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APPROXIMATIONS IN RECONSTRUCTING DISCONTINUOUS CONDUCTIVITIES IN THE CALDERÓN PROBLEM

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APPROXIMATIONS IN RECONSTRUCTING DISCONTINUOUS CONDUCTIVITIES IN THE CALDERÓN PROBLEM

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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ABSTRACT OF DISSERTATION

APPROXIMATIONS IN RECONSTRUCTING DISCONTINUOUS CONDUCTIVITIES IN THE CALDERÓN PROBLEM

In 2014, Astala, Päivärinta, Reyes, and Siltanen conducted numerical experiments reconstructing a piecewise continuous conductivity. The algorithm of the shortcut method is based on the reconstruction algorithm due to Nachman, which assumes a priori that the conductivity is Hölder continuous. In this dissertation, we prove that, in the presence of infinite-precision data, this shortcut procedure accurately recovers the scattering transform of an essentially bounded conductivity, provided it is constant in a neighborhood of the boundary. In this setting, Nachman's integral equations have a meaning and are still uniquely solvable.

To regularize the reconstruction, Astala et al. employ a high frequency cutoff of the scattering transform. We show that such scattering transforms correspond to Beltrami coefficients that are not compactly supported, but exhibit certain decay at infinity. For this class of Beltrami coefficients, we establish that the complex geometric optics solutions to the Beltrami equation exist and exhibit the same subexponential decay as described in the 2006 work of Astala and Päivärinta. This is a first step toward extending the inverse scattering map of Astala and Päivärinta to non-compactly supported conductivities.

KEYWORDS: inverse problem, Calderón problem, Beltrami equations, Complex Geometric Optics solutions, quasiconformal mappings

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Date: April 26, 2019

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Dedicated to my wife, M.E, who took a chance on a math nerd.

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Chapter 1 What's Calderón's Problem Anyway?

This dissertation is about an inverse problem. What's that, you may ask? Well, it turns out that there is no universal definition of an inverse problem. But suffice it to say that a lot of inverse problems have a similar setup to what follows. Suppose you have an object with a property on the interior that you would like to know. Since you can't crack open the object and look inside, you are restricted to making certain measurements on the boundary of the object. Usually if we know the property on the inside we could predict what measurement you would get (this is called the direct problem). The *inverse problem*, then, is to recover the unknown property from the boundary measurement.

1.1 A Layperson's Introduction to the Calderón Problem

We'll look at an inverse problem that was first posed by Alberto Calderón [16]. Let's think about the electrical conductivity of the human body. Tissues conduct electricity in different ways. Blood, for example, is highly conductive. When your heart pumps blood into the lungs, the conductivity of your lung tissue is much higher than the conductivity of the heart muscle. This sharp contrast in the conductivity gives us an image. In order to recover this image, we attach leads to the patient, run specific current patterns through them, and measure the corresponding voltages. This is the boundary measurement for the inverse problem. The specific medical imaging application is called *electrical impedance tomography.*[1] Let's get a little bit more specific.

Let Ω be a bounded domain in $\mathbb{R}^3$. Let $\sigma(x)$ be the conductivity at a point $x \in \Omega$. For now, we'll assume that $\sigma \in L^\infty(\mathbb{R}^3)$. Into this body, we have put an electric potential $u(x)$. The total electric field density is given by $\vec{J} = -\sigma(x)\nabla u(x)$. But in this case, there is no current flowing out of the boundary. In terms of calculus, this means

$$\int_{\partial\Omega} \vec{J}(x) \cdot \nu(x)\, ds = 0 \tag{1.1}$$

where $\nu(x)$ is the unit outward normal for $x \in \partial\Omega$. Employing the Divergence Theorem, this means that the electric potential u satisfies the following partial differential equation (PDE).

$$\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega \tag{1.2}$$

$$u|_{\partial\Omega} = f \tag{1.3}$$

If we specify the voltage on the boundary, we can then measure the corresponding current density, $\sigma \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$. In terms of the PDE, this is the Neumann data corresponding to the Dirichlet problem.

[1]See [25] for an example of the medical imaging procedure.

We can think of the measurement process as the action of a boundary operator. This operator, which is called the **Dirichlet-to-Neumann operator**, denoted as Λ_σ has a heuristic definition as:

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} \tag{1.4}$$

We'll come back to this a little later with a more rigorous treatment of Λ_σ. So, the inverse conductivity problem can be boiled down to this: If we know the operator Λ_σ, can we recover the function σ?

Whenever we talk about an inverse boundary problem, there are a couple of things we could mean when we say, "solve the inverse problem." We'll give a brief discussion of what these are and the mathematicians who are working on these issues.

Identifiablility

This is the question of uniqueness. That is, if two conductivities σ, and $\tilde{\sigma}$ have Dirichlet-to-Neumann operators that are the same (i.e. $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$), then are the two conductivities equal ($\sigma = \tilde{\sigma}$)? This question was the one originally considered by Calderón, and it was also studied by others [8, 15, 31, 32, 48].

Reconstruction

Is there an algorithm that allows me to move from the boundary operator Λ_σ and get to the conductivity σ? This question is taken up in [37] and we'll discuss it further in the next section.

Stability

Do small errors in the Dirichlet-to-Neumann map correspond to small errors in the conductivity or are they amplified? The study of this type of question is the subject of [10, 11, 17, 21] and others.

Numerics

Is there a way to use numerical algorithms to approximate σ based off of a representation of Λ_σ as experimental data? This question has been studied in numerous ways starting with [5, 6, 9, 29, 30, 44] to name a few.

Partial Data

If you only know the boundary operator on a subset of the boundary, is it still possible to answer any of the above questions? What type of subset of the boundary do you need? This question is beyond the scope of this dissertation, but one can look further at [26, 27, 28, 49].

1.2 Nachman's Reconstruction Procedure

We are considering the Calderón problem in dimension two, which we identify with the complex plane $\mathbb{C}$. The goal is to reconstruct the conductivity σ of a conducting body Ω from boundary measurements. The electrical potential u obeys the equation

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \\ u|_{\partial\Omega} &= f \end{aligned} \tag{1.5}$$

where $f \in H^{1/2}(\partial\Omega)$ is the potential on the boundary. The Dirichlet-to-Neumann map $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is given by

$$\langle g, \Lambda_\sigma f \rangle = \int_\Omega \sigma \nabla u \cdot \nabla u \; dz \tag{1.6}$$

where $v \in H^1(\Omega)$ with boundary trace g, and $\langle \cdot\,, \cdot \rangle$ denotes the dual pairing of $H^{1/2}(\partial\Omega)$ with $H^{-1/2}(\partial\Omega)$.[2] The unique identifiability problem was solved for piecewise analytic conductivities by Kohn and Vogelius in [31, 32].[3] The breakthrough came in 1996 when Nachman [37] showed unique identifiability via a reconstruction algorithm. He introduced a change of variable

$$q = \frac{\Delta\left(\sigma^{1/2}\right)}{\sigma^{1/2}} \qquad v = \sigma^{1/2} u \tag{1.7}$$

which transforms the conductivity equation (1.5) into the Schrödinger equation at zero energy. Thus Nachman assumes a priori that $\sigma \in W^{2,p}(\mathbb{C})$ for $1 < p < 2$. By Morrey's inequality, this implies that σ is Hölder continuous. The main component of Nachman's approach is the family of exponentially growing solutions to the Schrödinger equation first discovered by Faddeev [20],

$$\begin{aligned} (-\Delta + q)\psi(z,\xi) &= 0, \\ \lim_{|z|\to\infty} \psi(z,\xi)e^{-i\xi z} - 1 &= 0. \end{aligned} \tag{1.8}$$

where $\xi \in \mathbb{C}$ and ξz denotes $\mathbb{C}$-multiplication. These solutions are called *Complex Geometric Optics* (CGO) solutions.[4] To analyze these solutions, it is helpful to introduce the operators $\partial = \frac{1}{2}(\partial_{z_1} - i\partial_{z_2})$ and $\overline{\partial} = \frac{1}{2}(\partial_{z_1} + i\partial_{z_2})$, which are the derivatives with respect the the complex variables z and $\overline{z}$, respectively. In this notation, $-\Delta = -4\partial\overline{\partial}$. If we write $m(z,\xi) = e^{-i\xi z}\psi(z,\xi)$, then m is called the *normalized CGO solution.* Substituting the normalized solution into (1.8) gives

$$\begin{aligned} \overline{\partial}(\partial + i\xi)m(z,\xi) &= \frac{1}{4}q(z)m(z,\xi), \\ \lim_{z\to\infty} m(z,\xi) &= 1 \end{aligned} \tag{1.9}$$

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[2]You can arrive at this definition by taking the heuristic $\Lambda_\sigma f = \sigma \frac{\partial u}{\partial \nu}$ and integrating by parts.

