.O.G., since the signature of the forms is constant in each component) Then,

$$
\begin{aligned}
|\operatorname{sgn}(\lambda_1, \mu_1) - \operatorname{sgn}(\lambda_2, \mu_2)| &= |\operatorname{sgn}(\lambda_1, \mu_1) - \operatorname{sgn}((\lambda_0, \mu_0)) + \operatorname{sgn}(\lambda_0, \mu_0) - \operatorname{sgn}(\lambda_2, \mu_2)| \\
&\leq |\operatorname{sgn}(\lambda_1, \mu_1) - \operatorname{sgn}((\lambda_0, \mu_0))| + |\operatorname{sgn}(\lambda_0, \mu_0) - \operatorname{sgn}(\lambda_2, \mu_2)| \\
&\leq n - r + n - r \\
&= 2(n-r)
\end{aligned}
$$

This finishes the proof of the Proposition. □

2.3 Quadratic Forms over an Infinite Field.

Proposition 2.3.1. *Let f, g be quadratic forms in n variables over any infinite field $\mathbb{K}$ with* $\operatorname{char}(\mathbb{K}) \neq 2$ *, such that every form in the $\mathbb{K}$–pencil generated by f, g is singular.*

Then f, g have a common nontrivial singular zero over $\mathbb{K}$.

Proof. Let r denote the maximum of the ranks of all the forms in the $\mathbb{K}$–pencil. Since all the forms in the $\mathbb{K}$–pencil are singular, $r < n$. W.L.O.G., we may assume that f has rank r and by a change of variables over $\mathbb{K}$, we can put it into the form

$$f = a_1 X_1^2 + \cdots + a_r X_r^2,$$

where $a_i \neq 0$ for $1 \leq i \leq r$, and

$$g = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} X_i X_j.$$

Since $\mathrm{char}(\mathbb{K}) \neq 2$, we can write the symmetric matrix corresponding to $\lambda f + \mu g$,

$$\begin{array}{c} \\ r \\ \\ n-r \end{array} \begin{array}{c} \begin{array}{cc} \quad\quad r \quad\quad & \quad n-r \end{array} \\ \left(\begin{array}{ccc|c} \lambda a_1 + \mu b_{11} & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda a_r + \mu b_{rr} & \\ \hline & 0 & & \mu B \end{array}\right) \end{array}$$

where B is an $(n-r)\times(n-r)$ submatrix whose entries are b_{ij}, for $r+1 \leq i, j \leq n$. Since $\mathbb{K}$ is an infinite field and every form in the $\mathbb{K}$–pencil is singular, $P(\lambda, \mu) = \det(\lambda f + \mu g) \equiv 0$. Note that the coefficient of $\lambda^{r+1}\mu^{n-r}$ in $P(\lambda, \mu)$ is $a_1 \cdot a_2 \cdots a_r \cdot \det(B)$, which is zero and hence $\det(B) = 0$. Thus we can find a nontrivial $\vec{v} = (v_{r+1}, \ldots, v_n) \in \mathbb{K}^{n-r}$ such that $v^{\,t}Bv = 0$, *i.e*, $\vec{v}$ is a nontrivial zero of the quadratic form corresponding to matrix B, which is $g\big|_{X_i=0:1\leq i\leq r}$. We can extend $\vec{v}$ to $(0, \ldots, 0, v_{r+1}, \ldots, v_n) \in \mathbb{K}^n$ to get a nontrivial common zero of f and g over $\mathbb{K}$. □

Lemma 2.3.2. *A nonzero polynomial in m variables defined over an infinite field $\mathbb{F}$ is nonzero at infinitely many points in $\mathbb{F}^m$.*

Proof. Let $\mathbb{P}(X_1,\ldots,X_m)$ be any nonzero polynomial with coefficient in $\mathbb{F}$. Suppose that $\mathbb{P}$ is nonzero at only finitely many points $\xi_1,\ldots,\xi_k$ in $\mathbb{F}^m$. Let $P_1,\ldots,P_k$ be nonzero linear polynomials over $\mathbb{F}$ such that $P_i(\xi_i)=0$ for $1\le i\le k$.

Let $\mathbb{P}'=\prod_{i=1}^{k} P_i$. Then we get that $\mathbb{P}\cdot\mathbb{P}'$ is a nonzero polynomial over $\mathbb{F}$ that vanishes everywhere in $\mathbb{F}^m$. This is a contradiction because $\mathbb{F}$ is an infinite field. Therefore, we may conclude that $\mathbb{P}$ is nonzero at infinitely many points in $\mathbb{F}^m$. This completes the proof of Lemma 2.3.2. □

Lemma 2.3.3. *Let f be a quadratic form over an infinite field $\mathbb{F}$ in $n\ge 3$ variables such that* $\mathrm{rank}(f)\ge 3$*, and has a nonsingular zero in $\mathbb{F}^n$. Then f has infinitely many nonsingular zeros in $\mathbb{F}^n$ that avoid any given proper linear subspace of $\mathbb{F}^n$.*

Proof. We are given that $f=f(X_1,\ldots,X_n)$ has a nonsingular zero in $\mathbb{F}^n$ and $\mathrm{rank}(f)\ge 3$. After a nonsingular linear transformation, we may assume that $\vec{e_1}=(1,0,\ldots,0)$ is that zero, and f can we rewritten in the form

$$f=X_1\left(\sum_{i=2}^{n} b_iX_i\right)+q(X_2,\ldots,X_n). \tag{2.9}$$

Note that,

$$\frac{\partial f}{\partial X_i}(\vec{e_1})=\begin{cases}0, & \text{if } i=1\\ b_i, & \text{if } i\ge 2\end{cases}$$

Since $\vec{e_1}$ is a nonsingular zero, at least one of the b_i's is nonzero. W.L.O.G., let $b_2\ne 0$. Using another nonsingular linear transformation, we can assume that

$$f=X_1X_2+q'(X_2,\ldots,X_n). \tag{2.10}$$

Note that by Lemma 2.1.6,

$$\mathrm{rank}\left(f|_{\{X_2=0\}}\right)=\mathrm{rank}(q'(0,X_3,\ldots,X_n))\ge 1. \tag{2.11}$$

Let W be a linear subspace of $\mathbb{F}^n$ such that $\dim(W) = n-1$. Then

$$W = \{(X_1,\ldots,X_n) \in \mathbb{F}^n \,|\, L(X_1,\ldots,X_n) = 0\}\},$$

for some nonzero linear form over $\mathbb{F}$ denoted by

$$L(X_1,\ldots,X_n) = \sum_{i=1}^{n} c_i X_i, \text{ where not all } c_i\text{'s are zero.}$$

Case 1: Suppose that $\vec{e}_1 \in W$. We will show that there exist infinitely many nonsingular zeros of f that do not lie in W. Since $\vec{e}_1 \in W$, $c_1 = 0$ in L, and $L = c_2X_2 + \cdots + c_nX_n$ is a nonzero linear form over $\mathbb{F}$. This implies that at least one of the $c_i's$ is nonzero for $2 \le i \le n$. Taking $X_2 = 1$ in L gives us the following polynomial

$$P(X_3,\ldots,X_n) = c_2 + \sum_{i=3}^{n} c_i X_i. \tag{2.12}$$

$P(X_3,\ldots,X_n)$ is a nonzero polynomial over $\mathbb{F}$ because at least one of the $c_i's$ is nonzero for $2 \le i \le n$. Let $S(P) = \{(a_3,\ldots,a_n) \in \mathbb{F}^{n-2} \big| P(a_3,\ldots,a_n) \neq 0\}$. By Lemma 2.3.2, we get that $S(P)$ is an infinite set in $\mathbb{F}^{n-2}$. Note that for any two distinct choices $(a_3^{(1)},\ldots,a_n^{(1)})$ and $(a_3^{(2)},\ldots,a_n^{(2)})$ of vectors in S, the vectors $(1,a_3^{(1)},0,\ldots,0)$ and $(1,a_3^{(2)},0,\ldots,0)$ are linearly independent.

Using (2.10), for a particular choice of $(a_3,\ldots,a_n) \in S(P)$, we take

$$X_1 = -q'(1,a_3,\ldots,a_n).$$

Then $\vec{\alpha} = (-q'(1,a_3,\ldots,a_n),1,a_3,\ldots,a_n)$ is a zero of f in $\mathbb{F}^n$ such that

$$\frac{\partial f}{\partial X_1}(\vec{\alpha}) = 1 \neq 0.$$

i.e, α is a nonsingular zero of f. Note that $\alpha \notin W$ by construction. Since $S(P)$ is an infinite set in $\mathbb{F}^{n-2}$, we get infinitely many choices for $\vec{\alpha} \in \mathbb{F}^n$.

Case 2: Suppose that $\vec{e_1} \notin W$. This implies that $c_1 \neq 0$. W.L.O.G., let $c_1 = 1$ and

$$L = X_1 + c_2X_2 + \cdots + c_nX_n. \tag{2.13}$$

Using (2.10) and (2.13), we define the following quadratic form over $\mathbb{F}$:

$$h(X_2, \ldots, X_n) = -q'(X_2, \ldots, X_n) + c_2X_2^2 + c_3X_2X_3 + \cdots + c_nX_2X_n. \tag{2.14}$$

By Equation (2.11),

$$\operatorname{rank}(h|_{X_2=0}) = \operatorname{rank}(-q'(0, X_3, \ldots, X_n)) \geq 1.$$

This implies that h is a nonzero quadratic form over $\mathbb{F}$.

Lemma 2.3.2 implies that X_2h is nonzero at infinitely many points in $\mathbb{F}^{n-1}$. Therefore, h is nonzero at infinitely many points in $\mathbb{F}^{n-1}$ such that $X_2 \neq 0$.

Let $S(h, X_2) = \{(a_2, \ldots, a_n) \in \mathbb{F}^{n-1} \big| h(a_2, \ldots, a_n) \neq 0, a_2 \neq 0\}$ denote the infinite set of all the points in $\mathbb{F}^{n-1}$ such that X_2h is nonzero. Using equation 2.10 for any point $(a_2, \ldots, a_n)$ in S, we may take

$$X_1 = a_1 = \frac{-q'(a_2, a_3, \ldots, a_n)}{a_2}.$$

Then $\vec{\alpha} = \left(\frac{-q'(a_2, a_3, \ldots, a_n)}{a_2}, a_2, a_3, \ldots, a_n\right)$ is a nontrivial zero of f in $\mathbb{F}^n$ such that

$$\frac{\partial f}{\partial X_1}(\vec{\alpha}) = a_2 \neq 0.$$

i.e, $\vec{\alpha}$ is a nonsingular zero of f. Since the set $S(h, X_2)$ is infinite, there are infinitely many choices for $\vec{\alpha} \in \mathbb{F}^n$. Also note that

$$\begin{aligned}
L(\alpha) &= a_1 + c_2a_2 + \cdots + c_na_n \\
&= \frac{-q'(a_2, a_3, \ldots, a_n)}{a_2} + c_2a_2 + \cdots + c_na_n \\
&= \frac{-q'(a_2, a_3, \ldots, a_n) + c_2a_2^2 + \cdots + c_na_2a_n}{a_2} \\
&= \frac{h(a_2, \ldots, a_n)}{a_2} \neq 0
\end{aligned}$$

This implies that $\alpha \notin W$.

Therefore, we have shown that there are infinitely many nonsingular zeros of f in $\mathbb{F}^n$ that avoid W.

$\square$

Lemma 2.3.4. *Let f and g be a pair quadratic forms in $n \geq 5$ variables over an infinite field $\mathbb{F}$ such that there exists a form in the $\mathbb{F}$−pencil that contains at least two hyperbolic planes and has rank at least 5. Suppose that f and g have a nonsingular common zero in $\mathbb{F}^n$. Then f and g have infinitely many nonsingular zeros in $\mathbb{F}^n$.*

Proof. W.L.O.G., we may choose f to be that form in the pencil that contains at least two hyperbolic planes and has rank at least 5. By the hypothesis we know that f and g have a nonsingular common zero in $\mathbb{F}^n$. Therefore, after a nonsingular linear change of variables we can rewrite f and g as

$$f = X_1X_2 + q_1(X_2,\ldots,X_n),$$

$$g = X_1X_3 + q_2(X_2,\ldots,X_n).$$

Since f is a quadratic form over $\mathbb{F}$ that contains at least two hyperbolic planes and $\text{rank}(f) \geq 5$, we get that

$$f|_{\{X_2=0\}} = q_1(0,X_3,\ldots,X_n)$$

is isotropic over $\mathbb{F}$ and that $\text{rank}(f|_{\{X_2=0\}}) \geq 3$. By Lemma 2.3.3, $f|_{\{X_2=0\}}$ has infinitely many nonsingular zeros such that $X_3 \neq 0$. Let $Z(f|_{\{X_2=0\}},X_3)$ denote that infinite set of nonsingular zeros, *i.e*,

$$Z(f|_{\{X_2=0\}},X_3) = \{(a_3,\ldots,a_n) \in \mathbb{F}^{n-2} \big| a_3 \neq 0, (a_3,\ldots,a_n) \text{is a nonsingular zero of} f|_{\{X_2=0\}}\}.$$

For any point $(a_3,\ldots,a_n)$ in $Z(f|_{\{X_2=0\}},X_3)$, we may take $X_1 = a_1 = \dfrac{-q_2(0,a_3,\ldots,a_n)}{a_3}$. Then $\vec{\alpha} = (a_1,0,a_3,\ldots,a_n)$ is a nontrivial common zero of f and g over $\mathbb{F}$.

Note that

$$f = X_1X_2 + a_{22}X_2^2 + X_2(b_{23}X_3 + \cdots + b_{2n}X_n) + q_1'(X_3,\ldots,X_n),$$

$$f\big|_{\{X_2=0\}} = q_1'(X_3,\ldots,X_n),$$

where $q_1'(X_3,\dots,X_n)$ is a quadratic form over $\mathbb{F}$.

Hence, for $i \geq 3$

$$\frac{\partial f}{\partial X_i} = b_{2i}X_2 + \frac{\partial q_1'}{\partial X_i}$$
$$\frac{\partial f}{\partial X_i}\Big|_{\{X_2=0\}} = \frac{\partial q_1'}{\partial X_i} = \frac{\partial (f|_{\{X_2=0\}})}{\partial X_i}.$$

Therefore, we can make the following observations about the partial derivatives of f and g.:

1. For $i \geq 3$,

$$\frac{\partial f}{\partial X_i}(\vec{\alpha}) = \frac{\partial f}{\partial X_i}\Big|_{\{X_2=0\}}(a_3,\dots,a_n)$$
$$= \frac{\partial (f|_{\{X_2=0\}})}{\partial X_i}(a_3,\dots,a_n).$$

   Since $(a_3,\dots,a_n)$ is a nonsingular zero of $f|_{\{X_2=0\}}$, we get that $\frac{\partial f}{\partial X_i}(\vec{\alpha})$ is nonzero for at least one $i \geq 3$.

2. $\frac{\partial f}{\partial X_1}(\alpha) = 0$ and $\frac{\partial g}{\partial X_1}(\vec{\alpha}) = a_3 \neq 0$.

Therefore, the jacobian matrix shown below has full rank.

$$\begin{bmatrix} \frac{\partial f}{\partial X_1}(\vec{\alpha}) = 0 & \frac{\partial f}{\partial X_2}(\vec{\alpha}) & \frac{\partial f}{\partial X_3}(\vec{\alpha}) & \cdots & \frac{\partial f}{\partial X_n}(\vec{\alpha}) \\ \frac{\partial g}{\partial X_1}(\vec{\alpha}) = a_3 & \frac{\partial g}{\partial X_2}(\vec{\alpha}) & \frac{\partial g}{\partial X_3}(\vec{\alpha}) & \cdots & \frac{\partial g}{\partial X_n}(\vec{\alpha}) \end{bmatrix}.$$

This implies that $\vec{\alpha}$ is a nonsingular common zero of f and g. Note that the set $Z(f|_{\{X_2=0\}}, X_3)$ defined above is infinite and any distinct vector $(a_3,\dots,a_n) \in Z$ gives us a distinct common nonsingular zero $\vec{\alpha} \in \mathbb{F}^n$ of the quadratic forms f and g. Therefore, f and g have infinitely many common nonsingular zeros in $\mathbb{F}^n$.

□

Lemma 2.3.5 (Lemma 3.2 in [10]). *Let* $f = X_1A(X_2,\ldots,X_n) - B(X_2,\ldots,X_n)$*, and* $g(X_1,\ldots,X_n)$ *be homogeneous forms over an infinite field* $\mathbb{K}$ *of degrees* d*, and* e*, respectively with* $A \neq 0$*. Assume that* f *does not divide* g*, and* f *is irreducible. Then there exists a* $\mathbb{K}$*-rational zero of* f *which is not a zero of* g*.*

Proof. Assume that every zero $\mathbb{K}$–rational of f is a zero of g. Then

$$(A(X_2,\ldots,X_n))^e g(X_1,\ldots,X_n) = g(X_1A(X_2,\ldots,X_n),\ldots,X_nA(X_2,\ldots,X_n)) \qquad (*)$$
$$= g(B(X_2,\ldots,X_n), X_2A(X_2,\ldots,X_n),\ldots,X_nA(X_2,\ldots,X_n)) \mod f.$$

Define

$$h(X_2,\ldots,X_n) = g(B(X_2,\ldots,X_n), X_2A(X_2,\ldots,X_n),\ldots,X_nA(X_2,\ldots,X_n)).$$

h is a homogeneous form of degree de.

1. For all $\vec{a} = (a_2,\ldots,a_n) \in \mathbb{K}^{n-1}$ such that $A(\vec{a}) \neq 0$, we have

$$\begin{aligned} h(\vec{a}) &= g(B(\vec{a}), a_2A(\vec{a}),\ldots,a_nA(\vec{a})) \\ &= (A(\vec{a}))^e g\left(\frac{B(\vec{a})}{A(\vec{a})}, a_2,\ldots,a_n\right) \\ &= 0, \end{aligned}$$

   since $\left(\frac{B(\vec{a})}{A(\vec{a})}, a_2,\ldots,a_n\right)$ is a zero of f and thus also a zero of g.

2. For all $\vec{a} = (a_2,\ldots,a_n) \in \mathbb{K}^{n-1}$ such that $A(\vec{a}) = 0$, we have

$$\begin{aligned} h(\vec{a}) &= g(B(\vec{a}), a_2A(\vec{a}),\ldots,a_nA(\vec{a})) \\ &= g(B(\vec{a}), 0,\ldots,0) = 0, \end{aligned}$$

   since $(a, 0,\ldots,0)$ is a zero of f for all $a \in \mathbb{K}$, and hence a zero of g. We have shown that h vanishes on all of $\mathbb{K}^{n-1}$, and since $\mathbb{K}$ is an infinite field, h must be the zero polynomial. By (*), we see that f divides $A^e g$. But $\gcd(A, f) = 1$, since f is irreducible and $\deg(A) < \deg f$. Thus f must divide g.

$\square$

2.4 Approximation Theorems over an Arbitrary Number Field.

Proposition 2.4.1 and its proof is a generalization of [9, Theorem 1.2, page 467].

Proposition 2.4.1. *(Weak Approximation Theorem) Let $\mathbb{K}$ be a field, $|\quad|_1, \ldots, |\quad|_s$ nontrivial independent absolute values on $\mathbb{K}$, and $\mathbb{K}_1, \ldots, \mathbb{K}_s$ represent the completions of $\mathbb{K}$ with respect to $|\quad|_1, \ldots, |\quad|_s$, respectively. Let $x_i \in \mathbb{K}_i$ and $\varepsilon > 0$. Then there exists $x \in \mathbb{K}$ such that*

$$|x - x_i|_i < \varepsilon,$$

for all i.

Proof. Let us first consider $|\quad|_1$ and $|\quad|_s$. By the hypothesis, we can find $\alpha \in \mathbb{K}$ such that $|\alpha|_1 < 1$ and $|\alpha|_s \geq 1$. Note that $\alpha \neq 0$. Similarly, we can find $\beta \in \mathbb{K}$ such that $|\frac{\ }{-}_1 \geq 1$ and $|\frac{\ }{-}_s < 1$. Let $y = \dfrac{\beta}{\alpha}$. Then $|y|_1 > 1$ and $|y|_s < 1$.

Claim A. *For each i, there exists $\gamma_i \in \mathbb{K}$ such that $|\gamma_i|_i > 1$ and $|\gamma_i|_j < 1$ for all $j \neq i$.*

We will first show that there exists $z_1 \in \mathbb{K}$ such that

$$|\gamma_1|_1 > 1 \text{ and } |\gamma_1|_j < 1,\ j \neq 1.$$

We have already proved this when $s = 2$. Suppose that we have found $z_1 \in \mathbb{K}$ such that

$$|z_1|_1 > 1 \text{ and } |z_1|_j < 1,\ j = 2, \ldots, s-1.$$

1. If $|z_1|_s \leq 1$, then $|z_1^n y|_s < 1$ for any n, and there exists $N \in \mathbb{N}$ such that for $j = 2, \ldots, s-1$,

   $$|z_1^n y|_j < 1$$

   for all $n \geq N$.

   Then for $\gamma_1 = z_1^N y$,

$$|\gamma_1|_1 = |z_1^N y|_1 = |z_1|_1^N |y|_1 > 1,$$

and

$$|\gamma_1|_j < 1, j \neq 1.$$

2. If $|z_1|_s > 1$, then the sequence

$$t_n = \frac{z_1^n}{1 + z_1^n}$$

tends to 1 w.r.t $| \quad |_1, | \quad |_s$ and tends to 0 w.r.t $| \quad |_j$ for $j = 2, \ldots, s-1$.

Hence, for $j = 2, \ldots, s-1$, there exists $n_0 \in \mathbb{N}$ such that

$$|t_n y|_j < 1,$$

for all $n \geq n_0$.

For $j = s$, $|y|_s < 1$, and $|t_n|_s \to 1$ as $n \to \infty$. Therefore, there exists $n_s \in \mathbb{N}$ such that $|t_n y|_s < 1$ for all $n \geq n_s$.

For $j = 1$, $|y|_1 > 1$, and $|t_n|_1 \to 1$ as $n \to \infty$. Therefore,there exists $n_1 \in \mathbb{N}$ such that $|t_n y|_1 > 1$ for all $n \geq n_1$.

Choose $N \geq \max\{n_0, n_1, n_s\}$, and let $\gamma_1 = t_N y$. Then

$$|\gamma_1|_1 > 1,$$

and

$$|\gamma_1|_j < 1, j \neq 1.$$

A similar proof works for all $2 \leq i \leq s$.

This completes the proof of the claim above.

Since $\mathbb{K}$ is dense in $\mathbb{K}_i$ for all $i \in [s]$, for $\varepsilon > 0$, we can find $y_i \in \mathbb{K}$ such that

$$|y_i - x_i|_i < \frac{\varepsilon}{2}$$

for all i. Let $m = \max\{|y_i|_j, i, j \in [s]\}$.

For each $i \in [s]$, note that the sequence

$$\lim_{n\to\infty} \frac{\gamma_i^n}{1+\gamma_i^n} = 1$$

w.r.t $|\quad|_i$, and

$$\lim_{n\to\infty} \frac{\gamma_i^n}{1+\gamma_i^n} = 0$$

w.r.t $|\quad|_j$, $j \in [s], j \neq i$.

More precisely, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\left|\frac{\gamma_i^n}{1+\gamma_i^n} - 1\right|_i < \frac{\varepsilon}{4m},$$

and

$$\left|\frac{\gamma_i^n}{1+\gamma_i^n}\right|_j < \frac{\varepsilon}{4m(s-1)}, \; j \neq i,$$

for all $n \geq N_0$, and for all $i, j \in [s]$.

For each i, let $\Gamma_i = \dfrac{\gamma_i^{N_0}}{1+\gamma_i^{N_0}}$.

We define

$$x = \sum_{j=1}^{s} y_j \Gamma_j.$$

Note that

$$
\begin{aligned}
|x-y_i|_i &= \left|\sum_{j=1}^{s} y_j\Gamma_j - y_i\right|_i \\
&\le \left|\sum_{j=1, j\neq i}^{s} y_j\Gamma_j\right|_i + |y_i\Gamma_i - y_i|_i \\
&\le \sum_{j=1, j\neq i}^{s} \left|y_j\right|_i \left|\Gamma_j\right|_i + |y_i|_i|\Gamma_i - 1|_i \\
&\le (s-1)m\frac{\varepsilon}{4m(s-1)} + m\frac{\varepsilon}{4m} = \frac{\varepsilon}{2}.
\end{aligned}
$$

So for $\varepsilon > 0$,

$$
\begin{aligned}
|x-x_i|_i &= |x-y_i+y_i-x_i|_i \\
&\le |x-y_i|_i + |y_i-x_i|_i \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

This finishes the proof of the Proposition. □

The statement and proof of Proposition 2.4.2 is a generalization of [2, Lemma 2.8, page 62] to an arbitary complete field.

Proposition 2.4.2. *Let $\mathbb{K}$ be a complete field under a nontrivial absolute value denoted by $|\ \ |$. Let f be an isotropic nonsingular quadratic form in $n \geq 3$ variables over a field $\mathbb{K}$, and let*

$$L(X) = l_1X_1 + \cdots + l_nX_n$$

be a nonzero linear form over $\mathbb{K}$. Let $\vec{b} \in \mathbb{K}^n$ be a nontrivial zero of f. Then for each neighborhood U of $\vec{b}$, there exists a nontrivial $\vec{c} \in U$ such that $f(\vec{c}) = 0$ and $L(\vec{c}) \neq 0$.

Proof. By Proposition 2.2 in [9], page 470, we know that any two norms on $\mathbb{K}^n$, compatible with the absolute value on $\mathbb{K}$, are equivalent. So using the given absolute value on $\mathbb{K}$, we define the following norm on $\mathbb{K}^n$.

$$\begin{aligned} | \quad | : \mathbb{K}^n &\to \mathbb{R} \\ |(v_1, \ldots, v_n)| &\mapsto \sqrt{|v_1|^2 + \cdots + |v_n|^2} \end{aligned}$$

We may suppose W.L.O.G. that $\vec{b} = (1, 0, \ldots, 0)$ and after a linear transformation on the variables $X_2, \ldots, X_n$, that f can be rewritten as

$$f(X) = a_{12}X_1X_2 + f(0, X_2, \ldots, X_n)$$

with $a_{12} \neq 0$. Note that

$$f(0, X_2, \ldots, X_n) = X_2(a_{22}X_2 + \cdots + a_{2n}X_n) + g(X_3, \ldots, X_n),$$

where g is a quadratic form in $n-2$ variables. Now under the following nonsingular linear transformation

$$\begin{aligned} &X_1 \to X_1 + \frac{a_{22}}{a_{12}}X_2 + \cdots + \frac{a_{2n}}{a_{12}}X_n \\ &X_i \to X_i \quad ; i \neq 1, \end{aligned}$$

we can rewrite f as

$$f(X) = a_{12}X_1X_2 + g(X_3, \ldots, X_n)$$

Note that $\vec{b}$ stays the same under this transformation.

Now consider the given linear form

$$L(X) = l_1X_1 + \cdots + l_nX_n.$$

If $l_1 \neq 0$, then we may take $\vec{c} = \vec{b}$. Now we assume that $l_1 = 0$.

Case 1) Suppose one of $l_3, \ldots, l_n$ is nonzero. Choose $d_3, \ldots, d_n \in \mathbb{K}$ such that

$$l_3d_3 + \cdots + l_nd_n \neq 0. \tag{2.15}$$

Then for any $\lambda \in \mathbb{K}$, the point $\vec{b}_\lambda$ with coordinates

$$\vec{b}_\lambda = (1, \frac{-l^2 g(d_3, \ldots, d_n)}{a_{12}}, \lambda d_3, \ldots, \lambda d_n) \tag{2.16}$$

is a nontrivial zero of f.

(i) If $l_2 = 0$, then $L(\vec{b}_\lambda) \neq 0$ for all $\lambda \in \mathbb{K}^\times$

(ii) If $l_2 \neq 0$, then $L(\vec{b}_\lambda) = 0$ for at most 2 values of l.

Let $B_\varepsilon(\vec{b})$ be any open neighborhood of $\vec{b}$ of radius $\varepsilon > 0$. Then we may choose $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$, and

$$|\lambda| < \frac{\varepsilon}{\sqrt{\left|\frac{g(d_3, \ldots, d_n)}{a_{12}}\right|^2 + |d_3|^2 + \cdots + |d_n|^2}},$$

and $L(b_\lambda) \neq 0$.

Note that since not all d_is are zero,

$$\sqrt{\left|\frac{g(d_3, \ldots, d_n)}{a_{12}}\right|^2 + |d_3|^2 + \cdots + |d_n|^2}$$

is a nonzero element of $\mathbb{R}$.

