2020

Eigenvalue Statistics and Localization for Random Band Matrices with Fixed Width and Wegner Orbital Model

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Digital Object Identifier: https://doi.org/10.13023/etd.2020.204

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Eigenvalue Statistics and Localization for Random Band Matrices with Fixed Width and Wegner Orbital Model

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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2020

ABSTRACT OF DISSERTATION

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Abstract: We discuss two models from the study of disordered quantum systems. The first is the Random Band Matrix with a fixed band width and Gaussian or more general disorder. The second is the Wegner n-orbital model. We establish that the point process constructed from the eigenvalues of finite size matrices converge to a Poisson Point Process in the limit as the matrix size goes to infinity.

The proof is based on the method of Minami for the Anderson tight-binding model. As a first step, we expand upon the localization results by Schenker and Peled-Schenker-Shamis-Sodin to account for complex energies. We use the fractional moment method of Aizenman-Molchanov to derive these bounds. In addition, we establish convergence and smoothness of the density of states functions by modifying estimates of Dolai-Krishna-Mallick to allow for unbounded random variables. From there we follow the Daley and Vere-Jones criteria for establishing the convergence of the eigenvalue point process to the Poisson Point process.

The analysis is first presented for the band matrix with adjustments for the orbital model following after. Other properties of these models such as ergodicity and Lyapunov exponents are discussed.

KEYWORDS: Random Band Matrices, Eigenvalue Statistics, Wegner Orbital Model

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May 15, 2020

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Dedicated to my father, Marc Brodie, who was the first to show me the beauty of mathematics.

ACKNOWLEDGMENTS

I would first and foremost like to thank my advisor, Dr. Peter Hislop, who helped me to grow as a researcher and mathematician. I would also like to thank my numerous graduate school professors. In particular, I would like to acknowledge Dr. Francis Chung, Dr. Larry Harris, and Dr. Peter Perry who were instrumental to my mathematical development. I am also grateful for the friends I made during my time in the math department at UK. I would like to specifically thank George Lytle and Sam Herschenfeld for our conversations and mathematical discussions. Lastly, I would like to thank Darleen Perez-Lavin for supporting me and keeping me steady over the past several years.

TABLE OF CONTENTS

Acknowledgments iii

Chapter 1 Introduction and Preliminaries 1
1.1 Random Schrödinger Operators 1
1.1.1 Ergodicity and Spectrum 1
1.1.2 Anderson Localization 2
1.2 Random Matrices 3
1.3 Eigenvalue Statistics 4
1.3.1 Local Eigenvalue Statistics 5

Chapter 2 Models and Results 6
2.1 Random Band Matrices 6
2.2 Wegner Orbital Model 7
2.3 Results 7

Chapter 3 Density of States for Random Band Matrices 9
3.1 Ergodicity and Infinite Volume Density of States 9
3.2 Spectral Averaging 11
3.3 Wegner and Minami Estimates 12
3.3.1 Estimates for Matrices with Diagonal Disorder 12
3.3.2 Estimates for Random Band Matrices 15
3.4 Density of States Function 15

Chapter 4 Localization for Random Band Matrices 17
4.1 Fractional Moment Bound 17
4.2 *A priori* Bounds 18
4.3 Decoupling Lemmas 21
4.4 Proof of Localization at Extreme Energy 23
4.5 Localization Under Rank One Perturbation 25

Chapter 5 Eigenvalue Statistics for Random Band Matrices 27
5.1 Regularity Conditions 27
5.2 First Power to Fractional Power Estimate 28
5.3 Convergence of Density of States Functions 34
5.4 Smoothness of Density of States 36
5.5 Poisson Statistics 44
5.5.1 Reduction to Array of Independent Point Processes 45
5.5.2 Convergence of Array of Point Processes 48

Chapter 6 Analysis for Wegner Orbital Model 53
6.1 Model 53

6.2 Eigenvalue Counting Estimates 54
6.3 Density of States Measure 55
6.4 Localization and Convergence of Density of States 57
6.4.1 Fractional Moment Bounds 57
6.4.2 Convergence of Density of States 58
6.4.3 Smoothness of Density of States 62
6.5 Poisson Statistics 62

Chapter 7 Complete Localization for Orbital Model in 1D 69
7.1 Outline 69
7.2 Transfer Matrices 69
7.3 Lyapunov Exponents 71
7.4 Simplicity of the Lyapunov Spectrum 72
7.5 From Simplicity to Spectral Localization 77
7.5.1 Spectral Measure 77
7.5.2 The Hyperbolic Set and Pure Point Spectrum 80
7.5.3 Spectral Averaging 82

Appendices 85
Appendix A: Schur Formula 85
Application to Green's Functions of RBM 86
Appendix B: Technical Lemmas 87
Appendix C: Subharmonicity 89
Subharmonic Functions 89
Application to Green's Functions of RBM 90

Bibliography 92

Vita 94

Chapter 1 Introduction and Preliminaries

In this document, we examine two models that come up in the study of disordered systems: the Wegner Orbital Model and the Random Band Matrix with a fixed band width. Our main focus is to study the *local eigenvalue statistics* of these models. We will show that locally the eigenvalues of the random band matrix and the orbital model in the localization regime are distributed according to a Poisson Point Process.

Before studying these models we outline some of the important models and concepts from the study of disordered systems upon which much of the analysis is based.

1.1 Random Schrödinger Operators

The most commonly studied Random Schrödinger Operator comes from the Anderson Tight-Binding Model, which was introduced in [4] in 1958.

To define the Anderson Hamiltonian, we let $\{\omega_x\}_{x\in\mathbb{Z}^d}$ be a family of independent and identically distributed random variables ω_x with a compactly supported probability density function ρ.

Then, the Anderson tight-binding Hamiltonian is an operator h_ω on the lattice $\ell^2(\mathbb{Z}^d)$, which takes the form

$$h_\omega = h_0 + \lambda V_\omega \tag{1.1.1}$$

where $V_\omega = \sum_{x\in\mathbb{Z}^d} \omega_x P_x$. Here P_x is the projection onto site $\{x\}$, and h_0 is the discrete centered Laplacian. Explicitly, for $u \in \ell^2(\mathbb{Z}^d)$:

$$(h_\omega u)(x) = \sum_{y\colon \|y-x\|_1=1} u(y) + \lambda\omega_x u(x). \tag{1.1.2}$$

The constant $\lambda > 0$ is a coupling constant which tunes the strength of the disorder.

1.1.1 Ergodicity and Spectrum

Here h_ω is an operator-valued function on a measure space Ω. That is, for each configuration of random variables $\omega \in \Omega$, h_ω is a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$. Since the variables ω_j are independent, the measure space Ω is the infinite product of the measure space for each individual ω_j.

An important first result is that h_ω is *ergodic* under the action of translation in the probability space. That is, we have the following unitary equivalence:

If $(T_i\omega)_j = \omega_{j-i}$, and U_i is translation by i in $\ell^2(\mathbb{Z}^d)$, then

$$h_{T_i\omega} = U_i h_\omega U_i^*. \tag{1.1.3}$$

As a consequence of ergodicity, the spectrum $\sigma(h_\omega)$ is deterministic. That is, for almost every configuration of random variables ω, the spectrum a fixed set Σ. We can show that

$$\Sigma = [-2d, 2d] + \lambda \cdot \mathrm{supp}(\rho). \tag{1.1.4}$$

Note that $[-2d, 2d]$ is the spectrum of h_0 and $\text{supp}(\rho)$ the spectrum of V_ω. See [16] for an introduction to ergodic random operators and the implications for the spectrum.

We will also be able to make use of the Birkhoff Ergodic Theorem when discussing random operators:

Birkhoff Ergodic Theorem. *Let $T : X \to X$ be a measure preserving transformation on a measure space (X, Σ, μ). Then if $f \in L^1(d\mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int f \, d\mu. \tag{1.1.5}$$

Note that the translations T_i above are examples of measure preserving transformations on the measure space Ω.

1.1.2 Anderson Localization

The hallmark characteristic of Random Schrödinger operators is the phenomenon of *Anderson Localization*, which was first described physically by Anderson in the 1950's and has been studied in the mathematical literature extensively since the 1970's. Loosely speaking, the phenomenon of Anderson Localization is that in the presence of strong disorder ($\lambda >> 0$), solutions to the Schrödinger equation remain localized in some sense. This is usually characterized in one of the two following ways:

1. *Spectral Localization*: The operator h_ω exhibits spectral localization on a set I if
$$\sigma(h_\omega) \cap I \tag{1.1.6}$$
is the closure of a dense set of eigenvalues, and corresponding eigenfunctions decay exponentially.

2. *Dynamical Localization*: The operator h_ω exhibits dynamical localization on a set I if there exist constants C and γ such that
$$\sup_{t\in\mathbb{R}} \mathbb{E}\{\left|\left\langle \delta_x, e^{-ith_\omega} E_I(h_\omega)\delta_y \right\rangle\right|\} \leq Ce^{-\gamma|x-y|}. \tag{1.1.7}$$
Here $E_I(h_\omega)$ is the spectral projection of h_ω onto the set I.

We note that dynamical localization is the stronger condition and implies spectral localization.

For the Anderson Model in one dimension, it is known that h_ω exhibits localization at all energies and for any value of $\lambda > 0$. In dimension $d \geq 2$, localization has been proven at all energies for $\lambda >> 0$, or for any $\lambda > 0$ for intervals I near the edge of the spectrum Σ.

Based on physical and numerical results, we expect there to be absolutely continuous spectrum near the center of the spectrum for small λ in dimension $d \geq 3$, but this remains an open problem in the mathematical literature.

There are two main paths to proving Anderson Localization in arbitrary dimension: the Multiscale Analysis [12] and the Fractional Moment Method [1]. In this document, we focus on the latter method, which shows that a necessary condition for dynamical localization of a Random Schrödinger operator is exponential decay of the s power of the matrix elements of the resolvent, for some $0 < s < 1$. Explicitly, bounds of the form

$$\mathbb{E}\left|\left\langle \delta_x, (h_\omega - z)^{-1}\delta_y \right\rangle\right|^s \leq Ce^{-\gamma|x-y|} \tag{1.1.8}$$

suffice to prove dynamical localization in the form (1.1.7).

For a good introduction to Anderson Localization using the Fractional Moment Method, see [26], and for an introduction to Anderson Localization using the Multiscale Analysis, see [16].

1.2 Random Matrices

Another model in the study of disordered systems is the random matrix. The study of random matrices took off in the 1950's with Eugene Wigner, who proposed that statistics of energy level spacings of heavy nuclei could be modelled by studying the spacing between eigenvalues of a matrix with entries distributed according to a given probability distribution and independent up to a symmetry condition.

The set of random matrices with the most interest to us here is Gaussian Orthogonal Ensemble. An element G_L of the Gaussian Orthogonal Ensemble (GOE) of dimension L, takes the form

$$G_L = \frac{1}{2}(X_L + X_L^T) \tag{1.2.1}$$

where

$$X_L = \frac{1}{\sqrt{L}} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1L} \\ g_{21} & g_{22} & & \\ \vdots & & \ddots & \\ g_{L1} & & & g_{LL} \end{pmatrix} \tag{1.2.2}$$

and g_{ij} are independent Gaussian random variables with mean 0 and variance 1. The normalization factor $\frac{1}{\sqrt{L}}$ implies that $\mathbb{E}\|G_L\| = 1$ independent of L.

The probability density function of the GOE is given by

$$\rho(G_L) = \frac{1}{Z_L} e^{-\frac{L}{4}\operatorname{tr}(G_L)^2} \tag{1.2.3}$$

for a suitable constant Z_L. The Gaussian Orthogonal Ensemble is notable because the probability density is invariant under conjugation by orthogonal matrices.

For $\{\lambda_i\}_{i=1}^L$, the eigenvalues of a GOE matrix G_L, the joint probability density can be written explicitly as

$$\frac{1}{Z_L}\prod_{k=1}^{L} e^{-\frac{L}{4}\lambda_k^2} \prod_{i<j} |\lambda_i - \lambda_j|. \tag{1.2.4}$$

The $|\lambda_i - \lambda_j|$ term implies that the eigenvalues tend to repel each other, i.e. the probability that two eigenvalues are close together is small.

For a thorough review of the Gaussian Orthogonal Ensemble and other related random matrix ensembles, see [18].

1.3 Eigenvalue Statistics

The central theme of this document is to study the *eigenvalue statistics* of the random operators. To introduce the eigenvalue statistics of a random operator, we present the concepts in terms of the Anderson Hamiltonian

$$h_\omega = h_0 + \lambda \cdot V_\omega. \tag{1.3.1}$$

We first consider the sequence of boxes $\Lambda_N = \{-N, \ldots, N\}^d \subset \mathbb{Z}^d$ and consider the restriction of h_ω to the box:

$$h_\omega^{\Lambda_N} = P_{\Lambda_N} h_\omega P_{\Lambda_N}. \tag{1.3.2}$$

The operator $h_\omega^{\Lambda_N}$ is a rank $|\Lambda_N|$ matrix, and so its spectrum is a discrete set of $|\Lambda_N|$ eigenvalues which are each located inside Σ. The eigenvalue statistics describe how the eigenvalues "fill in" Σ as $\Lambda_N \nearrow \mathbb{Z}^d$.

A global description of the eigenvalue statistics is given by the *density of states*. The density of states *measure* for each N, is defined by

$$\nu_\omega^N(A) = \frac{1}{|\Lambda_N|}\mathbb{E}\{\#\text{ of eigenvalues of } h_\omega \text{ in } A\} = \frac{1}{|\Lambda_N|}\mathbb{E}\{\operatorname{tr} E_A(h_\omega)\}. \tag{1.3.3}$$

Another description of the density of states is through the distribution function for the measure, which we call the *integrated density of states*

$$N_\omega^N(E) = \nu_\omega^N\big((-\infty, E]\big) = \frac{1}{|\Lambda_N|}\mathbb{E}\{\#\text{ of eigenvalues of } h_\omega \leq E\}. \tag{1.3.4}$$

The density of states measures converge in the limit as $N \to \infty$, and from the Birkhoff Ergodic Theorem, we have the expression

$$\begin{aligned}
\nu_\omega^\infty(A) &= \lim_{N\to\infty} \nu_\omega^N(A) \\
&= \frac{1}{|\Lambda_N|}\mathbb{E}\{\operatorname{tr} E_A(h_\omega)\} \\
&= \frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N} \mathbb{E}\langle \delta_x, E_A(h_\omega)\delta_x\rangle \\
&= \mathbb{E}\{\langle \delta_0, E_A(h_\omega)\delta_0\rangle\}.
\end{aligned} \tag{1.3.5}$$

This is called the infinite volume density of states measure. The distribution for ν_ω^∞,

$$N_\omega^\infty(E) = \nu_\omega^N(-\infty, E] \tag{1.3.6}$$

is the infinite volume integrated density of states.

The derivative of N_ω^∞ is called the density of states function for h_ω:

$$n_\omega^\infty(E) = \frac{dN_\omega^\infty}{dE}(E). \tag{1.3.7}$$

Its existence follows from a Wegner Estimate:

$$\mathbb{E}\left\{\nu^N(A)\right\} \leq C_\rho |A|, \tag{1.3.8}$$

which we will discuss more thoroughly in Section 3.3. For more details on the density of states for the Anderson model see [16] and [9].

1.3.1 Local Eigenvalue Statistics

We are also interested in the *local* eigenvalue statistics. To study the local eigenvalue statistics, we pick an $E \in \Sigma$, and look at how the eigenvalues of $h_\omega^{\Lambda_N}$ are distributed in the interval $\left(E - \frac{1}{L|\Lambda_N|}, E + \frac{1}{L|\Lambda_N|}\right)$. Explicitly this involves defining a point measure defined on the re-scaled eigenvalues:

$$\xi_{\omega,E}^N(s)\, ds = \sum_{j=1}^{|\Lambda_N|} \delta_{|\Lambda_N|(\lambda_j^N - E)}(s)\, ds \tag{1.3.9}$$

where $\{\lambda_j\}$ are the eigenvalues of $h_\omega^{\Lambda_N}$.

Minami [19] showed that for E in the localized spectrum of h_ω, $\xi_{\omega,E}^N$ converges in the appropriate sense to the Poisson Point Process with intensity measure $n(E)\, ds$. Heuristically, we can think of this as saying the eigenvalues of $h_\omega^{\Lambda_N}$ locally behave like independent random variables.

This is in contrast to the eigenvalue statistics of the GOE, where the probability the eigenvalues are close together is small, despite the eigenvalues having the same average spacing.

We will see that the local eigenvalue statistics of the random band matrix and the Wegner Orbital Model align with those of the Anderson tight-binding model.

Chapter 2 Models and Results

In this chapter, we define the Random Band Matrix and Wegner Orbital Model. We then outline the major results that will be presented in following chapters.

2.1 Random Band Matrices

Let H_L^N be a $2N+1 \times 2N+1$ symmetric random band matrix with bandwidth L. We can define H_L^N through its matrix elements:

$$\langle e_i, H_L^N e_j \rangle = \frac{1}{\sqrt{L}} \begin{cases} v_{ij} & if \quad |i-j| \leq L \\ 0 & if \quad |i-j| > L \end{cases} \tag{2.1.1}$$

$$-N \leq i, j \leq N, \tag{2.1.2}$$

where the random variables v_{ij} within the band are independent and identically distributed up to symmetry ($v_{ij} = v_{ji}$). The indexing is chosen so that when the matrix is represented with respect to the standard basis, the top left entry corresponds to the index $(-N, -N)$, and the bottom right entry corresponds to the index (N, N). In this way the $(0,0)$ entry is always in the center of the matrix regardless of the choice of N.

We often take each v_{ij} to be Gaussian with mean 0 and variance 1. That is, the common probability density ρ is defined by

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{2.1.3}$$

In the case that L is held fixed as $N \to \infty$, H_L^N has a natural limiting operator H_L^∞ on $\ell^2(\mathbb{Z}^d)$, defined in terms of its matrix elements as above by

$$\langle e_i, H_L^\infty e_j \rangle = \frac{1}{\sqrt{L}} \begin{cases} v_{ij} & if \quad |i-j| \leq L \\ 0 & if \quad |i-j| > L \end{cases} \tag{2.1.4}$$

for i, j in $\mathbb{Z}$.

In this way, letting $\chi_{[-N,N]}$ be the orthogonal projection onto interval $[-N, N]$, H_L^N is equivalent

$$\chi_{[-N,N]} \, H_L^\infty \, \chi_{[-N,N]} \tag{2.1.5}$$

on its range.

In the following, we will only perform analysis for fixed bandwidth random matrices, but in the literature there is considerable interest in studying the band matrix H_L^N when L grows as $N \to \infty$. For example we can fix $0 < \alpha < 1$, and take $L \sim N^\alpha$. In this case, there is no limiting operator on $\ell^2(\mathbb{Z})$. In [5] and [20], the authors proved that the integrated density of states

$$N_L^N(E) = \frac{1}{2N+1} \mathbb{E} \, \# \left\{ \text{eigenvalues of } H_L^N \leq E \right\} \tag{2.1.6}$$

converges to the semi-circle law: for each $E \in \mathbb{R}$,

$$\lim_{N\to\infty} N_L^N(E) = \frac{1}{4\pi}\int_{-\infty}^{E} \sqrt{(4-x^2)_+}\, dx. \tag{2.1.7}$$

In order to better understand the *local* eigenvalue statistics for H_L^N, we would need to establish pointwise convergence of the density of states *function*,

$$n_L^n(E) = \frac{d}{dE} N_L^N(E) \tag{2.1.8}$$

to the semi-circle function, which remains an open problem.

In the fixed band case, we will show pointwise convergence of n_L^N to a limiting function n_L^∞ as $N \to \infty$. This result will play a central role in the analysis of the local eigenvalue statistics.

As shown in [5] the integrated density of states for the fixed width random band matrix is not given by the semi-circle law, but is modified by an $O(\frac{1}{L})$ correction.

2.2 Wegner Orbital Model

The Wegner Orbital Model was introduced in [21, 27] to describe a quantum particle on a lattice with L degrees of freedom per site, which are interpreted physically as electron orbitals. The Hamiltonian for this model is a Random Schrödinger Operator with random matrix disorder on the Hilbert Space $\ell^2(\mathbb{Z}^d : \mathbb{C}^L)$:

$$H_L = H_0 + V^{GOE(L)} = \Delta \otimes I + \sum_{x\in\mathbb{Z}^d} V(j)\, P_x, \tag{2.2.1}$$

where each $V(j)$ is independently sampled from the $L \times L$ Gaussian Orthogonal Ensemble (1.2.2), and P_x is the (rank L) projection onto the site $x \in \mathbb{Z}^d$. Explicitly, for $u \in \ell^2(\mathbb{Z}^d : \mathbb{C}^L)$,

$$[H_L u]\,(x) = \sum_{y:\, \|x-y\|_1=1} u(y) + V(x)u(x). \tag{2.2.2}$$

The major difference in the analysis between the Wegner Orbital Model and the Anderson Model is that the randomness at each point in the lattice is rank L. Thus, many of the techniques of rank one perturbation theory (see for example [25]), which are frequently used in the study of Random Schrödinger Operators, do not carry over directly.

2.3 Results

The main result of this document is the convergence of the local eigenvalue point process for the band matrix with fixed band width and the Wegner orbital model in the localization regime to a Poisson point process. We will make this statement more precise in Chapters 5 for the Random Band Matrix and Chapter 6 for the Orbital Model.

In order to establish this result, we must have localization estimates in the form of fractional moment bounds (1.1.8), eigenvalue counting estimates for the finite volume operators, and control over the density of states functions.

The necessary fractional moment bounds take the form of

$$\mathbb{E}\left|\langle e_x, (H-z)^{-1} e_y\rangle\right|^s \leq C e^{-\gamma|x-y|} \tag{2.3.1}$$

for suitable constants C and γ. The bounds for the random band matrix are obtained in Chapter 4, and build off the work of [23] and [3]. Fractional moment bounds for the Orbital Model are given in [22] and a modification is made in Chapter 6.

Two recent papers [2, 22] established key eigenvalue counting estimates for the orbital model:

$$\mathbb{E}\prod_{\ell=0}^{m-1}\left(\operatorname{tr} E_A(H_L^\Lambda) - \ell\right) \leq (CL|\Lambda||A|)^m, \tag{2.3.2}$$

where H_L^Λ is a the operator restricted to a finite box Λ, and C is independent of L and N.

A modification can be made to obtain a similar estimate for the band matrix with Gaussian disorder.

The two most important cases of this estimate are the $m = 1$ (Wegner-type Estimate) and $m = 2$ (Minami-type Estimate) cases, which are named after the authors who first proved similar estimates for the Anderson Model.

We will present a different proof of these estimates, based off of the methods in [9] and extended to include non-compactly supported random variables. This method gives less optimal dependence on L, but still suffices to establish convergence of the local eigenvalue point process of our models to a Poisson point process.

The control over the density of states functions results extend and improve on methods in [11] for Random Schrödinger Operators with rank one disorder and compactly supported random variables.

The analysis will be carried out in detail for the Random Band Matrix in Chapter 5, with the necessary adjustments being made for the Wegner Orbital Model being made in Chapter 6.

In this chapter and the following two chapters we study the fixed width random band matrix H_L^N defined in (2.1.1) with L held constant as $N \to \infty$. In Chapter 3, we outline the existence results for the density of states and derive Wegner and Minami eigenvalue counting estimates. To prove these estimates, we use the method of [9], although we mention an improvement for the case of Gaussian random variables made in [22]. The results in this chapter are standard for matrices with diagonal disorder.

3.1 Ergodicity and Infinite Volume Density of States

Let $\{\lambda_j(N)\}$ be the set of eigenvalues of H_L^N. For each N, we define the density of states measure for H_L^N by

$$\nu_L^N(A) = \frac{1}{2N+1}\mathbb{E}\#\{\,\lambda_j(N) \in A\} = \frac{1}{2N+1}\mathbb{E}\left\{\,\mathrm{tr}\, E_A(H_L^N)\right\}, \tag{3.1.1}$$

the expected proportion of the eigenvalues of H_L^N in the set A.

We could then define the infinite volume density of states measure by taking a limit of $\nu_L^N(A)$ as $N \to \infty$.

Another approach to defining the infinite volume density of states is to begin with the infinite volume operator, H_L^∞ and then restrict to finite volume. To accomplish this, let $E_A(H_L^\infty)$ be the spectral projection of H_L^∞ onto the set A.

Then

$$\nu_L^\infty(A) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{j=-N}^{N} \langle e_j, E_A(H_L^\infty) e_j\rangle. \tag{3.1.2}$$

To understand what happens in the limit, we note that H_L^∞ is ergodic under translation in $\mathbb{Z}$. Explicitly, if we take $\Omega \subset \mathbb{R}^{\mathbb{Z}^2}$ to be the probability space of random variables in H_L^∞, ω is a configuration of random variables in Ω, and T_k translation by $(k,k) \in \mathbb{R}^{\mathbb{Z}^2}$,

$$\langle e_i, H_L^\infty(T_k\omega)e_j\rangle = \langle e_{i-k}, H_L^\infty(\omega)e_{j-k}\rangle. \tag{3.1.3}$$

Thus, by the Birkhoff Ergodic Theorem,

$$\nu_L^\infty(A) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{j=-N}^{N} \langle e_j, E_A(H_L^\infty) e_j\rangle = \mathbb{E}\langle e_0, E_A(H_L^\infty)e_0\rangle. \tag{3.1.4}$$

Note that ν_L^∞ is deterministic–it does not depend on the configuration of random variables. For a more thorough exposition on ergodic operators and consequences for the density of states, see [16].

We now show that the finite volume density of states measures ν_L^N converge in the appropriate sense to the measure ν_L^∞ as defined in (3.1.4) almost surely.

Theorem 3.1.1. *Suppose the entries of H_L^N have finite first moment. Let ν_L^N, be the density of states measures for H_L^N. Then ν_L^N converges vaguely almost surely to the measure ν_L^∞ as defined in 3.1.4.*

In particular, for each bounded interval A,

$$\lim_{N\to\infty} \nu_L^N(A) = \nu_L^\infty(A). \tag{3.1.5}$$

Here vague convergence means that

$$\nu_L^N(\varphi) \to \nu_L^\infty(\varphi) \tag{3.1.6}$$

for all $\varphi \in \mathcal{C}_0$, continuous functions that vanish at ∞. We note that a basis for $\mathcal{C}_0$ is the set of functions of the form $\varphi_z = \frac{1}{x-z}$ with $z \in \mathbb{C}_+$, and so it suffices to check the property for functions of this form [16].

Proof. Let $\varphi_z(x) = \frac{1}{x-z}$ with $z \in \mathbb{C}_+$. Then

$$\begin{aligned}
&\lim_{N\to\infty} |\nu_L^N(\varphi_z) - \nu_L^\infty(\varphi_z)| \\
&= \lim_{N\to\infty} \frac{1}{2N+1}\mathbb{E}\sum_{j=-N}^{N} \left[\langle e_j, (H_L^N - z)^{-1} e_j\rangle - \langle e_j, (H_L^N - z)^{-1} e_j\rangle\right] \qquad (3.1.7)\\
&= \lim_{N\to\infty} \frac{1}{2N+1}\mathbb{E}\sum_{j=-N}^{N} \langle e_j, (H_L^N - z)^{-1}(H_L^\infty - H_L^N)(H_L^\infty - z)^{-1} e_j\rangle.
\end{aligned}$$

Because of the cancellation in $H_L^\infty - H_L^N$, we are left with a sum over a "boundary" index set

$$\mathcal{I} = \{(k,\ell) : |k| \le N,\ |\ell| > N,\ \text{and}\ |k-\ell| \le L\}. \tag{3.1.8}$$

Thus,

$$\begin{aligned}
\left|\nu_L^N(\varphi_z) - \nu_L^\infty(\varphi_z)\right| &= \left|\mathbb{E}\frac{1}{2N+1}\sum_{j=-N}^{N}\sum_{(k,\ell)\in\mathcal{I}} v_{k\ell}\langle e_j, (H_L^N - z)^{-1} e_k\rangle\langle e_\ell (H_L^\infty - z)^{-1} e_j\rangle\right| \\
&\le \mathbb{E}\frac{1}{2N+1}\sum_{(k,\ell)\in\mathcal{I}} |v_{k\ell}| \|(H_L^N - z)^{-1} e_k\|\, \|(H_L^\infty - z)^{-1} e_\ell\| \\
&\le \mathbb{E}\frac{C}{2N+1} 4L \frac{1}{(\operatorname{Im} z)^2} \\
&\longrightarrow 0. \qquad (3.1.9)
\end{aligned}$$

Thus, the two methods of defining the infinite volume density of states are equivalent. $\square$

3.2 Spectral Averaging

We will need the following *a priori* bound on the expectation of the imaginary part of Green's functions of matrices with i.i.d. random variables along the diagonal. This is called a *spectral averaging* estimate. The result is standard in the study of random operators, and relies only on having diagonal disorder.

Lemma 3.2.1 (Spectral Averaging). *Let H_ω be a self-adjoint matrix such that its diagonal entries $[H_\omega]_{jj} := \omega_j$ are absolutely continuous and distributed according to a common probability density function ρ with $\|\rho\|_\infty < \infty$.*

Then, for each standard basis vector e_j, and any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathbb{E}\,\mathrm{Im}\left\langle e_j, (H_\omega - z)^{-1} e_j\right\rangle \leq \pi\|\rho\|_\infty. \tag{3.2.1}$$

Proof. Let $z = E + i\varepsilon$. Using the Schur Complement Formula (Lemma A.1 in Appendix A) with $P = P_j$, as in Proposition A.1, we have:

$$\mathbb{E}\left\{\mathrm{Im}\left\langle e_j, (H_\omega - E - i\varepsilon)^{-1} e_j\right\rangle\right\} = \mathbb{E}\left\{\mathrm{Im}\,\frac{1}{\omega_j - E - i\varepsilon + a}\right\} \tag{3.2.2}$$

where a is independent of ω_j.

Let $\tilde{E} = E - \mathrm{Re}\,a$ and $\tilde{\varepsilon} = \varepsilon - \mathrm{Im}\,a$. Then,

$$\mathbb{E}\left\{\mathrm{Im}\,\frac{1}{\omega_j - E - i\varepsilon + a}\right\} = \mathbb{E}\left\{\frac{\tilde{\varepsilon}}{(\omega_j - \tilde{E})^2 + \tilde{\varepsilon}^2}\right\} = \mathbb{E}\left\{\frac{1}{\tilde{\varepsilon}} \cdot \frac{1}{1 + \left(\frac{\omega_j - \tilde{E}}{\tilde{\varepsilon}}\right)^2}\right\}. \tag{3.2.3}$$

Integrating in ω_j:

$$\begin{aligned}
\mathbb{E}_{\omega_j}\left\{\frac{1}{\tilde{\varepsilon}} \cdot \frac{1}{1 + \left(\frac{\omega_j - \tilde{E}}{\tilde{\varepsilon}}\right)^2}\right\} &= \int_{\mathbb{R}} \rho(\omega_j)\frac{1}{\tilde{\varepsilon}} \cdot \frac{1}{1 + \left(\frac{\omega_j - \tilde{E}}{\tilde{\varepsilon}}\right)^2}\, d\omega_j \\
&= \int_{\mathbb{R}} \rho\left(\tilde{E} + \tilde{\varepsilon}u\right) \frac{1}{1 + u^2}\, du \\
&\leq \|\rho\|_\infty \int_{\mathbb{R}} \frac{1}{1 + u^2}\, du \\
&= \pi\|\rho\|_\infty.
\end{aligned} \tag{3.2.4}$$

Note that the bound is independent of ε. □

Lemma 3.2.2 (Spectral Averaging Part 2). *Let A be a finite interval in $\mathbb{R}$. Then for each $j = -N, \ldots, N$,*

$$\mathbb{E}\langle e_j, E_A(H_\omega) e_j\rangle \leq \pi\|\rho\|_\infty |A|, \tag{3.2.5}$$

where $E_A(H_\omega)$ is the spectral projection of H_ω onto the set A.