[3]Interestingly, these authors consider the problem in terms of heat conductivity instead of electrical conductivity.

[4]The term CGO solution was first coined by Sylvester and Uhlmann [48] in the context of solving the unique identifiability problem in dimensions three and higher.

The Green's function for this equation was further studied in [43], and we will give a brief overview of its properties in the next chapter.

The CGO solutions can be used to define a nonphysical scattering transform of the potential q,

$$\mathbf{t}(\xi) = \int e_\xi(z) m(z,\xi) q(z)\ dz \tag{1.10}$$

where $e_\xi(z)$ is the phase

$$e_\xi(z) = \exp(i(\xi z + \overline{\xi z})). \tag{1.11}$$

This scattering transform can be thought of as a nonlinear Fourier transform of q. If you substitute $m = (m-1)+1$ into equation (1.10), you get the Fourier transform of q (up to a linear transformation) and a nonlinear correction. The nonlinearity is in fact quite profound since m itself depends on q via (1.9).

The "miracle" of CGO solutions is that they solve an equation in both z and ξ. This $\overline{\partial}$-equation was studied in the context of the two-dimensional Schrödinger problem at zero energy by Boiti, Lon, Manna, and Pempinelli [14]. For potentials originating from the inverse conductivity problem, Nachman showed that the normalized CGO solutions solve

$$\begin{aligned} \overline{\partial}_\xi m(z,\xi) &= \frac{\mathbf{t}(\xi)}{4\pi\overline{\xi}} e_{-\xi}(z)\overline{m(z,\xi)} \\ \lim_{\xi\to\infty} m(z,\xi) &= 1 \end{aligned} \tag{1.12}$$

To see where (1.12) comes from, we can recast (1.9) as an integral equation involving the Faddeev Green's function $g_\xi(z)$,

$$m = 1 + g_\xi * (qm) = 1 + T_\xi m. \tag{1.13}$$

We differentiate this with respect to ξ,

$$\overline{\partial}_\xi m = (\overline{\partial}_\xi T_\xi)m + T_\xi(\overline{\partial}_\xi m) \tag{1.14}$$

where $\overline{\partial}_\xi T_\xi$ is a "derivative" of the operator, which turns out to be convolution with the $\overline{\partial}_\xi$ derivative of $g_\xi(z)$.

The key point is that the scattering transform of q can be determined directly from the Dirichlet-to-Neumann operator. Nachman [37, Section 6] employs a reduction that allows us to make the a priori assumption that $\sigma(z) = 1$ in a neighborhood of $\partial\Omega$, and thus extend $\sigma \equiv 1$ outside Ω. Under the change of variable, this implies that q has compact support contained within Ω. Let Λ_q denote the Dirichlet-to-Neumann operator for the *Schrödinger problem*

$$\begin{aligned} (-\Delta + q)\psi &= 0, \\ \psi|_{\partial\Omega} &= f. \end{aligned} \tag{1.15}$$

Let Λ_0 be the Dirichlet-to-Neumann operator for harmonic functions on Ω, corresponding to $q(z) \equiv 0$ and $\sigma(z) \equiv 1$.[5] The reduction given by Nachman also implies that $\Lambda_q = \Lambda_\sigma$. The compact support of q in the reduction proves crucial, as one can reduce (1.8) and (1.10) to the boundary integral equations

$$\psi|_{\partial\Omega} = e^{i\xi z}\big|_{\partial\Omega} - S_\xi(\Lambda_q - \Lambda_0)\left(\psi|_{\partial\Omega}\right) \tag{1.16}$$

$$\mathbf{t}(\xi) = \int_{\partial\Omega} e^{i\overline{\xi z}}(\Lambda_q - \Lambda_0)\left(\psi|_{\partial\Omega}\right)\, ds. \tag{1.17}$$

Here S_ξ is convolution with the Faddeev Green's function on $\partial\Omega$. The properties of S_ξ will be addressed in Section 2.2. The boundary integral equations (1.16) were first introduced by R. Novikov [39]. Therefore, given $\mathbf{t}$, one can then solve the $\overline{\partial}$-problem (1.12) and recover σ from

$$\sigma(z) = \lim_{\xi \to 0} m(z,\xi)^2.$$

In what follows, we will use a standard reduction due to Nachman [37, Section 6]. Without loss, we may assume that Ω is the unit disc $\mathbb{D}$ and that $\sigma(z) = 1$ in a neighborhood of $\partial\mathbb{D}$. Since Ω is a bounded domain, you can use the constant extension of σ to compute the Dirichlet-to-Neumann operator for a disc, using the original Dirichlet-to-Neumann operator for σ. A scaling argument then allows you to consider $\mathbb{D}$.

1.3 Brown and Uhlmann's Contribution

After Nachman's breakthrough, the challenge was to reduce the a priori regularity assumptions on the conductivity σ. Brown and Uhlmann [15] made the change of variable

$$q = -\frac{1}{2}\partial \log \sigma \tag{1.18}$$

and define a matrix potential Q as

$$Q = \begin{bmatrix} 0 & q \\ \overline{q} & 0 \end{bmatrix}. \tag{1.19}$$

Let D be the operator

$$D = \begin{bmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{bmatrix}. \tag{1.20}$$

Then if u solves the conductivity equation (1.5), then the vector

$$\begin{bmatrix} v \\ w \end{bmatrix} = \sigma^{1/2} \begin{bmatrix} \partial u \\ \overline{\partial} u \end{bmatrix}$$

solves the system

$$D \begin{bmatrix} v \\ w \end{bmatrix} - Q \begin{bmatrix} v \\ w \end{bmatrix} = 0 \tag{1.21}$$

[5] Later, we will use Λ_1 to denote this operator when referring to the conductivity setting.

The system (1.21) was studied by Beals and Coifman [12, 13] in the context of the Davey-Stewartson-II equation. The approach of Brown and Uhlmann assumes a priori that $\sigma \in W^{1,p}(\mathbb{C})$ with $2 < p < \infty$.[6] We will return to this setup in a later chapter.

1.4 Astala and Päivärinta's Solution

The Calderón problem was solved in full generality for two dimensions by Astala and Päivärinta in 2006 [7, 8].

Theorem 1.4.1. *[8, Theorem 1] Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain and $\sigma_i \in L^\infty(\Omega)$, $i = 1, 2$. Suppose that there is a constant $c > 0$ such that $c^{-1} \leq \sigma_i \leq c$. If*

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$$

then $\sigma_1 = \sigma_2$.

They consider the case $\sigma \in L^\infty(\mathbb{D})$ where $0 < c \leq \sigma(z)$ for almost every $z \in \mathbb{D}$. They extend σ to be $\sigma(z) = 1$ for $z \in \mathbb{C} \setminus \mathbb{D}$. If u solves the conductivity equation (1.5), there is a function v that solves the companion equation

$$\nabla \cdot (\sigma^{-1} \nabla v) = 0 \tag{1.22}$$

called the σ-harmonic conjugate of u such that the function $f = u + iv$ solves the Beltrami equation

$$\overline{\partial} f = \mu \overline{\partial f} \tag{1.23}$$

where the Beltrami coefficient μ is

$$\mu(z) = \frac{1 - \sigma(z)}{1 + \sigma(z)}. \tag{1.24}$$

The assumptions on σ imply that μ is real-valued, $\|\mu\|_\infty \leq k < 1$, and the support of μ is in $\mathbb{D}$. Astala and Päivärinta show that the Beltrami equation (1.23) admits CGO solutions of the form

$$f_\mu(z, \xi) = e^{i\xi z} M_\mu(z, \xi). \tag{1.25}$$

These CGO solutions define a scattering transform analogous to Nachman's $\mathbf{t}$, and the transform remains well-defined under the weaker assumption that $\mu \in L^\infty(\mathbb{D})$. Their first result concerns the existence and uniqueness of CGO solutions.

Theorem 1.4.2. *[8, Theorem 4.2] Let $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_\infty \leq k < 1$. For each $\xi \in \mathbb{C}$, and each $p \in (2, 1 + k^{-1})$, there exists a unique solution $f_\mu \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ of* (1.23) *of the form* (1.25) *where $M_\mu(z, \xi) - 1 \in W^{1,p}(\mathbb{C})$.*

[6] Morrey's inequality again shows that such a σ is Hölder continuous.