For the above choice of l

$$\begin{aligned}
|\vec{b} - \vec{b}_\lambda| &= |l|\sqrt{|l|^2\left|\frac{g(d_3, \ldots, d_n)}{a_{12}}\right|^2 + |d_3|^2 + \cdots + |d_n|^2} \\
&\leq |l|\sqrt{\left|\frac{g(d_3, \ldots, d_n)}{a_{12}}\right|^2 + |d_3|^2 + \cdots + |d_n|^2} \\
&< \frac{\varepsilon}{\sqrt{\left|\frac{g(d_3, \ldots, d_n)}{a_{12}}\right|^2 + |d_3|^2 + \cdots + |d_n|^2}} \sqrt{\left|\frac{g(d_3, \ldots, d_n)}{a_{12}}\right|^2 + |d_3|^2 + \cdots + |d_n|^2} \\
&= \varepsilon
\end{aligned}$$

This shows that $b_\lambda \in B_\varepsilon(\vec{b})$ such that $f(\vec{b}_\lambda) = 0$ and $L(\vec{b}_\lambda) \neq 0$.

Case 2) Suppose $l_3 = \ldots = l_n = 0$. Then $l_2 \neq 0$. Since $\text{rank}(f) = n \geq 3$, we get that $\text{rank}(g) \geq 1$. Therefore, we can choose $d_3, \ldots, d_n \in \mathbb{K}$ such that $g(d_3, \ldots, d_n) \neq 0$. Then for $\lambda \neq 0$ and $\vec{b}_\lambda$ as defined in equation (2.16), $f(\vec{b}_\lambda) = 0$ and $L(\vec{b}_\lambda) \neq 0$. Let $B_\varepsilon(\vec{b})$ be any open neighborhood of $\vec{b}$ of radius $\varepsilon > 0$. By an argument similar to the one in Case 1, we can choose $\lambda \neq 0$ small enough such that $\vec{b}_\lambda \in B_\varepsilon(\vec{b})$.

This finishes the proof of the Proposition.

□

The statement and proof of Proposition 2.4.3 is a generalization of [2, Lemma 9.1, page 89] to an arbitary field with characteristic not 2.

Proposition 2.4.3. *Let $\mathbb{K}$ be a field with characteristic not 2. Let $f(x)$ be an isotropic quadratic form over $\mathbb{K}$ in $n \geq 3$ variables, $|\ \ |_1, \ldots, |\ \ |_s$ nontrivial independent absolute values on $\mathbb{K}$, and $\mathbb{K}_1, \ldots, \mathbb{K}_s$ represent the completions of $\mathbb{K}$ with respect to $|\ \ |_1, \ldots, |\ \ |_s$, respectively. Let $\varepsilon > 0$ and $\vec{b}_i \in \mathbb{K}_i^n$ be given with $f(b_i) = 0$, then there exists $\vec{b} \in \mathbb{K}^n$ such that $f(\vec{b}) = 0$ and*

$$|\vec{b} - \vec{b}_i|_i < \varepsilon,$$

for all i.

Proof. By the hypothesis there exists a nontrivial $\vec{c} \in \mathbb{K}^n$ such that $f(\vec{c}) = 0$. Let $B_f(\vec{u}, \vec{v})$ be the bilinear form corresponding to f.

$$2B_f(\vec{u}, \vec{v}) = f(\vec{u} + \vec{v}) - f(\vec{u}) - f(\vec{v})$$

Case 1) Suppose that $B_f(\vec{c}, \vec{b}_i) \neq 0$ for $1 \leq i \leq s$.

By Proposition 2.4.1 and continuity, we can choose $\vec{d} \in \mathbb{K}^n$ such that $\vec{d}$ is arbitrarily close to b_i for all i, and $B_f(\vec{c}, \vec{d}) \neq 0$.

We want to choose $\lambda \in \mathbb{K}$ such that

$$f(\lambda \vec{c} + \vec{d}) = 0.$$

Let

$$\lambda = \frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})}.$$

Then

$$\begin{aligned} f\left(\frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})}\vec{c}+\vec{d}\right) &= 2B_f\left(\frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})}\vec{c},\vec{d}\right)+\left(\frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})}\right)^2 f(\vec{c})+f(\vec{d}) \\ &= 2\frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})}B_f(\vec{c},\vec{d})+f(\vec{d}) \\ &= -f(\vec{d})+f(\vec{d}) = 0. \end{aligned}$$

Note that $\frac{-f(\vec{x})}{2B_f(\vec{c},\vec{x})}$ is continuous at $\vec{b}_i$ for all i. Given ε, there exists $\delta_i > 0$ such that if

$$|\vec{d}-\vec{b}_i|_i < \delta_i, \tag{2.17}$$

then

$$\begin{aligned} \left|\frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})} - \frac{-f(\vec{b}_i)}{2B_f(\vec{c},\vec{b}_i)}\right|_i &< \varepsilon \\ \left|\frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})} - 0\right|_i &< \varepsilon \\ |l|_i &< \varepsilon. \end{aligned}$$

This implies that as

$$\vec{d} \to \vec{b}_i \text{ w.r.t} \mid \quad |_i, \tag{2.18}$$

$$l = \frac{-f(\vec{d})}{2B_f(\vec{c},\vec{d})} \to \frac{-f(\vec{b}_i)}{2B_f(\vec{c},\vec{b}_i)} = 0. \tag{2.19}$$

Let $\delta = \min\{\delta_i | i \in [s]\}$.

On replacing δ_i by δ in (2.17), the limit in (2.18) and (2.19) can be achieved simultaneously in $\mathbb{K}_i$ for all $i \in [s]$.

Hence, we get that

$$\lambda\vec{c} + \vec{d} \to \vec{b_i},$$

w.r.t $|\quad|_i$ for all $i \in [s]$. To complete the proof of Case 1, we take $\vec{b} = \lambda\vec{c} + \vec{d}$.

Case 2) Suppose that $B_f(\vec{c}, \vec{b_k}) = 0$ for some $1 \le k \le n$. Then by Proposition 2.4.2, we can find $\vec{b}'_k$ arbitrarily close to $\vec{b_k}$ such that $f(b'_k) = 0$, and $B_f(\vec{c}, \vec{b}'_k) \neq 0$.

Then

$$b''_i = \begin{cases} b_i & if\, B_f(\vec{c}, \vec{b_i}) \neq 0, \\ b'_i & if\, B_f(\vec{c}, \vec{b_i}) = 0 \end{cases}$$

Note that for each $i \in [s]$, $B_f(\vec{c}, \vec{b}''_i) \neq 0$.

We replace $\vec{b_i}$ with $\vec{b}''_i$ in Case 1. As argued before, we can find l such that for each $i \in [s]$,

$$\lambda\vec{c} + \vec{d} \to \vec{b}''_i$$

w.r.t $|\quad|_i$, and $f(\lambda\vec{c} + \vec{d}) = 0$. To complete the proof of Case 2, we take $\vec{b} = \lambda\vec{c} + \vec{d}$.

This completes the proof of the Proposition.

□

2.5 Quadratic Forms over a Number Field and its Completions.

Notation. Below is a list of notation and terminology associated that is used extensively in this section:

- $\mathbb{K}$ will denote a number field.
- Ω is the set of all places on $\mathbb{K}$. Ω contains all the archimedean and nonarchimedean absolute values on $\mathbb{K}$ upto equivalence. We often use the word *'infinte prime'* to refer to an archimedean valuation and *'finte prime'* to refer to a nonarchimedean valuation on $\mathbb{K}$.

- If $\mathfrak{p} \in \Omega$, then $\mathbb{K}_\mathfrak{p}$ denotes the completion of $\mathbb{K}$ with respect to $\mathfrak{p}$.

- For archimedean places (or infinite primes) $\mathfrak{p}$, $\mathbb{K}_\mathfrak{p}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}_\mathfrak{p}$ is isomorphic to $\mathbb{R}$, then $\mathbb{K}_\mathfrak{p}$ is a called real completion of $\mathbb{K}$, $\mathfrak{p}$ is called a real place on $\mathbb{K}$ and the corresponding isomorphism $\theta_\mathfrak{p} : \mathbb{K}_\mathfrak{p} \to \mathbb{R}$ is called an ordering on $\mathbb{K}_\mathfrak{p}$.

- For nonarchimedean places (or finite primes) $\mathfrak{p}$, $\mathbb{K}_\mathfrak{p}$ is a local field, that is, *c.d.v. field* with a finite residue field, and $\nu_\mathfrak{p}$ denotes the corresponding discrete valuation on $\mathbb{K}$.

Proposition 2.5.1. *Let f and g be nonsingular quadratic forms in at least 9 variables over $\mathbb{K}$ such that every form in the $\mathbb{K}$–pencil generated by f and g has rank at least 5. Then there exists a nonsingular form in the $\mathbb{K}$– pencil that contains at least 3 hyperbolic planes over $\mathbb{K}$.*

Proof. Let $\mathbb{K}_\mathfrak{p}$ be a real completion of $\mathbb{K}$, and $\mathscr{C} = \{(\lambda,\mu) \big| \lambda,\mu \in \mathbb{R}, \lambda^2+\mu^2 = 1\}$. Consider the signature map

$$\begin{aligned} \mathrm{sgn} : \mathscr{C} &\to \mathbb{Z} \\ (\lambda,\mu) &\mapsto \mathrm{sgn}(\lambda f + \mu g) \end{aligned}$$

Note that image of sgn is contained in $[-n,n]$.
Assume that no nonsingular form $\lambda f + \mu g$ in the $\mathbb{K}_\mathfrak{p}$–pencil contains 3 hyperbolic planes. Then the signature of any nonsingular form, in the $\mathbb{K}_\mathfrak{p}$–pencil, has absolute value at least $(n-4)$. Let $\mathscr{C}_i$, $1 \le i \le t$, denote all the distinct connected intervals in $\mathscr{C} - \mathscr{S}$, where

$$\mathscr{S} = \{(\lambda,\mu) \in \mathscr{C} : \lambda f + \mu g \text{ is singular}\}.$$

Since sgn is an odd function, there will be two adjacent connected components on $\mathscr{C}$ where the signature jumps from being positive to negative or vice versa. Therefore, there must be a jump of at least $2(n-4)$ for the signature as (λ,μ) varies

on $\mathscr{C}$. By Proposition 2.2.10, we know that such a jump happens only when (λ, μ) passes through a point in $\mathscr{S}$, and the jump is bounded above by twice the nullity of the associated singular form. Let $\lambda_0 f + \mu_0 g$ be that singular form in the $\mathbb{K}_\mathfrak{p}$–pencil and let $r = \text{rank}(\lambda_0 f + \mu_0 g)$. Then the jump in the signature as we pass through (λ_0, μ_0) is bounded above by $2(n-r)$.
Therefore,

$$2(n-4) \leq 2(n-r)$$

$$-4 \leq -r$$

$$r \leq 4,$$

which is a contradiction since the rank of every form in the pencil $\mathbb{K}$–pencil is at least 5 and by Corollary 2.1.16, we know that the rank of any form in the $\mathbb{K}_\mathfrak{p}$– pencil is at least 5. Hence there exists a $(\lambda_\mathfrak{p}, \mu_\mathfrak{p}) \in \mathscr{C}$ such that $\lambda_\mathfrak{p} f + \mu_\mathfrak{p} g$ is nonsingular and contains 3 hyperbolic planes. Note that $(\lambda_\mathfrak{p}, \mu_\mathfrak{p})$ lies in a connected interval of $\mathscr{C} - \mathscr{S}$. Since there are only finitely many real completions of $\mathbb{K}$, by using Proposition 20, we can choose $\lambda_1, \mu_1 \in \mathbb{K}$ such that they are arbitrarily close to $\lambda_\mathfrak{p}, \mu_\mathfrak{p}$, and (λ_1, μ_1) avoids the points in $\mathscr{S}$ for each real completion $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$. This implies $\lambda_1 f + \mu_1 g$ is a nonsingular quadratic form in the $\mathbb{K}$–pencil such that

$$\text{sgn}(\lambda_1 f + \mu_1 g) = \text{sgn}(\lambda_\mathfrak{p} f + \mu_\mathfrak{p} g),$$

for each real completion $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$. Therefore, it contains 3 hyperbolic planes over $\mathbb{K}_\mathfrak{p}$ for each real completion of $\mathbb{K}$.

For the nonarchimedean places $\mathfrak{p}$, we know that any form over $\mathbb{K}_\mathfrak{p}$ in at least 5 variables variables is isotropic. Since $n \geq 9$, any nonsingular form in the $\mathbb{K}_\mathfrak{p}$ pencil automatically contains at least 3 hyperbolic planes.

Therefore, $\lambda_1 f + \mu_1 g$ contains 3 hyperbolic planes over $\mathbb{K}_\mathfrak{p}$ for each place $\mathfrak{p}$ over $\mathbb{K}$ and hence, by the Hasse-Minkowski Theorem, we can conclude that $\lambda_1 f + \mu_1 g$ contains at least 3 hyperbolic planes over $\mathbb{K}$.

□

Let Ω be the set of all archimedean(real) and non-archimedean places $\mathbb{K}$.

Lemma 2.5.2. *Let $\mathfrak{p} \in \Omega$. Let f, g be nonsingular quadratic forms in at least 9 variables over $\mathbb{K}_\mathfrak{p}$. Assume that all the forms in the $\mathbb{K}_\mathfrak{p}$–pencil are of rank at least 5, and every form in the $\mathbb{K}_\mathfrak{p}$-pencil is indefinite for each real completion $\mathbb{K}_\mathfrak{p}$. Then f, g have a nonsingular common zero over $\mathbb{K}_\mathfrak{p}$ for every $\mathfrak{p} \in \Omega$.*

Proof.

1. First we consider the case when $\mathfrak{p} \in \Omega$ is non-archimedean. Since the number of variables is at least 9, by Demyanov's Theorem in [5] we know that there exists a nontrivial common zero of $f = 0$ and $g = 0$ over $\mathbb{K}_\mathfrak{p}$. Let $P_{0\mathfrak{p}}$ denote a nontrivial common zero of f, g over $\mathbb{K}_\mathfrak{p}$.

2. Let $\mathfrak{p} \in \Omega$ be archimedian such that $\mathbb{K}_\mathfrak{p}$ is a real completion of $\mathbb{K}$. By the hypothesis, we know that every form in the $\mathbb{K}_\mathfrak{p}$-pencil is indefinite, hence we can use Proposition 2.2.7(b) to conclude that f, g have nontrivial common zero over $\mathbb{K}_\mathfrak{p}$. Let $P_{0\mathfrak{p}}$ denote a nontrivial common zero of f, g over $\mathbb{K}_\mathfrak{p}$.

Suppose that $P_{0\mathfrak{p}}$ is singular *i.e*, the tangent hyperplanes to $f = 0$ and $g = 0$ at $P_{0\mathfrak{p}}$ are the same. By a nonsingular linear change of variables over $\mathbb{K}_\mathfrak{p}$, we can take $P_{0\mathfrak{p}} = (1, 0, \dots, 0)^t$, and rewrite f and g in the following form,

$$f = X_1 L_1 + f_0(X_2, \dots, X_n),$$

and

$$g = X_1 L_2 + g_0(X_2, \dots, X_n),$$

where L_1, L_2 are linearly dependent linear forms in the variables $X_2, \dots, X_n$.

Since L_1 and L_2 are linearly dependent, we can find a nonzero $\lambda \in \mathbb{K}_\mathfrak{p}$ such that

$$L_1 = \lambda L_2.$$

Since every form in the $\mathbb{K}_{\mathfrak{p}}$−pencil has rank at least 5,

$$\operatorname{rank}(f-\lambda g)=\operatorname{rank}(f_0-\lambda g_0)\geq 5.$$

W.L.O.G., we may replace g by $f-\lambda g$, and consider the following

$$f=X_1L_1+f_0(X_2,\ldots,X_n),$$

and

$$g=g(X_2,\ldots,X_n),$$

where L_1 is linear form in the variables $X_2,\ldots,X_n$, and $\operatorname{rank}(g)\geq 5$. As such we can find a nonsingular zero of g. Note that this zero only involves $X_2,\ldots,X_n$.

By Lemma 2.1.7, we know that all the nonsingular zeros of a quadratic form do not lie in hyperplane. So, we can find one such nonsingular zero $(u_2,\ldots,u_n)$ of g in $\mathbb{K}_{\mathfrak{p}}^{n-1}$ such that $L_1(u_2,\ldots,u_n)\neq 0$. Since $(u_2,\ldots,u_n)$ is a nonsingular zero of g, W.L.O.G., we may assume that

$$\frac{\partial(g)}{\partial X_2}(u_2,\ldots,u_n)\neq 0, \qquad (**)$$

Now we may choose

$$u_1=-\frac{f_0(u_2,\ldots,u_n)}{L_1(u_2,\ldots,u_n)},$$

and let $\vec{u}=(u_1,u_2,\ldots,u_n)^t$. $\vec{u}$ is a nontrivial common zero of f,g in $\mathbb{K}_{\mathfrak{p}}^n$. Consider the jacobian matrix of f and g w.r.t to $\vec{u}$.

$$\begin{bmatrix} \frac{\partial f}{\partial X_1}(\vec{u})=L_1(\vec{u}) & \frac{\partial f}{\partial X_2}(\vec{u}) & \ldots & \frac{\partial f}{\partial X_n}(\vec{u}) \\ \frac{\partial g}{\partial X_1}(\vec{u}) & \frac{\partial g}{\partial X_2}(\vec{u}) & \ldots & \frac{\partial g}{\partial X_n}(\vec{u}) \end{bmatrix}$$

$$\begin{bmatrix} L_2(\vec{u}) & \frac{\partial f}{\partial X_2}(\vec{u}) & \ldots & \frac{\partial g}{\partial X_n}(\vec{u}) \\ 0 & \frac{\partial g}{\partial X_2}(\vec{u}) & \ldots & \frac{\partial g}{\partial X_n}(\vec{u}) \end{bmatrix}$$

By (**) and the fact that $L_2(\vec{u}) \neq 0$, the first 2×2 minor in the above matrix is

$$L_2(\vec{u})\left(\frac{\partial g}{\partial X_2}(\vec{u})\right) \neq 0.$$

This implies that the jacobian matrix of f and g w.r.t to $\vec{u}$ has full rank and therefore, $\vec{u} = (u_1, u_2, \ldots, u_n)^t$ is a common nonsingular zero of f, g in in $\mathbb{K}_{\mathfrak{p}}^n$. As a result, the corresponding tangent hyperplanes to $f = 0$ and $g = 0$ w.r.t to $\vec{u}$ are also distinct.

□

3.1 Introduction

Proposition 3.1.1. *[8, Proposition 6.16, page 403]*
For any field $\mathbb{F}$,

$$ru(\mathbb{F}) \leq u_{\mathbb{F}}(r) \leq \frac{r(r+1)}{2}u(\mathbb{F}), \tag{3.1}$$

for any $r \geq 1$.

In particular for $r = 2$, we get that

$$2u(\mathbb{F}) \leq u_{\mathbb{F}}(r) \leq 3u(\mathbb{F}). \tag{3.2}$$

If $\mathbb{F}$ is a field such that $u(\mathbb{F}) = 4$, (*i.e.* any quadratic form over $\mathbb{F}$ in at least 5 variables is always isotropic), then for $r = 2$, Proposition 3.1.1 implies that any two quadratic forms in more than 12 variables over $\mathbb{F}$ have a nontrivial common zero. However, if $\mathbb{F}$ is a $\mathfrak{p}$–adic field ($u(\mathbb{F}) = 4$,[8, Theorem 2.12, page 158]), from the classical work of Dem'yanov [5] (and Birch-Lewis-Murphy [1]), we know that the following sharper result is true.

Theorem 3.1.2. *Over a* $\mathfrak{p}$*–adic field* $\mathbb{F}$, *any two quadratic forms in more that 8 varibales over* $\mathbb{F}$ *have a non trivial common zero over* $\mathbb{F}$ *i.e,*

$$u_{\mathbb{F}}(2) = 2u(\mathbb{F}) \tag{3.3}$$

In 1962, B.J, Birch, D.J. Lewis and T.G. Murphy gave an alternative proof of Theorem 3.1.2 in [1]. Their proof naturally extends to an analogous result over any complete discretely valued field of characteristic different from 2.

Theorem 3.1.3. *Over a complete discretely valued (c.d.v.) field* $\mathbb{F}$ *with characteristic not* 2*, and* $u_{\overline{\mathbb{F}}}(1) < \infty$*,*

$$u_{\mathbb{F}}(2) = 2u_{\overline{\mathbb{F}}}(2) \tag{3.4}$$

In this chapter we give a detailed proof of the Main Theorem (3.1.3) that generalizes the argument in [1].

However, before we proceed to the proof of Main Theorem (3.1.3), we will discuss some definitions, terminology, and concepts that are frequently used in the proof.

We begin by stating the basic terminology of fields with a nonarchimedean valuation.

Definition 3.1.4 (Discretely Valued Field)**.** A *discretely valued field* (or a *d.v.* field for short) is a field $\mathbb{F}$ equipped with a discrete valuation *i.e.,* a surjective map

$$\nu : \mathbb{F}^{\times} \to \mathbb{Z}$$

such that

1. $\nu(ab) = \nu(a) + \nu(b)$, for all $a, b \in \mathbb{F}^{\times}$; and

2. $\nu(a+b) \geq min\{\nu(a), \nu(b)\}$, for all $a, b \in \mathbb{F}^{\times}$.

3. To extend this map to $\mathbb{F}$, we take $\nu(0) = \infty$.

Definition 3.1.5 (Valuation Ring)**.** The *valuation ring* of $\mathbb{F}$ is the subring of $\mathbb{F}$ defined by

$$\mathcal{O} = \{x \in \mathbb{F} : \nu(x) \geq 0\}.$$

The *valuation ring* $\mathcal{O}$ of $\mathbb{F}$ has the following properties:

4. The quotient field of $\mathcal{O}$ is $\mathbb{F}$.

5. $\mathcal{O}$ has a unique maximal ideal

$$\mathfrak{p} = \{x \in \mathbb{F} : \nu(x) \geq 1\},$$

including 0. $\mathfrak{p}$ is generated by any element $\pi \in \mathbb{F}$ such that $\nu(\pi) = 1$. Such an element π is determined up to a unit in $\mathcal{O}$, and is called a *uniformizer* of $\mathcal{O}$ or of $\mathbb{F}$.

6. The group of units of the valuation ring $\mathcal{O}$ is given by

$$\begin{aligned} U = U(\mathcal{O}) &= \{x \in \mathcal{O} : x \notin \mathfrak{p}\} \\ &= \{x \in \mathbb{F}^\times : \nu(x) = 0\}, \end{aligned}$$

and every element $x \in \mathbb{F}^\times$ can be written uniquely in the form

$$x = \mu\pi^{\nu(x)},$$

where $\mu \in U$ and π is a fixed uniformizer.

7. The field $\overline{\mathbb{F}} := \mathcal{O}/\mathfrak{p}$ is called the *residue field* of $\mathcal{O}$ relative to the valuation ν, and the projection of $\mathcal{O}$ onto $\overline{\mathbb{F}}$ is expressed as

$$x \in \mathcal{O} \mapsto \overline{x} = x + \mathfrak{p}.$$

Let $(\mathbb{F}, \nu)$ be a *d.v. field*. For a fixed real number c greater than 1, we define

$$d(x,y) = c^{-\nu(x-y)}, \quad x, y \in \mathbb{F}. \tag{3.5}$$

This gives a metric on $\mathbb{F}$ relative to the discrete valuation ν.

Definition 3.1.6 (Complete Discretely Valued Fields)**.** The pair $(\mathbb{F}, \nu)$ is a *complete discretely valued field* or *c.d.v. field* if $\mathbb{F}$ is complete with respect to ν. In other words, every Cauchy sequence in $\mathbb{F}$ converges to a point $\mathbb{F}$ with respect to the metric defined in (3.5).

3.2 Proof of the Main Theorem over a *c.d.v.* Field

We first assume that $\mathbb{F}$ is an arbitrary field with $\mathrm{char}(\mathbb{F}) \neq 2$. We state the following lemma from [1] without proof.

Lemma 3.2.1. *[1, Lemma 3] Let f, g be any two quadratic forms in n-variables over $\mathbb{F}$. There is a polynomial $\mathscr{I}(f,g)$ in the coefficients of f and g such that for $a,b,c,d \in \mathbb{F}$ and a nonsingular linear transformation T,*

$$\mathscr{I}(af_T + bg_T, cf_T + dg_T) = (ad-bc)^{n(n-1)} \det(T)^{4(n-1)} \mathscr{I}(f,g).$$

Let M_f, M_g be the symmetric matrices associated with the forms f, g, respectively and let

$$P(x,y) = \det |xM_g - yM_f|.$$

If $P(x,y)$ is not identically zero, then

$$P(x,y) = \prod_{i=1}^{n} (\lambda_i x - \mu_i y)$$

where λ_i, μ_i are in the algebraic closure of $\mathbb{F}$, and are not all zero.

Then we take,

$$\mathscr{I}(f,g) = \prod_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)^2 \tag{3.6}$$

It can be verified that $\mathscr{I}(f,g)$ satisfies equation in Lemma 3.2.1.

We now suppose that $(\mathbb{F}, v)$ is a *c.d.v. field*.

If

$$f = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

is a quadratic form with coefficients $a_{ij} \in \mathcal{O}$, then

$$\bar{f} = \sum_{i,j=1}^{n} (a_{ij} + \mathfrak{p}) x_i x_j$$

is a quadratic form with coefficients $a_{ij} + \mathfrak{p} \in \overline{\mathbb{F}}$.

Definition 3.2.2 (Primitive Vector)**.** We say that a vector $(x_1, \ldots, x_n) \in \mathcal{O}^n$ is primitive if there exists at least one i such that $\nu(x_i) = 0$.

Lemma 3.2.3. *If f, g are quadratic forms in n variables over the valuation ring $\mathcal{O}$ and if $\bar{f}, \bar{g}$ have a nonsingular common zero in the residue field $\overline{\mathbb{F}} := \mathcal{O}/\mathfrak{p}$, then f, g have a common primitive zero in $\mathbb{F}$.*

Proof. We will show that a primitive common zero $\mathscr{X}$ exists in $\mathbb{F}$ by constructing a Cauchy sequence of common zeros modulo powers of $\mathfrak{p}$ converging to it. More precisely, we will construct a sequence $(\mathscr{X}^{(i)})$ of primitive vectors in $\mathcal{O}^n$ such that

(**i**) $f(\mathscr{X}^{(i)}) \equiv g(\mathscr{X}^{(i)}) \equiv 0 \mod \mathfrak{p}^i$

(**ii**) $\mathscr{X}^{(i)} \equiv \mathscr{X}^{(i+1)} \mod \mathfrak{p}^i$

If a sequence satisfying the above conditions exists, then Condition (**ii**) implies that it is a Cauchy sequence in $\mathcal{O}^n \subset \mathbb{F}^n$. Since $\mathbb{F}$ complete, this sequence will converge in $\mathbb{F}^n$. In particular, let $\mathscr{X} = \lim_{i \to \infty} \mathscr{X}^{(i)}$. Since $\mathbb{F}$ is complete, we get that $\mathscr{X}$ exists and (**ii**) implies that $\mathscr{X}$ is a primitive vector in $\mathbb{F}^n$.

Since f, g are continuous over $\mathcal{O}$, by (**i**) we get that

$$f(\mathscr{X}) = f(\lim_{i \to \infty} \mathscr{X}^{(i)}) = \lim_{i \to \infty} f(\mathscr{X}^{(i)}) = 0$$

Similarly, $g(\mathscr{X}) = 0$.