Proof. This result follows from Stone's Formula:

$$\langle e_j, E_A(H_\omega)e_j\rangle = \lim_{\varepsilon\searrow 0} \operatorname{Im} \frac{1}{\pi}\int_A \langle e_j, (H_\omega - E - i\varepsilon)^{-1} e_j\rangle\, dE. \tag{3.2.6}$$

The expectation can be brought inside the ε limit by the Dominated Convergence Theorem.

□

In the case of the random band matrix, H_L^N, the diagonal random variables are normalized by the factor $\frac{1}{\sqrt{L}}$. Thus the spectral averaging result for the band matrix becomes:

Lemma 3.2.3. *Let H_L^N be a random band matrix with non-zero entries v_{ij} having probability density function ρ.*

Then, for each standard basis vector e_j, and any $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\mathbb{E}\operatorname{Im}\left\langle e_j, \left(H_L^N - z\right)^{-1} e_j\right\rangle \leq \pi\sqrt{L}\|\rho\|_\infty. \tag{3.2.7}$$

3.3 Wegner and Minami Estimates

The two most important eigenvalue counting estimates for establishing Poisson local eigenvalue statistics of a random operator H are the Wegner Estimate and the Minami Estimate. The Wegner Estimate bounds the probability of finding at least one eigenvalue of H in a given interval by a constant times the length of the interval. The Minami Estimate bounds the probability of finding at least two eigenvalues of H by the length of the interval squared.

3.3.1 Estimates for Matrices with Diagonal Disorder

We first state the Wegner and Minami Estimates for matrices with diagonal disorder. The proofs for the Wegner and Minami estimates with compactly supported probability density are borrowed from [9].

Theorem 3.3.1 (Wegner Estimate)**.** *Let H_ω be an $M\times M$ matrix with independent random variables ω_j with common probability density ρ along the diagonal. Suppose $\|\rho\|_\infty < \infty$. Then*

$$\mathbb{P}\{\operatorname{tr} E_A(H_\omega) \geq 1\} \leq \mathbb{E}\{\operatorname{tr} E_A(H_\omega)\} \leq \pi\|\rho\|_\infty|A| \tag{3.3.1}$$

where $E_A(\cdot)$ is spectral projection onto the set A.

Proof. The first inequality follows from Chebyshev's inequality. For the second inequality, we expand the trace in a basis and apply the spectral averaging result:

$$\mathbb{E}\{\operatorname{tr} E_A(H_\omega)\} = \mathbb{E}\left\{\sum_{i=1}^M \langle\delta_i, E_A(H_\omega)\delta_i\rangle\right\} \leq \pi M\|\rho\|_\infty|A|. \tag{3.3.2}$$

□

We now present a proof of the Minami Estimate from [9]. The proof works for random variables with a probability density function ρ with compact support and $\|\rho\|_\infty < \infty$. We will then present in a Corollary an extension which will allow give a Minami estimate for random variables with unbounded support, for example Gaussian random variables.

Theorem 3.3.2 (Minami Estimate)**.** *Let H_ω be an $M \times M$ matrix with independent random variables ω_j with common probability density ρ along the diagonal. Suppose the common probability distribution ρ is bounded with compact support. Then*

$$\mathbb{P}\{\operatorname{tr} E_A(H_\omega) \geq 2\} \leq \mathbb{E}\{\operatorname{tr} E_A(H_\omega)(\operatorname{tr} E_A(H_\omega) - 1)\} \leq (\pi M \|\rho\|_\infty |A|)^2 \tag{3.3.3}$$

.

Proof. For the first inequality, note that since $X = \operatorname{tr} E_A(H_\omega)$ is a random variable taking values in $\mathbb{N}$,

$$\begin{aligned}
\mathbb{E}\{X(X-1)\} &= \sum_{k=1}^{\infty} X(X-1)\mathbb{P}\{X=k\} \\
&= \sum_{k=2}^{\infty} X(X-1)\mathbb{P}\{X=k\} \\
&\geq \mathbb{P}\{X=2\}.
\end{aligned} \tag{3.3.4}$$

For the second inequality, suppose the support of ρ is contained in $[-R, R]$. We expand the first trace in a basis:

$$X(X-1) = \sum_{j=1}^{M} \langle \delta_j, E_A(H_\omega)\delta_j\rangle(\operatorname{tr} E_A(H_\omega) - 1) \tag{3.3.5}$$

We then estimate the terms

$$\mathbb{E}\{\langle \delta_j, E_A(H_\omega)\delta_j\rangle(\operatorname{tr} E_A(H_\omega) - 1)\} \tag{3.3.6}$$

one at a time. First we replace ω_j in the $(\operatorname{tr} E_A(H_\omega) - 1)$ term with a random variable $\tau_j = 2R + \tilde{\omega}_j$ where $\{\tilde{\omega}_j\}$ are independent and identically distributed with the same density ρ. We now average over the τ_j as well. Since this is a rank one perturbation, it shifts the trace by at most one [9, Lemma 4.1]. Thus,

$$\begin{aligned}
&\mathbb{E}_{\omega_j}\{\langle \delta_j, E_A(H_\omega)\delta_j\rangle(\operatorname{tr} E_A(H_\omega) - 1)\} \\
&\leq \mathbb{E}_{\omega_j,\tau_j}\{\langle \delta_j, E_A\left(H_\omega(\omega_j, \omega_j^\perp)\right)\delta_j\rangle \operatorname{tr} E_A(H_\omega(\tau_j, \omega_j^\perp)).
\end{aligned} \tag{3.3.7}$$

Now that the second term is independent of ω_j, we can apply the spectral averaging estimate by evaluating $\mathbb{E}_{\omega_j}$:

$$\begin{aligned}
&\mathbb{E}_{\omega,\tau}\{\langle \delta_j, E_A(H_\omega(\omega_j, \omega_j^\perp)\delta_j\rangle \operatorname{tr} E_A(H_\omega(\tau_j, \omega_j^\perp)) \\
&\leq \pi\|\rho\|_\infty |A| \mathbb{E}_{\omega_j^\perp,\tau_j}\{\operatorname{tr} E_A(H_\omega(\tau_j, \omega_j^\perp))\} \\
&\leq \pi\|\rho\|_\infty |A| \cdot M\pi\|\rho\|_\infty |A|
\end{aligned} \tag{3.3.8}$$

by the Wegner Estimate applied to $H_L(\tau_j, \omega_j^{\perp})$.

Finally, summing over j results in the Minami Estimate:

$$\mathbb{E}\{\operatorname{tr} E_A(H_\omega)(\operatorname{tr} E_A(H_\omega) - 1)\} \leq (\pi\|\rho\|_\infty |A| M)^2. \tag{3.3.9}$$

$\square$

We now extend the Minami Estimate to include unbounded random variables. We will accomplish this by approximating the random variables with compactly supported random variables, and noting that the Theorem 3.3.2 is independent of the size of the support of the density function ρ.

Corollary 3.3.1. *Let H_ω be an $M \times M$ matrix with independent random variables ω_j with common probability density ρ along the diagonal. Suppose ρ is bounded. Then*

$$\mathbb{E}\{\operatorname{tr} E_A(H_\omega)(\operatorname{tr} E_A(H_\omega) - 1)\} \leq (\pi\|\rho\|_\infty |A| M)^2. \tag{3.3.10}$$

Proof. We first begin by approximating the random variables ω with random variables ω_j^R taking values in $[-R, R]$. The density function of ω^R is defined to be

$$\rho^R(x) = \frac{1}{\int_{-R}^{R} \rho(y)\mathrm{d}y} \rho(x)\, \mathbb{1}_{[-R,R]}(x). \tag{3.3.11}$$

Note that the Minami estimate holds for H_{ω^R}, the matrix with diagonal disorder sampled independently with density ρ^R with

$$\|\rho^R\|_\infty = \frac{1}{\int_{-R}^{R} \rho(y)\mathrm{d}y} \|\rho \mathbb{1}_{[-R,R]}\|_\infty. \tag{3.3.12}$$

Here $\lim_{R\to\infty} \int_{-R}^{R} \rho(y)\mathrm{d}y = 1$, and so $\lim_{R\to\infty} \|\rho^R\|_\infty = \|\rho\|_\infty$.

Finally, suppose that $f : \mathbb{R}^M \to \mathbb{R}$ is in L^∞. To simplify notation, let

$$\tilde{\rho}(\omega) = \prod_{i=1}^{M} \rho(\omega_i). \tag{3.3.13}$$

Then

$$\begin{aligned}
&\lim_{R\to\infty} \left[\mathbb{E}_\omega\{f\} - \mathbb{E}_{\omega^R}\{f\}\right] \\
&= \lim_{R\to\infty} \int_{\mathbb{R}} f(\omega)\tilde{\rho}(\omega)\mathrm{d}\omega - \left(\frac{1}{\int_{-R}^{R} \rho(y)\mathrm{d}y}\right)^M \int_{\mathbb{R}} f(\omega)\tilde{\rho}(\omega)\mathbb{1}_{[-R,R]^M}(\omega)\mathrm{d}\omega \qquad (3.3.14) \\
&= \lim_{R\to\infty} \int_{\mathbb{R}} f(\omega)\tilde{\rho}(\omega)\left(1 - \left(\frac{1}{\int_{-R}^{R} \rho(y)\mathrm{d}y}\right)^M \mathbb{1}_{[-R,R]^M}(\omega)\right) d\omega.
\end{aligned}$$

Bringing the limit inside by the Dominated Convergence Theorem, we see that the limit goes to 0.

Taking $f(\omega) = \operatorname{tr} E_A(H_\omega)(\operatorname{tr} E_A(H_\omega) - 1) \in L^\infty(\mathbb{R}^M)$, and noting that (3.3.3) is independent of the size of the support R, we have the Minami estimate for any probability density ρ satisfying $\|\rho\|_\infty < \infty$. $\square$

3.3.2 Estimates for Random Band Matrices

As with the spectral averaging estimate 3.2.3, the normalization by $\frac{1}{\sqrt{L}}$ in the definition of the random band matrix adjusts the Wegner and Minami estimates.

For the band matrix, we have

Theorem 3.3.3 (RBM Wegner Estimate). *Let H_L^N be a random band matrix with entries having probability density ρ. Then*

$$\mathbb{P}\{\operatorname{tr} E_A(H_L^N) \geq 1\} \leq \mathbb{E}\{\operatorname{tr} E_A(H_L^N)\} \leq \pi\|\rho\|_\infty(2N+1)\sqrt{L}|A|. \tag{3.3.15}$$

and

Theorem 3.3.4 (RBM Minami Estimate). *Let H_L^N be a random band matrix with entries having probability density ρ. Then*

$$\mathbb{P}\{\operatorname{tr} E_A(H_L^N) \geq 2\} \leq \mathbb{E}\{\operatorname{tr} E_A(H_L^N)(\operatorname{tr} E_A(H_L^N) - 1)\} \leq (\pi(2N+1)\sqrt{L}|A|)^2 \tag{3.3.16}$$

.

By the methods of [2, 22], the factor of $\sqrt{L}$ can be removed in the case of Gaussian disorder, although the proof becomes significantly more complicated and the improvement is not essential for the remaining results.

3.4 Density of States Function

One important consequence of the Wegner Estimate is that it gives control over the density of states measure. Explicitly, for any interval A,

$$\nu_L^N(A) \leq \pi\|\rho\|_\infty|A|. \tag{3.4.1}$$

That is, ν_L^N is absolutely continuous with respect to Lebesgue measure. Furthermore, since the bound is uniform in N, the estimate carries through to the infinite volume density of states measure:

$$\nu_L^\infty(A) \leq \pi\|\rho\|_\infty|A|. \tag{3.4.2}$$

By the absolute continuity of these measures, their Radon-Nikodym derivatives exist almost everywhere:

$$n_L^N(E) := \frac{d\nu_L^N}{dE}(E) \tag{3.4.3}$$

$$n_L^\infty(E) := \frac{d\nu_L^\infty}{dE}(E). \tag{3.4.4}$$

We call these the density of states *functions* for N and ∞ respectively. In fact, we will be able to prove that the density of states functions exist everywhere and is differentiable given suitable smoothness conditions of ρ.

We will use the following representation for the density of states function:

Proposition 3.4.1.

$$n_L^N(E) = \lim_{\varepsilon \searrow 0} \mathbb{E} \frac{1}{2N+1} \frac{1}{\pi} \sum_{j=-N}^{N} \operatorname{Im} \langle e_j, (H_L^N - E - i\varepsilon)^{-1} e_j \rangle \tag{3.4.5}$$

and

$$n_L^\infty(E) = \lim_{\varepsilon \searrow 0} \mathbb{E} \operatorname{Im} \frac{1}{\pi} \left\langle e_0, (H_L^\infty - E - i\varepsilon)^{-1} e_0 \right\rangle . \tag{3.4.6}$$

Proof. From Stone's Formula,

$$\nu_L^N([E, E+h]) = \frac{1}{2N+1} \sum_j \mathbb{E} \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \operatorname{Im} \int_E^{E+h} \langle e_j, (H_L^N - \lambda - i\varepsilon)^{-1} e_j \rangle \, d\lambda. \tag{3.4.7}$$

By the Dominated Convergence Theorem, we can bring the ε limit inside of the λ integral. Thus,

$$\begin{aligned} n_L^N(E) &= \lim_{h \to 0} \frac{\nu_L^N([E, E+h]}{h} \\ &= \lim_{h \to 0} \frac{1}{2N+1} \sum_{j=-N}^{N} \frac{1}{h} \mathbb{E} \frac{1}{\pi} \operatorname{Im} \frac{1}{\pi} \int_E^{E+h} \lim_{\varepsilon \searrow 0} \langle e_j, (H_L^N - \lambda - i\varepsilon)^{-1} e_j \rangle \, d\lambda. \end{aligned} \tag{3.4.8}$$

The representation for the finite matrix then follows by the Lebesgue Differentiation Theorem.

For the infinite band matrix, we have from (3.1.2)

$$\begin{aligned} \nu_L^\infty(E, E+h) &= \mathbb{E} \langle e_0, E_{[E,E+h]}(H_L^\infty) e_0 \rangle \\ &= \mathbb{E} \frac{1}{\pi} \int_E^{E+h} \lim_{\varepsilon \searrow 0} \operatorname{Im} \langle e_0, (H_L^\infty - \lambda - i\varepsilon)^{-1} e_0 \rangle \, d\lambda, \end{aligned} \tag{3.4.9}$$

and so again by the Lebesgue Differentiation Theorem

$$n_L^\infty(E) = \lim_{h \to 0} \nu_L^\infty(E, E+h) = \mathbb{E} \frac{1}{\pi} \lim_{\varepsilon \searrow 0} \operatorname{Im} \langle e_0, (H_L^\infty - E - i\varepsilon)^{-1} e_0 \rangle. \tag{3.4.10}$$

□

We show that the fixed width random band matrix satisfies fractional moment bounds on the matrix elements of the resolvent. We extend existing results into the complex plane.

4.1 Fractional Moment Bound

In this section, we improve upon the localization bound of Schenker [23] for a random band matrix of finite dimension and band width L. The bound is stated in terms of decay of fractional moments of the Green's functions:

Theorem 4.1.1 (Schenker). *Let H_L^N be a symmetric random band matrix with $L << N$. Given $r > 0$ and $s \in (0,1)$, there are $\mu_{r,s}$ and $C_{r,s} < \infty$ and $\alpha_s > 0$ such that*

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - \lambda\right)^{-1} e_k\right\rangle\right|^s\right\} \leq C_{r,s} L^{s/2} e^{-\alpha_s \frac{|j-k|}{L^{\mu_{r,s}}}} \tag{4.1.1}$$

for all $\lambda \in [-r, r]$ and all $i, j = -N, \ldots, N$.

In order to complete the proofs of Poisson statistics, we will need to boost this result up to complex values of λ. To accomplish this, we use the following result.

Theorem 4.1.2 (Localization at Extreme Energies). *For some fixed value R and $0 < s < 1$, there exist constants C and $\alpha > 0$ such that for each $z \in \mathbb{C}$ with $|z| > R$,*

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq C e^{-\alpha|j-k|}. \tag{4.1.2}$$

Here,

$$\alpha = -\frac{1}{L} \log\left(\frac{2CL}{C(R)^s}\right) \tag{4.1.3}$$

where the constant C comes from the upper decoupling estimate Lemma 4.3.1 and the function $C(R)$ satisfying $\lim_{R\to\infty} \frac{1}{R} C(R) = 1$ comes from the lower decoupling estimate Lemma 4.3.1.

The proof follows the iteration procedure of Aizenman and Molchanov [1] and is carried out in sections 4.2-4.4.

For now taking Theorem 4.1.2 for granted, we can extend the fractional moment estimates into the complex plane.

Theorem 4.1.3 (Localization at all energies). *For any $z \in \mathbb{C}$, and $0 < s < 1$ there exist constants C and α depending on L and s such that*

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq C e^{-\alpha|j-k|}. \tag{4.1.4}$$

Proof. We follow the strategy suggested in [3] for boosting fractional moment estimates from real energies to energies with non-zero imaginary part by taking advantage of the subharmonicity of the absolute value of the Green's functions. For a summary on the results of subharmonic functions that we need, see Appendix C.

By combining the previous two theorems and there are constants C and α, uniform in $\lambda \in \mathbb{R}$ such that

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - \lambda\right)^{-1} e_k\right\rangle\right|^s\right\} \leq Ce^{-\alpha|j-k|}. \tag{4.1.5}$$

We now note that the function

$$f(z) = \mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \tag{4.1.6}$$

is subharmonic in the upper half-plane with boundary values that exist almost everywhere along the real axis by the upcoming *a priori* bound (4.2.1).

Thus, setting $z = x + iy$ with $y > 0$ and using the Poisson kernel representation for harmonic functions in the upper half plane, we have

$$f(z) \leq \frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\frac{y}{(x-\lambda)^2 + y^2}\, d\lambda. \tag{4.1.7}$$

Using the localization bound which is now uniform for $\lambda \in \mathbb{R}$, the above is bounded by

$$\frac{1}{\pi}\int_{\mathbb{R}} Ce^{-\alpha|j-k|}\frac{y}{(x-\lambda)^2 + y^2} = Ce^{-\alpha|j-k|}. \tag{4.1.8}$$

The estimate also holds for $y < 0$ using the Poisson kernel for the lower-half plane. □

4.2 *A priori* Bounds

An essential step towards deriving the fractional moment bounds is the *a priori* bound:

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq C \tag{4.2.1}$$

for some constant uniform in j, k, N, and z.

Since the results hold in general for self-adjoint matrices with diagonal disorder, the lemmas are stated that way.

We first state the result for diagonal matrix elements, that is $j = k$.

Proposition 4.2.1 (Diagonal *a priori* bound)**.** *Let H_ω be a self-adjoint matrix with independent and identically distributed random variables ω_{jj} with a probability density ρ satisfying $\|\rho\|_\infty < \infty$ along its diagonal. Suppose $0 < s < 1$. Then, for each standard basis vector e_j, and any $z \in \mathbb{C}_+$,*

$$\mathbb{E}\left\{\left|\left\langle e_j, (H_\omega - z)^{-1} e_j\right\rangle\right|^s\right\} \leq C_\rho. \tag{4.2.2}$$

Proof. Using the Schur Complement Formula as in Proposition A.1, we have:

$$\mathbb{E}\left\{\left|\left\langle e_j, (H_\omega - E - i\varepsilon)^{-1} e_j\right\rangle\right|^s\right\} = \mathbb{E}\left\{\left|\frac{1}{\omega_j - z + a}\right|^s\right\} \tag{4.2.3}$$

where a is independent of ω_j. The fractional power s and boundedness of ρ ensures the integral is finite on compact sets. Since ρ is a probability distribution, the integral is finite near infinity. We can define the constant C_ρ as

$$C_\rho := \sup_{a \in \mathbb{C}\setminus\mathbb{R}} \int_{\mathbb{R}} d\omega_j \rho(\omega_j) \left|\frac{1}{\omega_j + a}\right|^s.$$

Thus the constant C_ρ is independent of z and the dimension of the matrix N. □

In order to prove the off-diagonal *a priori* bound, we need the following proposition. The proof of the next proposition and the following lemma below are outlined in [23].

Proposition 4.2.2. *Let* $V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ *be a* 2×2 *diagonal random matrix with* v_1 *and* v_2 *independent with common density* ρ*, and let* A *be an arbitrary* 2×2 *self-adjoint matrix independent of* V*. Then there is a constant* C_ρ *such that*

$$\mathbb{P}\left\{\|(V + A)^{-1}\| > t\right\} < \frac{C_\rho}{t}. \tag{4.2.4}$$

Proof. Note that

$$\|(V + A)^{-1}\| > t \tag{4.2.5}$$

implies that $\sigma(A + V) \bigcap \left[-\frac{1}{t}, \frac{1}{t}\right] \neq \emptyset$.

This in turn implies that $\sigma((A + V)^2) \bigcap \left[0, \frac{1}{t^2}\right] \neq \emptyset$, and thus

$$\sigma\left((A + V)^2 + \frac{1}{t^2}\right) \bigcap \left[0, \frac{2}{t^2}\right] \neq \emptyset. \tag{4.2.6}$$

Thus, using the Chebyshev Inequality,

$$\begin{aligned} \mathbb{P}\left\{\|(V + A)^{-1}\| > t\right\} &\leq \mathbb{P}\left\{\left\|\left[(A + V)^2 + \frac{1}{t^2}\right]^{-1}\right\| > \frac{t^2}{2}\right\} \\ &\leq \frac{2}{t^2}\mathbb{E}\left\{\left\|\left[(A + V)^2 + \frac{1}{t^2}\right]^{-1}\right\|\right\} \end{aligned} \tag{4.2.7}$$

Since A and V are self-adjoint, the operator

$$\left[(A + V)^2 + \frac{1}{t^2}\right]^{-1} \tag{4.2.8}$$

is positive definite, and so the matrix norm is bounded above by the trace. Thus,

$$\frac{2}{t^2}\mathbb{E}\left\{\left\|\left[(A+V)^2+\frac{1}{t^2}\right]^{-1}\right\|\right\} \leq \frac{2}{t^2}\mathbb{E}\,\mathrm{tr}\left[(A+V)^2+\frac{1}{t^2}\right]^{-1}. \tag{4.2.9}$$

In addition, we note

$$\left[(A+V)^2+\frac{1}{t^2}\right]^{-1} = \frac{1}{t}\,\mathrm{Im}\,\frac{1}{A+V+i\frac{1}{t}}. \tag{4.2.10}$$

The result now follows from the spectral averaging estimate (Lemma 3.2.1:

$$\begin{aligned}\mathrm{tr}\,\mathbb{E}\left\{\mathrm{Im}\,\frac{1}{A+V+i\frac{1}{t}}\right\} &= \sum_{j=1,2}\mathbb{E}\left\{\left\langle e_j, \mathrm{Im}\,\frac{1}{A+V+i\frac{1}{t}}e_j\right\rangle\right\} \\ &\leq 2C_\rho. \end{aligned} \tag{4.2.11}$$

Combining the bound with equation 4.2.7, we have

$$\mathbb{P}\left\{\|(V+A)^{-1}\| > t\right\} < \frac{4C_\rho}{t}. \tag{4.2.12}$$

Updating the constant C_ρ, we have the desired upper bound.

□

Lemma 4.2.1 (Off diagonal *a priori* bound at real energy)**.** *Let H_ω be a self-adjoint matrix with independent and identically distributed random variables ω_{jj} with a bounded density ρ along its diagonal. Suppose $0 < s < 1$. Then, for each pair of standard basis vectors e_j and e_k, and any $\lambda \in \mathbb{R}$,*

$$\mathbb{E}\left\{\left|\left\langle e_j, (H_\omega - \lambda)^{-1} e_k\right\rangle\right|^s\right\} \leq C_\rho. \tag{4.2.13}$$

Proof. For $j, k \in \{-N, \ldots, N\}$, let $P_{\{j,k\}}$ be the orthogonal projection onto the span of e_j and e_k. Then,

$$\left|\left\langle e_j, (H_\omega - z)^{-1} e_k\right\rangle\right| \leq \|P_{\{j,k\}}\,(H_\omega - z)^{-1}\,P_{\{j,k\}}\| \tag{4.2.14}$$

where the right hand side is the operator norm of the 2×2 matrix. From the Schur Complement Formula (Lemma A.1 in Appendix A), we can write

$$P_{\{j,k\}}\,(H_\omega - z)^{-1}\,P_{\{j,k\}} = (V_{jk} + A(z))^{-1} \tag{4.2.15}$$

where

$$V_{jk} = \begin{pmatrix} \omega_{jj} & 0 \\ 0 & \omega_{kk} \end{pmatrix} \tag{4.2.16}$$

and $A(z)$ is independent of V_{jk}.

Using the layer cake representation of the expectation, we have

$$\mathbb{E}\left\{\|\,(V_{jk}+A(z))^{-1}\,\|^s\right\} = \int_0^\infty \mathbb{P}\left\{\|\,(V_{jk}+A(z))^{-1}\,\|^s > t\right\}\,dt. \tag{4.2.17}$$

By the previous proposition and the fact that the probability is pointwise bounded by 1, the integral is bounded above by

$$1 + \int_1^\infty \frac{C}{t^{1/s}}. \tag{4.2.18}$$

Since $s < 1$, the integral is finite.

□

We cannot immediately carry the result to complex λ with non-zero imaginary part since Proposition 4.2.2 may not hold if the matrices are not normal. Nevertheless, the bound is uniform for each $\lambda \in \mathbb{R}$, so we can boost the estimate from the real line to the complex plane using subharmonic properties of the Green's functions. The necessary properties of subharmonic functions and their applications to Green's functions are discussed in Appendix C.

Lemma 4.2.2 (Off diagonal *a priori* bound at non-real energy)**.** *Let H_ω be a self-adjoint matrix with i.i.d. random variables with bounded density ρ along its diagonal. Then for $0 < s < 1$, each pair of standard basis vectors e_j and e_k, and any $z \in \mathbb{C}$,*

$$\mathbb{E}\left\{\left|\left\langle e_j, (H_\omega - z)^{-1} e_k\right\rangle\right|^s\right\} \leq C_\rho. \tag{4.2.19}$$

Proof. The function

$$f(z) = \mathbb{E}\left\{\left|\left\langle e_j, (H_\omega - z)^{-1} e_k\right\rangle\right|^s\right\} \tag{4.2.20}$$

is subharmonic with boundary values that exist for every $\lambda \in \mathbb{R}$. Thus if $z = x + iy$ with $y > 0$,

$$\begin{aligned}
\mathbb{E}\left\{\left|\left\langle e_j, (H_\omega - z)^{-1} e_k\right\rangle\right|^s\right\} &\leq \frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\frac{y}{(x-\lambda)^2 - y^2}d\lambda \\
&\leq \frac{C}{\pi}\int_{\mathbb{R}} \frac{y}{(x-\lambda)^2 - y^2} \\
&= C.
\end{aligned} \tag{4.2.21}$$

□

4.3 Decoupling Lemmas

To derive a fractional moment bound for random band matrices which have both diagonal and off-diagonal randomness, we need both upper and lower decoupling estimates.

The following lemma is contained in [1, Lemma 3.1], see also [14] for a similar result.

Lemma 4.3.1 (Lower Decoupling)**.** *Let ρ be the density of a Lipschitz continuous probability measure and $0 < s < 1$. Then there is a $C > 0$ such that*

$$\int \frac{|v-\eta|^s}{|v-\beta|^s}\rho(v)\,dv \geq C_\rho(|\eta|)^s \int \frac{1}{|v-\beta|^s}\rho(v)\,dv. \tag{4.3.1}$$

If $\int |v|^\gamma \rho(v)\,dv < \infty$ for some $\gamma > s$, then $C_\rho(R)^s$ is an increasing function in $R > 0$ and

$$\lim_{R\to\infty} \frac{C_\rho(R)}{R} = 1. \tag{4.3.2}$$

In [1], the authors require less strict conditions on the probability density ρ. They require ρ to be locally uniformly τ-Hölder continuous, and obtain constants that depend on τ. We note that if ρ is Lipschitz, we can take $\tau = 1$.

The upper decoupling estimate is [1, Theorem III.2], which we again reproduce in a slightly less general form that suffices for the necessary application here.

Theorem 4.3.1. *[1, Theorem III.2] Let ρ be the density of a Lipschitz continuous probability measure on $\mathbb{R}$ with $\int |u|^\gamma \rho(u)\,du < \infty$ and $0 < s < 1$. Then for any polynomials p of degree n and q of degree k satisfying $s(n+k) < \gamma$ and $s \cdot k < 1$, then there exists C such that*

$$\frac{1}{\int \frac{|p(u)|^s}{|q(u)|^s}\rho(u)\,du} \int_{\mathbb{R}} |u|^{\gamma - s(n+k)} \frac{|p(u)|^s}{|q(u)|^s}\rho(u)\,du \leq C \int_{\mathbb{R}} |u|^\gamma \rho(u)\,du. \tag{4.3.3}$$

The pre-factor is chosen so that

$$\frac{1}{\int \frac{|p(u)|^s}{|q(u)|^s}\rho(u)\,du} \frac{|p(u)|^s}{|q(u)|^s}\rho(u)\,du \tag{4.3.4}$$

is a probability measure on $\mathbb{R}$.

With a suitable choice of s in relation to γ, n, and k the theorem becomes the upper decoupling estimate:

Lemma 4.3.2 (Upper Decoupling)**.** *Let ρ be the density of a Lipschitz continuous probability measure on $\mathbb{R}$ with $\int |u|^\gamma \rho(u)\,du < \infty$ and $0 < s < 1$, and let p be a polynomial of degree n and q a polynomial of degree k. Suppose $0 < s < \min\left[\frac{1}{k}, \frac{\gamma}{1+n+k}\right]$. Then there exists a constant C such that*

$$\int_{\mathbb{R}} |u|^s \frac{|p(u)|^s}{|q(u)|^s}\rho(u)\,du \leq C \int_{\mathbb{R}} \frac{|p(u)|^s}{|q(u)|^s}\rho(u)\,du. \tag{4.3.5}$$

In the application of the upper decoupling estimate to follow, we will take $n = 1$ and $k = 2$, so that we must have $s < \min\left[\frac{1}{2}, \frac{\gamma}{4}\right]$. Note that for the Gaussian random variables, we will be able to take γ as any finite positive real number.

4.4 Proof of Localization at Extreme Energy

In this section we prove Theorem 4.1.2. The basis of the proof is an iteration of the *a priori* bound proved in Lemma 4.2.2:

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq C \tag{4.4.1}$$

for some constant uniform in j, k, N, and z.

To derive the fractional moment bound, we begin with the identity

$$\left\langle e_j, \left(H_L^N - z\right)\left(H_L^N - z\right)^{-1} e_k\right\rangle = \delta_{jk}, \tag{4.4.2}$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.3}$$

Thus for $j \neq k$, we have

$$\sum_{j' : |j'-j| \leq L} (v_{jj'} - z\delta_{jj'}) \left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle = 0. \tag{4.4.4}$$

where v_{jk} is the entry $\langle e_j, H_L^N e_k\rangle$.

Rearranging, gives

$$(v_{jj} - z)\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle = -\sum_{\substack{j' : |j'-j| \leq L \\ j' \neq j}} v_{jj'} \left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle. \tag{4.4.5}$$

We now take note of the following inequality:

Inequality 4.4.1. *Suppose* $0 < s < 1$. *Then*

$$\left|\sum_{i=1}^N x_i\right|^s \leq \sum_{i=1}^N |x_i|^s. \tag{4.4.6}$$

Using this inequality, we have

$$\begin{aligned} &\mathbb{E}\left\{|v_{jj} - z|^s \left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \\ &\leq \sum_{\substack{j' : |j'-j| \leq L \\ j' \neq j}} \mathbb{E}\left\{|v_{jj'}|^s \left|\left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\}. \end{aligned} \tag{4.4.7}$$

Through the Schur complement formula, with $P = P_j$, we have

$$\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle = \frac{A}{B(v_{jj} - z) + C} \tag{4.4.8}$$

where A, B, and C are scalars independent of v_{jj}, although they do depend on the remaining random variables. Thus, we can see that the left side of (4.4.7) has the form

$$|v_{jj} - z|^s \left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s = |v_{jj} - z|^s \left|\frac{A}{B(v_{jj} - z) + C}\right|^s \tag{4.4.9}$$

and so the lower decoupling lemma bounds the expectation below by

$$C\left(|z|\right)^s \mathbb{E}\left|\frac{a}{v_{jj} - z + b}\right|^s = C\left(|z|\right)^s \mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \tag{4.4.10}$$

where $C\left(|z|\right)$ scales like $|z|$ for large $|z|$.