We refer to M_μ as the normalized CGO solution of (1.23) and denote by $M_{\pm\mu}$ the normalized solutions corresponding to μ and $-\mu$. The associated scattering transform τ_μ is given by

$$\overline{\tau_\mu(\xi)} = \frac{1}{2\pi} \int \overline{\partial}(M_\mu(z,\xi) - M_{-\mu}(z,\xi))\, dz. \tag{1.26}$$

There is a way to obtain corresponding CGO solutions to the conductivity equation $\nabla \cdot (\sigma \nabla u) = 0$ via the formula

$$2u = f_\mu(x,k) + \overline{f_\mu(x,k)} + f_{-\mu}(x,k) - \overline{f_{-\mu}(x,k)} \tag{1.27}$$

see [6, Equation (1.7)]. Thus both the solutions for μ and $-\mu$ are needed.

If μ is sufficiently regular, the scattering transforms $\mathbf{t}$ (1.10) and τ_μ are related by

$$\mathbf{t}(\xi) = -4\pi i \overline{\xi} \tau_\mu(\xi) \tag{1.28}$$

The CGO solutions to the Beltrami equation also satisfy a $\overline{\partial}$-problem in ξ, but the assumption $\mu \in L^\infty(\mathbb{D})$ does not provide τ with enough regularity for the $\overline{\partial}$-problem to be solved. Instead, the authors establish unique identifiability through a careful topological argument based on the behavior of the CGO solutions as the parameter $\xi \to \infty$. The CGO solutions satisfy what they call *subexponential growth*. If we write the solution f to (1.23) as

$$f_\mu(z,\xi) = e^{i\xi\varphi(z,\xi)} \tag{1.29}$$

then the function φ solves a nonlinear Beltrami equation.

Theorem 1.4.3. *[8, Theorem 7.2] For the CGO solution f_μ to* (1.23)*, let φ be the function in the representation* (1.29)*. Then as $\xi \to \infty$,*

$$\varphi(z,\xi) \to z \tag{1.30}$$

uniformly in $z \in \mathbb{C}$.

In the uniqueness proof of Astala and Päivärinta, it is shown first that Λ_σ uniquely determines τ_μ. The goal is to show that if $\tau_\mu = \tau_{\tilde{\mu}}$, then the CGO solutions must also be the same (and hence the conductivities are the same). Theorem 1.4.3 and the conversion formula (1.27) allow one to write down the asymptotics of the CGO solution u of the conductivity equation as $\xi \to \infty$. The asymptotics of u are then used to conclude uniqueness via a topological argument. In this way, subexponential growth is a first step toward solving the uniqueness question.

1.5 A Numerical Investigation for Discontinuous Conductivities

We will now turn our attention to the motivating questions for the following chapters. In [9], Astala, et al. implement numerical experiments on conductivities with jump discontinuities. The goal was to compare the algorithms based on the approaches of Nachman and Astala-Päivärinta. The *shortcut method* that they study is formally similar the to the Nachman theory-based regularized algorithm studied in Knudsen et al. [30].

A finite-dimensional approximation to the Dirichlet-to-Neumann operator is used to simulate experimental data. The shortcut method uses Nachman's integral equations (1.16) and (1.17) to represent the scattering transform of the conductivity. A high-frequency cutoff on the scattering transform is implemented to make efficient and tractable code. For smooth conductivities, the high-frequency cutoff can be understood as a regularization technique for the inversion process [30]. The frequency cutoff ensures that the scattering transform lies in the appropriate space for the $\bar{\partial}$-method of Nachman (1.12) to be used for the recovery of σ.

Surprisingly, the shortcut method produced qualitatively accurate results for *discontinuous* conductivities. Astala et al. then computed reconstructions for different cutoff levels in the nonlinear frequency domain. In practical situations, the cutoff radius cannot exceed 7, but the authors utilize more powerful computing to calculate reconstructions using a cutoff radius of 60. The results of these reconstructions give numerical evidence of a nonlinear analogue of Gibbs phenomenon. The Gibbs phenomenon in Fourier theory describes the oscillations that take place in a finite Fourier series approximation of a step function. As the number of terms in the series increases, the oscillations cluster together. For more information, see [50] A similar nonlinear Gibbs phenomenon for the nonlinear Schrödinger equation was studied in [18].

The rigorous analysis of the shortcut method is the motivation of this dissertation. Two questions arise in the presence of infinite-precision data (i.e. the full Dirichlet-to-Neumann operator).

- Can one accurately obtain the scattering transform of a discontinuous conductivity using Nachman's integral equations?

- Do the reconstructions obtained from a frequency cutoff at level R converge to the conductivity as $R \to \infty$?

Thematically, these questions are questions of continuity of the direct map ($\sigma \mapsto \tau$) and the inverse map ($\tau \mapsto \sigma$).

1.6 Main Results

Motivated by these questions, we present the main results of this dissertation. The first two theorems answer the first question in the affirmative.

Theorem 1.6.1. *Let $\sigma \in L^\infty(\mathbb{D})$ with $\sigma(z) \geq c$ for a fixed $c > 0$, and suppose that there is an $r_1 \in (0,1)$ such that $\sigma(z) = 1$ for $|z| \geq r_1$. For each $\xi \in \mathbb{C}$, there exists a unique $g \in H^{1/2}(\partial\mathbb{D})$ so that*

$$g = e^{i\xi z}\big|_{\partial\mathbb{D}} - S_\xi(\Lambda_\sigma - \Lambda_1)g.$$

Thus Nachman's integral equation (1.16) can be solved, even when σ is not continuous. We can use (1.17) to write

$$\mathbf{t}(\xi) = \int_{\partial\mathbb{D}} e^{i\overline{\xi z}}(\Lambda_\sigma - \Lambda_1)g \, ds \tag{1.31}$$

and know that the integral converges for each $\xi \in \mathbb{C}$. In fact, the next theorem shows how this function is related to the scattering transform of σ.

Theorem 1.6.2. *Suppose that σ is a fixed conductivity with strictly positive essential infimum, and that $\{\sigma_n\}$ is a sequence of smooth conductivities in $\mathbb{D}$ obeying*

(i) There is a fixed $r_1 \in (0,1)$ so that $\sigma_n(z) = 1$ for $|z| \geq r_1$ for all n (and for σ),

(ii) There is a fixed $c > 0$ so that $\sigma_n(z) \geq c$ for a.e. $z \in \mathbb{D}$ and for all n (and for σ),

(iii) For a.e. z, $\sigma_n(z) \to \sigma(z)$ as $n \to \infty$.

Denote by $\mathbf{t}_n$ (resp. $\mathbf{t}$) the scattering transform for σ_n (resp. σ) obtained from (1.16)-(1.17)*. Then $\mathbf{t}_n \to \mathbf{t}$ pointwise. Moreover, $\mathbf{t}$ is related to the Astala-Päivärinta scattering transform τ for σ by* (1.28)*.*

These theorems show that with infinite precision, the shortcut method in [9] accurately recovers the scattering transform of σ from the Dirichlet-to-Neumann operator Λ_σ. The proofs of these theorems are found in Chapter 3.

The second motivating question is much harder. The proper setting to study the convergence of the reconstructed conductivities is not evident. The first step is to extend the framework of Astala-Päivärinta to include conductivities that produce truncated scattering transforms. Chapter 4 is devoted to this process, which involves a new notion of principal solutions for Beltrami equations with non-compactly supported coefficient. The first theorem describes the conductivities that produce a cutoff scattering transform.

Theorem 1.6.3. *Suppose $\tau \in C_0^\infty(\mathbb{C})$ satisfies $\tau(0) = 0$ and $|\tau(\xi)| \leq 1$. Then τ corresponds to a continuous Beltrami coefficient μ that satisfies*

$$\|\mu\|_\infty \leq k < 1 \text{ and } \mu(z) \sim \mathcal{O}\left(\frac{1}{|z|^2}\right) \text{ as } z \to \infty \tag{1.32}$$

Consequently, $\mu \in L^r(\mathbb{C})$ for all $1 < r \leq \infty$. Moreover, $\partial\mu \in L^r(\mathbb{C})$ for all $1 < r \leq \infty$.

The first piece of the Astala-Päivärinta framework is to show the existence and uniqueness of CGO solutions to the Beltrami equation with our class of Beltrami coefficients.