Therefore, it is enough to show that such a sequence exists. Let $\overline{\mathscr{X}^{(1)}}$ be any nonsingular common zero of $\bar{f}$ and $\bar{g}$ in $\overline{\mathbb{F}}$. We may choose $\mathscr{X}^{(1)}$ to be any inverse image of $\overline{\mathscr{X}^{(1)}}$ in $\mathcal{O}^n$. Suppose that we have constructed $\mathscr{X}^{(r)}$. In order to construct $\mathscr{X}^{(r+1)}$, we need to find $\mathcal{Y} = (y_1, \ldots, y_n) \in \mathcal{O}^n$ such that

$$\mathscr{X}^{(r+1)} = \mathscr{X}^{(r)} + \pi^r \mathcal{Y}.$$

Then,

$$\begin{aligned} f(\mathscr{X}^{(r+1)}) &= f(\mathscr{X}^{(r)} + \pi^r \mathcal{Y}) \\ &= f(\mathscr{X}^{(r)}) + \pi^r \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathscr{X}^{(r)}) \cdot y_i \right) + \text{higher powers of } \pi \end{aligned}$$

We want to choose $\mathcal{Y}$ such that

$$f(\mathscr{X}^{(r+1)}) \equiv g(\mathscr{X}^{(r+1)}) \equiv 0 \mod \mathfrak{p}^{r+1}$$

Because of condition (**i**), π^r divides

$$f(\mathscr{X}^{(r)}) + \pi^r \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathscr{X}^{(r)}) \cdot y_i \right) + \text{higher powers of } \pi$$

So we want to choose $\mathcal{Y} \in \mathcal{O}^n$ such that

$$\begin{aligned} \pi^{-r} f(\mathscr{X}^{(r)}) + \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathscr{X}^{(r)}) \cdot y_i \right) &\equiv \pi^{-r} g(\mathscr{X}^{(r)}) + \left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\mathscr{X}^{(r)}) \cdot y_i \right) \\ &\equiv 0 \mod \mathfrak{p}^i \end{aligned}$$

Note that,

$$\pi^{-r} f(\mathscr{X}^{(r)}) + \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathscr{X}^{(r)}) \cdot y_i \right) \equiv 0 \mod \mathfrak{p},$$

and

$$\pi^{-r} g(\mathscr{X}^{(r)}) + \left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\mathscr{X}^{(r)}) \cdot y_i \right) \equiv 0 \mod \mathfrak{p},$$

are linear equations in variables $y_1, \ldots, y_n$ over $\overline{\mathbb{F}}$. The coefficient matrix for this system of linear equations over $\overline{\mathbb{F}}$ is given by

$$\begin{bmatrix} \frac{\partial \bar{f}}{\partial x_1}\left(\overline{\mathscr{X}^{(r)}}\right) & \cdots & \frac{\partial \bar{f}}{\partial x_n}\left(\overline{\mathscr{X}^{(r)}}\right) \\ & & \\ \frac{\partial \bar{g}}{\partial x_1}\left(\overline{\mathscr{X}^{(r)}}\right) & \cdots & \frac{\partial \bar{g}}{\partial x_n}\left(\overline{\mathscr{X}^{(r)}}\right) \end{bmatrix} \tag{3.7}$$

We know that $\overline{\mathscr{X}^{(1)}}$ is a nonsingular common zero of $\bar{f}, \bar{g}$ over $\overline{\mathbb{F}}$. Since $\mathscr{X}^{(r)} \equiv \mathscr{X}^{(1)} \mod \pi$ by construction,

$$\frac{\partial f}{\partial x_i}(\mathscr{X}^{(r)}) \equiv \frac{\partial f}{\partial x_i}(\mathscr{X}^{(1)}) \mod \mathfrak{p}$$

and

$$\frac{\partial g}{\partial x_i}(\mathscr{X}^{(r)}) \equiv \frac{\partial g}{\partial x_i}(\mathscr{X}^{(1)}) \mod \mathfrak{p},$$

we get that $\overline{\mathscr{X}^{(r)}}$ is also a nonsingular zero of $\bar{f}, \bar{g}$ and hence, the vectors $\frac{\partial \bar{f}}{\partial x}(\overline{\mathscr{X}^{(r)}})$, $\frac{\partial \bar{g}}{\partial x}(\overline{\mathscr{X}^{(r)}})$ are linearly independent over $\overline{\mathbb{F}}$.

As a result, the row rank of the matrix in (3.7) corresponding to the above system of linear equations is 2. Since we have a system of two linear equations where the coefficient matrix has full row rank, it must have at least one solution in $\overline{\mathbb{F}}^n$. Hence, we can choose $\mathcal{Y} \in \mathcal{O}^n$ to be any inverse of that solution. This completes the proof of the lemma. □

For the rest of this section, we make the following assumptions on the field $\mathbb{F}$.

- $(\mathbb{F}, \nu)$ is a *c.d.v.* field with char$(\mathbb{F}) \neq 2$,
- The u−invariant of the residue field $\overline{\mathbb{F}}$ is finite *i.e*, $u(\overline{\mathbb{F}}) < \infty$.

Lemma 3.2.4. *Let f, g be a pair of quadratic forms in n variables over $\mathcal{O}$. Assume that $P(x,y) = \det|xM_g - yM_f|$ is not identically zero. Then $\mathscr{I}(f,g)$ as defined in (3.6) is an element of $\mathcal{O}$.*

Proof. Since $P(x,y) = \det|xM_g - yM_f|$ is not identically zero, we get that

$$P(x,y) = \prod_{i=1}^{n}(\lambda_i x - \mu_i y)$$

is a homogeneous polynomial of degree n over the algebraic closure of $\mathbb{F}$. This implies that λ_i and μ_i cannot be simultaneously zero for the same subscript i. Note that if λ_i (or μ_i) is zero for more than one i , then (3.6) implies that $\mathscr{I}(f,g) = 0$. So W.L.O.G., we may assume that at least $n-1$ $\lambda_i s$ and at least $n-1$ $\mu_i s$ are nonzero.

Case 1: Suppose that λ_i and μ_i are nonzero for each i, $1 \le i \le n$. Then we can rewrite

$$P(x,y) = y^n \prod_{i=1}^{n} (\lambda_i \frac{x}{y} - \mu_i)$$

Let $Z = \frac{x}{y}$, and let

$$\begin{aligned} P_1(Z) &= \prod_{i=1}^{n} (\lambda_i Z - \mu_i), \\ &= \alpha_n Z^n + \cdots + \alpha_0, \\ &= \alpha_n \prod_{i=1}^{n} (Z - t_i), \end{aligned} \tag{3.8}$$

where $P_1(Z)$ is polynomial of degree n with coefficients in $\mathcal{O}$, $\alpha_n = \prod_{i=1}^{n} \lambda_i \in \mathcal{O} - \{0\}$, and $t_i = \frac{\mu_i}{\lambda_i}$. Let $P_1'(Z)$ denote the derivative of P_1 with respect Z. By [9, Proposition 8.5, page 204], we get that the resultant of P_1, P_1' is

$$\mathbf{Res}(P_1, P_1') = (-1)^{n(n-1)/2} \alpha_n D(P_1), \tag{3.9}$$

where

$$D(P_1(Z)) = \alpha_n^{(2n-2)} \prod_{i<j}^{n} (t_i - t_j)^2 \tag{3.10}$$

By the definition of resultant in [9, page 200], $\mathbf{Res}(P_1, P_1')$ is the determinant of the matrix A_{2n-1} (3.11) whose entries are determined by the coefficients of P_1 and P_1'. This implies that $\mathbf{Res}(P_1, P_1')$ is also an element of $\mathcal{O}$.

$$\begin{pmatrix} \alpha_n & \alpha_{n-1} & \dots & \alpha_0 & & & & \\ & \alpha_n & \alpha_{n-1} & \dots & \alpha_0 & & & \\ & & & \dots & & & & \\ & & & & \alpha_n & \alpha_{n-1} & \dots & \alpha_0 \\ \alpha_n' & \alpha_{n-1}' & \dots & \alpha_0' & & & & \\ & \alpha_n' & \alpha_{n-1}' & \dots & \alpha_0' & & & \\ & & & \dots & & & & \\ & & & & \alpha_n' & \alpha_{n-1}' & \dots & \alpha_0' \end{pmatrix} \tag{3.11}$$

The matrix A_{2n-1} in (3.11) is a $(n+(n-1))\times(n+(n-1))$ matrix over $\mathcal{O}$ where the blank spaces are supposed to be filled with zeros. Note that the first column of A_{2n-1} is divisible by α_n because $\alpha_n' = n\alpha_n$. Therefore, $\mathbf{Res}(P_1, P_1')$ is also divisible by α_n. This implies that $\alpha_n^{-1}\mathbf{Res}(P_1, P_1') \in \mathcal{O}$. Using this fact in (3.9) we get that $D(P_1) \in \mathcal{O}$. It the follows by (3.10) that

$$\prod_{i<j}(t_i - t_j)^2 \in \mathcal{O},$$

and since $\alpha_n = \prod_{i=1}^n \lambda_i \in \mathcal{O}$, we get that

$$\alpha_n^2 \prod_{i<j}^n (t_i - t_j)^2 \in \mathcal{O}$$

Next we note that

$$\begin{aligned}\alpha_n^2 \prod_{i<j}^n (t_i - t_j)^2 &= \left(\prod_{i=1}^n \lambda_i\right)^2 \prod_{i<j}^n \left(\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j}\right)^2 \\ &= \prod_{i<j}^n (\lambda_i \mu_j - \lambda_j \mu_i)^2 = \mathscr{I}(f,g).\end{aligned}$$

Therefore, we have shown that $\mathscr{I}(f,g)$ is an element of $\mathcal{O}$.

Case 2: W.L.O.G., suppose that $\mu_1 = 0$, then it follows that $\lambda_1 \neq 0$, and $\mu_i \neq 0$ for $i > 1$.

$$\begin{aligned}P(x,y) &= \lambda_1 x \prod_{i=2}^n (\lambda_i x - \mu_i y) \\ &= x \prod_{i=2}^n (\lambda_1 \lambda_i x - \lambda_1 \mu_i y) \\ &= x \prod_{i=2}^n (\Lambda_i x - \Gamma_i y),\end{aligned} \tag{3.12}$$

where $\Lambda_i = \lambda_1 \lambda_i$, and $\Gamma_i = \lambda_1 \mu_i \neq 0$ for any i such that $2 \leq i \leq n$. If $\Lambda_i \neq 0$ for each i, then we can show that $\mathscr{I}(f,g)$ is an element of $\mathcal{O}$ by taking

$$P_1(Z) = \prod_{i=2}^n (\Lambda_i Z - \Gamma_i)$$

in Case 1.

If $\Lambda_i = 0$ for some $i \geq 2$, say $\Lambda_2 = 0$, then $\Lambda_i \neq 0$ for $i \geq 3$. We can repeat the above process and take

$$P_1(Z) = \prod_{i=3}^{n} (\Gamma_2 \Lambda_i Z - \Gamma_2 \Gamma_i)$$

in Case 1.

This completes the proof of the lemma. $\square$

Theorem 3.1.3. *Let f, g be a pair of quadratic forms over $\mathbb{F}$ in $n \geq 2u_{\overline{\mathbb{F}}}(2)+1$ variables, then they have a primitive common zero in $\mathbb{F}$.*

Proof. $\mathbb{F}$ is a *c.d.v.* field, char($\mathbb{F}$) $\neq 2$, and $u(\mathbb{F}) < \infty$. Let $\overline{\mathbb{F}}$ denote the residue field of $\mathbb{F}$. Without loss of generality, we may assume that the coefficients of f, g are in $\mathcal{O}$ (*i.e.* integers) and hence $\mathscr{I}(f,g)$ is an element of $\mathcal{O}$.

First we assume that $\mathscr{I}(f,g) \neq 0$. The proof consists of three steps.

Step 1: Define

$$A := \left\{ (f', g') = (\mu f_S + \lambda g_S, \mu' f_S + \lambda' g_S) \;\middle|\; \begin{array}{c} f', g' \in \mathcal{O}[x_1, \ldots, x_n] \\ S \text{ is a nonsingular transformation over } \mathbb{F}, \\ \mu, \mu', l, \lambda' \in \mathbb{F} \text{ so that} \\ \mu\lambda' - \lambda\mu' \neq 0 \text{ in } \mathbb{F} \end{array} \right\}$$

By Lemma 3.2.1, we see that $\mathscr{I}(f',g') \neq 0$, for any $(f',g') \in A$. Note that:

- $(f',g') \in A$ is equivalent to (f,g) in the sense that there is a 1-1 correspondence between the common zeros of (f',g') and (f,g).
- We can choose a pair for which $\nu\left[\mathscr{I}(f',g')\right]$ is minimal.

Assume without loss of generality that (f,g) is that pair to begin with.

Step 2: We claim that

(i) $O\left(\bar{f}, \bar{g}\right) \geq u_{\overline{\mathbb{F}}}(2) + 1$, and

(ii) If $\bar{h} = \bar{\mu}\bar{f} - \bar{\lambda}\bar{g}$ for $\bar{\mu}$, $\bar{\lambda} \in \overline{\mathbb{F}}$, not both zero, then $O\left(\bar{h}\right) \geq u\left(\overline{\mathbb{F}}\right) + 1$.

Proof of Claim (i): Let $O\left(\bar{f}, \bar{g}\right) = m$. Then there is a unimodular-transformation U so that $\bar{f}_U$ and $\bar{g}_U$ involve at most the variables $x_1, \ldots, x_m$. Define another linear transformation R by

$$\begin{aligned} R : \mathbb{F}[x_1, \ldots, x_n] &\to \mathbb{F}[x_1, \ldots, x_n] \\ x_i &\mapsto \pi x_i, \quad 1 \leq i \leq m \\ x_i &\mapsto x_i, \quad m < i \leq n \end{aligned}$$

Then $\left(\pi^{-1} f_{UR}, \pi^{-1} g_{UR}\right) \in A$, and

$$\begin{aligned} \nu\left[\mathscr{I}\left(\pi^{-1} f_{UR}, \pi^{-1} g_{UR}\right)\right] &= \nu\left[\left(\pi^{-1}\right)^{n(n-1)} \left(\det(R)\right)^{4(n-1)} \mathscr{I}(f, g)\right] \\ &= -2n(n-1) + 4m(n-1) + \nu[\mathscr{I}(f, g)] \\ &= (4m - 2n)(n-1) + \nu[\mathscr{I}(f, g)] \\ &\geq \nu[\mathscr{I}(f, g)], \end{aligned}$$

which can only happen if $4m - 2n \geq 0$. This implies that

$$2m \geq n \geq 2u_{\overline{\mathbb{F}}}(2) + 1.$$

As such, since m is an integer, we have that

$$m \geq u_{\overline{\mathbb{F}}}(2) + 1,$$

as claimed.

Proof of Claim (ii): Let $O\left(\bar{h}\right) = m$ and assume that $\bar{\lambda} \neq 0$ in $\overline{\mathbb{F}}$. Then there is a unimodular-transformation U so that $\bar{h}_U$ involves at most the variables $x_1, \ldots, x_m$. Define another linear transformation R by

$$\begin{aligned} R : \mathbb{F}[x_1, \ldots, x_n] &\to \mathbb{F}[x_1, \ldots, x_n] \\ x_i &\mapsto \pi x_i, \quad 1 \leq i \leq m \\ x_i &\mapsto x_i, \quad m < i \leq n \end{aligned}$$

Then $\left(f_{UR}, \pi^{-1}h_{UR}\right) \in A$, and

$$\begin{aligned}
\nu\left[\mathscr{I}\left(f_{UR}, \pi^{-1}h_{UR}\right)\right] &= \nu\left[\left(\pi^{-1}\right)^{n(n-1)}\left(\det(R)\right)^{4(n-1)}\mathscr{I}(f,h)\right] \\
&= -n(n-1) + 4m(n-1) + \nu[\mathscr{I}(f,h)] \\
&= (4m-n)(n-1) + \nu\left[(-l)^{n(n-1)}\mathscr{I}(f,g)\right] \\
&= (4m-n)(n-1) + \nu[\mathscr{I}(f,h)] \\
&\geq \nu[\mathscr{I}(f,g)]
\end{aligned}$$

which can only happen if $4m - n \geq 0$. Hence,

$$4m \geq n \geq 2u_{\overline{\mathbb{F}}}(2) + 1 \geq 4u(\overline{\mathbb{F}}) + 1,$$

because $u_{\overline{\mathbb{F}}}(2) \geq 2u(\overline{\mathbb{F}})$.

As such, since m is an integer, we have that

$$m \geq u(\overline{\mathbb{F}}) + 1,$$

as claimed.

We now proceed to the third step:

Step 3: By **C**laim (i), $\bar{f}, \bar{g}$ have a at least one common nontrivial zero in $\overline{\mathbb{F}}$. If one of these zeros is nonsingular, then by using Lemma 3.2.3, we have a primitive common zero for the pair f, g in $\mathbb{F}$.

If $\bar{f}$, $\bar{g}$ have no nonsingular common zero, then by Lemma 2.1.9, there exist a form $\bar{h} = \bar{\mu}\bar{f} - \bar{\lambda}\bar{g}$, which has singular zeros. Using **C**laim (ii) and Lemma 2.1.11, we get a contradiction.

Hence, f, g must have a primitive common zero in $\mathbb{F}$.

Finally, we assume that that $\mathscr{I}(f,g) = 0$.

Claim 3.2.5. *We can find a sequence $f^{(J)}, g^{(J)}$ of pairs of quadratic forms with $\mathscr{I}(f^{(J)}, g^{(J)})$ nonzero which converges to f, g over $\mathbb{F}$ as $J \to \infty$.*

Proof of Claim 3.2.5:

We define $f^{(J)}$, and $g^{(J)}$ such that

$$M_{f^{(J)}} = M_f + \pi^J I,$$

and

$$M_{g^{(J)}} = M_g + \pi^J D,$$

where I is the identity matrix and D is a diagonal matrix with all its diagonal entries distinct. Let d_i denote the diagonal entries of D.

Then

$$\mathscr{I}(f^{(J)}, g^{(J)}) = \mathscr{I}\left(f + \alpha \sum_{i=1}^{n} X_i^2, g + \alpha \sum_{i=1}^{n} d_i X_i^2\right) \tag{3.13}$$

is a polynomial in which α which is not identically zero because the coefficient of the highest power of α is

$$\mathscr{I}\left(\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} d_i X_i^2\right) = \prod_{i<j=1}^{n} (d_i - d_j) \neq 0.$$

Therefore, we can find a sufficient large J such that $\alpha = \pi^J$ is not a root of (3.13). This implies that for a sufficiently large J, $\mathscr{I}(f^{(J)}, g^{(J)}) \neq 0$ and therefore each pair of quadratic forms $f^{(J)}, g^{(J)}$ has a primitive zero. Let $\mathscr{X}^{(J)}$ denote that primitive zero.

This implies that

$$f(\mathscr{X}^{(J)}) \equiv 0(\mod \pi^J),$$

and

$$g(\mathscr{X}^{(J)}) \equiv 0(\mod \pi^J).$$

Then by Proposition 5.24, [7], Page 67, we get that there exists $\mathscr{X} \in \mathbb{F}^n$ such that $\mathscr{X}$ is primitive common zero of f and g aribitrarily close to $\mathscr{X}^{(J)}$ for all sufficiently large values of J.

This completes the proof of Theorem 3.1.3. □

CHAPTER 4. A SYSTEM OF TWO QUADRATIC FORMS IN $N \geq 11$ VARIABLES OVER A NUMBER FIELD

4.1 Introduction.

In the previous chapter, we looked at zeros of a system of two quadratic forms over a *c.d.v. field* with a finite class. In this chapter we will shift our main focus to a class of global fields called the *Number Fields* (i.e. finite extensions of $\mathbb{Q}$). However, before we discuss existence of rational zeros of a system of two quadratic forms over an arbitrary number field, we will take sometime to discuss the motivation behind the assumptions and techniques used in giving a proof the result stated above.

In 1959, an American-born British mathematician, Louis J. Mordell, known for pioneering research in number theory, proved the following theorem which states that if $n \geq 13$ and f, g are quadratic forms over $\mathbb{Q}$ that satisfy certain number theoretic conditions, then they have infinitely many common rational zeros:

Theorem 4.1.1 (Mordell). *Let $f(x) = f(X_1, \ldots, X_n)$ and $g(x) = g(X_1, \ldots, X_n)$ be two quadratic forms with rational coefficients, in n variables. Suppose that for real l, μ, non both zero, each form in the pencil is indefinite and has rank at least 5. If $n \geq 13$, we assume that at least one form in the pencil has the absolute value of its signature bounded above by $(n-10)$. Then $f(X) = g(X) = 0$ have infinitely many nontrivial common rational zeros.*

In his proof of Theorem 4.1.1, Mordell works with a quadratic form f over $\mathbb{Q}$ that is nonsingular in $n(\geq 13)$ variables,($i.e$, $rank(f) = n$) and the absolute value of the signature of f is at most three. After a nonsingular rational change of variables, f

can be rewritten as

$$f = \sum_{i=1}^{5} x_i x_{i+5} + f_1(x_{11}, x_{12}, x_{13}).$$

Note that if we set $x_i = 0$ for all i with $6 \leq i \leq 13$, then we automatically have a zero of f where the last 8 coordinates are zero, and we end up reducing g to a quadratic form g_1 in the 5 variables $x_1, \ldots, x_5$.
By the Hasse-Minkowski Theorem([8], Theorem 3.1, page 170), an indefinite rational quadratic form in at least 5 variables has nontrivial rational integer solutions. Then, it can be concluded that g_1 has nontrivial rational integer solutions provided g_1 is indefinite. Once we have a nontrivial integer solution of g_1, it can be naturally extended to a common integer solution of the pair f, g.

Using the ideas presented in Mordell's paper [11], Peter Swinnerton-Dyer, an English Mathematician showed in [14] that Theorem 4.1.1 holds for $n = 11$. In Mordell's paper [11], the reduction of g to g_1 resulted in an indefinite quadratic form in 5 variables, but a similar reduction in the case of a pair of forms in 11 variable results in a form g_1 in 4 variables. While it is known that any indefinite quadratic form in 5 variables has a nontrivial rational integer solution, there are many examples of indefinite quadratic forms in 4 variables which do not have nontrivial rational integer solutions. In this context, the crux of the argument presented by Swinnerton-Dyer in [14] is a method for reducing f in a way that ensures that the resulting 4–dimensional quadratic form g_1 is isotropic over $\mathbb{Q}$. The result proved in [14] is stated below:

Theorem 4.1.2. *(Swinnerton-Dyer) Let f, g be homogeneous quadratic forms in 11 variables defined over the rationals $\mathbb{Q}$. Suppose that for all real λ, μ, each form in the pencil $\lambda f + \mu g$ is indefinite and has rank at least 5. Then, f, g have a nontrivial common rational zero.*

The goal of this chapter is to discuss how the techniques used in [14] be generalized to system of two quadratic forms over an arbitrary number field with more than one independent archimedean absolute value associated with it. Over $\mathbb{Q}$, since there is only one archimedean absolute value, it is enough to reduce to a quadratic form g_1 in 4 variables which is indefinite with respect to this unique archimedean absolute value and has a nontrivial zero with respect to each $\mathfrak{p}-$adic absolute value. This sets the stage to use the Hasse-Mikowski Theorem to conclude the desired result. However, the major challenge over an arbitrary number field is that there can be more than one independent archimedean absolute value defined on it. In this case, in order to get to the point, where we can use the Hasse-Minkowski Theorem, we need to make sure that g_1 is indefinite with respect to each of these independent archimedean absolute values. To that end, in section 4.5, we give a self contained proof of the following generalization of theorem 4.1.2 to an arbitrary number field.

Theorem 4.1.3. *Let $\mathbb{K}$ be a number field with s distinct real places denoted by $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Let f, g be quadratic forms in at least 11 variables, defined over $\mathbb{K}$; Suppose that every form in the $\mathbb{K}-$pencil has rank at least 5 and if $s \geq 1$, suppose that every nonzero quadratic form $\lambda f + \mu g$ in the $\mathbb{K}_{\mathfrak{p}_i}-$pencil is indefinite for all $1 \leq i \leq s$. Then f, g have infinitely many nontrivial common zeros over $\mathbb{K}$.*

Throughout this chapter, we will adhere to the following notation and terminology (unless stated otherwise)

- $\mathbb{K}$ denotes a number field.
- Ω is the set of all places on $\mathbb{K}$. Ω contains all the archimedean and nonarchimedean absolute values on $\mathbb{K}$ upto equivalence. We often use the word *'infinte prime'* to refer to an archimedean place and *'finte prime'* to refer to a nonarchimedean place on $\mathbb{K}$.

- If $\mathfrak{p} \in \Omega$, then $\mathbb{K}_{\mathfrak{p}}$ denotes the completion of $\mathbb{K}$ with respect to $\mathfrak{p}$.
- For archimedean places (or infinite primes) $\mathfrak{p}$, $\mathbb{K}_{\mathfrak{p}}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}_{\mathfrak{p}}$ is isomorphic to $\mathbb{R}$, then $\mathbb{K}_{\mathfrak{p}}$ is a called real completion of $\mathbb{K}$, $\mathfrak{p}$ is called a real place on $\mathbb{K}$ and the corresponding isomorphism $\theta_{\mathfrak{p}} : \mathbb{K}_{\mathfrak{p}} \to \mathbb{R}$ is called an ordering on $\mathbb{K}_{\mathfrak{p}}$.
- For nonarchimedean places (or finite primes) $\mathfrak{p}$, $\mathbb{K}_{\mathfrak{p}}$ is a local field, that is, *c.d.v. field* with a finite residue class, and $\nu_{\mathfrak{p}}$ denotes the corresponding discrete valuation on $\mathbb{K}$.

4.2 A Result over Local Fields.

The next lemma is a generalization of Lemma 4 in [14] and is needed to ensure that in the final stage of reduction we get a 4–dimensional quadratic form that is isotropic. In this lemma, 0 is treated as a $\mathfrak{p}$–adic square so the set of squares over $\mathbb{K}_{\mathfrak{p}}$ can be treated as a closed set.

Proposition 4.2.1. *Let $\mathfrak{p}$ be a finite prime and let f, g be linearly independent quadratic forms in $n \geq 5$ variables defined over a $\mathfrak{p}$-adic field $\mathbb{K}_{\mathfrak{p}}$. Suppose that f is a nonsingular quadratic form such that*

$$f \cong \mathbb{H} \perp \mathbb{H} \perp h,$$

where h is a qudratic form of rank at least 1. Assume that $g'(\vec{v}) \in \mathbb{K}^2$ whenever $f'(\vec{v}) = 0$. Then there is a form of rank at most 1 in the pencil generated by f, g over $\mathbb{K}_{\mathfrak{p}}$.

Proof. Let L be any two-dimensional subspace of zeros of f defined over $\mathbb{K}$. Consider $g\big|_L : L \to \mathbb{K}$. Then $g\big|_L$ has rank at most 1 because any quadratic form of rank at least 2 represents non-squares over $\mathbb{K}_{\mathfrak{p}}$. Since $\operatorname{rank}(g\big|_L) \leq 1$, we get that $\operatorname{rad}(g\big|_L) \subset L$ has dimension at least 1. Note that any nontrivial element in the radical of $g\big|_L$ is a nontrivial common zero of f, g over $\mathbb{K}$.

Let L_0 be a two-dimensional subspace of zeros of f defined over $\mathbb{K}$, and let $\vec{u} \in \operatorname{rad}(g\big|_{L_0}) \subset L_0$. W.L.O.G., after a nonsingular linear change of variables, we can assume that $\vec{u} = \vec{e}_1$, is a nontrivial common zero of f, g over $\mathbb{K}_\mathfrak{p}$, and we can rewrite f as

$$f = X_1X_2 + X_3X_4 + h(X_5, \ldots, X_n),$$

where where h is a qudratic form of rank at least 1, and

$$g = X_1(b_2X_2 + \cdots + b_nX_n) + Q(X_2, \ldots, X_n),$$

where Q is a quadratic form. Let L_1 denote that two-dimensional space of zeros of f given by $X_2 = X_4 = \cdots = X_n = 0$. Then

$$g\big|_{L_1} = b_3X_1X_3 + \beta X_3^2 = X_3(b_3X_1 + \beta X_3).$$

Since $g'\big|_{L_1}$ has rank at most 1, it follows that $b_3 = 0$. Similarly, by considering another two-dimesional subspace of zeros of f given $X_2 = X_3 = X_5 = \cdots = X_n = 0$, we can conclude that $b_4 = 0$. We can replace g by $-b_2 f + g$. This lets us assume that $b_2 = 0$. We now have

$$f = X_1X_2 + X_3X_4 + h(X_5, \ldots, X_n),$$

and

$$g = X_1(b_5X_5 + \cdots b_nX_n) + Q(X_2, \ldots, X_5).$$

Let $\vec{w} = (0, 0, -h(1, \ldots, 1), 1, 1, \ldots, 1)$, then $f(\vec{w}) = 0$ and $g\big|_{k\vec{e}_1 + k\vec{w}_1}$ has rank at most 1. This implies that $b_5 = \cdots = b_n = 0$. Therefore, we get that

$$f = X_1X_2 + X_3X_4 + h(X_5, \ldots, X_n),$$

and

$$g = Q(X_2, \ldots, X_n).$$

Thus $\vec{e}_1$ is a singular zero of the pair f, g.

We will now show that g has rank at most 1. To do this, suppose that the rank of g is at least 2. Thus, there exist $c_2, \ldots, c_n \in \mathbb{K}_\mathfrak{p}$ such that $Q(c_2 \ldots, c_n) = \alpha$, where $\alpha \notin \mathbb{K}^2$. Then $Q \perp \langle -\alpha \rangle$ is isotropic and has rank at least 3. Now we can complete the proof in the following way:

Since $Q \perp \langle -\alpha \rangle$ has rank at least 3, we get $(Q \perp \langle -\alpha \rangle) \nmid X_2 X_{n+1}$. Hence by Lemma 2.3.5, we can find a nonsingular zero $\vec{Z} = (z_2, \ldots, z_{n+1})$ of $Q \perp \langle -\alpha \rangle$ such that $z_2 \neq 0$ and $z_{n+1} \neq 0$.

Let $z_1 = \frac{-z_3 z_4 - h(z_5, \ldots, z_n)}{z_2}$. Then

$$f(z_1, \ldots, z_n) = 0.$$

and

$$\begin{aligned} g((z_2, \ldots, z_n) &= Q(z_2, \ldots, z_n) \\ &= \alpha z_{n+1}^2 \\ &\neq 0 \end{aligned}$$

where αz_{n+1}^2 is not a square, which is a contradiction to the hypothesis that $g(\vec{v})$ is a square whenever $f(\vec{v}) = 0$.. This implies that $\operatorname{rank}(g) \leq 1$.

□

In Proposition 4.2.1, when $n = 5$, the condition that kernel of f has dimension 1 *i.e*

$$f \cong \mathbb{H} \perp \mathbb{H} \perp aX_5^2, \quad a \neq 0,$$

over $\mathbb{K}_\mathfrak{p}$, is vital. The following example shows that without this condition Proposition 4.2.1 would be false for every prime $\mathfrak{p}$.