We can also use the Schur formula with $P = P_{j'}$ to write

$$\left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle = \frac{Av_{jj'} + B}{C{v_{jj'}}^2 + Dv_{jj'} + E} \tag{4.4.11}$$

for $j \neq j'$ where A, B, C, D, and E are all independent of $v_{jj'}$. Thus we can use the upper decoupling lemma on each term in the sum on the right side of (4.4.7) to bound

$$\begin{aligned}\mathbb{E}\left\{|v_{jj'}|^s \left|\left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} &= \mathbb{E}\left\{|v_{jj'}|^s \left|\frac{Av_{jj'} + B}{C{v_{jj'}}^2 + Dv_{jj'} + E}\right|^s\right\} \\ &\leq C\mathbb{E}\left\{\left|\frac{Av_{jj'} + B}{C{v_{jj'}}^2 + Dv_{jj'} + E}\right|^s\right\} \\ &= C\mathbb{E}\left\{\left|\left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\}.\end{aligned} \tag{4.4.12}$$

Rewriting 4.4.7 with the new lower and upper bounds gives

$$C\left(|z|\right)^s \mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq \sum_{\substack{j' : |j'-j| \leq L \\ j' \neq j}} C\mathbb{E}\left\{\left|\left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \tag{4.4.13}$$

or equivalently

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq \frac{C}{C(|z|)^s} \sum_{\substack{j' : |j'-j| \leq L \\ j' \neq j}} \mathbb{E}\left\{\left|\left\langle e_{j'}, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\}. \tag{4.4.14}$$

This estimate can now be iterated $|j-k|/L$ times (until at least one of the indices in the sum overlaps with k). At the end of the iteration process, we use the *a priori* bounds to bound each expectation by an absolute constant. Thus, we have

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq \left(\frac{C \cdot 2L}{C(|z|)^s}\right)^{\frac{|x-y|}{L}}. \tag{4.4.15}$$

We now take R such that $C(R)^s > C \cdot 2L$ and note that for each z with $|z| > R$, we have exponential decay. $\square$

4.5 Localization Under Rank One Perturbation

As a technical result we show that if we change one of the diagonal random variables in a matrix with diagonal disorder satisfying fractional moment bounds, then the resulting matrix still satisfies fractional moment bounds with updated constants depending on the moments of the random variables.

Theorem 4.5.1. *Suppose H_ω is a matrix with independent random variables ω_j with density function ρ having finite second moment along the diagonal. Further, let $0 < s < 1$, and suppose that there exist constants C and γ such that for each standard basis vector e_j and e_k and $z \in \mathbb{C}_+$,*

$$\mathbb{E}\left|\langle e_j, (H_\omega - z)^{-1} e_k \rangle\right|^s \le C e^{-\gamma|j-k|}. \tag{4.5.1}$$

Now suppose $H_{\tilde\omega}$ is obtained by replacing the ω_j with another random variable $\tilde\omega_j$ with probability distribution $\tilde\rho$. Then for $0 < s < 1/4$, there exist updated constants $\tilde C$ and $\tilde\gamma$ such that

$$\mathbb{E}\left|\langle e_j, (H_{\tilde\omega} - z)^{-1} e_k \rangle\right|^s \le \tilde C e^{-\tilde\gamma|j-k|}. \tag{4.5.2}$$

Proof. In this proof we let $\mathbb{E}$ denote the expectation with respect to the random variables $\{\omega_j\}$ as well as the additional random variable $\tilde\omega_j$. Since this forms a product measure of probability measures, we are free to take the integrals in any order.

We first note that H_ω and $H_{\tilde\omega}$ differ from each other by $(\tilde\omega_i - \omega_i)P_i$, and iterate the resolvent identity to get

$$\begin{aligned}
&\langle e_j, (H_{\tilde\omega} - z)^{-1} e_k \rangle \\
&= \langle e_j, (H_\omega - z)^{-1} e_k \rangle + \langle e_j, (H_\omega - z)^{-1} e_i \rangle (\tilde\omega_i - \omega_i) \langle e_i, (H_{\tilde\omega} - z)^{-1} e_k \rangle \\
&= \langle e_j, (H_{\tilde\omega} - z)^{-1} e_k \rangle \\
&\quad + \langle e_j, (H_\omega - z)^{-1} e_i \rangle (\tilde\omega_i - \omega_i) \langle e_i, (H_\omega - z)^{-1} e_k \rangle \\
&\quad - (\tilde\omega_i - \omega_i)^2 \langle e_j, (H_\omega - z)^{-1} e_i \rangle \langle e_i, (H_{\tilde\omega} - z)^{-1} e_i \rangle \langle e_i, (H_\omega - z)^{-1} e_k \rangle.
\end{aligned} \tag{4.5.3}$$

Now by Inequality 1 and a generalized Hölder inequality, we get the bound

$$\begin{aligned}
&\mathbb{E}\left|\langle e_j, (H_{\tilde\omega} - z)^{-1} e_k \rangle\right|^s \\
&\le \mathbb{E}\left|\langle e_j, (H_\omega - z)^{-1} e_k \rangle\right|^s \\
&\quad + \left(\mathbb{E}\left|\tilde\omega_i - \omega_i\right|^{3s} \mathbb{E}\left|\langle e_j, (H_\omega - z)^{-1} e_i \rangle\right|^{3s} \mathbb{E}\left|\langle e_i, (H_\omega - z)^{-1} e_k \rangle\right|^{3s}\right)^{1/3} \\
&\quad + \Big(\mathbb{E}\left|(\tilde\omega_i - \omega_i)^2\right|^{4s} \mathbb{E}\left|\langle e_j, (H_\omega - z)^{-1} e_i \rangle\right|^{4s} \\
&\qquad \times \mathbb{E}\left|\langle e_i, (H_{\tilde\omega} - z)^{-1} e_i \rangle\right|^{4s} \mathbb{E}\left|\langle e_i, (H_\omega - z)^{-1} e_k \rangle\right|^{4s}\Big)^{1/4}.
\end{aligned} \tag{4.5.4}$$

The first term is immediately bounded exponentially in $|j-k|$ by (4.5.1).

For the second term, we use the (4.5.1) and the moment condition.

$$\begin{aligned}\mathbb{E}\left|\tilde{\omega}_i-\omega_i\right|^{3s}\mathbb{E}\left|\langle e_j,(H_\omega-z)^{-1}e_i\rangle\right|^{3s}\mathbb{E}\left|\langle e_i,(H_\omega-z)^{-1}e_k\rangle\right|^{3s}&\\ \leq C'\cdot Ce^{-\gamma_{3s}|j-i|}Ce^{-\gamma_{3s}|i-k|}\leq \tilde{C}e^{-\tilde{\gamma}|j-k|}.&\end{aligned} \tag{4.5.5}$$

For the third term, we also use Lemma 4.2.2 to bound

$$\mathbb{E}\left|\langle e_j,(H_{\tilde{\omega}}-z)^{-1}e_k\rangle\right|^s\leq C_{\tilde{\rho}}, \tag{4.5.6}$$

so that

$$\begin{aligned}&\Big(\mathbb{E}\left|(\tilde{\omega}_i-\omega_i)^2\right|^{4s}\mathbb{E}\left|\langle e_j,(H_\omega-z)^{-1}e_i\rangle\right|^{4s}\\ &\qquad\times\mathbb{E}\left|\langle e_i,(H_{\tilde{\omega}}-z)^{-1}e_i\rangle\right|^{4s}\mathbb{E}\left|\langle e_i,(H_\omega-z)^{-1}e_k\rangle\right|^{4s}\Big)^{1/4}\\ &\leq C'\cdot Ce^{-\gamma_{4s}|j-i|}\cdot C_{\tilde{\rho}}\cdot Ce^{-\gamma_{4s}|i-k|}\\ &\leq \tilde{C}e^{-\tilde{\gamma}|j-k|}.\end{aligned} \tag{4.5.7}$$

Combining the bounds with updated constants, we have the desired exponential decay. □

Chapter 5 Eigenvalue Statistics for Random Band Matrices

In this chapter we take advantage of the localization estimates derived in Chapter 4 to study the eigenvalue statistics of the Random Band Matrix. We show that if the probability density of the entries is smooth enough, the density of states functions converge to a limiting function n_L^∞. Further, we show that function n_L^∞ is differentiable. We use these results to establish convergence of the re-scaled eigenvalue point process to a Poisson point process.

5.1 Regularity Conditions

For the results in this chapter, we must have certain regularity conditions on the probability density the entries of H_L^N. We collect them here.

Hypothesis 1. *The probability density ρ of each entry v_{ij} has finite third moment. That is,*

$$\int_{\mathbb{R}} |x|^3 \rho(x)\, dx < \infty. \tag{5.1.1}$$

We require Hypothesis 1 since we will need to evaluate terms of the form $\mathbb{E}\{|v_{ij}|^{3s}\}$, which arrive from use of the Hölder inequality.

Hypothesis 2. *The probability density ρ of each entry v_{ij} has a second derivative in $L^2(\mathbb{R})$. That is,*

$$\int_{\mathbb{R}} |\rho_{ij}''(x)|^2\, dx < \infty. \tag{5.1.2}$$

Hypothesis 2 implies that the Fourier transform, $\hat{\rho}$, of the probability density satisfies

$$|\hat{\rho}(\lambda)| \le \frac{C}{1+\lambda^2} \tag{5.1.3}$$

for some constant C. It is the decay property of $\hat{\rho}$ that is used in the proof of convergence of the density of states functions.

To establish differentiability of the density of states we will need a slightly stronger hypothesis:

Hypothesis 3. *The derivative of ρ is integrable, and the third derivative is in L^2. That is,*

$$\int_{\mathbb{R}} |\rho'(x)|\, dx < \infty \tag{5.1.4}$$

and

$$\int_{\mathbb{R}} |\rho'''(x)|^2\, dx < \infty. \tag{5.1.5}$$

Note that the Gaussian probability density

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{5.1.6}$$

satisfies each of the above hypotheses.

5.2 First Power to Fractional Power Estimate

For this and the following section, we assume H_L^N satisfies hypotheses 1 and 2.

Theorem 5.2.1. *Suppose $M > N >> L$ and $\varepsilon > 0$. Then for any $0 < s < 1$ and j such that $|j| < N - L$, there exists a constant C such that*

$$\begin{aligned} &\left|\mathbb{E}\left\{\left\langle e_j, \left(H_L^N - E - i\varepsilon\right)^{-1} e_j\right\rangle - \left\langle e_j, \left(H_L^M - E - i\varepsilon\right)^{-1} e_j\right\rangle\right\}\right| \\ &\leq C\mathbb{E}\left|\left\langle \Psi_j^T, \left[\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} - \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1}\right] \Psi_j\right\rangle\right|^s \end{aligned} \tag{5.2.1}$$

where $\tilde{H}_L^N$ and $\tilde{H}_L^M$ are sub-matrices of H_L^N and H_L^M respectively, and Ψ_j is a (random) vector in $\mathbb{R}^{2M+1}$ with non-zero entries occurring only on indices between $j - L$ and $j + L$.

Here A^{-1} denotes the inverse of the matrix A on its range with all other matrix elements set to 0.

Proof. First we use the Schur Complement Formula (see Appendix A) with $P = P_j$ and $Q = 1 - P$, to formally write

$$\left\langle e_j, \left(H_L^N - E - i\varepsilon\right)^{-1} e_j\right\rangle \tag{5.2.2}$$

$$= \frac{1}{P\left(H_L^N - E - i\varepsilon\right)P - P\,H_L^N\,Q\left(Q\left(H_L^N - E - i\varepsilon\right)Q\right)^{-1} Q\,H_L^N\,P}. \tag{5.2.3}$$

Note that

$$P\,H_L^N\;P = \left[H_L^N\right]_{jj} =: v_j(N). \tag{5.2.4}$$

On the other hand, $Q\,H_L^N\,P$ is the j^{th} column of H_L^N, and has the form

$$Q\,H_L^N\,P = \sum_{\substack{i=j-L,\ldots,j+L \\ i\neq j}} \left[H_L^N\right]_{ji} e_i =: \Psi_j(N). \tag{5.2.5}$$

In addition, we will relabel

$$Q\left(H_L^N - E - i\varepsilon\right)Q =: \tilde{H}_L^N - E - i\varepsilon, \tag{5.2.6}$$

where $\tilde{H}_L^N$ is an $2N \times 2N$ submatrix of H_L^N.

When we make the same computation for H_L^M with $M > N$ coming from the same sample of random variables and $|j| < N - L$, then $v_j(M) = v_j(N)$ and $\Psi_j(M) =$

$\Psi_j(N)$. $\tilde{H}_L^M$ only differs from $\tilde{H}_L^N$ for entries with one index j satisfying $|j| > N$. Thus, we can drop the extra notation and write the previous application of the Schur formula as

$$\left\langle e_j, \left(H_L^N - E - i\varepsilon\right)^{-1} e_j \right\rangle = \frac{1}{v_j - E - i\varepsilon - \Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j}. \tag{5.2.7}$$

The denominator has imaginary part equal to

$$-\varepsilon - \Psi_j^T \frac{\varepsilon}{\left(\tilde{H}_L^N - E\right)^2 + \varepsilon^2} \Psi_j < 0, \tag{5.2.8}$$

so we may apply Lemma B.1 (Appendix B):

$$\frac{1}{v_j - E - i\varepsilon - \Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j} = \int_0^\infty e^{-i\left(v_j - E - i\varepsilon - \Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} \, d\lambda. \tag{5.2.9}$$

Thus we have:

$$\begin{aligned}
&\left| \mathbb{E}\left\{ \left\langle e_j, \left(H_L^N - E - i\varepsilon\right)^{-1} e_j \right\rangle - \left\langle e_j, \left(H_L^M - E - i\varepsilon\right)^{-1} e_j \right\rangle \right\} \right| \\
&= \left| \mathbb{E} \int_0^\infty e^{-i(\,v_j - E - i\varepsilon)\lambda} \left(e^{i\left(\Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} - e^{i\left(\Psi_j^T \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} \right) d\lambda \right| \\
&\leq \left| \mathbb{E} \int_0^\infty e^{-i\, v_j\, \lambda} \left(e^{i\left(\Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} - e^{i\left(\Psi_j^T \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} \right) d\lambda \right|.
\end{aligned} \tag{5.2.10}$$

Noting that the second exponential is independent of v_j, we use Fubini's Theorem to bring the expectation with respect to v_j inside the λ integral, to get

$$\left| \mathbb{E}_{v_j^\perp} \int_0^\infty \mathbb{E}_{v_j} \left\{ e^{-i(v_j - E)\lambda} \right\} \left(e^{i\left(\Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} - e^{i\left(\Psi_j^T \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right)\lambda} \right) d\lambda \right|. \tag{5.2.11}$$

Let ρ be the density function for the random variable v_j. Then

$$\mathbb{E}_{v_j} \left\{ e^{-i\, v_j\, \lambda} \right\} = \int e^{-i\, v_j\, \lambda} \rho(v_j) \, dv_j, \tag{5.2.12}$$

which is the Fourier transform of $\rho(v_j)$. Since by assumption, ρ has an L^2 second derivative, its Fourier transform has the pointwise bound

$$\hat{\rho}(\lambda) \leq \frac{C}{1 + |\lambda|^2} \tag{5.2.13}$$

for some constant C.

Thus,

$$\begin{aligned}
&\left|\mathbb{E}\left\{\left\langle e_j, \left(H_L^N - E - i\varepsilon\right)^{-1} e_j\right\rangle - \left\langle e_j, \left(H_L^M - E - i\varepsilon\right)^{-1} e_j\right\rangle\right\}\right| \\
&\le \int_0^\infty \frac{C}{1+|\lambda|^2} \left| \left(e^{i\left(\Psi_j^T\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1}\Psi_j\right)\lambda} - e^{i\left(\Psi_j^T\left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1}\Psi_j\right)\lambda} \right) \right| d\lambda. \qquad (5.2.14)
\end{aligned}$$

Now applying Lemma B.1, we have an upper bound of

$$\int_0^\infty \frac{C}{1+|\lambda|^2} 2^{1-s}\lambda^s \left|\Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j - \Psi_j^T \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right|^s d\lambda. \tag{5.2.15}$$

This expression is integrable in λ, and so updating the constant, we obtain the desired bound

$$C\mathbb{E}_{v_j^\perp} \left|\Psi_j^T \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j - \Psi_j^T \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right|^s. \tag{5.2.16}$$

□

Theorem 5.2.2. *Suppose $M > N >> L$. Let $P = |e_j\rangle\langle e_j|$ and $Q = 1 - P$. Set*

$$\begin{aligned}
\Psi_j &= QH_L^N P = QH_L^M P \\
\tilde{H}_L^N &= QH_L^N Q \\
\tilde{H}_L^M &= QH_L^M Q
\end{aligned} \tag{5.2.17}$$

as in the previous theorem.

Finally, suppose $s < 1/9$. Then there exist constants C and $\gamma > 0$ depending on s and L such that

$$\mathbb{E}\left|\left\langle \Psi_j, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j\right\rangle - \left\langle \Psi_j, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right\rangle\right|^s \le Ce^{-\gamma(N-|j|+2L)}. \tag{5.2.18}$$

Proof. Applying the resolvent identity, we get

$$\begin{aligned}
&\mathbb{E}\left|\left\langle \Psi_j, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j\right\rangle - \left\langle \Psi_j, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right\rangle\right|^s \\
&= \mathbb{E}\left|\left\langle \Psi_j, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \left(\tilde{H}_L^M - \tilde{H}_L^N\right) \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j\right\rangle\right|^s. \qquad (5.2.19)
\end{aligned}$$

Here all matrix elements of $\tilde{H}_L^M - \tilde{H}_L^N$ are 0 for indices k where $|k| \le N$. Further, recall that

$$\Psi_j = \sum_{\substack{i=j-L,\dots,j+L \\ i\neq j}} \left[H_L^N\right]_{ji} e_i. \tag{5.2.20}$$

Then

$$
\begin{aligned}
&\left\langle \Psi_j, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \left(\tilde{H}_L^M - \tilde{H}_L^N\right) \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j \right\rangle \\
&= \sum_{\substack{j-L\le i,k\le j+L \\ i,k\neq j}} v_{ij} v_{jk} \left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \left(\tilde{H}_L^M - \tilde{H}_L^N\right) \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} e_i \right\rangle \\
&= \sum_{(i,k,\ell,m)\in\mathcal{S}} v_{ij} v_{jk} v_{\ell m} \left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle \left\langle e_m, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} e_i \right\rangle.
\end{aligned}
$$

where $\mathcal{S}$ is the set of indices $\{(i,k,\ell,m)\}$ satisfying:

$$
\begin{cases}
i,k \neq j \\
\max(j-L,-N) \le i,k \le \max(j+L,N) \\
-N \le \ell \le -N+L \text{ or } N-L < \ell \le N \\
-N-L \le m < -N \text{ or } N < m \le N+L.
\end{cases} \tag{5.2.21}
$$

We now take recall the following inequality valid for $0 < s < 1$ (Inequality 1):

$$
\left|\sum_{i=1}^{N} a_i\right|^s \le \sum_{i=1}^{N} |a_i|^s. \tag{5.2.22}
$$

With this inequality, we have the bound

$$
\begin{aligned}
&\mathbb{E}\left|\left\langle \Psi_j, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j \right\rangle - \left\langle \Psi_j, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j \right\rangle\right|^s \\
&\le \sum_{(i,k,\ell,m)\in\mathcal{S}} \mathbb{E}\left| v_{ij} v_{jk} v_{\ell m} \left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle \left\langle e_m, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} e_i \right\rangle\right|^s.
\end{aligned} \tag{5.2.23}
$$

Using a generalized Holder inequality bounds each term in the sum by the $\frac{1}{3}$ power of

$$
\mathbb{E}|v_{ij} v_{jk} v_{\ell m}|^{3s} \mathbb{E}\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s} \mathbb{E}\left|\left\langle e_m, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} e_i \right\rangle\right|^{3s}. \tag{5.2.24}
$$

The first term is bounded by a constant under the conditions on the moments of the random variables. For the second and third terms, we obtain exponential bounds

$$
\mathbb{E}\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s} \le C_{L,s} e^{-\gamma_{L,s}(N-|j|+2L)}, \tag{5.2.25}
$$

which we will prove in the following proposition.

Thus combining the bounds and updating any constants that depend on only L and s, we have the desired bound:

$$\mathbb{E}\left|\left\langle \Psi_j, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \Psi_j \right\rangle - \left\langle \Psi_j, \left(\tilde{H}_L^M - E - i\varepsilon\right)^{-1} \Psi_j \right\rangle\right|^s \tag{5.2.26}$$

$$\leq C_{L,s} e^{-\gamma_{L,s}(N-|j|+2L)}. \tag{5.2.27}$$

□

To prove 5.2.25, we would like to apply the fractional moment bounds of Theorem 4.1.3, to the reduced matrices $\tilde{H}_L^N$ and $\tilde{H}_L^M$. Unfortunately, $\tilde{H}_L^N$ and $\tilde{H}_L^M$ are not i.i.d. band matrices themselves, (even under a change of basis) since there is a "defect" where the row and column were removed. Instead we will need to compare the Green's functions with the Green's function for the original band matrices H_L^N and H_L^M.

Proposition 5.2.1. *Let $\tilde{H}_L^N := \tilde{H}_L^N(j)$ be the sub-matrix of H_L^N obtained by setting the j^{th} row and column equal to 0. Suppose $|k-j| \leq L$ and $|\ell \pm N| \leq L$. Further, suppose $k, \ell \neq j$. Then, there exist constants $C_{L,s}$ and $\gamma_{L,s}$ such that*

$$\mathbb{E}\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s} \leq C_{L,s} e^{-\gamma_{L,s}(N-|j|+2L)} \tag{5.2.28}$$

Proof. By the triangle inequality and Inequality 1,

$$\mathbb{E}\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s}$$

$$\leq \mathbb{E}\left|\left\langle e_k, \left(H_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s} \tag{5.2.29}$$

$$+ \mathbb{E}\left|\left\langle e_k, \left[\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} - \left(H_L^N - E - i\varepsilon\right)^{-1}\right] e_\ell \right\rangle\right|^{3s}. \tag{5.2.30}$$

(5.2.29) decays exponentially like $|k-\ell|$ by Theorem 4.1.3.

For (5.2.30) we have:

$$\mathbb{E}\left|\left\langle e_k, \left[\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} - \left(H_L^N - E - i\varepsilon\right)^{-1}\right] e_\ell \right\rangle\right|^{3s}$$

$$= \mathbb{E}\left|\left\langle e_j, \left[\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \left(H_L^N - \tilde{H}_L^N\right) \left(H_L^N - E - i\varepsilon\right)^{-1}\right] e_k \right\rangle\right|^{3s}. \tag{5.2.31}$$

Above, $\left(H_L^N - \tilde{H}_L^N\right)$ is the matrix consisting only of the j^{th} row and j^{th} column of H_L^N. In particular,

$$\left[\left(H_L^N - \tilde{H}_L^N\right)\right]_{k\ell} = \begin{cases} v_{k\ell} & if \quad k = j, \quad \leq j - L \leq \ell \leq j + L \\ v_{k\ell} & if \quad \ell = j, \quad j - L \leq k \leq L \\ 0 & otherwise \end{cases} \tag{5.2.32}$$

Thus, carrying out the matrix multiplication, we have

$$\begin{aligned}&\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \left(H_L^N - \tilde{H}_L^N\right) \left(H_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle \\ &= \sum_{(m,n)\in\mathcal{T}} \left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_m \right\rangle v_{mn} \left\langle e_n, \left(H_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle \end{aligned} \tag{5.2.33}$$

where

$$\mathcal{T} = \left\{ (m,n) \; : \; \begin{array}{rl} & m = j \textit{ and } j - L \le n \le j + L \\ \textit{or} & n = j \textit{ and } j - L \le m \le j + L \end{array} \right\}. \tag{5.2.34}$$

Applying the triangle inequality and Inequality 1 again, gives the bound:

$$\begin{aligned}&\mathbb{E}\left|\left\langle e_k, \left[\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} \left(H_L^N - \tilde{H}_L^N\right) \left(H_L^N - E - i\varepsilon\right)^{-1}\right] e_\ell \right\rangle\right|^{3s} \\ &\le \sum_{(m,n)\in\mathcal{T}} \mathbb{E}\left\{\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_m \right\rangle\right|^{3s} |v_{mn}|^{3s} \left|\left\langle e_n, \left(H_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s}\right\}. \end{aligned} \tag{5.2.35}$$

With another use of the generalized Holder inequality, we bound each term in the sum by

$$\left(\mathbb{E}\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_m \right\rangle\right|^{9s} \mathbb{E}\,|v_{mn}|^{9s} \left|\left\langle e_n, \left(H_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{9s}\right)^{1/3}. \tag{5.2.36}$$

Note that since e_j is in the kernel of $\tilde{H}_L^N$,

$$\left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_j = -\frac{1}{z} e_j, \tag{5.2.37}$$

and since $k \neq j$,

$$\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_j \right\rangle = 0. \tag{5.2.38}$$

Thus we can bound the first in the product (5.2.36) by 0 if $m = j$, and a constant using the *a priori* bound from Lemma 4.2.2 otherwise.

The second term in the product is bounded by the first moment of the random variables. The third term decays exponentially in $|n - \ell|$ by Theorem 4.1.3.

Thus, updating the constant $C_{L,s}$ we bound

$$\begin{aligned}&\left(\mathbb{E}\left|\left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon\right)^{-1} e_\ell \right\rangle\right|^{3s}\right)^{1/3} \\ &\qquad \le \sum_{(m,n)\in\mathcal{T}} C_{L,s} e^{-\gamma_{L,s}|n-\ell|}. \end{aligned} \tag{5.2.39}$$

Now recall that by the definition of $\mathcal{T}$,

$$\begin{aligned} |n - |j|| &\leq L \\ ||\ell| - N| &\leq L \end{aligned} \tag{5.2.40}$$

and so

$$\begin{aligned} |n - \ell| &\leq |n - N| + |N - \ell| \\ &\leq |n - j| + |j - N| + |N - \ell| \\ &\leq N - |j| + 2L. \end{aligned} \tag{5.2.41}$$

Thus, updating the constants $C_{L,s}$ and $\gamma_{L,s}$, we obtain the upper bound

$$C_{L,s} e^{-\gamma_{L,s}(N-|j|+2L)}. \tag{5.2.42}$$

□

Note that we also get the bound for $\tilde{H}_L^M(j)$ for $M > N$ in terms of the distance from N to j:

Proposition 5.2.2. *Suppose $N > M$. Let $\tilde{H}_L^N := \tilde{H}_L^N(j)$ be the sub-matrix of H_L^N obtained by setting the j^{th} row and column equal to 0. Suppose $|k - j| \leq L$ and $|\ell \pm M| \leq L$. Then, there exist constants $C_{L,s}$ and $\gamma_{L,s}$ such that*

$$\mathbb{E} \left| \left\langle e_k, \left(\tilde{H}_L^N - E - i\varepsilon \right)^{-1} e_\ell \right\rangle \right|^{3s} \leq C_{L,s} e^{-\gamma_{L,s}(M-|j|+2L)} \tag{5.2.43}$$

In fact, we can strengthen this result somewhat–at a penalty of further dependence on L in the exponent–to a full result on exponential decay of the matrix elements of the Green's functions using the technique from the proof of Theorem 4.5.1, but we do not carry out this analysis since it is not necessary for our results.

5.3 Convergence of Density of States Functions

We can now prove convergence of the density of states functions n_L^N to n_L^∞. Recall that from Proposition 3.4.1, we have the representations

$$n_L^N(E) = \lim_{\varepsilon \searrow 0} \frac{1}{2N+1} \sum_{j=-N}^{N} \mathbb{E} \operatorname{Im} \left\langle e_j, (H_L^N - E - i\varepsilon)^{-1} e_j \right\rangle. \tag{5.3.1}$$

and

$$n_L^\infty(E) = \lim_{\varepsilon \searrow 0} \frac{1}{2N+1} \sum_{j=-N}^{N} \mathbb{E} \operatorname{Im} \left\langle e_0, (H_L^\infty - E - i\varepsilon)^{-1} e_0 \right\rangle. \tag{5.3.2}$$

In fact, by the Birkhoff Ergodic Theorem, we can write

$$n_L^\infty(E) = \lim_{\varepsilon \searrow 0} \frac{1}{2N+1} \sum_{j=-N}^{N} \frac{1}{2N+1} \sum_{j=-N}^{N} \mathbb{E} \operatorname{Im} \left\langle e_j, (H_L^\infty - E - i\varepsilon)^{-1} e_j \right\rangle \tag{5.3.3}$$

for each N.

Theorem 5.3.1. *Let H_L^N be a random symmetric band matrix with fixed bandwidth L and entries satisfying Hypotheses 1, 2, and 3. Then for each $E \in \mathbb{R}$,*

$$n_L^N(E) \longrightarrow n_L^\infty(E). \tag{5.3.4}$$

Furthermore, the convergence is uniform in E.

Proof. Let $z \in \mathbb{C}_+$. Consider the difference

$$\left| \frac{1}{2N+1} \mathbb{E} \sum_{j=-N}^{N} \operatorname{Im} \left\langle e_j, \left(H_L^N - z\right)^{-1} e_j \right\rangle - \frac{1}{2N+1} \mathbb{E} \sum_{j=-N}^{N} \operatorname{Im} \left\langle e_j, \left(H_L^\infty - z\right)^{-1} e_j \right\rangle \right|. \tag{5.3.5}$$

By Theorems 5.2.1 and 5.2.2,

$$\mathbb{E} \left| \operatorname{Im} \left\langle e_j, \left(H_L^N - z\right)^{-1} e_j \right\rangle - \operatorname{Im} \left\langle e_j, \left(H_L^\infty - z\right)^{-1} e_j \right\rangle \right| \le C e^{-\gamma |N - |j| + 2L|}. \tag{5.3.6}$$

In addition, by Lemma 3.2.1 (Spectral Averaging),

$$\mathbb{E} \left| \operatorname{Im} \left\langle e_j, \left(H_L^N - z\right)^{-1} e_j \right\rangle - \operatorname{Im} \left\langle e_j, \left(H_L^\infty - z\right)^{-1} e_j \right\rangle \right| \le C. \tag{5.3.7}$$

In each case, the constant C is independent of N and z.

We can thus take $0 < \alpha < 1$ and split the sum of the differences into two pieces:

$$\begin{aligned}
&\frac{1}{2N+1} \mathbb{E} \sum_{j=-N}^{N} \left| \operatorname{Im} \left\langle e_j, \left(H_L^N - z\right)^{-1} e_j \right\rangle - \operatorname{Im} \left\langle e_j, \left(H_L^\infty - z\right)^{-1} e_j \right\rangle \right| \\
&= \frac{1}{2N+1} \mathbb{E} \sum_{|j| \le N - N^\alpha} \left| \left\langle e_j, \left(H_L^N - z\right)^{-1} e_j \right\rangle - \left\langle e_j, \left(H_L^\infty - z\right)^{-1} e_j \right\rangle \right| \\
&\quad + \frac{1}{2N+1} \mathbb{E} \sum_{N - N^\alpha < |j| \le N} \left| \operatorname{Im} \left\langle e_j, \left(H_L^N - z\right)^{-1} e_j \right\rangle - \operatorname{Im} \left\langle e_j, \left(H_L^\infty - z\right)^{-1} e_j \right\rangle \right| \\
&\le \frac{1}{2N+1} \sum_{|j| \le N - N^\alpha} C e^{-\gamma (N - (N - N^\alpha) - 2L)} + \frac{1}{2N+1} \sum_{N - N^\alpha < |j| \le N} C \\
&= C \frac{2(N - N^\alpha) + 1}{2N+1} e^{-\gamma (N^\alpha - 2L)} + \frac{2N^\alpha}{2N+1} \\
&\longrightarrow 0.
\end{aligned} \tag{5.3.8}$$

Since the constants are independent of z, the convergence is uniform for $z \in \mathbb{C}_+$.