Theorem 1.6.4. *Suppose that μ is a real-valued measurable function with $\|\mu\|_\infty \leq k < 1$ and that $\mu \in L^r(\mathbb{C})$ for all r with $1 < r < \infty$. Let $2 < p < 1 + k^{-1}$. Then for each $\xi \in \mathbb{C}$ there exists a unique solution $f \in W^{1,p}_{loc}(\mathbb{C})$ to the equation*

$$\overline{\partial} f = \mu \overline{\partial f} \tag{1.33}$$

where f can be written as $f(z,\xi) = e^{i\xi z} M(z,\xi)$ and

$$M(\cdot,\xi) - 1 \in W^{1,p}(\mathbb{C}) \tag{1.34}$$

In a similar fashion to [8], we consider the phenomenon of subexponential growth. We write $f(z,\xi) = e^{i\xi\varphi(z,\xi)}$. This implies that φ solves

$$\overline{\partial}\varphi(z,\xi) = -\frac{\bar{\xi}}{\xi}\mu(z) e_{-\xi}(\varphi(z,\xi)) \overline{\partial\varphi(z,\xi)}. \tag{1.35}$$

In [8], the authors can appeal to the theory of quasiconformal maps to conclude that φ exists and is a homeomorphism. For our class of Beltrami coefficients, we need a new notion of principal solution to Beltrami equations. We are grateful to Kari Astala for his correspondence helping us make a sensible definition. We will discuss this new definition later in Chapter 4. We proved that equation (1.35) has a principal solution in this context.

Theorem 1.6.5. *Suppose that μ is a real-valued measurable function with $\|\mu\|_\infty \leq k < 1$, and that $\mu \in L^r(\mathbb{C})$ for $1 < r < \infty$. Let $1 + k < q < 2 < p < 1 + k^{-1}$. Then, for each $\xi \in \mathbb{C}$ there exists a unique solution φ to the nonlinear Beltrami equation* (1.35) *such that $\varphi - z \in W^{1,p}(\mathbb{C})$.*

The continuity of the solution to (1.35) comes from Morrey's inequality, but we can actually show more.

Theorem 1.6.6. *For each fixed $\xi \in \mathbb{C}$, the solution $\varphi(\cdot,\xi)$ from Theorem 1.6.5 is a global homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$.*

After these properties of the solution to (1.35) are established, we show that the CGO solutions for our class of Beltrami coefficients exhibit subexponential growth.

Theorem 1.6.7. *Suppose that μ is a real-valued measurable function with $\|\mu\|_\infty \leq k < 1$, $\mu \in L^r(\mathbb{C})$, and $\partial\mu \in L^r(\mathbb{C})$ for all r with $1 < r < \infty$. Suppose $2 < p < 1 + k^{-1}$. Let φ be the solution to equation* (1.35) *from Theorem 1.6.5. Then*

$$\varphi(z,\xi) \to z \tag{1.36}$$

uniformly in $z \in \mathbb{C}$ as $\xi \to \infty$.

The analysis of the Astala-Päivärinta framework for conductivities with truncated scattering transform is taken up in Chapter 4. Subsequent work to this dissertation will examine continuity of the inverse map in this new setting. Unique identifiability for this class of Beltrami coefficients has a ways to go yet. The following steps in Astala and Päivärinta's framework would have to be proved.

- Starting with a Beltrami coefficient that is decaying like $\mathcal{O}\left(|z|^{-2}\right)$ as $z \to \infty$, is the corresponding τ compactly supported?
- Does the τ generated by our μ satisfy $|\tau(\xi)| \leq 1$?
- Once uniqueness is shown in this setting, can we show the convergence of reconstructions as the cutoff radius goes to ∞?

These questions and more will be taken up in future work. The next chapter is a compendium of mathematical tools that will be used in the subsequent analysis. Chapter 3 is devoted to the proofs of Theorem 1.6.1 and Theorem 1.6.2. In Chapter 4, we take up the analysis of Beltrami coefficients with truncated scattering transforms and prove Theorem 1.6.3, Theorem 1.6.4, Theorem 1.6.5, Theorem 1.6.6, and Theorem 1.6.7.

Chapter 2 Preliminaries

The following is a collection of the common mathematical tools that we'll use in proving our results. Here and in what follows, we use the notation $f \lesssim_c g$ to mean that $f \leq Cg$ where the implied constant C depends on the quantities c.

2.1 H^s Spaces, Fourier Basis, Harmonic extensions

An L^2 function $f \in L^2(\partial\mathbb{D})$ admits a Fourier series expansion $f(\theta) \sim \sum_n b_n \varphi_n(\theta)$, where

$$\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}.$$

The equation

$$(P_j f)(\theta) = \sum_{|n| \leq j} b_n \varphi_n(\theta) \tag{2.1}$$

for $j \in \mathbb{N}$ defines a finite-rank projection. For $s \in \mathbb{R}$, we denote by $H^s(\partial\mathbb{D})$ the completion of $C^\infty(\partial\mathbb{D})$ in the norm

$$\|f\|_{H^s(\partial\mathbb{D})} = \left(\sum_{n=-\infty}^{\infty} (1 + |n|)^{2s} |b_n|^2 \right)^{1/2}.$$

It is easy to see that the embedding

$$H^s(\partial\mathbb{D}) \hookrightarrow H^{s'}(\partial\mathbb{D}) \tag{2.2}$$

is compact provided $s > s'$.

The harmonic extension of $f \in L^2(\partial\mathbb{D})$ to $\mathbb{D}$ is given by

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} b_n \varphi_n(\theta).$$

It is easy to see that for any $r_1 \in (0, 1)$, the estimate

$$\|u\|_{L^2(|z|<r_1)} \lesssim_{m, r_1} \|f\|_{H^{-m}} \tag{2.3}$$

holds for the harmonic extension.

2.2 Faddeev's Green's Function and the operator S_ξ

The Faddeev Green's function is the convolution kernel $G_\xi(z - y)$ where

$$G_\xi(z) = \frac{e^{i\xi z}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\eta}}{|\eta|^2 + 2\xi(\eta_1 + i\eta_2)} \, d\eta \tag{2.4}$$

where $z \cdot \eta = z_1\eta_1 + z_2\eta_2$ and $\xi \in \mathbb{C}$. This is the natural Green's function for the elliptic problem (1.9). Writing $G_\xi(z) = e^{i\xi z} g_\xi(z)$ we see that $g_\xi(z)$ differs from the Green's function $G_0(z) = -(2\pi)^{-1} \log|z|$ of the Laplacian by a function which is smooth and harmonic on all of $\mathbb{R}^2$ and, in particular, is regular at 0 (see, for example, [43, Section 3.1] for further discussion and estimates).

In the reduction of (1.9) to the boundary integral equation (1.16), the operator S_ξ is the corresponding single layer

$$\left(S_\xi f\right)(z) = \int_{\partial\Omega} G_\xi(z-y) f(y)\, dy. \tag{2.5}$$

For $p \in (1,\infty)$ and any $f \in L^p(\partial\Omega)$, the function $S_\xi f$ is smooth and harmonic on $\mathbb{R}^2 \setminus \partial\Omega$. We will assume $\Omega \subset \mathbb{R}^2$ is bounded and simply connected with smooth boundary (since our application is to $\Omega = \mathbb{D}$) even though the assertions below are known in greater generality. Moreover, since the convolution kernel G_ξ is at most logarithmically singular, $S_\xi f$ restricts to a well-defined a function on $\partial\Omega$. When restricted to $\partial\Omega$,

$$S_\xi : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), \quad s \in [-1, 0] \tag{2.6}$$

(see [37, Lemma 7.1]), even if Ω only has Lipschitz boundary.

It follows from the form of $G_\xi(z)$ and classical potential theory that, if $\nu(z)$ is the unit normal to $\partial\Omega$ at $z \in \partial\Omega$, the identities

$$\lim_{\substack{y\to z\\ z\in\mathbb{R}^2\setminus\Omega}} \langle \nu(z), \left(\nabla S_\xi f\right)(y)\rangle = -\left(\frac{1}{2}I - S_\xi\right) f(z) \tag{2.7}$$

$$\lim_{\substack{y\to z\\ y\in\Omega}} \langle \nu(z), \left(\nabla S_\xi f\right)(y)\rangle = -\left(\frac{1}{2}I + S_\xi\right) f(z) \tag{2.8}$$

hold.