We first assume that $\mathfrak{p}$ is an odd prime and u is a non-square $\mathfrak{p}$–adic integer.

Let

$$\begin{aligned} f' &= X_1 X_2 - uX_3^2 + \mathfrak{p} X_4^2 - \mathfrak{p} u X_5^2 \\ g' &= (X_1 - X_2)^2 + \mathfrak{p} X_3^2 \end{aligned}$$

Note that the kernel of f' has dimension 3 and there is no form in the $\overline{\mathbb{Q}}_{\mathfrak{p}-}$pencil generated by f', g' that has rank 1. We will show that Proposition 4.2.1 fails in this case. Let $(a_1,\ldots,a_5)$ be a nontrivial zero of f defined over $\mathbb{Q}_{\mathfrak{p}}$. W.L.O.G., we may assume that each $a_i \in \mathbb{Z}_{\mathfrak{p}}$, and that one of them is a $\mathfrak{p}-$adic unit.

Case (1) If $\nu_{\mathfrak{p}}(a_1 - a_2) = 0$, then

$$g(a_1, a_2, a_3) = (a_1 - a_2)^2 + pa_3^2$$

is a square modulo $\mathfrak{p}$, and hence is a square in $\mathbb{Z}_{\mathfrak{p}}$.

Case (2) If $\nu_{\mathfrak{p}}(a_1 - a_2) \geq 1$, then we get the following subcases

a) $\nu_{\mathfrak{p}}(a_1) = \nu_{\mathfrak{p}}(a_2) = 0$ *i.e,* a_1 and a_2 are $\mathfrak{p}-$ adic units. Since $\nu_{\mathfrak{p}}(a_1 - a_2) \geq 1$,

$$\begin{aligned} a_1 - a_2 &\equiv 0 \mod (\mathfrak{p}) \\ a_1 &\equiv a_2 \mod (\mathfrak{p} \\ a_1 a_2 &\equiv a_2^2 \mod (\mathfrak{p}), \end{aligned}$$

i.e, $a_1 a_2$ is a square in $\mathbb{Z}_{\mathfrak{p}}$. Now consider the following quadratic form in 4 variables

$$h = X_1^2 - uX_2^2 + pX_3^2 - \mathfrak{p}uX_4^2.$$

Note that $(\sqrt{a_1 a_2}, a_3, a_4, a_5)$ is a nontrivial $\mathfrak{p}-$adic zero of h, which is a contradiction as h is anisotropic over $\mathbb{Q}_{\mathfrak{p}}$.

b) If $\nu_{\mathfrak{p}}(a) \geq 1$, then since $\nu_{\mathfrak{p}}(a_1 - a_2) \geq 1$, we get that $\nu_{\mathfrak{p}}(a_2) \geq 1$. This implies

that

$$
\begin{aligned}
a_1 a_2 &\equiv 0 \mod (\mathfrak{p}^2) \\
-u a_3^2 + \mathfrak{p} a_4^2 - u \mathfrak{p} a_5^2 &\equiv 0 \mod (\mathfrak{p}^2) \\
-u a_3^2 &\equiv 0 \mod (\mathfrak{p}) \\
a_3^2 &\equiv 0 \mod (\mathfrak{p}) \\
a_3 &\equiv 0 \mod (\mathfrak{p})
\end{aligned}
$$

Now once we have that, we get that

$$a_4^2 - u a_5^2 \equiv 0 \mod (\mathfrak{p})$$

Since u is a non-square unit in $\mathbb{Z}_{\mathfrak{p}}$, we get that $a_4 \equiv a_5 \equiv 0 \mod (\mathfrak{p})$. So we have shown that $\mathfrak{p}$ divides all the a_i's, which is a contradiction as at least one of the a_i's is a unit in $\mathbb{Z}_{\mathfrak{p}}$.

Now we assume that $\mathfrak{p} = 2$. Let

$$
\begin{aligned}
f' &= X_1 X_2 + X_3^2 + X_4^2 + X_5^2 \\
g' &= (X_1 - X_2)^2 + 128 X_3^2
\end{aligned}
$$

Note that the kernel of f' has dimension 3 and there is no form in the $\overline{\mathbb{Q}}_2$−pencil generated by f', g' that has rank 1. Let $(a_1, \ldots, a_5)$ be a nontrivial zero of f defined over $\mathbb{Q}_2$. W.L.O.G., we may assume that each $a_i \in \mathbb{Z}_2$, and that one of them is a 2−adic unit.

Case (1) If $\nu_2(a_1 - a_2) = 0$, then

$$g(a_1, a_2, a_3) = (a_1 - a_2)^2 + 128 a_3^2$$

is a square modulo 2, and hence is a square in $\mathbb{Q}_2$.

Case (2) If $\nu_2(a_1 - a_2) \geq 1$, then we get the following subcases:

a) Suppose $\nu_2(a_1) = \nu_2(a_2) = 0$ and $\nu_2(a_1 - a_2) \geq 3$. Then $a_1 \equiv a_2 \mod (8)$, and hence $a_1 a_2 \in \mathbb{Q}_2^2$.

As a result, we get that $(\sqrt{a_1 a_2}, a_3, a_4, a_5)$ is a nontrivial 2– adic zero of the quadratic form

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = \langle 1, 1, 1, 1 \rangle,$$

which is a contradiction.

b) Suppose $\nu_2(a_1) = \nu_2(a_2) = 0$ and $\nu_2(a_1 - a_2) = 1$ or 2. Then $\nu_2\left((a_1 - a_2)^2\right) =$ 2 or 4. We claim that $(a_1 - a_2)^2 + 128a_3^2 \in \mathbb{Q}_2^2$.

proof: Note that

$$(a_1 - a_2)^2 + 128a_3^2 = (a_1 - a_2)^2 \left(1 + \frac{128}{(a_1 - a_2)^2} a_3^2\right) \in \mathbb{Q}_2^2,$$

because $\nu_2\left(\frac{128}{(a_1 - a_2)^2} a_3^2\right) \geq 7 - 4 = 3$. Therefore, $(a_1 - a_2)^2 + 128a_3^2 \equiv (a_1 - a_2)^2 \mod (8)$ which proves the claim.

c) Now suppose that $\nu_2(a_1) \geq 1$. This implies that $\nu_2(a_2) \geq 1$. Then $4 \mid a_1 a_2$, and therefore

$$a_3^2 + a_4^2 + a_5^2 \equiv 0 \mod (4).$$

This implies that 2 must divide a_3, a_4, a_5 and hence $\nu_2(a_i) \geq 1$ for all $1 \leq i \leq 5$, which is a contradiction.

4.3 Some Results on $\mathbb{K}$–Rational Zeros

In this section we prove a Lemma that will be needed in the proof of the Main Theorem 4.1.3. Lemma 4.3.2 is a generalization of Lemma 3 in [14]. It deals with a special case which requires an argument that is different from the main line of proof of Theorem 4.1.3, and hence has been presented in this section in order to avoid any confusion. Proposition 4.3.1 is a result over real completion of $\mathbb{K}$ and is used in the proof of Lemma 4.3.2 to deal with the mutliple archimedean places associated with $\mathbb{K}$.

Proposition 4.3.1. *Under the conditions of the main theorem 4.1.3, if for each $i \in [s]$ there are $\mathbb{K}_{\mathfrak{p}_i}$–points on $f = 0$ that give either sign to g under the ordering $\theta_{\mathfrak{p}_i}$, then there exists a $\mathbb{K}$–rational point $\vec{w}$ on $f = 0$ that $g(\vec{w})$ has an arbitrarily given sign at each ordering of $\mathbb{K}$.*

Proof. For each $i \in [s]$, let $t_i \in \{-1, 1\}$. Let $\vec{v_i} \in \mathbb{K}^n_{\mathfrak{p}_i}$ such that $\vec{v_i} \neq (0, \ldots, 0)$, $f(\vec{v_i}) = 0$, and $g(\vec{v_i})t_i > 0$. Since g is continuous, for each $i \in [s]$, there exists $\varepsilon_i > 0$ such that if $\vec{v} \in \mathbb{K}^n_{\mathfrak{p}_i}$, and $|\vec{v} - \vec{v_i}|_i < \varepsilon_i$, then $g(\vec{v})t_i > 0$.

Let $\varepsilon = \min\{\varepsilon_i | i \in [s]\}$. Then by Proposition 2.4.3, there exists $\vec{w} \in \mathbb{K}^n$ such that

$$|\vec{w} - \vec{v_i}|_i < \varepsilon,$$

for all $i \in [s]$ and $f(\vec{w}) = 0$.

Hence, $\vec{w}$ is a $\mathbb{K}$–rational point such that $f(\vec{w}) = 0$ and $g(\vec{w})t_i > 0$, for each $i \in [s]$.

□

Proposition 4.3.2. *Let f and g be quadratic forms in at least 11 variables over $\mathbb{K}$. Assume that in the $\mathbb{K}$–pencil has rank at least 5, and assume that for each real completion of $\mathbb{K}_{\mathfrak{p}}$ of $\mathbb{K}$, every form in the $\mathbb{K}_{\mathfrak{p}}$–pencil is indefinite. Suppose that there exist $\lambda, \mu \in \mathbb{K}$, not both zero such that $\lambda f + \mu g$ has rank at most 6. Then f, g have a nontrivial common $\mathbb{K}$–rational zero.*

Proof. [Case 1.] Suppose that the $\mathbb{K}$–pencil generated by f, g contains a form of rank 5 or 6. By a $\mathbb{K}$–rational change of basis, we may assume that this form is f; and then by a $\mathbb{K}$–rational change of variables we can rewrite it as

$$f = f(X_1, \ldots, X_6).$$

We can assume that the coefficient of X_i^2 in g for all $7 \leq i \leq n$ is nonzero, because otherwise we will have a nontrivial common $\mathbb{K}$–rational zero of f, g. Let b_n be the coefficient of X_n^2 in g. For each ordering $\theta_{\mathfrak{p}_i}$, $i \in [s]$, by Proposition 2.2.7b, there are $\mathbb{K}_{\mathfrak{p}_i}$−points on $f = 0$ which give either sign to g. For each $i \in [s]$, let $\vec{P_i} \in \mathbb{K}_{\mathfrak{p}_i}^n$ denote a point such that $f(\vec{P_i}) = 0$, and $g(\vec{P_i})$ and $\theta_{\mathfrak{p}_i}(b_n)$ are opposite in sign. Hence by Proposition 4.3.1, there exists $\vec{P_0} \in \mathbb{K}^n$ such that $f(\vec{P_0}) = 0$ and $g(\vec{P_0})$ and b_n are opposite in sign w.r.t to $\theta_{\mathfrak{p}_i}$ for each $i \in [s]$.

By a $\mathbb{K}$−rational change of variables only on $X_1, \ldots, X_6$, we may assume that $\vec{P_0}$ lies on $X_1 = \cdots = X_5 = 0$. This implies that the coefficient of X_6^2 in f is zero. Let g_1 be form obtained from g by setting $X_1 = \cdots = X_5 = 0$. Note that g_1 is a quadratic form in the at least 6 variables $X_6, \ldots, X_n$ such that $g_1(\vec{e_n}) = b_n$ and $g_1(\vec{P_0}) = g(\vec{P_0})$ have opposite signs under each ordering $\theta_{\mathfrak{p}_i}$, $i \in [s]$. Hence, g_1 is an indefinite form in at least 6 variables with respect to to each ordering on $\mathbb{K}$, and thus by the Hasse-Minkowski Theorem has a nontrivial $\mathbb{K}$−rational zero. Let $(v_6, \ldots, v_n)$ denote a nontrivial $\mathbb{K}$−rational zero of g_1. Then $(0,0,0,0,0,v_6,\ldots,v_n)$ is a nontrivial common $\mathbb{K}$−rational zero of f and g.

□

4.4 Process of Splitting Off a Hyperbolic Plane.

In this section we assume that $f(X)$, $g(X)$ are nonsingular quadratic forms in variables over an infinite field $\mathbb{F}$ with characteristic not 2. We also assume that rank(f) ≥ 3, and that f and g are independent quadratic forms. By Lemma 2.2.8, we know that there are finitely many singular forms in the $\overline{\mathbb{F}}$−pencil generated by f and g. Suppose that the number of singular forms in the $\overline{\mathbb{F}}$−pencil is l. For $1 \leq j \leq l$, let $h_j(X)$ represent a singular form $\overline{\mathbb{F}}$−pencil generated by f, g.

Let $q(X)$ be the quadratic form whose symmetric matrix is given by $M_f M_g^{-1} M_f$.

Claim A. *The quadratic forms f and q are independent, and hence f does not divide q.*

Proof. Suppose that q and f are dependent as quadratic forms over $\mathbb{F}$. Note that f is irreducible as $\operatorname{rank}(f) \geq 3$. This implies that f and q have no nonconstant common factor. Therefore, if q and f are dependent, then q must be a constant multiple of f. In other words,

$$q(X) = Cf(X), \text{ where } C \in \mathbb{F}^{\times}$$

and hence,

$$\begin{aligned} & M_f M_g^{-1} M_f = C M_f \\ \Longrightarrow \quad & M_f M_g^{-1} = C I_n \\ \Longrightarrow \quad & M_f = C M_g \\ \Longrightarrow \quad & f(X) = C g(X), \end{aligned}$$

which is a contradiction as $f(X)$ and $g(X)$ are independent quadratic forms. $\square$

Recall. In Chapter 2, section 2.1, we gave a definition of a polar hyperplane (2.1.2) and a tangent hyperplane (2.1.3) to a quadratic form over a field $\mathbb{F}$. When $\operatorname{char}(\mathbb{F}) \neq 2$, we have the following representation for the polar hyperplane and/or tangent hyperplane to f at a vector in $\mathbb{F}^n$.

Observation 1. *Let* $\operatorname{char}(\mathbb{F}) \neq 2$, $f(X_1, \ldots, X_n)$ *a quadratic form in n variables over* $\mathbb{F}$, *and* $\vec{P} \in \mathbb{F}^n$. *Then the polar hyperplane to f at $\vec{P}$, denoted by* $\mathbb{H}_f^{\vec{P}}$, *is the set of vectors that satisfy the equation*

$$\vec{P}^t M_f \vec{X} = 0,$$

where $\vec{X} = (X_1 \cdots X_n)^t$.

Notation. $\mathbb{H}_f^{\vec{P}} : \vec{P}^t M_f \vec{X} = 0$

If $\vec{P}$ is an isotropic vector of $\mathbb{F}$, $\mathbb{H}_f^{\vec{P}}$ is called the tangent hyperplane to f at $\vec{P}$.

Notation. $\mathbb{T}_f^{\vec{P}} : \vec{P}^t M_f \vec{X} = 0$

Let $\vec{P} = (p_1, \ldots, p_n)^t$ and $f = \sum_{i,j=1}^{n} a_{ij} X_i X_j$. Now observe that

$$\frac{\partial f}{\partial X_j} = 2a_{jj}X_j + \sum_{i=1, i\neq j}^{n} a_{ij}X_i,$$

and

$$\begin{aligned}\frac{\partial f}{\partial X_i}(\vec{P}) &= 2a_{jj}p_j + \sum_{i=1, i\neq j}^{n} a_{ij}p_j \\ &= [p_1 \cdots p_n] \begin{bmatrix} a_{1j} \\ \vdots \\ 2a_{jj} \\ \vdots \\ a_{nj} \end{bmatrix} \\ &= 2\,[p_1 \cdots p_n] \begin{bmatrix} \frac{a_{1j}}{2} \\ \vdots \\ a_{jj} \\ \vdots \\ \frac{a_{nj}}{2} \end{bmatrix}\end{aligned} \tag{4.1}$$

Note that $\begin{bmatrix} \frac{a_{1j}}{2} \\ \vdots \\ a_{jj} \\ \vdots \\ \frac{a_{nj}}{2} \end{bmatrix}$ is the j–th column in the symmetric matrix M_f associated with f. It then follows that,

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\vec{P})X_i = 2\,[p_1 \cdots p_n]\, M_f \vec{X} \\ = 2\vec{P}^t M_f \vec{X} \tag{4.2}$$

By definition 2.1.2, $\mathbb{H}_f^{\vec{P}}$ is the kernel of the linear form

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\vec{P})X_i \tag{4.3}$$

Since char($\mathbb{F}$) $\neq 2$, equation (4.2) implies the kernel of the linear form (4.3) is the same as the kernel of $\vec{P}^t M_f \vec{X}$. Therefore, $\mathbb{H}_f^{\vec{P}} : \vec{P}^t M_f \vec{X} = 0$. If $\vec{P}$ is an isotropic vector of f, then $\mathbb{T}_f^{\vec{P}} : \vec{P}^t M_f \vec{X} = 0$.

Now we are in the position to describe the process of splitting off a hyperbolic plane in f. There are two main steps involved in this process.

Step 1. We want to choose a $\mathbb{F}-$rational point $\vec{P_1}$ such that it satisfies the following properties:

a) $f(\vec{P_1}) = 0$,

b) $g(\vec{P_1}) \neq 0$,

c) $q(\vec{P_1}) \neq 0$,

d) For $1 \leq j \leq l$, the polar hyperplane to $g = 0$ at $\vec{P_1}$, denoted by ξ_1, does not contain all the singular zeros of h_j, and

e) For $1 \leq j \leq l$, if $\dim\big(\text{rad}(h_j)\big) > 1$, then there exists a nonzero $(\vec{w}^j)' \in \text{rad}(h_j) \cap \xi_1$ such that it does not lie on the tangent hyperplane to $f = 0$ at $\vec{P_1}$ *i.e* $\vec{P_1}^t M_f (\vec{w}^j)' \neq 0$,

Step 2. For a fixed $\vec{P_1}$, we choose another $\mathbb{F}-$rational point $\vec{P_2}$ satisfying the following properties:

a) $f(\vec{P_2}) = 0$,

b) $\vec{P}_2$ does not lie on the tangent hyperplane to f at $\vec{P}_1$.

To complete Step 1, we first choose a nonzero $(\vec{w}^j) \in \text{rad}(h_j)$, for $1 \leq j \leq l$, Let $(\vec{w}^j)^{\,t} = (w_1^j, \ldots, w_n^j)$, and define

$$\mathscr{L}_j(X) = \sum_{i=1}^{n} w_i^j \frac{\partial g}{\partial X_i},$$

a linear form in the variables $X_1, \ldots, X_n$, for $1 \leq j \leq l$.

Now note that

- f is irreducible over $\mathbb{F}$, since f is a nonsingular quadratic form in $n \geq 3$ variables.
- f does not divide $gq\prod_{j=1}^{l}\mathscr{L}_j$, which is a homogeneous form over $\mathbb{F}$ of degree $4+l$,
- if $\dim(\text{rad}(h_j)) > 1$, then we choose any nonzero $(\vec{w}^j)' \in \text{rad}(h) \cap \xi_1$. Let

$$\mathscr{L}_j'(X) = \begin{cases} X^t M_f (\vec{w}^j)', \text{ if } \dim(\text{rad}(h_j)) > 1 \\ 1, \qquad\qquad \text{if } \dim(\text{rad}(h_j)) = 1 \end{cases}.$$

  f does not divide $gq\prod_{j=1}^{l}\mathscr{L}_j\mathscr{L}_j'$, which is a homogeneous form over $\mathbb{F}$ of degree at most $4+2l$.

Hence, by Lemma 2.3.5, there exists a $\mathbb{F}$−rational zero of f which is not a zero of gqL (or $gqLL'$, if $\dim(\text{rad}(h)) > 1$).

Let $\vec{P}_1$ denote a $\mathbb{F}$−rational zero such that

(a) $f(\vec{P}_1) = 0$,

(b) $g(\vec{P}_1) \neq 0$,

(c) $q(\vec{P_1}) \neq 0$,

(d) For $1 \leq j \leq l$, $\mathscr{L}_j(\vec{P_1}) = \sum_{i=1}^{n} w_i^j \frac{\partial g}{\partial X_i}(\vec{P_1}) \neq 0$, which implies that $(\vec{w}^j)$ does not lie on the polar hyperplane to g at $\vec{P_1}$.

(e) for $1 \leq j \leq l$, if $\dim\Big(\mathrm{rad}(h_j)\Big) > 1$, then $\mathscr{L}_j'(\vec{P_1}) = P_1^t M_f (\vec{w}^j)' \neq 0$. This implies that $(\vec{w}^j)'$ does lie on the tangent hyperplane to f at $\vec{P_1}$.

This completes step 1.

Before we go to Step 2, we will define some notation as well as dicuss some important consequences of Step 1.

Let $L_1(X) = \vec{P}_1^t M_f X$, and $L_2(X) = \vec{P}_1^t M_g X$.

Recall. The tangent hyperplane to f at $\vec{P_1}$ is denoted by

$$\mathbb{T}_f^{\vec{P_1}} : L_1 = 0,$$

and the polar hyperplane to g at $\vec{P_1}$ is denoted by the

$$\xi_1 : L_2 = 0.$$

As consequence of step 1,

1. ξ_1 and $\mathbb{T}_f^{\vec{P_1}}$ do not coincide.

   Proof. By (a) and (b), $\vec{P_1}$ lies on $\mathbb{T}_f^{\vec{P_1}}$ but it does not lie of ξ_1. □

2. the restriction of g to ξ_1 gives a nonsingular quadratic form of rank $n-1$.

   Proof. By (b) and Lemma 2.1.12, ξ_1 is not tangent to $g = 0$, and therefore by Lemma 2.1.6, $\mathrm{rank}(g\big|_{\xi_1}) = n-1$. □

3. the restriction of g to $\xi_1 : L_2 = 0$ and $\mathbb{T}_f^{\vec{P_1}} : L_1 = 0$ is a nonsingular quadratic form of rank $n-2$.

Proof. Let $\vec{P}_1$ be a nonsingular point on $f = 0$. We may suppose W.L.O.G. that $\vec{P}_1 = (1, 0, \ldots, 0)$ and after a linear transformation on the variables $X_2, \ldots, X_n$, that f can be rewritten as

$$f(X) = a_{12}X_1X_2 + f(0, X_2, \ldots, X_n)$$

with $a_{12} \neq 0$. Note that

$$f(0, X_2, \ldots, X_n) = X_2(a_{22}X_2 + \cdots + a_{2n}X_n) + f_1(X_3, \ldots, X_n),$$

where f_1 is a quadratic form in $n-2$ variables. Now under the following nonsingular linear transformation

$$X_1 + \frac{a_{22}}{a_{12}}X_2 + \cdots + \frac{a_{2n}}{a_{12}}X_n \to X_1$$
$$X_i \to X_i \quad ; i \neq 1,$$

we can rewrite f as

$$f(X) = a_{12}X_1X_2 + f_1(X_3, \ldots, X_n) \tag{4.4}$$

Note that $\vec{P}_1$ stays the same under this transformation and the tangent hyperplane to $f = 0$ at $\vec{P}_1$ is given by $X_2 = 0$.

By multiplying by a nonzero scalar if necessary, we can write

$$L_2 = X_1 - c_2X_2 - \cdots - c_nX_n. \tag{4.5}$$

Claim B. *We can rewrite g as*

$$g = b_{11}L_2^2 + g_1(X_2, \ldots, X_n),$$

where $b_{11} \neq 0$.

Proof. Since $\vec{P}_1$ is not a zero of g, the coefficient of X_1^2 in g must be be nonzero. Let $b_{11} \in \mathbb{F}^\times$ denote the coefficient of X_1^2 in g. Note that

$$g(X) = b_{11}\left(X_1^2 + \frac{b_{12}}{b_{11}}X_1X_2 + \cdots + \frac{b_{1n}}{b_{11}}X_1X_n) + g(0, X_2, \ldots, X_n\right) \qquad (*)$$
$$= b_{11}\left(X_1 + \frac{b_{12}}{2b_{11}}X_2 + \cdots + \frac{b_{1n}}{2b_{11}}X_n\right)^2 + g_1(X_2, \ldots, X_n)$$

Then the polar hyperplane to $g = 0$ at $\vec{P}_1$ is given by the kernel of

$$\sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(\vec{P}_1)X_i = 2b_{11}X_1 + 2b_{11}\frac{b_{12}}{2b_{11}}X_2 + \cdots + 2b_{11}\frac{b_{1n}}{2b_{11}}X_n$$
$$= 2b_{11}X_1 + b_{12}X_2 + \cdots + b_{1n}X_n$$

Dividing by $2b_{11}$, we get that the polar hyperplane to $g = 0$ at $\vec{P}_1$ is given by the kernel of the linear form

$$X_1 + \frac{b_{12}}{2b_{11}}X_2 + \cdots + \frac{b_{1n}}{2b_{11}}X_n.$$

By comparing the coefficients with L_2 in equation (4.5), we get that for all $2 \le i \le n$,

$$\frac{b_{1i}}{2b_{11}} = -c_i.$$

Using this in (*), we get that

$$g = b_{11}L_2^2 + g_1(X_2, \ldots, X_n),$$

□

Hence,

$$g\big|_{\xi_1 : L_2 = 0} = g_1(X_2, \ldots, X_n).$$

Suppose that $\text{rank}(g_1\big|_{\{\mathbb{T}_f^{\vec{P}_1} : X_2 = 0\}}) < n - 2$.

By Lemma 2.1.6, $\mathbb{T}_f^{\vec{P}_1}$ is tangent to g_1, and hence there exists a nonzero vector $(u_2,\dots,u_n) \in \mathbb{F}^{n-1}$ such that

$$g_1(u_2,\dots,u_n) = 0,$$

and the tangent hyperplane to g_1 at $(u_2,\dots,u_n)$ is

$$\mathbb{T}_f^{\vec{P}_1} : X_2 = 0,$$

i.e,

$$\frac{\partial g_1}{\partial X_i}(u_2,\dots,u_n) = \begin{cases} 1\,;\ i = 2, \\ 0\,;\ i \neq 2. \end{cases}$$

Now using equation (4.5), let $u_1 = c_2u_2 + \cdots + c_nu_n$, and $\vec{u} = (u_1, u_2, \dots, u_n)$. Then $L_2(\vec{u}) = 0$. Hence

$$g(\vec{u}) = b_{11}L_2^2(\vec{u}) + g_1(\vec{u}) = 0.$$

Note that

$$\begin{aligned} \frac{\partial g}{\partial X_i}(\vec{u}) &= 2L_2(\vec{u})\frac{\partial L_2}{\partial X_i}(\vec{u}) + \frac{\partial g_1}{\partial X_i}(\vec{u}) \\ &= 0 + \frac{\partial g_1}{\partial X_i}(u_2,\dots,u_n) \\ &= \begin{cases} 1\,;\ i = 2, \\ 0\,;\ i \neq 2 \end{cases} \end{aligned}$$

This implies that $\mathbb{T}_f^{\vec{P}_1} : X_2 = 0$ is tangent to g at $\vec{u}$.

Therefore,

$$
\begin{aligned}
\vec{u}^t M_g X &= \vec{P}_1^t M_f X \\
\implies \quad \vec{u}^t M_g &= \vec{P}_1^t M_f \\
\implies \quad \vec{u}^t &= \vec{P}_1^t M_f M_g^{-1} \\
\implies \quad g(\vec{u}) &= (\vec{P}_1^t M_f M_g^{-1}) M_g (\vec{P}_1^t M_f M_g^{-1})^t \\
\implies \quad g(\vec{u}) &= \vec{P}_1^t (M_f M_g^{-1} M_f) P_1 = q(\vec{P}_1) \neq 0, \text{ by step 1(c),}
\end{aligned}
$$

which is a contradiction. Therefore, $\mathbb{T}_f^{\vec{P}_1}$ is not tangent to g_1. By Lemma 2.1.6, $g_1\big|_{\{\mathbb{T}_f^{\vec{P}_1}:X_2=0\}}$ is a nonsingular quadratic form of rank $n-2$. □

4. For $1 \leq j \leq l$, if $\dim(\text{rad}(h_j)) > 1$, then by step 1(d) $\mathbb{T}_f^{\vec{P}_1}$ does not contain all those singular points of $h_j = 0$ which lie on ξ_1 *i.e*,

$$\text{rad}(h_j) \cap \xi_1 \not\subset \mathbb{T}_f^{\vec{P}_1}.$$

Let $\Gamma^{(j)} = \dim(\text{rad}(h_j))$ and $\Gamma_1^{(j)} = \dim\left(\text{rad}\left(h_j\big|_{\{\xi_1=0,X_2=0\}}\right)\right)$
Note that

- by step 1(d) we know that $\text{rad}(h_j) \not\subset \xi_1$, and hence, we can use Lemma 2.1.6 to conclude that

$$\text{rank}(h_j|_{\xi_1=0}) = \text{rank}(h_j).$$

Using Lemma 2.1.1 ,

$$\text{rank}\left(h_j|_{\{\xi_1=0,X_2=0\}}\right) \geq \text{rank}\left(h_j|_{\xi_1=0}\right) - 2 = \text{rank}(h_j) - 2.$$

This implies that

$$
\begin{aligned}
\Gamma_1^{(j)} &= (n-2) - \text{rank}\left(h_j\big|_{\{\xi_1=0,X_2=0\}}\right) \\
&\leq (n-2) - (\text{rank}(h_j) - 2) \\
&\leq n - \text{rank}(h_j) \\
&= \Gamma^{(j)}.
\end{aligned}
$$

- if $\Gamma^{(j)} > 1$, *i.e*, $\text{rank}(h_j) < n-1$, then by (4), $\text{rad}(h_j|_{\xi_1=0}) \not\subset \mathbb{T}_f^{\vec{P}_1}$, and hence by Lemma 2.1.6

$$\text{rank}\left(h_j|_{\{\xi_1=0,X_2=0\}}\right) = \text{rank}\left(h_j|_{\xi_1=0}\right) = \text{rank}(h_j).$$

This implies that

$$\begin{aligned}\Gamma_1^{(j)} &= (n-2) - \text{rank}\left(h_j\big|_{\{\xi_1=0,X_2=0\}}\right)\\ &= (n-2) - \text{rank}(h_j)\\ &n - \text{rank}(h_j) - 2\\ &= \Gamma^{(j)} - 2.\end{aligned}$$

Step 2. For a fixed $\vec{P}_1$, we choose another $\mathbb{F}$−rational point $\vec{P}_2$ satisfying the following properties:

a) $f(\vec{P}_2) = 0$,

b) $L_1(\vec{P}_2) \neq 0$, *i.e*, $\vec{P}_2$ does not lie on $\mathbb{T}_f^{\vec{P}_1}$,

Using equation (4.4), we may choose $\vec{P}_2 = (0,1,0,\ldots,0)$. Note that

- $\vec{P}_2$ is a nonsingular zero of f,
- $\vec{P}_2$ does not lie on $X_2 = 0$, and
- the tangent hyperplane to $f = 0$ at $\vec{P}_2$ is $X_1 = 0$.