Thus, taking $z = E + i\varepsilon$, we can interchange the ε and N limits in the following and apply (5.3.8):

$$\begin{aligned}
\lim_{N\to\infty} n_L^N(E) &= \lim_{N\to\infty}\lim_{\varepsilon\searrow 0} \frac{1}{2N+1}\frac{1}{\pi}\sum_{j=-N}^{N} \operatorname{Im}\mathbb{E}\left\{\left\langle e_j, (H_L^N - E - i\varepsilon)^{-1} e_j\right\rangle\right\} \\
&= \lim_{\varepsilon\searrow 0}\lim_{N\to\infty} \frac{1}{2N+1}\frac{1}{\pi}\sum_{j=-N}^{N} \operatorname{Im}\mathbb{E}\left\{\left\langle e_j, (H_L^N - E - i\varepsilon)^{-1} e_j\right\rangle\right\} \\
&= \lim_{\varepsilon\searrow 0}\lim_{N\to\infty} \frac{1}{2N+1}\frac{1}{\pi}\sum_{j=-N}^{N} \operatorname{Im}\mathbb{E}\left\{\left\langle e_j, (H_L^\infty - E - i\varepsilon)^{-1} e_j\right\rangle\right\} \\
&= n_L^\infty(E). \qquad (5.3.9)
\end{aligned}$$

By (5.3.8), the limit is uniform in E, and so we have uniform convergence of the finite matrix density of states function n_L^N to the infinite volume density of states function n_L^∞.

□

5.4 Smoothness of Density of States

We now prove that the density of states is smooth in the case of Gaussian random variables, or in general has as a number of derivatives depending on the number of derivatives of the common probability density of the random variables. Inspiration and some of the techniques for the following proofs came from [11]. We first address the case of the finite N density of states.

Theorem 5.4.1. *Let H_L^N be a $2N+1 \times 2N+1$ random band matrix with band width L, and random variables having common density ρ which satisfies*

$$\int_{\mathbb{R}} \left|\rho^{(K)}(x)\right| \, dx < \infty. \qquad (5.4.1)$$

Then the integrated density of states function $N_L^N(E) \in \mathcal{C}^K(\mathbb{R})$.

Proof. Let $E_{(-\infty,E]}(H_L^N)$ be the spectral projection for H_L^N onto the set $(-\infty, E]$. Then

$$\begin{aligned}
N_L^N(E) &= \frac{1}{2N+1}\mathbb{E}\left\{ \operatorname{tr} E_{(-\infty,E]}(H_L^N)\right\} \\
&= \frac{1}{2N+1}\mathbb{E}\left\{ \operatorname{tr} E_{(-\infty,0]}(H_L^N - EI)\right\}. \qquad (5.4.2)
\end{aligned}$$

Absorbing E into the diagonal random variables, we have for $k = 1, \ldots, K$:

$$\begin{aligned}
&\left(\frac{d}{dE}\right)^k N_L^N(E) \\
&= \frac{d^k}{dE^k}\frac{1}{2N+1}\int \prod_{i=-N}^{N} dv_{ii}\rho(v_{ii} - E) \prod_{\substack{i<j:\\ |i-j|\le L}} dv_{ij}\rho(v_{ij}) \operatorname{tr} E_{(-\infty,0]}\left(H_L^N\right). \qquad (5.4.3)
\end{aligned}$$

The integral is the convolution of the K times differentiable function $\prod_{i=-N}^{N} \rho(v_{ii})$, with the L^1 function $\operatorname{tr} E_{(-\infty,0]}\left(H_L^N\right)$. Thus, we can bring the derivatives onto the product of the probability densities of the diagonal random variables. From the Leibniz Rule for the derivatives of products:

$$\frac{d^k}{dE^k}\prod_{i=-N}^{N}\rho(v_{ii}-E)=\sum_{i_{-N}+\cdots+i_N=k}\frac{k!}{i_{-N}!\cdots i_N!}\prod_{j=-N}^{N}\frac{d^{i_j}}{dE^{i_j}}\rho(v_{jj}-E). \tag{5.4.4}$$

we have

$$\frac{1}{2N+1}\sum_{i_{-N}+\cdots+i_N=k}\frac{k!}{i_{-N}!\cdots i_N!}\int\prod_{j=-N}^{N}dv_{jj}\frac{d^{i_j}}{dE^{i_j}}\rho(v_{jj}-E) \times\prod_{\substack{i<j:\\|i-j|\le L}}dv_{ij}\rho(v_{ij})\operatorname{tr} E_{(-\infty,0]}\left(H_L^N\right). \tag{5.4.5}$$

By (5.4.1), each derivative exists and result is integrable in each v_{ii}. □

Unfortunately, the above proof of existence does not give us uniform estimates on the derivatives in either the dimension N or band width L. Therefore, we will need more work to establish smoothness for the infinite volume density of states N_L^∞.

As a first step, we use the following Lemma which is stated as Lemma A.1 in [11].

Lemma 5.4.1. *Consider a positive function $f \in L^1(\mathbb{R}, dx)$ and $J \subset \mathbb{R}$ an interval. Let $F(z) = \int \frac{1}{x-z} f(x)dx$. Then, for any $m \in \mathbb{N}$,*

$$ess\ sup_{x\in J}\left|\frac{d^m}{dx^m}f(x)\right|<\infty \tag{5.4.6}$$

whenever

$$\sup_{z\in\mathbb{C}^+,\ \mathrm{Re}(z)\in J}\left|\frac{d^m}{dz^m}\,\mathrm{Im}(F(z))\right|<\infty. \tag{5.4.7}$$

We will apply this Lemma with $f = n_L^\infty$, the infinite volume density function, and so we will need to bound the derivatives of Borel transform of the density of states measure, uniformly in $\mathbb{C}_+$.

We now prove the smoothness for the infinite volume density of states. Recall that from the ergodicity of the infinite volume operator, we have the representation:

$$n_L^\infty(E)=\lim_{\varepsilon\searrow 0}\mathbb{E}\,\mathrm{Im}\langle e_0,(H_L^\infty-E-i\varepsilon)^{-1}e_0\rangle. \tag{5.4.8}$$

To simplify the presentation, we prove $n_L^\infty \in \mathcal{C}^1(\mathbb{R})$ first. We will treat higher order derivatives in a corollary.

Theorem 5.4.2. *Let H_L^∞ be an infinite by infinite symmetric random band matrix on $\ell^2(\mathbb{Z})$ with entries satisfying Hypothesis 3. Then the infinite volume density of states function $n_L^\infty \in \mathcal{C}^1(\mathbb{R})$.*

Proof. We note that $n_L^\infty(E)$ is the boundary value of

$$\mathbb{E}\left\{\operatorname{Im}\langle e_0, (H_L^\infty - E - i\varepsilon)^{-1} e_0\rangle\right\} \tag{5.4.9}$$

as $\varepsilon \searrow 0$.

From the previous Lemma, the result follows once we show that

$$\operatorname{ess\,sup}_{z\in\mathbb{C}^+} \left| \frac{d^m}{dz^m} \mathbb{E}\operatorname{Im}\langle e_0, (H_L^\infty - z)^{-1} e_0\rangle \right| < \infty. \tag{5.4.10}$$

From Theorems 5.2.1 and 5.2.2, we have that

$$\mathbb{E}\langle e_0, (H_L^N - z)^{-1} e_0\rangle \to \mathbb{E}\langle e_0, (H_L^\infty - z)^{-1} e_0\rangle \tag{5.4.11}$$

as $N \to \infty$ uniformly for z in compact subsets of $\mathbb{C}_+$. Furthermore, since the Green's functions are analytic for $z \in \mathbb{C}^+$, this implies that the derivatives also converge uniformly on compact subsets of the upper half-plane.

We can therefore write the infinite volume Green's function at fixed $z \in \mathbb{C}^+$ as the following telescoping series:

$$\begin{aligned}
&\mathbb{E}\langle \operatorname{Im} e_0, (H_L^\infty - z)^{-1} e_0\rangle \\
&= \sum_{M=N}^{\infty} \left[\mathbb{E}\operatorname{Im}\langle e_0, (H_L^{M+1} - z)^{-1} e_0\rangle - \mathbb{E}\operatorname{Im}\langle e_0, (H_L^M - z)^{-1} e_0\rangle\right] \\
&\qquad + \mathbb{E}\operatorname{Im}\langle e_0, (H_L^N - z)^{-1} e_0\rangle.
\end{aligned} \tag{5.4.12}$$

From Theorem 5.4.1,

$$\operatorname{ess\,sup}_{z\in\mathbb{C}^+} \frac{d}{dz} \mathbb{E}\operatorname{Im}\langle e_0, (H_L^N - z)^{-1} e_0\rangle \tag{5.4.13}$$

is finite. Thus it remains to bound

$$\frac{d}{dz}\mathbb{E}\operatorname{Im}\left[\langle e_0, (H_L^{M+1} - z)^{-1} e_0\rangle - \mathbb{E}\operatorname{Im}\langle e_0, (H_L^M - z)^{-1} e_0\rangle\right]. \tag{5.4.14}$$

The Cauchy-Riemann equations

$$u_x = v_y \tag{5.4.15}$$
$$u_y = -v_x \tag{5.4.16}$$

hold for $f = u + iv$ analytic. Thus, for $z = E + i\varepsilon$,

$$\frac{d}{dz}\operatorname{Im}\langle e_0, (H_L^N - z)^{-1} e_0\rangle \tag{5.4.17}$$

can be written in terms of derivative with respect to E of the real and imaginary part of the Green's function. Thus it suffices to obtain estimates for

$$\frac{d}{dE}\left[\mathbb{E}\left\{\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \mathbb{E}\langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right\}\right]. \tag{5.4.18}$$

For these terms, we have

$$\begin{aligned}
&\mathbb{E}\left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right] \\
&= \mathbb{E}_{off} \int \prod_{i=-M-1}^{M+1} dv_{ii}\rho(v_{ii}) \left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right]
\end{aligned} \tag{5.4.19}$$

where ρ is the probability density function of the diagonal random variables, and $\mathbb{E}_{off}$ is the expectation with respect to the off-diagonal elements. Now proceeding as in the proof of Theorem 5.4.1, we have:

$$\begin{aligned}
&\mathbb{E}\left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right] \\
&= \mathbb{E}_{off} \int \prod_{i=-M-1}^{M+1} dv_{ii}\rho(v_{ii}) \left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right] \\
&= \mathbb{E}_{off} \int \prod_{i=-M-1}^{M+1} dv_{ii}\rho(v_{ii} - E) \left[\langle e_0, (H_L^{M+1} - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - i\varepsilon)^{-1} e_0\rangle\right].
\end{aligned} \tag{5.4.20}$$

Thus

$$\begin{aligned}
&\frac{d}{dE}\mathbb{E}\left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right] \\
&= \mathbb{E}_{off} \int \frac{d}{dE} \prod_{i=-M-1}^{M+1} dv_{ii}\rho(v_{ii} - E) \left[\langle e_0, (H_L^{M+1} - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - i\varepsilon)^{-1} e_0\rangle\right] \\
&= -\mathbb{E}_{off} \int \sum_{i=-M-1}^{M+1} \rho'(v_{ii} - E) \prod_{j\neq i} \rho(v_{jj} - E) \\
&\qquad\qquad \times \left[\langle e_0, (H_L^{M+1} - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - i\varepsilon)^{-1} e_0\rangle\right] \\
&= -\mathbb{E}_{off} \int \sum_{i=-M-1}^{M+1} \rho'(v_{ii}) \prod_{j\neq i} \rho(v_{jj}) \\
&\qquad\qquad \times \left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right] \\
&= -\sum_{i=-M-1}^{M+1} \mathbb{E}_{v_{ii}^{\perp}} \int \rho'(v_{ii}) \left[\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0\rangle - \langle e_0, (H_L^M - E - i\varepsilon)^{-1} e_0\rangle\right],
\end{aligned} \tag{5.4.21}$$

where $\mathbb{E}_{v_{ii}^{\perp}}$ is the expectation of all random variables except for v_{ii}.

We now proceed as in the proof of first power to s power bound (Theorem 5.2.1), replacing the expectation with respect to v_{ii} with an integral against $\rho'(v_{ii})$. In particular, in analogy to (5.2.10), we have

$$\mathbb{E}_{v_{ii}{}^\perp}\int dv_{ii}\rho'(v_{ii})\left[\langle e_0,(H_L^{M+1}-E-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^{M}-E-i\varepsilon)^{-1}e_0\rangle\right]$$
$$=\mathbb{E}_{v_{ii}{}^\perp}\int dv_{ii}\rho'(v_{ii})\int_0^\infty e^{-i(\,v_0\,-E-i\varepsilon)\lambda}$$
$$\times\left(e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M+1}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}-e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}\right)d\lambda. \tag{5.4.22}$$

As in the proof of Theorem 5.2.1, we take the integral with respect to v_{00} first. If $i=0$, we obtain the Fourier Transform of ρ':

$$\mathbb{E}_{v_{00}{}^\perp}\int_0^\infty\widehat{\rho'}(\lambda)e^{i\varepsilon\lambda}\left(e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M+1}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}-e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}\right)d\lambda. \tag{5.4.23}$$

Now applying Lemma B.3, we have

$$\left|\mathbb{E}_{v_{00}{}^\perp}\int_0^\infty\hat{\rho}'(\lambda)e^{i\varepsilon\lambda}\left(e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M+1}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}-e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}\right)d\lambda\right|$$
$$\le\mathbb{E}_{v_{00}{}^\perp}\int_0^\infty\Bigg(\left|\hat{\rho}'(\lambda)\right|2^{1-s}\lambda^s$$
$$\times\left|\Psi_0^T\left(\tilde{H}_L^{M+1}-E-i\varepsilon\right)^{-1}\Psi_0-\Psi_0^T\left(\tilde{H}_L^{M}-E-i\varepsilon\right)^{-1}\Psi_0\right|^s\Bigg)d\lambda. \tag{5.4.24}$$

Since $\int_{\mathbb{R}}|\rho'''|<\infty$ by Hypothesis 3, $\hat{\rho}'(\lambda)$ decays fast enough for the λ integral to be finite. Thus for some constant C, the above is bounded by

$$C\mathbb{E}_{v_{00}{}^\perp}\left|\Psi_0^T\left(\tilde{H}_L^{M+1}-E-i\varepsilon\right)^{-1}\Psi_0-\Psi_0^T\left(\tilde{H}_L^{M}-E-i\varepsilon\right)^{-1}\Psi_0\right|^s. \tag{5.4.25}$$

We can now directly apply Theorem 5.2.2 to bound

$$\left|\mathbb{E}_{v_{00}{}^\perp}\int dv_{00}\rho'(v_{00})\langle e_0,(H_L^{M+1}-E-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^{M}-E-i\varepsilon)^{-1}e_0\rangle\right|$$
$$\le Ce^{-\gamma(M-2L)}. \tag{5.4.26}$$

If, on the other hand, $i\neq 0$, we have by first integrating over v_{00}, that (5.4.22) is equal to

$$\mathbb{E}_{(v_{00},v_{ii})^\perp}\int dv_{ii}\rho'(v_{ii})\int_0^\infty\hat{\rho}(\lambda)e^{i\varepsilon\lambda}$$
$$\times\left(e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M+1}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}-e^{i\left(\Psi_0^T\left(\tilde{H}_L^{M}-E-i\varepsilon\right)^{-1}\Psi_0\right)\lambda}\right)d\lambda. \tag{5.4.27}$$

Again using Lemma B.3, we bound (5.4.27) by

$$\mathbb{E}_{(v_0,v_{ii})^\perp} \int dv_{ii}\, |\rho'(v_{ii})| \left| \Psi_0^T \left(\tilde{H}_L^{M+1} - E - i\varepsilon \right)^{-1} \Psi_0 - \Psi_0^T \left(\tilde{H}_L^{M} - E - i\varepsilon \right)^{-1} \Psi_0 \right|^s \times \int_0^\infty |\hat{\rho}(\lambda)|\, 2^{1-s} \lambda^s \, d\lambda \qquad (5.4.28)$$

which is bounded by a constant times

$$\mathbb{E}_{(v_0,v_{ii})^\perp} \int dv_{ii}\, |\rho'(v_{ii})| \left| \Psi_0^T \left(\tilde{H}_L^{M+1} - E - i\varepsilon \right)^{-1} \Psi_0 - \Psi_0^T \left(\tilde{H}_L^{M} - E - i\varepsilon \right)^{-1} \Psi_0 \right|^s . \qquad (5.4.29)$$

We now let

$$C' = \int_{\mathbb{R}} |\rho'(x)|\, dx, \qquad (5.4.30)$$

and define a probability density

$$\tilde{\rho} = \frac{1}{C'} \rho. \qquad (5.4.31)$$

Thus we can view (5.4.29) as the expectation of

$$C' \left| \Psi_0^T \left(\tilde{H'}_L^{M} - E - i\varepsilon \right)^{-1} \Psi_0 - \Psi_0^T \left(\tilde{H'}_L^{M+1} - E - i\varepsilon \right)^{-1} \Psi_0 \right|^s \qquad (5.4.32)$$

where $\tilde{H'}_L^M$ is obtained by replacing the entry v_{ii} with a random variable with probability density $\tilde{\rho}$. Since we have fractional moment bounds for $\tilde{H}_L^M$ and $\tilde{H}_L^{M+1}$, and $\tilde{H'}_L^M$ and $\tilde{H'}^{M+1}$ are rank one perturbations to $\tilde{H}_L^M$ and $\tilde{H}_L^{M+1}$, we can apply Theorems 4.5.1 and 5.2.2, to bound

$$\mathbb{E}_{(v_{00},v_{ii})^\perp} \int dv_{ii}\, |\rho'(v_{ii})| \left| \Psi_0^T \left(\tilde{H}_L^{M+1} - E - i\varepsilon \right)^{-1} \Psi_0 - \Psi_0^T \left(\tilde{H}_L^{M} - E - i\varepsilon \right)^{-1} \Psi_0 \right|^s \leq \tilde{C} e^{-\tilde{\gamma}(M-2L)} \qquad (5.4.33)$$

for some constants, $\tilde{C}$ and $\tilde{\gamma}$.

Therefore, combining the bounds (5.4.26) and (5.4.33) and updating the constants, we have from (5.4.21),

$$\begin{aligned} \left| \frac{d}{dE} \mathbb{E}\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1} e_0 \rangle - \frac{d}{dE} \mathbb{E}\langle e_0, (H_L^{M} - E - i\varepsilon)^{-1} e_0 \rangle \right| &\leq \sum_{i=-M-1}^{M+1} C e^{-\gamma(M-2L)} \\ &\leq C(1+2M) e^{-\gamma(|M|-2L)}. \end{aligned} \qquad (5.4.34)$$

From the telescoping series expansion (5.4.12) and for $\mathbb{E}\,\mathrm{Im}\langle e_0, (H_L^\infty - z)^{-1}e_0\rangle$, we obtain the bound

$$\begin{aligned}\left|\frac{d}{dz}\mathbb{E}\,\mathrm{Im}\langle e_0, (H_L^\infty - z)^{-1}e_0\rangle \right. & \left. -\frac{d}{dz}\mathbb{E}\,\mathrm{Im}\langle e_0, (H_L^N - z)^{-1}e_0\rangle\right| \\ & \leq \sum_{M=N}^{\infty} C(1+2M)e^{-\gamma(M-2L)} \\ & < \infty. \end{aligned} \tag{5.4.35}$$

Since this bound holds uniformly for z in compact subsets of $\mathbb{C}_+$, we have smoothness of the density of states function by Lemma 5.4.1. □

We can now extend the methods above to prove the existence of K^{th} order derivatives in the case that the probability density ρ satisfies the stricter conditions:

Hypothesis 4. *The K^{th} derivative of ρ is integrable, and the $K+2$ derivative is in L^2. That is,*

$$\int_{\mathbb{R}} |\rho^{(K)}(x)|\, dx < \infty \tag{5.4.36}$$

and

$$\int_{\mathbb{R}} |\rho^{(K+2)}(x)|^2\, dx < \infty. \tag{5.4.37}$$

Note that Gaussian variables satisfy the hypothesis for all $K \in \mathbb{N}$.

Corollary 5.4.1. *Let H_L^∞ be an infinite by infinite symmetric random band matrix on $\ell^2(\mathbb{Z})$ satisfying Hypothesis 4. Then the infinite volume density of states function $n_L^\infty \in \mathcal{C}^K(\mathbb{R})$.*

Proof. From Lemma 5.4.1, we need to satisfy equation (5.4.18) with $\frac{d}{dE}$ replaced by a derivative of order K:

$$\frac{d^K}{dE^K}\left[\mathbb{E}\,\mathrm{Im}\langle e_0, (H_L^{M+1} - E - i\varepsilon)^{-1}e_0\rangle - \mathbb{E}\,\mathrm{Im}\langle e_0, (H_L^{M} - E - i\varepsilon)^{-1}e_0\rangle\right] \tag{5.4.38}$$

We now use the Leibniz formula on the probability densities:

$$
\begin{aligned}
&\frac{d^K}{dE^K}\mathbb{E}\left|\langle e_0,(H_L^{M+1}-E-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^M-E-i\varepsilon)^{-1}e_0\rangle\right|\\
&=\mathbb{E}_{off}\int\frac{d^K}{dE^K}\prod_{i=-M-1}^{M+1}dv_{ii}\rho(v_{ii}-E)\left|\langle e_0,(H_L^{M+1}-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^M-i\varepsilon)^{-1}e_0\rangle\right|\\
&=\mathbb{E}_{off}\sum_{i_{-N}+\cdots+i_N=K}\left(\frac{k!}{i_{-N}!\cdots i_N!}\int\prod_{j=-N}^{N}dv_{jj}\frac{d^{i_j}}{dE^{i_j}}\rho(v_{jj}-E)\prod_{\ell\neq i_j}\rho(v_{\ell\ell}-E)\right.\\
&\qquad\qquad\left.\times\left|\langle e_0,(H_L^{M+1}-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^M-i\varepsilon)^{-1}e_0\rangle\right|\right)\\
&=\mathbb{E}_{off}\sum_{i_{-N}+\cdots+i_N=K}\left(\frac{k!}{i_{-N}!\cdots i_N!}\int\prod_{j=-N}^{N}dv_{jj}\frac{d^{i_j}}{dE^{i_j}}\rho(v_{jj})\prod_{\ell\neq i_j}\rho(v_{\ell\ell})\right.\\
&\qquad\qquad\left.\times\left|\langle e_0,(H_L^{M+1}-E-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^M-E-i\varepsilon)^{-1}e_0\rangle\right|\right).
\end{aligned}
\tag{5.4.39}
$$

We proceed as in the proof of Theorem 5.4.2, noting that by Hypothesis 4, the Fourier Transform of $\rho^{(n)}$ satisfies

$$\int_0^\infty|\widehat{\rho^{(n)}}(\lambda)|\lambda^s\,d\lambda<\infty \tag{5.4.40}$$

for each $n=1,\ldots,K$.

Further, we can define probability measures by

$$\tilde{\rho}^{(n)}(x)=\left(\frac{1}{\int_{\mathbb{R}}\left|\frac{d^n}{dy^n}\rho(y)\right|\,dy}\right)\left|\frac{d^n}{dx^n}\rho(x)\right|. \tag{5.4.41}$$

Thus, we can follow the steps in the proof of Theorem 5.4.2, using the localization under rank one perturbation (Theorem 4.5.1) at most K times, to obtain exponential decay for each of the terms in (5.4.39). Thus

$$
\begin{aligned}
&\frac{d^K}{dE^K}\mathbb{E}\left|\langle e_0,(H_L^{M+1}-E-i\varepsilon)^{-1}e_0\rangle-\langle e_0,(H_L^M-E-i\varepsilon)^{-1}e_0\rangle\right|\\
&\leq\sum_{i_{-N}+\cdots+i_N=K}\frac{k!}{i_{-N}!\cdots i_N!}Ce^{-\gamma(M-2L)}.
\end{aligned}
\tag{5.4.42}
$$

for some constants C and γ independent of M.

We return to the telescoping series representation

$$
\begin{aligned}
&\mathbb{E}\,\mathrm{Im}\langle e_0,(H_L^\infty-z)^{-1}e_0\rangle\\
&=\sum_{M=N}^{\infty}\left[\mathbb{E}\,\mathrm{Im}\langle e_0,(H_L^{M+1}-z)^{-1}e_0\rangle-\mathbb{E}\,\mathrm{Im}\langle e_0,(H_L^M-z)^{-1}e_0\rangle\right]\\
&\qquad+\mathbb{E}\,\mathrm{Im}\langle e_0,(H_L^N-z)^{-1}e_0\rangle,
\end{aligned}
\tag{5.4.43}
$$

to conclude that for $z = E + i\varepsilon$.

$$\left|\frac{d^K}{dE^K}\mathbb{E}\langle \operatorname{Im} e_0, (H_L^\infty - z)^{-1} e_0\rangle\right| \le \sum_{M=N}^{\infty} Ce^{-\gamma(M-2L)} + \left|\frac{d^K}{dE^K}\mathbb{E}\operatorname{Im}\langle e_0, (H_L^N - z)^{-1} e_0\rangle\right| \tag{5.4.44}$$

which is finite uniformly in $z \in \mathbb{C}^+$. Thus by Lemma 5.4.1, $n_L^\infty \in \mathcal{C}^K(\mathbb{R})$. □

5.5 Poisson Statistics

The Poisson Point Process with intensity measure κ is a random measure

$$\xi_\kappa^P \tag{5.5.1}$$

on $\mathbb{R}$ with the following properties:

1. For each interval $A \subset \mathbb{R}$, $\xi_\kappa^P(A)$ is a Poisson random variable with mean $\kappa(A)$:

$$\mathbb{P}\left\{\xi_\kappa^P(A) = k\right\} = e^{-\kappa(A)}\frac{(\kappa(A))^k}{k!}, \tag{5.5.2}$$

2. If A and B are disjoint, $\xi_\mu^P(A)$ and $\xi_\mu^P(B)$ are independent random variables.

We will show that the *eigenvalue point process* for H_L^N converges in the appropriate sense to a Poisson process.

To define the eigenvalue point process, we let $\{\lambda_j^N\}_{j=-N}^N$ denote the eigenvalues of H_L^N and pick a point $E \in \mathbb{R}$. We then consider the re-scaled eigenvalues centered at E:

$$\lambda^{*N}_j := (2N+1)\left(\lambda_j^N - E\right). \tag{5.5.3}$$

The normalization $(2N+1)$ is chosen according to the average eigenvalue spacing as determined by the Wegner estimate.

The eigenvalue point process for H_L^N centered at E is a random point measure supported on the re-scaled eigenvalues:

$$\mu_L^N(s)\,ds = \sum_{j=-N}^{N} \delta_{(2N+1)\left(\lambda_j^N - E\right)}(s)\,ds. \tag{5.5.4}$$

Theorem 5.5.1. *Let H_L^N be a random band matrix with fixed bandwidth L, and let $\{\lambda_j^N\}_{j=-N}^N$ denote the eigenvalues of H_L^N and $E \in \mathbb{R}$. Then the re-scaled eigenvalue point process $\mu_L^N(s)\,ds = \sum_{j=-N}^N \delta_{(2N+1)(\lambda_j^N - E)}(s)\,ds$ converges weakly in expectation to the Poisson point process with intensity given by the density of states at E times Lebesgue measure.*

Explicitly, weak convergence in expectation means that for each bounded and measurable function f on $\mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{E}\mu_L^N(f) = \mathbb{E}\xi_{n(E)dx}^P(f). \tag{5.5.5}$$

The proof follows the same strategy as that of Minami [19] for the Anderson tight-binding model. Lemma 1 of [19] provides a useful reduction:

Lemma 5.5.1 (Minami)**.** *Suppose μ_N is a sequence of measures satisfying*

$$\mu_N(A) \le C|A| \tag{5.5.6}$$

and μ a measure satisfying

$$\mu(A) \le C|A|. \tag{5.5.7}$$

Then a sufficient condition for weak convergence of μ_N to μ is

$$\mathbb{E}e^{-\mu_N(\varphi_z)} \longrightarrow \mathbb{E}e^{-\mu(\varphi_z)} \tag{5.5.8}$$

uniformly for all functions $\varphi_z(x) = \operatorname{Im} \dfrac{1}{x-z}$ with $z \in \mathbb{C}_+$, i.e. $\operatorname{Im} z > 0$.

The conditions (5.5.6) and (5.5.7) follow from the Wegner-type estimate. By the lemma, we can replace an arbitrary bounded and continuous function f with φ_z. This reduction is useful because we can now represent the eigenvalue point process on φ_z in terms of the trace of the resolvent:

$$\begin{aligned}
\mu_L^N(\varphi_z) &= \sum_{j=-N}^{N} \varphi_z\left((2N+1)\left(\lambda_j^N - E\right)\right) \\
&= \sum_{j=-N}^{N} \operatorname{Im} \frac{1}{\left((2N+1)\left(\lambda_j^N - E\right) - z\right)} \\
&= \frac{1}{2N+1} \sum_{j=-N}^{N} \operatorname{Im} \frac{1}{\left(\lambda_j^N - E\right) - \frac{z}{2N+1}} \\
&= \frac{1}{2N+1} \operatorname{tr}\left(H_L^N - E - \frac{z}{2N+1}\right)^{-1}.
\end{aligned} \tag{5.5.9}$$

5.5.1 Reduction to Array of Independent Point Processes

To begin the proof of Poisson Statistics, we replace the full eigenvalue point process by a sum of smaller independent point processes. We divide the set of indices $[-N, N]$ into sub-intervals of length n where $n \sim N^\alpha$ for some $0 < \alpha < 1$. We will label each subset of indices N_p for $p = 1, \dots, N^{1-\alpha}$

When N is large enough so that $N >> n >> L$, this produces $N^{1-\alpha}$ sub-matrices $H_L^{N,p}$ with band width L, and indices ranging over $N_p \times N_p$. The sub-matrices are independent of each other, and thus the eigenvalue statistics of each sub-matrix is independent of the statistics of any other sub-matrix.

The first step of the proof of Poisson statistics is to show that in the limit as $N \to \infty$ we can replace the eigenvalue point process of H_L^N with the sum of the eigenvalue point processes for $H_L^{N,p}$. To accomplish this we use fractional moment bounds derived in Theorem 4.1.3.

Lemma 5.5.2. *Let $\{\lambda_j^{N,p}\}_{j\in N_p}$ denote the eigenvalues of $H_L^{N,p}$. Let $E\in\mathbb{R}$. Define the eigenvalue point process for $H_L^{N,p}$ by*

$$\mu_L^{N,p}(s)\,ds=\sum_{j\in N_p}\delta_{(2N+1)(\lambda_j^{N,p}-E)}(s)\,ds. \tag{5.5.10}$$

Then for $\varphi_z(x)=\operatorname{Im}\dfrac{1}{x-z}$ with $z\in\mathbb{C}_+$,

$$\lim_{N\to\infty}\left|\mathbb{E}\left\{\mu_L^N(\varphi_z)\right\}-\mathbb{E}\left\{\sum_p\mu_L^{N,p}(\varphi_z)\right\}\right|=0. \tag{5.5.11}$$

Note that the scaling for the sub-matrix point processes retains the scaling of the point process for the full matrix.