2.3 Alessandrini Identity

We will make extensive use of the following identity [2] which is an easy consequence of Green's theorem. Suppose that u solves (1.5) and that $v \in H^1(\Omega)$ with boundary trace $g \in H^{1/2}(\partial\Omega)$. Then

$$\langle g, \Lambda_\sigma f\rangle = \int_\Omega \sigma(z)(\nabla u)(z) \cdot (\nabla v)(z)\, dz \tag{2.9}$$

where $\langle g, h\rangle$ denotes the dual pairing of $g \in H^{1/2}(\partial\Omega)$ with $h \in H^{-1/2}(\partial\Omega)$.

2.4 A Priori Estimates and Uniqueness Theorems

We'll need the following results from [4] which we state here for the reader's convenience. First, we need the following *a priori* estimate on solutions of Beltrami's equation. This estimate will allow us to analyze convergence of CGO solutions to the Beltrami equations assuming that the Beltrami coefficients converge pointwise.

Theorem 2.4.1. *[4, Theorem 5.4.2] Let* $f \in W^{1,q}_{loc}(\Omega)$, *for some* $q \in (1+k, 1+\frac{1}{k})$, *satisfy the distortion inequality*

$$\left|\overline{\partial} f\right| \leq k \left|\partial f\right|$$

for almost every $z \in \Omega$. *Then* $f \in W^{1,p}_{loc}(\Omega)$ *for every* $p \in (1+k, 1+\frac{1}{k})$. *In particular,* f *is continuous, and for every* $s \in (1+k, 1+\frac{1}{k})$, *the critical interval, we have the Caccioppoli estimate*

$$\left\|\eta \nabla f\right\|_s \leq C_s(k) \left\|f \nabla \eta\right\|_s \tag{2.10}$$

whenever η *is a compactly supported Lipschitz function in* Ω.

We will use a Liouville-type theorem taken from [4].

Theorem 2.4.2 (Theorem 8.5.1 in [4])**.** *Suppose* $F \in W^{1,2}_{loc}(\mathbb{C})$ *satisfies the homogeneous distortion inequality*

$$|\overline{\partial} F| \leq k|\partial F| + \alpha(z)|F|$$

where $\alpha \in L^p \cap L^q$ *for some* $1 < q < 2 < p < \infty$ *and* $0 \leq k < 1$. *Then* F *is continuous. Moreover, if*

$$\lim_{z \to \infty} F(z) = 0$$

then $F \equiv 0$.

The following uniqueness theorem for CGO solutions of the conductivity equation will help establish the unique solvability of the integral equation (1.16).

Theorem 2.4.3. *[4, Corollary 18.1.2] Suppose that* $\sigma, 1/\sigma \in L^\infty(\mathbb{D})$ *and that* $\sigma(z) \equiv 1$ *for* $|z| \geq 1$. *Then the equation* $\nabla \cdot (\sigma \nabla u) = 0$ *admits a unique weak solution* $u \in W^{1,2}_{loc}(\mathbb{C})$ *such that*

$$\lim_{|z| \to \infty} \left(e^{-i\xi z} u(z, \xi) - 1\right) = 0. \tag{2.11}$$

2.5 Useful Operators and Estimates

The Hardy-Littlewood-Sobolev inequality is one of the tools used to study fractional integrals. We will use it to study the decay of the Cauchy transform of a function. It also plays a fundamental role in the analysis of $\overline{\partial}$-problems. For a proof, see for example [34, Section 2.2]. A sharp constant for the Hardy-Littlewood-Sobolev inequality together with an explicit maximizer is given in [33]; see [22] for a simplified proof of the optimal inequality.

Theorem 2.5.1 (Hardy-Littlewood-Sobolev Inequality)**.** *Suppose that* $0 < \alpha < n$, $1 < p < q < \infty$, *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

If $f \in L^p(\mathbb{R}^n)$, *the integral*

$$(I_\alpha f)(x) = \int \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

converges absolutely for a.e. x, and the estimate

$$\|I_\alpha(f)\|_q \lesssim_{\alpha,p,n} \|f\|_p \tag{2.12}$$

The solid Cauchy transform P is convolution with the fundamental solution for $\overline{\partial} u = f$.

$$(Pf)(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-w} f(w)dw \tag{2.13}$$

The following estimates are standard (see, e.g. [4], Theorems 4.3.8, 4.3.11, and 4.3.13).

Lemma 2.5.2. *For any $p > 2$ and $1 < q < 2 < p < \infty$, the solid Cauchy transform P obeys the following estimates:*

$$\|Pf\|_p \lesssim_p \|f\|_{2p/(p+2)}\,, \tag{2.14}$$

$$\|Pf\|_\infty \lesssim_{p,q} \|f\|_{L^p\cap L^q}\,, \tag{2.15}$$

$$\sup_{z\neq w} \frac{|(Pf)(z)-(Pf)(w)|}{|z-w|^{1-2/p}} \lesssim_p \|f\|_p\,. \tag{2.16}$$

Moreover, if $f \in L^p \cap L^q$ for $1 < q < 2 < p < \infty$, $(Pf)(z) \to 0$ as $|z| \to \infty$.

The Beurling transform S is a singular integral operator of Calderón-Zygmund type. It is defined as the principal value-integral on $C_0^\infty(\mathbb{C})$

$$(Sf)(z) = \text{p.v.} \int \frac{f(w)}{(z-w)^2}\, dw. \tag{2.17}$$

The Beurling transform can be extended to a bounded operator from $L^p(\mathbb{C})$ to itself for all $p \in (1,\infty)$, and it is an isometry on $L^2(\mathbb{C})$. Another important property of S is

$$(Sf)(z) = \partial(Pf)(z), \tag{2.18}$$

so that symbolically, $S = \partial\overline{\partial}^{-1}$. For more information, see Chapter 4 of [4]. We define

$$(\bar{S}f)(z) = \overline{\partial \overline{Pf}}(z) \tag{2.19}$$

Lemma 2.5.3. *For any $p \in (1,\infty)$, $\|\bar{S}(f)\|_p \lesssim_p \|f\|_p$. If $\mu \in L^\infty$ with $\|\mu\|_\infty \leq k < 1$, then $\|\mu\bar{S}\|_{L^p\to L^p} < 1$ for all $p \in (1+k, 1+k^{-1})$.*

2.6 Compactness and Fixed Point Theorems

To study the nonlinear Beltrami equation, we will need the following classical fixed point theorem.

Theorem 2.6.1 (Schauder Fixed Point Theorem [42])**.** *Let $\mathcal{D}$ be a closed convex subset of a Banach space X and let $T : \mathcal{D} \to \mathcal{D}$ be a continuous map. If the image $T(\mathcal{D})$ has compact closure, then T has a fixed point.*

We will also need the following facts about linear operators between L^p-spaces.

Definition 2.6.2. *A Banach space X has the **approximation property** if every compact operator is the norm-limit of a sequence of finite-rank operators.*

Proposition 2.6.3 (see [41])**.** *Suppose $1 \leq p, q < \infty$. Then the Banach space $\mathcal{L}(L^p(\mathbb{C}), L^q(\mathbb{C}))$ has the approximation property.*

The following characterization of pre-compact sets in $L^p(\mathbb{C})$ will come in handy in Chapter 4.

Theorem 2.6.4 (Kolmogorov-Riesz Theorem)**.** *Let $K \subset L^q(\mathbb{C})$ be a bounded set. Then K is relatively compact if and only if*

1. *Uniform Decay:* $\displaystyle\lim_{R\to\infty} \int_{|x|\geq R} |f|^q \, dx = 0$ *uniformly in K.*
2. *Uniform L^q Continuity:* $\displaystyle\lim_{a\to 0} \|f(\cdot - a) - f(\cdot)\|_q = 0$ *uniformly in K.*

Another resource on the Kolmogorov-Riesz theorem can be found in the review papers by Hanche-Olsen and Holden [23, 24].

We frequently refer to the Fredholm alternative, a method to show existence and uniqueness for equations involving compact operators.

Theorem 2.6.5. *[40, Corollary to Theorem VI.14] Suppose A is a compact operator on $L^p(\mathbb{C})$ for $1 < p < \infty$. Then either $(I - A)^{-1}$ exists or $A\psi = \psi$ has a solution.*

Chapter 3 Nachman's Reconstruction Method for Discontinuous Conductivities

In this chapter, we will study the shortcut method described in [9] to obtain the scattering transform of a piecewise continuous conductivity. We will show that Nachman's integral equations are still uniquely solvable for the Dirichlet-to-Neumann operator of a positive, essentially bounded conductivity with strictly positive essential lower bound. Moreover, we identify the resulting scattering transform as a natural analogue of Nachman's scattering transform which is, in fact, a limit of scattering transforms obtained through pointwise approximation by smooth functions. A key ingredient in our analysis is the Beltrami equation of Astala-Päivärinta and the associated scattering transform, which provides a way of identifying the 'scattering transform' that arises from the limit of Nachman's equations. The results in this chapter can also be found in [35].