This completes step 2.

Summary 4.4.1.

(I) After a nonsingular linear change of variables, we may assume that the tangent hyperplane to $f = 0$ at $\vec{P}_1$ is given by

$$\mathbb{T}_f^{\vec{P}_1} : X_2 = 0,$$

and the tangent hyperplane to $f = 0$ at $\vec{P}_2$ is given by

$$\mathbb{T}_f^{\vec{P}_2} : X_1 = 0.$$

(II) We can then split off a hyperplane X_1X_2 from f and rewrite f and g as

$$f(X) = X_1X_2 + f_1(X_3, \ldots, X_n),$$

and

$$g(X) = b_{11}L_2^2(X) + g_1(X_2, \ldots, X_n),$$

where $b_{11} \neq 0$ and $L_2(X) = X_1 - c_2X_2 - \cdots - c_nX_n$.

(III) The restriction of g to $\xi_1 : L_2 = 0$ and $X_2 = 0$ is nonsingular. In other words the restriction of g_1 to $X_2 = 0$ is nonsingular.

(IV) For $1 \leq j \leq l$, $\Gamma_1^{(j)} \leq \Gamma^{(j)}$ always and $\Gamma_1^{(j)} = \Gamma^{(j)} - 2$ if $\Gamma^{(j)} > 1$.

4.5 Proof of the Main Theorem for $n \geq 11$ Variables

Before we embark on the proof of the main theorem 4.1.3, let us look at the assumptions we are now in a position to make based on the results from the previous sections.

1. By Proposition 2.3.1, we know that if every form in the $\mathbb{K}$–pencil generated by f and g is singular, then f and g have a nontrivial common $\mathbb{K}$–rational zero, and by Proposition 4.3.2, we know that if every form in the $\mathbb{K}$–pencil has rank at least 5 and there exists a form with rank at most 6 in the $\mathbb{K}$–pencil, then f and g have a nontrivial common $\mathbb{K}$– rational zero.

   As a consequence, we may assume that the $\mathbb{K}$– pencil generated by f and g contains at least one nonsingular quadratic form and every nonzero form in the $\mathbb{K}$−pencil has rank at least 7. This implies that the determinant polynomial $\det(\lambda f + \mu g)$ is not the zero polynomial, and hence the polynomial $\det(\lambda f + \mu g)$ has at most finitely many zeros. This implies that the $\mathbb{K}$−pencil generated by f and g contains only finitely many singular forms.

Therefore, W.L.O.G., we may assume that the $\mathbb{K}$−pencil generated by f and g contains nonsingular quadratic forms and every nonzero form in the pencil has rank at least 7.

2. By Proposition 2.5.1, we know that there is a nonsingular form in the $\mathbb{K}$−pencil generated by f and g that contains at least three hyperbolic planes.

   Therefore, we may assume that f is a nonsingular form with at least 3 *hyperbolic planes over $\mathbb{K}$, and since there are infinitely many nonsingular forms in the $\mathbb{K}$−pencil, we may choose g to also be nonsingular.*

Let $\mathscr{P}$ be the set of all nonarchimedean places $\mathfrak{p}$ for which the kernel of f over $\mathbb{K}_{\mathfrak{p}}$ has dimension 3, if n is odd or dimension 4, if n is even.

Claim A. *The set $\mathscr{P}$ is finite.*

Proof.

1. Suppose that n odd. Then $\mathscr{P}$ is the set of all nonarchimedean places for which the kernel of f over $\mathbb{K}_{\mathfrak{p}}$ has dimension 3. If there exists a non-dyadic place $\mathfrak{p} \in \mathscr{P}$, then the dimension of the kernel of f as a quadratic form over $\mathbb{K}_{\mathfrak{p}}$ is 3 *i.e*, f is not a unit form over $\mathbb{K}_{\mathfrak{p}}$, and hence $\nu_{\mathfrak{p}}(\det f) \neq 0$. This implies that the set $\mathscr{P}$ must be finite.

2. Suppose that n is even. Then $\mathscr{P}$ is the set of all nonarchimedean places for which the kernel of f over $\mathbb{K}_{\mathfrak{p}}$ has dimension 4. If there exists a non-dyadic place $\mathfrak{p} \in \mathscr{P}$, then the dimension of the kernel of f as a quadratic form over $\mathbb{K}_{\mathfrak{p}}$ is 4. By [8, Theorem 2.2(3), page 152], the $\det(f)$ is a square

in $\mathbb{K}_{\mathfrak{p}}^{\times}$, and $\nu_{\mathfrak{p}}(det f) \neq 0$. This implies $\mathscr{P}$ is a finite set.

This completes the proof of the claim.

□

Let $\mathbb{S} := \mathscr{P} \cup \{\mathfrak{p}_i \in \Omega \big| \mathfrak{p}_i \text{ is archimedean}\}$. Since there are only finitely many archimedean places on $\mathbb{K}$, let s denote the number the of archimedean places on $\mathbb{K}$.

The next step is to split off two hyperbolic planes from f, taking it into the form

$$f = X_1X_2 + X_3X_4 + f'(X_5, \ldots, X_n),$$

in such a way that the quadratic form obtained from g by putting $X_i = 0$ for all $i \neq 1, 3$ is indefinite and represents zero in each $\mathbb{K}_{\mathfrak{p}}$ for which $\mathfrak{p} \in \mathscr{P}$.

To accomplish this, we use Lemma 2.5.2 to choose for each

- $\mathfrak{p} \in \mathscr{P}$, a $\mathbb{K}_{\mathfrak{p}}$-point $\vec{P}_{0\mathfrak{p}}$ on $f = 0$ and $g = 0$ such that $\vec{P}_{0\mathfrak{p}}$ is a nonsingular common zero of f and g over $\mathbb{K}_{\mathfrak{p}}$.
- $i \in [s]$, a $\mathbb{K}_{\mathfrak{p}_i}$-point $\vec{P}_{0i}$ on $f = 0$ and $g = 0$ such that $\vec{P}_{0i}$ is nonsingular common zero of of f, g over $\mathbb{K}_{\mathfrak{p}_i}$.

Claim B. *Let $\mathfrak{p} \in \mathbb{S}$. Given $\vec{P}_{0\mathfrak{p}}$, we can choose a nonsingular zero $\vec{P}_{1\mathfrak{p}}$ on $f = 0$ and $\mathbb{T}_f^{\vec{P}_{0\mathfrak{p}}} = 0$, such that $\vec{P}_{1\mathfrak{p}}$ does not lie on $\mathbb{T}_g^{\vec{P}_{0\mathfrak{p}}} = 0$.*

Proof. W.L.O.G, let $\vec{P}_{0\mathfrak{p}} = (1, 0, \ldots, 0)^t$ and let the tangent hyperplane to $f = 0$ and $g = 0$ at $\vec{P}_{0\mathfrak{p}}$ be $X_2 = 0$ and $X_3 = 0$, respectively. This implies that f and g are of the form

$$f = X_1X_2 + f_0(X_2, \ldots, X_n),$$

and

$$g = X_1X_3 + g_0(X_2, \ldots, X_n).$$

Since $X_2 = 0$ is tangent to $f = 0$, by Lemma 2.1.6

$$\operatorname{rank}(f\big|_{X_2=0}) = \operatorname{rank}(f_0(0, X_3, \dots, X_n) = n - 2 \geq 9.$$

For $i \geq 3$, choose $u_i \in \mathbb{K}_\mathfrak{p}$ such that $f_0(0, u_3, \dots, u_n) = 0$, and $(u_3, \dots, u_n)$ is a nonsingular zero of $f\big|_{X_2=0}$, since $f\big|_{X_2=0}$ is a nonsingular quadratic form of rank of at least 9.

By Lemma 2.1.7, we can choose $(u_3, \dots, u_n) \in \mathbb{K}_\mathfrak{p}^{n-2}$ such that $u_3 \neq 0$.

Let $\vec{P}_{1p} = (1, 0, u_3, u_4, \dots, u_n) \in \mathbb{K}_\mathfrak{p}^n$. Then

- $f(\vec{P}_{1\mathfrak{p}}) = 0$
- $\vec{P}_{1\mathfrak{p}}$ lies on $X_2 = 0$ but does not lie on $X_3 = 0$.

This completes proof of Claim B. □

Claim C. *For each $\mathfrak{p} \in \mathbb{S}$, the line $\vec{P}_{0\mathfrak{p}}\vec{P}_{1\mathfrak{p}}$ lies entirely in $f = 0$.*

Proof. W.L.O.G, let $\vec{P}_{0\mathfrak{p}} = (1, 0, \dots, 0)^t$ and let the tangent hyperplane to $f = 0$ and $g = 0$ at $\vec{P}_{0\mathfrak{p}}$ be $X_2 = 0$ and $X_3 = 0$, respectively. This implies that f and g are of the form

$$f = X_1X_2 + f_0(X_2, \dots, X_n),$$

and

$$g = X_1X_3 + g_0(X_2, \dots, X_n).$$

Since $\vec{P}_{1\mathfrak{p}}$ does not lie on $X_3 = 0$, it must be of the form $\vec{P}_{1\mathfrak{p}} = (u_1, 0, 1, u_4, \dots, u_n)^t$. So any point on the line $\vec{P}_{0\mathfrak{p}}\vec{P}_{1\mathfrak{p}}$ is of the form $(tu_1 + s, 0, t, tu_4, \dots, tu_n), s, t \in \mathbb{K}_\mathfrak{p}$, and

$$
\begin{aligned}
f(tu_1+s,0,t,tu_4,\ldots,tu_n) &= 0+f_0(0,t,tu_4,\ldots,tu_n)\\
&= t^2 f_0(0,1,u_4,\ldots,u_n)\\
&= 0.
\end{aligned}
$$

This completes proof of Claim C. □

For $\mathfrak{p} \in \mathbb{S}$, by Lemma 2.1.7 and Proposition 2.1.10, we choose a nonsingular zero $\vec{P}_{2\mathfrak{p}}$ on $f=0$ such that it does not lie on $\mathbb{T}_f^{\vec{P}_{1\mathfrak{p}}}=0$, and $\vec{P}_{1\mathfrak{p}}$ does not lie on $\mathbb{T}_f^{\vec{P}_{2\mathfrak{p}}}=0$. Let $\vec{P}_{3\mathfrak{p}}$ be the point where the line $\vec{P}_{0\mathfrak{p}}P_{1\mathfrak{p}}$ meets the tangent hyperplane $\mathbb{T}_f^{\vec{P}_{2\mathfrak{p}}}=0$. Since $\vec{P}_{1\mathfrak{p}}$ does not lie on $\mathbb{T}_f^{\vec{P}_{2\mathfrak{p}}}=0$, this point $\vec{P}_{3\mathfrak{p}}$ is different from $\vec{P}_{1\mathfrak{p}}$. Since the tangent hyperplanes $\mathbb{T}_f^{\vec{P}_{0\mathfrak{p}}}=0$, $\mathbb{T}_g^{\vec{P}_{0\mathfrak{p}}}=0$ to $f=0$, $g=0$, respectively, are distinct, the line $\vec{P}_{0\mathfrak{p}}\vec{P}_{1\mathfrak{p}}$ cannot be tangent to $g=0$ at $\vec{P}_{0\mathfrak{p}}$, and hence must meet $g=0$ in two distinct points in $\mathbb{K}_{\mathfrak{p}}$. As a result, the restriction of g to the line $\vec{P}_{0\mathfrak{p}}\vec{P}_{1\mathfrak{p}}$ in any convenient coordinates will result in an isotropic quadratic form over $\mathbb{K}_{\mathfrak{p}}$, and therefore the determinant of this quadratic form will be minus a nonzero square in $\mathbb{K}_{\mathfrak{p}}$.

Since $(\mathbb{K}_{\mathfrak{p}}^{\times})^2)$ forms an open set in the in $\mathbb{K}_{\mathfrak{p}}$, g restricted to any line sufficiently close to $\vec{P}_{0\mathfrak{p}}\vec{P}_{1\mathfrak{p}}$, will also result in an isotropic quadratic form over $\mathbb{K}_{\mathfrak{p}}$.

Using the argument from Section 4.4 along with Proposition 2.4.3, we can choose a $\mathbb{K}$–rational point $\vec{P}_1$ on $f=0$ near $\vec{P}_{1i}$ for each $i \in [s]$, and $\vec{P}_{1\mathfrak{p}}$ for each $\mathfrak{p}$ in $\mathscr{P}$, and another $\mathbb{K}$– rational point $\vec{P}_2$ on $f=0$ near $\vec{P}_{2i}$ for each $i \in [s]$, and $\vec{P}_{2\mathfrak{p}}$ for each $\mathfrak{p}$ in $\mathscr{P}$, such that

$$
\begin{aligned}
f &= X_1X_2 + f_2(X_3,\ldots,X_n)\\
g &= b_{11}\xi_1^2 + g_1(X_2,\ldots,X_n),
\end{aligned}
$$

where for each $i \in [s]$, $\theta_i(g(\vec{P}_1)) = \theta_i(b_{11}) \neq 0$, $\xi_1 = X_1 + c_{12}X_2 + \cdots + c_{1n}X_n$, and if g_2

is the restriction of g to $\xi_1 = 0$ and $X_2 = 0$, (or restriction of g_1 to $X_2 = 0$) then f_2, g_2 are nonsingular forms in $(n-2)$ variables.

Claim D. *Each form in the* $\mathbb{K}-$ *pencil generated by* f_2 *and* g_2 *has rank at least 7.*

Proof. Let $\lambda, \mu \in \mathbb{K}$, not both zero.

(a) If $\lambda f + \mu g$ is nonsingular i.e, $\operatorname{rank}(\lambda f + \mu g) = n$, then by Lemma 2.1.1

$$\begin{aligned}
\operatorname{rank}(\lambda f_2 + \mu g_2) &= \operatorname{rank}\left(\lambda f + \mu g\big|_{\xi_1=0, X_2=0}\right) \\
&= \operatorname{rank}\left(\lambda f + \mu g_1\big|_{X_2=0}\right) \\
&\geq \operatorname{rank}(\lambda f + \mu g) - 2(n-(n-2)) \\
&\geq n - 2(2) = n - 4 \geq 7
\end{aligned}$$

(b) If $\lambda f + \mu g$ is singular such that $\operatorname{rank}(\lambda f + \mu g) = n - 1 \geq 10$, then by (IV) in Summary 4.4.1 in section 4.4,

$$\Upsilon(\lambda f_2 + \mu g_2) \leq \Upsilon(\lambda f + \mu g),$$

Hence,

$$\begin{aligned}
\operatorname{rank}(\lambda f_2 + \mu g_2) &= (n-2) - \Upsilon(\lambda f_2 + \mu g_2) \\
&\geq n - 2 - \Upsilon(\lambda f + \mu g) \\
&= n - 2 - 1 = n - 3 > 7
\end{aligned}$$

(c) If $\lambda f + \mu g$ is singular such that $\operatorname{rank}(\lambda f + \mu g) < (n-1)$, then by (IV) in Summary 4.4.1 in section 4.4,

$$\Upsilon(\lambda f_2 + \mu g_2) = \Upsilon(\lambda f + \mu g) - 2,$$

Hence,

$$\begin{aligned}\operatorname{rank}(\lambda f_2 + \mu g_2) &= (n-2) - \Upsilon(\lambda f_2 + \mu g_2)\\ &(n-2) - \Upsilon(\lambda f + \mu g) + 2 = n - \Upsilon(\lambda f + \mu g)\\ &= \operatorname{rank}(\lambda f + \mu g) \geq 7\end{aligned}$$

□

This implies that f_2 and g_2 are linearly independent. Since $f_2 = f\big|_{X_2=0}$, and $\operatorname{rank}(f_2) = n-2$, by Proposition 2.2.5, we get that $\operatorname{sgn}(f_2) = \operatorname{sgn}(f)$, and hence f_2 remains indefinite because $|\operatorname{sgn}(f_2)| \leq (n-6)$.

Now we repeat this reduction process with f_2, g_2. We choose another rational point $\vec{P}_3$ on $f_2 = 0$ which is near $\vec{P}_{3i}$, for all $i \in [s]$ and $\vec{P}_{3\mathfrak{p}}$ for all $\mathfrak{p} \in \mathscr{P}$. $\vec{P}_3$ is initially defined in the space of $X_3, \ldots, X_n$; but we can extend it to $\mathbb{K}^n$ by setting $X_1 = X_2 = 0$. Now we choose $\vec{P}_4$ on $f_2 = 0$ such that $\vec{P}_4$ does not lie on the tangent hyperplane to $f_2 = 0$ at $\vec{P}_3$. Using the argument from Section 2.3,

$$\vec{P}_3 = (0,0,1,0,\ldots,0)^t,$$

$$\mathbb{T}_f^{\vec{P}_3} : X_4 = 0,$$

and

$$\vec{P}_4 = (0,0,0,1,0,\ldots,0)^t,$$

$$\mathbb{T}_f^{\vec{P}_4} : X_3 = 0.$$

Then we get that

$$f_2 = X_3X_4 + f_4(X_5,\ldots,X_n)$$

$$g_2 = b_{33}\xi_3^2 + g_3(X_4,\ldots,X_n),$$

where $\xi_3 = X_3 + c_{34}X_4 + \cdots + c_{3n}X_n$;

We may assume that $b_{33} \neq 0$, because otherwise we can obtain a common non-trivial zero of f, g over $\mathbb{K}$ by setting $X_3 = 1, \xi_1 = 0, X_2 = X_4 = X_5 = \cdots = X_n = 0$. This implies that for each $i \in [s]$, $\theta_i(b_{33}) \neq 0$. Moreover if g_4 is the restriction of g_3 to $X_4 = 0$, then f_4, g_4 are nonsingular forms in $(n-4) \geq 7$ variables. In claim D, by replacing

- f, and g by f_2, and g_2, respectively,
- f_2, and g_2 by f_4, and g_4, respectively,
- n by $n-2$,
- $n-1$ by $n-3$,
- $n-2$ by $n-4$,

we get that every form in the $\mathbb{K}$–pencil generated by f_4, and g_4 has rank at least 5.

This implies that f_4 and g_4 are linearly independent. Since $f_4 = f_2\big|_{X_4=0}$, and $\mathrm{rank}(f_4) = n-4$, by Proposition 2.2.5, we get that $\mathrm{sgn}(f_4) = \mathrm{sgn}(f_2)$, and hence f_4 remains indefinite because $|\mathrm{sgn}(f_4)| \leq n-6$.

Remark 2. For each $\mathfrak{p} \in \mathbb{S}$, note that $\vec{P}_1\vec{P}_3$ can be made arbitrarily close to $\vec{P}_{1\mathfrak{p}}\vec{P}_{3\mathfrak{p}}$. Since $\vec{P}_{1\mathfrak{p}}\vec{P}_{3\mathfrak{p}}$ meets $g = 0$ in two distinct $\mathbb{K}_{\mathfrak{p}}$– points, it implies that

$$g\big|_{\vec{P}_1\vec{P}_3} = b_{11}\xi_1^2 + b_{33}\xi_3^2$$

is isotropic over $\mathbb{K}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathbb{S}$.

In particular for each $i \in [s]$, $b_{11}\xi_1^2 + b_{33}\xi_3^2$ is indefinite over $\mathbb{K}_{\mathfrak{p}_i}$ and hence $\theta_{\mathfrak{p}_i}(b_{11})$ and $\theta_{\mathfrak{p}_i}(b_{33})$ have opposite sign.

Now we repeat the reduction process one more time with f_4, g_4. We recall that f_4 and g_4 are nonsingular quadratic form in $n-4 \geq 7$ variables, and the local conditions at the places $\mathfrak{p} \in \mathbb{S}$ have been satisfied (see *Remark* 2).

We can find $\mathbb{K}$–rational points $\vec{P}_5$ and $\vec{P}_6$ on $f = 0$ such that

$$\vec{P_5} = (0,0,0,0,1,0,\ldots,0),$$
$$\mathbb{T}_f^{\vec{P_5}} : X_6 = 0,$$

$$\vec{P_6} = (0,0,0,0,0,1,0,\ldots,0),$$
$$\mathbb{T}_f^{\vec{P_6}} : X_5 = 0,$$

and

$$f_4 = X_5X_6 + f_6(X_7,\ldots,X_n)$$
$$g_4 = b_{55}\xi_5^2 + g_5(X_6,\ldots,X_n),$$

where $\xi_5 = X_5 + c_{56}X_6 + \cdots + c_{5n}X_n$. If g_6 is the restriction of g_5 to $X_6 = 0$, then f_6, g_6 are nonsingular forms in $n-6 \geq 5$ variables. In claim D, by replacing

- f, and g by f_4, and g_4, respectively,
- f_4, and g_4 by f_6, and g_6, respectively,
- n by $n-4$,
- $n-1$ by $n-5$,
- $n-2$ by $n-6$,

we get that every form in the $\mathbb{K}$–pencil generated f_6 and g_6 has rank at least 3.

We may also assume that $b_{55} \neq 0$, because otherwise we can obtain a common $\mathbb{K}$–rational zero of f and g by putting $X_5 = 1$, $\xi_1 = \xi_3 = 0$ and $X_2 = X_4 = X_6 = X_7 = \cdots = X_n = 0$.

Let $\mathscr{P}'$ be the set of all nonarchimedean places $\mathfrak{p}$ such that $b_{11}\xi_1^2 + b_{33}\xi_3^2 + b_{55}\xi_5^2$ does not have a nontrivial zero over the $\mathbb{K}_\mathfrak{p}$ with respect to $|\,.\,|_\mathfrak{p}$.

Claim E. *$\mathscr{P}'$ is a finite set.*

Proof. Let $\mathfrak{p}$ be a nondyadic place such that $\nu_{\mathfrak{p}}(b_{11}b_{33}b_{55}) = 0$. This implies that b_{11}, b_{33}, b_{55} are units in $\mathbb{K}_{\mathfrak{p}}$. Then by [8], page 153, corollary 2.5(2), we know that $b_{11}\xi_1^2 + b_{33}\xi_3^2 + b_{55}\xi_5^2$ has a nontrivial zero in $\mathbb{K}_{\mathfrak{p}}$. Hence $\mathfrak{p} \notin \mathscr{P}'$. So if $\mathfrak{p} \in \mathscr{P}'$, then $\nu_{\mathfrak{p}}(b_{11}b_{33}b_{55}) \neq 0$. Therefore, $\mathscr{P}'$ is a finite set. □

Remark 3.

1. We have arranged $\mathscr{P}'$ such that it does not contain $\mathbb{S}$. If $\mathfrak{p} \in \mathbb{S}$, by Remark (2), we know that $b_{11}\xi_1^2 + b_{33}\xi_3^2 + b_{55}\xi_5^2$ is isotropic over $\mathbb{K}_{\mathfrak{p}}$, and hence $\mathfrak{p} \notin \mathscr{P}'$. This implies that if $\mathfrak{p} \in \mathscr{P}'$, then $\mathfrak{p}$ is a nonarchimedean place on $\mathbb{K}$ such that the dimension of the kernel of f over $\mathbb{K}_{\mathfrak{p}}$ is 1, if n is odd and 2, if n is even.

2. Since every form in the pencil generated by f_6, and $b_{11}b_{33}b_{55}g_6$ has rank at least 3, by using Proposition 4.2.1, we may conclude that there exists a $\mathbb{K}_{\mathfrak{p}}$–point $\vec{P}_{7\mathfrak{p}}$ on $f_6 = 0$ such that $b_{11}b_{33}b_{55}g_6(\vec{P}_{7\mathfrak{p}})$ is not a square in $\mathbb{K}_{\mathfrak{p}}$.

3. By Lemma 2.3.5, Proposition 2.4.3, and Proposition 4.2.1, we can find a $\mathbb{K}$–rational point $\vec{P}_7$ on $f_6 = 0$ sufficiently close to $\vec{P}_{7\mathfrak{p}}$ for each $\mathfrak{p} \in \mathscr{P}'$ such that $b_{11}b_{33}b_{55}g_6(\vec{P}_7) \neq 0$, and hence is not a square in $\mathbb{K}_{\mathfrak{p}}$. .

   By a $\mathbb{K}$– rational change of variables we may assume that $\vec{P}_7 = (1, 0, \ldots, 0)$. Let $b_{77} = g_6(\vec{P}_7) \neq 0$, *i.e*, the coefficient of X_7^2 in g_6 is nonzero and consider the linear subspace $\mathcal{W}$ given by

$$X_2 = X_4 = X_6 = X_8 = \cdots = X_n = 0.$$

4. Since we have arranged the quadratic form f in the form $f = X_1X_2 + X_3X_4 + X_5X_6 + f_6$, note that $f = 0$ identically on $\mathcal{W}$.

5. The restriction of g to $\mathcal{W}$ is given by

$$b_{11}\xi_1^2 + b_{33}\xi_3^2 + b_{55}\xi_5^2 + b_{77}X_7^2 \tag{4.6}$$

Claim F. *The form (4.6) is an indefinite quadratic form with respect to each real completion of $\mathbb{K}$ and has a nontrivial zero in $\mathbb{K}_\mathfrak{p}$ for each nonarchimedean place $\mathfrak{p}$ on $\mathbb{K}$.*

Proof. Note that $\theta_i(b_{11})$ and $\theta_i(b_{33})$ are opposite in signs and hence the form in (4.6) is indefinite with respect to each θ_i, $i \in [s]$.

Next we show that the form in (4.6) has a nontrivial zero in $\mathbb{K}_\mathfrak{p}$ for each nonarchimedean place $\mathfrak{p}$ on $\mathbb{K}$.

1. If $\mathfrak{p} \notin \mathscr{P}'$, then

$$b_{11}\xi_1^2 + b_{33}\xi_3^2 + b_{55}\xi_5^2$$

has a nontrivial zero over $\mathbb{K}_\mathfrak{p}$. If we set $X_7 = 0$, the form in equation (4.6) will have a nontrivial zero if $\mathfrak{p} \notin \mathscr{P}'$.

2. Suppose that for some $\mathfrak{p} \in \mathscr{P}'$, the form in (4.6) does not have a nontrivial zero.

   By [8, Theorem 2.2(3), page 152], we get that the determinant of this form must be a square in $\mathbb{K}_\mathfrak{p}$. The determinant of the form in (4.6) is is $b_{11}b_{33}b_{55}b_{77}$ and by *Remark*(3), we know that if $\mathfrak{p} \in \mathscr{P}'$, then $b_{11}b_{33}b_{55}b_{77}$ is not a square in $\mathbb{K}_\mathfrak{p}$. This gives us a contradiction. Thus the form in (4.6) must have a nontrivial zero in $\mathbb{K}_\mathfrak{p}$ for each $\mathfrak{p} \in \mathscr{P}'$.

This completes the proof of Claim F.

□

At this point, by the Hasse-Minkowski Theorem ([8, Theorem 3.1, page 170]) we may conclude that the form in (4.6)

$$b_{11}\xi_1^2 + b_{33}\xi_3^2 + b_{55}\xi_5^2 + b_{77}X_7^2$$

has a nontrivial zero over $\mathbb{K}$.

Let $\vec{\alpha} = (\alpha_1, \alpha_3, \alpha_5, \alpha_7) \in \mathbb{K}^4$ represent that $\mathbb{K}$–rational zero.

Then $(\alpha_1, 0, \alpha_3, 0, \alpha_5, 0, \alpha_7, 0, \ldots, 0) \in \mathcal{W}$ is a common $\mathbb{K}$–rational zero of both f and g.

