Periodic Points on Tori: Vanishing and Realizability

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Periodic Points on Tori: Vanishing and Realizability

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
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ABSTRACT OF DISSERTATION

Periodic Points on Tori: Vanishing and Realizability

Let X be a finite simplicial complex and $f\colon X \to X$ be a continuous map. A point $x \in X$ is a fixed point if $f(x) = x$. Classically fixed point theory develops invariants and obstructions to the removal of fixed points through continuous deformation. The Lefschetz Fixed number is an algebraic invariant that obstructs the removal of fixed points through continuous deformation.

$$L(f) = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr}\left(f_i : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})\right).$$

The Lefschetz Fixed Point theorem states if $L(f) \neq 0$, then f (and therefore all $g \simeq f$) has a fixed point. In general, the converse is not true. However, Lefschetz Number is a complete invariant for describing the minimum set of fixed points for continuous maps of tori. That is, if T is the d dimensional torus and $f\colon T \to T$ is continuous, then there exists a map g homotopic to f so that

$$|\operatorname{Fix}(g)| = |L(f)|$$

A point $x \in T$ is a periodic point of order n if $f^n(x) = x$. In this paper we realize the minimum set of periodic points of an endomorphism of tori by studying the sequence of Lefschetz numbers for the iterates of f, $\{L(f^m)\}_{m|n}$. More specifically, there exists a map g homotopic to f so that

$$|\operatorname{Fix}(g)| = \sum_{m|n}(-1)^m \left|L(f^m)\right|.$$

Furthermore, provided the sequence $\{L(f^m)\}_{m|n}$ is not identically zero, it also provides a complete lower bound for maps of tori parameterized by S^1. Therefore, if $F\colon S^1 \to \operatorname{end}(T)$ satisfies $L(F_x) \neq 0$ for all $x \in S^1$, then $F \simeq G$ so that

$$|\operatorname{Fix}(G_x)| = \sum_{m|n}(-1)^m \left|L(F_x^m)\right|.$$

This behavior is very particular to maps of tori and is not expected to generalize to endomorphisms of other spaces, even manifolds. In fact, our extra requirement that $L(F_x) \neq 0$ for $F\colon S^1 \to \mathrm{end}(T)$ is evidence for the requirements of the parameterized invariant suggested in [MP18].

KEYWORDS: Topology, Algebra, Fixed Point Theory, Category Theory

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Periodic Points on Tori: Vanishing and Realizability

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1.1 Introduction

If X is a topological space and $f\colon X \to X$ is an endomorphism of X, then $x \in X$ is a *fixed point* if $f(x) = x$. Fixed point theory is the study of such points and fixed point theorems aim to classify, count, and minimize the number of fixed points. These questions become more tractable when considering homotopy classes of maps instead a single endomorphism. After making this simplification, we can leverage the tools of algebraic topology to define invariants and obstructions to the existence of fixed points. The main questions we address in fixed point theory can be summarized:

i. *Vanishing:* Is f homotopic to a map, g, so that g has no fixed points?

ii. *Realizability*: What map $g \simeq f$ realizes the minimum number of fixed points?

A classical invariant that gives a partial answer to these questions is the Lefschetz number. If X is a space so that $H_i(X;\mathbb{Q})$ is finite dimensional and nonzero for only finitely many i, then the **Lefschetz number** of an endomorphism $f\colon X \to X$ is

$$L(f) = \sum_i (-1)^i \operatorname{tr}(f_i\colon H_i(T;\mathbb{Q}) \to H_i(T;\mathbb{Q})).$$

The Lefschetz Fixed Point theorem states if $L(f) \neq 0$, then f must have a fixed point.

When restricted to tori, the Lefschetz number become a much stronger invariant. We fix some notation for maps between tori since this is the family of spaces we will focus.

Notation 1.1.1. Let $T = \prod_1^d S^1$. Since $T = B\mathbb{Z}^d$ a map $f\colon T \to T$ is homotopic to the linear representative given by $f_*\colon \pi_1(T) \to \pi_1(T)$. Throughout we let

$$M_f\colon \mathbb{R}^d \to \mathbb{R}^d$$

be the matrix representing f_* and $\overline{M}_f\colon T \to T$ is the map induced by M_f. Recall that $f \simeq \bar{M}_f$. More generally for any matrix $M \in \mathbb{Z}^{d\times d}$ we let $\overline{M}$ denote the induced map on the d dimensional torus.

Brooks, Brown, Pak, and Taylor [BBPT75] have shown that the Lefschetz number of a map between tori is

$$L(f) = \det(I - M_f),$$

and f is homotopic to a map g so that

$$|\operatorname{Fix}(g)| = |\det(I - M_f)|.$$

In fact, if $L(f) \neq 0$, then M_f has exactly $|\det(I - M_f)|$ fixed points. Otherwise $L(f) = 0$ and 1 is an eigenvalue of M_f. Then the minimum number of fixed points can be given by a type of "rotation" within the eigenspace of 1. If v is in the eigenspace of 1, then the map $M_f + \pi v$ has fixed points. Therefore the Lefschetz number is a complete invariant for the removal and minimization of fixed points for endomorphisms of tori.

Given a map $f\colon X \to X$ and a natural number $n \in \mathbb{N}$ we say that $x \in X$ is a *periodic point* of order n if $f^n(x) = x$. We can ask the following questions vanishing and realizability of periodic points.

i. Vanishing: Is f homotopic to a map, g, so that g^n has no fixed points?

ii. Realizability: What map $g \simeq f$ realizes the minimum number of periodic points?

Since the Lefschetz number is a complete invariant for maps between tori, we can hope that it is also complete for periodic points. Unfortunately, the Lefschetz number of the n-th iterate of f, $L(f^n)$, is not a strong enough invariant for periodic points.

Example 1.1.2. Let $f\colon S^1 \to S^1$ be the degree -1 map. Then $L(f) = 2$, $f^2 = \mathrm{id}$, and $L(f^2) = 0$. Since $L(f) = 2$ all maps $g \simeq f$ must have at least 2 fixed points and therefore $|\operatorname{Fix}(g^2)| \geqslant 2$.

In this paper we show that the sequence of Lefshcetz numbers, $\{L(f^m)\}_{m|n}$, is enough to determine a lower bound to the number of periodic points of a map $f\colon T \to T$. Furthermore, under certain hypothesis, these Lefschetz numbers are enough to determine a minimum set of periodic points for endomorphisms of tori parameterized by S^1. This is motivated by the work of C.Y. You, Halpern, Heath, Wong, Jeziersky [You95, Jia83, Hea99, BBPT75].

Theorem 1.1.3. *Let f be an endomorphism of a torus and n be an integer. If M_f is the linear map homotopic f, then f is homotopic to a map g with*

$$\operatorname{Fix}(g^n) = \bigcup_{m|n, L(f^m)\neq 0} \operatorname{Fix}(M_f^m).$$

Further, if k is homotopic to f, then

$$|\operatorname{Fix}(g^n)| \leqslant |\operatorname{Fix}(k^n)|.$$

In the described periodic point set, $\bigcup_{m|n, L(f^m)\neq 0} \operatorname{Fix}(M_f^m)$, we are onnly considering $m|n$ so that $L(f^m) \neq 0/$ If there are no $m|n$ so that $L(f^m) = 0$, then

$$\bigcup_{\substack{m|n,\\ L(f^m)\neq 0}} \operatorname{Fix}(M_f^m) = \operatorname{Fix}(M_f^n)$$

and Theorem 1.1.3 asserts that the linear map h achieves a minimum number of periodic points. On the other hand, if every $m|n$ is satisfies $L(f^m) = 0$, then

$$\bigcup_{\substack{m|n,\\ L(f^m)\neq 0}} \operatorname{Fix}(M_f^m) = \varnothing$$

and Theorem 1.1.3 would imply that there exists a $g \simeq f$ with no periodic points. This particular minimum is never achieved by the linear map. Of course there are many intermediary steps between these cases. They are detected by the vanishing of the Lefschetz numbers in the sequence $\{L(f^m)\}_{m|n}$. We highlight some of these cases in the next examples.

Example 1.1.4. Consider the map $\overline{B}\colon T \to T$ induced by the linear map on $\mathbb{R}^2$ given by the matrix

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

For this map we note that $B^6 = I$ and we want to minimize the set $\mathrm{Fix}(\overline{B}^6)$. Calculating these periodic points is an exercise in modular arithmetic and linear algebra, we summarize them Table 1.1.

Table 1.1: Table of Periodic Points

i	1	2	3	4	5	6
$L(\overline{B}^i)$	1	3	4	3	1	0
$\mathrm{Fix}(\overline{B}^i)$	$(0,0)$	$(0,0), \left(\frac{1}{3}, \frac{2}{3}\right)$ $\left(\frac{2}{3}, \frac{1}{3}\right)$	$(0,0), \left(0, \frac{1}{2}\right)$ $\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right)$	$(0,0), \left(\frac{1}{3}, \frac{2}{3}\right)$ $\left(\frac{2}{3}, \frac{1}{3}\right)$	$(0,0)$	(x,y)

In this example every point in T is fixed by $\overline{B}^6$, which is far from optimal! Theorem 1.1.3 asserts that we can remove all periodic points of order exactly 6, however, periodic points of order $1, 2$, and 3 must remain.

Minimizing a set of periodic points for a single endomorphism has been a question of interest in fixed point theory for quite some time. However, it is the first part of a much more broad question. That is, we would like to understand fixed and periodic points for families of endomorphisms parameterized by a topological space. In general, understanding classes of maps parameterized by a space can be very difficult. To first tackle this problem we will rephrase our previous results. Currently we have been working to understand homotopy classes of maps

$$* \to \mathrm{end}(T).$$

Rephrasing Theorem 1.1.3 in this perspective gives us the following corollary.

Corollary 1.1.5. *Let $F\colon S^0 \to \mathrm{end}(T)$ and n be an integer. If $M_{F(x)}$ is the linear map homotopic to $F(x)$ for $x \in S^0$, then F is homotopic to a map G so that*

$$\mathrm{Fix}(G(x)^n) = \bigcup_{m|n, L(G(x)^m)\neq 0} \mathrm{Fix}(M_{F(x)}^m).$$

Further, if K is homotopic to F, then

$$|\operatorname{Fix}(G(-,t)^n)| \leqslant |\operatorname{Fix}(K(-,t)^n)|.$$

Stated in this form, we are led to the following theorem which describes the minimum set of periodic points for families of maps that are parameterized by S^0 and S^1.

Theorem 1.1.6. *Let $l = \{0,1\}$, $F\colon S^l \to \operatorname{end}(T)$ be continuous so that $L(F(0)) \neq 0$, and n be an integer. If M_f is the linear map homotopic $F(0)$ then F is homotopic to a map G with*

$$\operatorname{Fix}(G(-,t)^n) = \bigcup_{m|n, L(M_f^m)\neq 0} \operatorname{Fix}(M_f^m).$$

Further, if K is homotopic to F, then

$$|\operatorname{Fix}(G(-,t)^n)| \leqslant |\operatorname{Fix}(K(-,t)^n)|.$$

1.2 Literature Review

To those unfamiliar with classical fixed point theory, the Lefschetz number does not seem to be related to fixed points. The Lefschetz-Hopf Theorem [Bro71] shows that the Lefschetz number is a weighted sum of the fixed points of f

$$L(f) = \sum_{x\in\operatorname{Fix}(f)} i(x).$$

Where $i(x) \in \mathbb{Z}$ is the *fixed point index* of x [Dol65]. As mentioned in Section 1.1, the Lefschetz number gives a partial answer to the vanishing of fixed points. In order to have a more complete answer we require a more complete invariant. There are many invariants which generalize the Lefschetz Number. In this section, we will introduce some of these invariants and their implications for the vanishing and realizability questions for fixed and periodic points.

1.2.1 The Reidemeister Trace

The *Reidemeister Trace* of f, $R(f)$, generalizes the Lefschetz fixed point theorem by recording the fixed point indices and their corresponding fixed point class. Define the f twisted loop space of X

$$\Lambda_f X := \{\gamma \in X^I | \gamma(0) = f(\gamma(1))\}.$$

For each $x \in \operatorname{Fix}(f)$, the constant path at x, c_x, is an element of $\Lambda_f X$. The homotopy class $[c_x] \in \pi_0(\Lambda_f X)$ is the *fixed point class* of x. The Reidemeister Trace of f is the element of $\mathbb{Z}\pi_0(\Lambda_f X)$ given by

$$R(f) = \sum_{x\in\operatorname{Fix}(f)} i(x)[c_x].$$

If X is a closed smooth manifold of dimension$\geqslant 3$, the Reidemeister Trace of f is a complete invariant for the vanishing of fixed points. Therefore $R(f) = 0$ if and only if there exists a g homotopic to f and $\mathrm{Fix}(g) = \varnothing$ [Gh66, Wec42, Jia83]. Forgetting the fixed point classes in $R(f)$ defines a map $\mathbb{Z}\pi_0(\Lambda_f X) \to \mathbb{Z}$ and the image of $R(f)$ is $L(f)$. Furthermore, the number of fixed point classes of f with non-zero indices recovers the Nielsen number of f, $N(f)$. This gives a lower bound to the number of fixed points of f [Jia83, Gh66]. These invariants will be discussed further in Section 1.2.2.

Example 1.2.1. For $f\colon T \to T$ the Lefschetz Number is $L(f) = \det(I - M_f)$ [BBPT75]. If $L(f) \neq 0$, then there are $|L(f)|$ fixed point classes. Each of which have index $sign(\det(I - f_*)) = \pm 1$. If $L(f) = 0$, then the index of each fixed point class is zero. Therefore, for maps between tori, $R(f)$ can be recovered from $L(f)$.

Like the Lefschetz number, the Reidemeister trace of f^n gives us an obstruction to the removal of periodic points. However, it is usually far from an optimal lower bound, see Example 1.1.2. As an attempt to refine the Reidemeister trace to handle periodic points, we recall the C_n equivariant n^{th} *Fuller Map of f*

$$X \times \ldots \times X \xrightarrow{\Psi_n f} X \times \ldots \times X$$
$$(x_1, \ldots, x_n) \longmapsto (f(x_n), f(x_1), \ldots, f(x_{n-1}))$$

The set of fixed points of $\Psi_n f$ is isomorphic to the set of n periodic points of f, $\mathrm{Fix}(f^n)$. Further, $\mathrm{Fix}(\Psi_n f)/C_n$ decomposes the periodic points into orbits and taking C_m fixed points recovers the $\frac{n}{m}$ fuller map of f $(\Psi_n f)^{C_m} = (\Psi_{\frac{n}{m}}) f^{\times m}$ [Dol83, Fer99, Kom88, Won93]. There is a C_n equivariant analog of the Reidemeister trace of $\Psi_n f$, $R_{C_n}(\Psi_n f)$ which serves as an obstruction to the removal of n periodic points [MP18, Hea99, HY92, Won93]. Malkiewich and Ponto have shown that $R_{C_n}(\Psi_n f)$ recovers $R_{C_m}(\Psi_m f)$ and $R(f^m)$ for $m|n$ [MP18]. Theorems 1.1.3 and 1.1.6 and Corollary 1.1.5 then show that $R_{C_n}(\Psi_n f)$ is a complete invariant for the removal and minimization of periodic points for maps between tori and families of maps parameterized by S^1 with non-trivial Reidemeister trace at the basepoint.

1.2.2 Nielsen Fixed Point Theory

Another invariant that address the questions of vanishing and realizability for periodic points. One is a generalization from Nielsen fixed point theory. Much like the Lefschetz fixed point theorem, Nielsen periodic point theory assigns a number to an endomorphism f. The *Periodic Nielsen Number* [Jia83], $\mathrm{NF}_n(f)$, is an algebraic invariant which satisfies the inequality

$$\mathrm{NF}_n(f) \leqslant \min\{\#\,\mathrm{Fix}(g^n) \,|\, g \simeq f\}.$$

This invariant gives us a lower bound of periodic points but is difficult to compute in general. Jezierski has shown that NF_n is a strict lower bound for endomorphisms of compact manifolds of dimension $\geqslant 3$. Our focus is for families of endomorphisms which requires a more precise construction of such maps for maps between tori. For maps between tori, it is enough to consider the sequence of Lefschetz numbers

$$\{L(f^m)\colon m|n\}.$$

In his survey paper [Hea99], Heath shows that in the case of tori this invariant is

$$\mathrm{NF}_n(f) = \sum_{\varnothing\neq\mu\subset M(f,n)} (-1)^{\#\mu-1}\left|L((f)^{gcd(\mu)})\right|$$

where $M(f,n)$ is the set of maximal divisors of n with non vanishing Lefschetz numbers. However, once we restrict to maps between tori the question of vanishing is not interesting

Vanishing: Is f homotopic to a map g so that g^n has no fixed points?

Answer: $f\colon T\to T$ is homotopic to one with no periodic points if and only if $L(f)=0$. Then f is homotopic to a map with no fixed points [BBPT75, You95, Hal79, Hea99, HY92]. This follows immediately from the fact that if $L(f)=0$, then $L(f^n)=0$ for all n.

However, the question of realizability is still interesting for tori. C.Y. You has proven that this invariant is complete for maps between tori with some additional hypotheses [You95]. Jezierski revisited this result using [Jez06] to show periodic points between tori can be minimized by a smooth map [Jez16]. Halpern [Hal79] shows Periodic Nielsen number is a complete invariant for maps $f\colon T\to T$ so that $f^n=\mathrm{id}$ and $L(f^m)\neq 0$ for proper divisors of n [Hal79]. Corollary 1.1.5 and Theorems 1.1.3 and 1.1.6 recover the aforementioned results for tori using techniques motivated by the aforementioned work of Halpern and Cheng'ye You.

Chapter 2 Outline of the Proof

In [BBPT75], Brooks, Brown, Pak, and Taylor observed that for maps of tori

$$L(f) = (\mathrm{char}(f_*))(1) = \det(I = f_*).$$

This description of the Lefschetz number will be more convenient for our purposes since it will allow us to easily distinguish between endomorphisms of the torus that require different proofs for Theorem 1.1.3. The cases are described by Table 2.1. We will prove their equivalence in the later sections.

Table 2.1: Comparison of Lefschetz numbers and characteristic polynomials

	Lefschetz Property	Characteristic Polynomial Property
1	$L(f^n) \neq 0$	$\mathrm{char}(f_*)$ is not divisible by any cyclotomic polynomial of order $m\|n$.
2	$L(f^n) = 0$, $f^n \simeq \mathrm{id}$, and $L(f^m) \neq 0$ for all $m\|n$	$\mathrm{char}(f_*)$ is a power of a single cyclotomic polynomial and $f_*^n = \mathrm{id}$.
3	$L(f^n) = 0$, $f^n \simeq \mathrm{id}$ and $L(f^m) = 0$ for at least one $m\|n$, $m < n$	$\mathrm{char}(f_*)$ is a product of distinct cyclotomic polynomials of order $m\|n$ and $f_*^n = \mathrm{id}$.
	$L(f^n) = 0$ and $f^n \not\simeq \mathrm{id}$	$\mathrm{char}(f_*)$ is divisible by a cyclotomic polynomial of order $m\|n$ and $f_*^n \neq \mathrm{id}$

Halpern, Brooks-Brown-Pak-Taylor, and C.Y. You proved a version of case 1 of Theorem 1.1.3 [Hal79, BBPT75, You95]. Since its proof is illuminating we will recall it in Theorem 2.0.1. Halpern proved a generalization of case 2 of Theorem 1.1.3 using Morse theory [Hal79]. C.Y. You proved a restricted version of the third case of Theorem 1.1.3 in [You95]. We give new proofs of these results that use similar techniques and provide greater control over the behavior of these maps.

We fix the notation

$$\overline{\mathcal{P}}_n(f) := \bigcup_{m|n, L(f^m)\neq 0} \mathrm{Fix}(\overline{M}_f^m)$$

if M_f is the linear map homotopic f. If $m_1|m_2|n$ then $\mathrm{Fix}(f^{m_1}) \subset \mathrm{Fix}(f^{m_2})$ and the same point may appear in multiple sets $\mathrm{Fix}(\overline{M}_f^m)$. Each such point will only count once in the union. This is in contrast to the periodic Nielsen number [Jia83] that must carefully account for this redundancy.

Theorem 2.0.1 (Theorem 1.1.3, $L(f^n) \neq 0$)**.** *Let f be an endomorphism of a torus and n be an integer so that $L(f^n) \neq 0$. Then*

$$\mathrm{Fix}(\overline{M}_f^n) = \overline{\mathcal{P}}_n(f).$$

Proof. This is a consequence of the following result of Halpern [Hal79]. If $f\colon T \to T$ is covered by a linear map $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and 1 is not a characteristic root of A (equivalently, $L(f) \neq 0$ [BBPT75]), then the fixed points are isolated, they all have the same index, and the number of them is $|L(f)|$.