To describe our results, we first recall a standard reduction due to Nachman [37, Section 6]. Without loss, we may assume that Ω from Chapter 1 is the unit disc $\mathbb{D}$ and that $\sigma(x) \equiv 1$ in a neighborhood of $\mathbb{D}$. We make the second assumption more precise:

$$\text{There is an } r_1 \in (0,1) \text{ so that } \sigma(z) = 1 \text{ for } |z| \geq r_1. \tag{3.1}$$

Nachman's integral equations for the trace of CGO solutions to the Schrödinger problem can also be expressed by

$$\psi|_{\partial\mathbb{D}} = e^{i\xi z}\big|_{\partial\mathbb{D}} - S_\xi \left(\Lambda_\sigma - \Lambda_1\right)\left(\psi|_{\partial\mathbb{D}}\right), \tag{3.2}$$

$$\mathbf{t}(\xi) = \int_{\partial\mathbb{D}} e^{i\bar{\xi}\bar{z}} \left(\Lambda_\sigma - \Lambda_1\right)\left(\psi|_{\partial\mathbb{D}}\right) ds \tag{3.3}$$

where by abuse of notation we write Λ_1, the Dirichlet-to-Neumann operator with $\sigma(z) = 1$. Observe that, under our assumption (3.1), a solution ψ of the Schrödinger problem(1.15) generates a solution of the conductivity equation (1.5) via $u(z) = \sigma(z)^{-1/2}\psi(z)$, and the Dirichlet-to-Neumann operators for (1.15) and (1.5) are in fact *identical.* Thus, under the assumption (3.1), we can recast (1.16) and (1.17) in terms of the Dirichlet-to-Neumann operators for the original conductivity problem, taking Ω to be the unit disc $\mathbb{D}$. Our first result is that (3.2) is uniquely solvable for $\sigma \in L^\infty$ with strictly positive essential infimum and any ξ.

Theorem 3.0.1. *Let $\sigma \in L^\infty(\mathbb{D})$ with $\sigma(z) \geq c$ for a fixed $c > 0$, and suppose that* (3.1) *holds. For each $\xi \in \mathbb{C}$, there exists a unique $g \in H^{1/2}(\partial\mathbb{D})$ so that*

$$g = e^{i\xi z} - S_\xi \left(\Lambda_\sigma - \Lambda_1\right) g.$$

As we will see, (3.1) implies that $\Lambda_\sigma - \Lambda_1$ is smoothing even though σ may be nonsmooth. One can then mimic Nachman's original argument from Fredholm theory to prove the unique solvability. We will show that the "scattering transform" generated by (3.3) is a natural limit of smooth approximations, and remains related to the Astala-Päivärinta scattering transform τ by

$$\mathbf{t}(\xi) = -4\pi i\bar{\xi}\tau_\mu(\xi), \tag{3.4}$$

which holds for smooth conductivities [9, Equation (2.10)]. In this context, even though the Schrödinger problem now involves a distribution potential, we can use this to represent the scattering transform of σ.

To make this connection, we consider approximation of $\sigma \in L^\infty$ by smooth conductivities. In particular, suppose that σ is a fixed conductivity obeying (3.1) with strictly positive essential infimum, and that $\{\sigma_n\}$ is a sequence of smooth conductivities in $\mathbb{D}$ obeying

(i) There is a fixed $r_1 \in (0,1)$ so that $\sigma_n(z) = 1$ for $|z| \geq r_1$ and for all n,

(ii) There is a fixed $c > 0$ so that $\sigma_n(z) \geq c$ for a.e. $z \in \mathbb{D}$ and for all n,

(iii) For a.e. z, $\sigma_n(z) \to \sigma(z)$ as $n \to \infty$.

Theorem 3.0.2. *Suppose that $\{\sigma_n\}$ obeys (i)–(iii), and denote by $\mathbf{t}_n$ (resp. $\mathbf{t}$) the scattering transform for σ_n (resp. σ) obtained from* (3.2)–(3.3)*. Then $\mathbf{t}_n \to \mathbf{t}$ pointwise. Moreover, $\mathbf{t}$ is related to the Astala-Päivärinta scattering transform τ for σ by* (3.4)*.*

We will prove Theorem 3.0.2 by studying convergence of the operators $(\Lambda_{\sigma_n} - \Lambda_1)$ to $(\Lambda_\sigma - \Lambda_1)$ as $n \to \infty$. An important ingredient in the proof will be the fact that the operators $\Lambda_{\sigma_n} - \Lambda_1$ are *uniformly* compact in a sense to be made precise, so that weak convergence (which is relatively easy to prove) can be "upgraded" to norm convergence.

3.1 Boundary Integral Equation

In this section we prove Theorem 3.0.1. Our strategy is to show that the integral operator

$$T_\xi \coloneqq S_\xi(\Lambda_\sigma - \Lambda_1)$$

is compact on $H^{1/2}(\partial\mathbb{D})$ and then mimic Nachman's argument in [37, Section 8] to show that $I + T_\xi$ is injective. The following simple lemma reduces the compactness statement to interior elliptic estimates plus the property (2.3) of harmonic extensions.

Lemma 3.1.1. *For any f and g belonging to $H^{1/2}(\partial\mathbb{D})$, the identity*

$$\langle g, (\Lambda_\sigma - \Lambda_1)f\rangle = \int_{\mathbb{D}} (\sigma - 1)\nabla v \cdot \nabla u \, dz \tag{3.5}$$

holds, where u solves (1.5) *and v is the harmonic extension of g to $\mathbb{D}$ and $\langle g, h\rangle$ denotes the dual pairing of $g \in H^{1/2}(\partial\mathbb{D})$ with $h \in H^{-1/2}(\partial\mathbb{D})$.*

Proof. Let w be the harmonic extension of f to $\mathbb{D}$. It follows from Alessandrini's identity (2.9) that

$$\begin{aligned}\langle g, (\Lambda_\sigma - \Lambda_1) f\rangle &= \int_{\mathbb{D}} \sigma \nabla v \cdot \nabla u \, dz - \int_{\mathbb{D}} \nabla v \cdot \nabla w \, dz \\ &= \int_{\mathbb{D}} (\sigma - 1) \nabla v \cdot \nabla u \, dz + \int_{\mathbb{D}} \nabla v \cdot \nabla (u - w) \, dz\end{aligned}$$

The second term vanishes since v is harmonic and $(u - w)|_{\partial\mathbb{D}} = 0$. □

Next, we note the following interior elliptic estimate. This estimate apears in various forms in the context of stability analysis of the inverse problem. See, e.g. [19, 21]. Our proof was written independently of that work.

Lemma 3.1.2. *Suppose that σ satisfies* (3.1)*, let $f \in H^{1/2}(\partial\mathbb{D})$, and let u denote the unique solution of* (1.5) *for the given f. For any $m > 0$, the estimate*

$$\|\nabla u\|_{L^2(|z|<r_1)} \lesssim \|f\|_{H^{-m}(\partial\mathbb{D})} \tag{3.6}$$

holds, where the implied constant depends only on m, r_1, $\operatorname{ess\,inf} \sigma$*, and* $\operatorname{ess\,sup} \sigma$*.*

Proof. As before, let w be the harmonic extension of f into $\mathbb{D}$. Let r_1 be the radius defined in (3.1), and let $0 < r_1 < r_2 < 1$. Choose $\chi \in C^\infty(\overline{\mathbb{D}})$ so that

$$\chi(z) = \begin{cases} 0, & 0 \leq |z| \leq r_1 \\ 1, & r_2 \leq |z| \leq 1 \end{cases} \tag{3.7}$$

Let $h(z) = \chi(z) w(z)$. Note that h has support where $\sigma(z) = 1$. We compute

$$\begin{aligned}\nabla \cdot (\sigma \nabla (u - h)) &= \nabla \cdot (\sigma \nabla u) - \nabla \cdot (\sigma \nabla (h)) \\ &= -(\Delta \chi) w - 2 \nabla \chi \cdot \nabla w\end{aligned}$$

By construction, we know $(u - h)|_{\partial\mathbb{D}} = 0$.