Next, will prove the following claim:

Claim 4.5.1. *f and g have a nonsingular $\mathbb{K}$–rational zero.*

Proof. If all common zeros of f and g over $\mathbb{K}$ are singular, then by Lemma 2.1.9 there is a form $\lambda_1 f + \mu_1 g$ in the $\mathbb{K}$– pencil generated by f and g that has only singular zeros. This is implies that $\text{rank}(\lambda_1 f + \mu_1 g) < 5$ or it not indefinite with respect to some real place on $\mathbb{K}$. This is a contradiction to the hypotheses in Theorem **??** that every form in the $\mathbb{K}$–pencil generated by f and g has rank at least 5 and is indefinite with respect to all real places on $\mathbb{K}$. Therefore, f and g have a nonsingular $\mathbb{K}$–rational zero. □

By Lemma 2.3.4, f and g have infinitely many nonsingular $\mathbb{K}$– rational zeros. This completes the proof of the theorem for the case when the number of variables n is at least 11.

CHAPTER 5. A SYSTEM OF TWO QUADRATIC FORMS IN $N \geq 9$ VARIABLES OVER AN ARBITRARY NUMBER FIELD

5.1 Introduction

In this chapter, we give a proof of the following main theorem.

Theorem 5.1.1. *Let $\mathbb{K}$ be a number field with s distinct real places denoted by $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Let f, g be quadratic forms in at least 9 variables, defined over $\mathbb{K}$; Suppose that every form in the $\mathbb{K}$–pencil has rank at least 5 and if $s \geq 1$, suppose that every nonzero quadratic form $\lambda f + \mu g$ in the $\mathbb{K}_{\mathfrak{p}_i}$–pencil is indefinite for all $1 \leq i \leq s$. Then f, g have infinitely many nontrivial common zeros over $\mathbb{K}$.*

In [4, Theorem 10.1], the authors Colliot-Thélène, Sansuc, Swinnerton-Dyer prove a more general result about a system of two quadratic forms over a number field and Theorem 5.1.1 is presented as a corollary to that theorem. The proof of the result in [4, Theorem 10.1] requires prior knowledge of several key results that are often very geometric and/or analytic in nature. Therefore, the work in this chapter is aimed towards simplifying as well as clarifying the details of the proof in [4, Theorem 10.1] using primarily number-theoretic arguments. We would also like to point out that we use the technique of splitting off hyperbolic planes described in section 4.4 of Chapter 4 in order to complete the first step of the proof, which is different from the proof given in [4].

The notation used in Chapter 5 is same as the notation in Chapter 4. However, before we embark on the proof of Theorem 5.1.1, we would like to restate the notation used in the proof as well as discuss the reasons behind the specific assumption made in the statement of Theorem 5.1.1.

- $\mathbb{K}$ will denote a number field.

- Ω is the set of all places on $\mathbb{K}$. Ω contains all the archimedean and nonarchimedean absolute values on $\mathbb{K}$ upto equivalence. We often use the word *'infinte prime'* to refer to an archimedean valuation and *'finte prime'* to refer to a nonarchimedean valuation on $\mathbb{K}$.

- If $\mathfrak{p} \in \Omega$, then $\mathbb{K}_{\mathfrak{p}}$ denotes the completion of $\mathbb{K}$ with respect to $\mathfrak{p}$.

- For archimedean places (or infinite primes) $\mathfrak{p}$, $\mathbb{K}_{\mathfrak{p}}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}_{\mathfrak{p}}$ is isomorphic to $\mathbb{R}$, then we call $\mathfrak{p}$ to be real place on $\mathbb{K}$ and the corresponding isomorphism $\theta_{\mathfrak{p}} : \mathbb{K}_{\mathfrak{p}} \to \mathbb{R}$ is called an ordering on $\mathbb{K}_{\mathfrak{p}}$.

- For nonarchimedean places (or finite primes) $\mathfrak{p}$, $\mathbb{K}_{\mathfrak{p}}$ is a local field, that is, *c.d.v. field* with a finite residue class, and $\nu_{\mathfrak{p}}$ denotes the corresponding discrete valuation on $\mathbb{K}$.

Remark. 1. We require all the forms in the $\mathbb{K}_{\mathfrak{p}_i}$−pencil generated by f and g to be indefinite for each $i \in [s]$ because

   a) If there exists an $i \in [s]$ such that there is a form in the $\mathbb{K}_{\mathfrak{p}_i}$−pencil that is definite, then f and g cannot have a nontrivial common zero over $\mathbb{K}_{\mathfrak{p}_i}$ and hence, they cannot have a nontrivial common zero over $\mathbb{K}$.

   b) If there exists an $i \in [s]$ such that there is a nonzero form $\lambda f + \mu g$ in the $\mathbb{K}_{\mathfrak{p}_i}$−pencil that is semi-definite, then the $\mathbb{K}_{\mathfrak{p}_i}$ points on $\lambda f + \mu g = 0$ form a $\mathbb{K}_{\mathfrak{p}_i}$−linear subspace, and the $\mathbb{K}$−rational points are those of the maximal $\mathbb{K}$−rational linear space contained in it. It is possible that $\lambda f + \mu g$ does not have any nontrivial $\mathbb{K}$−rational zero. So in this case the maximal $\mathbb{K}$−rational linear subspace is the trivial subspace.

2. We also require every form in the $\mathbb{K}$−pencil to have rank at least 5 because otherwise, we can find counterexamples. One such counterexample is given below:

Choose a nonarchimedian place $\mathfrak{p}$ on $\mathbb{K}$. Let π be an element of $\mathbb{K}$ such that $\nu_{\mathfrak{p}}(\pi) = 1$. By Theorem 2.4.1, we can choose π such that $\theta_{\mathfrak{p}_i}(\pi) > 0$ for each $i \in [s]$. Let $u \in \mathbb{K}$ be a unit in the valuation ring $(\mathbb{K}, \mathfrak{p})$ such that $u \notin (\mathbb{K}_{\mathfrak{p}}^{\times})^2$. Consider the following system of quadratic forms over $\mathbb{K}$.

$$f = X_1^2 - \pi X_2^2 - uX_3^2 + \pi u X_4^2$$
$$g = -X_4^2 + X_5^2 + X_6^2 + X_7^2 + X_8^2 + X_9^2.$$

- If $\lambda = 0$, then for any $\mu \in \mathbb{K}_{\mathfrak{p}_i} - \{0\}$, μg is an indefinite form in $\mathbb{K}_{\mathfrak{p}_i}$ for each $i \in [s]$

  Proof. Let $i \in [s]$. The coefficient of X_4^2 in μg is $-\mu$, and the coefficient of X_5^2 in μg is μ. Note that $\theta_{\mathfrak{p}_i}(\mu)$ and $\theta_{\mathfrak{p}_i}(-\mu)$ are opposite in sign because

  $$\theta_{\mathfrak{p}_i}(-\mu) = -\theta_{\mathfrak{p}_i}(\mu).$$

  □

- If $\lambda \neq 0$, then for $\mu \in \mathbb{K}_{\mathfrak{p}_i}$, $\lambda f + \mu g$ is an indefinite form in $\mathbb{K}_{\mathfrak{p}_i}$ for each $i \in [s]$.

  Proof. Let $i \in [s]$. The coefficient of X_1^2 in $\lambda f + \mu g$ is λ and the coefficient of X_2^2 in $\lambda f + \mu g$ is $-\lambda\pi$. Note that

  $$\theta_{\mathfrak{p}_i}(-\lambda\pi) = -\theta_{\mathfrak{p}_i}(\lambda)\theta_{\mathfrak{p}_i}(\pi).$$

  Since $\theta_{\mathfrak{p}_i}(\pi) > 0$, it then follows that $\theta_{\mathfrak{p}_i}(-\lambda\pi)$ and $\theta_{\mathfrak{p}_i}(\lambda)$ are opposite in signs. □

We first consider f as a form in the four variables X_1, X_2, X_3, X_4. By [8, Theorem 2.1 c], we know that f is anisotropic over $\mathbb{K}_{\mathfrak{p}}$. This implies that f does not have a nontrivial zero over $\mathbb{K}$, when considered as a form in the four variables X_1, X_2, X_3, X_4.

Now consider the forms f, g as forms in 9 variables over $\mathbb{K}$. If there exists a common $\mathbb{K}$–rational zero of f, g, then it must be of the following form:

$$\vec{a} = (0,0,0,0,a_5,\dots,a_9) \in \mathbb{K}^9$$

because $f = f\big|_{\{X_j=0:5\le j\le 9\}}$ does not have any nontrivial $\mathbb{K}$–rational zeros. Since $g(\vec{a}) = 0$, we get

$$-0^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 = 0,$$

which implies that $a_j = 0$ for all $5 \le j \le 9$. Therefore, f, g have no nontrivial common $\mathbb{K}$–rational zero.

5.2 A Result over Completions of a Number Field $\mathbb{K}$.

Lemma 5.2.1. *Let $\mathfrak{p}$ be any place on $\mathbb{K}$. Let f,g be quadratic forms in at least $n \ge 7$ variables over $\mathbb{K}_\mathfrak{p}$ such that they have a nonsingular common zero over $\mathbb{K}_\mathfrak{p}$, and the $\mathbb{K}_\mathfrak{p}$–pencil generated by f and g contains at least one nonsingular form. Let h be another quadratic form over $\mathbb{K}_\mathfrak{p}$ of rank 4, and L be any linear form over $\mathbb{K}_\mathfrak{p}$ in variables $X_1,\dots,X_n$. Then there exists a nontrivial common zero $\vec{P}_\mathfrak{p}$ of f, g over $\mathbb{K}_\mathfrak{p}$ such that $h(\vec{P}_\mathfrak{p}) \neq 0$ and $L(\vec{P}_\mathfrak{p}) \neq 0$.*

Proof. W.L.O.G, we may assume that the rank$(g) = 7$. We consider the following cases:

Case 1: Suppose that h does not vanish on any nontrivial common zero of f and g over $\mathbb{K}_\mathfrak{p}$. By the hypothesis, we know that f,g have a nonsingular common zero over $\mathbb{K}_\mathfrak{p}$. W.L.O.G., we can assume that $\vec{e_1}$ is that nonsingular common zero and that we can write

$$f = X_1X_2 + f_1(X_2,\dots,X_n)$$

$$g = X_1X_3 + g_1(X_2, \ldots, X_n),$$

$$L = l_1X_1 + \cdots + l_nX_n$$

where f_1 and g_1 are quadratic forms and L is a linear form over $\mathbb{K}_\mathfrak{p}$. If $L(\vec{e}_1) \neq 0$, then we are done. Therefore, we can assume that $L(\vec{e}_1) = 0$, which implies that $L = l_2X_2 + \cdots + l_nX_n$. If necessary, we can interchange f and g, to assume that X_3 does not divide L. Let g_3 denote the quadratic form obtained by setting $X_3 = 0$ in g.

Note that g_3 is an isotropic quadratic form of rank 5 over $\mathbb{K}_\mathfrak{p}$, because

a) if $\mathfrak{p}$ is nonarchimedean, then $u(\mathbb{K}_\mathfrak{p}) \geq 4$.

b) if $\mathfrak{p}$ is archimedean, then by Proposition 2.2.5 we know that

$$\operatorname{sgn}(g) = \operatorname{sgn}(g_3).$$

Also note that g_3 does not divide $X_2 \cdot L\big|_{X_3=0}$ because $\operatorname{rank} g_3 = 5$. Therefore by Lemma 2.3.5, we can conclude that there exists a nontrivial zero of g_3 such that it is not a zero of $X_2 \cdot L\big|_{X_3=0}$. Let $(u_2, 0, u_4, \ldots, u_n)$ represent that zero over $\mathbb{K}_\mathfrak{p}$, where $u_2 \neq 0$. Therefore, W.L.O.G., we can assume that $u_2 = 1$. Now let $u_1 = -f_1(1, 0, u_4, \ldots, u_n)$ and let $\vec{P}_\mathfrak{p} = (u_1, 1, 0, u_4, \ldots, u_n)$. Then $\vec{P}_\mathfrak{p}$ is a common zero of f and g such that $h(\vec{P}_\mathfrak{p}) \neq 0$ an'd $L(\vec{P}_\mathfrak{p}) \neq 0$.

Case 2: Suppose that h vanishes on at least one nonsingular zero of f and g over $\mathbb{K}_\mathfrak{p}$. W.L.O.G., we can assume that $\vec{e}_1$ is that a nonsingular zero of f and g. We can write

$$f = X_1X_2 + f_1(X_2, \ldots, X_n),$$

$$g = X_1X_3 + g_1(X_2, \dots, X_n),$$

$$h = X_1M(X_2, \dots, X_n) + h_1(X_2, \dots, X_n),$$

$$L = l_1X_1 + \cdots + l_nX_n$$

where $f_1,\ g_1,\ h_1$ are quadratic forms and $M,\ L$ are linear form over $\mathbb{K}_{\mathfrak{p}}$.

We can assume X_3 does not divide L and X_3 does not divide $h_1|_{M=0}$. Since $\mathbb{K}$ is an infinite field, there are only finitely many linear forms that could divide either L or $h_1|_{M=0}$. Therefore, we can choose $\lambda \in \mathbb{K}$ such that $(\lambda X_2 + X_3)$ does not divide L and $(\lambda X_2 + X_3)$ does not divide $h_1|_{M=0}$. If X_3 divides L or $h_1|_{M=0}$, then we may replace g by $\lambda f + g$. So after replacing g by $\lambda f + g$ for some appropriate choice of λ, if necessary, and a nonsingular linear change of variables, we can assume that X_3 does not divide L, and X_3 does not divide $h_1|_{M=0}$.

Also as in Case 1, $g_3 = g|_{X_3=0}$ is an isotropic quadratic form over $\mathbb{K}_{\mathfrak{p}}$.

a) If $L(\vec{e}_1) = 0$, then $L = l_2X_2 + \cdots + l_nX_n$.

Claim 5.2.2. *g_3 does not divide*

$$L\Big|_{X_3=0} \cdot \Big(-f_1\Big|_{X_3=0} M\Big|_{X_3=0} + X_2h_1\Big|_{X_3=0}\Big).$$

Proof:

If g_3 divides $L\Big|_{X_3=0} \cdot \Big(-f_1\Big|_{X_3=0} M\Big|_{X_3=0} + X_2h_1\Big|_{X_3=0}\Big)$, then since g_3 is irreducible ($\operatorname{rank} g_3 = 5$), the following statements must be true.

- g_3 divides $\left(-f_1\big|_{X_3=0}M\big|_{X_3=0}+X_2h_1\big|_{X_3=0}\right)$

- $g_3\big|_{M=0}$ divides $(X_2h_1|_{\{X_3=0,\ M=0\}})$,

- $g_3\big|_{M=0}$ divides $h_1|_{\{X_3=0,\ M=0\}}$.

which is a contradiction as $\text{rank}(g_3\big|_{M=0}) \geq 3$, and $h_1|_{\{X_3=0,\ M=0\}})$ is a nonzero quadratic form of rank at most 2.

This completes the proof of Claim 5.2.2.

Therefore, by using Lemma 2.3.5, we can conclude that g_3 has a non-trivial zero $P'_\mathfrak{p} = (u_2, 0, u_4, \ldots, u_n)$ of g_3 over $\mathbb{K}_\mathfrak{p}$ such that

(i) $L(P'_\mathfrak{p}) \neq 0$,

(ii) $u_2 \neq 0$,

(iii) $-f_1(P'_\mathfrak{p})M(P'_\mathfrak{p}) + u_2h_1(P'_\mathfrak{p}) \neq 0$

W.L.O.G., we can assume that $u_2 = 1$. Now let $u_1 = -f_1(1, 0, u_4, \ldots, u_n)$ and let $\vec{P}_\mathfrak{p} = (u_1, 1, 0, u_4, \ldots, u_n)$. Then $\vec{P}_\mathfrak{p}$ is a common zero of f and g such that $h(\vec{P}_\mathfrak{p}) \neq 0$ amd $L(\vec{P}_\mathfrak{p}) \neq 0$.

b) If $L(\vec{e}_1) \neq 0$, then $L = l_1X_1 + l_2X_2 + \cdots + l_nX_n$, where $l_1 \neq 0$. By multiplying by a constant if necessary, we can assume that $L = X_1 + l_2X_2 + \cdots + l_nX_n$.

Claim 5.2.3. *g_3 does not divide*

$$\left(-f_1\big|_{X_3=0} + X_2(l_2X_2 + l_4X_4 + \cdots + l_nX_n)\right) \cdot \left(-f_1\big|_{X_3=0}M\big|_{X_3=0} + X_2h_1\big|_{X_3=0}\right).$$

Proof:

Since g_3 is an irreducible quadratic form of rank 7, if g_3 divides

$$\left(-f_1\big|_{X_3=0}+X_2(l_2X_2+l_4X_4+\cdots+l_nX_n)\right)\cdot\left(-f_1\big|_{X_3=0}M\big|_{X_3=0}+X_2h_1\big|_{X_3=0}\right),$$

then g_3 must divide at least one of

$$\left(-f_1\big|_{X_3=0}+X_2(l_2X_2+l_4X_4+\cdots+l_nX_n)\right)$$

or

$$\left(-f_1\big|_{X_3=0}M\big|_{X_3=0}+X_2h_1\big|_{X_3=0}\right).$$

Suppose that $g_3 = c\left(-f_1\big|_{X_3=0}+X_2(l_2X_2+l_4X_4+\cdots+l_nX_n)\right)$ for some $c \in \mathbb{K}_\mathfrak{p}$.

Then $\operatorname{rank}((g+cf)\big|_{X_3=0})$

$$\begin{aligned}
&= \operatorname{rank}(g_3+cX_1X_2+cf_1\big|_{X_3=0})\\
&= \operatorname{rank}\left(c\left(-f_1\big|_{X_3=0}+X_2(l_2X_2+l_4X_4+\cdots+l_nX_n)\right)+cX_1X_2+cf_1\big|_{X_3=0}\right)\\
&= \operatorname{rank}(c(X_1X_2+X_2(l_2X_2+l_4X_4+\cdots+l_nX_n)))\\
&= \operatorname{rank}(cX_2(X_1+l_2X_2+l_4X_4+\cdots+l_nX_n)) = 2
\end{aligned}$$

This is a contradiction since that $\operatorname{rank}((g+cf)\big|_{X_3=0})$ is at least 3. Therefore g_3 cannot divide $\left(-f_1\big|_{X_3=0}+X_2(l_2X_2+l_4X_4+\cdots+l_nX_n)\right)$.

Using the argument from part (i), g_3 also does not divide

$$\left(-f_1\big|_{X_3=0}M\big|_{X_3=0}+X_2h_1\big|_{X_3=0}\right).$$

This completes the proof of Claim 5.2.3.

Therefore, by using Lemma 2.3.5, we can conclude that g_3 has a nontrivial zero $P'_\mathfrak{p} = (u_2, 0, u_4, \ldots, u_n)$ of g_3 over $\mathbb{K}_\mathfrak{p}$ such that

(i) $u_2 \neq 0$,

(ii) $-f_1(P'_\mathfrak{p}) + u_2(l_2u_2 + \cdots + l_nu_n) \neq 0$

(iii) $-f_1(P'_\mathfrak{p})M(P'_\mathfrak{p}) + u_2h_1(P'_\mathfrak{p}) \neq 0$

W.L.O.G., we can assume that $u_2 = 1$. Now let $u_1 = -f_1(1, 0, u_4, \ldots, u_n)$ and let $\vec{P}_\mathfrak{p} = (u_1, 1, 0, u_4, \ldots, u_n)$.

Then $\vec{P}_\mathfrak{p}$ is a common zero of f and g such that

$$h(\vec{P}_\mathfrak{p}) = -f_1(1,0,u_4,\ldots,u_n)M(1,0,u_4,\ldots,u_n) + u_2h_1(1,0,u_4,\ldots,u_n) \neq 0$$

$$L(\vec{P}_\mathfrak{p}) = -f_1(1,0,u_4,\ldots,u_n) + u_2(l_2u_2 + \cdots + l_nu_n) \neq 0.$$

Case3: Suppose that $\vec{e_1}$ is a singular zero of f and g. We can rewrite f, g and h as

$$f = X_1X_2 + f_1(X_2, \ldots, X_n),$$

$$g = g_1(X_2, \ldots, X_n),$$

and

$$h = X_1M(X_2, \ldots, X_n) + h_1(X_2, \ldots, X_n),$$

where f_1, g_1, h_1 are quadratic forms and M is a linear form in variables $X_2, \ldots, X_n$ over $\mathbb{K}_\mathfrak{p}$.

Using an argument analogous to the one in Case 2, we can show that there exists a nontrivial common zero $\vec{P}_\mathfrak{p}$ of f, g over $\mathbb{K}_\mathfrak{p}$ such that $h(\vec{P}_\mathfrak{p}) \neq 0$ and $L(\vec{P}_\mathfrak{p}) \neq 0$.

This completes the proof of the lemma. □

Case 1 in the proof of Lemma 5.2.1 leads to the following corollary:

Corollary 5.2.4. *Let f, g be quadratic forms in at least $n \geq 7$ variables over $\mathbb{K}_\mathfrak{p}$ such that they have a nonsingular common zero over $\mathbb{K}_\mathfrak{p}$, and the $\mathbb{K}_\mathfrak{p}$–pencil generated by f and g contains at least one nonsingular form. Then the nonsingular common zeros of f and g over $\mathbb{K}_\mathfrak{p}$ do not lie in a hyperplane.*

5.3 Additional Results.

In this section we give rigorous algebraic proofs of some propositions from [3] that are used in the proof of Theorem 5.1.1.

Lemma 5.3.1. *Let $P(X_1, \ldots, X_m)$ be any polynomial with coefficients in $\mathbb{K}$, and $\mathfrak{p}$ be any place of $\mathbb{K}$. Suppose that P has a nonsingular zero over $\mathbb{K}_\mathfrak{p}$. Then P represents all square classes of $\mathbb{K}_\mathfrak{p}^\times$.*

Proof. W.L.O.G., after a nonsingular linear change of variables over $\mathbb{K}_\mathfrak{p}$, we may assume that $\vec{0}$ is a nonsingular zero of P, *i.e.*, $P(\vec{0}) = 0$ and $\frac{\partial P}{\partial X_i}(\vec{0}) \neq 0$ for some i. This implies that the constant term in P is zero and P has at least one nonzero linear term. Therefore, after another linear change of variables, we may assume that

$$P = X_1 + P_1(X_1, \ldots, X_m),$$

where P_1 is a polynomial over $\mathbb{K}_\mathfrak{p}$ such that the degree of each term is at least 2.

Case 1: We first assume that $\mathfrak{p}$ is a nonarchimedean place of $\mathbb{K}$. Multiplying by $\pi_\mathfrak{p}^s$, where s is sufficiently large, we may assume that $P_2 = \pi_\mathfrak{p}^s P_1$ has coefficients in the valuation ring $\mathcal{O}_\mathfrak{p}$. Therefore, we can rewrite P as shown below:

$$P = X_1 + \pi_\mathfrak{p}^{-s} P_2(X_1, \ldots, X_m),$$

where P_2 is a polynomial over $\mathcal{O}_\mathfrak{p}$ such that the degree each term is at least 2. Let α be a representative from any square class of $\mathbb{K}_\mathfrak{p}^\times$. Set $X_1 = \alpha\pi_\mathfrak{p}^{2t}$, where t is sufficiently large, and set $X_i = 0$ for all $i \geq 2$.

$$\begin{aligned} P(\alpha\pi_\mathfrak{p}^{2t}, 0, \ldots, 0) &= \alpha\pi_\mathfrak{p}^{2t} + \pi_\mathfrak{p}^{-s} P_2(\alpha\pi_\mathfrak{p}^{2t}, 0, \ldots, 0) \\ &= \alpha\pi_\mathfrak{p}^{2t}(1 + \beta\pi_\mathfrak{p}^{2t-s}), \end{aligned}$$

where $\beta \in \mathbb{K}_\mathfrak{p}$. By Theorem 2.18 in [8], page 161, we can choose t sufficiently large so that $(1 + \beta\pi_\mathfrak{p}^{2t-s})$ is a square in $\mathbb{K}_\mathfrak{p}^\times$. This implies that $\alpha\pi_\mathfrak{p}^{2t}(1 + \beta\pi_\mathfrak{p}^{2t-s})$

is in the same square class as α. Therefore, P represents all square classes of $\mathbb{K}_{\mathfrak{p}}^{\times}$.

Case 2: Now suppose that $\mathfrak{p}$ is an archimedean place. Let $\theta_{\mathfrak{p}}$ represent the associated ordering. We will show that P represents both positive and negative values over $\mathbb{K}_{\mathfrak{p}}$.

We recall that in the polynomial $P_1(X_1,\ldots,X_n)$ each term has degree at least 2. Therefore, if $P(X_1,0,\ldots,0)$ is not the zero polynomial, then the degree of X_1 in each term must be at least 2 and

$$\lim_{X_1\to 0}\frac{P_1(X_1,0,\ldots,0)}{X_1}=0$$

Hence, we can choose $X_1=\alpha\neq 0$, sufficiently small, such that

$$\theta_{\mathfrak{p}}\left(1+\frac{P_1(\alpha,0,\ldots,0)}{\alpha}\right)>0.$$

Then

$$\begin{aligned}P(\alpha,0,\ldots,0)&=\alpha+P_1(\alpha,0,\ldots,0)\\&=\alpha\left(1+\frac{P_1(\alpha,0,\ldots,0)}{\alpha}\right)\end{aligned}$$

Therefore, the sign of $P(\alpha,0,\ldots,0)$ is the same as the sign of α with respect to $\theta_{\mathfrak{p}}$. Hence, P represents both postive and negative values over $\mathbb{K}_{\mathfrak{p}}$.

□

Proposition 5.3.2 ([3], Proposition 3.12). *Let $\mathbb{K}$ be an arbitrary number field. Let $Q(Y_1,\ldots,Y_n)$ be a quadratic form with coefficients in $\mathbb{K}$ and rank at least 3, and let $P(X_1,\ldots,X_m)$ be an arbitrary polynomial in $\mathbb{K}[X_1,\ldots,X_m]$. Suppose that for each $\mathfrak{p}\in\Omega$*

$$Q(Y_1,\ldots,Y_n)-P(X_1,\ldots,X_m) \tag{5.1}$$

has a nonsingular zero over $\mathbb{K}_{\mathfrak{p}}$. Then it has a nontrivial zero over $\mathbb{K}$.

Proof. We consider the following two cases:

Case 1. Suppose that $P(X_1,\ldots,X_m)$ is the zero polynomial. Note that $Q(Y_1,\ldots,Y_n)$ is a quadratic form over $\mathbb{K}$ having rank at least 3 such that it has nontrivial zero for each $\mathfrak{p} \in \Omega$. Therefore, by the Hasse-Minkowski Theorem (Theorem 3.1 in [8], page 170), we can conclude that it has nontrivial zero over $\mathbb{K}$.

Case2. Suppose that $P(X_1,\ldots,X_m)$ is not identically zero. Let

$$S_0 = \{\mathfrak{p} \in \Omega \Big| Q(Y_1,\ldots,Y_n) \text{ is anisotropic over } \mathbb{K}_{\mathfrak{p}}\}.$$

Since $\text{rank}(Q(Y_1,\ldots,Y_n)) \geq 3$, S_0 is a finite set.

If S_0 is empty, then by the Hasse-Minkowski Theorem Q is isotropic over $\mathbb{K}$. Let $\vec{y} = (y_1,\ldots,y_n)$ be any nontrivial zero of Q over $\mathbb{K}$. Then $\left(\vec{y},\vec{0}\right)$ is a nontrivial zero of equation (5.1).

Therefore, we assume that S_0 is a nonempty finite set. For each $\mathfrak{p} \in S_0$, let $(\vec{Y}_{\mathfrak{p}}, \vec{X}_{\mathfrak{p}})$ denote a nonsingular zero of equation(5.1) over $\mathbb{K}_{\mathfrak{p}}$ where

$$\vec{Y}_{\mathfrak{p}} = (Y_{1\mathfrak{p}},\ldots,Y_{n\mathfrak{p}}),$$

and

$$\vec{X}_{\mathfrak{p}} = (X_{1\mathfrak{p}},\ldots,X_{m\mathfrak{p}}).$$

Claim. *For each $\mathfrak{p} \in S_0$, we can choose a nonsingular zero of equation (5.1) such that $\vec{Y}_{\mathfrak{p}} \neq \vec{0}$.*

Proof: If for some $\mathfrak{p} \in S_0$, $\vec{Y}_{\mathfrak{p}} = \vec{0}$, then we get that $P(\vec{X}_{\mathfrak{p}}) = 0$, where $\vec{X}_{\mathfrak{p}} \neq \vec{0}$. This implies that $\vec{X}_{\mathfrak{p}}$ is a nonsingular zero of P over $\mathbb{K}_{\mathfrak{p}}$. By lemma 5.3.1, we get that P respresents all square classes of $\mathbb{K}_{\mathfrak{p}}$. Therefore, we can choose $\vec{X}'_{\mathfrak{p}} = (X'_{1\mathfrak{p}},\ldots,X'_{m\mathfrak{p}})$ such that $Q(Y_1,\ldots,Y_n)$ represents $P(\vec{X}'_{\mathfrak{p}})$ over $\mathbb{K}_{\mathfrak{p}}$. This implies that $P(\vec{X}'_{\mathfrak{p}}) \neq 0$ and

$$Q(Y_1,\ldots,Y_n) - P(\vec{X}'_{\mathfrak{p}})Z^2 \tag{5.2}$$

is an isotropic quadratic form of rank at least 4 over $\mathbb{K}_\mathfrak{p}$. Therefore, it has a nonsingular zero over $\mathbb{K}_\mathfrak{p}$ such that $Z \neq 0$.