Therefore, If $f\colon T \to T$ is covered by a linear map so that $L(f^n) \neq 0$, then 1 is not a characteristic root of f^n, and $|\operatorname{Fix}(f^n)| = |L(f^n)|$ [BBPT75]. ☐

Remark 2.0.2. The above proof contains the first equivalence in Table 2.1. That is, 1 is not a root of $\operatorname{char}(f^n)$ if and only if ζ_m is not a root of $\operatorname{char}(f)$ for $m|n$. Furthermore, $\operatorname{char}(f) \in \mathbb{Z}[x]$, Φ_m is irreducible over $\mathbb{Z}[x]$, and ζ_m is not a characteristic root if and only if $\operatorname{char}(f)$ is not divisible by any Φ_m for $m|n$.

The proofs of cases 2 and 3 are more elaborate so we postpone technical aspects of the proofs until Chapter 3 and Section 4.2. In this section we will state the core technical lemmas and show how they imply the remaining cases of Theorem 1.1.3.

Let $\pi\colon \mathbb{R}^d \to T$ be the quotient map. Then we define

$$\mathcal{P}_n(f) := \pi^{-1}(\overline{\mathcal{P}}_n(f)).$$

Proposition 2.0.3. *Let A be an integer matrix so that*

i. $A^n = I$ and

ii. $\det(A^m - I) \neq 0$ for proper divisors of n.

There is a $\mathbb{Z}^d$-homeomorphism $\Theta_n(A)\colon \mathbb{R}^d \to \mathbb{R}^d$ which is homotopic to the identity, commutes with A, and

$$\operatorname{Fix}(\Theta_n(A)^n) = \mathcal{P}_n(A).$$

The construction of $\Theta_n(A)$ is motivated by Halpern's work in [Hal79]. Using Morse Theory, Halpern shows:

Theorem (Theorem 4 [Hal79])**.** *If M is a closed manifold of dimension $\geqslant 2$ and maps $f\colon M \to M$ so that $f^n = \mathrm{id}$ and*

$$|P| := \left| \bigcup_{\substack{m|n \\ m<n}} \operatorname{Fix}(f^m) \right| < \infty.$$

Then $f \simeq g$ so that $\operatorname{Fix}(g^n) = P$ and $f|_P = g|_P$.

When $M = T$ and $f\colon T \to T$ is linear, then $P = \overline{\mathcal{P}}_n(f)$ and Theorem 2.0.4 is a corollary. Chapter 4 requires more precise control than is offered in [Hal79], we achieve this by restricting attention to tori.

Theorem 2.0.4 (Theorem 1.1.3, $L(f^n) = 0$, $f^n \simeq \mathrm{id}$, and $L(f^m) \neq 0$ for all $m|n$)**.** *Let f be an endomorphism of a torus and n be an integer so that*

i. $f^n \simeq \mathrm{id}$ (and so $L(f^n) = 0$) and *ii.* $L(f^m) \neq 0$ *for proper divisors of* n.

Then f *is homotopic to a map* g *with*

$$\operatorname{Fix}(g^n) = \overline{\mathcal{P}}_n(f).$$

Proof. Since $f^n \simeq \mathrm{id}$ and $L(f^m) \neq 0$ for proper divisors of n, $A_f^n = I$ and $\det(I - A_f^m) \neq 0$ for $m|n$ [BBPT75]. Therefore A_f is as in Proposition 2.0.3 and there exists a map $\Theta_n(A_f)$ which commutes with A_f. Consider the map $A_f \circ \Theta_n(A_f) \colon \mathbb{R}^d \to \mathbb{R}^d$. Note that $A_f \simeq A_f \circ \Theta_n(A_f)$ since $\Theta_n(A_f) \simeq \mathrm{id}$. Then

$$\begin{aligned} \operatorname{Fix}((A_f \circ \Theta_n(A_f))^n) &= \operatorname{Fix}(A_f^n \circ \Theta_n(A_f)^n) \\ &= \operatorname{Fix}(\Theta_n(A_f)^n) \\ &= \mathcal{P}_n(f) \end{aligned}$$

Let $\widetilde{A}_f \colon T \to T$ be the map induced by $A_f \circ \Theta_n(A_f)$. Then

$$\operatorname{Fix}(\widetilde{A}_f^n) = \overline{\mathcal{P}}_n(f)$$

which completes the proof. □

The remaining case captures all cases not contained in the first two. While it is not obvious at the moment, the proof technique for this final case will heavily rely on the previous two cases. As in case i and ii, we start by stating a proposition which will imply the desired result when descending to tori.

Proposition 2.0.5. *Let* M *be a square matrix and* n *be an integer so that*

i. $\det(I - M^n) = 0$,

ii. $\operatorname{char}(M) = \Phi_{l_1}^{\alpha_1}\Phi_{l_2}^{\alpha_2}\dots\Phi_{l_s}^{\alpha_s} g$ *where* $l_i|n$, Φ_m *is the cyclotomic polynomial of order* m, *and* $g(\zeta_n) \neq 0$ *for any* n^{th} *root of unity* ζ_n.

Then, up to an conjugation, M *is of homotopic to a map* $\widetilde{M}$ *so that* $\mathcal{P}_n(\overline{M}) \subseteq \operatorname{Fix}(\widetilde{M}^n)$ *and* $\left|\operatorname{Fix}(\widetilde{M}^n) \backslash \mathbb{Z}^d\right| < \infty$. *If the containment is proper, then* $\widetilde{M} \simeq \widehat{M}$ *so that*

$$\mathcal{P}_n(\overline{M}) \subseteq \operatorname{Fix}(\widehat{M}^n) \subsetneq \operatorname{Fix}(\widetilde{M}^n)$$

Theorem 2.0.6 (Theorem 1.1.3, $L(f^n) = 0$, and $f^n \not\simeq \mathrm{id}$ or $L(f^m) = 0$ for at least one $m|n$)**.** *Let* f *be an endomorphism of a torus and* n *is an integer so that*

i. $L(f^n) = 0$ *and*

ii. one of the following two conditions holds:

a) $f^n \simeq \mathrm{id}$ *and* $L(f^m) = 0$ *for some* $m|n$ *and* $m < n$ *or*

b) $f^n \not\simeq \mathrm{id}$.

Then f is homotopic to a map g so that

$$\mathrm{Fix}(g^n) = \overline{\mathcal{P}}_n(f).$$

Proof. We first show that if $f\colon T \to T$ is as in the statement of the theorem, then the matrix defining f_*, M_f, satisfies they hypothesis of Proposition 2.0.5 and case 3 of Table 2.1:

i. $\det(I - M_f^n) = 0$,

ii. $\mathrm{char}(M_f) = \Phi_{l_1}^{\alpha_1}\Phi_{l_2}^{\alpha_2}\dots\Phi_{l_s}^{\alpha_s} g$ where $l_i|n$, Φ_m is the cyclotomic polynomial of order m, and $g(\zeta_n) \neq 0$ where ζ_n is an n^{th} root of unity.

We have already seen that if $f^n \simeq \mathrm{id}$ and $L(f^m) \neq 0$ for proper divisors of n, then $\mathrm{char}(M_f)$ is a power of Φ_n and $f_*^n = \mathrm{id}$. We consider the other two cases independently.

i. Assume $L(f^n) = \det(M_f^n - I) = 0$, $f^n \simeq \mathrm{id}$, and there exists a proper divisor of n so that $L(f^m) = 0$. Since $L(f^m) = \mathrm{char}(M_f^m)(1) = 0$ for proper divisors of n the characteristic polynomial, $\mathrm{char}(M_f)$, will be divisible by Φ_m for proper divisors of n. Furthermore since $f^n \simeq \mathrm{id}$ the induced matrix M_f is such that $M_f^n = \mathrm{id}$.

ii. Otherwise, $L(f^n) = 0$ and $f^n \not\simeq \mathrm{id}$. Then $\det(I - M_f^n) = 0$ and $\mathrm{char}(M_f)$ is divisible by at least one characteristic polynomial of order dividing n. Furthermore, since $f^n \not\simeq \mathrm{id}$ the induced map f_*^n is not the identity.

Next we apply Proposition 2.0.5 to the map M_f. Since $\mathcal{P}_n(f)\backslash\mathbb{Z}^d \cong \overline{\mathcal{P}}_n(f)$ and $\mathrm{Fix}(\widetilde{M_f}^n)\backslash\mathbb{Z}^d$ are finite, Proposition 2.0.5 can be applied finitely many times to produce a map $G\colon \mathbb{R}^d \to \mathbb{R}^d$ with periodic points $\mathcal{P}_n(f)$. Therefore, up to conjugation, the induced map $\overline{G} := g\colon T \to T$ has periodic points $\overline{\mathcal{P}}_n(f)$. □

Together Theorems 2.0.1, 2.0.4 and 2.0.6 complete the proof of the realizability statement of Theorem 1.1.3. Up to this point we have not addressed the minimiality statements of this periodic fixed point set. This follows as a consequence of results of Heath and Brooks-Brown-Pak-Taylor [Hea99, BBPT75].

Corollary 2.0.7 ([Hea99])**.** *Let f be an endomorphism of a torus and n be an integer. Then*

$$\left|\overline{\mathcal{P}}_n(f)\right| \leqslant \min\{|\mathrm{Fix}(g^n)| : g \simeq f\}$$

Proof. Let$\{m_1, m_2, \dots m_t\}$ be the set of maximal divisors, with respect to divisibility, of n such that $L(f^{m_i}) \neq 0$ for $1 \leqslant i \leqslant t$. Then

$$|\overline{\mathcal{P}}_n(f)| = \left|\bigcup_{m|n;L(f^m)\neq 0} \mathrm{Fix}(f^m)\right| = \left|\bigcup_1^t \mathrm{Fix}(f^{m_i})\right|$$

Using an inclusion/exclusion argument we have the equalities.

$$\begin{aligned}\left|\bigcup_1^t \mathrm{Fix}(f^{m_i})\right| &= \sum_{i=1}^t |\mathrm{Fix}(f^{m_i})| - \sum_{1\leqslant i\leqslant j\leqslant t} |\mathrm{Fix}(f^{m_i}) \cap \mathrm{Fix}(f^{m_j})| \\ &+ \sum_{1\leqslant i\leqslant j\leqslant k\leqslant t} |\mathrm{Fix}(f^{m_i}) \cap \mathrm{Fix}(f^{m_j}) \cap \mathrm{Fix}(f^{m_k})| - \dots \\ &+ (-1)^{t-1} |\mathrm{Fix}(f^{m_1}) \cap \mathrm{Fix}(f^{m_2}) \cap \dots \cap \mathrm{Fix}(f^{m_t})| \\ &= \sum_{\varnothing\neq\mu\subset\{m_1,m_2,\dots m_t\}} (-1)^{|\mu|-1} \left|\mathrm{Fix}(f^{\gcd\mu})\right|\end{aligned}$$

The last equality follows from the fact that $\mathrm{Fix}(f^a) \cap \mathrm{Fix}(f^b) = \mathrm{Fix}(f^{\gcd(a,b)})$. For each i, $\mathrm{Fix}(f^{m_i}) = |L(f^{m_i})| \neq 0$ for endomorphisms of tori in [BBPT75],

$$\sum_{\varnothing\neq\mu\subset\{m_1,m_2,\dots m_t\}} (-1)^{|\mu|-1} \left|\mathrm{Fix}(f^{\gcd\mu})\right| = \sum_{\varnothing\neq\mu\subset\{m_1,m_2,\dots m_t\}} (-1)^{|\mu|-1} \left|L(f^{\gcd\mu})\right|$$

Heath showed this is a lower bound [Hea99]. $\square$

The goal of this section is to prove Proposition 2.0.3. In doing so, we translate the questions of periodic fixed point theory into a study of equivariance and points with non trivial isotropy. The first step is to define an action of C_n on $\mathbb{R}^d$ and extend it to an action of $\mathbb{Z}^d \rtimes C_n$ on $\mathbb{R}^d$. Finally we construct a $\mathbb{Z}^d \rtimes C_n$ equivariant simplicial complex on $\mathbb{R}^d$ which allows us to define the map $\Theta_n(A)$.

We begin at the end, and consider an example which we will carry through this chapter.

Example 3.0.1. Consider the following integral matrix

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that $B^6 = I$, $\det(I - B) = 1$, $\det(I - B^2) = 3$, and $\det(I - B^3) = 4$. Calculating the periodic points of B is an exercise in modular arithmetic and linear algebra, we summarize them in Table 3.1.

Table 3.1: Table of Periodic Points

i	1	2	3	4	5	6
$L(\overline{B}^i)$	1	3	4	3	1	0
$\mathrm{Fix}(\overline{B}^i)$	$(0,0)$	$(0,0), \left(\frac{1}{3}, \frac{2}{3}\right)$ $\left(\frac{2}{3}, \frac{1}{3}\right)$	$(0,0), \left(0, \frac{1}{2}\right)$ $\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right)$	$(0,0), \left(\frac{1}{3}, \frac{2}{3}\right)$ $\left(\frac{2}{3}, \frac{1}{3}\right)$	$(0,0)$	(x,y)

For the decomposition K in Figure 3.1a, the map $\overline{B}\colon T \to T$ is a cellular map and the vertices of K, $V(K)$, are precisely the periodic points of $\overline{B}$ in $\mathcal{P}_6(\overline{B})$. Using the orientation of these cells, we define a (cellular) map $\Theta_6(\overline{B}) \simeq \mathrm{id}$ so that $\mathrm{Fix}(\Theta_6(\overline{B})^6) = V(K)$. This is done inductively by fixing the vertices, sliding the points on the interior of the 1-cells towards the larger vertex, and then continuously extending through the 2-dimensional cells. The explicit construction is given in Proposition 3.4.2. Then $\overline{B}$ commutes with $\Theta_6(\overline{B})$ and $\widetilde{B} := \overline{B} \circ \Theta_6(\overline{B})$ satisfies

$$\widetilde{B} \simeq \overline{B} \text{ and}$$
$$\mathrm{Fix}(\widetilde{B}^6) = V(K).$$

These types of maps between tori are the focus of Chapter 3. As such, we give them a name.

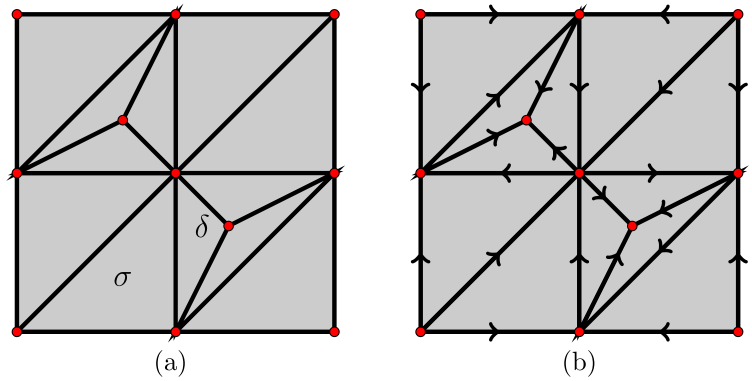


Figure 3.1: Beginning at the End Example 3.0.1

Definition 3.0.2. An integer matrix A is n**-cyclotomic** if

- $A^n = \text{id}$ and
- $\det(I - A^m) \neq 0$ for proper divisors m of n.

Matrices naturally define an action on euclidean space, and possible actions of n-cyclotomic matrices are quite limited. The following observation gives an example of their limitations.

Observation 1. $\langle A \rangle$ acts freely on $\mathbb{R}^d$ away from the origin.

If $A^i v = v$ for some $v \in \mathbb{R}^d \backslash \{\vec{0}\}$, then $A^i \in \langle A \rangle_v \subset \langle A \rangle$. Since $\langle A \rangle$ is cyclic, the group $\langle A \rangle_v = \langle A^j \rangle$ for some $1 \leqslant j \leqslant n$. Then $A^j v = v$ and $char(A^j)(1) = 0$. This observation along with the assumption that $\det(I - A^m) \neq 0$ for all proper divisors of n imply $n|j$ and $\langle A^j \rangle = \langle I \rangle$. Therefore $(\mathbb{R}^d)^{\langle A^m \rangle} = \{0\}$ for all proper divisors of m and the set of points $v \in \mathbb{R}^d$ which $\langle A \rangle$ does not act freely on is the origin, $\{\vec{0}\}$.

The remainder of this chapter is devoted to decompositions similar to the one in Example 3.0.1 for every n-cyclotomic matrix.

3.1 Properties of Cyclotomic Matrices

If A is a n-cyclotomic $d \times d$ integer matrix, then the group generated by A,

$$\langle A \rangle := \{A^i : i \in \mathbb{Z}\},$$

is a cyclic group of order n. The left action of integer matrices on $\mathbb{R}^d$ restricts to a left action of $\langle A \rangle$.

Definition 3.1.1. [Sch86] A nonempty set C of points in $\mathbb{R}^d$ is a **convex cone** if $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \geqslant 0$. A cone C is a **polyhedral cone** if

$$C = \{x | Bx \leqslant 0\}$$

for some matrix B, i.e. if C is the intersection of finitely many linear halfspaces. The cone **generated** by the vectors $S = \{x_1, \ldots, x_m\}$ is the set

$$\operatorname{cone}(S) = \left\{ \sum_{u \in S} \lambda_u u | \lambda_u \in \mathbb{R}_{\geqslant 0} \right\}$$

i.e. it is the smallest cone containing $x_1, \ldots, x_m$. If $\{x_1, \ldots, x_m\}$ is minimal, then the rays $\overrightarrow{0, x_i}$ are called called **extremal rays** of $\operatorname{cone}\{x_1, \ldots, x_m\}$. A cone arising in this way is called **finitely generated**. We assume $\operatorname{cone}(\varnothing) = \{0\}$.

There are many equivalent ways to generate the same polyhedral cone [Sch86]. In fact, if $\operatorname{conv}(S)$ is the convex hull of a finite set S and $V(\operatorname{conv}(S))$ is the vertex set of $\operatorname{conv}(S)$, then $V(\operatorname{conv}(S))$ is finite and

$$\operatorname{cone}(V(\operatorname{conv}(S))) = \operatorname{cone}(S).$$

The following theorem of Farkas-Minkowski-Weyl shows that the concepts of "polyhedral" and "finitely generated" are equivalent. We will make significant use of the transitions among the descriptions it provides.

Theorem 3.1.2 (Farkas-Minkowski-Weyl [Sch86])**.** *A convex cone is polyhedral if and only if it is finitely generated.*

For our purposes it will be enough to understand the action of $\langle A \rangle$ on a set polyhedral cones rather than the action of $\langle A \rangle$ on $\mathbb{R}^d$. We require a set of polyhedral cones which assemble to cover $\mathbb{R}^d$ and respect the action of $\langle A \rangle$. This notion leads us to the following definition.

Definition 3.1.3. A **$\langle A \rangle$-polyhedral cone decomposition** of $\mathbb{R}^d$ is a collection, $\mathcal{U} = \{U_1, \ldots, U_l\}$, of polyhedral cones so that:

i. $\dim(U_i \cap U_j) < d$ for $i \neq j$,

ii. $\bigcup_i U_i = \mathbb{R}^d$, and

iii. the action of A on $\mathbb{R}^n$ defines a free action of $\langle A \rangle$ on $\{U_1, \ldots, U_l\}$.

There are a few properties of n-cyclotomic matrices that we will take advantage of often. We summarize them below. To separate the matrix from this action we identify $\langle A \rangle$ with C_n.

Properties.

I- C_n acts by homeomorphims.

II- The C_n action respects scalars. That is $g(\lambda x) = \lambda g x$ for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $g \in C_n$.

III- The C_n action respects addition. That is $g(x + y) = \lambda g x + g y$ for all $x, y \in \mathbb{R}^d$ and $g \in C_n$.

IV- The C_n action restricts to an action on $\mathbb{Z}^d$ by homeomorphisms.

V- C_n acts freely on a polyhedral conic decomposition of $\mathbb{R}^d$

VI- $(\mathbb{R}^d)^{mC_n} = \{0\}$ for all $m|n$.

VII- If σ is convex, then $g\sigma$ is convex for all $g \in C_n$.

As with all things, it is important to note that such a thing exists. The next lemma constructs a polyhedral cone decomposition for any n-cyclotomic matrix A.