The unique solution $v \in H_0^1(\mathbb{D})$ of

$$\nabla \cdot (\sigma \nabla v) = g$$

obeys the bound

$$\|\nabla v\|_{L^2(\mathbb{D})} \lesssim \|g\|_{L^2(\mathbb{D})}$$

where the implied constants depend only on $\operatorname{ess\,inf} \sigma$ and $\operatorname{ess\,sup} \sigma$. Hence

$$\|\nabla u\|_{L^2(|z|<r_1)} = \|\nabla (u - w)\|_{L^2(|z|<r_1)} \lesssim \|-(\Delta \chi) h - 2 \nabla \chi \cdot \nabla h\|_{L^2(\mathbb{D})}$$

We obtain the desired estimate using (2.3). □

Next, we prove an operator bound on $(\Lambda_\sigma - \Lambda_1)$ with a uniformity that will be useful later.

Lemma 3.1.3. *Let $\sigma \in L^\infty(\mathbb{D})$ with $\sigma(z) \geq c > 0$ a.e. for some constant c. Suppose, moreover, that σ obeys* (3.1)*. Then for any $m > 0$, the operator $(\Lambda_\sigma - \Lambda_1)$ is bounded from $H^{-m}(\partial\mathbb{D})$ to $H^m(\partial\mathbb{D})$ with constants depending only on r_1, m, $\operatorname{ess\,inf}\sigma$, and $\operatorname{ess\,sup}\sigma$.*

Proof. We will begin with $f, g \in H^{1/2}(\partial\mathbb{D})$ and show that the pairing

$$|\langle g, (\Lambda_\sigma - \Lambda_1)f\rangle|$$

can be bounded in terms of $\|f\|_{H^{-m}}$ and $\|g\|_{H^{-m}}$. Then a density argument will establish the lemma.

Let v be a harmonic extension of g into $\mathbb{D}$. Then by Lemma 3.1.1 we obtain

$$\begin{aligned} |(g, (\Lambda_\sigma - \Lambda_1)f)| &= \left| \int_{\mathbb{D}} (\sigma - 1)\nabla v \cdot \nabla u \, dz \right| \\ &\lesssim_\sigma \|\nabla u\|_{L^2(|z|<r_1)} \|\nabla v\|_{L^2(|z|<r_1)} \\ &\lesssim_{\sigma, r_1, m} \|f\|_{H^{-m}} \|g\|_{H^{-m}} \end{aligned}$$

where we used Lemma 3.1.2 to estimate $\|\nabla u\|_{L^2(|z|<r_1)}$ and we used (2.3) again to estimate $\|\nabla v\|_{L^2(|z|<r_1)}$. The implied constants depend only on $\operatorname{ess\,inf}\sigma$ and $\operatorname{ess\,sup}\sigma$. □

It now follows from Lemma 3.1.3 and the compact embedding (2.2) that T_ξ is compact as an operator from $H^{1/2}(\partial\mathbb{D})$ to $H^{1/2}(\partial\mathbb{D})$. Thus, to show that (3.2) is uniquely solvable, it suffices by Fredholm theory to show that the only vector $g \in H^{1/2}(\partial\mathbb{D})$ with $g = -T_\xi g$ is the zero vector. We will show that any such g generates a global solution to the problem

$$\begin{gathered} \nabla \cdot (\sigma \nabla u) = 0, \\ \lim_{|z|\to\infty} e^{-i\xi z} u(z, \xi) = 0. \end{gathered} \tag{3.8}$$

We will then appeal to Theorem 2.4.3 to conclude that $g = 0$.

Proof of Theorem 3.0.1. We follow the proof of Theorem 5 in [37, Section 7]. Fix $\xi \in \mathbb{C}$, suppose that $g \in H^{1/2}(\partial\mathbb{D})$ satisfies $T_\xi g = -g$, let $h = (\Lambda_\sigma - \Lambda_1)g$ and let $v = S_\xi h$ on $\mathbb{R}^2 \setminus \partial\mathbb{D}$. The function v is harmonic on $\mathbb{R}^2 \setminus \partial\mathbb{D}$ and continuous across $\partial\mathbb{D}$. Thus, if v_+ and v_- are the respective boundary values of v from $\mathbb{R}^2 \setminus \partial\mathbb{D}$ and from $\partial\mathbb{D}$, $v_+ = v_- = g$. It follows from (2.7)–(2.8) and the fact that $g = -T_\xi g$ that

$$\frac{\partial v_+}{\partial \nu} - \frac{\partial \nu_-}{\partial \nu} = h = \Lambda_\sigma g - \Lambda_1 g. \tag{3.9}$$

Since $\partial\nu_-/\partial\nu = \Lambda_1 g$, we conclude that $\partial\nu_+/\partial\nu = \Lambda_\sigma g$. Now define

$$u(z) = \begin{cases} v(z), & x \in \mathbb{R}^2 \setminus \Omega \\ u_i(z), & x \in \Omega \end{cases}$$

where u_i is the unique solution to the problem

$$\nabla \cdot (\sigma \nabla u_i) = 0, \quad u_i|_{\partial \mathbb{D}} = g.$$

In this case $u_+ = u_-$ and $\partial u_+ / \partial \nu = \partial u_- / \partial \nu$, so u extends to a solution of (3.8) as claimed. It now follows from Theorem 2.4.3 that $u = 0$. Since g is the boundary trace of u, we conclude that $g = 0$. □

3.2 Convergence of Scattering Transforms

In this section we prove Theorem 3.0.2 in two steps. First, we show that the Dirichlet-to-Neumann operators Λ_{σ_n} associated to the sequence $\{\sigma_n\}$ converge in norm to Λ_σ. We then use this fact to conclude that the corresponding scattering transforms converge. The second step uses Astala-Päivärinta's scattering transform to identify the limit.

We begin with a simple result on weak convergence that exploits Alessandrini's identity and convergence of positive quadratic forms.

Lemma 3.2.1. *Suppose that $\{\sigma_n\}$ is a sequence of positive $L^\infty(\mathbb{D})$ obeying conditions (i)–(iii) of Theorem 3.0.2. Then $\Lambda_{\sigma_n} \to \Lambda_\sigma$ in the weak operator topology on $\mathcal{L}(H^{1/2}(\partial \mathbb{D}), H^{-1/2}(\partial \mathbb{D}))$.*

Proof. For any σ, it follows from (2.9) that Λ_σ defines a positive quadratic form

$$\langle f, \Lambda_\sigma f \rangle = \int_{\mathbb{D}} \sigma \left| \nabla u \right|^2 \, dz$$

on $H^{1/2}(\partial \mathbb{D})$. Note that $(\Lambda_\sigma - \Lambda_{\sigma_n})$ are self-adjoint operators. If we can show that

$$\lim_{n \to \infty} \langle f, (\Lambda_\sigma - \Lambda_{\sigma_n}) \, f \rangle = 0 \tag{3.10}$$

it will then follow by polarization that $\Lambda_{\sigma_n} \to \Lambda_\sigma$ in the weak operator topology. But

$$\langle f, (\Lambda_\sigma - \Lambda_{\sigma_n}) \, f \rangle = \int_{\mathbb{D}} (\sigma - \sigma_n) \left| \nabla u \right|^2 \, dz + \int_{\mathbb{D}} \sigma_n \left(\left| \nabla u \right|^2 - \left| \nabla u_n \right|^2 \right) \, dz. \tag{3.11}$$

The first right-hand term in (3.11) goes to zero by dominated convergence. Since the $\{\sigma_n\}$ are uniformly bounded, it suffices to show that $\nabla u_n \to \nabla u$ in L^2. A straightforward computation shows that

$$0 = \nabla \cdot (\sigma_n \nabla (u_n - u)) + \nabla \cdot ((\sigma_n - \sigma) \nabla u) \, .$$

Multiplying through by $v_n = u_n - u$ and integrating over $\mathbb{D}$, we obtain

$$\int_{\mathbb{D}} \sigma_n \left| \nabla v_n \right|^2 \, dz = - \int_{\mathbb{D}} (\sigma_n - \sigma) \nabla v_n \cdot \nabla u \, dz. \tag{3.12}$$

Since σ_n is bounded below by a fixed positive constant c independent of n, we can use the Cauchy-Schwarz inequality to conclude that

$$\frac{c}{2}\int_{\mathbb{D}} |\nabla v_n|^2 \, dz \leq \frac{1}{2c}\int_D (\sigma - \sigma_n) |\nabla u|^2 \, dz$$

and conclude that $\nabla u_n \to \nabla u$ in L^2 by dominated convergence. □

From Lemma 3.1.3 we obtain the following uniform approximation property for the operators

$$A_n := \Lambda_{\sigma_n} - \Lambda_1. \tag{3.13}$$

Lemma 3.2.2. *Suppose that $\{\sigma_n\}$ is a sequence of conductivities obeying hypotheses (i)–(iii) of Theorem 3.0.2, and let A_n be defined as in* (3.13)*. Given any $\varepsilon > 0$ there is a $k \in \mathbb{N}$ independent of n so that*

$$\|(I - P_k)A_n\|_{H^{1/2}\to H^{-1/2}} < \varepsilon, \qquad \|A_n(I - P_k)\|_{H^{1/2}\to H^{-1/2}} < \varepsilon,$$

where P_k is the finite rank projection operator from (2.1)