W.L.O.G., we can take $Z = 1$, and let $(Y'_{1\mathfrak{p}}, \ldots, Y'_{n\mathfrak{p}}, 1)$ represent that nonsingular zero of the quadratic form in (5.2). Let $\vec{Y}'_\mathfrak{p} = (Y'_{1\mathfrak{p}}, \ldots, Y'_{n\mathfrak{p}})$. Then $(\vec{Y}'_\mathfrak{p}, \vec{X}'_\mathfrak{p})$ is a nonsingular zero of (5.1) over $\mathbb{K}_\mathfrak{p}$ such that $\vec{Y}'_\mathfrak{p} \neq \vec{0}$.

This completes the proof of the claim.

Therefore, W.L.O.G, we can can assume that for all $\mathfrak{p} \in S_0$, $(\vec{Y}_\mathfrak{p}, \vec{X}_\mathfrak{p})$ denotes a nonsingular zero of (5.1) over $\mathbb{K}_\mathfrak{p}$ such that $\vec{Y}_\mathfrak{p} \neq \vec{0}$.

Since Q is anisotropic over $\mathbb{K}_\mathfrak{p}$ for each $\mathfrak{p} \in S_0$, we get that $Q(\vec{Y}_\mathfrak{p}) \neq 0$. However, $Q(\vec{Y}_\mathfrak{p}) - P(\vec{X}_\mathfrak{p}) = 0$ implies that

$$Q(\vec{Y}_\mathfrak{p}) = P(\vec{X}_\mathfrak{p}) \neq 0.$$

Using Proposition 2.4.1, we can choose $\vec{\mathcal{X}} \in \mathbb{K}^m$ arbitrarily close to $\vec{X}_\mathfrak{p}$ for each $\mathfrak{p}$ in S_0. This implies that we can choose $\vec{\mathcal{X}} \in \mathbb{K}^m$ such that $0 \neq P(\vec{\mathcal{X}})$ is in the same square class as $P(\vec{X}_\mathfrak{p})$ for each $\mathfrak{p}$ in S_0.

Next we consider

$$Q(Y_1, \ldots, Y_n) - P(\vec{\mathcal{X}})Y_{n+1}^2,$$

which is a quadratic form over $\mathbb{K}$ having rank at least 4.

We will show that $Q(Y_1, \ldots, Y_n) - P(\vec{\mathcal{X}})Y_{n+1}^2$ is isotropic over $\mathbb{K}_\mathfrak{p}$ for every place $\mathfrak{p}$ over $\mathbb{K}$.

Since $P\left(\vec{\mathcal{X}}\right)$ and $P\left(\vec{X}_\mathfrak{p}\right)$ are in the same square class for each $\mathfrak{p} \in S_0$, we get that $Q(Y_1, \ldots, Y_n) - P(\vec{\mathcal{X}})Y_{n+1}^2$ is isotropic for every $\mathfrak{p}$ in S_0. It is also isotropic for all $\mathfrak{p} \notin S_0$ because $Q(Y_1, \ldots, Y_n)$ is isotropic for all $\mathfrak{p} \notin S_0$. Therefore, $Q(Y_1, \ldots, Y_n) - P(\vec{\mathcal{X}})Y_{n+1}^2$ is isotropic over $\mathbb{K}_\mathfrak{p}$ for all places $\mathfrak{p}$ on $\mathbb{K}$. Therefore, by the Hasse-Minkowski Theorem, we can find a global zero $\vec{\mathcal{Y}} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_n, \mathcal{Y}_{n+1})$ of

$$Q(Y_1, \ldots, Y_n) - P(\vec{X})Y_{n+1}^2$$

where $\mathcal{Y}_{n+1} \neq 0$. This also implies that at least one of the $\mathcal{Y}_i$'s for $1 \leq i \leq n$ must also be nonzero. W.L.O.G., we take $\mathcal{Y}_{n+1} = 1$. Then $(\vec{\mathcal{Y}}, \vec{\mathcal{X}})$ is a nontrivial zero of (5.1) over $\mathbb{K}$.

□

Lemma 5.3.3. *Let $\mathbb{K}$ be any infinite field and let V be an n–dimensional vector space over $\mathbb{K}$. Let $\vec{v}_1 \in V$, and for $1 \leq i \leq t$, let $\vec{w}_i \in \mathbb{K}^n$. Then there exists a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of V over $\mathbb{K}$ such that $Span\{\vec{v}_2, \ldots, \vec{v}_n\}$ over $\mathbb{K}$, does not contain $\vec{w}_i$ for any i.*

Proof. We will show that there exists an $(n-1)$-dimensional subspace of V such that $\vec{v}_1, \vec{w}_1, \ldots, \vec{w}_t$ are not contained in that subspace. Let $\vec{v}_1 = \vec{w}_0$ and let $\vec{w}_i = (a_{i1}, \ldots, a_{in})$ for $0 \leq i \leq t$, with respect to the standard basis of V over $\mathbb{K}$. Since $\mathbb{K}$ is infinite, we can choose $b_1, \ldots, b_n \in \mathbb{K}$ such that

$$a_{i1}b_1 + \cdots + a_{in}b_n \neq 0,$$

for each $0 \leq i \leq t$, Let $L = b_1X_1 + \cdots + b_nX_n$. Let $W := \{\vec{v} \in V \big| L(\vec{v}) = 0\}$. Then for each $0 \leq i \leq t$, $w_i \notin W$. Let $\{\vec{v}_2, \ldots, v_n\}$ be a basis of W over $\mathbb{K}$. Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis of V over $\mathbb{K}$ such that $Span\{\vec{v}_2, \ldots, \vec{v}_n\}$ over $\mathbb{K}$, does not contain $\vec{w}_i$ for any i. □

Corollary 5.3.4. *Let $\mathbb{K}$ be any number field and let V be an n–dimensional vector space over $\mathbb{K}$. For $\mathfrak{p} \in \Omega$, $\mathbb{K}_\mathfrak{p}$ represent the completion of $\mathbb{K}$ at $\mathfrak{p}$. Let $\vec{v}_1 \in V$, and for $1 \leq i \leq t$, let $\vec{w}_i \in \mathbb{K}^n_{\mathfrak{p}_i}$. Then there exists a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V over $\mathbb{K}$ such that $Span\{\vec{v}_2, \ldots, \vec{v}_n\}$ over $\mathbb{K}_{\mathfrak{p}_i}$, does not contain $\vec{w}_i$ for any i.*

Proof. Let $\vec{v}_1 = \vec{w}_0$ and let $\vec{w}_i = (a_{i1}, \ldots, a_{in})$ for $0 \leq i \leq t$, with respect to the standard basis of V over $\mathbb{K}_{\mathfrak{p}_i}$. Since $\mathbb{K}_{\mathfrak{p}_i}$ is infinite, we can choose $b_{i1}, \ldots, b_{in} \in \mathbb{K}_{\mathfrak{p}_i}$ such that

$$a_{i1}b_{i1} + \cdots + a_{in}b_{in} \neq 0,$$

for each $0 \le i \le t$. For any given $\epsilon > 0$, using Proposition 2.4.1, we can choose $b_1, \ldots, b_n \in \mathbb{K}$ such that

$$|(b_1, \ldots, b_n) - (b_{i1}, \ldots, b_{in})|_{\mathfrak{p}_i} < \epsilon.$$

This implies that, we can choose $b_1, \ldots, b_n \in \mathbb{K}$ such that

$$a_{i1}b_1 + \cdots + a_{in}b_n \neq 0,$$

for each $0 \le i \le t$.

Let $L = b_1X_1 + \cdots + b_nX_n$. Let $W := \{\vec{v} \in V \big| L(\vec{v}) = 0\}$. W is a $(n-1)$-dimensional subspace of the vector space V over $\mathbb{K}$. Let $\{\vec{v}_2, \ldots, v_n\}$ be a basis of W over $\mathbb{K}$. $\vec{v}_1 \notin W$, because $L(\vec{v}_1) = L(\vec{w}_0) \neq 0$. Hence, $\{\vec{v}_1, \vec{v}_2, \ldots, v_n\}$ is a basis of V over $\mathbb{K}$. Note that for $0 \le i \le t$, $\vec{w}_i \notin Span_{\mathbb{K}_{\mathfrak{p}_i}}\{\vec{v}_2, \ldots, \vec{v}_n\}$, since $L(\vec{w}_i) \neq 0$. Therefore, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of V over $\mathbb{K}$ such that $Span\{\vec{v}_2, \ldots, \vec{v}_n\}$ over $\mathbb{K}_{\mathfrak{p}_i}$ does not contain $\vec{w}_i$ for any i. □

Proposition 5.3.5. *[3, Proposition 3.13] Let $\mathbb{K}$ be a number field and let $Q(Y_1, \ldots, Y_n)$, $Q_1(Y_{n+1}, \ldots, Y_m)$, and $Q_2(Y_{n+1}, \ldots, Y_m)$ be quadratic forms with coefficients in $\mathbb{K}$ such that Q and Q_2 have rank at least 3. Suppose that the following quadratic forms*

$$Q(Y_1, \ldots, Y_n) + Q_1(Y_{n+1}, \ldots, Y_m) = 0, \quad Q_2(Y_{n+1}, \ldots, Y_m) = 0 \tag{5.3}$$

have a common nonsingular zero in $\mathbb{K}_\mathfrak{p}$ for each $\mathfrak{p} \in \Omega$. Then they have a common nontrivial zero over $\mathbb{K}$.

Proof. If $(Y_1, \ldots, Y_m)$ is any nonsingular zero of the given quadratic forms, then the point $(Y_{n+1}, \ldots, Y_m)$ is not $(0, \ldots, 0)$ and is a nonsingular zero of Q_2. Thus if the given system has a nonsingular common zero of the form $(\mathcal{Y}_{1\mathfrak{p}}, \ldots, \mathcal{Y}_{n\mathfrak{p}}, \mathcal{Y}_{n+1\mathfrak{p}}, \ldots, \mathcal{Y}_{m\mathfrak{p}})$ over $\mathbb{K}_\mathfrak{p}$ for all places $\mathfrak{p}$ of $\mathbb{K}$, then $(\mathcal{Y}_{n+1\mathfrak{p}}, \ldots, \mathcal{Y}_{m\mathfrak{p}})$ is a nonsingular zero of Q_2 for all the places $\mathfrak{p}$ of $\mathbb{K}$. Therefore, by the Hasse-Minkowski Theorem, Q_2 has a nonsingular zero over $\mathbb{K}$ as well.

Let

$$S_0 = \{\mathfrak{p} \in \Omega \Big| Q(Y_1, \ldots, Y_n) \text{ is anisotropic over } \mathbb{K}_{\mathfrak{p}}\}.$$

Since $\operatorname{rank}(Q) \geq 3$, we get that S_0 is finite. By Lemma 5.3.3, for each place $\mathfrak{p} \in S_0$ of $\mathbb{K}$, we can choose $(\mathcal{Y}_{1\mathfrak{p}}, \ldots, \mathcal{Y}_{n\mathfrak{p}}, \mathcal{Y}_{n+1\mathfrak{p}}, \ldots, \mathcal{Y}_{m\mathfrak{p}})$ such that $\mathcal{Y}_{n+1\mathfrak{p}} = 1$. Therefore by using Proposition 2.4.3, we can find a nontrivial zero $\vec{b} = (b_{n+1}, \ldots, b_m)$ of Q_2 over $\mathbb{K}$ that is arbitrarily close to $(\mathcal{Y}_{n+1\mathfrak{p}}, \ldots, \mathcal{Y}_{m\mathfrak{p}})$ for each $\mathfrak{p} \in S_0$. This implies that $b_{n+1} \neq 0$. Therefore, after $\mathbb{K}-$ linear change of variables involving only the variables $Y_{n+1}, \ldots, Y_m$, we may assume the above system of quadratic forms reduces to

$$Q(Y_1, \ldots, Y_n) + Q_1(Y_{n+1}, \ldots, Y_m) = 0$$

$$Q_2 = Y_{n+1}Y_{n+2} + h(Y_{n+3}, \ldots, Y_m),$$

where h is a quadratic form with coefficients in $\mathbb{K}$, and $\vec{b} = (1, b_{n+2}, \ldots, b_m)$.

Case 1: Suppose that S_0 is empty. By the Hasse-Minkowski Thoerem, we get that Q is isotropic over $\mathbb{K}$. Then any nontrivial zero of Q can be extended to a nontrivial zero of the given system by setting the remaining variables $Y_{n+1}, \ldots, Y_m$ equal to zero.

Case 2: Suppose that S_0 is a nonempty finite set *i.e.*, Q is anisotropic over $\mathbb{K}$. We will show that the above system of quadratic forms has a nontrivial common zero over $\mathbb{K}$. Take $Y_{n+1} = 1$ and set $Y_{n+2} = -h(Y_{n+3}, \ldots, Y_m)$. Consider the equation defined by

$$Q(Y_1, \ldots, Y_n) + Q_1(1, -h(Y_{n+3}, \ldots, Y_m), Y_{n+3}, \ldots, Y_m) = 0 \tag{5.4}$$

where $Q(Y_1, \ldots, Y_n)$ is a quadratic form with coefficients in $\mathbb{K}$ and has rank at least 3, and

$$Q_1(1, -h(Y_{n+3}, \ldots, Y_m), Y_{n+3}, \ldots, Y_m)$$

is a polynomial in $\mathbb{K}[Y_{n+3},\dots,Y_m]$.

Note that for each place $\mathfrak{p}\in S_0$, $(\mathcal{Y}_{1\mathfrak{p}},\dots,\mathcal{Y}_{n\mathfrak{p}},1,\mathcal{Y}_{(n+2)\mathfrak{p}},\dots,\mathcal{Y}_{m\mathfrak{p}})$ is a zero of Q_2 as well as $Q+Q_1$. This implies that

$$\mathcal{Y}_{(n+2)\mathfrak{p}} = -h(Y_{(n+3)\mathfrak{p}},\dots,\mathcal{Y}_{m\mathfrak{p}})$$

$$\begin{aligned} Q(\mathcal{Y}_{1\mathfrak{p}},\dots,\mathcal{Y}_{n\mathfrak{p}}) + Q_1(1,\mathcal{Y}_{(n+2)\mathfrak{p}},\dots,\mathcal{Y}_{m\mathfrak{p}}) &= Q(\mathcal{Y}_{1\mathfrak{p}},\dots,\mathcal{Y}_{n\mathfrak{p}}) + \\ &\quad Q_1(1,-h(Y_{(n+3)\mathfrak{p}},\dots,\mathcal{Y}_{m\mathfrak{p}}),Y_{(n+3)\mathfrak{p}},\dots,\mathcal{Y}_{m\mathfrak{p}}) \\ &= 0. \end{aligned}$$

This implies that for all $\mathfrak{p}\in S_0$,

$$Q(Y_1,\dots,Y_n) + Q_1(1,-h(Y_{n+3},\dots,Y_m),Y_{n+3},\dots,Y_m) = 0$$

is isotropic over $\mathbb{K}_{\mathfrak{p}}$.

For $\mathfrak{p}\notin S_0$, we get the following two cases:

1. If $Q_1(\vec{b}) = 0$, then $\vec{b}$ is a common zero of Q_1 and Q_2 over $\mathbb{K}$. Since Q is an isotropic form of rank at least 3 over $\mathbb{K}_{\mathfrak{p}}$, let $(\mathcal{Y}_1,\dots,\mathcal{Y}_n)$ be a nonsingular of Q over $\mathbb{K}_{\mathfrak{p}}$. Then $(\mathcal{Y}_1,\dots,\mathcal{Y}_n,\vec{b})$ is a nonsingular zero of the polynomial in equation (5.4).

2. If $Q_1(\vec{b}) = c\in\mathbb{K}^\times$, then we consider the quadratic form $Q(Y_1,\dots,Y_n)+cZ^2$ with rank at least 4 over $\mathbb{K}$ in variables $Y_1,\dots,Y_n,\ Z$. Since Q is an isotropic form over $\mathbb{K}_{\mathfrak{p}}$, by Theorem 3.4 in [8], page 10, we know that Q is universal over $\mathbb{K}_{\mathfrak{p}}$. Therefore, we can find a nonsingular zero of $Q(Y_1,\dots,Y_n)+cZ^2$, where $Z\neq 0$. W.L.O.G., we can take $Z=1$, and let $(\mathcal{Y}_1,\dots,\mathcal{Y}_n,1)$ represent that nonsingular zero. Then

$$(\mathcal{Y}_1,\dots,\mathcal{Y}_n,1,b_{n+3},\dots,b_m) = (\mathcal{Y}_1,\dots,\mathcal{Y}_n,\vec{b})$$

is a nonsingular zero of the polynomial in equation (5.4).

This implies that

$$Q(Y_1,\dots,Y_n)+Q_1(1,-h(Y_{n+3},\dots,Y_m),Y_{n+3},\dots,Y_m)=0$$

has a nonsingular zero over $\mathbb{K}_\mathfrak{p}$ for all places $\mathfrak{p}$ of $\mathbb{K}$, and hence, it satisfies the hypothesis of Proposition 5.3.2. Therefore, we can choose a $(\mathcal{Y}_1,\dots,\mathcal{Y}_n,\mathcal{Y}_{n+3},\dots,\mathcal{Y}_m)$ over $\mathbb{K}$ such that

$$Q(\mathcal{Y}_1,\dots,\mathcal{Y}_n)+Q_1(1,-h(\mathcal{Y}_{n+3},\dots,\mathcal{Y}_m),\mathcal{Y}_{n+3},\dots,\mathcal{Y}_m)=0.$$

Let $\mathcal{Y}_{n+2}=-h(\mathcal{Y}_{n+3},\dots,\mathcal{Y}_m)$. Then

$$(\mathcal{Y}_1,\dots,\mathcal{Y}_n,1,\mathcal{Y}_{n+2},\mathcal{Y}_{n+3},\dots,\mathcal{Y}_m))$$

is the required common zero over $\mathbb{K}$.

□

Proposition 5.3.6. *[3, Proposition 3.14] Let $\mathbb{K}$ be a number field and f,g be two quadratic forms in $n\geq 6$ variables such that (f,g) is a nondegenerate pair. Assume that every form in the $\mathbb{K}$-pencil has rank at least three. Suppose that there exists a form of rank at most $n-3$ in the $\mathbb{K}$-pencil generated by f, and g, and the forms f,g have a nonsingular common zero over $\mathbb{K}_\mathfrak{p}$ for each $\mathfrak{p}\in\Omega$. Then they have a nonsingular common zero over $\mathbb{K}$.*

Proof. W.L.O.G., we can assume that $r_f=\operatorname{rank}(f)\leq n-3$, $r_f\geq 3$, and

$$f=\sum_{i=1}^{r_f}a_iX_i^2,$$

where $a_i\in\mathbb{K}^\times$. Let

$$H=\operatorname{rad}(f)=\{\vec{v}\in\mathbb{K}^n|X_1=\dots=X_{r_f}=0\}.$$

If $g|_H$ is either identically zero or $0<\operatorname{rank}(g|_H)<n-r_f$, *i.e.*, $g|_H$ is singular, then there exists a $\mathbb{K}$–rational zero of g on H, and hence f, g have a nontrivial $\mathbb{K}$–rational zero.

So we assume that $\mathrm{rank}(g|_H) = n - r_f$ *i.e.,* $g|_H$ is a nonsingular quadratic form in the variables $X_{r_f+1}, \ldots, X_n$.

A $\mathbb{K}$–linear change of variables involving only the variables $X_{r_f+1}, \ldots, X_n$ reduces g to the form

$$g = \sum_{i=r_f+1}^{n} b_i X_i^2 + \sum_{i=r_f+1}^{n} X_i L_i(X_1, \ldots, X_{r_f}) + Q(X_1, \ldots, X_{r_f}),$$

where $b_i \in \mathbb{K}^{\times}$, L_i is a linear form for each i, and Q is a quadratic form in variables $X_1, \ldots, X_{r_f}$ with coefficients in $\mathbb{K}$.

We define a nonsingular linear change of variables over $\mathbb{K}$ as follows:

$$\begin{aligned} X_i &\mapsto X_i; \qquad 1 \le i \le r_f \\ X_i &\mapsto X_i - \frac{L_i}{2b_i}; \qquad r_f + 1 \le i \le n \end{aligned}$$

Note that under this linear change of variables f stays fixed and g reduces the following form

$$\begin{aligned} g &= \sum_{i=r_f+1}^{n} b_i \left(X_i - \frac{L_i}{2b_i} \right)^2 + \sum_{i=r_f+1}^{n} \left(X_i - \frac{L_i}{2b_i} \right) L_i(X_1, \ldots, X_{r_f}) + Q(X_1, \ldots, X_{r_f}) \\ &= \sum_{i=r_f+1}^{n} \left(b_i X_i^2 - X_i L_i + \frac{L_i^2}{4b_i} \right) + \sum_{i=r_f+1}^{n} \left(X_i L_i - \frac{L_i^2}{2b_i} \right) + Q(X_1, \ldots, X_{r_f}) \\ &= \sum_{i=r_f+1}^{n} \left(b_i X_i^2 - \frac{L_i^2}{4b_i} \right) + Q(X_1, \ldots, X_{r_f}) \\ &= \sum_{i=r_f+1}^{n} b_i X_i^2 + Q_1(X_1, \ldots, X_{r_f}) \end{aligned}$$

As a result of the above linear change of variables f and g are reduced to

$$g = \sum_{i=r_f+1}^{n} b_i X_i^2 + Q_1(X_1, \ldots, X_{r_f})$$

$$f = f(X_1, \ldots, X_{r_f}),$$

which satisfies the hypothesis of Proposition 5.3.5 (take $Q_2 = f$ and $Q = \sum_{i=r_f+1}^{n} b_i X_i^2$).

Therefore, f and g have a nontrivial common zero over $\mathbb{K}$. □

5.4 Proof of the Main Theorem for $n \geq 9$ Variables.

We assume that every form in the $\mathbb{K}$−pencil generated by f and g has rank at least 5 and that for each real completion $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$, every form in the $\mathbb{K}_\mathfrak{p}$−pencil is indefinite.
By Proposition 2.3.1, we now know that if every form in the $\mathbb{K}$− pencil generated by f and g is singular, then f and g have a nontrivial common $\mathbb{K}$−rational zero.

By Lemma 2.5.2 and Proposition 5.3.6, if there exists a form in the $\mathbb{K}$−pencil with rank at most 6, then f and g have a nontrivial common $\mathbb{K}$−rational zero.
Therefore, we may assume that the $\mathbb{K}$− pencil generated by f and g contain at least one nonsingular quadratic form and every nonzero form in the $\mathbb{K}$−pencil has rank at least 7. This implies that the determinant polynomial $\det(\lambda f + \mu g)$ is not the zero polynomial, and hence the polynomial $\det(\lambda f + \mu g)$ has at most finitely many zeros. This implies that the $\mathbb{K}$−pencil generated by f and g contains only finitely many singular forms.
Therefore, W.L.O.G., we may assume that the $\mathbb{K}$−pencil generated by f and g contains nonsingular quadratic forms and every nonzero form in the pencil has rank at least 7.

By Proposition 2.5.1, we know that there exists a nonsingular form in the $\mathbb{K}$−pencil that contains at least 3 hyperbolic planes over $\mathbb{K}$.. Therefore, W.L.O.G, we may assume that f is a nonsingular quadratic form over $\mathbb{K}$ such that it contains at least 3 hyperbolic planes and g is another nonsingular quadratic form over $\mathbb{K}$ such that every nonzero form in the $\mathbb{K}$–pencil generated by f and g has rank at least 7.

Hence using the technique for splitting off hyperplane as described in Section 4.4, and after a nonsingular linear change of variables we may rewrite f and g over $\mathbb{K}$ as follows

$$\begin{aligned} f &= X_1X_2 + X_3X_4 + X_5X_6 + f_0(X_7, \dots, X_n), \\ g &= g(X_1, \dots, X_n). \end{aligned} \tag{5.5}$$

Let the space H_0 be defined by

$$\begin{aligned} X_2 = X_4 = X_6 = X_7 = \cdots = X_n = 0 \\ g(X_1, 0, X_3, 0, X_5, 0, \dots, 0) = 0 \end{aligned} \tag{5.6}$$

we like to recall that using the techniques that are demonstrated in Section 4.4 to split off hyperbolic planes in f guarantees that rank of $g(X_1, 0, X_3, 0, X_5, 0, \dots, 0)$ is exactly 3. Let $\mathscr{P}' = \{\mathfrak{p} \big| g(X_1, 0, X_3, 0, X_5, 0, \dots, 0) \text{ is anisotropic over } \mathbb{K}_{\mathfrak{p}}.\}$
Since $\operatorname{rank}(g(X_1, 0, X_3, 0, X_5, 0, \dots, 0)) = 3$, we get that $\mathscr{P}'$ is a finite set.

Let $\mathbb{F}$ be any overfield of $\mathbb{K}$. We define $H_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{F}$ to be a space given by

$$\begin{aligned} X_4 &= \alpha X_2 \\ X_6 &= \beta X_2. \end{aligned} \tag{5.7}$$

For each place $\mathfrak{p}$ on $\mathbb{K}$, let $\mathbb{V}_{\mathfrak{p}}(f, g)$ denote the set of all common zeros of f, g over $\mathbb{K}_{\mathfrak{p}}$ and $\mathbb{V}(f, g) = \cup_{\mathfrak{p}} \mathbb{V}_{\mathfrak{p}}(f, g)$. By Lemma 2.5.2, we know that $\mathbb{V}_{\mathfrak{p}}$ contains nonsingular common zeros of f, g over $\mathbb{K}_{\mathfrak{p}}$ for each place $\mathfrak{p}$ on $\mathbb{K}$.

Consider the Jacobian matrix for the system

$$(X_4 - \alpha X_2, X_6 - \beta X_2, f, g).$$

$$\begin{bmatrix} 0 & -\alpha & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\beta & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ X_2 & X_1 & X_4 & X_3 & X_6 & X_5 & f'_{X_7} & \cdots & f'_{X_n} \\ g'_{X_1} & g'_{X_2} & g'_{X_3} & g'_{X_4} & g'_{X_5} & g'_{X_6} & g'_{X_7} & \cdots & g'_{X_n} \end{bmatrix} \tag{5.8}$$

For $i = 3, 5, 7, \dots, n$, we let $h_i = X_2 g'_{X_i} - f'_{X_i} g'_{X_1}$. Note that h_i is a quadratic form of rank at most 4 over $\mathbb{K}$.

Claim 5.4.1. *There exists at least one h_i that is not identically zero.*

Proof. Suppose $h_i = 0$ identically for each $i = 3, 5, 7, \dots, n$. Then

$$h_3 = 0 \implies X_2 g'_{X_3} = X_4 g'_{X_1} \tag{5.9}$$

This implies that there exists a $\gamma \in \mathbb{K}$ such that

$$g'_{X_1} = \gamma X_2 \tag{5.10}$$

Therefore,

$$\begin{aligned} h_5 = 0 &\implies X_2 g'_{X_5} = X_6 g'_{X_1} \implies g'_{X_5} = \gamma X_6 \\ h_7 = 0 &\implies X_2 g'_{X_7} = f'_{X_7} g'_{X_1} \implies g'_{X_7} = \gamma f'_{X_7} \\ h_8 = 0 &\implies X_2 g'_{X_8} = f'_{X_8} g'_{X_1} \implies g'_{X_8} = \gamma f'_{X_8} \\ h_9 = 0 &\implies X_2 g'_{X_9} = f'_{X_9} g'_{X_1} \implies g'_{X_9} = \gamma f'_{X_9} \end{aligned} \tag{5.11}$$

Thus the partial derivatives of $\gamma f - g$ w.r.t X_i for $i = 3, 5, 7, \dots, n$ are identically 0. Hence, the quadratic form $\gamma f - g$ being a function of X_1, X_2, and X_4 only is of rank at most 3, which is a contradiction as every form in the $\mathbb{K}$–pencil generated by f, and g has rank at most 7. This completes the proof of Claim 5.4.1. □

W.L.O.G., we may assume that h_3 is not identically 0.