Lemma 3.1.4. *If $A \in \mathbb{Z}^{d\times d}$ is a n-cyclotomic matrix then there exist a $\langle A\rangle$ polyhedral cone decomposition of $\mathbb{R}^d$.*

Proof. Let $\{e_1, \dots, e_d\}$ be a basis of $\mathbb{R}^d$ and $H := \{H_1, \dots H_d\}$ be the corresponding coordinate hyperplanes in $\mathbb{R}^d$. Let

$$\langle A\rangle H := \{A^j H_i \colon H_i \in H,\ 0 \leqslant j \leqslant n-1\}.$$

If $\{u_1, \dots, u_l\}$ are the components of $\mathbb{R}^d \backslash (\langle A\rangle H)$ then the closure of a component, $U_i := \bar{u}_i$, is a polyhedral cone . The set $\{U_1, \dots, U_l\}$ satisfies

$$\dim(U_i \cap U_j) < d, \tag{3.1}$$

$$U_i^\circ \cap U_j^\circ = \varnothing \tag{3.2}$$

for $i \neq j$, and

$$\bigcup_i U_i = \mathbb{R}^d.$$

By Theorem 3.1.2, each polyhedral cone U_i is defined by a matrix inequality

$$B_i\vec{x} \leqslant \vec{0}$$

and AU_i is defined by the matrix innequality

$$AB_i\vec{x} \leqslant \vec{0}.$$

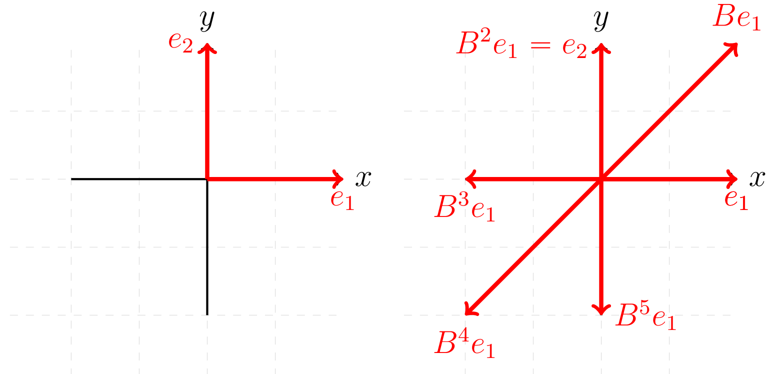
Since the rows of AB_i are the images of the vectors defining U_i under the map A, AU_i is a polyhedral cone in $\{U_1, \dots, U_l\}$. Therefore $\langle A\rangle$ acts on the set $\{U_1, \dots, U_l\}$ by restricting the action of $\langle A\rangle$ on $\mathbb{R}^d$.

It remains to verify that $\langle A\rangle$ acts freely on $\mathcal{U}$. Assume that

$$A^m U_j = U_j.$$

(Note that by Displays (3.1) and (3.2) this is only way these sets can have a nontrivial intersection.) Suppose $\delta_j := \operatorname{conv}(S_i)$ is a $d-1$ simplex which defines the region U_j (Theorem 3.1.2). By Observation 1 the action of A is a linear automorphism with $\ker(A) = \{0\}$ and A acts freely away from 0. Since $0 \notin \delta_j$ the restriction of the action of A on δ_j induces a map map

$$A^m|_{\delta_j} \colon \delta_j \to A^m\delta_j$$



(a) Choosing e_1, e_2 (b) Considering $\langle B\rangle\{e_1, e_2\}$

Figure 3.2: Constructing Polyhedral Conic Decomposition (1)

which is an linear isomorphism of two $(d-1)$ cells contained in $U_1\backslash 0$. Since every point $x \in U_j$ can be written as $s\vec{v}$ for some $s \in \mathbb{R}_{>0}$ and $v \in \delta_j$ (Theorem 3.1.2), we define a retraction

$$r\colon U_j\backslash\{0\} \to \delta_j$$

by $r(s\vec{v}) = v$. By the Brouwer fixed point theorem there exists a point $x \in \delta_j$ so that

$$r|_{A^m\delta_j} \circ A^m|_{\delta_j}(x) = x.$$

Therefore the vector through x is an eigenvector of A^m with real eigenvalue. Since A is n-cyclotomic, the eigenvalues of A are primitive n^{th} roots of unity. Since $m|n$ and A^m has a real eigenvalue, $m = n$ and $A^m = \text{id}$. □

Throughout this section there is a medley of notation and new definitions. To help illuminate these ideas we will carry Example 3.0.1 throughout.

Example 3.1.5. Consider the following integral matrix:

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that $B^6 = I$ and therefore $\langle B\rangle \cong C_6$. We construct the polyhedral cone decomposition by choosing the standard basis vectors, $\{e_1, e_2\}$, of $\mathbb{R}^2$. (The choice of standard basis vectors is not necessary and any basis will yield a polyhedral cone decomposition of $\mathbb{R}^2$.) Once we have this basis, we consider all vectors in the set $\langle B\rangle\{e_1, e_2\}$. This yields Figures 3.2a and 3.2b.

Next, consider all the hyperplanes (in this case lines) spanned by set $\langle B\rangle\{e_1, e_2\}$. We define the polyhedral conic regions by considering the closure of the components in $\mathbb{R}^2\backslash\left(\langle B\rangle\{e_1, e_2\}\right)$. Thus the polyhedral cone decomposition of $\mathbb{R}^2$ is the set $\mathcal{U} = \{U_1, U_2, U_3, U_4, U_5, U_6\}$ in Figure 3.3b.

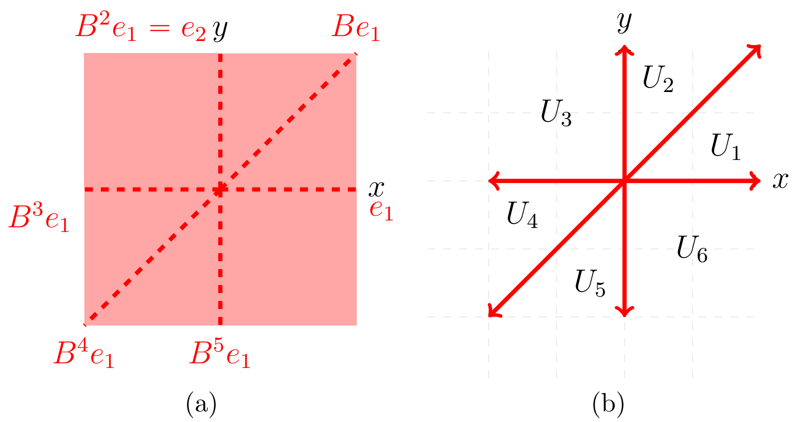


Figure 3.3: Constructing Polyhedral Conic Decomposition (2)

Lemma 3.1.4 reduces the understanding of $\langle A \rangle$ on $\mathbb{R}^d$ to the action of $\langle A \rangle$ on $\mathcal{U}$ and some compatability information. Since the action of A is linear, it is enough to understand the action of $\langle A \rangle$ close to 0 and extend linearly. For the purposes of Sections 3.2 and 3.3, we will be explicit about the restrictions to a neighborhood of 0. Since $\overline{\mathcal{P}}_n(\overline{A})$ is a finite collection of points in T, it is discrete. Then the set $\mathcal{P}_n(\overline{A})$ is also discrete and there exists an $\varepsilon > 0$ so that

$$\varepsilon < \min \left\{ \frac{\min\{d(y_1, y_2) | y_1, y_2 \in \mathcal{P}_n(\overline{A});\ y_1 \neq y_2\}}{2}, \frac{1}{2} \right\},$$

then

$$B(x, \epsilon) \cap \mathcal{P}_n(\overline{A}) = \{x\}$$

for all $x \in \mathcal{P}_n(\overline{A})$.

Proposition 3.1.6. *Let $A \in \mathbb{Z}^{d \times d}$ be n-cyclotomic matrix and $\mathcal{U} = \{U_1, \ldots, U_l\}$ be a polyhedral conic decomposition of $\mathbb{R}^d$ for A. There exist generating sets S_i of U_i so that the collection $\{S_1, \ldots, S_l\}$ satisfies:*

i. $S_i \subset B(\varepsilon, 0)$ for all i,

ii. if $U_i \cap U_j \neq \{0\}$, then $Cone(S_i \cap S_j) = U_i \cap U_j$, and

iii. if $AU_i = U_j$, then $S_i = S_j$.

Proof. By Lemma 3.1.4, there exists a $\langle A \rangle$ polyhedral cone decomposition, $\mathcal{U} := \{U_1, \ldots U_l\}$, of $\mathbb{R}^d$. Let $\{V_1, \ldots, V_k\}$ be a choice of representative for each coset of $\mathcal{U}$ under the action of $\langle A \rangle$.

We proceed inductively by first creating the generating set for V_1 (and therefore all polyhedral cones in $[V_1] \in \mathcal{U}/\langle A \rangle$.) Choose a non zero point, x_1, on an extremal ray of V_1 so that $\langle A \rangle \{x_1\} \subset B(\varepsilon, 0)$. Let $\widetilde{S}_1(0) = \{x_1\}$ and define $\widetilde{S}_1(1)$

$$\widetilde{S}_1(1) = \widetilde{S}_1(0) \bigcup \left(\bigcup_j \left(A^j x_1 \cap V_1 \right) \right).$$

$\widetilde{S}_1(1)$ may not generate V_1. If not, there exists an extremal ray of V_1 in $V_1\backslash\mathrm{conv}(\widetilde{S}_1(1))$. Let x_2 be a non zero point on that extremal ray so that $\langle A\rangle\{x_2\} \subset B(\varepsilon, 0)$. Then define $\widetilde{S}_1(2)$

$$\widetilde{S}_1(2) = \widetilde{S}_1(1)\bigcup\left(\bigcup_j \left(A^j x_2 \cap V_1\right)\right).$$

Since V_1 is a polyhedral cone, there exists an $m \in \mathbb{N}$ so that $\widetilde{S}_1(m)$ generates V_1. Define $S_1 := \widetilde{S}_1(m)$.

By construction, $A^j S_1$ generates $A^j V_1$. If U_i is not in the coset of V_1 the set $\cup_j A^j S_1$ may still intersect U_i. While constructing the generating sets for the remaining V_i we record this intersection.

Assume that generating sets has been constructed for $V_1, \ldots, V_{i-1}$. Let $\widetilde{S}_i(0)$ be the intersection of all previous generating sets and V_i,

$$\widetilde{S}_i(0) := \left(\bigcup_j A^j \left(\bigcup_{l=1}^{i-1} S_l\right)\right)\bigcap V_i.$$

If $\widetilde{S}_i(0)$ is a generating set, then $S_i := \widetilde{S}_i(0)$. Otherwise, there exists an extremal ray in $V_i\backslash\,\mathrm{conv}(\widetilde{S}_i(0))$. Let $y_1 \in B(\varepsilon, 0)$ be some non zero point on that extremal ray so that $\langle A\rangle\{y_1\} \subset B(\varepsilon, 0)$. Then we define $\widetilde{S}_i(1)$

$$\widetilde{S}_i(1) = \widetilde{S}_i(0)\bigcup\left(\bigcup_j \left(A^j y_1 \cap V_i\right)\right).$$

and repeat as in the case of V_1. Since V_i is a polyhedral cone, there exists some $m' \in \mathbb{N}$ so that $\widetilde{S}_i(m')$ generates V_i. Define $S_i := \widetilde{S_i(m')}$. $\square$

Example 3.1.7. We return to Example 3.1.5 to give an example of this construction. Recall that

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The polyhedral decomposition from Example 3.1.5 is depicted in Figure 3.4a. Pick a point x_1, on an extremal ray of U_1. As in the proof of Proposition 3.1.6, let $\widetilde{S}_1(0) = \{x_1\}$. See Figure 3.4a.

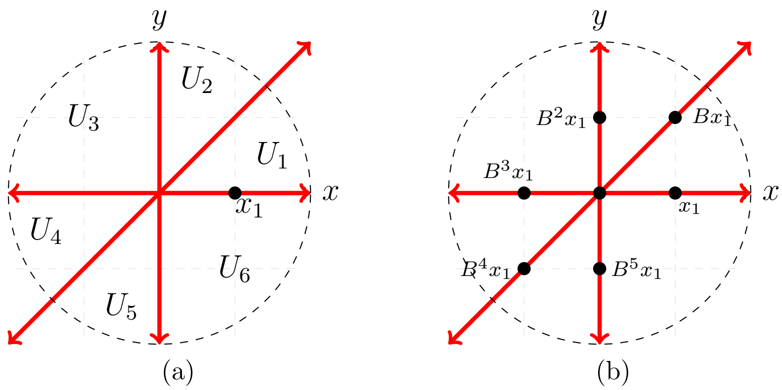


Figure 3.4: Generating Sets in Example 3.1.7

Next, we act by $\langle B \rangle$ on x_1 and add points to the generating set

$$\widetilde{S}_1(1) = \widetilde{S}_1(0) \bigcup \left(\bigcup_{1}^{6} (B^j x_1 \cap U_1) \right) = \{x_1, Bx_1\}.$$

The set $\widetilde{S}_1(1)$ generates U_1 and so $S_1 := \widetilde{S}_1(1)$. Using Figure 3.4b we see that the images of $\langle B \rangle \{x_1\}$ generate each $U_i \in \mathfrak{U}$.

$$\widetilde{S}_i(0) := \left(\bigcup_{j=1}^{5} B^j \left(\bigcup_{l=1}^{i-1} S_l \right) \right) \bigcap V_i S_i = \{B^{i-1}x_1, B^i x_1\}$$

In this example our generating sets are $S_i = \{B^{i-1}x_1, B^i x_1\}$ for $1 \leqslant i \leqslant 6$. We can also take the convex hull of each S_i, $\delta_i := \mathrm{conv}(S_i)$, as the generating set of U_i as in Theorem 3.1.2.

Example 3.1.8. Figure 3.5 is the generating set and corresponding neighborhood for a different choice of basis. This changes the number of polyhedral cones and their shape slightly and, if the reader chooses, can be carried through the later sections. For this version, we choose $e_1 = (2, 0)$ and $e_2 = (3, 1)$.

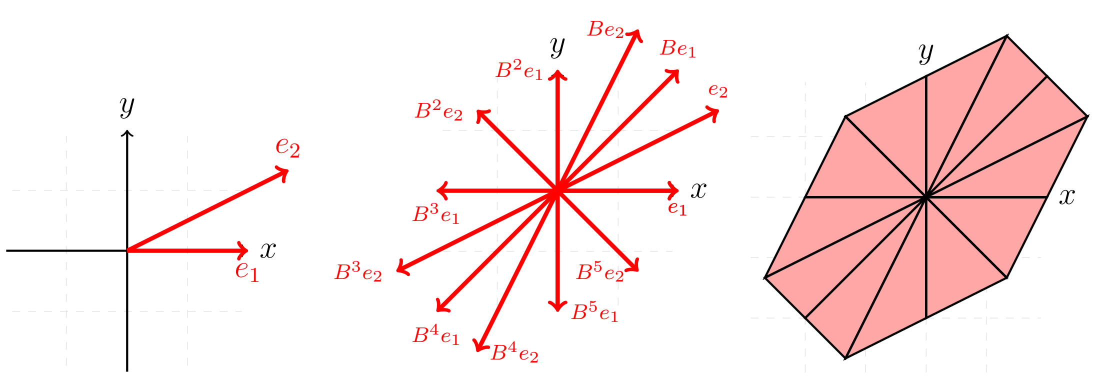


Figure 3.5: Supplemental Generating Set in Example 3.1.7

3.2 Extending to $\mathbb{Z}^d \rtimes \langle A\rangle$ Action on $\mathbb{R}^d$

We are not interested in just understanding the action of $\langle A\rangle$ on $\mathbb{R}^d$. Up to this point, we have not made any mention of descending to the torus. We will avoid this descent by using the $\mathbb{Z}^d$ action on $\mathbb{R}^d$ by translation. Since A descends to a map of tori, the action of $\langle A\rangle$ and $\mathbb{Z}^d$ define an action of $\mathbb{Z}^d \rtimes \langle A\rangle$ on $\mathbb{R}^d$. To define this action we regard the n-cyclotomic matrix as an automorphism of $\mathbb{Z}^d$ and consider

$$\phi\colon \langle A\rangle \to \mathrm{aut}(\mathbb{Z}^d).$$

The action of $\mathbb{Z}^d \rtimes_\phi \langle A\rangle$ on $\mathbb{R}^d$ is defined as

$$(\vec{z}, A^i) \cdot \vec{r} = A^i\vec{r} + \vec{z}$$

This action respects the identity since $(\vec{0}, I) \cdot \vec{r} = I\vec{r} + \vec{0} = \vec{r}$. It is associative since

$$\begin{aligned}
(\vec{z}_1, A^j) \cdot [(\vec{z}_2, A^i) \cdot \vec{r}] &= (\vec{z}_1, A^j) \cdot (A^i\vec{r} + \vec{z}_2) \\
&= A^{i+j}\vec{r} + A^j\vec{z}_2 + \vec{z}_1 \\
&= (A^i\vec{z}_2 + \vec{z}_1, A^j) \cdot \vec{r} \\
&= [(\vec{z}_1, A^j) \cdot (\vec{z}_2, A^i)] \cdot \vec{r}
\end{aligned}$$

This action translates the question of periodic points in T to a question of equivariant fixed point theory on $\mathbb{R}^d$.

Lemma 3.2.1. *The points of $\mathbb{R}^n$ with nontrivial isotropy under the action of $\mathbb{Z}^d \rtimes \langle A\rangle$ are $\mathcal{P}_n(A)$.*

Proof. A $\vec{r} \in \mathbb{R}^d$ is fixed by $(\vec{z}, A^i) \in \mathbb{Z}^d \rtimes \langle A\rangle$ if and only if $A^i\vec{r} + \vec{z} = \vec{r}$. Therefore $\pi(\vec{r}) \in T$ is a fixed point of $\overline{A}\colon T \to T$. Thus $i|n$ and $\vec{r} \in \mathcal{P}_n(\overline{A})$ and

$$\{\vec{r} \in \mathbb{R}^d | (\vec{z}, A^i) \cdot \vec{r} = \vec{r} \text{ for some } (\vec{z}, A^i) \in \mathbb{Z}^d \rtimes \langle A\rangle\} = \mathcal{P}_n(\overline{A}). \qquad \square$$

Example 3.2.2. Returning to Examples 3.1.5 and 3.1.7 we recall

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

$\langle B\rangle \cong C_6$, and therefore $\mathbb{Z}^2 \rtimes \langle B\rangle \cong \mathbb{Z}^2 \rtimes C_6$. By Lemma 3.2.1, the points in $\mathbb{R}^2$ with non trivial isotropy subgroups are precisely the inverse images under $\mathbb{R}^2 \to T$ of the periodic points of $\overline{B}\colon T \to T$ of order $6 \nmid n$. Calculating these periodic points is an exercise in modular arithmetic and linear algebra. We summarize them Table 3.2.

$$\mathcal{P}_n(\overline{B}) = \left(\mathbb{Z}^2 \rtimes \langle B\rangle\right) \left\{(0,0), \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

We now make a definition combining the polyhedral cone decomposition of $\mathbb{R}^d$, the "closeness" criteria of Proposition 3.1.6, and the action of $\mathbb{Z}^d \rtimes \langle A\rangle$.

Table 3.2: Table of Periodic Points for Example 3.2.2

i	1	2	3	4	5	6
$L(\overline{B}^i)$	1	3	4	3	1	0
$\mathrm{Fix}(\overline{B}^i)$	$(0,0)$	$(0,0),(\frac{1}{3},\frac{2}{3})$ $(\frac{2}{3},\frac{1}{3})$	$(0,0),(0,\frac{1}{2})$ $(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2})$	$(0,0),(\frac{1}{3},\frac{2}{3})$ $(\frac{2}{3},\frac{1}{3})$	$(0,0)$	(x,y)

Definition 3.2.3. An **extendable linear neighborhood** of $x \in \overline{\mathcal{P}}_n(\overline{A})$ is a collection of simplices $\Delta_x = \{\delta_1, \ldots, \delta_l\}$ satisfying:

i. For all $\delta \in \Delta_x$, $x \in \delta$ and $\bigcup_{\delta \in \Delta_x} \delta$ is a connected neighborhood of x,

ii. if $\delta_i, \delta_j \in \Delta_x$ with vertex sets $V(\delta_i)$ and $V(\delta_j)$, then

$$\mathrm{conv}(V(\delta_i) \cap V(\delta_j)) = \delta_i \cap \delta_j$$

iii. the action of $(\mathbb{Z}^d \rtimes \langle A \rangle)_x$ induces an automorphism of Δ_x (and so a one-to-one and onto map).