Proof. Then

$$\begin{aligned}\mathbb{E}\left\{\mu_L^{N,p}(\varphi_z)\right\}&=\mathbb{E}\left\{\operatorname{tr}\varphi_z\left((2N+1)\left(H_L^{N,p}-E\right)\right)\right\}\\&=\mathbb{E}\left\{\sum_{j\in N_p}\left\langle e_j,\ \operatorname{Im}\frac{1}{(2N+1)\left(H_L^{N,p}-E\right)-z}e_j\right\rangle\right\}\\&=\frac{1}{2N+1}\mathbb{E}\left\{\sum_{j\in N_p}\left\langle e_j,\ \operatorname{Im}\frac{1}{\left(H_L^{N,p}-E\right)-\frac{z}{2N+1}}e_j\right\rangle\right\}.\end{aligned} \tag{5.5.12}$$

Similarly,

$$\mathbb{E}\left\{\mu_L^N(\varphi_z)\right\}=\frac{1}{2N+1}\mathbb{E}\left\{\sum_{j=-N}^{N}\left\langle e_j,\ \operatorname{Im}\frac{1}{\left(H_L^{N,p}-E\right)-\frac{z}{2N+1}}e_j\right\rangle\right\}. \tag{5.5.13}$$

Note that

$$\begin{aligned}&\mathbb{E}\left|\left\langle e_j,\left(H_L^{N,p}-E-\frac{z}{2N+1}\right)^{-1}e_j\right\rangle-\left\langle e_j,\left(H_L^{N}-E-\frac{z}{2N+1}\right)^{-1}e_j\right\rangle\right|\\=&\mathbb{E}\left|\left\langle e_j,\left(H_L^{N}-E-\frac{z}{2N+1}\right)^{-1}(H_L^N-H_L^{N,p})\left(H_L^{N,p}-E-\frac{z}{2N+1}\right)^{-1}e_j\right\rangle\right|\\\le&\mathbb{E}\sum_{\substack{|\ell-\partial N_p|\le L\\|k-\ell|\le L}}\left|\left\langle e_j,\left(H_L^{N}-E-\frac{z}{2N+1}\right)^{-1}e_k\right\rangle v_{k\ell}\right.\\&\qquad\qquad\left.\times\left\langle e_\ell,\left(H_L^{N,p}-E-\frac{z}{2N+1}\right)^{-1}e_j\right\rangle\right|.\end{aligned} \tag{5.5.14}$$

For j such that $|j-\ell| > \beta\log(N)$–for sufficiently large β to be chosen later–we can apply the Cauchy-Schwarz inequality to bound the quantity by

$$\sum_{\substack{|\ell-\partial N_p|\leq L\\ |k-\ell|\leq L}} \mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - E - \frac{z}{2N+1}\right)^{-1} e_k\right\rangle v_{k\ell}\right|^2\right\}^{1/2}$$
$$\times\mathbb{E}\left\{\left|\left\langle e_\ell, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right|^2\right\}^{1/2}. \tag{5.5.15}$$

Using the elementary resolvent bound $\left\|\left(H_L^N - z\right)^{-1}\right\| \leq \frac{1}{|\operatorname{Im} z|}$ on both terms, we have the bound

$$\left(\frac{2N+1}{\operatorname{Im} z}\right)^{2-\frac{s}{2}} \sum_{\substack{|\ell-\partial N_p|\leq L,\\ |k-\ell|\leq L}} \mathbb{E}\left\{|v_{k\ell}|^2\right\}^{\frac{1}{2}} \mathbb{E}\left\{\left|\left\langle e_\ell, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right|^{2s}\right\}^{\frac{1}{2}}. \tag{5.5.16}$$

We can now apply the localization estimate and the finite moment condition to bound above:

$$\left(\frac{2N+1}{\operatorname{Im} z}\right)^{2-s/2} C \sum_{\substack{|\ell-\partial N_p|\leq L,\\ |k-\ell|\leq L}} \left(Ce^{-\gamma(|k-\ell|-2L)}\right)^{1/2}$$
$$\leq \left(\frac{2N+1}{\operatorname{Im} z}\right)^{2-s/2} 2\left((2N+1)-\beta\log N\right) Ce^{2\gamma L} N^{-\beta\cdot\gamma} \tag{5.5.17}$$

Taking β large enough, we get that

$$\sum_j \mathbb{E}\left|\left\langle e_j, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle - \left\langle e_j, \left(H_L^{N} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right| \tag{5.5.18}$$

converges to 0 as $N\to\infty$, where the sum is over indices j such that the distance from j to the edge of their respective interval is greater than $\beta\log N$.

For the remaining indices, we apply the same resolvent bound as well as the *a* *priori* bound:

$$\begin{aligned}
&\mathbb{E}\left|\left\langle e_j, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle - \left\langle e_j, \left(H_L^{N} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right| \\
&\leq \mathbb{E}\left|\left\langle e_j, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right| + \mathbb{E}\left|\left\langle e_j, \left(H - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right| \\
&\leq \left(\frac{2N+1}{\operatorname{Im} z}\right)^{1-s} \left(\mathbb{E}\left|\left\langle e_j, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right|^s \right. \\
&\qquad\qquad\qquad\qquad \left. + \mathbb{E}\left|\left\langle e_j, \left(H - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle\right|^s\right) \\
&\leq \left(\frac{2N+1}{\operatorname{Im} z}\right)^{1-s} \frac{CL^{s/2}}{1-s}.
\end{aligned} \tag{5.5.19}$$

Thus, for indices $j(p)$ within $\beta \log N$ of boundary of each N_p, we have

$$\begin{aligned}
&= \frac{1}{2N+1} \operatorname{Im} \sum_p \sum_{j(p)} \left\langle e_j, \left(H_L^{N,p} - E - \frac{z}{2N+1}\right)^{-1} e_j\right\rangle \\
&\qquad\qquad \leq \frac{1}{2N+1} \beta \log N \left(\frac{2N+1}{\operatorname{Im} z}\right)^{1-s} C_{L,s} \to 0.
\end{aligned} \tag{5.5.20}$$

Combining with the decay for indices at distance greater than $\beta \log N$ from the boundary of each N_p, we have

$$\lim_{N\to\infty} \left| \mathbb{E}\left\{\mu_L^N(\varphi_z)\right\} - \mathbb{E}\left\{\sum_p \mu_L^{N,p}(\varphi_z)\right\}\right| = 0. \tag{5.5.21}$$

□

From this lemma, the identity

$$\left|e^X - e^Y\right| \leq |X - Y|, \tag{5.5.22}$$

and Lemma 5.5.1, we can conclude that the point process μ_L^N and sum of the array of point processes $\sum_p \mu_L^{N,p}$ have the same weak limit.

5.5.2 Convergence of Array of Point Processes

We now can prove Poisson statistics for the random band matrix by showing that the sum of independent point processes for its diagonal sub-matrices converge in expectation to a Poisson process.

To show that the independent array

$$\sum_p \mu_L^{N,p} \tag{5.5.23}$$

of point processes converges in expectation to a Poisson process it suffices to demonstrate the following properties [10]:

Theorem 5.5.2 (Daley, Vere-Jones Criteria)**.**

1. *The array is uniformly asymptotically negligible:*
   For each bounded interval A in $\mathbb{R}$,

$$\lim_{N\to\infty} \sup_p \mathbb{P}\left\{\mu_L^{N,p}(A) \geq 1\right\} = 0. \tag{5.5.24}$$

2. *Convergence to the density of states:*

$$\lim_{N\to\infty} \sum_p \mathbb{P}\left\{\mu_L^{N,p}(A) \geq 1\right\} = n(E)|A|. \tag{5.5.25}$$

3. *Minami estimate:*

$$\lim_{N\to\infty} \sum_p \mathbb{P}\left\{\mu_L^{N,p}(A) \geq 2\right\} = 0. \tag{5.5.26}$$

Before moving to the proof we note the following consequence of the Minami-type estimate (Theorem 3.3.4) which appeared [19].

Proposition 5.5.1. *Let X_N be a sequence of random variables taking values in $\mathbb{N}$. Further suppose $\mathbb{E}\{X_N(X_N-1)\} \to 0$ as $N \to \infty$. Then*

$$\lim_{N\to\infty} |\mathbb{E}\{X_N\} - \mathbb{P}\{X_N \geq 1\}| = 0. \tag{5.5.27}$$

Proof. Since X_N takes values in $\mathbb{N}$, we can write

$$\mathbb{E}\{X_N\} = \sum_{k=1}^{\infty} k \cdot \mathbb{P}\{X_N = k\}, \tag{5.5.28}$$

so

$$\begin{aligned}
&\mathbb{E}\{X_N\} - \mathbb{P}\{X_N \geq 1\} \\
&= \sum_{k=1}^{\infty} k \cdot \mathbb{P}\{X_N = k\} - \sum_{k=1}^{\infty} \mathbb{P}\{X_N = k\} \\
&= \sum_{k=1}^{\infty} (k-1)\mathbb{P}\{X_N = k\} \\
&\leq \sum_{k=1}^{\infty} k(k-1)\mathbb{P}\{X_N = k\} \\
&\leq E\{X_N(X_N-1)\} \to 0.
\end{aligned} \tag{5.5.29}$$

□

Lemma 5.5.3. *With the above definitions,*

$$\mathbb{E}\left\{\sum_p \mu_L^{N,p}\right\} \tag{5.5.30}$$

converges to a Poisson point process with intensity measure $n_L^\infty(E)dx$, where $n_L^\infty(E)$ is the pointwise limit of the density of states functions evaluated at E.

Proof. We prove this by demonstrating items (1), (2), and (3) of the Daley, Vere-Jones Criteria. Items (1) and (3) follow from the Wegner (Theorem 3.3.3) and Minami (Theorem 3.3.4) estimates respectively. Item (2) will require uniform convergence and smoothness of the density of states functions (Theorems 5.3.1 and 5.4.2).

Proof of (1): For any p,

$$\begin{aligned}\mathbb{P}\left\{\mu_L^{N,p}(A) \geq 1\right\} &\leq \mathbb{E}\left\{\operatorname{tr}\chi_A\left((2N+1)(H_L^{N,p}-E)\right)\right\}\\ &= \mathbb{P}\left\{\operatorname{tr}\chi_{E+\frac{A}{2N+1}}\left((H_L^{N,p})\right)\right\}\\ &\leq C\frac{|A|}{2N+1}N^\alpha \to 0.\end{aligned} \tag{5.5.31}$$

Proof of (3): We have

$$\begin{aligned}&\sum_p \mathbb{P}\left\{\mu_L^{N,p}(A) \geq 2\right\}\\ &\quad= \sum_p \mathbb{P}\left\{\operatorname{tr}\chi_A\left((2N+1)(H_L^{N,p}-E)\right) \geq 2\right\}\\ &\quad= \sum_p \mathbb{P}\left\{\operatorname{tr}\chi_{E+\frac{A}{2N+1}}\left(H_L^{N,p}\right) \geq 2\right\}\\ &\quad\leq \sum_p \frac{1}{2N+1}\mathbb{E}\left\{\operatorname{tr}\chi_{E+\frac{A}{2N+1}}\left(H_L^{N,p}\right)\operatorname{tr}\chi_{E+\frac{A}{2N+1}}\left(H_L^{N,p}\right)-1\right\}\\ &\quad\leq \frac{2N+1}{N^\alpha}\cdot\left(C\frac{|A|}{2N+1}N^\alpha\right)^2 \longrightarrow 0.\end{aligned} \tag{5.5.32}$$

Proof of (2): Suppose A is a bounded interval. By the previous Proposition, we can substitute

$$\mathbb{P}\{\mu_L^{N,p}(A) \geq 1\} \tag{5.5.33}$$

with

$$\mathbb{E}\{\mu_L^{N,p}(A)\} \tag{5.5.34}$$

in the $N \to \infty$ limit.

Further, from Lemma 5.5.2 we can replace $\sum_p \mathbb{E}\{\mu_L^{N,p}(A)\}$ with $\mathbb{E}\left\{\mu^N(A)\right\}$ in the $N \to \infty$ limit.

For $\varphi_z(x) = \operatorname{Im}\frac{1}{x-z}$,

$$
\begin{aligned}
\mathbb{E}\left\{\mu_L^N(\varphi_z)\right\} &= \mathbb{E}\left\{\sum_{j=-N}^{N} \delta_{(2N+1)\left(\lambda_j^N - E\right)}(\varphi_z)\right\} \\
&= \mathbb{E}\left\{\operatorname{tr}\varphi_z\left((2N+1)\left(H_L^N - E\right)\right)\right\} \\
&= \mathbb{E}\left\{\sum_{j=-N}^{N}\left\langle e_j,\ \operatorname{Im}\frac{1}{(2N+1)\left(H_L^N - E\right) - z} e_j\right\rangle\right\} \\
&= \mathbb{E}\left\{\frac{1}{2N+1}\sum_{j=-N}^{N}\operatorname{Im}\left\langle e_j,\ \frac{1}{\left(H_L^N - E\right) - \frac{z}{2N+1}} e_j\right\rangle\right\} \\
&= \int \operatorname{Im}\frac{1}{x - E - \frac{z}{2N+1}} d\nu_L^N(x)
\end{aligned}
$$

where ν_L^N is the density of states measure for H_L^N. Let $z(N) = \frac{z}{2N+1}$. Then, evaluating the imaginary part and re-writing the measure in terms of the density of states function, we get

$$
\int \frac{1}{\left(\frac{x-E-\operatorname{Re} z(N)}{\operatorname{Im} z(N)}\right)^2 + 1}\frac{1}{\operatorname{Im} z(N)} n_L^N(x)\, dx. \tag{5.5.35}
$$

Making a change of variables this becomes

$$
\int \frac{1}{u^2+1} n_L^N\left(E + u\operatorname{Im} z(N) + \operatorname{Re} z(N)\right)\, du. \tag{5.5.36}
$$

Note that n_L^N is pointwise uniformly bounded in N, by the density of states representation (3.4.1) and the spectral averaging estimate Lemma 3.2.1. Thus, there exists a constant C such that

$$
\frac{1}{u^2+1} n_L^N\left(E + u\operatorname{Im} z(N) + \operatorname{Re} z(N)\right) \le \frac{C}{u^2+1} \in L^1(\mathbb{R}). \tag{5.5.37}
$$

We can therefore apply the Dominated Convergence Theorem to bring the limit inside the integral:

$$
\begin{aligned}
&\lim_{N\to\infty}\int \frac{1}{u^2+1} n_L^N\left(E + u\operatorname{Im} z(N) + \operatorname{Re} z(N)\right)\, du \\
&= \int \frac{1}{u^2+1}\lim_{N\to\infty} n_L^N\left(E + u\operatorname{Im} z(N) + \operatorname{Re} z(N)\right)\, du.
\end{aligned} \tag{5.5.38}
$$

We have

$$
\begin{aligned}
&\lim_{N\to\infty} \left|n_L^N\left(E+u\operatorname{Im} z(N)+\operatorname{Re} z(N)\right)-n_L^\infty(E)\right| \\
&\leq \lim_{N\to\infty}\Bigg(\left|n_L^N\left(E+u\operatorname{Im} z(N)+\operatorname{Re} z(N)\right)-n_L^\infty\left(E+u\operatorname{Im} z(N)+\operatorname{Re} z(N)\right)\right| \\
&\qquad\qquad + \left|n_L^\infty\left(E+u\operatorname{Im} z(N)+\operatorname{Re} z(N)\right)-n_L^\infty(E)\right|\Bigg) \\
&= 0, \qquad (5.5.39)
\end{aligned}
$$

where the first term goes to 0 by the uniform convergence of the density of states functions n_L^N to n_L^∞ (Theorem 5.3.1), and the second term by the continuity of n_L^∞ (Theorem 5.4.2),

Thus evaluating the limit in (5.5.38), we have

$$
\int \frac{1}{u^2+1} n_L^\infty(E)\,du = \pi n_L^\infty(E) \qquad (5.5.40)
$$

where $\pi = \|\varphi_z\|_1$. Since the convergence is uniform for φ_z with $z \in \mathbb{C}_+$, the convergence holds holds for characteristic functions of intervals. Thus we have condition (2).

□

By combining Lemmas 5.5.2 and 5.5.3, and applying Lemma 5.5.1, we have Theorem 5.5.1, the weak convergence of the eigenvalue point process to the Poisson Point Process.

Chapter 6 Analysis for Wegner Orbital Model

We now turn our attention to the Wegner Orbital Model defined in (2.2.1). We will be able to achieve results similar to those for the Random Band Matrix when the strength of the disorder is high enough. In particular, we will use fractional moment bounds and eigenvalue counting estimates to achieve convergence of the re-scaled eigenvalue point process to a Poisson point process. In many cases, we refer directly to proofs in Chapters 3, 4, and 5 instead of redoing the analysis in this new context.

6.1 Model

Recall the Hamiltonian for the Wegner Orbital is an operator H_L on the Hilbert Space $\ell^2(\mathbb{Z}^d : \mathbb{C}^L)$, given by

$$H_L(\lambda) = H_0 + \lambda \cdot V^{GOE(L)} = \Delta \otimes I + \lambda \sum_{x \in \mathbb{Z}^d} V(x)\, P_x, \tag{6.1.1}$$

where each $V(x)$ is independently sampled from the $L \times L$ Gaussian Orthogonal Ensemble (1.2.2). Explicitly, for $u \in \ell^2(\mathbb{Z}^d : \mathbb{C}^L)$,

$$\left[H_L(\lambda)u\right](x) = \sum_{y\colon \|x-y\|=1} u(y) + \lambda V(x)u(x). \tag{6.1.2}$$

Here we include a coupling constant λ, which measures the strength of the disorder. we will see that for λ above a certain threshold, $H_L(\lambda)$ becomes localized at all energies.

The standard basis for $\ell^2(\mathbb{Z}^d : \mathbb{C}^L)$, can be written as a tensor product of a delta function on $\mathbb{Z}^d$ with a standard basis vector in $\mathbb{C}^L$:

$$\left\{\delta_x \otimes e_j \,:\, x \in \mathbb{Z}^d,\ j = 1, \ldots, L\right\}. \tag{6.1.3}$$

We shorten the notation where convenient as

$$\delta_{x,j} := \delta_x \otimes e_j. \tag{6.1.4}$$

Since we will not be changing the number of orbitals L, we will drop the L from the notation and let

$$H := H_L(\lambda). \tag{6.1.5}$$

Note, however, that there is considerable interest in taking limit as the number of orbitals goes to infinity in the literature [15, 27].

If Λ is a subset of $\mathbb{Z}^d$, and P_Λ the orthogonal projection onto sites in Λ, then we let

$$H^\Lambda := P_\Lambda H P_\Lambda \tag{6.1.6}$$

be the restriction of H to Λ with simple boundary conditions.

6.2 Eigenvalue Counting Estimates

In this section, we collect several estimates that are used in the analysis in this chapter. The first are the spectral averaging estimates which first appeared as Lemma 3.2.1 and Lemma 3.2.2 in Chapter 3.

Lemma 6.2.1 (Spectral Averaging)**.** *Let $x \in \mathbb{Z}^d$ and $j \in \{1, \ldots, L\}$. Further let Λ be any subset of $\mathbb{Z}^d$.*

1. *For each $z \in \mathbb{C}$, there exists a constant C such that*

$$\mathbb{E}\,\mathrm{Im}\left\langle \delta_{x,j}(H^{\Lambda} - z)^{-1}\delta_{x,j}\right\rangle \leq C. \tag{6.2.1}$$

2. *Let A be a finite interval in $\mathbb{R}$ and $E_A(H^{\Lambda})$ the spectral projection of H^{Λ} onto A. Then there is a constant C such that*

$$\mathbb{E}\left\langle \delta_{x,j}, E_A(H^{\Lambda})\delta_{x,j}\right\rangle \leq C|A|. \tag{6.2.2}$$

Note that we can take $\Lambda = \mathbb{Z}^d$ above.

We also will need a Wegner Estimate and a Minami Estimate.

Theorem 6.2.1 (Wegner Estimate)**.** *Let Λ be a finite subset of $\mathbb{Z}^d$, and A an interval in $\mathbb{R}$. Then there exists a constant C such that*

$$\mathbb{P}\{\,\mathrm{tr}\,E_A(H^{\Lambda}) \geq 1\} \leq \mathbb{E}\{\,\mathrm{tr}\,E_A(H^{\Lambda})\} \leq CL|\Lambda||A|. \tag{6.2.3}$$

Theorem 6.2.2 (Minami Estimate)**.** *Let Λ be a finite subset of $\mathbb{Z}^d$, and A an interval in $\mathbb{R}$. Then there exists a constant C such that*

$$\mathbb{P}\{\,\mathrm{tr}\,E_A(H^{\Lambda}) \geq 2\} \leq \mathbb{E}\{\,\mathrm{tr}\,E_A(H^{\Lambda})(\,\mathrm{tr}\,E_A(H^{\Lambda}) - 1)\} \leq (\pi M\|\rho\|_{\infty}|A|)^2 \tag{6.2.4}$$

.

The Wegner and Minami estimates follow from [22, Theorem 3], which provides the more general eigenvalue counting estimate:

$$\mathbb{E}\prod_{\ell=0}^{m-1}\left(\,\mathrm{tr}\,E_A(H_L^{\Lambda}) - \ell\right) \leq (CL|\Lambda||\,A|)^m. \tag{6.2.5}$$

We note that the Wegner and Minami estimates derived in Chapter 3 for an operator with diagonal disorder also apply to the Orbital Model, although with less optimal dependence on L.

6.3 Density of States Measure

The density of states measure is given by

$$\nu(\varphi) := \lim_{N\to\infty} \nu_N(\varphi) := \lim_{N\to\infty} \frac{1}{L|\Lambda_N|} \operatorname{tr}\left(\chi_{\Lambda_N}\varphi(H)\chi_{\Lambda_N}\right) \tag{6.3.1}$$

for $\varphi \in C_0^\infty(\mathbb{R})$. Note that we can identify the probability space of operators with the probability space $\Omega \cong GOE(L)^{\mathbb{Z}^d}$. Recall $GOE(L)$ is the probability space of symmetric $L \times L$ matrices with probability density function proportional to $e^{-\frac{1}{4}\operatorname{tr} X^2}$. We let $\mathbb{P}_{GOE(L)}$ be the probability measure of $GOE(L)$, and $\mathbb{P}$ the probability measure of Ω, which is an infinite product of $\mathbb{P}_{GOE(L)}$ measures.

Lemma 6.3.1. *The random variables $\langle \delta_{x,j}, \varphi(H)\delta_{x,j}\rangle$ form an ergodic family under a semi-direct product of translation in $\mathbb{Z}^d$ with rotation in $\mathbb{C}^L$.*

Proof. Consider the family of operators $S_{i,\theta} := T_i \otimes R_\theta$ where T_i is a shift by $i \in \mathbb{Z}^d$ and R_θ is conjugation by the matrix, r_θ, that rotates by the angle $\theta \in \mathbf{S}^{N-1}$. Note that $S_{i,\theta}$ is unitary on $\ell^2(\mathbb{Z}^d : \mathbb{C}^L)$.

Since the GOE is invariant under conjugation by orthogonal matrices, and the samples from the GOE are taken independently of their location in the lattice, $\{S_{i,\theta}\}$ is measure-preserving on $\Omega \cong GOE(L)^{\mathbb{Z}^d}$.

Now suppose $Q \subset \Omega$ is invariant under the family $\{S_{i,\theta}\}$. Let P_x be the (rank L) projection onto the random variables at site x in $\mathbb{Z}^d$. Then we can write

$$\mathbb{P}\{Q\} = \mathbb{P}\{{S_{i,\theta}}^{-1} Q\} = \mathbb{P}\left\{\sum_x P_x\left({S_{i,\theta}}^{-1} Q\right)\right\}. \tag{6.3.2}$$

Since the samples from the GOE are taken independently at each site, fixing θ, $P_x\left({S_{i,\theta}}^{-1} Q\right) = P_0\left({S_{i-x,\theta}}^{-1} Q\right)$ which holds for each x in $\mathbb{Z}^d$. Note that

$$P_0\left({S_{i-x,\theta}}^{-1} Q\right) = {r_\theta}^* P_0(Q) r_\theta \tag{6.3.3}$$

which can be interpreted as a set in the probability space $GOE(L)$.

Thus,

$$\begin{aligned}
\mathbb{P}\{Q\} &= \mathbb{P}\left\{\sum_{x\in\mathbb{Z}^d} P_0({S_{i,\theta}}^{-1} Q)\right\} \\
&= \prod_{x\in\mathbb{Z}^d} \mathbb{P}_{GOE(L)}\left\{{r_\theta}^* P_0(Q) r_\theta\right\} \\
&= \prod_{x\in\mathbb{Z}^d} \mathbb{P}_{GOE(L)} P_0(Q).
\end{aligned} \tag{6.3.4}$$

In the infinite product we thus have $\mathbb{P}\{Q\} = 1$ if $\mathbb{P}_{GOE(L)}\{P_0(Q)\} = 1$, and $\mathbb{P}\{Q\} = 0$ if $\mathbb{P}_{GOE(L)}P_0(Q) < 1$. Thus each invariant set under the family $\{S_{i,\theta}\}$ has either 0 or full measure.

Finally, note that

$$\begin{aligned}\langle \delta_{x,j}, S_{i,\theta}(H)\delta_{x,j}\rangle &= \langle \delta_x \otimes e_j, R_\theta \otimes T_i(H)\delta_x \otimes e_j\rangle \\ &= \langle T_i^*(\delta_x) \otimes r_\theta(e_j), HT_i^*(\delta_x) \otimes r_\theta(e_j)\rangle \\ &= \langle \delta_{x-i} \otimes \theta, H\delta_{x-i} \otimes \theta\rangle \\ &= \langle \delta_{S_{i,\theta}(x,j)}, H\delta_{S_{i,\theta}(x,j)}\rangle\end{aligned} \tag{6.3.5}$$

and so the family $\langle \delta_{x,j}, H\delta_{x,j}\rangle$ is ergodic under the action of $\{S_{i,\theta}\}$. By Lemma 4.5 of [16], this can be extended to

$$\langle \delta_{x,j}, f(H)\delta_{x,j}\rangle \tag{6.3.6}$$

for bounded measurable f by treating the expression as integral of f with respect to the spectral measure of H corresponding to $\delta_{x,j}$. □

From this, we can prove:

Lemma 6.3.2. *The density of states measure $\nu(\cdot)$ is equal to the measure*

$$\mathbb{E}\{\langle \delta_{0,1}, E_H(\cdot)\delta_{0,1}\rangle\} \tag{6.3.7}$$

where $E_H(\cdot)$ is the resolution of the identity on H.

Proof.

$$\frac{1}{L|\Lambda_N|} \operatorname{tr}\,(P_{\Lambda_N}\varphi(H)P_{\Lambda_N}) = \frac{1}{L|\Lambda_N|}\sum_{j=1}^{N}\sum_{x\in\mathbb{Z}^d}\langle \delta_{x,j}, \varphi(H)\delta_{x,j}\rangle. \tag{6.3.8}$$

Since $\{\langle \delta_{x,j}, \varphi(H)\delta_{x,j}\rangle\}$ form an ergodic family of random variables, the Birkhoff Ergodic Theorem (1.1.5) implies the sum converges to

$$\mathbb{E}\langle \delta_{0,1}, \varphi(H)\delta_{0,1}\rangle \tag{6.3.9}$$

as $N \to \infty$. Thus the density of states measure is given by

$$\nu(\varphi) = \mathbb{E}\langle \delta_{0,1}, \varphi(H)\delta_{0,1}\rangle. \tag{6.3.10}$$

□

The above formulation of the density of states began with a function of the infinite volume operator H, and then restricted to finite volume. As with the random band matrix, a different formulation of the density of states begins with finite volume operators. These two formulations provide an equivalent infinite volume density of states defined in the limit as $\Lambda_N \nearrow \mathbb{Z}^d$.

Lemma 6.3.3 (Density of States for the Orbital Model)**.** *Let $\Lambda_N = \{-N \ldots, N\}^d \subset \mathbb{Z}^d$. Define a measure for each N by*

$$\tilde{\nu}_N(\varphi) = \frac{1}{L|\Lambda_N|}\mathbb{E}\operatorname{tr}\varphi(H^{\Lambda_N}). \tag{6.3.11}$$

Then

$$\lim_{N\to\infty}\tilde{\nu}_N(\varphi) = \nu(\varphi) = \mathbb{E}\langle \delta_{0,1}, \varphi(H)\delta_{0,1}\rangle. \tag{6.3.12}$$

for each $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$.

The proof is similar to the proof of Theorem 3.1.1 in Chapter 3.

Lemma 6.3.4. *The density of states measure ν is absolutely continuous with respect to Lebesgue measure.*

Proof. As with the band matrix, this result follows from the Wegner-type estimate (3.3.1).

For any E in $\mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\langle \delta_{0,1}, \chi_{[E-\varepsilon,E+\varepsilon]}(H)\delta_{0,1}\rangle \\
&= \lim_{N\to\infty} \frac{1}{L|\Lambda_N|}\mathbb{E}\operatorname{tr}\chi_{[E-\varepsilon,E+\varepsilon]}(H^{\Lambda_N}) \\
&\leq \lim_{N\to\infty} \frac{1}{L|\Lambda_N|}\cdot CL|\Lambda_N|\varepsilon \\
&= C\varepsilon.
\end{aligned} \tag{6.3.13}$$

□

As with the band matrix, we note that since the density of measure is a.c. with respect to Lebesgue measure, the integrated density of states defined by $N(E) = \nu((-\infty, E])$ is differentiable almost everywhere. Its derivative, the density of states function,

$$n(E) = \frac{dN}{dx}(E) \tag{6.3.14}$$

will be the intensity of the Poisson point process describing the eigenvalue point process centered at E. Although the Wegner Estimate only guarantees n is defined almost everywhere, it is differentiable as in the band matrix with Gaussian disorder case.

6.4 Localization and Convergence of Density of States

We now go through the major results on fractional moment bounds for H_L and the density of states functions.

6.4.1 Fractional Moment Bounds

Let B be a bounded linear operator on $\ell^2(\mathbb{Z}^d : \mathbb{C}^L)$. We then let $B(x, y)$ denote the x, y block of B. That is, if P_x and P_y are the rank L projections onto the sites $x, y \in \mathbb{Z}^d$ respectively, then $B(x, y)$ the $L \times L$ matrix which is the non-trivial block of $P_x B P_y$. Thus, in terms of matrix elements, we have for $j, k \in \{1, \ldots, L\}$,

$$[B(x, y)]_{jk} = \langle \delta_{x,j}, B\delta_{y,k}\rangle \,. \tag{6.4.1}$$

We let $\|\cdot\|$ be the matrix norm of such an $L \times L$ matrix.

In [22], the authors established the following fractional moment bound for the Wegner Orbital Model:

Theorem 6.4.1. *[22, Theorem 1] Suppose $H = H_L(\lambda) = H_0 + \lambda \cdot V^{GOE(L)}$ is the Wegner Orbital Hamiltonian on $\ell^2(\mathbb{Z}^d; \mathbb{C}^L)$ and $\Lambda \subset \mathbb{Z}^d$, a box of finite size. Define*

$$H^\Lambda = P_\Lambda \, H \, P_\Lambda. \tag{6.4.2}$$

Suppose $0 < s < 1$. Then there exists a λ_0 such that for all $\lambda > \lambda_0$, $x, y \in \Lambda$, $j, k = 1, \ldots, L$, and $E \in [-r, r]$

$$\mathbb{E}\left\{\left\|(H^\Lambda - E)^{-1}(x, y)\right\|^s\right\} \leq Ce^{-\gamma|x-y|}, \tag{6.4.3}$$

for come constants C and γ.

The constants C and γ in the theorem depend on s, λ_0, d, and r. As with the random band matrix, in order to perform our analysis on the local eigenvalue statistics, we will need to push this estimate into the upper complex half-plane.

Theorem 6.4.2. *Suppose $0 < s < 1$ and $\Lambda \subset \mathbb{Z}^d$ a finite box. Then there exists a λ_0 such that for all $\lambda > \lambda_0$, $x, y \in \mathbb{Z}^d$, $j, k = 1, \ldots, L$, and $z \in \mathbb{C}$.*

$$\mathbb{E}\left\{\left\|(H^\Lambda - z)^{-1}(x, y)\right\|^s\right\} \leq Ce^{-\gamma|x-y|}, \tag{6.4.4}$$

for come constants C and γ.