Claim 5.4.2. *We can choose $\alpha_\mathfrak{p}, \beta_\mathfrak{p} \in \mathbb{K}_\mathfrak{p}$ such that $H_{\alpha_\mathfrak{p},\beta_\mathfrak{p}}$ contains nonsingular common zero of f, g over $\mathbb{K}_\mathfrak{p}$ such that $X_2 \neq 0$.*

Proof. By Lemma 5.2.1, for each place $\mathfrak{p} \in \mathscr{P}'$, we can choose a common zero $\vec{P}_\mathfrak{p}$ of f, g over $\mathbb{K}_\mathfrak{p}$ such that $X_2 \neq 0$, and $h_3(\vec{P}_\mathfrak{p}) \neq 0$. This implies that the jacobian matrix (5.8) evaluated at $\vec{P}_\mathfrak{p}$ has full rank, and hence $\vec{P}_\mathfrak{p}$ is a nonsingular common zero of f, g over $\mathbb{K}_\mathfrak{p}$ such that it does not lie on $X_2 = 0$.
Let $\vec{P}_\mathfrak{p} = (u_{1\mathfrak{p}}, \ldots, u_{n\mathfrak{p}})$, where $u_{2\mathfrak{p}} \neq 0$. Then we can take $\alpha_\mathfrak{p} = \dfrac{u_{4\mathfrak{p}}}{u_{2\mathfrak{p}}}$ and $\beta_\mathfrak{p} = \dfrac{u_{6\mathfrak{p}}}{u_{2\mathfrak{p}}}$.
Therefore, for each place $\mathfrak{p} \in \mathscr{P}'$, we can choose $\alpha_\mathfrak{p}, \beta_\mathfrak{p} \in \mathbb{K}_\mathfrak{p}$ such that $H_{\alpha_\mathfrak{p},\beta_\mathfrak{p}}$ contains nonsingular common zero of f, g over $\mathbb{K}_\mathfrak{p}$. This completes the proof of Claim 5.4.2. □

Since $\mathscr{P}'$ is a finite set, by using Proposition 2.4.1, we can find $\alpha, \beta \in \mathbb{K}$ such that for each place $\mathfrak{p} \in \mathscr{P}'$, there exists a nonsingular $\mathbb{K}_\mathfrak{p}$–zero of f, g that lies in $H_{\alpha,\beta}$.

Let $V_1(f,g) = \mathbb{V}(f,g) \cap H_{\alpha,\beta}$. Then,

$$\begin{aligned} f_1 &= f\big|_{H_{\alpha,\beta}} = X_2(X_1 + \alpha X_3 + \beta X_5) + q_1(X_7, \ldots, X_n) \\ g_1 &= g\big|_{H_{\alpha,\beta}} = g(X_1, X_2, X_3, \alpha X_2, X_5, \beta X_2, X_7, \ldots, X_n), \end{aligned} \tag{5.12}$$

are quadratic forms in the $(n-2) \geq 7$ variables $X_1, X_2, X_3, X_5, X_7, \ldots, X_n$.

Claim 5.4.3. *f_1, g_1 are independent quadratic forms over $\mathbb{K}$, that is, they do not have non constant common factor over $\mathbb{K}$. .*

Proof. Note that $\operatorname{rank}(f_1) = n - 4 \geq 5$.

Given any $\lambda \in \overline{\mathbb{K}}$, note that

$$g_1 + \lambda f_1\big|_{H_0} = g_1 + \lambda f_1\big|_{X_2 = X_7 = \cdots = X_n = 0} = g + \lambda f\big|_{H_0}$$

Since $\operatorname{rank}(g\big|_{H_0}) = 3$ and $f\big|_{H_0} = 0$,

$$\operatorname{rank}(g_1 + \lambda f_1\big|_{H_0}) = \operatorname{rank}(g + \lambda f\big|_{H_0}) = 3.$$

Thus f_1, g_1 have no common nonconstant factors. □

If f_1 and g_1 can be expressed in less that $n-2$ variables over $\mathbb{K}$, then we can find a common singular point of f_1 and g_1 in V_1 with coordinates in $\mathbb{K}$. So we assume that the system of quadratic forms f_1 and g_1 cannot be expressed in less that $n-2$ variables over $\mathbb{K}$ *i.e,* f_1, g_1 is a nondegenerate system of quadratic forms over $\mathbb{K}$ in variables $X_1, X_2, X_3, X_5, X_7, \ldots, X_n$.

Claim 5.4.4. *V_1 has nonsingular $\mathbb{K}_\mathfrak{p}$–points for all places $\mathfrak{p}$ on $\mathbb{K}$.*

Proof.

1. By Claim 5.4.2 we know that V_1 contains nonsingular $\mathbb{K}_\mathfrak{p}$–points of f and g for each $\mathfrak{p} \in \mathscr{P}'$.
2. For $\mathfrak{p} \notin \mathscr{P}'$, note that $\mathbb{V}(f,g) \cap H_0 \subset V_1$. By Proposition 2.4.2, we can choose a point in $\mathbb{V}(f,g) \cap H_0$ such that $X_1 + \alpha X_3 + \beta X_5 \neq 0$. As a result, this point will be a nonsingular $\mathbb{K}_\mathfrak{p}$–point of f, g.

Therefore, we conclude that V_1 has nonsingular $\mathbb{K}_\mathfrak{p}$–points for all places $\mathfrak{p}$ on $\mathbb{K}$. □

In other words, Claim 5.4.4 implies that quadratic forms defined in (5.12)

$$\begin{aligned} f_1 &= f\big|_{H_{\alpha,\beta}} = X_2(X_1 + \alpha X_3 + \beta X_5) + q_1(X_7, \ldots, X_n) \\ g_1 &= g\big|_{H_{\alpha,\beta}} = g(X_1, X_2, X_3, \alpha X_2, X_5, \beta X_2, X_7, \ldots, X_n), \end{aligned}$$

have nonsingular common zeros over $\mathbb{K}_\mathfrak{p}$ for each place $\mathfrak{p}$ of $\mathbb{K}$. We consider the following two cases:

Case 1. Since $(n-2) \geq 7$, suppose that there exists a form in the $\mathbb{K}$–pencil defined by f_1 and g_1 that has rank at most $((n-2)-3) = (n-5)$. In this case we can apply

Proposition 5.3.6 to conclude that f_1 and g_1 have a nontrivial common zero over $\mathbb{K}$.

Case 2. Suppose that every form in the $\mathbb{K}$-pencil defined by f_1 and g_1 has rank at least $n-4$. Since f_1 does not have full rank ($\text{rank}(f_1) = n-4 \geq 5$), it has singular zeros over $\mathbb{K}$. Let $H = \text{rad}(f_1)$. Note that the dimension of H is two. Therefore, $0 \leq \text{rank}(g_1\big|_H) \leq 2$. After a nonsingular linear transformation over $\mathbb{K}$, we can assume that

$$\begin{aligned} f_1 &= X_1X_2 + q_1(X_7,\dots,X_n) \\ g_1 &= g_1(X_1,X_2,X_3,X_5,X_7,\dots,X_n), \end{aligned} \tag{5.13}$$

and $H = \{\vec{v} \in \mathbb{K}^{n-2}\big| X_1 = X_2 = X_7 = \cdots = X_n = 0\}$.
This leads us to the following 3 subcases:

A. If $\text{rank}(g_1\big|_H) < 2$, then $g_1\big|_H$ has a nontrivial zero in H. This would imply the f_1 and g_1 have a common nontrivial zero in H.

B. Suppose that $\text{rank}(g_1\big|_H) = 2$ and g_1 is a product of two linear forms over $\mathbb{K}$, that is, $g_1\big|_H = L_1 \cdot L_2$, where L_1, L_2 are linear forms over $\mathbb{K}$ in variables X_3, X_5. Therefore g_1 has a nontrivial zero when restricted to H and hence f_1 and g_1 have nontrivial common zero over H.
Therefore, we may assume that H does not contain any nontrivial zero of g_1.

C. Suppose that $\text{rank}(g_1\big|_H) = 2$ and g_1 is not a product of two linear forms. This implies that $g_1\big|_H$ is a nonsingular quadratic form of rank 2 that is of the form

$$L_1^2 - aL_2^2$$

where a is not a square in $\mathbb{K}$, and L_1, L_2 are linearly independent linear forms over $\mathbb{K}$ in variables X_3, X_5. Therefore $g_1\big|_H$ has a pair of conjuagate nontrivial zeros $\vec{Z}_1$ and $\vec{Z}_2$. Note that $\vec{Z}_1$ and $\vec{Z}_2$ are singular common zeros of the pair f_1 and g_1 over $\overline{\mathbb{K}}$ because

$$\frac{\partial f}{\partial X_i}(\vec{Z}_1) = 0, \quad \frac{\partial f}{\partial X_i}(\vec{Z}_2) = 0 \tag{5.14}$$

for each i. After a nonsingular $\mathbb{K}$–linear change of variables, we may assume that

$$\vec{Z}_1 = (0, 0, \sqrt{a}, 1, 0, 0, 0),$$

and

$$\vec{Z}_2 = (0, 0, -\sqrt{a}, 1, 0, 0, 0),$$

where a is a nonsquare point in $\mathbb{K}$. Since $\vec{Z}_1$ and $\vec{Z}_2$ are singular common zeros of f_1 and g_1, W.L.O.G., we may assume that

$$\begin{aligned} f_1 &= \gamma(X_3^2 - aX_5^2) + X_3L_3 + X_5L_5 + f_2 \\ g_1 &= \delta(X_3^2 - aX_5^2) + X_3M_3 + X_5M_5 + g_2, \end{aligned} \tag{5.15}$$

where $\gamma,\ \delta \in \mathbb{K}$, with $L_i,\ M_i$ linear forms in variables $X_1, X_2, X_7, \ldots, X_n$ and with $f_2,\ g_2$ quadratic forms in variables $X_1, X_2, X_7, \ldots, X_n$.
Note that

- $\delta \neq 0$ because $g_1\big|_H$ is of rank 2, and
- $\gamma = 0$ because $f_1\big|_H$ is identically 0 since $H = \mathrm{rad}(f_1)$.

Therefore, W.L.O.G., we make take $\delta = 1$. Using the following nonsingular $\mathbb{K}$–linear change of variables

$$\begin{aligned} X_3 &\mapsto X_3 + \frac{1}{2}M_3 \\ X_5 &\mapsto X_5 - \frac{1}{2a}M_5 \\ X_i &\mapsto X_i \ (i = 1, 2, 7, \ldots, n) \end{aligned} \tag{5.16}$$

which does not affect the conjugate singular points $\vec{Z}_1$ and $\vec{Z}_2$, and we may W.L.O.G., asssume that $M_3 = M_5 = 0$, we get that

$$\begin{aligned} f_1 &= X_3L_3 + X_5L_5 + Q_1(X_1, X_2, X_7, \ldots, X_n) \\ g_1 &= X_3^2 - aX_5^2 + Q_2(X_1, X_2, X_7, \ldots, X_n). \end{aligned} \tag{5.17}$$

Let

$$\begin{aligned} L_3 &= l_{31}X_1 + l_{32}X_2 + l_{37}X_7 + \cdots l_{3n}X_n \\ L_5 &= l_{51}X_1 + l_{52}X_2 + l_{57}X_7 + \cdots l_{5n}X_n, \end{aligned} \tag{5.18}$$

where all the coefficients are in $\mathbb{K}$.

By 5.14, we know that for each $i = 1, 2, 3, 5, 7, \ldots, n$

$$\frac{\partial f}{\partial X_i}(\vec{Z}_1) = 0, \quad \frac{\partial f}{\partial X_i}(\vec{Z}_2) = 0$$

This implies that for $i = 1, 2, 7, \ldots, n$

$$\frac{\partial f}{\partial X_i}(\vec{Z}_1) = l_{3i}\sqrt{a} + l_{5i} = 0, \frac{\partial f}{\partial X_i}(\vec{Z}_2) \quad = -l_{3i}\sqrt{a} + l_{5i} = 0, \tag{5.19}$$

Since a is not a square in $\mathbb{K}$, 5.19 implies that

$$l_{3i} = l_{5i} = 0, \text{for } i = 1, 2, 7, \ldots, n.$$

It then follows that L_3 and L_5 are identically 0. As a result, the pair of quadratic forms f_1 and g_1 now read as

$$\begin{aligned} f_1 &= Q_1(X_1, X_2, X_7, \ldots, X_n) \\ g_1 &= X_3^2 - aX_5^2 + Q_2(X_1, X_2, X_7, \ldots, X_n). \end{aligned} \tag{5.20}$$

We recall that from Claim 5.4.4 that f_1 and g_1 have a nonsingular common zero over $\mathbb{K}_\mathfrak{p}$ for each place $\mathfrak{p}$ of $\mathbb{K}$. It then follows that f_1 has a nonsingular zero in all completions $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$. Hence, by the Hasse-Minkowski Theorem, f_1 has a nonsingular zero over $\mathbb{K}$ as well. Therefore, after a nonsingular linear change only on the variables $X_1, X_2, X_7, \ldots, X_n$, we can rewrite f_1 and g_1 as

$$\begin{aligned} f_1 &= X_1X_2 + q_1(X_7,\ldots,X_n) \\ g_1 &= X_3^2 - aX_5^2 + Q_2'(X_1,X_2,X_7,\ldots,X_n), \end{aligned} \tag{5.21}$$

where q_1 is quadratic form over $\mathbb{K}$. Since $\operatorname{rank}(f_1) = n-4 \geq 5$, by Lemma 2.1.6 it follows that $\operatorname{rank}(q_1) = n-6 \geq 3$. This also implies that q_1 is an irreducible quadratic form over $\mathbb{K}$. If f_1 and Q_2' have a nontrivial common $\mathbb{K}$–rational zero, then we are done because that zero can be extended to a common nontrivial zero of f_1 and g_1 by setting $X_3 = X_5 = 0$. Therefore, for the rest of the proof, we assume that f_1 and Q_2' do not have a common nontrivial $\mathbb{K}$– rational zero. This implies that coefficients of X_1^2 and X_2^2 in Q_2' are nonzero, otherwise $\vec{e}_1$ or $\vec{e}_2$ will be a common zero of f_1 and Q_2' over $\mathbb{K}$.

Now we set $X_1 = 1$ and $X_2 = -q_1(X_7,\ldots,X_n)$ in f_1 and g_1. Under this substitution f_1 is identically zero and g_1 gets transformed to

$$X_3^2 - aX_5^2 + Q_2'(1,-q_1(X_7,\ldots,X_n),X_7,\ldots,X_n), \tag{5.22}$$

where $Q_2'(1,-q_1(X_7,\ldots,X_n),X_7,\ldots,X_n)$ a polynomial over $\mathbb{K}$ of total degree 4 because the coefficient of X_2^2 in $Q_2'(X_1,X_2,X_7,\ldots,X_n)$ is nonzero.

Claim 5.4.5. *The polynomial in 5.22*

$$X_3^2 - aX_5^2 + Q_2'(1,-q_1(X_7,\ldots,X_n),X_7,\ldots,X_n)$$

has a nonsingular zero in each completion $\mathbb{K}_\mathfrak{p}$ *of* $\mathbb{K}$.

Proof. Note f_1 and g_1 are quadratic forms in $(n-2) \geq 7$ variables such every form in the $\mathbb{K}$–pencil generated by f_1 and g_1 is at least 5, and they have a nonsingular common zero over each completion $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$. By Corollary 5.2.4, we know that all nonsingular common zeros over $\mathbb{K}_\mathfrak{p}$ do not lies in a hyperplane. Therefore, it follows that f_1 and g_1 have a nonsingular common zero where $X_1 \neq 0$ in each completion $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$. Let $\vec{P}'_\mathfrak{p} = \left(p_{1\mathfrak{p}}, p_{2\mathfrak{p}}, p_{3\mathfrak{p}}, p_{5\mathfrak{p}}, p_{7\mathfrak{p}}, \ldots, p_{n\mathfrak{p}}\right)$ denote that nonsingular common zero over $\mathbb{K}_\mathfrak{p}$. By multiplying by a constant if necessary, we may assume that $X_1 = 1$ in $\vec{P}'_\mathfrak{p}$ for each $\mathfrak{p}$. Since $\vec{P}'_\mathfrak{p}$ is a nonsingular zero of f_1, it implies that

$$X_2 = p_{2\mathfrak{p}} = -q_1\left(p_{7\mathfrak{p}}, \ldots, p_{n\mathfrak{p}}\right)$$

Since $\vec{P}'_\mathfrak{p}$ is also a nonsingular zero of g_1, it then follows that

$$g(\vec{P}'_\mathfrak{p}) = p_{3\mathfrak{p}}^2 - a p_{5\mathfrak{p}}^2 + Q'_2\left(1, -q_1\left(p_{7\mathfrak{p}}, \ldots, p_{n\mathfrak{p}}\right), p_{7\mathfrak{p}}, \ldots, p_{n\mathfrak{p}}\right) = 0 \quad (5.23)$$

Therefore, the polynomial in 5.22 has a nonsingular zero in each completion $\mathbb{K}_\mathfrak{p}$ of $\mathbb{K}$. □

In order to complete the proof of the main theorem, we require the following lemma, which we will prove later.

Lemma 5.4.6. *Let $Q(X_1, X_2, \ldots, X_n)$ be a quadratic form in $n \geq 4$ variables over any field $\mathbb{F}$ such that the* rank(Q) *is rank at least 3, let $q(X_3, \ldots, X_n)$ be an irreducible quadratic form over $\mathbb{F}$ such that every form in the $\mathbb{F}$−pencil generated by Q and $Q' = X_1X_2 - q(X_3, \ldots, X_n)$ is at least 3.*

Suppose that

$$Q(1, q(X_3, \ldots, X_n), X_3 \ldots, X_n) \quad (5.24)$$

is a nonzero polynomial of degree 4 *over $\mathbb{K}$. Then it is irreducible over $\mathbb{F}$.*

Next we show that the quadratic forms $Q'_2(X_1, X_2, X_7, \ldots, X_n)$, $q_1(X_7, \ldots, X_n)$ as defined in (5.21), and the polynomial $Q'_2(1, -q_1(X_7, \ldots, X_n), X_7, \ldots, X_n)$ satisfy the hypotheses of Lemma 5.4.6 over $\mathbb{K}$.

i. $Q_2'(X_1, X_2, X_7, \ldots, X_n)$ has rank at least 3:

Since that rank(g_1) is $n-2 \geq 7$, we get that rank(Q_2) is at least 5.

ii. $q_1(X_7, \ldots, X_n)$ is irreducible:

Since rank(f_1) $= n-4 \geq 5$, we get that rank(q_1) ≥ 3. This implies that $q_1$is irreducible over $\mathbb{K}$.

iii. rank($\lambda f_1 + \mu Q_2'$) ≥ 3, for all $\lambda, \mu \in \mathbb{K}$:

Note that $f_1 = X_1X_2 + q_1(X_7, \ldots, X_n)$. If there exists $\lambda, \mu \in \mathbb{K}$ such that rank($\lambda f_1 + \mu Q_2'$) < 3, then this would imply that rank($\lambda f_1 + \mu g_1$) < 5. This is a contradiction to the assumption that every form in the $\mathbb{K}$–pencil generated by f_1, g_1 is at least $n-4 \geq 5$.

iv. $Q_2'(1, -q_1(X_7, \ldots, X_n), X_7, \ldots, X_n)$ is a nonzero polynomial of degree 4:

By an earlier assumption, we know that the coefficient of X_2^2 in $Q_2'(X_1, X_2, X_7, \ldots, X_n)$ is nonzero. Therefore $Q_2'(1, -q_1, X_7, \ldots, X_n)$ is a nonzero polynomial of total degree 4.

Then by using [4, Theorem 9.3], we get that the polynomial in 5.22 has a nontrivial $\mathbb{K}$–rational zero. Let $(p_3, p_5, p_7, \ldots, p_n)$ denote that nontrivial $\mathbb{K}$– rational zero. Then $P = (1, -q(p_7, \ldots, p_n), p_3, p_5, p_7, \ldots, p_n)$ is a nontrivial common zero of f_1 and g_1 over $\mathbb{K}$, and hence

$$P' = (1, -q(p_7, \ldots, p_n), p_3, -\alpha q(p_7, \ldots, p_n), p_5, -\beta q(p_7, \ldots, p_n), p_7, \ldots, p_n)$$

is a nontrivial common zero of f and g over $\mathbb{K}$.

Next, will prove the following claim:

Claim 5.4.7. *f and g have a nonsingular common zero over* $\mathbb{K}$.

Proof. If all common zeros of f and g over $\mathbb{K}$ are singular, then by Lemma 2.1.9 there is a form $\lambda_1 f + \mu_1 g$ in the $\mathbb{K}$– pencil generated by f and g that has only singular zeros over $\mathbb{K}$. This is implies that rank$(\lambda_1 f + \mu_1 g) < 5$ or it not indefinite with respect to some real place on $\mathbb{K}$. This is a contradiction to the hypotheses in Theorem 5.1.1 that every form in the $\mathbb{K}-$pencil generated by f and g has rank at least 5 and is indefinite with respect to all real places on $\mathbb{K}$. Therefore, f and g have a nonsingular common zero over $\mathbb{K}$. □

By Lemma 2.3.4, f and g have infinitely many nonsingular common zeros over $\mathbb{K}$..

For the sake of completeness, we state Theorem 9.3 from [4] using the terminology and notation followed in this dissertation:

Theorem. *[4, Theorem 9.3] Let $\mathbb{K}$ be a number field, let a be in $\mathbb{K}^{\times}$ and let $P(x_1,\ldots,x_n)$ be a nonzero irreducible polynomial of total degree at most 4 with coefficients in $\mathbb{K}$. If*

$$y^2 - az^2 = P(x_1,\ldots,x_n)$$

has a nonsingular zero in each completion $\mathbb{K}_{\mathfrak{p}}$ of $\mathbb{K}$, then it has nontrivial $\mathbb{K}-$rational zero.

This completes the proof of the main theorem for the case when the number of variables is at least 9. To this end, we will give a proof of Lemma 5.4.6.

Proof. Note that the coefficient of X_2^2 is nonzero because we are given that $Q(1, q(X_3,\ldots,X_n), X_3,\ldots,X_n)$ is a nonzero polynomial of total degree 4.

We will show that $Q(1, q(X_3,\ldots,X_n), X_3,\ldots,X_n)$ is irreducible over $\mathbb{K}$. Note that

$$\begin{aligned} Q(X_1, X_2, X_3,\ldots,X_n) = {} & \alpha_{22}X_2^2 + \alpha_{21}X_2X_1 + \alpha_{11}X_1^2 \\ & + X_2M_2(X_3,\ldots,X_n) + X_1M_1(X_3,\ldots,X_n) \\ & + h(X_3,\ldots,X_n), \end{aligned} \tag{5.25}$$

where $M_1(X_3,\ldots,X_n)$, $M_2(X_3,\ldots,X_n)$ are linear forms over $\mathbb{K}$, $h(X_3,\ldots,X_n)$ is a quadratic form over $\mathbb{K}$, and $\alpha_{22} \neq 0$. Now substituting $X_1 = 1$ and $X_2 = q(X_3,\ldots,X_n)$ in Q and rearranging the terms we get that

$$\begin{aligned} Q(1,q(X_3,\ldots,X_n),X_3,\ldots,X_n) = \alpha_{22}q^2 + qM_2 \\ + (h(X_3,\ldots,X_n) + \alpha_{21}q) + M_1 + \alpha_{11} \end{aligned} \tag{5.26}$$

Suppose that Q is reducible over $\mathbb{K}$. This gives us the following two cases: **<u>Case 1.</u>** $Q(1,q(X_3,\ldots,X_n),X_3,\ldots,X_n) = L(X_3,\ldots,X_n)Q_3(X_3,\ldots,X_n)$, where L and Q_3 are polynomials of degree 1 and 3, respectively. Let $L^{(i)}$, $Q_3^{(i)}$ represent the homogeneous polynomial of degree i in L, Q_3, respectively. Then

$$L = L^{(1)} + L^{(0)},$$

and

$$Q_3 = Q_3^{(3)} + Q_3^{(2)} + Q_3^{(1)} + Q_3^{(0)},$$

where $L^{(1)}$ and $Q_3^{(3)}$ are nonzero homogeneous polynomials of degree 1 and 3, respectively. Using (5.26), we get that

$$\alpha_{22}q^2 = L^{(1)} \cdot Q_3^{(3)}. \tag{5.27}$$

This implies that L_1 divides q^2, and since L_1 is a linear form, it divides q_1. This is a contradiction as q an irreducible quadratic form over $\mathbb{K}$.

<u>Case 2.</u> $Q(1,q(X_3,\ldots,X_n),X_3,\ldots,X_n) = h_1(X_3,\ldots,X_n) \cdot h_2(X_3,\ldots,X_n)$, where h_1 and h_2 are nonzero polynomials of degree 2 over $\mathbb{K}$. Let $h_1^{(i)}$, $h_2^{(i)}$ represent the homogeneous polynomial of degree i in h_1, h_2, respectively. Then

$$\begin{aligned} h_1 &= h_1^{(2)} + h_1^{(1)} + h_1^{(0)}, \\ h_2 &= h_2^{(2)} + h_2^{(1)} + h_2^{(0)}, \end{aligned} \tag{5.28}$$

where $h_1^{(2)}$, and $h_2^{(2)}$ are nonzero homogeneous polynomials of degree 2 over $\mathbb{K}$. As in the previous case, we use equation (5.26) to observe that

$$\alpha_{22}q^2 = h_1^{(2)}h_2^{(2)}. \tag{5.29}$$

Since q is irreducible, we get that

$$\begin{aligned} h_1^{(2)} &= c_1 q \\ h_2^{(2)} &= c_2 q, \end{aligned}$$

where $c_1c_2 = \alpha_{22}$. W.L.O.G., we take $c_1 = 1$ and $c_2 = \alpha_{22}$ Therefore, we get that

$$\begin{aligned} h_1^{(2)} &= q \\ h_2^{(2)} &= \alpha_{22}q, \end{aligned} \tag{5.30}$$

On comparing the homogeneous polynomial of degree 3 on both sides, we get that

$$\begin{aligned} qM_2 &= h_1^{(2)}h_2^{(1)} + h_2^{(2)}h_1^{(1)} \\ &= qh_2^{(1)} + \alpha_{22}qh_1^{(1)} \end{aligned} \tag{5.31}$$

Therefore,

$$M_2 = h_2^{(1)} + \alpha_{22}h_1^{(1)}. \tag{5.32}$$

On comparing the homogeneous polynomial of degree 2 on both sides, we get that

$$\begin{aligned} h(X_3,\ldots,X_n) + \alpha_{21}q &= h_2^{(0)}h_1^{(2)} + h_1^{(0)}h_2^{(2)} + h_1^{(1)}h_2^{(1)} \\ &= h_2^{(0)}q + h_1^{(0)}\alpha_{22}q + h_1^{(1)}h_2^{(1)} \end{aligned} \tag{5.33}$$

On comparing the homogeneous polynomial of degree 1 on both sides, we get that

$$M_1 = h_2^{(0)} h_1^{(1)} + h_1^{(0)} h_2^{(1)} \tag{5.34}$$

On comparing the constant term on both sides, we get that

$$h_1^{(0)} h_2^{(0)} = \alpha_{11}. \tag{5.35}$$

Note that

$$\begin{aligned}
&\Big[(\alpha_{21} - h_2^{(0)} - \alpha_{22} h_1^{(0)})(X_1X_2 - q(X_3, \dots, X_n)) \\
&\qquad + \left(\alpha_{22}X_2 + h_2^{(0)}X_1 + h_2^{(1)}\right)\left(X_2 + h_1^{(0)}X_1 + h_1^{(1)}\right)\Big] \\
&= \alpha_{22}X_2^2 + \alpha_{21}X_1X_2 + h_1^{(0)}h_2^{(0)}X_1^2 \\
&\qquad + X_2\left(\alpha_{22}h_1^{(1)} + h_2^{(1)}\right) + X_1\left(h_2^{(0)}h_1^{(1)} + h_1^{(0)}h_2^{(1)}\right) \\
&\qquad + \left(q(h_2^{(0)} + \alpha_{22}h_1^{(0)} - \alpha_{21}) + h_1^{(1)}h_2^{(1)}\right)
\end{aligned} \tag{5.36}$$

Substituting information from equations (5.32), (5.33), (5.34), and (5.35),

$$\begin{aligned}
&\Big[(\alpha_{21} - h_2^{(0)} - \alpha_{22} h_1^{(0)})(X_1X_2 - q(X_3, \dots, X_n)) \\
&\qquad + \left(\alpha_{22}X_2 + h_2^{(0)}X_1 + h_2^{(1)}\right)\left(X_2 + h_1^{(0)}X_1 + h_1^{(1)}\right)\Big] \\
&= \alpha_{22}X_2^2 + \alpha_{21}X_1X_2 + \alpha_{11}X_1^2 \\
&\qquad + X_2M_2 + X_1M_1 + h(X_3, \dots, X_n) \\
&\overset{(5.25)}{=} Q(X_1, X_2, \dots, X_n)
\end{aligned} \tag{5.37}$$

Therefore,

$$\begin{aligned}
&Q(X_1, X_2, X_3, \dots, X_n) - C(X_1X_2 - q(X_3, \dots, X_n)) \\
&= \left(\alpha_{22}X_2 + h_2^{(0)}X_1 + h_2^{(1)}\right)\left(X_2 + h_1^{(0)}X_1 + h_1^{(1)}\right)
\end{aligned} \tag{5.38}$$

where $C = (\alpha_{21} - h_2^{(0)} - \alpha_{22}h_1^{(0)})$ is a constant in $\mathbb{F}$. This shows that a there exists a form in the pencil generated by $Q(X_1, X_2, X_3, \dots, X_n)$ and $Q' = X_1X_2 -$

$q(X_3,\ldots,X_n)$ that has rank at 2 which is a contradiction to the assumption that rank of every form in the pencil is at least 3.

Therefore, we have shown that $Q(1, q(X_3,\ldots,X_n), X_3,\ldots,X_n)$ is a nonzero irreducible polynomial over $\mathbb{K}$.

□

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