We let $\mathcal{N}\Delta_x$ denote the neighborhood given by the union of simplices in Δ

$$\mathcal{N}\Delta_x = \bigcup_{\delta \in \Delta_x} \delta.$$

Proposition 3.1.6 implies the existence of extendable linear neighborhoods.

Proposition 3.2.4. *Let $A \in \mathbb{Z}^{d \times d}$ be n-cyclotomic. Every $x \in \mathcal{P}_n(\overline{A})$ has an extendable linear neighborhood.*

Proof. Let $x \in \mathcal{P}_n(\overline{A})$. By Lemma 3.2.1, x is a periodic point of $\overline{A}$ of period $m|n$. Therefore there exists $(\vec{z}, A^m) \in \mathbb{Z}^d \rtimes \langle A \rangle$ so that $(\vec{z}, A^m)x = x$.

If A is n-cyclotomic, then A^m is $\frac{n}{m}$-cyclotomic. By Proposition 3.1.6 there exists a polyhedral conic decomposition of $\mathbb{R}^d$, $\mathcal{U} = \{U_1, \ldots, U_l\}$, corresponding to A^m with generating sets in $B(\varepsilon, 0)$. Let S_i be a generating set of U_i and $r_x : \mathbb{R}^d \to \mathbb{R}^d$ be the linear translation $r_x(y) = y + x$. The cells of an extendable linear neighborhood of x are

$$\delta_i(x) := \mathrm{conv}\left(\{x\} \cup r_x(S_i)\right) = r_x(\mathrm{conv}(\{0\} \cup S_i).$$

Next, we show that the set $\Delta_x = \{\delta_1(x), \ldots, \delta_l(x)\}$ is an extendable linear neighborhood of x. Let $\delta_i, \delta_j \in \Delta_x$. By Proposition 3.1.6.*ii* we have the equalities.

$$\begin{aligned}\mathrm{conv}(V(\delta_i) \cap V(\delta_j)) &= \mathrm{conv}\left((\{x\} \cup r_x(S_i)) \cap (\{x\} \cup r_x(S_i))\right)\\ &= \mathrm{conv}\left(\{x\} \cup (r_x(S_i) \cap r_x(S_j))\right)\\ &= \mathrm{conv}\left(\{x\} \cup (r_x(S_i \cap S_j))\right)\\ &= \delta_i \cap \delta_j\end{aligned}$$

Therefore, Δ_x satisfies Definition 3.2.3.*iii*. Furthermore, since A^m is an automorphism and $\mathcal{U}(x)$ satisfies Proposition 3.1.6.*iii*, the subgroup $(\mathbb{Z}^d \rtimes \langle A \rangle)_x$ acts by automorphisms. Therefore $\Delta(x)$ is an extendable linear neighborhood around x. $\square$

We illustrate the proof of Proposition 3.2.4 using the matrix in Examples 3.1.5, 3.1.7 and 3.2.2.

Example 3.2.5. For simplicity, we build the extendable linear neighborhood around the origin and its orbits, this illuminates the general case and differs only by a translation. Figure 3.6a is the $\mathbb{Z}^2$ translations of the polyhedral conic regions around the origin.

Since the origin corresponds to an orbit of period one, consider the $\mathbb{Z}^2 \rtimes \{e\} \cong \mathbb{Z}^2$ translates of the polyhedral conic regions. This is not a problem since the ability to linearly scale is the motivation for the name "extendable linear neighborhoods".

To finish the construction of the extendable linear neighborhood of the origin and its translates we follow Proposition 3.1.6 around the origin and translate using the $\mathbb{Z}^2 \rtimes \langle e \rangle$ action. Then the shaded regions in Figure 3.6c assemble to become the extendable linear neighborhood of the origin and its $\mathbb{Z}^2$ orbits.

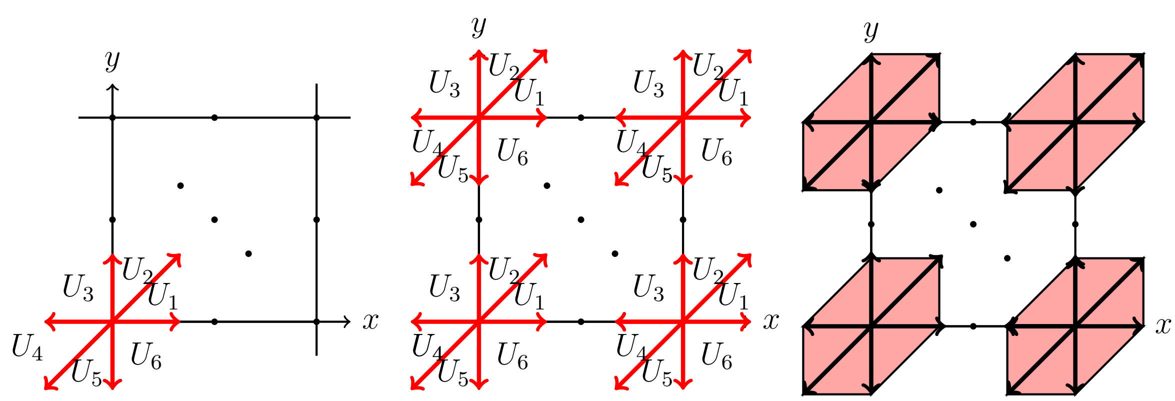


Figure 3.6: Generating extendable linear neighborhoods Example 3.2.5

3.3 Constructing $\mathbb{Z}^d \rtimes \langle A \rangle$ Decomposition of $\mathbb{R}^d$

Definition 3.3.1. A simplicial complex $K \subset \mathbb{R}^d$ is ***n*-vacant** if for each simplex σ in K

i. $\sigma \bigcap \mathcal{P}_n(\overline{A}) = V(\sigma)$,

ii. if $(z, A^i)\sigma \cap \sigma \neq \varnothing$, then $(z, A^i)\sigma \cap \sigma$ is a subcomplex (face) of σ.

If K is n-vacant, we say that $\sigma \in K$ is **maximal** if there is no $\sigma' \in K$ so that σ is a proper face of σ'. Note that if $K \subset \mathbb{R}^d$ is n-vacant, then $\left(\mathbb{Z}^d \rtimes \langle A \rangle\right) K$ is n-vacant. The study of n-vacant simplicies and the construction a n-vacant complex of $\mathbb{R}^d$ is central to the proof of Proposition 2.0.3.

Proposition 3.3.2. *If A is n-cyclotomic, then there is an n-vacant $\mathbb{Z}^d \rtimes \langle A \rangle$-simplicial decomposition of $\mathbb{R}^d$.*

The proof of Proposition 3.3.2 is an inductive proof and the first step is to create a n-vacant extendable linear neighborhood around each point $x \in \mathcal{P}_{\overline{A}}(n)$. First, Lemma 3.3.3 shows us that we can refine extendable linear neighborhoods by adding more points equivariantly.

Lemma 3.3.3. *Let K be a n-vacant complex containing x. There exists an extendable linear neighborhood, Δ_x, of x so that, for all $\delta \in \Delta_x$, if $V(\delta) \subset K$, then $\delta \subset K$.*

Proof. By Proposition 3.2.4 there exists a extendable linear neighborhood, Δ_x, of x. If every $\delta \in \Delta_x$ such that $V(\delta) \subset K$ is contained in K, then we are done. Otherwise, let $\delta \in \Delta_x$ such that $V(\delta) \in K$ where $\delta \not\subset K$. Let $V(\delta) = \{x, u_1, \ldots, u_d\}$. Since $\delta \not\subset K$, there exists $u \in \operatorname{conv}(u_1, \ldots, u_d)^\circ \backslash K$. Define the simplices $\delta_i(u) = \{x, u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_d\}$. Then $V(\delta_i(u)) \not\subset K$, $\delta_i(u) \not\subset K$, the collection $\{\delta_i(u)\}$ satisfies Definition 3.2.3*ii*, and

$$\delta = \bigcup_i \delta_i(u).$$

Define the extendable linear neighborhood Δ'_x

$$\Delta'_x = (\Delta \backslash [\delta]) \bigcup \left(\left(\mathbb{Z}^d \rtimes \langle A \rangle\right) \{\delta_i(u)\}\right).$$

Since Δ_x is finite, there are finitely many cosets which require such replacement. $\square$

Sections 3.1 and 3.2 culminate in producing these extendable linear neighborhoods which are contained in $B_\varepsilon(x)$ for each $x \in \mathcal{P}_{\overline{A}}(n)$. In Section 3.3 we introduced the concept of n-vacant simplices. These two topics are similar, but incompatible at a first glance. For example, let $\mathcal{N}\Delta_x$ be an extendable linear neighborhood around $x \in \mathcal{P}_{\overline{A}}(n)$ and σ be an n-vacant simplex containing x and a point $y, x \in \mathcal{P}_{\overline{A}}(n)$, $y \neq x$. By construction σ cannot be contained in $\mathcal{N}\Delta_x$. Lemma 3.3.4 shows that these two concepts can be made compatible using the "extendable" nature of extendable linear neighborhoods.

Lemma 3.3.4. *Let K be a n-vacant complex in $\mathbb{R}^d$ containing $x \in \mathcal{P}_n(\overline{A})$ such that every maximal simplex $\sigma \in K$ contains $x \in \mathcal{P}_{\overline{A}}(n)$. Then there is an extendable linear neighborhood $\mathcal{N}\Delta_x(\sigma, t_0)$ of x containing σ satisfying:*

$$\mathcal{N}\Delta_x(K, t_0) \bigcap \mathcal{P}_n(\overline{A}) = V(K).$$

Proof. Suppose $x \in \mathcal{P}_n(\overline{A})$ is a representative of an order $m|n$ periodic point. Let Δ_x be the the extendable linear neighborhood. Each $\delta \in \Delta_x$ is a translation of a cell around the origin. Therefore, it is enough to work around 0 and translate appropriately.

We proceed by induction using the dimension of $\sigma \in K$. If σ is zero dimensional, then $\sigma = \{0\}$ and Δ_x is sufficient. Assume K is of dimension $1 \leqslant k \leqslant d$. We extend Δ_x to include the K by extending each $\delta \in \Delta_x$ where it intersects K. Choose an arbitrary $\delta \in \Delta_x$ and let S be the vertex set defining δ. We extend δ by defining a function, ϕ, on the set

$$T := (V(\delta) \cup V(\delta \cap K))\backslash\{0\}.$$

Define ϕ as the piecewise function:

i. If $u \in V(\delta)\backslash V(\delta \cap K)$, then $\phi(u) = u$. See Figure 3.7a. Define

$$\widetilde{S}_1 := \phi(V(\delta)\backslash V(\delta \cap K)) = \{v_1, \ldots v_\beta\}.$$

ii. If $u \in V(\delta \cap K)$, then there exists a positive t_u so that $t_u\overrightarrow{u} \in \partial K$. Define $\phi(u) = t_u\overrightarrow{u}$. See Figure 3.7b. Define

$$\widetilde{S}_2 := \phi(V(\delta \cap K)) = \{v_{\beta+1}, \ldots, v_\alpha\}.$$

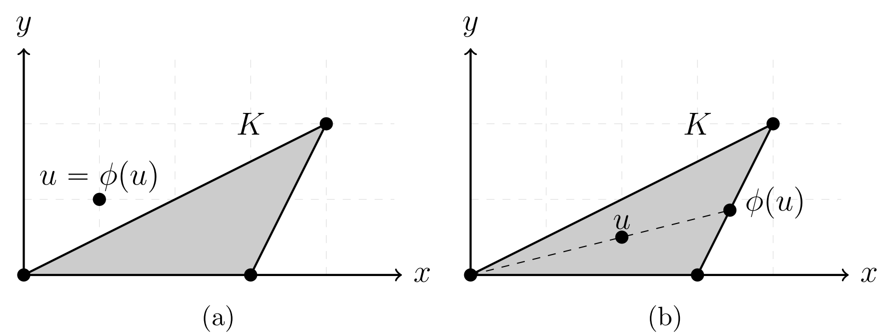


Figure 3.7: Redefining Generating Set Lemma 3.3.4

Define $\widetilde{S} := \phi(T) = \widetilde{S}_1 \cup \widetilde{S}_2$ and For $t > 0$, we define a new generating set

$$S(K,t) := \{v_1, \ldots v_\beta, tu_{\beta+1}, \ldots, tu_\alpha\}.$$

Like to Proposition 3.2.4, we let $\delta(K,t)$ denote the translation of convex hull of $S(K,t) \cup \{0\}$ to x and we define the corresponding neighborhood

$$\Delta_x(\sigma, t) := \{\delta_i(\sigma, t)\}.$$

Since K is n-vacant and $\mathcal{P}_n(\overline{A})$ is discrete there is a $t_0 > 0$ so that $\mathcal{N}_0(\sigma_k, t_0)$ satisfies

$$\mathcal{N}\Delta_x(K, t_0) \bigcap \mathcal{P}_n(\overline{A}) = V(K). \qquad \square$$

We illustrate the proof by continuing with Example 3.0.1.

Example 3.3.5. To fully describe the example we would need to consider the $\mathbb{Z}^d$ translates, but we will start with a more simple figure. Recall the extendable linear neighborhood of 0 in Example 3.2.5 (Figure 3.8a) and consider the n-vacant simplex $K = \left(\mathbb{Z}^d \rtimes \langle A\rangle\right) \sigma_1$ Figure 3.8b.

The complex K contains the vertices to the cells of the extendable linear neighborhood Δ_x. However, K is one dimensional and each $\delta \in \Delta$ is not contained in K. By Lemma 3.3.3, we refine the extendable linear neighborhood of 0 so that for each $\delta \in \Delta'$ if $V(\delta) \subset K$, then $\delta \subset K$. In this example we add $\mathbb{Z}^d \rtimes \langle B\rangle \left\{\frac{1}{3}, \frac{2}{9}\right\}$ to the generating sets. See Figure 3.8c. As in the proof of Lemma 3.3.4 we then define a function on the set $V(\delta) \cup V(\delta \cap K)$. The function is given by

$$
\begin{aligned}
\left(0, \frac{1}{3}\right) &\longmapsto \left(0, \frac{1}{2}\right) & \qquad \left(0, -\frac{1}{3}\right) &\longmapsto \left(0, -\frac{1}{2}\right) \\
\left(\frac{1}{3}, \frac{1}{3}\right) &\longmapsto \left(\frac{1}{2}, \frac{1}{2}\right) & \qquad \left(-\frac{1}{3}, -\frac{1}{3}\right) &\longmapsto \left(-\frac{1}{2}, -\frac{1}{2}\right) \\
\left(\frac{1}{3}, 0\right) &\longmapsto \left(\frac{1}{2}, 0\right) & \qquad \left(-\frac{1}{3}, 0\right) &\longmapsto \left(-\frac{1}{2}, 0\right)
\end{aligned}
$$

and is the identity elsewhere. The extendable linear neighborhood containing K is shown in Figure 3.8d

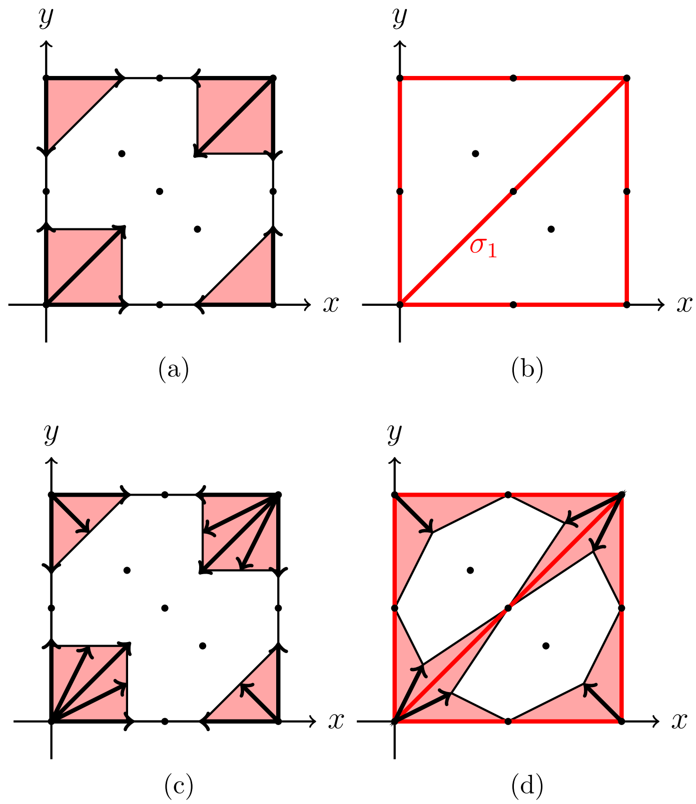


Figure 3.8: Constructing extendable linear neighborhood Example 3.3.5

We have constructed an extendable linear neighborhood containing a n-vacant simplex, but this neighborhood is not n-vacant since the vertices are not entirely contained in $\mathcal{P}_n(\overline{A})$.

Lemma 3.3.6. *Let $x \in \mathcal{P}_n(\overline{A})$ be a periodic point of order $m|n$ and $\Delta_x = \{\delta_1, \ldots, \delta_l\}$ be an extendable linear neighborhood of x so that $x \in V(\delta_i) \subset \mathcal{P}_n(\overline{A})$. Then there exists a n-vacant complex, K, so that*

$$x \in B_\varepsilon(x) \subset K \subseteq N.$$

Proof. Suppose $\Delta_x = \{\delta_1, \ldots, \delta_l\}$ is an extendable linear neighborhood of x is a collection of simplices so that $x \in V(\delta_i) \subset \mathcal{P}_n(\overline{A})$. Let $\mathcal{U}$ be a polyhedral conic decomposition corresponding to A^m. Using the vertex description defined in Proposition 3.2.4, we have

$$\delta_i := \operatorname{conv}\left(\{x\} \cup r_x(S_i)\right) = r_x(\operatorname{conv}(\{0\} \cup S_i).$$

If each δ_i is n-vacant, then the neighborhood given by their union is also n-vacant. If not, then there exists an i so that $\delta_i(x)$ is not n-vacant.

Let δ be a simplex δ_i that is not n-vacant. Then there exists an $y \in \delta \cap \mathcal{P}_n(\overline{A})$ so that $y \notin V(\delta)$. Without loss of generality, we may assume that for each $v \in V(\delta)$, the

$1-$simplex $\text{conv}(x, v)$ is n-vacant. Let $\sigma_k \subset \delta$ be the highest dimensional n-vacant subcomplex of δ. If more than one exists, then choose one such simplex σ_k.

By Lemma 3.3.4 there exists an extendable linear neighborhood, $\mathcal{N}\Delta_x(\sigma_k, t)$ containing $(\mathbb{Z}^d \rtimes \langle A \rangle)_x \sigma_k$ so that

$$\mathcal{N}\Delta_x(\sigma_k, t) \bigcap \mathcal{P}_n(\overline{A}) = V(K).$$

More precisely, the extendable linear neighborhood $\Delta(x, t) := \{\delta_i(\sigma_k, t)\}$. Let the vertex description of $\delta_i(\sigma_k, t)$ be

$$S_i(\sigma_k, t) := \{v_1, \ldots v_\beta, tu_{\beta+1}, \ldots, tu_\alpha\}.$$

Extend these neighborhoods by increasing t. Since δ has finite volume, there is a smallest $t_1 > 0$ so that $\mathcal{N}\Delta_x(\sigma_k, t_1)$ intersects some $v \in \mathcal{P}_n(\overline{A}) \cap \delta$ which is not a vertex of δ and $\sigma_{k+1} = \text{conv}(V(\sigma_k \cup \{v\})$ is n-vacant . We continue to add points in this fashion to create a d dimensional n-vacant simplex $\sigma \subset \delta$. Then replace δ with a triangulation of δ containing σ.

The non n-vacant simplex δ has been replaced with a collection of cells within δ that contains at least one n-vacant simplex, σ. Repeat this process for all non n-vacant cells in this collection. Then δ has been replaced by a collection of cells which are n-vacant. The neighborhood $N = \bigcup_1^l \delta_i$ is compact and the set $\mathcal{P}_n(\overline{A}) \cap N$ is finite. Therefore, we can iterate this refinement finitely many times to create a n-vacant neighborhood of x. □

Continuing the theme of the chapter, we continue the construction Examples 3.2.5 and 3.3.5 with this new result.