Proof. As with the band matrix, the expectation of the fractional moments of the matrix norm is subharmonic in z, and so by the Poisson kernel representation, the bound along the real line carries through to the upper half plane. $\square$

We have an immediate corollary in terms of matrix elements:

Corollary 6.4.1. *Suppose $0 < s < 1$ and $\Lambda \subset \mathbb{Z}^d$ a finite box. Then there exists a λ_0 such that for all $\lambda > \lambda_0$, $x, y \in \mathbb{Z}^d$, $j, k = 1, \ldots, L$, and $z \in \mathbb{C}$.*

$$\mathbb{E}\left\{\left|\left\langle \delta_{x,j}, (H^\Lambda - z)^{-1}\delta_{y,k}\right\rangle\right|^s\right\} \leq Ce^{-\gamma|x-y|}, \tag{6.4.5}$$

for come constants C and γ.

For the remainder of the chapter we assume λ is large enough so that fractional moment bounds (6.4.4) hold. Because of this, we treat H as though $\lambda = 1$ in the computations, although in practice λ may have to much larger, and so each constant depends non-trivially on λ.

6.4.2 Convergence of Density of States

The proof of convergence of the density of states functions for the orbital model follows the same pathway as the proof for the fixed width band matrix. Because of this, we only emphasize the differences and do not carry through the full analysis.

Theorem 6.4.3 (Fractional Power Estimate)**.** *Suppose $M > N$ and $\varepsilon > 0$. Then for any $0 < s < 1$, $x \in \Lambda_N$, $j \in \mathbb{C}^L$, there is a constant C such that*

$$\begin{aligned}&\mathbb{E}\left|\langle\delta_{x,j}, \left[(H^{\Lambda_N} - z)^{-1} - H^{\Lambda_M}\right]\delta_{x,j}\rangle\right| \\ \leq & C\,\mathbb{E}\left|\Psi_{x,j}^T\left[\left(\tilde{H}^{\Lambda_M} - z\right)^{-1} - \left(\tilde{H}^{\Lambda_N} - z\right)^{-1}\right]\Psi_{x,j}\right|^s\end{aligned} \tag{6.4.6}$$

where $\Psi_{x,j}$, $\tilde{H}^N$, and $\tilde{H}^M$ are defined through the Schur Complement Formula:
For $P = |\delta_{x,j}\rangle\langle\delta_{x,j}|$, and $Q = 1 - P$,

$$\begin{aligned}\Psi_{x,j} &= Q\,H^{\Lambda_N}\,P\delta_{x,j} \\ \tilde{H}^{\Lambda_N} &= Q\,H^{\Lambda_N}\,Q.\end{aligned} \tag{6.4.7}$$

The inverse is taken over the range of the operator with all other matrix elements set to 0*.*

Proof. Since the proof relies only on averaging along Gaussian random variable $V(x)_{jj}$, the only difference in the analysis between the orbital model and the band matrix is the definition of $\Psi_{x,j}$ and the reduced matrices H^{Λ_N} and H^{Λ_M}, and so the result follows directly from the proof of Theorem 5.2.1. □

What remains is to show that the we obtain sufficient exponential decay with these newly formed $\Psi_{x,j}$ defined for the Orbital Model.

Proposition 6.4.1. *For $\tilde{H}^{\Lambda_M} := \tilde{H}^{\Lambda_M}(x,j)$ be defined as in the previous theorem and $0 < s < 1/3$, there exist constants $\tilde{C}$ and $\tilde{\gamma}$, depending on d and L, such that*

$$\mathbb{E}|\langle\delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle|^s \leq \tilde{C}e^{-\tilde{\gamma}|x-y|}. \tag{6.4.8}$$

Proof. We proceed as in the proof of the corresponding Proposition for the band matrix, by comparing to the original operator H^{Λ_M}. By the resolvent identity

$$\begin{aligned}\langle\delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle &= \langle\delta_{x,j}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle \\ &\quad + \langle\delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}(\tilde{H}^{\Lambda_M} - H^{\Lambda_M})(H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle.\end{aligned} \tag{6.4.9}$$

Here the only non-zero entries of the matrix $(H^{\Lambda_M} - \tilde{H}^{\Lambda_M})$ are given by:

$$\begin{aligned}H^{\Lambda_M}(x,x)_{j\ell} &= V(x)_{j\ell} \\ H^{\Lambda_M}(x,x)_{\ell j} &= V(x)_{\ell j} \\ H^{\Lambda_M}(x,v)_{jj} &= 1 \quad \text{if } |x-v| = 1 \\ H^{\Lambda_M}(v,x)_{jj} &= 1 \quad \text{if } |x-v| = 1.\end{aligned} \tag{6.4.10}$$

Setting $\mathcal{T} := \mathcal{T}(j,x)$ equal to the set of indices corresponding to the non-zero entries above, we have

$$\begin{aligned}\langle \delta_{x,j}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle + \langle \delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}(H^{\Lambda_M} - \tilde{H}^{\Lambda_M})(H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle \\ = \sum_{i,\ell,v,w\in\mathcal{T}} v_{k\ell}\langle \delta_{x,j}, (H^{\Lambda_M} - z)^{-1}\delta_{v,i}\rangle\langle \delta_{w,\ell}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle.\end{aligned} \tag{6.4.11}$$

Now taking the expectation and s power, we have

$$\begin{aligned}&\mathbb{E}\left|\langle \delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle\right|^s \\ &\le \mathbb{E}\left|\langle \delta_{x,j}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle\right|^s \\ &\qquad + \sum_{i,\ell,v,w\in\mathcal{T}} \mathbb{E}\left|v_{k\ell}\langle \delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{v,i}\rangle\langle \delta_{w,\ell}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle\right|^s \qquad (6.4.12) \\ &\le \mathbb{E}\left|\langle \delta_{x,j}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle\right|^s \\ &\quad + \sum_{i,\ell,v,w\in\mathcal{T}} \left(\mathbb{E}|v_{k\ell}|^{3s}\right)^{1/3} \\ &\qquad \times \left(\mathbb{E}|\langle \delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{v,i}\rangle|^{3s}\right)^{1/3} \left(\mathbb{E}|\langle \delta_{w,\ell}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle|^{3s}\right)^{1/3}.\end{aligned} \tag{6.4.13}$$

From the spectral averaging bound, $\mathbb{E}|\langle \delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}\delta_{v,i}\rangle|^{3s} \le C$ for some constant C (see also the proof of Prosition 5.2.2. Since $|w - x| \le 1$ for $w \in \mathcal{T}$, we have the bound

$$\mathbb{E}|\langle \delta_{w,\ell}, (H^{\Lambda_M} - z)^{-1}\delta_{y,k}\rangle|^{3s} \le C_{3s}e^{-\gamma_{3s}|x-y|+1}. \tag{6.4.14}$$

Thus combining all constants and noting there are at most $2d + 2L - 1$ terms in $\mathcal{T}$, we have the desired bound,

$$\mathbb{E}|\langle \delta_{x,j}, (\tilde{H}^{\Lambda_M} - z)^{-1}, \delta_{y,k}\rangle|^s \le \tilde{C}e^{-\tilde{\gamma}|x-y|}. \tag{6.4.15}$$

□

Define the *boundary* $\partial\Lambda_M$ of $\Lambda_M \subset \mathbb{Z}^d$ as the set of pairs $(x, y) \in \mathbb{Z}^d$ such that $x \in \Lambda_M$, $y \in {\Lambda_M}^c$ and $|x-y| = 1$. We define the *boundary operator* Γ_M on $\ell^2(\mathbb{Z}^d : \mathbb{C}^L)$ by setting the (x, y) block equal to I_L if $(x, y) \in \partial\Lambda_M$ or $(y, x) \in \partial\Lambda_M$.

We note that the boundary operator takes an input $\varphi \in \ell^2(\mathbb{Z}^d : \mathbb{C}^L)$, projects onto $\partial\Lambda_M$, and then interchanges blocks corresponding to the layer inside Λ_M and the layer outside Λ_M.

Theorem 6.4.4. *For $N > M$ and $\Psi_{x,j}$ as defined in the previous theorem and $0 < s << 1$, there exist constants C and γ such that*

$$\mathbb{E}\left|\Psi_{x,j}^T\left[(\tilde{H}^{\Lambda_N} - E - i\varepsilon)^{-1} - (\tilde{H}^{\Lambda_M} - E - i\varepsilon)^{-1}\right]\Psi_{x,j}\right|^s \le CM^{d-1}e^{-\gamma M}. \tag{6.4.16}$$

Proof. From the previous proposition, $\tilde{H}^{\Lambda_N}$ and $\tilde{H}^{\Lambda_M}$ satisfy fractional moment bounds as well.

From there, we have

$$\begin{aligned}
&\left|\Psi_{x,j}^T\left[(H^{\Lambda_N}-E-i\varepsilon)^{-1}-(H^{\Lambda_M}-E-i\varepsilon)^{-1}\right]\Psi_{x,j}\right| \\
&\leq \left|\sum_{i,\ell=1}^{N} V(x)_{ij}V(x)_{j\ell}\langle\delta_{x,i},(H^{\Lambda_N}-E-i\varepsilon)^{-1}(H^{\Lambda_M}-H^{\Lambda_N})(H^{\Lambda_M}-E-i\varepsilon)^{-1}\delta_{x,\ell}\rangle\right| \\
&\qquad + \left|\sum_{|x'-x|=1}\langle\delta_{x',j},(H^{\Lambda_N}-E-i\varepsilon)^{-1}(H^{\Lambda_M}-H^{\Lambda_N})(H^{\Lambda_M}-E-i\varepsilon)^{-1}\delta_{x,j}\rangle\right| \\
&= \left|\sum_{i,\ell=1}^{N} V(x)_{ij}V(x)_{j\ell}\langle\delta_{x,i},(H^{\Lambda_N}-E-i\varepsilon)^{-1}\Gamma_M(H^{\Lambda_M}-E-i\varepsilon)^{-1}\delta_{x,\ell}\rangle\right| \\
&\qquad + \sum_{|x'-x|=1}\left|\langle\delta_{x',j},(H^{\Lambda_N}-E-i\varepsilon)^{-1}\Gamma_M(H^{\Lambda_M}-E-i\varepsilon)^{-1}\delta_{x,j}\rangle\right| \\
&\leq \sum_{i,\ell=1}^{N}\sum_{y,y'\in\partial\Lambda_M}|V(x)_{ij}||V(x)_{j\ell}|\|(H^{\Lambda_N}-E-i\varepsilon)^{-1}(x,y)\|\|(H^{\Lambda_M}-E-i\varepsilon)^{-1}(x,y)\| \\
&\qquad + \sum_{\substack{|x'-x|=1\\|x''-x|=1}}\sum_{y,y'\in\partial\Lambda_M}\|(H^{\Lambda_N}-E-i\varepsilon)^{-1}(x',y)\|\|(H^{\Lambda_M}-E-i\varepsilon)^{-1}(y',x'')\|
\end{aligned} \tag{6.4.17}$$

When taking the expectation and the s power, we take another use of the generalized Hölder Inequality to achieve the bound

$$\begin{aligned}
&\mathbb{E}\left|\Psi_{x,j}^T\left[(\tilde{H}^{\Lambda_N}-E-i\varepsilon)^{-1}-(\tilde{H}^{\Lambda_M}-E-i\varepsilon)^{-1}\right]\Psi_{x,j}\right|^s \\
&\leq \sum_{\substack{|x'-x|=1\\|x''-x|=1}}\sum_{\substack{y,y'\\\in\partial\Lambda_M}}\left(\mathbb{E}\|(H^{\Lambda_N}-E-i\varepsilon)^{-1}(x',y)\|^{2s}\right)^{\frac{1}{2}}\left(\mathbb{E}\|(H^{\Lambda_M}-E-i\varepsilon)^{-1}(y',x'')\|^{2s}\right)^{\frac{1}{2}} \\
&+ \sum_{y,y'\in\partial\Lambda_M}\left(\mathbb{E}\|V(x)\|^{6s}\right)^{1/3}\left(\mathbb{E}\|(H^{\Lambda_N}-E-i\varepsilon)^{-1}(x,y)\|^{3s}\right)^{1/3} \\
&\qquad\qquad\times\left(\mathbb{E}\|(H^{\Lambda_M}-E-i\varepsilon)^{-1}(x,y)\|^{3s}\right)^{1/3}.
\end{aligned} \tag{6.4.18}$$

Since the minimal distance between sites in the above formula is $M-1$, we can use Theorem 6.4.2 and combine constants and sum to achieve an upper bound of

$$C(2d+1)(M^{d-1})e^{-\gamma(M-1)}, \tag{6.4.19}$$

where M^{d-1} is proportional to the size of $\partial\Lambda_M$. □

6.4.3 Smoothness of Density of States

Theorem 6.4.5. *The density of states functions $n^N(E)$ and $n(E)$ are smooth in E.*

The smoothness of the density of states in finite volume follows exactly as in the band matrix case, since this involved only taking derivatives along the diagonal random variables and conditioning on the other variables.

For the infinite volume case, we also refer to Lemma 5.4.1 and define the telescoping series

$$n(E) = \mathbb{E} \sum_{N=M}^{\infty} \left[\langle \delta_{0,1}, (H^{M+1} - E - i\varepsilon)^{-1} - \delta_{0,1} \rangle - \langle \delta_{0,1}, (H^M - E - i\varepsilon)^{-1} \delta_{0,1} \rangle \right] \\ + \mathbb{E} \langle \delta_{0,1}, (H^N - E - i\varepsilon)^{-1} \delta_{0,1} \rangle. \tag{6.4.20}$$

Since we have Gaussian random variables along the diagonals, we can take derivatives of all orders in E and the telescoping series is still summable following the process in the proof of Theorems 5.4.1 and 5.4.2, and so smoothness follows from Lemma 5.4.1.

6.5 Poisson Statistics

We now prove the main result following the strategy of Minami and in parallel to the proof for the band matrix.

Theorem 6.5.1. *Let x and y be points in $\mathbb{Z}^d$, and $j, k \in \{1, \dots, L\}$. Let $E \in \mathbb{R}$ be such that $n(E) > 0$, and λ such that*

$$\mathbb{E} \left\{ |\langle \delta_{x,j}, (H_L - z)^{-1} \delta_{y,k} \rangle|^s \right\} \leq C e^{-\gamma |x-y|} \tag{6.5.1}$$

holds for each $z \in \mathbb{C}_+$ given $0 < s < 1$, $C > 0$, and $\gamma > 0$. Let $\{E_{m,N}\}_{m=1}^{L|\Lambda_N|}$ denote the eigenvalues of H^{Λ_N}.

Then the point process

$$\mu^N(s) = \sum_{m=1}^{L|\Lambda_N|} \delta_{L|\Lambda_N|(E_{m,N} - E)}(s)\, ds \tag{6.5.2}$$

converges in expectation to the Poisson point process with intensity measure $n(E)\, dx$.

As with the band matrix, instead of directly working with the operators H^{Λ_N}, we will divide Λ_N into sub-boxes $\Lambda_{N,p}$ with side length on the scale of N^α with $0 < \alpha < 1$ to be specified later. We then study smaller point processes on each $\Lambda_{N,p}$ that retain the original scaling:

$$\mu^{N,p} = \sum_{m=1}^{L|\Lambda_{N,p}|} \delta_{L|\Lambda_N|(E_{m,L,p} - E)}. \tag{6.5.3}$$

Recall that the set of linear combinations of functions of the form $\varphi_z(x) = \operatorname{Im} \frac{1}{x-z}$ with $z \in \mathbb{C}^+$ is dense in $\mathcal{C}_0(\mathbb{R})$.

Lemma 6.5.1. *Let μ^N be the eigenvalue point process for H^{Λ_N} and $\{\mu^{N,p}\}$ the eigenvalue point processes on the sub-boxes $\Lambda_{N,p}$ as defined in (6.5.3). Then for φ_z as defined above,*

$$\lim_{N\to\infty} \mathbb{E}\left|\mu^N(\varphi_z) - \sum_p \mu^{N,p}(\varphi)\right| = 0. \tag{6.5.4}$$

This will show that the limiting point process of μ^N is the same as the limiting process for the sum of the triangular array of smaller point processes.

Proof:

$$\begin{aligned}
&\mu^N(\varphi_z) - \mu^{N,p}(\varphi_z) \\
&= \sum_{m=1}^{L|\Lambda_N|} \delta_{L|\Lambda_N|(E_{m,L}-E)}(\varphi_z) - \sum_p \sum_{n=1}^{L|\Lambda_{N,p}|} \delta_{L|\Lambda_N|(E_{n,L,p}-E)}(\varphi_z) \\
&= \sum_{m=1}^{L|\Lambda_N|} \varphi_z\big(L|\Lambda_N|(E_{m,L}-E)\big) - \sum_p \sum_{n=1}^{L|\Lambda_{N,p}|} \varphi_z\big(L|\Lambda_N|(E_{n,L,p}-E)\big) \\
&= \operatorname{tr} \varphi_z\big(L|\Lambda_N|(H^{\Lambda_N}-E)\big) - \sum_p \operatorname{tr} \varphi_z\big((L|\Lambda_N|(H^{\Lambda_{N_p}}-E)\big) \\
&= \operatorname{tr}\, \mathrm{Im}\, \frac{1}{L|\Lambda_N|(H^{\Lambda_N}-E)-z} - \sum_p \operatorname{tr}\, \mathrm{Im}\, \frac{1}{L|\Lambda_N|(H^{\Lambda_{N_p}}-E)-z} \\
&= \frac{1}{L|\Lambda_N|}\left[\operatorname{tr}\, \mathrm{Im}\left(H^{\Lambda_N} - E - \frac{z}{L|\Lambda_N|}\right)^{-1} - \sum_p \operatorname{tr}\, \mathrm{Im}\left(H^{\Lambda_{N_p}} - E - \frac{z}{L|\Lambda_N|}\right)^{-1}\right] \\
&= \frac{1}{L|\Lambda_N|}\, \mathrm{Im} \sum_p \sum_{j=1}^{L} \sum_{n\in\Lambda_{N,p}} \left[\langle \delta_{x,j}, \left(H^{\Lambda_N} - E - \frac{z}{L|\Lambda_N|}\right)^{-1} \delta_{x,j}\rangle\right. \\
&\qquad\qquad \left. - \langle \delta_{x,j}, \left(H^{\Lambda_{N_p}} - E - \frac{z}{L|\Lambda_N|}\right)^{-1} \delta_{x,j}\rangle.\right]
\end{aligned} \tag{6.5.5}$$

For each p, define $\Lambda_{N,p}^{\text{int}}$ to be the sub-cube of $\Lambda_{N,p}$ with the same center and side length on the scale of $N^\alpha - \beta \log N$ for β to be specified later.

Set $\Lambda_{N,p}^{\text{ext}}$ equal to the exterior layer $\Lambda_{N,p} \backslash \Lambda_{N,p}^{\text{int}}$. Also for convenience, set $z_L^E = E + \frac{z}{L|\Lambda_N|}$. Then

$$
\begin{aligned}
&\mu^N(\varphi_z) - \mu^{N,p}(\varphi_z) \\
&= \frac{1}{L|\Lambda_N|} \operatorname{Im} \sum_p \sum_{j=1}^N \sum_{n \in \Lambda_{N,p}^{\text{int}}} \left[\langle \delta_{x,j}, \left(H^{\Lambda_N} - z_L^E \right)^{-1} \delta_{x,j} \rangle - \langle \delta_{x,j}, \left(H^{\Lambda_{N_p}} - z_L^E \right)^{-1} \delta_{x,j} \rangle \right] \\
&+ \frac{1}{L|\Lambda_N|} \operatorname{Im} \sum_p \sum_{j=1}^N \left[\sum_{n \in \Lambda_{N,p}^{\text{ext}}} \langle \delta_{x,j}, \left(H^{\Lambda_N} - z_L^E \right)^{-1} \delta_{x,j} \rangle - \langle \delta_{x,j}, \left(H^{\Lambda_{N_p}} - z_L^E \right)^{-1} \delta_{x,j} \rangle \right] \qquad (6.5.6) \\
&=: A_L + B_L.
\end{aligned}
$$

We will see that $\mathbb{E}\{|A_L|\}$ and $\mathbb{E}\{|B_L|\}$ converge to 0 separately.

For B_L, we use the *a priori* bound (4.2.1), and the following elementary resolvent estimate:

$$
\|(H^{\Lambda_N} - z_L^E)^{-1}\| \leq (\operatorname{Im} z_L^E)^{-1} = \left(\frac{\operatorname{Im} z}{L|\Lambda_N|} \right)^{-1}. \qquad (6.5.7)
$$

We have

$$
\begin{aligned}
&\mathbb{E}\{|B_L|\} \\
&\leq \frac{1}{L|\Lambda_N|} \mathbb{E} \left\{ \sum_p \sum_{j=1}^N \sum_{x \in \Lambda_{N,p}^{\text{ext}}} \left| \langle \delta_{x,j}, (H^{\Lambda_N} - z_L^E)^{-1} \delta_{x,j} \rangle \right| + \left| \langle \delta_{x,j}, (H^{\Lambda_{N_p}} - z_L^E)^{-1} \delta_{x,j} \rangle \right| \right\} \\
&= \frac{1}{L|\Lambda_N|} \sum_p \sum_{j=1}^N \sum_{x \in \Lambda_{N,p}^{\text{exp}}} \mathbb{E} \left\{ \left| \langle \delta_{x,j}, (H^{\Lambda_N} - z_L^E)^{-1} \delta_{x,j} \rangle \right|^s \left| \langle \delta_{x,j}, (H^{\Lambda_N} - z_L^E)^{-1} \delta_{x,j} \rangle \right|^{1-s} \right\} \\
&+ \frac{1}{L|\Lambda_N|} \sum_p \sum_{j=1}^N \sum_{x \in \Lambda_{N,p}^{\text{exp}}} \mathbb{E} \left\{ \left| \langle \delta_{x,j}, (H^{\Lambda_{N_p}} - z_L^E)^{-1} \delta_{x,j} \rangle \right|^s \left| \langle \delta_{x,j}, (H^{\Lambda_{N_p}} - z_L^E)^{-1} \delta_{x,j} \rangle \right|^{1-s} \right\} \\
&\leq \frac{1}{L|\Lambda_N|} \sum_p \sum_{j=1}^N \sum_{x \in \Lambda_{N,p}^{\text{ext}}} \left(\frac{L|\Lambda_N|}{|z|} \right)^{1-s} \mathbb{E} \left\{ \left| \langle \delta_{x,j}, (H^{\Lambda_N} - z_L^E)^{-1} \delta_{x,j} \rangle \right|^s \right\} \\
&+ \frac{1}{L|\Lambda_N|} \sum_p \sum_{j=1}^N \sum_{x \in \Lambda_{N,p}^{\text{ext}}} \left(\frac{L|\Lambda_N|}{|z|} \right)^{1-s} \mathbb{E} \left\{ \left| \langle \delta_{x,j}, (H^{\Lambda_{N_p}} - z_L^E)^{-1} \delta_{x,j} \rangle \right|^s \right\}. \qquad (6.5.8)
\end{aligned}
$$

Now using the *a priori* bound (4.2.1), we bound the above by

$$\frac{2}{L|\Lambda_N|}\sum_p \sum_{j=1}^{N} \sum_{n=\Lambda_{N,p}^{\text{ext}}} \left(\frac{L|\Lambda_N|}{|z|}\right)^{1-s} C = \frac{C}{|\Lambda_N|}\frac{|\Lambda_N|}{|\Lambda_{N,p}|}(|\Lambda_{N,p}| - |\Lambda_{N,p}^{\text{int}}|)|\Lambda_N|^{1-s}$$
$$= C\frac{1}{N^{\alpha d}}(N^{\alpha d} - (N^\alpha - \beta \log N)^d)N^{d(1-s)}$$
$$\leq C\frac{1}{N^{\alpha d}}(N^{\alpha(d-1)}\log N)N^{d(1-s)}$$
$$= C(\log N)N^{-\alpha d+\alpha(d-1)+d(1-s)}$$
$$= C(\log N)N^{-\alpha+d(1-s)} \tag{6.5.9}$$

where we have updated the constant C throughout. This converges to 0 as $N \to \infty$ as long as s and α are chosen so that $-\alpha + d(1-s) < 0$, i.e. $s > 1 - \frac{\alpha}{d}$. Note that our choice of s here is independent of the necessity of s to be small for the convergence of the density of states.

For A_L we take advantage of the geometric resolvent identity and the fractional moment bounds (Theorem 6.4.2).

$$\mathbb{E}\left\{|A_L|\right\}$$
$$= \frac{1}{L|\Lambda_N|}\mathbb{E}\left|\text{Im}\sum_p\sum_{j=1}^{L}\sum_{\substack{x\in\\ \Lambda_{N,p}^{\text{int}}}}\langle\delta_{x,j},(H^{\Lambda_N}-z_L^E)^{-1}\delta_{x,j}\rangle - \langle\delta_{x,j},(H^{\Lambda_{N_p}}-z_L^E)^{-1}\delta_{x,j}\rangle\right|$$
$$= \frac{1}{L|\Lambda_N|}\mathbb{E}\left|\text{Im}\sum_p\sum_{j,k=1}^{L}\sum_{\substack{x\in\\ \Lambda_{N,p}^{\text{int}}}}\sum_{\substack{(v,w)\in\\ \partial\Lambda_(N,p)}}\langle\delta_{x,j},(H^{\Lambda_N}-z_L^E)^{-1}\delta_{v,k}\rangle\langle\delta_{w,k},(H^{\Lambda_{N_p}}-z_L^E)^{-1}\delta_{x,j}\rangle\right|$$
$$\leq \frac{1}{L|\Lambda_N|}\sum_p\sum_{j,k=1}^{N}\sum_{\substack{x\in\\ \Lambda_{N,p}^{\text{int}}}}\sum_{\substack{(v,w)\in\\ \partial\Lambda_(N,p)}}\mathbb{E}\left|\langle\delta_{x,j},(H^{\Lambda_N}-z_L^E)^{-1}\delta_{v,k}\rangle\right|\left|\langle\delta_{w,k},(H^{\Lambda_{N_p}}-z_L^E)^{-1}\delta_{x,j}\rangle\right|$$
$$\leq \frac{1}{L|\Lambda_N|}\sum_p\sum_{j,k=1}^{L}\sum_{\substack{x\in\\ \Lambda_{N,p}^{\text{int}}}}\sum_{\substack{(v,w)\in\\ \partial\Lambda_{N,p}}}\mathbb{E}\left\{\left|\langle\delta_{x,k},(H^{\Lambda_N}-z_L^E)^{-1}\delta_{v,k}\rangle\right|^{\frac{s}{2}}\right.$$
$$\left.\times\left|\langle\delta_{w,k},(H^{\Lambda_{N_p}}-z_L^E)^{-1}\delta_{x,j}\rangle\right|^{\frac{s}{2}}\right\}\left(\frac{L|\Lambda_N|}{|z|}\right)^{2-s}$$

$$\leq \frac{(L|\Lambda_N|)^{1-s}}{|z|} \sum_p \sum_{j,k=1}^{L} \sum_{\substack{x\in \\ \Lambda_{N,p}^{\mathrm{int}}}} \sum_{\substack{(v,w)\in \\ \partial\Lambda_{N,p}}} \mathbb{E}\left\{\left|\langle \delta_{x,k}, (H^{\Lambda_N} - z_L^E)^{-1}\delta_{v,k}\rangle\right|^s\right\}^{\frac{1}{2}}$$

$$\times\, \mathbb{E}\left\{\left|\langle \delta_{w,k}, (H^{\Lambda_{N_p}} - z_L^E)^{-1}\delta_{x,k}\rangle\right|^s\right\}^{\frac{1}{2}}$$

$$\leq \frac{(L|\Lambda_N|)^{1-s}}{|z|} \sum_p \sum_{j,k=1}^{L} \sum_{\substack{x\in \\ \Lambda_{N,p}^{\mathrm{int}}}} \sum_{\substack{(v,w)\in \\ \partial\Lambda_{N,p}}} Ce^{-\frac{\gamma}{2}\mathrm{dist}(\Lambda_{N,p}^{\mathrm{int}}, \partial\Lambda_{N,p})} \tag{6.5.10}$$

where the spectral averaging bound (3.2.5) was used on the H^{Λ_N} term, and the fractional moment bound (Theorem 6.4.2) was used on the $H^{\Lambda_{N_p}}$ term. Thus we have

$$\begin{aligned}
\mathbb{E}\{|B_L|\} &= \frac{(L|\Lambda_N|)^{1-s}}{|z|} \sum_p \sum_{j=1}^{N} \sum_{\substack{x\in \\ \Lambda_{N,p}^{\mathrm{int}}}} \sum_{\substack{(v,w)\in \\ \partial\Lambda_(N,p)}} e^{-\frac{1}{2}\gamma\beta\log N} \\
&= C|\Lambda_N|^{1-s}\frac{|\Lambda_N|}{|\Lambda_{N,p}|}|\Lambda_{N,p}^{\mathrm{int}}||\partial\Lambda_{N,p}|e^{-\frac{1}{2}\gamma\beta\log N} \\
&= CN^{d(1-s)}N^{d-\alpha d}(N^\alpha - \beta\log N)^d N^{d-1}N^{-\gamma\beta}
\end{aligned} \tag{6.5.11}$$

which converges to 0, so long as β is chosen so that $3d - s - 1 - \frac{1}{2}\gamma\beta < 0$.

Thus we have $\mathbb{E}\{A_L + B_L\} \to 0$, and so

$$\left|\mu^N(\varphi_z) - \sum_p \mu^{N,p}(\varphi_z)\right| \to 0 \tag{6.5.12}$$

for each φ_z.

□

We complete the proof of Theorem 6.4.4 by showing that $\sum_p \mu^{N,p}$ follows the Daley and Vere-Jones Critera (Theorem 5.5.2). Since we have Wegner and Minami Estimates, and similar convergence and smoothness for the density of states results, the proofs of these estimates follows as in the proof for the band matrix.

1. The array is uniformly asymptotically negligible:
   For each bounded interval A in $\mathbb{R}$,

$$\lim_{N\to\infty} \sup_p \mathbb{P}\left\{\mu_L^{N,p}(A) \geq 1\right\} = 0. \tag{6.5.13}$$

Proof.

$$\begin{aligned}
\mathbb{P}\{\mu^{N,p}(A) \geq 1\} \leq \mathbb{E}\{\mu^{N,p}(A)\} &= \mathbb{E}\left\{\sum_m \chi_A\big(L|\Lambda_N|(E_{L,p,m} - E)\big)\right\} \\
&= \mathbb{E}\left\{\# \text{ of eigenvalues of } H^{\Lambda_{N_p}} \text{ in } E + \frac{A}{L|\Lambda_N|}\right\} \\
&\leq C_N |\Lambda_{N,p}| \frac{|A|}{L|\Lambda_N|} \\
&\to 0 \qquad (6.5.14)
\end{aligned}$$

by Wegner estimate (Theorem 6.2.1). □

2. Convergence to the density of states:

$$\lim_{N\to\infty} \sum_p \mathbb{P}\left\{\mu_L^{N,p}(A) \geq 1\right\} = n(E)|A|. \qquad (6.5.15)$$

Proof. As with the band matrix, we use Proposition 5.5.1 to instead prove that

$$\lim_{N\to\infty} \sum_p \mathbb{E}\left\{\mu^{N,p}(A)\right\} = n(E)|A|. \qquad (6.5.16)$$

We first follow the steps in the proof of Lemma 6.5.1 replacing $\sum_p H^{N,p}$ with the full operator H^{Λ_N} to show that for any z in $\mathbb{C}_+$,

$$\begin{aligned}
\mathbb{E}\Bigg\{\Bigg|&\frac{1}{L|\Lambda_N|}\sum_{j=1}^{N}\sum_{x\in\Lambda_N} \operatorname{Im}\langle \delta_{x,j}, (H^{\Lambda_N} - z_L^E)^{-1}\delta_{x,j}\rangle \\
&-\frac{1}{L|\Lambda_N|}\sum_p\sum_{j=1}^{N}\sum_{x\in\Lambda_N} \operatorname{Im}\langle \delta_{x,j}, (H^{\Lambda_{N_p}} - z_L^E)^{-1}\delta_{x,j}\rangle\Bigg|\Bigg\} \qquad (6.5.17)
\end{aligned}$$

converges to 0 as $N \to \infty$.

Then writing

$$\mathbb{E}\left\{\left|\frac{1}{L|\Lambda_N|}\sum_{j=1}^{N}\sum_{x\in\Lambda_N} \operatorname{Im}\langle \delta_{x,j}, (H^{\Lambda_N} - z_L^E)^{-1}\delta_{x,j}\rangle\right|\right\} \qquad (6.5.18)$$

as an integral with respect to the finite volume density of states measure, we get:

$$\operatorname{Im}\int \frac{d\nu_N(x)}{x - z_L^E} = \operatorname{Im}\int \frac{n_N(x)}{x - z_L^E}\,dx. \qquad (6.5.19)$$

Recall that $z_L^E = E + \dfrac{z}{L|\Lambda_N|}$, so that the imaginary part of the integrand is equal to

$$\frac{(\operatorname{Im} z/|\Lambda_L|)\, n(x)}{(x - E - \operatorname{Re} z/|\Lambda_N|)^2 + (\operatorname{Im} z/|\Lambda_N|)^2} \qquad (6.5.20)$$

Changing variables and taking a limit as $N \to \infty$, we can write the integral as

$$\lim_{N\to\infty} \int \frac{1}{u^2+1} n_N \left(\frac{\text{Im } z}{|\Lambda_N|} u + \frac{\text{Re } z}{|\Lambda_N|} + E \right) \mathrm{d}u. \tag{6.5.21}$$

The integrand is dominated by

$$\frac{\|\rho\|_\infty}{u^2+1}, \tag{6.5.22}$$

and so we can bring the limit inside the integral by the Dominated Convergence Theorem.

As with the band matrix, the uniform convergence of the finite volume density of states functions to the infinite volume density of states function $n(E)$ and the smoothness of the infinite volume density of states implies that

$$\lim_{N\to\infty} n_N \left(\frac{\text{Im } z}{|\Lambda_N|} u + \frac{\text{Re } z}{|\Lambda_N|} + E \right) = n(E) \tag{6.5.23}$$

and so the integral (6.5.21) evaluates to $n(E)\pi$.

Since linear combinations of functions of the form $\varphi_z = \text{Im } \frac{1}{x-z}$ for $z \in \mathbb{C}^+$ are dense in L^2, and $\|\varphi_z\| = \pi$, we have that $\mathbb{E}\{\langle \delta_{0,1}, E_H(A)\delta_{0,1}\rangle\} = n(E)|A|$ and in particular, condition (2) is satisfied. □

3. Minami estimate:

$$\lim_{N\to\infty} \sum_p \mathbb{P}\left\{ \mu_L^{N,p}(A) \geq 2 \right\} = 0. \tag{6.5.24}$$

Proof.

$$\begin{aligned}
\sum_p \mathbb{P}\left\{\mu^{N,p}(A) \geq 2\right\} &= \sum_p \mathbb{P}\left\{ \operatorname{tr} \chi_{E+\frac{A}{N|\Lambda_N|}}(H^{\Lambda_{N_p}}) \geq 2 \right\} \\
&\leq \sum_p \mathbb{E}\left\{ \operatorname{tr} \chi_{E+\frac{A}{N|\Lambda_N|}}(H^{\Lambda_{N_p}}) \left(\operatorname{tr} \chi_{E+\frac{A}{N|\Lambda_N|}}(H^{\Lambda_{N_p}} - 1) \right) \right\} \\
&\leq \frac{|\Lambda_N|}{|\Lambda_{N,p}|} \left(\frac{|A|}{N|\Lambda_N|} C |\Lambda_{N,p}| \right)^2 \\
&= C \frac{|\Lambda_{N,p}|}{|\Lambda_N|} \to 0
\end{aligned} \tag{6.5.25}$$

by Minami estimate. □

Chapter 7 Complete Localization for Orbital Model in 1D

In this chapter, we prove that one dimensional Wegner Orbital Model has pure point spectrum at all energies at any strength of disorder. The analysis is accomplished through a study of the Lyapunov exponents, and follows techniques established for a related model, the Anderson model on a strip. Although results of this type exist in the literature already, the goal here is to present a more or less self-contained exposition in the context of the Wegner orbital model in 1D. For a sample of existing results, see for example [7, 8, 13, 17].

7.1 Outline

Here we consider the Wegner Orbital model with GOE disorder in one dimension:

$$\begin{aligned} H_L &: \ell^2(\mathbb{Z} \otimes \mathbb{C}^L) \longmapsto \ell^2(\mathbb{Z} \otimes \mathbb{C}^L) \\ H_L u(n) &= u(n+1) + u(n-1) + V(n)u(n) \end{aligned} \tag{7.1.1}$$

where $\{V(n)\}$ are sampled independently from the Gaussian Orthogonal Ensemble, and the addition is vector addition in $\mathbb{C}^L$. The goal is to show complete spectral localization of the operator H_L. That is for almost every configuration of $\{V(n)\}_{n\in\mathbb{Z}}$, the spectrum of H_L is pure point with exponentially decaying eigenfunctions. This is accomplished by an examination of the Lyapunov exponents.

Notably, this result should hold independently of the variance of the Gaussian random matrices in the potential. Results of this type have been proven for a related model, the Anderson model on the strip by [17] and [7]. The Anderson model on the strip is given by an operator h on $\ell^2(\mathbb{Z} \otimes \mathbb{C}^N)$ characterized by the fact that its restriction (with appropriate boundary conditions) to an interval $[-N, N] \subset \mathbb{Z}$ is equivalent to a 2-D Anderson Hamiltonian restricted to the box $[-N, N] \times [1, L] \subset \mathbb{Z}^2$. Random Schrodinger operators on the strip were studied in the 1980's with the hope of taking the width of the strip to ∞ in order to apply 1-D techniques to 2-D problems, but unfortunately the bounds on the limits were rarely good enough to provide new insights into multi-dimensional random operators.

Many of the techniques and results developed for studying Anderson models on the strip are useful in studying random operators with matrix valued potentials in one spatial dimension. In particular, we will follow much of the exposition in the textbook by Carmona and Lacroix [8] in the following.

7.2 Transfer Matrices

The second order difference equation given by $H_L u = Eu$

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{7.2.1}$$

can be written in terms of a $2L \times 2L$ *transfer matrix* as follows:

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \begin{pmatrix} E - V(n) & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}. \tag{7.2.2}$$

If we let $g(n) = \begin{pmatrix} E - V(n) & -I \\ I & 0 \end{pmatrix}$ and $\overline{u}(n) = \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$ we have

$$\overline{u}(n) = g(n)g(n-1)\cdots g(1)\overline{u}(1) \tag{7.2.3}$$

and similarly

$$\overline{u}(-n) = g(-n)^{-1}\cdots g(-1)^{-1}g(0)^{-1}g(1)^{-1}\overline{u}(1). \tag{7.2.4}$$

The matrix $U_E(n) = g(n)\cdots g(1)$ is called the *propagator* of H corresponding to E. When it is clear from context, we drop the subscript E from the notation. In this way, we shift the study of the difference equation (7.5.15) to a study of the products of transfer matrices.

Each transfer matrix is a $2L \times 2L$ *symplectic* matrix, meaning that for any transfer matrix g and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$,

$$g^T J g = J. \tag{7.2.5}$$

The set of all $2L \times 2L$ real symplectic matrices forms a Lie group denoted $\mathrm{Sp}(L, \mathbb{R})$. Several properties of $\mathrm{Sp}(L, \mathbb{R})$ are important to note:

1. Let $g \in \mathrm{Sp}(L, \mathbb{R})$. If λ is and eigenvalue of g, then λ^{-1} is also and eigenvalue of g. The same holds true for the singular values of g.

2. The Lie algebra associated to $\mathrm{Sp}(L, \mathbb{R})$, denoted $\mathfrak{sp}(L, \mathbb{R})$, is the set of $2L \times 2L$ matrices of the form
$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \tag{7.2.6}$$
where B and C are symmetric. It has dimension $L(2L + 1)$.

3. The symplectic lie algebra is a simple algebra, i.e. it has no non-trivial ideals.

From the group structure of $\mathrm{Sp}(L, \mathbb{R})$, each propagator matrix $U(n)$ is also in $\mathrm{Sp}(L, \mathbb{R})$.

We note the following about the Lie Algebra $\mathfrak{sp}(L, \mathbb{R})$. Let E_{ij} be the $L \times L$ matrix with the ij-entry set to 1 and all other entries 0.

Lemma 7.2.1. $\mathfrak{sp}(L, \mathbb{R})$ *is generated by the set of matrices*

$$X_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ij}^T \\ 0 & 0 \end{pmatrix} \tag{7.2.7}$$

$$Y_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ij}^T & 0 \end{pmatrix}. \tag{7.2.8}$$

with $|i - j| \leq 1$.

7.3 Lyapunov Exponents

Let $\wedge^p \mathcal{V}$ be the p^{th} exterior product of a vector space $\mathcal{V}$. That is, the elements of $\wedge^p \mathcal{V}$ are of the form

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p \tag{7.3.1}$$

for $v_1, \ldots, v_p \in \mathcal{V}$. If A is a bounded linear operator on $\mathcal{V}$, then $\wedge^p A$ is defined by the action

$$\wedge^p A(v_1 \wedge v_2 \wedge \cdots \wedge v_p) = Av_1 \wedge Av_2 \wedge \cdots \wedge Av_p. \tag{7.3.2}$$

For a brief introduction to the exterior product, see [25, Section 1.5].

We are now ready to define the Lyapunov exponents of a multiplicative system.

Definition 7.3.1 (Lyapunov Exponents)**.** Let $U(n) = g(n)\cdots g(1)$ be the product of n transfer matrices of the system defined in (7.2.1) and (7.2.2). Then for $p = 1, \ldots, 2L$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\| \wedge^p U(n)\|) = \gamma_1 + \gamma_2 + \cdots + \gamma_p. \tag{7.3.3}$$

The numbers γ_i are the *Lyapunov exponents* of multiplicative system of random matrices U. The Lyapunov exponents are ordered so that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{2L}$.

The Lyapunov exponents will give us a measure of the expected growth/decay rate of the products of transfer matrices $U(n)$ as $n \to \infty$.

We have the following relation between Lyapunov exponents of H_L.

Lemma 7.3.1. *The Lyapunov exponents for the transfer matrices of $U(n)$ satisfy H_L*

$$\gamma_p = -\gamma_{2L-p+1}. \tag{7.3.4}$$

for $p = 1, \ldots, L$.

Proof. Note that $\| \wedge^p U(n)\|$ is the product of the first p singular values, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ of $U(n)$ [8]. Since each $U(n)$ is symplectic (7.2), these satisfy the relation:

$$\mu_i = \frac{1}{\mu_{2L-i+1}}. \tag{7.3.5}$$

Thus

$$\log(\| \wedge^p U(n)\|) = \log \mu_1 + \cdots + \log(\mu_p) = -\log(\mu_{2L}) - \cdots - \log(\mu_{2L-p+1}). \tag{7.3.6}$$

Since this holds for each p, we have

$$\log(\mu_p) = -\log(\mu_{2L-p+1}). \tag{7.3.7}$$

This relation carries through in the limit as $n \to \infty$, so $\gamma_p = -\gamma_{2L-p+1}$. □

The Lyapunov exponent γ_i gives the exponential growth/decay rate of the action of $U(n)$ on vectors in certain subspaces. We will make this more explicit when we state the Oseledec Theorem in section 7.5.2. This corresponds to the growth/decay of the solutions to the difference equation

$$H_L u = Eu. \tag{7.3.8}$$

If we can show that all solutions to the above equation grow or decay exponentially in each direction, then all eigenfunctions must decay exponentially. To accomplish this, we must rule out the possibility of having a Lyapunov exponent of 0. By the previous Lemma, this can by accomplished by proving the Lyapunov spectrum is simple, that is no Lyapunov exponent occurs twice.

In the next section we prove the Lyapunov spectrum is simple, and we discuss the implication on the spectral properties of H_L in the section following.

7.4 Simplicity of the Lyapunov Spectrum

We say that the Lyapunov spectrum is *simple*, if the Lyapunov exponents obey the strict inequalities $\gamma_1 > \gamma_2 > \cdots > \gamma_{2L}$. Note that by Lemma 7.3.1, this implies that $\gamma_L > 0 > \gamma_{L+1}$, and in particular no Lyapunov exponent can be 0.

Let T_L be the smallest closed semi-group containing the transfer matrices of the 1-D orbital model. That is T_L is the set of finite products of matrices $g(V_i)$ with V_i symmetric. Note that each propagator matrix $U(n) \in T_L$ and that $T_L \subset \mathrm{Sp}(L, \mathbb{R}) \subset \mathrm{GL}(2L, \mathbb{R})$.

We define some important terms in the study of products of random matrices here. The definitions below are taken from [8]. Let $\mathcal{S}$ be a subset of $GL(2L, \mathbb{R})$.

Definition 7.4.1. $\mathcal{S}$ is *contractive* if there exists a sequence $s_n \in \mathcal{S}$ such that $\frac{1}{\|s_n\|} s_n$ converges to a rank one operator.

Definition 7.4.2. $\mathcal{S}$ is *strongly irreducible* if there is no finite union $W = \cup_{i=1}^r V_i$ of proper subspaces V_i of $\mathbb{R}^{2L}$ such that $gW = W$ for all $g \in \mathcal{S}$.

Using these definitions, we have the following Theorem, which we state in its application to the semi-group T_L, although it can be applied to more general matrix semi-groups.

Theorem 7.4.1. *[8, Theorem IV.4.15] Suppose T_L is contractive and strongly irreducible. Then the first two Lyapunov exponents obey the strict inequality $\gamma_1 > \gamma_2$.*

In order to prove strict inequalities for further exponents we must extend Definitions 7.4.1 and 7.4.2 somewhat. The following definitions are taken from [6].

Definition 7.4.3. Let p be in $\{1, \ldots, L\}$.

- Let L_p be the subspace of $\wedge^p(\mathbb{R}^{2L})$ spanned by

$$Me_1 \wedge Me_2 \wedge \cdots \wedge Me_p \ : \ M \in \mathrm{Sp}(L, \mathbb{R}). \tag{7.4.1}$$

- A subset $\mathcal{S}$ is *p-contracting* if there exists a sequence $\{M_n\}$ in T such that $\| \wedge^p M_n\|^{-1} \wedge^p M_n$ converges to a rank one matrix.
- A subset $\mathcal{S}$ of $\wedge^p \mathrm{Sp}(L, \mathbb{R})$ is L_p *strongly irreducible* if there does not exist a finite union W of proper linear subspaces of L_p such that $\wedge^p M(W) = W$ for any M in $\mathcal{S}$.

Using these new definitions we can extend Theorem 7.4.1 to the following:

Proposition 7.4.1. *[6, Proposition IV.3.4]*
Suppose the semi-group T_L is p-contracting and L_p strongly irreducible. Then $\gamma_p > \gamma_{p+1}$.

Instead of checking these properties by hand for T_L generated by transfer matrices of the orbital model, we have the following reduction which is Proposition IV.3.5 in [6] (see also Proposition IV.4.22 in [8]).

Proposition 7.4.2. *Suppose T_L contains an open subset of* $\mathrm{Sp}(L, \mathbb{R})$*. Then T_L is p-contractive and L_p strongly irreducible for each $p \in \{1, \ldots, L\}$. Consequently the Lyapunov exponents are distinct.*

The rest of the this section will be dedicated to proving that we can use the proposition for the orbital model. In particular, we will prove

Theorem 7.4.2. *The semi-group T_L generated by transfer matrices of H_L contains an open subset of* $\mathrm{Sp}(L, \mathbb{R})$*. From this it follows that the Lyapunov exponents of H_L are distinct.*

This was proven for the Anderson model on the strip given random variables with a probability density function ρ over $\mathbb{R}$ by Lacroix in [17]. The case with discrete random variables was treated in [7].

Before showing that the semi-group T_L contains an open subset of $\mathrm{Sp}(L, \mathbb{R})$, we first look at the smallest closed group generated by the transfer matrices, G_L. That is, we look first at the set of finite products of transfer matrices and their inverses.

Lemma 7.4.1. *The group G_L generated by transfer matrices of H_L is equal to* $\mathrm{Sp}(L, \mathbb{R})$*.*

Proof. We will prove this by showing the Lie Algebra of the subgroup generated by the transfer matrices is equal to $\mathfrak{sp}(L, \mathbb{R})$. Since each transfer matrix is symplectic, the containment of the subgroup in $\mathrm{Sp}(L, \mathbb{R})$ follows immediately.

For the reverse containment, recall that $\mathfrak{sp}(L, \mathbb{R})$ is generated by the matrices X_{ij} and Y_{ij} as defined in (7.2.7) and (7.2.8).

The set of transfer matrices contains all matrices of the form

$$g(V) = \begin{pmatrix} V & -I \\ I & 0 \end{pmatrix} \tag{7.4.2}$$

with V symmetric. Thus for V_1 and V_2 symmetric,

$$g(V_1)g(V_2)^{-1} = \begin{pmatrix} V_1 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & V_2 \end{pmatrix} = \begin{pmatrix} I & V_1 - V_2 \\ 0 & I \end{pmatrix}. \tag{7.4.3}$$

Since the Lie algebra of the subgroup is the tangent space to the Lie group at the identity, and $V_1 - V_2$ can take the form of any symmetric matrix for appropriate choices of V_1 and V_2, we can see that by differentiating the right side of (7.4.3), each X_{ij} is in the Lie algebra for appropriate choices of V_1 and V_2.

By considering

$$g(V_1)^{-1}g(V_2) \begin{pmatrix} 0 & I \\ -I & V_1 \end{pmatrix} \begin{pmatrix} V_2 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ V_1 - V_2 & I \end{pmatrix}, \tag{7.4.4}$$

we also get each Y_{ij} is in the Lie algebra. Thus the Lie algebra generated by the group is $\mathfrak{sp}(L, \mathbb{R})$, and so the group is given by $\mathrm{Sp}(L, \mathbb{R})$. □

Although we have that the subgroup generated by transfer matrices gives the symplectic group, to draw conclusions about the Lyapunov spectrum, we need the *semi-group* generated by the transfer matrices to produce an open set in the symplectic group. To accomplish this, we need the following technical lemmas, which are similar to results for a Random Schrödinger Operator on the strip in [8, Section IV.4].

Lemma 7.4.2. *Let* $g_i = \begin{pmatrix} V_i & -I \\ I & 0 \end{pmatrix}$ *where* V_i *is a symmetric matrix. Let* X_{ij} *be defined as before. Then for some* p*, there exist symmetric matrices* $V_1 \ldots, V_p$ *such that*

$$\left\{g_1^{-1} X_{ij}\, g_1,\ g_1^{-1} g_2^{-1} X_{ij}\, g_2 g_1, \ldots,\ g_1^{-1} g_2^{-1} \cdots g_p^{-1} X_{ij} g_p \cdots g_2 g_1 \quad : \quad 1 \leq i,j \leq L\right\} \tag{7.4.5}$$

spans $\mathfrak{sp}(L, \mathbb{R})$.

Proof. Let $\omega = \{h_1, \ldots, h_q\}$ be an ordered set of GOE(L)-type transfer matrices, and let

$$V_\omega = \mathrm{Span}\left\{h_1^{-1} X_{ij}\, h_1,\ h_1^{-1} h_2^{-1} X_{ij}\, h_2 h_1, \ldots,\ h_1^{-1} h_2^{-1} \cdots h_q^{-1} X_{ij} h_p \cdots h_2 h_1 \ : \right.$$
$$\left. 1 \leq i,j \leq N\right\}. \tag{7.4.6}$$

Let $\omega_0 = \{g_1, \ldots, g_p\}$ be a collection that is "maximal" in the sense that

$$\dim V_{\omega_0} \geq \dim V_\omega \tag{7.4.7}$$

for any ordered set ω. The existence of such a set is guaranteed since the dimension of any V_ω is at most the dimension of the symplectic group: $L(2L+1)$.

Note that by concatenating ω_0 with another set ω,

$$V_{\omega_0, \omega} = V_{\omega_0} + g_1^{-1} g_2^{-1} \cdots g_p^{-1} V_\omega g_p \cdots g_2 g_1. \tag{7.4.8}$$

Since V_{ω_0} is maximal, $V_{\omega_0,\omega} = V_{\omega_0}$ and thus the above equality implies

$$g_1^{-1} \cdots g_p^{-1} V_\omega g_p \cdots g_1 \quad \subset \quad V_{\omega_0} \tag{7.4.9}$$

or equivalently

$$V_\omega \quad \subset \quad g_1 \cdots g_p \, V_{\omega_0} \, g_p^{-1} \cdots g_1^{-1} \tag{7.4.10}$$

which holds for any ω.

First taking $\omega = \omega_0$ in (7.4.10), and comparing dimensions of each side, we have

$$V_{\omega_0} = g_1 \cdots g_p \, V_{\omega_0} \, g_p^{-1} \cdots g_1^{-1}. \tag{7.4.11}$$

Thus equations (7.4.10) and (7.4.11) tell us that

$$V_\omega \subset V_{\omega_0} \tag{7.4.12}$$

for any ω.

In particular, given any transfer matrix h,

$$V_{h,\omega_0} \subset V_{\omega_0}, \tag{7.4.13}$$

so by (7.4.8), for each h,

$$h^{-1} V_{\omega_0} h \subset V_{\omega_0}. \tag{7.4.14}$$

By again comparing dimensions of each side we in fact get equality:

$$h^{-1} V_{\omega_0} h = V_{\omega_0} = h V_\omega h^{-1}. \tag{7.4.15}$$

Since finite products of matrices of type h and their inverses generate the symplectic group (Lemma 7.4.1), we have that

$$s^{-1} V_{\omega_0} s = V_{\omega_0} \tag{7.4.16}$$

for any symplectic matrix s.

By taking $g = \exp(tY)$ for an arbitrary $Y \in \mathfrak{sp}(L, \mathbb{R})$, and then taking the differential at $t = 0$, (7.4.16) implies that V_{ω_0} is an ideal in $\mathfrak{sp}(L, \mathbb{R})$. Since the Lie Algebra is simple, and the ideal is clearly non-trivial, it must be equal to $\mathfrak{sp}(L, \mathbb{R})$ itself. $\square$

Note that the probability space of GOE(L) matrices is equivalent to $\mathbb{R}^{\left(\frac{L(L+1)}{2}\right)}$ with the appropriate probability measure.

Lemma 7.4.3. *Let $g(V)$ be a transfer matrix corresponding to a symmetric $L \times L$ matrix V.*

There exists a finite collection of points $A_1, \ldots, A_p \in \mathbb{R}^{\left(\frac{L(L+1)}{2}\right)}$ such that the map

$$\begin{aligned} \Phi : \left(\mathbb{R}^{L(L+1)/2}\right)^p &\longrightarrow \mathrm{Sp}(L, \mathbb{R}) \\ (V_1, \ldots, V_p) &\longmapsto g(V_p) \cdots g(V_1) \end{aligned} \tag{7.4.17}$$

is a submersion near $(A_1, \ldots, A_p)$.

Proof. If we choose a fixed point in our domain, $(V_1, \ldots, V_p)$ and vary each point coordinate slightly by a matrix T_i with $T_i(j,k) = t^{(i)}_{j,k} = T_i(k,j)$, then we can write

$$\Phi(V_1 + T_1, \ldots, V_p + T_p) = \begin{pmatrix} I & T_p \\ 0 & I \end{pmatrix} \begin{pmatrix} V_p & -I \\ I & 0 \end{pmatrix} \cdots \begin{pmatrix} I & T_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} V_1 & -I \\ I & 0 \end{pmatrix}. \tag{7.4.18}$$

$$= \begin{pmatrix} I & T_p \\ 0 & I \end{pmatrix} g(V_p) \cdots \begin{pmatrix} I & T_1 \\ 0 & I \end{pmatrix} g(V_1). \tag{7.4.19}$$

Thus

$$\left.\frac{d\Phi}{dt^{(i)}_{j,k}}\right|_0 = g(V_p) \cdot g(V_{i+1}) X_{jk} g(V_i) \cdots g(V_1) \tag{7.4.20}$$

where X_{jk} is as in the proof of Lemma 7.2.1.
Each

$$\left.\frac{d\Phi}{dt^{(i)}_{j,k}}\right|_0 \tag{7.4.21}$$

lives in the tangent space to $\mathrm{Sp}(L, \mathbb{R})$ at $\Phi(V_1, \ldots, V_p)$. We want to show that the set $\left\{ \left.\frac{d\Phi}{dt^{(i)}_{j,k}}\right|_0 \right\}$ spans this tangent space. Instead of directly showing this, we shift to the tangent space at the identity $\mathfrak{sp}(L, \mathbb{R})$ by left multiplying by the fixed matrix

$$\Phi(V_1, \ldots, V_p)^{-1} = g(V_1)^{-1} g(V_2)^{-1} \cdots g(V_p)^{-1}. \tag{7.4.22}$$

We now have

$$\begin{aligned} &g(V_1)^{-1} g(V_2)^{-1} \cdots g(V_p)^{-1} \left.\frac{d\Phi}{dt^{(i)}_{j,k}}\right|_0 \\ &= g(V_1)^{-1} g(V_2)^{-1} \cdots g(V_p)^{-1} g(V_p) \cdot g(V_{i+1}) X_{j,k} g(V_i) \cdots g(V_1). \end{aligned} \tag{7.4.23}$$

Thus the set $\left\{ \left.\frac{d\Phi}{dt^{(i)}_{j,k}}\right|_0 \right\}$ spans $\mathfrak{sp}(L, \mathbb{R})$ for the points $(A_1, \ldots, A_p)$ corresponding to the transfer matrices from Lemma 7.4.2. □

We are now ready to prove Theorem 7.4.2 on simplicity of the Lyapunov spectrum.

Proof. Since the map Φ as defined in (7.4.17) is a submersion from $\mathbb{R}^{\frac{L(L+1)}{2}}$ to the symplectic group near $(g(A_1), \ldots, g(A_p))$, the image of Φ contains an open set in $\mathrm{Sp}(L, \mathbb{R})$ near $\Phi(A_1, \cdots, A_p) = g(A_1) \cdots g(A_p)$. Since the range of Φ is contained in T_L, we conclude that the semi-group contains an open set of $\mathrm{Sp}(L, \mathbb{R})$.

Thus the semi-group T_L is contractive and strongly irreducible, and so the Lyapunov exponents are distinct.

We conclude that $\gamma_1 > \gamma_2 > \cdots > \gamma_L > 0$. □

7.5 From Simplicity to Spectral Localization

In the previous section, we showed that the Lyapunov exponents of H_L are non-zero. We now use this fact to show that the operator H_L exhibits spectral localization almost surely. That is, with probability one, the spectrum of H_L is pure point with exponentially decaying eigenfunctions. The outline is contained in Chapters 3 and 8 of [8].

For the following we write $H = H_L$.

7.5.1 Spectral Measure

Let $E_H(\cdot)$ be the spectral projection for the operator H. Consider the complex measures constructed from matrix elements of the spectral projector

$$\sigma_{m,n}^{i,j}(\cdot) = \langle \delta_{m,i}, E_H(\cdot)\delta_{n,j}\rangle. \tag{7.5.1}$$

Proposition 7.5.1. *The cyclic subspace generated by the L pairs of vectors*

$$\{\delta_{0,i}, \delta_{-1,i}\}_{i=1}^{L} \tag{7.5.2}$$

under the action of H is all of $\ell^2(\mathbb{Z}; \mathbb{C}^L)$.

Proof. We first note that for each $\lambda \in \mathbb{R}$, we can write the propagator $U(n)$ corresponding to the difference equation

$$Hu = \lambda u \tag{7.5.3}$$

by

$$U_\lambda(n) = \begin{pmatrix} P_n(\lambda) & Q_n(\lambda) \\ P_{n-1}(\lambda) & Q_{n-1}(\lambda) \end{pmatrix} \tag{7.5.4}$$

where each P_n and Q_n is polynomial in λ.

Recall that a solution u to (7.5.3) obeys

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = U_n(\lambda) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}. \tag{7.5.5}$$

We then claim that for each $i = 1, \ldots, L$

$$\delta_{n+1,i} = P_n(H)\delta_{0,i} + Q_n(H)\delta_{-1,i}. \tag{7.5.6}$$

We prove this by induction. In the base case

$$U_1(\lambda) = \begin{pmatrix} P_0(\lambda) & Q_0(\lambda) \\ P_{-1}(\lambda) & Q_{-1}(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda - V(0) & -I \\ I & 0 \end{pmatrix} \tag{7.5.7}$$

so that in the equation

$$\begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = U_0(\lambda) \begin{pmatrix} \delta_{0,i} \\ \delta_{-1,i} \end{pmatrix}. \tag{7.5.8}$$

Thus we have

$$P_0(H)Q\delta_{0,1} + Q_0(H)\delta_{-1,i} = (H - V_0)\delta_{0,i} - \delta_{-1,i} = \delta_{1,i}. \tag{7.5.9}$$

We start with (7.5.4) and proceed inductively. Carrying through the matrix multiplication, we have

$$P_{n+1}(\lambda) = (\lambda - V(n+1))P_n(\lambda) - P_{n-1}(\lambda) \tag{7.5.10}$$

and

$$Q_{n+1}(\lambda) = (\lambda - V(n+1))Q_n(\lambda) - Q_{n-1}(\lambda). \tag{7.5.11}$$

Thus

$$\begin{aligned} &P_{n+1}(H)\delta_{0,i} + Q_{n+1}(H)\delta_{-1,i} \\ &= (H - V(n+1))\left(P_n(\lambda)\delta_{0,i} + Q_n(\lambda)\delta_{-1,i}\right) - \left(P_{n-1}(\lambda)\delta_{0,i} + Q_{n-1}(\lambda)\delta_{-1,i}\right), \end{aligned} \tag{7.5.12}$$

which by induction is equal to

$$(H - V(n))\delta_{n,i} - \delta_{n-1,i} = \delta_{n+1,i}. \tag{7.5.13}$$

Thus each $\delta_{n,i}$ is in the cyclic subspace generated by the action of H on set $\{\delta_{0,i}, \delta_{-1,i}\}_{i=1}^{L}$. □

The above proposition could likely be improved by reducing the number of generating vectors. For example, Simon proved that each vector δ_n is a cyclic vector for the d-dimensional Anderson model on $\ell^2(\mathbb{Z}^d)$ [24]. Nevertheless, the proposition will suffice for the applications below.

Corollary 7.5.1. *Each complex measure $\sigma_{m,n}^{i,j}(\cdot)$ is absolutely continuous with respect to the measure*

$$\sigma(\cdot) := \sum_{i=1}^{L} \left[\sigma_{0,0}^{i,i}(\cdot) + \sigma_{-1,-1}^{i,i}(\cdot)\right]. \tag{7.5.14}$$

For this reason we will refer to σ as the spectral measure of H, although the measure is not unique.

Theorem 7.5.1. *For σ-a.e. λ, there is a polynomially bounded solution to the second order difference equation*

$$Hu = \lambda u. \tag{7.5.15}$$

Proof. Our proof closely follows the proof for the Anderson model that is presented in Chapter 7 of [16].

To begin we define a new measure

$$\tilde{\sigma} = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{L} \alpha_{n,i}\sigma_{n,n}^{i,i} \tag{7.5.16}$$

with $\alpha_{n,i} > 0$ chosen so that $\tilde{\sigma}(\mathbb{R}) = 1$. We note that

$$\alpha_{n,i} \leq \frac{c}{n^{1+\varepsilon}}. \tag{7.5.17}$$

By the corollary, $\tilde{\sigma}$ is absolutely continuous with respect to σ.