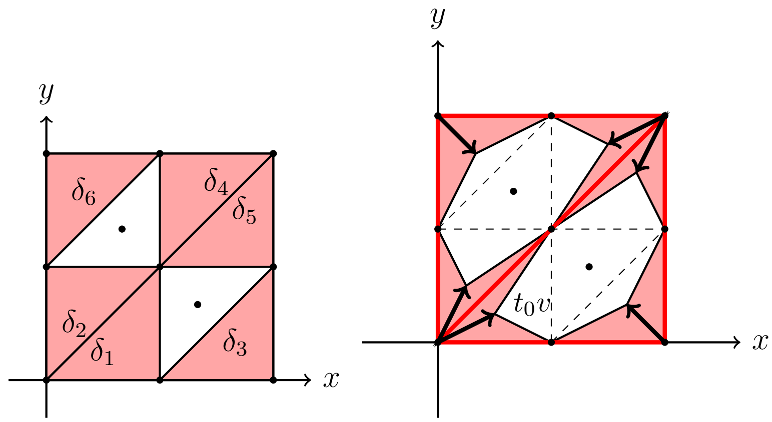
Example 3.3.7. We continue Examples 3.2.5 and 3.3.5. Unfortunately, if we consider the standard neighborhood of simplices which have vertices in $\mathcal{P}_n(\overline{B})$ from basis vectors, the corresponding set $\{\delta_1, \ldots, \delta_6\}$ defines an extendable linear neighborhood and we do not have to refine any simplex. See Figure 3.9a.

Of course, we cannot assume that this is always the case. In order to better illustrate the technique of Lemma 3.3.6 we let $\sigma_1 = \text{conv}\left((0,0), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ and recall Figure 3.8d and the neighborhood $\mathcal{N}\Delta_x(\sigma_1, t_0)$ constructed in Example 3.3.5. See Figure 3.9b.

We extend the bolded vectors by increasing t_0 until our neighborhood intersects a point in $\mathcal{P}_n(\overline{B})$ or until we reach the dashed lines in Figure 3.9b recovering the neighborhood $\{\delta_1, \ldots, \delta_6\}$.

Lemmas 3.3.3, 3.3.4 and 3.3.6 define a n-vacant neighborhood for a single point $x \in \mathcal{P}_n(\overline{A})$. However, constructing a n-vacant neighborhood of a single x is most likely not enough to cover $\mathbb{R}^d$, see Example 3.3.7. We now prove Proposition 3.3.2 by using Lemma 3.3.6 around each x and Lemma 3.3.4 to account for already existing n-vacant cells.

Proof of Proposition 3.3.2. By the Lemma 3.3.6 there exists a n-vacant neighborhood of $[0]$, $K_0 = \mathbb{Z}^d \rtimes \langle A \rangle \cdot \bigcup_i \sigma$. If K_0 covers $\mathbb{R}^d$, then we are done.



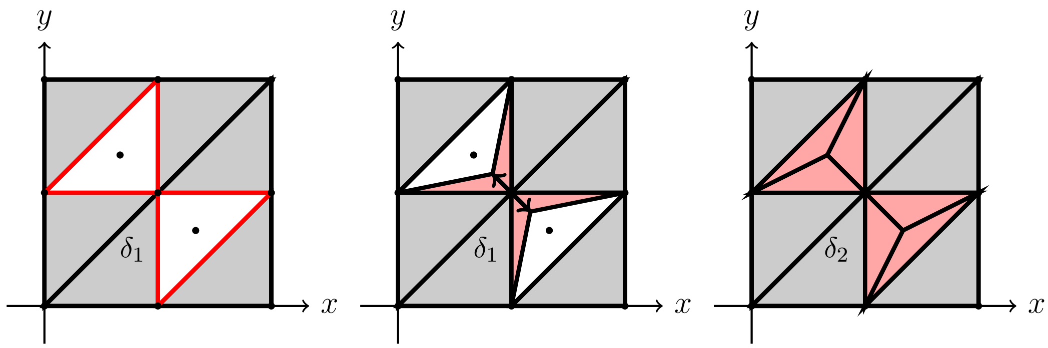
(a) (b) $\mathcal{N}\Delta_x(\sigma_1, t_0)$

Figure 3.9: n-vacant neighborhood of 0 Example 3.3.7

Otherwise, there exists a $x \in \mathcal{P}_n(\overline{A}) \cap K_0$ so that K_0 does not contain a neighborhood of x. Let $K_0(x) = \bigcup_1^l \sigma_i$ be the subcomplex of K_0 consisting of n-vacant simplices containing x. By assumption, $K_0(x)$ is not a neighborhood of x. However, there exists an extendable linear neighborhood of x, Δ_x, so that for each $\delta \in \Delta_x$ $x \in V(\delta) \subset \mathcal{P}_n(\overline{A})$ and $\operatorname{conv}(x, v)$ is n-vacant for each $v \in V(\delta_i)$.

Then apply Lemma 3.3.6 to $\{\sigma_1, \ldots, \sigma_l, \delta_1, \ldots, \delta_\alpha\}$ to construct a n-vacant simplicial complex which contains a neighborhood of the orbit $[x]$ and $[0]$ which contains the simplices $\sigma_1, \ldots, \sigma_l$. Since $\mathcal{P}_n(A)/(\mathbb{Z}^d \rtimes \langle A \rangle) \cong \overline{\mathcal{P}}_n(A)$ is finite we must do this finitely many times to cover $\mathbb{R}^d$. $\square$

Example 3.3.8. Recall the n-vacant complex, K_0, for B generated in Examples 3.2.5, 3.3.5 and 3.3.7. This n-vacant complex does not cover the torus and K_0 does not contain a neighborhood of $\left(\frac{1}{2}, \frac{1}{2}\right)$. Let δ_1 and its translates define K' which contain $\left(\frac{1}{2}, \frac{1}{2}\right)$, δ_1. See Figure 3.10a. Figure 3.10b shows how to generate an extendable linear neighborhood around $\left(\frac{1}{2}, \frac{1}{2}\right)$ containing K'. Extend this neighborhood until we encounter a point in $\mathcal{P}_6(\overline{B})$. For this example these points are $\left(\frac{2}{3}, \frac{1}{3}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}\right)$. See Figure 3.10c.



(a) Choose δ_1 (b) Construct $\mathcal{N}_{(\frac{1}{2},\frac{1}{2})}(\delta_1, t)$ (c) Final Covering

Figure 3.10: n-vacant Neighborhood of $\left(\frac{1}{2}, \frac{1}{2}\right)$

After completing this step, the n-vacant complex covers $I \times I$ and, after integral translations, we have generated the desired n-vacant triangulation of $\mathbb{R}^2$.

3.4 Constructing $\Theta_n(A)$

After constructing the n-vacant decomposition of $\mathbb{R}^d$, we define the map $\Theta_n(A)$ and prove Proposition 2.0.3. The results in this section are an alternative proof of the following result of Halpern from [Hal79].

Theorem (Theorem 4 [Hal79]). *If M is a manifold of dimension $\geqslant 2$ and $f \colon M \to M$ is a map so that $f^n = \mathrm{id}$ and*

$$|P| := \left| \bigcup_{\substack{m|n \\ m<n}} \mathrm{Fix}(f^m) \right| < \infty.$$

Then $f \simeq g$ so that $\mathrm{Fix}(g^n) = P$ and $f|_P = g|_P$.

The proof requires producing a map H which is homotopic to the identity, commutes with f, and has fixed points $P = \bigcup_{\substack{m|n \\ m<n}} \mathrm{Fix}(f^m)$. This is the motivation for Proposition 2.0.3. However, we required more structure than given in [Hal79] for Chapter 4, which is why we developed this approach.

Lemma 3.4.1. *Let σ be the simplex with vertices $\{v_0, \ldots, v_d\}$. There is a simplicial homeomorphism $\theta \colon \sigma \to \sigma$ so that $\theta \simeq \mathrm{id}$ and*

$$\mathrm{Fix}(\theta) = \mathrm{Fix}(\theta^n) = \{v_0, \ldots, v_d\}.$$

Proof. We proceed by inducting on the dimension of σ. Choose an ordering on σ so that $v_i < v_j$ when $i < j$ and $p \approx 1$ so that $1 > p > 0$.

Base Case: If $\sigma = \{v_0\}$ is a 0 simplex then

$$\theta(\sigma) = \sigma.$$

If $\sigma = \{v_i, v_j\}$ and $v_i < v_j$ then every point in σ can be written uniquely as $r\overrightarrow{\{v_i, v_j\}} + \vec{v}_i$ for $r \in [0,1]$. Then $\theta \colon \sigma \to \sigma$ is

$$\theta(r\overrightarrow{\{v_i, v_j\}} + \vec{v}_i) = r^p\overrightarrow{\{v_i, v_j\}} + \vec{v}_i.$$

Increase dimension: Assume θ is defined on all $k-1 \leqslant d$ simplices in σ.

Let $\sigma = \{v_0, v_1, \ldots, v_{k+1}\}$ be a k dimensional simplex so that $v_i \leqslant v_j$ if and only if $i \leqslant j$. For $x \in \sigma$, let v_x be the point given by the intersection of the line through x and v_0 and $\sigma_0 = \{v_1, \ldots, v_{k+1}\}$. Then each point $x \in \sigma \backslash v_0$ can be represented uniquely as $x = r\overrightarrow{\{v_0, v_x\}} + \vec{v}_0$ for $r \in (0,1]$. Then define

$$\theta(r\overrightarrow{\{v_0, v_x\}} + \vec{v}_0) = r^p\overrightarrow{\{v_0, v_{\theta(x)}\}} + \vec{v}_0$$

By construction θ has fixed points $\mathrm{Fix}(\theta) = v(\sigma) = \{v_0, \ldots, v_d\}$.

Homeomorphism and Homotopic to id**:** The construction of θ is depended on a choice of p. When this choice is relevant, we notate the map as $\theta(p)$. The inverse of $\theta(p)$ is given by $\theta(\frac{1}{p})$. Define the homotopy $H \colon \sigma \times I \to \sigma$ as $H(x,t) = \theta(t+(1-t)p)(x)$. Then $H(1) = \theta(1) = \mathrm{id}$ and $H(0) = \theta(p)$ □

Proposition 3.4.2. *For a matrix A that is n-cyclotomic and n-vacant decomposition of $\mathbb{R}^d$, there is an equivariant homeomorphism $\Theta_n(A,p) \colon \mathbb{R}^d \to \mathbb{R}^d$ which is homotopic to the identity, commutes with A and so that*

$$\mathrm{Fix}(\Theta_n(A,p)^n) = \mathcal{P}_n(\overline{A})$$

Proof. Let K be a n-vacant equivariant decomposition of $\mathbb{R}^d$ as in Proposition 3.3.2. We define the map $\Theta_n(A,p)$ by induction on the simplices in K.

Pick an order on $V(K)/\mathbb{Z}^d$ so that if $x \leqslant y$ then

$$(z, A^i)x \leqslant (z, A^i)y$$

for all $x, y \in V(K)/\mathbb{Z}^d$. Choose a $p \approx 1$ so that $1 > p$.

For each $\sigma \in K$ we let $\theta(\sigma) \colon \sigma \to \sigma$ be the map resulting from Lemma 3.4.1. By construction $\Theta_n(A,p)$ is $\mathbb{Z}^d \rtimes \langle A \rangle$ equivariant and $\mathrm{Fix}(\Theta_n(A,p)) = v(K) = \mathcal{P}_n(\overline{A})$.

Homeomorphim: Using the same order on $\mathcal{P}_n(\overline{A})$ the construction above produces $\Theta_n(A,p)^{-1} = \Theta_n(A, \frac{1}{p})$.

Homotopic to the identity: Similar to the homeomorphism, we define the homotopy by changing the p value while fixing the order on $\mathcal{P}_n(A)$. Recall the straight line homotopy between the constant maps at p and 1.

$$H \colon \mathbb{R}^d \times I \to \mathbb{R}^d$$

$$H(x,t) = \Theta_n(A, t + (1-t)p)$$

and note that $\Theta_n(A,1)$ is the identity map. □

The map $\Theta_n(A, p)$ is not unique and depends on the choice of $\mathbb{Z}^d \rtimes \langle A \rangle$ simplicial structure on the $\mathbb{R}^d$, the ordering on the points in $\overline{\mathcal{P}}_n(A)$, and the choice of $p \approx 1$. When these choices are irrelevant we denote these maps as $\Theta_n(A)$.

Chapter 4 Induction and Bundles

In this chapter we prove Proposition 2.0.5. This result requires using classical results for fixed points, fiberwise fixed point theory, and the equivariant tools developed in Chapter 3. Before the proof of of Proposition 2.0.5, lets take a moment to reflect on what we have accomplished so far.

First, we considered the case where $L(f^n) \neq 0$. This case was classically known and the minimal set of periodic points is achieved by the linear approximation of f. Second, we considered the class of maps studied by Halpern. For tori this restricted to understanding cyclotomic matrices.

Finally, in this section we combine all previous cases to consider maps between tori whose linear representation is given by:

i. block diagonal matrices whose blocks are n_i-cyclotomic matrices and $n_i|n$ or

ii. block upper triangular matrices whose diagonal blocks are n_i-cyclotomic matrices where $n_i|n$ or matrices B such that $L(B^n) \neq 0$.

4.1 Block Diagonal Cyclotomic Matrices

In the previous chapter, we studied cyclotomic matrices, $A \in \mathbb{Z}^{d\times d}$, their action on $\mathbb{R}^d$, and the induced action of $\mathbb{Z}^d \rtimes \langle A\rangle$ on $\mathbb{R}^d$ in order to understand the periodic points of $\overline{A}: T \to T$. For this section, we extend these results to block diagonal matrices whose blocks are cyclotomic matrices.

In this chapter we assume $A_i \in \mathbb{Z}^{d_i\times d_i}$ is n_i-cyclotomic. Let $n = lcm(n_i)$ and $\mathcal{A}$ be the block sum of A_i; we write $\mathcal{A} = \bigoplus_1^\alpha A_i$. This means $\mathcal{A}$ is not n_i-cyclotomic if there exists an i so that n_i is a proper divisor of n. However, $\langle \mathcal{A}\rangle$ is a subgroup of $\bigoplus_1^\alpha C_{n_i}$ which is isomorphic to C_n.

Theorem 4.1.1. *Let $A_1, \ldots, A_\alpha$ be n_i-cyclotomic matrices of dimension d_i where $n_i|n$. Let $d = \sum d_i$ and $\mathcal{A} = \bigoplus A_i : \mathbb{R}^d \to \mathbb{R}^d$. Then $\mathcal{A}$ is homotopic to a map g so that*

$$\mathrm{Fix}(g^n) = \mathcal{P}_n(\overline{\mathcal{A}}).$$

We begin by showing $\mathcal{A}$ is homotopic to a map with discrete fixed points. Before we begin the proof of Theorem 4.1.1, we give a motivating example that highlights how the periodic points of a single cyclotomic matrix differs from a block sum of cyclotomic matrices.

Lemma 4.1.2. *Let $A_1, \ldots, A_\alpha$ be n_i-cyclotomic matrices of dimension d_i where $lcm(n_i) = n$. Let $d = \sum d_i$ and $\mathcal{A} = \bigoplus A_i : \mathbb{R}^d \to \mathbb{R}^d$ be the map induced by the block sum the A_i's. Then $\mathcal{A}$ is homotopic to a map $\widetilde{\mathcal{A}}$ so that*

$$\mathrm{Fix}(\widetilde{\mathcal{A}}^n) = \prod \mathcal{P}_n(\overline{A}_i)$$

Proof. By Theorem 2.0.4 $A_i \simeq \widetilde{A_i}$ so that $\text{Fix}(\widetilde{A_i}^n) = \mathcal{P}_n(\overline{A})$. Define $\widetilde{\mathcal{A}} = \bigoplus \widetilde{A_i}$. $\square$

This set of periodic points is not always minimal. Example 4.1.3 highlights this discrepancy.

Example 4.1.3. Let f be the endomorphism of T^4 represented by the matrix

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

M can be written as a block diagonal matrix where $A_1^4 = I$, $A_2^6 = 1$. We are interested in the periodic points of f of order 12 since $lcm(4,6) = 12$. Then A_1 and A_2 have eigenvalues which are primitive roots of order 4 and 6 respectively. We calculate the Lefschetz numbers and periodic points of the iterates of A_1 and A_2 using linear algebra and modular arithmetic. See Table 4.1.

Table 4.1: Table of Periodic Points for Example 4.1.3 (1)

i	1	2	3	4	5	6
$L(A_1^i)$	2	4	2	0	2	4
$\text{Fix}(A_1^i)$	$(0,0), (\frac{1}{2},\frac{1}{2})$	$(0,0), (0,\frac{1}{2})$ $(\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$	$(0,0), (\frac{1}{2},\frac{1}{2})$	(x,y)	$(0,0), (\frac{1}{2},\frac{1}{2})$	$(0,0), (0,\frac{1}{2})$ $(\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$
$L(A_2^i)$	1	3	4	3	1	0
$\text{Fix}(A_2^i)$	$(0,0)$	$(0,0), (\frac{1}{3},\frac{2}{3})$ $(\frac{2}{3},\frac{1}{3})$	$(0,0), (0,\frac{1}{2})$ $(\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$	$(0,0), (\frac{1}{3},\frac{2}{3})$ $(\frac{2}{3},\frac{1}{3})$	$(0,0)$	(x,y)

By Theorem 2.0.4, A_1 and A_2 are homotopic to maps $\widetilde{A_1}$ and $\widetilde{A_2}$ with sets of periodic points

$$\text{Fix}\left(\widetilde{A_1}^{12}\right) = \left\{(0,0), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right)\right\}$$
$$\text{Fix}\left(\widetilde{A_2}^{12}\right) = \left\{(0,0), \left(\frac{1}{3},\frac{2}{3}\right), \left(\frac{2}{3},\frac{1}{3}\right), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

Then M is homotopic to the map $\widetilde{M} = \widetilde{A_1} \times \widetilde{A_2}$. The map $\widetilde{M}$ has 24 periodic points given by $\text{Fix}\left(\widetilde{A_1}^{12}\right) \times \text{Fix}\left(\widetilde{A_2}^{12}\right)$. We list these periodic points and their orders in Table 4.2.

Table 4.2: Table of Periodic Points for Example 4.1.3 (2)

A_1 \ A_2	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$
$(0,0)$	1	2	2	1
$(\frac{1}{3},\frac{2}{3})$	2	2	2	2
$(\frac{2}{3},\frac{1}{3})$	2	2	2	2
$(0,\frac{1}{2})$	3	6	6	3
$(\frac{1}{2},0)$	3	6	6	3
$(\frac{1}{2},\frac{1}{2})$	3	6	6	3

The Lefschetz number is multiplicative [HMP87, PS14a] and the matrix is block diagonal so we can recover Lefschetz numbers and periodic points of $\mathcal{A}$ from the corresponding computations for A_1 and A_2. The periodic points and Lefschetz numbers of M (and therefore $\widetilde{M}$) are in Table 4.3. Using only the results from Chapter 3 we

Table 4.3: Table of Periodic Points for Example 4.1.3 (3)

i	1	2	3	4	5	6
$L(M^i)$	2	12	8	0	2	0
$\mathrm{Fix}(M^i)$	$(0,0,0,0), (\frac{1}{2},\frac{1}{2},0,0)$	$(0,0,0,0), (0,\frac{1}{2},0,0), (\frac{1}{2},0,0,0), (\frac{1}{2},\frac{1}{2},0,0)$ $(0,0,\frac{1}{3},\frac{2}{3}), (0,\frac{1}{2},\frac{1}{3},\frac{2}{3}), (\frac{1}{2},0,\frac{1}{3},\frac{2}{3}), (\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3})$ $(0,0,\frac{2}{3},\frac{1}{3}), (0,\frac{1}{2},\frac{2}{3},\frac{1}{3}), (\frac{1}{2},0,\frac{2}{3},\frac{1}{3}), (\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{1}{3})$	$(0,0,0,0), (\frac{1}{2},\frac{1}{2},0,0)$ $(0,0,0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2},0,\frac{1}{2})$ $(0,0,\frac{1}{2},0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$ $(0,0,\frac{1}{2},\frac{1}{2}), (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$		$(0,0,0,0), (\frac{1}{2},\frac{1}{2},0,0)$	

can deform M into a map with discrete (24) fixed points. One could hope that it is enough to work in each component that the set of periodic points for $\widetilde{M}$ is minimum. However, if we compare with the cardinality for the periodic point set given in Theorem 1.1.3 we will see an issue.

$$\min_{g\simeq f}\{\left|\mathrm{Fix}\left(g^{12}\right)\right|\} = L(f^3) + L(f^2) - L(f^1) = 18$$

Upon further investigation, we discover that $\widetilde{M}$ has an orbit of order 6 even though $L(f^6) = 0$. This orbit is given by the product of an order 2 and order 3 periodic point of A_1 and A_2. This orbit accounts for the discrepancy between the 24 periodic points of $\widetilde{M}$ and the desired minimum of 18 periodic points. The remainder of this section will describe how to remove the periodic points this orbit. In order to remove these unnecessary periodic points, we require Proposition 4.1.4.

Proposition 4.1.4. *There is a homotopy, $H : \mathbb{R}^d \times I \to \mathbb{R}^d$, from $\Theta_n(A,p)$ to $\Theta_{\frac{n}{m}}(A^m,p)$ so that* $\mathrm{Fix}(H(-,t))$ *is finite for each $t \in I$.*

As with the previous sections, this proposition will require extra work along the way. First, we recall that the maps $\Theta_n(\overline{A})$ are not unique. They depend on the n-vacant decomposition of $\mathbb{R}^d$, a value p, and an ordering $\mathcal{O}$. Lemma 4.1.5 allows us to adjust the ordering through continuous deformation while adding discrete fixed points.

Lemma 4.1.5. *Let $\sigma := \mathrm{conv}\{v_0, \dots, v_d\}$ be a simplex with ordering $\mathcal{O}$ so that $v_i < v_j$ if $i < j$. Let $\mathcal{O}'$ be the ordering on $V(\sigma)$*

$$v_0 < \dots < v_{i-1} < v_{i+1} < v_i < v_{i+2} < \dots < v_d$$

Let θ and θ' be the maps from Lemma 3.4.1 with orderings $\mathcal{O}$ and $\mathcal{O}'$ respectively. There is a homotopy $H : \sigma \times I \to \sigma$ from θ to θ' so that

$$|\,\mathrm{Fix}(H(-,t))| < \infty$$

Proof. For $t \in I$ let $v(t) = t\overrightarrow{\{v_i, v_{i+1}\}} + v_i$. for each $0 < t < 1$, σ is subdivided into two simplices $\sigma_1(t)$ and $\sigma_2(t)$ with vertex orderings

$$\sigma_1(t) := \mathrm{conv}\{v_0, \dots v_{i-1}, v(t), v_i, v_{i+2}, \dots v_d\}$$

$$\sigma_2(t) := \mathrm{conv}\{v_0, \dots v_{i-1}, v(t), v_{i+1}, \dots v_d\}$$

Lemma 3.4.1 defines a map $\theta(t)$ on the simplices $\sigma_1(t) \cup \sigma_2(t)$ so that

$$\mathrm{Fix}(\theta(t)) = V(\sigma_1(t)) \cup V(\sigma_2(t)) = \{v_0, \dots, v_d, v(t)\}.$$

Define the homotopy

$$\begin{aligned} H\colon \sigma \times I &\to \sigma \\ (x,t) &\mapsto \theta(t)(x) \end{aligned}$$

Under this homotopy, as $t \to 1$ the vertex $v(t)$ is sent to v_{i+1} and $\theta(1) = \theta'$ $\square$

Remark 4.1.6. The dimension of the simplex and whether the simplex is n-vacant are not mentioned in this construction.

Example 4.1.7. Let σ be the standard 2-simplex with vertices $\{(0,0),(1,0),(0,1)\}$ and ordering $(0,0) < (0,1) < (1,0)$. See Figure 4.1a.

We add a new point $u(t)$ to the cell $\mathrm{conv}\{(0,1),(1,0)\}$ to subdivide σ into two simplices $\sigma_1(t)$ and $\sigma_2(t)$ with ordering given in Figure 4.1b. Figure 4.1b. Then collapse $u(t)$ to $(1,0)$ by sending $t \to 1$ Figure 4.1c.

We strategically add periodic points to $\Theta_n(\overline{A})$. This may be surprising since our goal is to eventually remove periodic points. We will quickly remedy this issue in the proof of Proposition 4.1.4.

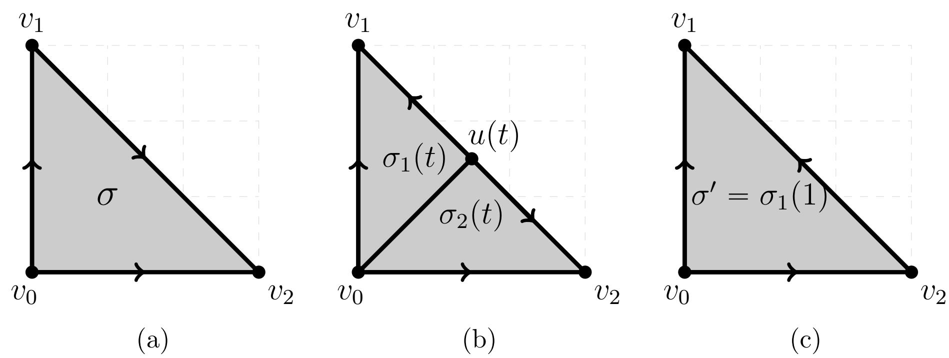


Figure 4.1: Change Ordering on a Simplex

Lemma 4.1.8. *Let σ be n-vacant and δ be $\frac{n}{m}$-vacant so that $\sigma \cap \delta \neq \emptyset$. There is a $\mathbb{Z}^d \rtimes \langle A^m \rangle$ equivariant homotopy $H\colon \sigma \times I \to \sigma$ so that:*

i. $H(-,0) = \Theta_n(A)$,

ii. $\mathrm{Fix}(H(-,t))$ *is discrete,*

iii. for $H(\delta \cap \sigma, 1) = \delta \cap \sigma$, *and*

iv. $\mathrm{Fix}(H(-,1)) = V(\delta) \cup V(\sigma) \cup V(\sigma \cap \delta)$

The proof of Lemma 4.1.8 is a more delicate version of the proof in Proposition 4.1.4.

Proof. Let σ be n-vacant and δ be $\frac{n}{m}$-vacant so that $\sigma \cap \delta \neq \emptyset$. If $\sigma \subset \delta$, H is the constant homotopy on $\Theta_n(A)$. Assume σ is not contained in δ and therefore, $\Theta_n(A)(\sigma)$ may not be contained in δ.

We proceed by induction on the dimension of σ. Let $\sigma_0 = \{v_0\}$ be n-vacant so that $\sigma_0 \cap \delta \neq \emptyset$. Then $v_0 \in \mathcal{P}_n(\overline{A})$ and therefore $\Theta_n(\overline{A})(v_0) = v_0 \in \delta$. Define H as the constant homotopy on v_0.

Let $\sigma_1 = \{v_0, v_1\}$ be n-vacant so that $v_0 < v_1$. We refine the simplex σ_1 by adding a point $u_1(t) = t\overrightarrow{\{v_0, v_1\}} + v_0$ with ordering $v_0 < u_1(t) < v_1$. There exists a smallest $t_1 \in I$ so that $u_1(t_1) \in \partial\delta$. Proposition 3.4.2 defines a map $\theta(t)$ on the simplices $\mathrm{conv}(v_0, v(t))$ and $\mathrm{conv}(v(t), v_1)$ with fixed points $\mathrm{Fix}(\theta(t)) = \{v_0, u_1(t), v_1\}$. Define H as

$$\begin{aligned} H\colon \sigma \times I &\to \sigma \\ (x,t) &\mapsto \theta(t \cdot t_1) \end{aligned}$$

If $\mathrm{conv}(v_0, v(t))$ and $\mathrm{conv}(v(t), v_1)$ are completely contained in δ, then we are done. If not, we repeat this process and add a second point $u_2(t)$ to $\mathrm{conv}(v_0, u_1(t))$ or $\mathrm{conv}(u_1(t), v_1)$.

Next, we consider $\sigma = \{v_0, \ldots v_k\}$ so that σ is not contained in δ. Then there exists a facet of σ that is not entirely contained in δ. By induction, each such facet can be refined into simplices that are either entirely contained in δ or intersect δ with co-dimension 2 by adding finitely many points $u_1(t), \ldots, u_\beta(t)$. Furthermore, for each $u_i(t)$ there exists a smallest t_i so that $u_i(t_1) \in \partial\delta$. Let $\overrightarrow{t}(\sigma)$ be the collections of all such t values. For each nonzero vector $\overrightarrow{t}$, this decomposes σ as the union of convex sets. Call this decomposition $K(\overrightarrow{t})$. By Lemma 3.4.1, there exists an $\mathbb{Z}^d \rtimes \langle A\rangle$ homeomorphism $\theta\left(\overrightarrow{t}\right) : \sigma \to \sigma$ so that $\theta(t) \simeq \mathrm{id}$ and $\mathrm{Fix}(\theta(t)) = V(K(\overrightarrow{t}))$. Define the corresponding homotopy

$$\begin{aligned} H\colon \sigma \times I &\to \sigma \\ (x,t) &\mapsto \theta(t \cdot \overrightarrow{t}(\delta)) \end{aligned} \qquad \square$$

The map $H(x,1)$ from Lemma 4.1.8 is $\mathbb{Z}^d \rtimes \langle A\rangle$ equivariant and therefore $\mathbb{Z}^d \rtimes \langle A^m\rangle$ equivariant. However, it has far to many fixed points. We have replaced the n-vacant simplex σ with a decomposition K' which is no longer n-vacant. For each simplex in $\tau \in K'$ and $\frac{n}{m}$-vacant simplex δ, either $\tau \subset \delta$ or $\tau \cap \sigma$ is co-dimension $\geqslant 1$ in τ. Furthermore, we have constructed a map $\theta\colon \sigma \to \sigma$ that is homotopic to $\Theta_n(A)$ and has fixed points given by $V(K')$ and have added discrete fixed points through this deformation.

Example 4.1.9. Suppose $\sigma = \mathrm{conv}(v_0, v_1, v_2)$ is n-vacant and $\delta = \mathrm{conv}(x_0, x_1, x_2)$ is $\frac{n}{m}$-vacant are given in Figure 4.2a. Note that δ is $\frac{n}{m}$-vacant and $v_0 \in \delta^\circ$, therefore $v_0 \in \mathcal{P}_n(A) \backslash \mathcal{P}_{\frac{n}{m}}(A^m)$. We add a point $v_0 < u_1(t) < v_1$ to $\mathrm{conv}(v_0, v_1)$ to partition σ into two simplices $\sigma_1(t)$ and $\sigma_2(t)$. There exists a point t_1 so that $u_1(t_1) \in \partial\delta$. The partition $\sigma_1(t_1) = \mathrm{conv}(v_0, u_1(t_1), v_2)$ and $\sigma_2(t_1) = \mathrm{conv}(u(t_1), v_1, v_2)$ are given in Figure 4.2b.

Note that $\sigma_2(t_0)$ satisfies the conditions of Example 4.1.9 since it only intersects δ at the point $u_1(t_0)$. However, $\sigma_1(t_0)$ is not contained in δ and does not satisfy property *iii* of Lemma 4.1.8. This is because the 1-simplex $\mathrm{conv}(v_0, v_2)$. Therefore we add a point $u_2(t)$ to the cell $\mathrm{conv}(v_0, v_2)$. There exists a t_2 so that $u_2(t_2) \in \partial\delta$. Then $\sigma_1(t_1)$ is partitioned into two simplices which we will call $\sigma_3(t_2)$ and $\sigma_4(t_2)$. This partition is shown in Figure 4.2c. Then $\sigma_2(t_1)$, $\sigma_3(t_2)$, and $\sigma_3(t_2)$ satisfy the properties Lemma 4.1.8.

In the proof of Lemma 4.1.8 the points $u_1(t)$ and $u_2(t)$ are added simultaneously which results in the same decomposition in Figure 4.2c. However, we added one point at a time to have more examples of how this refinement works.

Our next goal is to apply Proposition 4.1.4 to a n-vacant decomposition of $\mathbb{R}^d$ and then remove these unnecessary fixed points through a continuous deformation.

Proof of Proposition 4.1.4. Let $K(A)$ and $K(A^m)$ be the n-vacant and $\frac{n}{m}$-vacant decompositions of $\mathbb{R}^d$ corresponding to $\mathbb{Z}^d \rtimes \langle A\rangle$ and $\mathbb{Z}^d \rtimes \langle A^m\rangle$ respectively. By Lemma 4.1.8 there exists a $\mathbb{Z}^d \rtimes \langle A\rangle$ decomposition of $\mathbb{R}^d$, $K'(A)$, so that each $\frac{n}{m}$-vacant cell is a union of cells in $K'(A)$. By Lemma 4.1.8 $\Theta_n(\overline{A})$ is homotopic to a simplicial map $\theta\colon \mathbb{R}^d \to \mathbb{R}^d$ with fixed points $V(K'(A))$. We partition $V(K'(A))$ into sets $V_k(K'(A))$. For $k > 0$ we define

$$V_k(K'(A)) := \{x \in V(K'(A))\colon \exists \delta_k \in K(A^m) \text{ so that } x \in \delta_k^\circ\}.$$

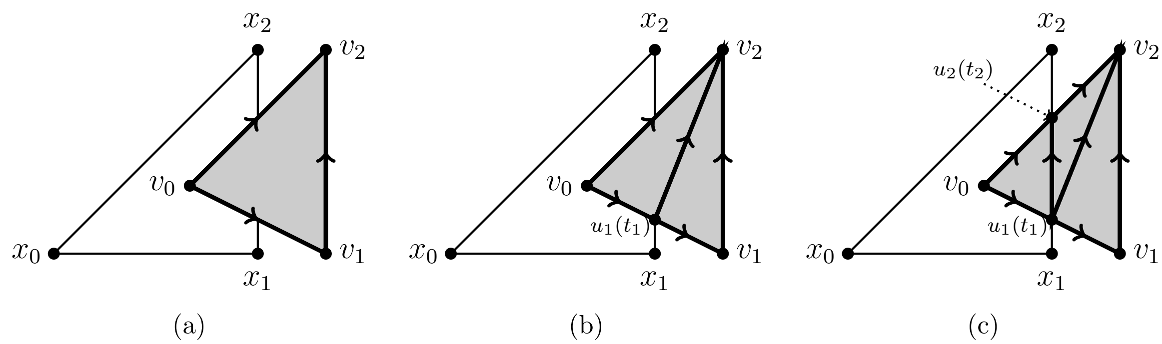


Figure 4.2: Changing Decomposition in Example 4.1.9

We define $V_0(K'(A)) = V(K(A^m))$. If $V_l(K'(A) = \varnothing$ for all $k > 0$, then $K'(A) = K(A^m)$ and $\theta = \Theta_{\frac{n}{m}}(\overline{A^m})$. If there is a $k > 0$ so that $V_k(K'(A)) \neq \varnothing$, then our goal is to continuously deform $\theta\colon \mathbb{R}^d \to \mathbb{R}^d$ into $\Theta_{A^m}(\frac{n}{m})$ by removing these fixed points. Since $\mathrm{Fix}(\Theta_{A^m}(\frac{n}{m})) = V(K(A^m)) = V_0(K'(A))$, we will remove $V_k(K'(A))$ for $k > 0$ through continuous deformation. First, we construct a homotopy and decomposition, $K''(A)$, so that $V_k(K''(A))$ contain, at most, one point for all $k > 0$.

Through finitely many applications of Lemma 4.1.5, we may assume that if $l > k$ and $v \in V_l(K'(A))$ and $w \in V_k(K'(A))$, then $v > w$. Pick $\frac{n}{m}$-vacant l-simplex $\delta \in K(A^m)$ and $y_\delta \in \delta^\circ$. For each $v \in V_l(K'(A)) \cap \delta$ define $v(t) = t\overrightarrow{\{v, y_\delta\}} + y_\delta$. This defines a decomposition for the simplices $\mathbb{Z}^d \rtimes \langle A^m \rangle \cdot \delta$. Let $K'(A, t_1, \ldots t_d)$ be the decomposition of $\mathbb{R}^d$ and $\theta'(t_1, \ldots, t_d)$ be the map from Lemma 3.4.1. Note that $K'(A, 1, \ldots, 1) = K'(A)$. For $\frac{i-1}{d} \leqslant t \leqslant \frac{i}{d}$ define

$$\begin{aligned} \widehat{H}_i\colon \mathbb{R}^d \times \left[\frac{i-1}{d}, \frac{i}{d}\right] &\to \mathbb{R}^d \\ (x,t) &\mapsto \theta'(1, \ldots, 1, dt - (i-1), 0, \ldots, 0)(x) \end{aligned}$$

Define a homotopy, $\widehat{H}$ by concatenating $\widehat{H}_i$ for all $1 \leqslant i \leqslant d$. Let $K''(A) = K'(A, 0, \ldots, 0)$ be the final decomposition and $\theta''\colon \mathbb{R}^d \to \mathbb{R}^d$ be the map given by $\widehat{H}(-, 0)$. For each $\delta \in K''(A)$ the set $\delta \cap V_k(K''(A))$ contains, at most one point. Recall, our next goal is to remove all points in $V_k(K''(A))$ for $k > 0$, we do so through a second homotopy.

For each $v \in V_l(K''(A))$, there exists an l dimensional simplex $\delta_l \in K(A^m)$ so that $v \in \delta_l^\circ$. Let $y_0 \in V(\delta_l)$ be smallest vertex in δ_l. Define $v(t) := t\overrightarrow{\{v, y_0\}} + y_0$. This defines a decomposition for all simplices $\mathbb{Z}^d \rtimes \langle A^m \rangle \cdot \delta$. Let $K''(A, t_1, \ldots t_d)$ be the decomposition of $\mathbb{R}^d$ and $\theta''(t_1, \ldots, t_d)$ be the map resultant from Lemma 3.4.1. Note that $K''(A, 1, \ldots, 1) = K''(A)$.

For $\frac{i-1}{d} \leqslant t \leqslant \frac{i}{d}$ we define

$$\begin{aligned} \widetilde{H}_i\colon \mathbb{R}^d \times \left[\frac{i-1}{d}, \frac{i}{d}\right] &\to \mathbb{R}^d \\ (x,t) &\mapsto \theta'(0, \ldots, 0, dt - (i-1), 1, \ldots, 1)(x) \end{aligned}$$

Define the homotopy, $\widetilde{H}$, by concatenating $\widetilde{H}_i$ for all $1 \leqslant i \leqslant d$. Then $K''(A, 0, \ldots, 0) = K(A^m)$ and $\theta''(0, \ldots, 0) = \Theta_{\frac{n}{m}}(A^m)$. To finish the proof we concatenate the homotopies $\widehat{H}$ and $\widetilde{H}$ to produce the desired map $H\colon \mathbb{R}^d \times I \to \mathbb{R}^d$. $\square$

Example 4.1.10. Recall the 6-cyclotomic matrix and 6-vacant decomposition of T given in Example 3.0.1.

We can now prove the main result of this chapter. The main idea of the proof is that the vanishing of the Lefschetz number of the m^{th} iterate of $\mathcal{A}$, $L(\mathcal{A}^m) = 0$, serves as a marker that $\mathcal{A}$ is homotopic to a map with no fixed points of order m. We begin by isolating an orbit of order m such that $L(\mathcal{A}^m) = 0$. Since $\mathcal{A}$ is a block sum, there exists an i so that $L(A_i^m) = 0$. We then take a "slice" around the i^{th} component and use Proposition 4.1.4 within this "slice" to remove these periodic points.

Proof of Theorem 4.1.1. Suppose $A_1, \ldots, A_\alpha$ are n_i-cyclotomic matrices of dimension d_i where $n_i | n$. Let $d = \sum d_i$ and $\mathcal{A} = \bigoplus A_i\colon \mathbb{R}^d \to \mathbb{R}^d$. By Lemma 4.1.2 $\mathcal{A}$ is homotopic to a map with discrete periodic points. If $\Theta_n(A_i)$ is the map from Proposition 3.4.2 for $A_i \in \mathbb{Z}^{d_i \times d_i}$, define

$$\widetilde{\mathcal{A}} = \bigoplus (A_i \circ \Theta_n(A_i)) = \bigoplus \widetilde{A_i}$$

Then $\mathrm{Fix}((\widetilde{\mathcal{A}})^n) = \bigoplus \mathcal{P}_n(\overline{A_i})$ and $\mathcal{P}_n(\overline{\mathcal{A}}) \subseteq \mathrm{Fix}((\widetilde{\mathcal{A}})^n)$. If this containment is proper, there exists a periodic point $\overrightarrow{x} := (x_1, \ldots, x_\alpha)$ of order m such that $L(\mathcal{A}^m) = 0$ and $n_i | m$. If x_i is of order l_i, then $lcm(l_1, \ldots l_\alpha) = m$. Since $L(\mathcal{A}^m) = 0$ there exists an A_i so that $L(A_i^m) = 0$. By conjugation by a transposition homeomorphism we assume that the order of A_1, n_1, is minimal amongst these orders and $l = lcm(l_2, \ldots, l_\alpha)$.