Note also that

$$\left|\sigma^{i,j}_{m,n}(A)\right| = |\langle \delta_{m,i}, E_H(A), \delta_{n,j}\rangle| \leq (\sigma^{i,i}_{m,m}(A))^{\frac{1}{2}}(\sigma^{j,j}_{n,n}(A))^{\frac{1}{2}} \tag{7.5.18}$$

and so the mixed spectral measures are also absolutely continuous with respect to $\tilde{\sigma}$. Thus, there is a Radon-Nikodym derivative $f^{i,j}_{m,n}$ such that

$$\sigma^{i,j}_{m,n}(A) = \int_A f^{i,j}_{m,n}(\lambda) d\tilde{\sigma}(\lambda). \tag{7.5.19}$$

We will see that the function $(m, i) \mapsto f^{i,j}_{m,n}(\lambda)$ for $(m, i) \in \mathbb{Z} \otimes \{1, \ldots, L\}$ and (n, j) fixed gives a polynomially bounded pointwise solution to $Hu = \lambda u$.

To achieve the polynomial bound, note that for a Borel set A,

$$\begin{aligned} \tilde{\sigma}(A) &= \int_A \sum_{n\in\mathbb{Z}} \sum_{i=1}^{L} \alpha_{n,i}\sigma^{i,i}_{n,n}(A) \\ &= \int_A \sum_{n\in\mathbb{Z}} \sum_{i=1}^{L} \alpha_{n,i} f^{i,i}_{n,n}(\lambda) d\tilde{\sigma}(\lambda) \end{aligned} \tag{7.5.20}$$

and so

$$\sum_{n\in\mathbb{Z}} \sum_{i=1}^{L} \alpha_{n,i} f^{i,i}_{n,n}(\lambda) = 1 \tag{7.5.21}$$

for $\tilde{\sigma}$ almost every λ, and so $f^{i,i}_{n,n}(\lambda) < \frac{1}{\alpha_{n,i}}$. By (7.5.18), this carries through to

$$\begin{aligned} &\left|\int_A f^{i,j}_{m,n} d\tilde{\sigma}(\lambda)\right| \\ &\leq \left|\int_A f^{i,i}_{m,m} d\tilde{\sigma}(\lambda)\right|^{\frac{1}{2}} \left|\int_A f^{j,j}_{n,n} d\tilde{\sigma}(\lambda)\right|^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\alpha_{m,i}}\right)^{\frac{1}{2}} \left(\frac{1}{\alpha_{n,j}}\right)^{\frac{1}{2}}. \end{aligned} \tag{7.5.22}$$

Thus $(m, i) \mapsto f^{i,j}_{m,n}(\lambda)$ with n, j fixed is bounded above by

$$\left(\frac{1}{\alpha_{m,i}}\right)^{\frac{1}{2}} \left(\frac{1}{\alpha_{n,j}}\right)^{\frac{1}{2}} \leq C \left(m^{1+\varepsilon}\right)^{1/2} \tag{7.5.23}$$

using equation (7.5.17).

To see that $(m,i) \mapsto f^{i,j}_{m,n}(\lambda)$ provides a solution to the difference equation (7.5.15), we let $g(\lambda)$ by an arbitrary measurable function of compact support. Then

$$\begin{aligned}
&\int_{\mathbb{R}} \lambda g(\lambda) f^{i,j}_{m,n}(\lambda) d\tilde{\sigma}(\lambda) \\
&\qquad = \langle \delta_{m,i}, Hg(H)\delta_{n,j}\rangle \\
&\qquad = \langle H\delta_{m,i}, g(H)\delta_{n,j}\rangle \\
&\qquad = \langle V(m)\delta_{m,i}, g(H)\delta_{n,j}\rangle + \sum_{|m'-m|=1} \langle \delta_{m',i}, g(H)\delta_{n,j}\rangle \\
&\qquad = \sum_{k=1}^{L} V(m)_{ik} \langle \delta_{m,k}, g(H)\delta_{n,j}\rangle + \sum_{|m'-m|=1} \langle \delta_{m',i}, g(H)\delta_{n,j}\rangle \\
&\qquad = \sum_{k=1}^{L} V(m)_{ik} \int_{\mathbb{R}} g(\lambda) f^{k,j}_{m,n}(\lambda) d\tilde{\sigma}(\lambda) + \sum_{|m'-m|=1} \int_{\mathbb{R}} g(\lambda) f^{i,j}_{m',n}(\lambda) d\tilde{\sigma}(\lambda) \\
&\qquad = \int_{\mathbb{R}} g(\lambda) \left[\sum_{k=1}^{L} V(m)_{ik} f^{k,j}_{m,n}(\lambda) + \sum_{|m'-m|=1} f^{i,j}_{m',n}(\lambda) \right] d\tilde{\sigma}(\lambda). \qquad (7.5.24)
\end{aligned}$$

In the last line we note that

$$\left[\sum_{k=1}^{L} V(m)_{ik} f^{k,j}_{m,n}(\lambda) + \sum_{|m'-m|=1} f^{i,j}_{m',n}(\lambda) \right] \tag{7.5.25}$$

is H applied to the function $(m,i) \mapsto f^{i,j}_{m,n}$. Since the relation

$$\int_{\mathbb{R}} g(\lambda)\lambda f^{i,j}_{m,n}(\lambda) d\tilde{\sigma}(\lambda) = \int_{\mathbb{R}} g(\lambda) H f^{i,j}_{m,n}(\lambda \tag{7.5.26}$$

holds for all compactly supported and measurable g, we see that

$$u(x,i) = \sum_{m\in\mathbb{Z}} \sum_{i=1}^{L} f^{i,j}_{m,n} \delta_{m,i} \tag{7.5.27}$$

is a pointwise solution to the difference equation $Hu = \lambda u$ which is polynomially bounded. □

7.5.2 The Hyperbolic Set and Pure Point Spectrum

Recall we have $2L$ Lyapunov exponents for H satisfying $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{2L}$ and $\gamma_p = \gamma_{2L-p+1}$ and defined by

$$\lim_{n\to\infty} \frac{1}{n} \log(\| \wedge^p U(n)\|) = \gamma_1 + \gamma_2 + \cdots + \gamma_p. \tag{7.5.28}$$

We have so far taken the existence of the Lyapunov exponents for granted. We now state the Oseledec Theorem, also called the Multiplicative Ergodic Theorem, (Theorem IV.2.6 in [8]), which gives the existence of Lyapunov exponents and more.

Theorem 7.5.2 (Oseledec Theorem)**.** *Let $U_\lambda(n)$ be the propogator matrix of H for fixed $\lambda \in \mathbb{R}$. Then for a.e. configuration of random variables the Lyapunov exponents as given by* (7.5.28) *exist.*

Further, let r be the number of distinct Lyapunov exponents. Then there exists a strictly decreasing sequence of measurable subspaces of $\mathbb{R}^{2L}$ denoted by $\mathcal{V}^i$ such that

1. $\mathbb{R}^{2L} = \mathcal{V}^1 \supset \mathcal{V}^2 \supset \cdots \supset \mathcal{V}^r \supset \mathcal{V}^{r+1}$
2. $v \in \mathcal{V}^i \backslash \mathcal{V}^{i+1}$ *if and only if* $\lim_{n\to\infty} \frac{1}{n} \log(\|U(n)v\|) = \gamma_i$, $i = 1, \ldots, r$.
3. $\dim \mathcal{V}^i - \dim \mathcal{V}^{i+1}$ *is the multiplicity of* γ_i.

Definition 7.5.1. *The number λ is hyperbolic with respect to the operator H if there exist numbers $\gamma_1^\pm(\lambda) \geq \gamma_2^\pm(\lambda) \geq \cdots \geq \gamma_L^\pm(\lambda) > 0$ such that for $p = 1, \cdots, L$*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \| \wedge^p U_\lambda(n)\| \geq \gamma^\pm = \gamma_1^\pm(\lambda) + \gamma_2^\pm(\lambda) + \cdots + \gamma_p^\pm(\lambda) \tag{7.5.29}$$

where U_λ is the propagator of H corresponding to the value λ.

The set of hyperbolic values of H is denoted $\mathrm{hyp}(H)$. Notice that in the case where the Lyapunov exponents exist and are non-zero, we can take $\gamma_p^+(\lambda) = \gamma_p^-(\lambda) = \gamma_p(\lambda)$ where $\gamma_p(\lambda)$ is the p^{th} Lyapunov exponent.

By showing that H does not have a Lyapunov exponent of 0 in section 7.4, we showed that for $\mathbb{P}-$ a.e. configuration of the random potential V^{GOE}, each λ is in $\mathrm{hyp}(H)$.

From this and Theorem 7.5.2, we have

Proposition 7.5.2. *[8, Theorem III.5.16] If $\lambda \in hyp(H)$, then there exist two L-dimensional subspaces $\mathcal{V}^+$ and $\mathcal{V}^-$ of $\mathbb{R}^{2L}$ such that $v \in \mathcal{V}^\pm$ if and only if*

$$\lim_{n\to\pm\infty} \log \|U_\lambda(n)v\| \leq -\gamma_L^\pm(\lambda) \tag{7.5.30}$$

and $v \notin \mathcal{V}^\pm$ if and only if

$$\lim_{n\to\pm\infty} \log \|U_\lambda(n)v\| \geq \gamma_L^\pm(\lambda). \tag{7.5.31}$$

We can now conclude the following.

Proposition 7.5.3. *[8, III.5.17] Let $\lambda \in hyp(H)$. Then λ is an eigenvalue if and only if*

$$\mathcal{V}^+ \cap \mathcal{V}^- \neq \emptyset. \tag{7.5.32}$$

Proof. For any $v \in \mathbb{R}^{2L}$, we can construct a pointwise solution to $Hu = \lambda u$ with $\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = v$. If $v \in \mathcal{V}^+ \cap \mathcal{V}^-$, then the corresponding solution u decays exponentially and is thus in ℓ^2.

On the other hand, if we know λ is an eigenvalue, then the corresponding eigenfunction u is in $\ell^2(\mathbb{Z};\mathbb{C}^L)$, and so it cannot grow exponentially at $\pm\infty$ as in (7.5.31). Thus by Proposition 7.5.2, u decays exponentially at $\pm\infty$, and so

$$v = \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \in \mathcal{V}^+ \cap \mathcal{V}^-. \tag{7.5.33}$$

□

Theorem 7.5.3. *[8, III.5.18/III.5.19] If the support of σ is contained in hyp(H), then σ is pure point.*

Proof. Let Σ denote the set of eigenvalues of H. Since Σ is countable, for any continuous measure m, $m(\Sigma) = 0$. Thus for m-a.e. λ, $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$, and so any (pointwise) solution to the eigenvalue equation $Hu = \lambda u$ must have $\|u(n)\|$ growing exponentially fast in at least one direction.

On the other hand, we know that for σ-a.e. λ, there exists a solution to $Hu = \lambda u$ which is polynomially bounded in n by Theorem 7.5.1, and thus if $\lambda \in \mathrm{hyp}(H)$, u is exponentially decaying in n in both directions. Thus $\sigma(\Sigma^c \cap \mathrm{hyp}(H)) = 0$. But his implies that σ is orthogonal to any continuous measure m on hyp(H) and hence $\sigma \perp \sigma_c$. □

Thus, to prove spectral localization for H, we will need to show that $\mathbb{P}$- almost surely,

$$\sigma(\mathrm{hyp}(H)^c) = 0. \tag{7.5.34}$$

7.5.3 Spectral Averaging

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space of configurations of the GOE potential, and let $\mathcal{B}$ be the Borel $\sigma-$ algebra of $\mathbb{R}$. Let $\omega = \{V(n)\}_{n=-\infty}^{\infty}$ be in Ω and X a subset of $\mathcal{F} \times \mathcal{B}$.

Definition 7.5.2. *Let $W = \{(\omega, \lambda) : \lambda \in hyp(H_\omega)\}$. Let ω be in Ω and denote by $W_\omega \subset \mathbb{R}$ denote the slice of W corresponding to ω.*

We can now rephrase condition (7.5.34) as

$$\sigma_\omega({W_\omega}^c) = 0 \tag{7.5.35}$$

for $\mathbb{P}-$ a.e. ω.

Proposition 7.5.4. *Let m be a measure that is absolutely continuous with respect to Lebesgue measure. Then, the set W is $\mathbb{P} \otimes m$ measurable.*

Proof. First consider the functions $\Phi_n(\omega, \lambda) : \Omega \otimes \mathbb{R} \to \mathbb{R}$ given by

$$\Phi_n(\omega, \lambda) = \frac{1}{n} \log \| \wedge^L U_\lambda(n)\|. \tag{7.5.36}$$

Each Φ_n is polynomial in ω and λ and thus jointly measurable. Thus the limiting function

$$\Phi(\omega, \lambda) = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^L U_\lambda(n) \| \tag{7.5.37}$$

is measurable. Finally, note that

$$W = \Phi^{-1}\{(0, \infty,)\} \tag{7.5.38}$$

and is thus measurable. □

Lemma 7.5.1. *Let m be a measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure. Then*

$$\mathbb{P} \otimes m(W^c) = 0. \tag{7.5.39}$$

Proof. By Fubini's Theorem, we can write

$$\int_{\Omega\otimes\mathbb{R}} \mathbb{1}_{W^c}(\omega, \lambda) \mathrm{d}\mathbb{P} \otimes m = \int_{\mathbb{R}} \int_{\Omega} \mathbb{1}_{W^c}(\omega, \lambda) \mathrm{d}\mathbb{P}\, \mathrm{d}m. \tag{7.5.40}$$

The inner integral evaluates to 0 by the almost sure positivity of the Lyapunov exponent for fixed λ.

□

We are now ready for our spectral averaging result. This formulation of spectral averaging will follow from Lemma 3.2.2.

Proposition 7.5.5. *Let $V(0)$ and $V(-1)$ be the GOE matrices at sites 0 and -1 respectively. Then for any interval A,*

$$\mathbb{E}_{V(0),V(-1)} \sigma_\omega(A) \le 2L|A| \tag{7.5.41}$$

where $|A|$ is the Lebesgue measure of A.

Proof. From (7.5.14)

$$\begin{aligned}
&\mathbb{E}_{V(0),V(-1)} \sigma(A) \\
&\quad = \mathbb{E}_{V(0),V(-1)} \left\{ \sum_{i=1}^{L} \sigma_{0,0}^{i,i}(A) + \sigma_{-1,-1}^{i,i}(A) \right\} \\
&\quad = \mathbb{E}_{V(0),V(-1)} \sum_{i=1}^{L} \langle \delta_{0,i}, E_H(A), \delta_{0,i} \rangle + \mathbb{E}_{V(0),V(-1)} \sum_{i=1}^{L} \langle \delta_{-1,i}, E_H(A), \delta_{-1,i} \rangle \\
&\quad \le 2L|A|
\end{aligned} \tag{7.5.42}$$

where in each term in each sum, we integrated first with respect to the variable $V(n)_{ii}$ and applied the spectral averaging estimate (3.2.5). □

The result is independent of the configuration of the remaining random variables.

Let Ω^* be the sub-σ algebra of the probability space Ω corresponding to sites outside of 0 and -1. Note that the Lyapunov exponents of H are independent of the random variables at sites 0 and -1. Thus the set W^c is measurable on $\Omega^* \times \mathbb{R}$.

Theorem 7.5.4. *For $\mathbb{P}$-a.e. ω,*

$$\sigma_\omega(W^c) = 0. \tag{7.5.43}$$

Proof. Since the set W^c is measurable on $\Omega^* \times \mathbb{R}$, its characteristic function is measurable on this set as well. Thus,

$$\begin{aligned}&\int_\Omega \int_\mathbb{R} \mathbb{1}_{W^c}(\omega^*, \lambda) d\sigma(\lambda) d\mathbb{P}(\omega)\\ &= \int_{\Omega^*} \int_\mathbb{R} \mathbb{1}_{W^c}(\omega^*, \lambda) \int_{\{V_0, V_{-1}\}} d\sigma_\omega(\lambda) d\mathbb{P}(V_0, V_{-1}) d\mathbb{P}\Omega^* \end{aligned}\tag{7.5.44}$$

By Proposition 7.5.5, the inner integral

$$\int_{\{V_0, V_{-1}\}} d\sigma_\omega(\lambda) d\mathbb{P}\omega \tag{7.5.45}$$

defines a measure that is absolutely continuous with respect to Lebesgue measure and indexed by the random variable ω^*. That is for each ω^*, there is a continuous function on $\mathbb{R}$, $\frac{d\sigma_{\omega^*}}{d\lambda}(\lambda)$ such that

$$\begin{aligned}&\int_{\Omega^*} \int_\mathbb{R} \mathbb{1}_{W^c}(\omega^*, \lambda) \int_{\{V_0, V_{-1}\}} d\sigma_\omega(\lambda) d\mathbb{P}V_0, V_{-1} d\mathbb{P}\Omega^*\\ &= \int_{\Omega^*} \int_\mathbb{R} \mathbb{1}_{W^c}(\omega^*, \lambda) \frac{d\sigma_{\omega^*}(\lambda)}{d\lambda}(\lambda) d\lambda d\mathbb{P}(\omega^*) = 0 \end{aligned}\tag{7.5.46}$$

since by Lemma 7.5.1,

$$\int_{\Omega^*} \int_\mathbb{R} \mathbb{1}_{W^c}(\omega^*, \lambda) d\lambda d\mathbb{P}(\omega^*) = 0. \tag{7.5.47}$$

Thus we have proven

$$\int_\Omega \int_\mathbb{R} \mathbb{1}_{W^c}(\omega^*, \lambda) d\sigma(\lambda) d\mathbb{P}(\omega) = 0 \tag{7.5.48}$$

which implies that for $\mathbb{P}$-a.e. ω, $\sigma_\omega(W^c) = 0$. This completes the theorem. □

Theorem 7.5.5. *Let $H = H_0 + V^{GOE}$. For $\mathbb{P}$-a.e. configuration of V^{GOE}, the spectral measure σ of H is pure point.*

Proof. By Theorem 7.5.4, for $\mathbb{P}$-a.e. V^{GOE}, $\mathrm{supp}(\sigma) \subseteq \mathrm{hyp}(H)$. Thus by Theorem 7.5.3, σ has no continuous component. □

Appendices

Appendix A: Schur Formula

Lemma A.1 (Schur Complement Formula)**.** *Suppose M is matrix that can be written in block form as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{A.1}$$

That is for some orthogonal projection P, and $Q = 1 - P$, we have

$$\begin{aligned} A &= PMP \\ B &= PMQ \\ C &= QMP \\ D &= QMQ. \end{aligned} \tag{A.2}$$

Further, suppose M is invertible, and D is invertible on its range. Then

$$PM^{-1}P = \left(PMP - PMQ(QMQ)^{-1}QMP\right)^{-1} \tag{A.3}$$

$$= \left(A - BD^{-1}C\right)^{-1}. \tag{A.4}$$

The Schur Complement Formula relates the blocks of the inverse of a matrix to the inverse of the corresponding blocks in the original matrix. We also obtain representations for the other blocks of the inverse matrix $PM^{-1}Q$, $QM^{-1}P$, and $QM^{-1}Q$.

The Schur Complement Formula can also be applied to infinite rank operators on a separable Hilbert space.

Proof. Note that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}. \tag{A.5}$$

Thus

$$M^{-1} = \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}^{-1} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^{-1} \tag{A.6}$$

$$= \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}^{-1} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}^{-1} \tag{A.7}$$

$$= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1} + D^{-1} \end{pmatrix}. \tag{A.8}$$

□

Application to Green's Functions of RBM

We now demonstrate the applications of the Schur Formula to expressions of the form

$$\langle e_j, (H_L^N - z)^{-1} e_k \rangle \tag{A.9}$$

where H_L^N is a symmetric band matrix with entries v_{jk} that are used extensively in Chapter 4. We are particularly interested in how the jk matrix element of the resolvent depends on the various random variables $v_{i\ell}$.

The first application of the Schur Formula is a version of the rank one perturbation formula used for isolating the dependence of a diagonal matrix element of the resolvent on the random variable v_{jj} that is more commonly acquired through the resolvent identity.

Proposition A.1. *Let $P = P_j$ and $Q = 1 - P$. Then*

$$\langle e_j, (H_L^N - z)^{-1} e_j \rangle = \frac{1}{v_{jj} - z - P\, H_L^N\, Q(Q\, H_L^N\, Q - z)^{-1} Q\, H_L^N P} = \frac{1}{v_{jj} - z + a(z)}. \tag{A.10}$$

Here $(Q\, H_L^N\, Q - z)^{-1}$ is the inverse of the sub-matrix on its range. The constant $a(z)$ depends on z and all random variables in H_L^N except for v_{jj}.

In the proof of Lemma 4.2.1, we took P to be the rank two projection $P_j + P_k$ to obtain a rank two perturbation formula.

Proposition A.2. *Let $P = P_j + P_k =: P_{\{j,k\}}$ and $Q = 1 - P$. Then*

$$\begin{aligned} P_{\{j,k\}}(H_L^N - z)^{-1} P_{\{j,k\}} \\ &= \left(\begin{pmatrix} v_{jj} - z & 0 \\ 0 & v_{kk} - z \end{pmatrix} - P\, H_L^N\, Q(Q\, H_L^N\, Q - z)^{-1} Q\, H_L^N P \right)^{-1} \\ &= \left(\begin{pmatrix} v_{jj} - z & 0 \\ 0 & v_{kk} - z \end{pmatrix} - A(z) \right)^{-1}. \end{aligned} \tag{A.11}$$

Here $A(z)$ is a 2×2matrix which depends on z and all random variables except for v_{jj} and v_{kk}.

We also use the Schur Formula, to get information about off-diagonal matrix elements of the resolvent, although the formulas get more complicated. In this case we will need to to look at the upper-right block in A.8 instead of the upper-left block. We first look at how an off-diagonal matrix element of the resolvent depends on a diagonal random variable.

Proposition A.3. *Let $P = P_j$ and $k \neq j$. Then there exist constants a and b that are independent of v_{jj} such that*

$$\left\langle e_j, (H_L^N - z)^{-1} e_k \right\rangle = \frac{a}{(v_{jj} - z) + b}. \tag{A.12}$$

Note that the constants depend on the remaining random variables.

Proof. With A, B, C, and D as defined in A.2 with $P = P_j$, and referring back to A.8, we note that

$$\left\langle e_j, (H_L^N - z)^{-1} e_k \right\rangle \tag{A.13}$$

is the entry corresponding to index k in the first row:

$$\left(A - BD^{-1}C\right)^{-1} \quad -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \ . \tag{A.14}$$

Since $k \neq j$, this is an entry in the $1 \times 2N$ matrix

$$-\left(A - BD^{-1}C\right)^{-1} BD^{-1}. \tag{A.15}$$

Notice that $\left(A - BD^{-1}C\right)^{-1}$ is a scalar quantity and that $A = v_{jj} - z$ is the only term with dependence on v_{jj}.

Thus each entry in A.15 is obtained by taking the corresponding entry in the $1 \times 2N$ matrix BD^{-1} and dividing by a linear function in v_{jj}. □

We now consider how the off-diagonal matrix elements of the resolvent depend on the off-diagonal entries of H_L^N. To obtain a convenient formula, it is necessary to choose the projection to match one of the indices of the entry.

Proposition A.4. *Let $P = P_j$ and $k, \ell \neq j$. Then there exist constants a, b, c, d, and e such that*

$$\left\langle e_j, (H_L^N - z)^{-1} e_k \right\rangle = \frac{a v_{j\ell} + b}{c v_{j\ell}^2 + d v_{j\ell} + e}. \tag{A.16}$$

Note the constants depend on the remaining random variables and z.

Proof. As in the previous proposition, $\left\langle e_j, (H_L^N - z)^{-1} e_k \right\rangle$ is an entry of the $1 \times 2N$ matrix

$$-\left(A - BD^{-1}C\right)^{-1} BD^{-1}. \tag{A.17}$$

Since H_L^N is symmetric, $B = C^T = Q H_L^N P$. The entry $v_{j\ell}$ of H_L^N only occurs in B and C. We see therefore BD^{-1} is linear in $v_{j\ell}$ in each entry, and $A - BD^{-1}C$ is quadratic in $v_{j\ell}$. Thus $\left\langle e_j, (H_L^N - z)^{-1} e_k \right\rangle$ is a rational function in $v_{j\ell}$ with degrees one and two in the numerator and denominator respectively.

□

Appendix B: Technical Lemmas

The following technical lemmas are also found in [11].

Lemma B.1. *Let A be a self-adjoint matrix, $E \in \mathbb{R}$, and $\varepsilon > 0$. Then*

$$\frac{1}{A - E - i\varepsilon} = i \int_0^\infty e^{-i(A - E - i\varepsilon)\lambda} \, d\lambda \tag{B.1}$$

Proof. Evaluating the integral on the right hand side, we get

$$i \int_0^{\infty} e^{-i(A-E-i\varepsilon)\lambda}\, d\lambda = \frac{i}{-i(A-E-i\varepsilon)} e^{-i(A-E-i\varepsilon)\lambda} \Big|_{\lambda=0}^{\lambda=\infty}$$
$$= -\frac{1}{(A-E-i\varepsilon)} e^{-i(A-E)\lambda} e^{-\varepsilon\lambda} \Big|_{\lambda=0}^{\lambda=\infty}$$
$$= \frac{1}{A-E-i\varepsilon}. \tag{B.2}$$

□

Lemma B.2. *Let A and B be matrices.*

$$e^{itA} - e^{itB} = i \int_0^t e^{i(t-s)A}(A-B)e^{isB}\, ds. \tag{B.3}$$

Proof.

$$\begin{aligned} e^{itA} - e^{itB} &= e^{itA}\left(1 - e^{-itA}e^{itB)}\right) \\ &= -e^{itA} \int_0^t \frac{d}{ds} e^{-isA} e^{isB}\, ds \\ &= -e^{itA} \int_0^t e^{-isA}(-iA)e^{isB} + e^{-isA}(iB)e^{isB}\, ds \\ &= -e^{itA} \int_0^t e^{-isA}\left(-iA + iB\right) e^{isB}\, ds \\ &= i \int_0^t e^{-i(s-t)A}\left(A - B\right) e^{isB}\, ds. \end{aligned} \tag{B.4}$$

□

Lemma B.3. *Let A and B be matrices with positive imaginary part. Then for each* $0 < s < 1$,

$$\left\| e^{itA} - e^{itB} \right\| \le 2^{1-s} |t|^s \|A - B\|^s. \tag{B.5}$$

Proof. From the previous lemma, we have the bound

$$\left\| e^{itA} - e^{itB} \right\| \le |t| \|A - B\|. \tag{B.6}$$

From the triangle inequality, we also have

$$\left\| e^{itA} - e^{itB} \right\| \le 2. \tag{B.7}$$

Combining these two estimates, we get

$$\left\| e^{itA} - e^{itB} \right\| = \left\| e^{itA} - e^{itB} \right\|^{1-s} \left\| e^{itA} - e^{itB} \right\|^{s} \le 2^{1-s} |t|^s \|A - B\|^s. \tag{B.8}$$

□

Note that the above lemmas also hold in the case where A and B are scalars.

Appendix C: Subharmonicity

We will recall here some of the necessary facts about subharmonic functions that are needed to extend the fractional moment bounds into the complex plane.

Subharmonic Functions

Recall that a function $f \in C^2(U)$ is defined to be *subharmonic* on a domain U if it satisfies the relation

$$-\Delta f(z) \leq 0 \tag{C.1}$$

for each $z \in U$. For $z = x + iy \in \mathbb{C}$, the Laplacian is taken to be $\partial_x^2 + \partial_y^2$.

Equivalently, $f \in C^2(U)$ is subharmonic if it satisfies the *mean value property*:

$$f(z) \leq \int_{\Gamma_z} f(w) \frac{dw}{|\Gamma_z|} \tag{C.2}$$

for any circular contour Γ_z centered at z and contained in U.

We now collect some of the necessary results about subharmonic functions.

Lemma C.1. *Suppose u is subharmonic on a bounded domain D. Then, $u \leq v$ for every harmonic function v on D with $u = v$ on ∂D.*

If D is unbounded, the statement remains is valid as long as we extend the boundary conditions to include

$$\lim_{|z|\to\infty} u(z) = \lim_{|z|\to\infty} v(z) = C \neq \pm\infty. \tag{C.3}$$

Proof. Suppose u is subharmonic and v is harmonic with $u(z) = v(z)$ on ∂D. Then $v - u$ satisfies the mean value property (1) with $v - u \equiv 0$ on ∂D. Since the mean value property precludes the possibility of having a local maximum on D, we must have $v - u \leq 0$. □

Lemma C.2. *Suppose $f(z)$ is analytic on a neighborhood $U \subset \mathbb{C}$. Then*

$$\log|f(z)| \tag{C.4}$$

is subharmonic.

Proof. If $f(z) \neq 0$, then $\log|f(z)|$ is the real part of the complex logarithm of $f(z)$ which is analytic in any neighborhood of z that avoids a branch cut. Thus $\log|f(z)|$ is harmonic in a neighborhood of z.

On the other hand, if $f(z) = 0$, then by definition $\log|f(z)| = -\infty$ and so the identity

$$f(z) \leq \int_{\Gamma_z} f(w) \frac{dw}{|\Gamma_z|} \tag{C.5}$$

holds trivially for any contour Γ_z. □

Corollary C.1. *If f is analytic, then $s \log|f(z)| = \log|f(z)|^s$ is subharmonic for any $s > 0$.*

Corollary C.2. *If f is analytic, then $|f(z)|^s$ is subharmonic for any $s > 0$.*

Proof. By the previous corollary, for a suitable circular contour Γ_z,

$$\log |f(z)|^s \leq \int_{\Gamma_z} \log |f(w)|^s \frac{dw}{|\Gamma_z|}. \tag{C.6}$$

Then, by the Mean Value Property and Jensen's inequality,

$$\begin{aligned} |f(z)|^s &= e^{\log |f(z)|^s} \\ &\leq \exp\left(\int_{\Gamma_z} \log |f(w)|^s \frac{dw}{|\Gamma_z|}\right) \\ &\leq \int_{\Gamma_z} \exp\left(\log |f(w)|^s\right) \frac{dw}{|\Gamma_z|} \\ &= \int_{\Gamma_z} |f(w)|^s \frac{dw}{|\Gamma_z|}. \end{aligned} \tag{C.7}$$

□

Application to Green's Functions of RBM

Lemma C.3. *For every configuration of random variables, the function*

$$f(z) = \left\langle e_j, \left(H_L^N - z\right)^{-1} e_k \right\rangle \tag{C.8}$$

is analytic in the upper half-plane.

Proof. Since H_L^N is self-adjoint, the Green's functions are well-defined and smooth as a function of z for $\operatorname{Im} z > 0$. □

Theorem C.1. *Let $0 < s < 1$ and let*

$$\bar{f}(z) = \mathbb{E}\left\{ \left| \left\langle e_j, \left(H_L^N - z\right)^{-1} e_k \right\rangle \right|^s \right\}. \tag{C.9}$$

Then

$$\bar{f}(x + iy) \leq \frac{1}{\pi} \int_{\mathbb{R}} \bar{f}(t) \frac{y}{(x-t)^2 + y^2}\, dt. \tag{C.10}$$

Proof. The un-averaged fractional moments of the Green's functions

$$f(z) = \left| \left\langle e_j, \left(H_L^N - z\right)^{-1} e_k \right\rangle \right|^s \tag{C.11}$$

are subharmonic on $\mathbb{C}_+$. Since f has at most $2N + 1$ (integrable) poles which are located on the real axis, for any fixed configuration of random variables, f is continuous up to real axis almost everywhere, and

$$\lim_{|z| \to \infty} f(z) = 0. \tag{C.12}$$

Thus, by taking a convolution with the Poisson kernel for the upper half plane, the function

$$u(x+iy) = \frac{1}{\pi} \int_{\mathbb{R}} f(t) \frac{y}{(x-t)^2 + y^2} \, dt \tag{C.13}$$

is harmonic on $\mathbb{C}_+$ with the same boundary values as $f(z)$. Thus by Lemma C.1,

$$f(z) \leq u(z) \tag{C.14}$$

throughout the upper half plane.

We can now take expectations to get

$$\bar{f}(z) \leq \mathbb{E} \left\{ \frac{1}{\pi} \int_{\mathbb{R}} f(t) \frac{y}{(x-t)^2 + y^2} \, dt \right\}, \tag{C.15}$$

and so the result carries through to the averaged functions by Fubini's Theorem.

□

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