We define two spaces around this periodic point. Since $\mathrm{Fix}(\widetilde{\mathcal{A}}^n) = \prod \mathcal{P}_n(\overline{A}_i)$ is discrete, there exists an $\varepsilon > 0$ so that

$$\varepsilon < \min \left\{ \frac{\min\{d(y_1, y_2) | y_1, y_2 \in \mathrm{Fix}(\mathcal{A}^n);\ y_1 \neq y_2\}}{2}, \frac{1}{2} \right\}$$

and

$$B(x, \epsilon) \cap \mathrm{Fix}(\mathcal{A}^n) = \{x\}$$

for all $x \in \mathcal{P}_n(\overline{\mathcal{A}})$. Let $d = \sum d_i$ and $B_\varepsilon(x_2, \ldots, x_\alpha) \subset \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_\alpha}$ be the ball of radius ε. For $0 \leqslant i \leqslant l$

$$\begin{aligned} \mathbb{R}^d(\overrightarrow{z}, i, x_1) &:= \mathbb{R}^{d_1} \times \{A_2^i x_2 + z_2, \ldots, A_\alpha^i x_\alpha + z_\alpha\} \\ U(\overrightarrow{z}, i, x_1) &:= \mathbb{R}^{d_1} \times B_\varepsilon(A_2^i x_2 + z_2, \ldots, A_\alpha^i x_\alpha + z_\alpha). \end{aligned}$$

The neighborhoods $U(\overrightarrow{z}, i, x_1)$ are disjoint. Using Proposition 3.4.2 we define $\widetilde{\mathcal{A}}' \simeq \widetilde{\mathcal{A}}$ by

$$\widetilde{\mathcal{A}}'(U(\overrightarrow{z}, i, x_1)) = \begin{cases} \bigoplus_1^\alpha A_j \Theta_n(A_j) & i = 0 \\ A_1 \times \bigoplus_2^\alpha A_j \Theta_n(A_j) & 0 < i \leqslant l \end{cases}$$

We will now use deformations from Proposition 4.1.4 on each of the $U(\overrightarrow{z}, 0, x_1)$. Let $H_{1,l}\colon \mathbb{R}^{d_1} \times I \to \mathbb{R}^{d_1}$ be the homotopy from $\Theta_n(A_1)$ to $\Theta_{\frac{n}{l}}(A_1^l)$ given by Proposition 4.1.4. For $w \in B_\varepsilon(x_2, \ldots, x_\alpha)$, let $|w|$ be the distance from z to the origin. For $(y, w) \in \mathbb{R}^{d_1} \times B_\varepsilon(x_2, \ldots, x_\alpha)$, let

$$g(y, w) = H_{1,l}\left(y, 1 - \frac{|w|}{\varepsilon}\right) \times \bigoplus_2^\alpha A_j \Theta_n(A_j).$$

The periodic points of g have not been changed away from the $U(\overrightarrow{z}, i, x_1)$. It is enough to compute the periodic points within U_0. For $(x, z) \in U(\overrightarrow{z}, 0, x_1) \backslash \mathbb{R}^d(\overrightarrow{z}, 0, x_1)$ the point z is not fixed by $\bigoplus_2^\alpha A_j^n$ and $(x, z) \notin \mathrm{Fix}(g^n)$. For $(x, z) \in \mathbb{R}^{d_1}(\overrightarrow{z}, 0, x_1)$, z is fixed by $\bigoplus_2^\alpha A_j^n$. Therefore, $(x, z) \in \mathrm{Fix}(g^n) \cap \mathbb{R}^{d_1}(\overrightarrow{z}, 0, x_1)$ if and only if $x \in \mathrm{Fix}((A_1^l \circ \Theta_{\frac{n}{l}}(A_1^l)^{\frac{n}{l}})$. By construction, $A_1^l \circ \Theta_{\frac{n}{l}}(A_1^l)$ has no periodic points of order k such that $L((A_1^l)^k) = 0$, and the point x_1 is not fixed. $\square$

4.2 Inductive and Non-trivial Bundle Maps

We start by considering the following motivating example from [You95].

Example 4.2.1. Let $\overline{M}\colon S^1 \times S^1 \to S^1 \times S^1$ be an endomorphism of the torus represented by the matrix

$$M = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \quad M^2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

Fixed points for M^2 are pairs (x, y) so that

$$x - 4y \equiv x \text{ and } y \equiv y.$$

The second equation does not depend on x. By Theorem 2.0.4 there is a map g, homotopic to multiplication by -1 on $\mathbb{R}$, with periodic points $\mathbb{Z} + \{y_1, y_2\}$. Then $\overline{M}$ is homotopic to the map

$$\widetilde{M}(x, y) = (g(x) + 2y, g(y)).$$

If $\{v_1, v_2\} + \mathbb{Z}$ are the fixed points of g, then fixed points of $\widetilde{M}^2$ are the points

$$\{(x, y_i) \in \mathbb{R} \times (\{y_1, y_2\} + \mathbb{Z}) | g(x + 2y_i) + 2g(y_i) \equiv x\}.$$

This is difficult to compute even though we started with the well behaved map g. We address this problem through homotopy extension. Let $L_{y_1}(t)$, and $L_{y_2}(t)$ be the lines from y_i to 0 in the unit interval and H be the homotopy defined by

$$H\colon\ (\mathbb{Z} + \{y_1, y_2\}) \times I \to \mathbb{R}$$

$$H(z + y_i, t) = z + L_{y_i}(t).$$

Consider the following homotopy extension square.

$$\begin{tikzcd}
\mathbb{Z} + \{y_1, y_2\} \arrow[r, "H"] \arrow[d, "i"'] & (\mathbb{R})^I \arrow[d, "p_0"] \\
\mathbb{R} \arrow[r, "2"'] \arrow[ur, dashed, "\widetilde{H}"] & \mathbb{R}
\end{tikzcd}$$

Define $q := \widetilde{H}(-, 1)$. Then $\widetilde{M}$ is homotopic to a map on tori, $\widehat{M}$, of the form:

$$\widehat{M}(x, y) = (g(x) + q(y), g(y))$$

Furthermore, the fixed points of $\widehat{M}^2$ are the points (x, y) so that

$$\big(g(g(x) + q(y)) + q(g(y)), g^2(y)\big) \equiv (x, y).$$

All fixed points must first satisfy $g^2(y) \equiv y$. These are the points $\{y_1, y_2\} + \mathbb{Z}$ (which are also the fixed points of g). This allows us to calculate the fixed points of $\widehat{M}^2$ by solving

$$\begin{aligned}
g(g(x) + q(y_i)) + q(g(y_i)) &\equiv x \\
g(g(x) + q(y_i)) + q(y_i) &\equiv x \\
g^2(x) &\equiv x
\end{aligned}$$

This results in four total points $\{x_1, x_2\} + \mathbb{Z}$. We can see here that we have removed the noise caused by the off diagonal position in the matrix,

$$\mathrm{Fix}(\widetilde{M}^2) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) + \mathbb{Z}^2\},$$

and the induced map on tori has precisely 4 periodic points.

In this example we leveraged many great facts about Tori and $\mathbb{R}^d$. First, the map M is a map of torus bundles.

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r, "M"] \arrow[d, "\pi_2"] & S^1 \times S^1 \arrow[d, "\pi_2"] \\
S^1 \arrow[r, "-1"'] & S^1
\end{tikzcd}$$

Further, the map on the base and on the fibers is compatible with the equivariant structure in Theorem 2.0.4. For this map of bundles, it is enough to first consider the map on the base and then homotope that map on each fiber. Using this motivation, we prove Proposition 2.0.5 in two steps. First, we show that M is conjugate to a map of torus bundles if:

i. $\det(I - M^n) = 0$, and

ii. $char(M) = \Phi_{l_1}^{\alpha_1} \Phi_{l_2}^{\alpha_2} \dots \Phi_{l_s}^{\alpha_s} g$ where $l_i | n$, Φ_m is the cyclotomic polynomial of order m, and $g(\zeta_n) \neq 0$ for any n^{th} root of unity ζ_n.

This approach is motivated by work of You [You95].

Lemma 4.2.2. *Let M be a square matrix n be an integer so that:*

i. $\det(I - M^n) = 0$,

ii. $char(M) = \Phi_{l_1}^{\alpha_1}\Phi_{l_2}^{\alpha_2}\dots\Phi_{l_s}^{\alpha_s}g$ *where* $l_i|n$*,* Φ_m *is the cyclotomic polynomial of order* m*, and* $g(\zeta_n) \neq 0$ *where* ζ_n *is any* n^{th} *root of unity.*

then, up to an conjugation, M is of the form

$$\begin{pmatrix} A_1 & C_{1,1} & C_{1,2} & \dots & C_{1,j} \\ 0 & A_2 & C_{2,2} & \dots & C_{2,j} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & A_j & C_{j,j} \\ 0 & 0 & 0 & 0 & B \end{pmatrix}$$

where A_i is l_i-cyclotomic and $char(B)$ is not divisible by any Φ_m for $m|n$.

Proof. Since $\det(I - M^n) = 0$ the eigenvalues of M are roots of unity of order l_i. Assume l_1 is the smallest of the l_i and define

$$K = \ker(M^{l_1} - I).$$

Since $\det(I - M^{l_1}) = 0$, the subspace K is non empty and a direct summand of $\mathbb{Z}^d$. Then there is an isomorphism

$$\rho\colon K \bigoplus V \to \mathbb{Z}^d.$$

The matrix representing $\rho^{-1} \circ M \circ \rho\colon \mathbb{Z}^d \to \mathbb{Z}^d$ has the form

$$M' = \begin{pmatrix} A & C \\ 0 & B' \end{pmatrix}.$$

Since $M^l|_K = A^l = \mathrm{id}$ and l is minimal, $\det(I - A^m) \neq 0$ for proper divisors m of l. Therefore that A is l_1-cyclotomic.

If $L((B')^n) = 0$, then map B' has eigenvalues which are roots of n^{th} roots of unity and therefore we can apply the above to the pair (B', V). Repeating this process s times produces a matrix of the desired form. □

After reducing to a map of torus bundles we can inductively remove fixed points on each fiber by using Theorem 2.0.4 on each matrix A_i. This culminates in a map homotopic to M which has finitely many fixed points and prescribed behavior in a neighborhood of those fixed points.

Proposition 4.2.3. *Let M be a square matrix and n be an integer so that:*

i. $\det(I - M^n) = 0$,

ii. $char(M) = \Phi_{l_1}^{\alpha_1}\Phi_{l_2}^{\alpha_2}\dots\Phi_{l_s}^{\alpha_s} g$ *where* $l_i | n$, Φ_m *is the cyclotomic polynomial of order* m, *and* $g(\zeta_n) \neq 0$ *where* ζ_n *is any* n^{th} *root of unity.*

Then M *is* $\mathbb{Z}^d$*-homotopic to a map* $\widetilde{M}$ *so that*

$$\mathrm{Fix}(\widetilde{M}^n) = \mathcal{P}_n(\overline{B}) \times \left(\Pi_{i=1}^{j}\mathcal{P}_n(\overline{A}_i)\right).$$

Further, for each $x \in \mathrm{Fix}(\widetilde{M}^n)$, *the map* $\widetilde{M}$ *restricted to* $B(\varepsilon, x)$ *is diagonal. That is*

$$\widetilde{M}|_{B(\varepsilon,(x_1,\dots,x_j,x_{j+1}))} = (\widetilde{A}_1 x_1, \dots, \widetilde{A}_j x_j, B x_{j+1})$$

where A_i *is* l_i*-cyclotomic and* $L(B^n) \neq 0$

Proof. Assume M is as in Lemma 4.2.2. Since $\det(I - B^n) \neq 0$, the set of fixed points of $\overline{B}^n$, $\mathcal{P}_n(\overline{B})$, is finite. Each map A_i is l_i-cyclotomic and Theorem 2.0.4 implies there is a map $\widetilde{A}_i \simeq A_i \colon \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$ so that

$$\mathrm{Fix}(\widetilde{A}_i^n) = \mathcal{P}_n(\overline{A}_i).$$

This set is discrete. For each $y_i \in \mathcal{P}_n(\overline{B})$ let $L_{y_i}(t)$ be the straight line from $C_{i,j}y_i$ to the origin in the unit cube. Then define the homotopy

$$H_{i,j} \colon \left(\mathcal{P}_n(\overline{B})x\right) \times I \to \mathbb{R}^d$$

$$H_{i,j}(z + y_i, t) = z + L_{y_i}(t).$$

Let $H_{i,j}$ the the homotopy which drags each point in $\mathcal{P}_n(\overline{B})$ to the origin along a straight line. Consider the following homotopy extension square

$$\begin{tikzcd}
\mathcal{P}_n(\overline{B}) \arrow[r, "H_{i,j}"] \arrow[d, "i"'] & (\mathbb{R}^{d_i})^I \arrow[d, "p_0"] \\
\mathbb{R}^{d_j} \arrow[r, "C_{i,j}"'] \arrow[ur, dashed, "\widetilde{H}_{i,j}"] & \mathbb{R}^{d_i}
\end{tikzcd}$$

$\mathbb{Z}^d$ homotopy extension defines the lift $\widetilde{H}_{i,j}$ [GM95]. Define

$$\widetilde{C}_{i,j} := \widetilde{H}_{i,j}(-, 1) \simeq C_{i,j}.$$

Then $\widetilde{C}_{i,j}|_{\mathcal{P}_n(\overline{B})} = 0$ for $i \leqslant j$. Similarly $\bigcup_{x \in \mathcal{P}_n(A_k)} \overline{B}(x, \varepsilon)$ is disjoint and we can produce maps $\widetilde{C}_{i,j-k} \simeq C_{i,j-k}$ so that $\widetilde{C}_{i,j-k}|_{\bigcup_{x \in \mathrm{Fix}(\widetilde{A}_k^n)} \overline{B}(x,\varepsilon)} = 0$ for $i > 0$.

Then M is homotopic to the map

$$\widetilde{M} = \begin{pmatrix} \widetilde{A}_1 & \widetilde{C}_{1,1} & \widetilde{C}_{1,2} & \dots & \widetilde{C}_{1,j} \\ 0 & \widetilde{A}_2 & \widetilde{C}_{2,2} & \dots & \widetilde{C}_{2,j} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \widetilde{A}_j & \widetilde{C}_{j,j} \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \tag{4.1}$$

Next we calculate the fixed points of $\widetilde{M}^n$. Assume the point $\vec{x} = (x_1, \ldots, x_j, y)$ in $\mathbb{R}^d$ is fixed by $\widetilde{M}^n$, then

$$B^n y = y.$$

Therefore, the fixed points of $\widetilde{M}^n$ are a subset of $T_1 \times \ldots \times T_j \times \mathcal{P}_n(\overline{B})$. The fixed points in the j^{th} component must satisfy the equation

$$\widetilde{A}_j(\widetilde{A}_j(\ldots \widetilde{A}_j(\widetilde{A}_j(x_j + \widetilde{C}_{j,j}y) + \widetilde{C}_{j,j}By) + \ldots) + \widetilde{C}_{j,j}B^{n-2}y) + \widetilde{C}_{j,j}B^{n-1}y = x_j \quad (4.2)$$

Since $y \in \mathcal{P}_n(\overline{B})$ and $\widetilde{C}_{j,j}$ is 0 on $\mathcal{P}_n(\overline{B})$. Display (4.2) simplifies to

$$\widetilde{A}_j^n(x_j) = x_j.$$

Therefore, $x_j \in \mathcal{P}_n(\overline{A}_j)$ and the fixed points of $\widetilde{M}^n$ are a subset of $T_1 \times \ldots \times T_{j-1} \times \mathcal{P}_n(\overline{A}_j) \times \mathcal{P}_n(\overline{B})$. Similarly $x_i \in \mathcal{P}_n(\overline{A}_i)$ and $\mathrm{Fix}(\widetilde{M}^n) = \mathcal{P}_n(\overline{B}) \times \big(\Pi_i \mathcal{P}_n(\overline{A}_i)\big)$, which is discrete. $\square$

Proposition 4.2.4. *Let M be a square matrix and n be an integer so that:*

i. $\det(I - M^n) = 0$,

ii. *$char(M) = \Phi_{l_1}^{\alpha_1}\Phi_{l_2}^{\alpha_2} \ldots \Phi_{l_s}^{\alpha_s} g$ where $l_i | n$, Φ_m is the cyclotomic polynomial of order m, and $g(\zeta_n) \neq 0$ where ζ_n is any n^{th} root of unity.*

Then, up to an conjugation, M is of homotopic to a map $\widetilde{M}$ so that $\overline{\mathcal{P}}_n(M) \subseteq$ $\mathrm{Fix}(\widetilde{M}^n)$. *If the containment is proper, then $\widetilde{M} \simeq \widehat{M}$ so that*

$$\overline{\mathcal{P}}_n(M) \subseteq \mathrm{Fix}(\widehat{M}^n) \subsetneq \mathrm{Fix}(\widetilde{M}^n).$$

Proof. Assume $\widetilde{M} \colon \mathbb{R}^d \to \mathbb{R}^d$ is of the form in Proposition 4.2.3. Let $P_i(\widetilde{M})$ denote the set of periodic points of $\widetilde{M}$ which are exactly period i. Assume m is the minimal proper divisor of n so that $L(f^m) = 0$ and $P_m(\widetilde{M}) \neq \varnothing$. Then $x \in P_m(\widetilde{M})$ can be written as

$$x := (x_1, x_2, \ldots, x_j, x_{j+1})$$

where each point x_i is a periodic point of $\widetilde{A}_i$ of order k_i, and $[x]$ denote the orbit of x under $\widetilde{M}$. By construction, within a neighborhood of x, M is of the form in Proposition 4.1.4.

Since x is of order m we have

$$lcm(k_i | 1 \leqslant i \leqslant j + 1) = m.$$

Since M is a map of bundles and $M_{x_{j+1}}$ is of the form in Proposition 4.1.4. Apply Proposition 4.1.4 to the fiber over x_{j+1}, the map M is homotopic to a map $\widetilde{M}$ so that $\widehat{M}|_{x_{j+1}}$ has no points of order $\frac{m}{k_{j+1}}$. Therefore x is no longer fixed by $\widehat{M}$ and

$$\overline{\mathcal{P}}_n(M) \subseteq \mathrm{Fix}(\widehat{M}^n) \subsetneq \mathrm{Fix}(\widetilde{M}^n). \qquad \square$$

Chapter 5 Periodic Points for Families of Endomorphisms

5.1 Klein and Williams: Pullback Diagram

Let $\mathrm{end}(M)$ denote the space of self maps of T, and $\mathrm{end}(M^{\times n})^{C_n}$ the space of C_n equivariant self maps of $M^{\times n}$. The Fuller construction defines a map

$$\Psi_n \colon \mathrm{end}(T) \to \mathrm{end}_{C_n}(T^{\times n})$$

by $\Psi_n(f)(x_1, \dots x_n) = (f(x_n), f(x_1), f(x_2) \dots, f(x_{n-1}))$, then $\Psi_n(f)$ is C_n equivariant and $\mathrm{Fix}(\Psi_n(f)) = \mathrm{Fix}(f^n)$. This relates the question of periodic points to the study of equivariant fixed point theory. Let $\mathrm{end}^{\flat_n}(T) \subset \mathrm{end}(T)$ denote the subspace of maps that have no n- periodic points, and $\mathrm{end}^{\flat}_{C_n}(T^{\times n}) \subset \mathrm{end}_{C_n}(T^{\times n})$ denote the subspace of maps that are C_n fixed point free.

Following the work of Klein and Williams [KW10], we wish to consider the following pullback square.

$$\begin{tikzcd}
\mathrm{end}^{\flat_n}(T) \arrow[r] \arrow[d, "\Psi_n"'] & \mathrm{end}(T) \arrow[d, "\Psi_n"] \\
\mathrm{end}^{\flat}_{C_n}(T^{\times n}) \arrow[r] & \mathrm{end}_{C_n}(T^{\times n})
\end{tikzcd}$$

The vertical maps are given by the Fuller transform and the horizontal ones are inclusions. Theorem 1.1.3 gives us the following corollary.

Corollary 5.1.1. *For a map of tori $f \colon T \to T$, $f \simeq g$ so that $\mathrm{Fix}(g^n) = \varnothing$ if and only if $L(f) = L(f^m) = 0$ for all $m|n$.*

Equivalently, this follows from the fact that tori are essentially irreducible as in [Hea99]. This is an extremely strong reduction. To understand the pullback diagram as in Klein and Williams, we start by considering homotopy classes of families of maps in $\mathrm{end}(T)$ parameterized by spheres.

5.2 Families of Maps Parameterized by S^1

In general, understanding families of maps parametrized by a space can be very difficult. To highlight how difficult, we restate Theorem 1.1.3 in terms of families of maps parameterized by a point. That is, minimizing the periodic points of a single endomorphism is equivalent to minimizing periodic points within the homotopy classes of maps

$$* \to \mathrm{end}(T).$$

This immediately leads us to the Corollary 5.2.1.

Corollary 5.2.1. *Let $F\colon S^0 \to \mathrm{end}(T)$ and n be an integer. If h_x is the linear map homotopic to $F(x)$ for $x \in S^0$, then F is homotopic to a map G so that*

$$\mathrm{Fix}(G(x)^n) = \bigcup_{m|n, L(G(x)^m)\neq 0} \mathrm{Fix}(h_x^m).$$

Further, if K is homotopic to F then

$$|\,\mathrm{Fix}(G(-,t)^n)| \leqslant |\,\mathrm{Fix}(K(-,t)^n)|.$$

When stated in this manner it becomes clear that we would like to make similar statements for spheres, S^d, for $d \geqslant 1$. Consider the following diagrams.

$$\begin{tikzcd}
{[S^1, \mathrm{end}^{\flat_n}(T)]} \arrow[r] \arrow[d, "\Psi_n"'] & {[S^1, \mathrm{end}(T)]} \arrow[d, "\Psi_n"] \\
{[S^1, \mathrm{end}^{\flat}_{C_n}(T^{\times n})]} \arrow[r] & {[S^1, \mathrm{end}_{C_n}(T^{\times n})]}
\end{tikzcd}$$

Since S^1 is connected, $F\colon S^1 \to \mathrm{end}(T)$ must land in one path component of $\mathrm{end}(T)$, which are in one to one correspondence with matrices $\mathbb{Z}^{d\times d}$. This can be thought of as a "souped up" homotopy between a map $f \in \mathrm{end}(T)$ and itself. By applying the mapping space adjuction this diagram is equivalent to the following.

$$\begin{tikzcd}
{[T \times S^1, T \times S^1]_{\flat_n, S^1}} \arrow[r] \arrow[d, "\Psi_n"'] & {[T \times S^1, T \times S^1]_{S^1}} \arrow[d, "\Psi_n"] \\
{[T^{\times n} \times S^1, T^{\times n} \times S^1]_{\flat, C_n, S^1}} \arrow[r] & {[T^{\times n} \times S^1, T^{\times n} \times S^1]_{C_n, S^1}}
\end{tikzcd}$$

Since these are maps between tori, these sets have been studied in the previous chapters. We are considering maps between torus bundles over the identity map of the circle. However, the constructions and results Section 4.2 are designed to be compatible with bundles.

A first hope is that the sequence of Lefschetz numbers, $\{L(F(-,t)^m)\}_{m|n}$, is enough to minimize the periodic points the corresponding map $F\colon T \times S^1 \to T \times S^1$. Example 5.2.2 shows this is incorrect.

Example 5.2.2. Let T be the d-torus and $\varphi\colon S^1 \to \mathrm{end}(T)$ be the rotation map $[r] \mapsto [id_r]$ where $id_t(x) = x + r * e$ and $e = (1, 1, \ldots 1)$. Passing through the adjunction this gives us the following

$$\Phi = \begin{pmatrix} I & e^T \\ 0 & 1 \end{pmatrix}.$$

The Intermediate Value Problem shows this map is not a fiberwise fixed point free. Since Φ decends to a map of tori so that the last component is always fixed, we regard the map on the fiber over x as a map $\Phi\colon I^{\times d} \times I \to \mathbb{R}^d$ and consider the homotopy

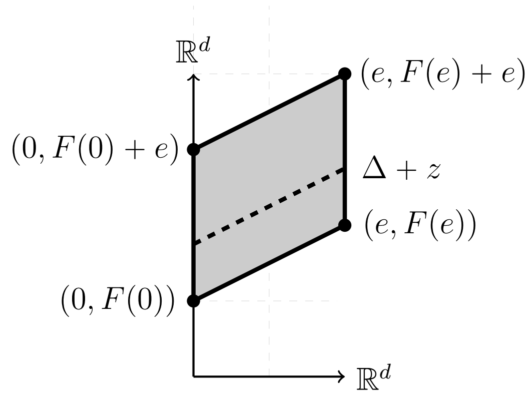


Figure 5.1: Example 5.2.2 Intermediate Value Theorem

class $[\Phi]$. For all $F \simeq \Phi$ must span the corners of the box given by $(0, F(0,0))$ and $(e, F(e,1) + e)$ and $e = (1, \ldots, 1)$. By the Intermediate Value Theorem, regardless of how much we "wiggle" the map it must intersect the diagonal $\Delta + z$ for some $z \in \mathbb{Z}^d$. Though high dimensions are difficult to visualize we have included a figure to illustrate this counter example. See Figure 5.1. Therefore at least one map of on the fibers must have a fixed point.

Theorem 5.2.3. *Let $d = \{0, 1\}$, $F \colon S^d \to \mathrm{end}(T)$ be continuous so that $L(F(0)) \neq 0$, and n be an integer. If h is the linear map homotopic to $F(0)$, then F is homotopic to a map G with*

$$\mathrm{Fix}(G(-,t)^n) = \bigcup_{m|n, L(h^m) \neq 0} \mathrm{Fix}(h^m).$$

Further, if K is homotopic to F then

$$|\,\mathrm{Fix}(G(-,t)^n)| \leqslant |\,\mathrm{Fix}(K(-,t)^n)|.$$

The restriction that $L(F(0)) \neq 0$ in Theorem 5.2.3 removes the map considered in Example 5.2.2. Recall that the study of maps $F' \colon S^1 \to \mathrm{end}(T)$ is equivalent to studying maps $F \colon T \times S^1 \to T$ or maps of bundles $F \colon T \times S^1 \to T \times S^1$. Since F is a map between tori it is homotopic to a linear map. We use all of these descriptions interchangeably. Without loss of generality, assume that F is of the form

$$F = \begin{pmatrix} M & r \end{pmatrix}$$

or

$$F = \begin{pmatrix} M & r \\ 0 & 1 \end{pmatrix}$$

where $M \simeq F(0) \in \mathrm{end}(T)$ and $r \in \mathbb{Z}^{d \times 1}$. For $t \in I$, $F(-,t) = [M(-) + rt, t]$ and

$$F^n(-,t) = [M^n(-) + M^{n-1}r + M^{n-2}r + \ldots Mr + r, t].$$

To understand fixed points of F we must understand how these "rotations" change the fixed points of M^n.

We reduce this problem to the following three cases in Table 5.1. In Example 5.2.2 we saw a problem with fixed points of maps so that $L(F-,0) = 0$. Remarkably, this issue is not applicable for fixed or periodic points $L(f(-,0)) \dotplus 0$ (aka we already have fixed points).

Table 5.1: Comparison of Lefschetz numbers and characteristic polynomials

	Lefschetz Property	Characteristic Polynomial Property
1	$L(F(-,0)^n) \neq 0$	$\mathrm{char}(F(-,0)_*)$ is not divisible by any cyclotomic polynomial of order $m\|n$.
2	$L(F(-,0)^n) = 0$, $F(-,0)^n \simeq \mathrm{id}$, and $L(F(-,0)^m) \neq 0$ for all $m\|n$	$\mathrm{char}(F(-,0)_*)$ is a power of a single cyclotomic polynomial and $F(-,0)^n_* = \mathrm{id}$.
3	$L(F(-,0)^n) = 0$, $f^n \simeq \mathrm{id}$ and $L(F(-,0)^m) = 0$ for at least one $m\|n$, $m < n$	$\mathrm{char}(F(-,0)_*)$ is a product of distinct cyclotomic polynomials of order $m\|n$ and $F(-,0)^n_* = \mathrm{id}$.
	$L(F(-,0)^n) = 0$ and $F(-,0)^n \not\simeq \mathrm{id}$	$\mathrm{char}(F(-,0)_*)$ is divisible by a cyclotomic polynomial of order $m\|n$ and $F(-,0)^n_* \neq \mathrm{id}$

Theorem 5.2.4. *Given a map $F(x,t)\colon T \times S^1 \to T \times S^1$ such that $L(F(-,t)) \dotplus 0$ then there exists a map $G \simeq F$ such that $\mathrm{Fix}(G(-,t)) = |L(F(-,t))|$*

Proof. A map $F(x,t)$ is uniquely determined up to homotopy by the map $F_*\colon \pi_1(T \times S^1) \to \pi_1(T)$, thus without loss of generality we may assume that F is linear. Since S^1 is connected, $L(F(-,t)) = L(F(-,t'))$ for all t and t'. By [BBPT75], if $L(M) \neq 0$ then $\overline{M}$ has precisely $|L(M)|$ fixed points. By the same calculation, $(M+r)$ has precisely $|L(M)| \neq 0$ fixed points. This completes the proof since $\mathrm{Fix}(F(-,t)^n) = |L(F(-,t)| \neq 0$ □

This completes the first case, which was recovered from classical fixed point theory.

Theorem 5.2.5. *Let $F\colon T \times S^1 \to T \times S^1$ is as above, $n > 1$ be and integer so that*

i. $F(-,t)^n \simeq \mathrm{id}$ *(and so $L(F(-,t)^n) = 0$) and*

ii. $L(F(-,t)^m) \neq 0$ *for proper divisors of n.*

Then $F \simeq G$ with periodic points

$$|\mathrm{Fix}(G(-,t)^n)| = \left|\overline{\mathcal{P}}_n(F(-,t))\right|$$

We have already done the majority of the work required to prove Theorem 5.2.5. The proof simply requires realizing that all the results we need come from Chapter 3.

Proof. Without loss of generality assume that F is linear and $F(-,t) \simeq A$. For this proof, it is best to consider the non-bundle map F' given by

$$F = \begin{pmatrix} A & r \end{pmatrix}.$$

Since $L(A^n) = 0$ for and n is minimal A has eigenvalues that are primitive n^{th} roots of unity. Therefore $char(A)$ is a power of the n^{th} cyclotomic polynomial. Since A satisfies its characteristic polynomial we have

$$F^n(x,t) = [A^n x + A^{n-1}rt + \dots Art] = [A^n x] = [x].$$

Therefore the map $(A + rt)$ is n-cyclotomic for all t. By Proposition 2.0.3, there exists a map $\Theta_n(A+rt)$ which commutes with $A+rt$ and has fixed points $\mathcal{P}_n(A+rt)$. Defining $\widetilde{F}(x,t) = [(A+rt) \circ \Theta_n(A+rt)]$ completes the proof. $\square$

Theorem 5.2.6. *Let $F\colon T \times S^1 \to T \times S^1$ be an endomorphism of a torus which fibers over S^1 and n an integer so that:*

i. $L(F(-,t)^n) = 0$ and

ii. one of the following two conditions holds:

a) $F(-,t)^n \simeq \mathrm{id}$ and $L(F(-,t)^m) = 0$ for some $m|n$ and $m < n$ or

b) $F(-,t)^n \not\simeq \mathrm{id}$.

Then F is homotopic to a map G so that

$$|\operatorname{Fix}(G(-,t)^n)| = |\overline{\mathcal{P}}_n(F(-,t))|$$

Similarly to Theorem 5.2.5, we have already done the majority of the work required to prove Theorem 5.2.6. The proof simply requires realizing that all the results we need come from Chapter 4.

Proof. First consider the map $F'\colon T \times S^1 \to T$ resultant of a map $S^1 \to \mathrm{end}(T)$. Without loss of generality F' can be assumed to be linear, and by Lemma 4.2.2 F' is given by the upper triangular matrix M.

$$M = \begin{pmatrix} A_1 & C_{1,1} & C_{1,2} & \dots & C_{1,j} & r_1 \\ 0 & A_2 & C_{2,2} & \dots & C_{2,j} & r_2 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & A_j & C_{j,j} & r_j \\ 0 & 0 & 0 & 0 & B & r \end{pmatrix}$$

where A_i is l_i-cyclotomic and $char(B)$ is not divisible by any Φ_m for $m|n$. We now make the observation that addition is commutative and rewrite M as

$$M(t) = \begin{pmatrix} (A_1 + r_1(t)) & C_{1,1} & C_{1,2} & \cdots & C_{1,j} \\ 0 & (A_2 + r_2(t)) & C_{2,2} & \cdots & C_{2,j} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & (A_j + r_j(t)) & C_{j,j} \\ 0 & 0 & 0 & 0 & (B + r(t)) \end{pmatrix}$$

By Theorems 5.2.4 and 5.2.5 maps $A_i + r(t)$ is n_i-cyclotomic and $L((B + r(t))^n) \neq 0$. Therefore M is of the form in Theorem 1.1.3 and homotopic to a map $M'(t)$ so that $\mathrm{Fix}(M(-,t)^n) = \overline{\mathcal{P}}_n(\overline{M(-,t)})$. $\square$

This completes the section for periodically vanishing Lefschetz numbers but I would like to return to the case where $L(M) = 0$ and Example 5.2.2 to see how catastrophic this failure is. After some careful thought, once can see that we can remove the requirement that $L(M) \neq 0$ of Theorem 5.2.3 after adding some restricting the maps $S^1 \to \mathrm{end}(T)$.

Theorem 5.2.7. *Let $F\colon T \times S^1 \to T \times S^1$ be the linear map represented by*

$$F = \begin{pmatrix} I & r \\ 0 & 1 \end{pmatrix}.$$

i. If $R = \Pi_i r_i = 0$, then F is homotopic to a G so that $\mathrm{Fix}(G(-,x)) = \varnothing$ for all $x \in S^1$.

ii. If $R = \Pi_i r_i \neq 0$, then for all G homotopic to F there exists an $x \in S^1$ so that $\mathrm{Fix}(G(-,x)) \neq \varnothing$.

Proof. Let $F\colon T \times S^1 \to T \times S^1$ be the linear map represented by

$$F' = \begin{pmatrix} I & r \\ 0 & 1 \end{pmatrix}.$$

Case i: Let $r := (r_1, \ldots, r_d)$, Since $R = \Pi_i r_i = 0$ there exists a j so that $r_j = 0$. Isolate the j^{th} component of F.

$$F(x_1, \ldots x_d, t) = (x_1 + r_1 t, \ldots, x_{j-1} + r_{j-1} t, x_j, x_{j+1} + r_{j+1} t \ldots, x_d + r_d t, t)$$

Define $G \simeq F$ as

$$G(x_1, \ldots x_d, t) = (x_1 + r_1 t, \ldots, x_{j-1} + r_{j-1} t, x_j + \pi, x_{j+1} + r_{j+1} t \ldots, x_d + r_d t, t),$$

then $\mathrm{Fix}(G(x,t)^n) = \varnothing$.

Case ii: This case follows from Example 5.2.2. $\square$

This leads us to the following corollary. This is quite unsatisfactory, which is why it was left until the end of this section.

Corollary 5.2.8. *Let $F\colon T \times S^1 \to T \times S^1$ be the linear map represented by*

$$F = \begin{pmatrix} M & r \\ 0 & 1 \end{pmatrix}.$$

where $L(M) = 0$. For certain r, F is homotopic to a G so that $\mathrm{Fix}(G^n) = \varnothing$

Proof. By Proposition 4.2.3 and Lemma 4.2.2, F is conjugate to a map F' of the form:

$$F = \begin{pmatrix} I & \widetilde{C} & r' \\ 0 & \widetilde{B} & s \\ 0 & 0 & 1 \end{pmatrix},$$

where $L(\widetilde{B}) \neq 0$, $\mathrm{Fix}(\widetilde{B})$ is discrete, and $\widetilde{C}(\mathrm{Fix}(\widetilde{B}) = 0$. Let $r' = (r_1, \ldots, r_l)$. If $R = \prod_i r_i = 0$, then by Theorem 5.2.7 F' is homotopic to a map G with no periodic points. □

Chapter 6 Future Directions

The results presented in this document are a narrow presentation of a broader story. In this chapter, we take a moment to reflect on the directions that can be taken after the completion of this dissertation. Some very clearly related to this document. Others take a further dive into homotopy theory and category theory.

6.1 Families of Endomorphisms

First and foremost, we state the general version of Theorem 5.2.5 for higher dimensional spheres. This work is in progress and will hopefully be added to the published version of this document.

Conjecture 6.1.1. *Let $F\colon S^d \to \mathrm{end}(T)$ be continuous and n be an integer. If h is the linear map homotopic to $F(0)$, then F is homotopic to a map G with*

$$\mathrm{Fix}(G(-,t)^n) = \bigcup_{m|n, L(h^m)\neq 0} \mathrm{Fix}(h^m).$$

Further, if K is homotopic to F then

$$|\,\mathrm{Fix}(G(-,t)^n)| \leqslant |\,\mathrm{Fix}(K(-,t)^n)|.$$

Conjecture 6.1.1 is a jumping off point to ask a similar question for $F\colon X \to \mathrm{end}(T)$ any X. We would ask for an extension of Conjecture 6.1.1 for X a CW complex.

Conjecture 6.1.2. *Let X be a finite CW-complex, $F\colon X \to \mathrm{end}(T)$ be continuous, and n be an integer. If h is the linear map homotopic to $F(0)$, then F is homotopic to a map G with*

$$\mathrm{Fix}(G(-,t)^n) = \bigcup_{m|n, L(h^m)\neq 0} \mathrm{Fix}(h^m).$$

Further, if K is homotopic to F then

$$|\,\mathrm{Fix}(G(-,t)^n)| \leqslant |\,\mathrm{Fix}(K(-,t)^n)|.$$

Proving the special case of a relative map $F\colon (D, \partial D) \to (\mathrm{end}(T), f)$ would be an important first step. I expect the techniques in this document should generalize to these examples and to a cellular argument for all CW complexes.

6.2 Bordism and Klein and Williams:

As in Section 5.1, let $\mathrm{end}(T^{\times n})^{C_n}$ be the space of C_n equivariant self maps of $T^{\times n}$ and $\mathrm{end}^\flat(T^{\times n})^{C_n} \subset \mathrm{end}(T^{\times n})^{C_n}$ denote the subspace of maps that are C_n fixed point free. The Fuller construction defines a map

$$\Psi_n\colon\ \mathrm{end}(T) \to \mathrm{end}(T^{\times n})^{C_n}.$$

The fixed points of $\Psi_n(f)$ are precisely n periodic points of f. Klein and Williams [KW10] wish to consider the following pullback square.

$$\begin{tikzcd}
\mathrm{end}^{\flat_n}(T) \arrow[r] \arrow[d, "\Psi_n"'] & \mathrm{end}(T) \arrow[d, "\Psi_n"] \\
\mathrm{end}^{\flat}(T^{\times n})^{\mathbb{Z}_n} \arrow[r] & \mathrm{end}(T^{\times n})^{\mathbb{Z}_n}
\end{tikzcd}$$

The vertical maps are given by the Fuller construction and the horizontal maps are inclusions. The pullback square records a transition from the study of periodic points of f to equivariant fixed points of $\Psi_n(f)$. It is not clear how much information is retained through this transition. That is, can we fully understand the vanishing and realizability of periodic points of f by understanding vanishing and realizability of equivariant fixed points of $\Psi_n(f)$? This brings us to the following question.

Conjecture 6.2.1 ([KW10]). *The above pullback square is a homotopy pullback.*

Our previous theorems provide evidence that the above pullback square is a homotopy pullback. To show that this is a homotopy pullback we must prove Conjecture 6.1.1.

The next question is to ask whether the pullback of Klein and William is a homotopy pullback for other classes of spaces. It is our hope that we can show that the Klein and Williams pullback diagram is a homotopy pullback for Lie groups or compact manifolds.

6.3 Bicategorical Trace

It has been shown in work by my advisor, Dr. Kate Ponto [Pon10, Pon15, Pon11], and her collaborators [PS13, PS14a, PS12, PS18, PS14b, MP18] that fixed point and periodic point invariants and theorems naturally arise as traces in categories.

This machinery can then be transported into various categories. However, the implications of transporting this information to other categories is still widely unknown. It is my goal to understand these categorical trace invariants and their implications in various fields. This project can be explained to undergraduate students in familiar categories.

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