Proof. From Lemma 3.1.3 we have the uniform operator bound $\|A_n\|_{H^{-m}\to H^m} \lesssim_m 1$ since the σ_n have uniformly bounded essential infima and suprema and all obey (3.1). If A_n' denotes the Banach space adjoint of A_n, we have the same bound on A_n' by duality. The second inequality in the lemma is equivalent to the bound

$$\|(I - P_k)A_n'\|_{H^{1/2}\to H^{-1/2}} < \varepsilon$$

by duality, so we'll only prove the first bound. We write

$$\begin{aligned}\|(I - P_k)A_n\|_{H^{1/2}\to H^{-1/2}} &\leq \|(I - P_k)\|_{H^m\to H^{-1/2}} \|A_n\|_{H^{-m}\to H^m} \\ &\lesssim_m k^{1/2-m}\end{aligned}$$

with constants uniform in n. □

Now let $A = \Lambda_\sigma - \Lambda_1$ where $\sigma_n \to \sigma$.

Proposition 3.2.3. *Suppose that $\{\sigma_n\}$satisfies hypotheses (i)–(iii) of Theorem 3.0.2. Then $A_n \to A$ in the norm topology on the bounded operators from $H^{1/2}$ to $H^{-1/2}$.*

Proof. Write

$$A_n - A = P_k(A_n - A)P_k + (I - P_k)(A_n - A) + (A_n - A)(I - P_k). \tag{3.14}$$

Since A is a fixed compact operator, we can choose $N \in \mathbb{N}$ so $\|(I - P_k)A\|_{H^{1/2}\to H^{-1/2}}$ and $\|A(I - P_k)\|_{H^{1/2}\to H^{-1/2}}$ are small for any $k \geq N$. Combining this observation with Proposition 3.2.3, we can choose $k \in \mathbb{N}$, uniformly in n, so that the first and and third right-hand terms of (3.14) are small uniformly in n. The middle term vanishes for any fixed k and $n \to \infty$ by Lemma 3.2.1. □

As an easy consequence:

Proposition 3.2.4. *Fix $\xi \in \mathbb{C}$. Suppose that $\{\sigma_n\}$ is a sequence obeying hypotheses (i)–(iii) of Theorem 3.0.2, and denote by $g_n(\,\cdot\,,\xi)$ and $g(\,\cdot\,,\xi)$ the respective solutions of* (3.2) *corresponding to σ_n and σ. Then, for each fixed ξ, $g_n \to g$ in $H^{1/2}(\partial\mathbb{D})$. Moreover, the scattering transforms $\mathbf{t}_n$ of σ_n converge pointwise to $\mathbf{t}$ given by* (3.3)*.*

Proof. By a slight abuse of notation, denote by T_n the operator $S_\xi\left(\Lambda_{\sigma_n} - \Lambda_1\right)$ and by T the operator $S_\xi\left(\Lambda_\sigma - \Lambda_1\right)$. It follows from (2.6) and Proposition 3.2.3 that $T_n \to T$ in $\mathcal{L}(H^{1/2}, H^{1/2})$. Since

$$g_n = (I - T_n)^{-1}\left(e^{i\xi z}\big|_{\partial\mathbb{D}}\right), \quad g = (I - T)^{-1}\left(e^{i\xi z}\big|_{\partial\mathbb{D}}\right),$$

it follows from the second resolvent identity that $g_n \to g$ in $H^{1/2}(\partial\mathbb{D})$. Convergence of $\mathbf{t}_n$ to $\mathbf{t}$ follows from the norm convergence of g_n to g and of $\Lambda_{\sigma_n} - \Lambda_1$ to $\Lambda_\sigma - \Lambda_1$. □

In the remainder of this section, we will identify what $\mathbf{t}$ actually is. In order to do so we need to prove a convergence theorem for the Astala-Päivärinta scattering transforms τ_n of the Beltrami coefficients $\mu_n = (1-\sigma_n)/(1+\sigma_n)$ to the transform τ of σ that is of some interest in itself.

Proposition 3.2.5. *Suppose that $\{\mu_n\}$ is a sequence of Beltrami coefficients with $0 \leq \mu_n(z) \leq k < 1$ for a.e. z. Suppose further that $\mu_n(z) \to \mu(z)$ pointwise where $\mu \in L^\infty(\mathbb{D})$ has the same properties. Finally, fix $\xi \in \mathbb{C}$ and let $M_{\pm\mu_n}(z,\xi)$ be the normalized CGO solution for the Beltrami equation from Theorem 1.4.2 with Beltrami coefficients $\pm\mu_n$, and let $M_{\pm\mu}$ be the normalized CGO solution for $\pm\mu$. Then, for a single choice of sign, $M_{\pm\mu_n} - 1 \to M_{\pm\mu} - 1$ weakly in $W^{1,p}(\mathbb{R}^2)$ for any $p \in (2, 1+k^{-1})$.*

We will prove Proposition 3.2.5 in several steps. First we show how to conclude the proof of Theorem 3.0.2 given its result.

Proof of Theorem 3.0.2, given Proposition 3.2.5. Recall that τ can be written as

$$\overline{\tau(\xi)} = \frac{1}{2\pi}\int \overline{\partial}(M_\mu - M_{-\mu})\, dz \tag{3.15}$$

Note that the CGO solutions $M_{\pm\mu_n}$ satisfy

$$\overline{\partial} M_{\pm\mu_n} = \pm\mu_n \overline{\partial(e_\xi M_{\pm\mu_n})}$$

and hence $\overline{\partial} M_{\pm\mu_n}$ are supported in $\mathbb{D}$. Proposition 3.2.5 and (3.15) show that $\tau_{\mu_n} \to \tau$ pointwise as $n \to \infty$ since the integral in (3.15) may be regarded as integrating the derivatives of $M_{\pm\mu_n}$ against a smooth, compactly supported function which is identically 1 in a neighborhood of $\mathbb{D}$.[1] Since τ_n converges pointwise to τ and $\mathbf{t}_n(\xi) = -4\pi i\overline{\xi}\tau_{\mu_n}(\xi)$, we conclude that $\mathbf{t}(\xi) = -4\pi i\overline{\xi}\tau(\xi)$. □

To establish the weak convergence, we first need a uniform bound on $M_{\pm\mu_n} - 1$ in $W^{1,p}(\mathbb{R}^2)$.

[1] That function is a mollified version of $\chi_{\mathbb{D}}(z)$, the characteristic function of $\mathbb{D}$.

Lemma 3.2.6. *Suppose that $\{\mu_n\}$ is a sequence of Beltrami coefficients obeying the hypothesis of Proposition 3.2.5, and let $M_n = M_{\mu_n}$. Then there exists a constant C such that*

$$\sup_n \|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)} < C. \tag{3.16}$$

Proof. Let $c_n = \|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)}$. If $\limsup_n c_n = +\infty$, set $v_n = c_n^{-1}(M_n - 1)$. Since $\{v_n\}$ is bounded in $W^{1,p}$, by passing to a subsequence we can assume that $\{v_n\}$ has a weak limit, v. Note that $\|v_n\|_{W^{1,p}(\mathbb{R}^2)} = 1$.

We first claim that, if such a limit exists, it is nonzero. Suppose, on the other hand, that $v_n \to 0$ weakly in $W^{1,p}(\mathbb{R}^2)$. It follows from the Rellich-Kondrachov Theorem[2] [1, Theorem 6.2] that $v_n \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^2)$. A short computation shows that

$$\overline{\partial} v_n = \frac{\mu_n}{c_n}\overline{\partial} e_\xi + \mu_n \overline{\partial(e_\xi v_n)} \tag{3.17}$$

and, since $v_n \in W^{1,p}(\mathbb{R}^2)$ we may invert the $\overline{\partial}$ operator using the Cauchy transform and use equation (2.14) of Lemma 2.5.2 to conclude that

$$\begin{aligned} \|v_n\|_{L^p(\mathbb{R}^2)} \lesssim_p & \left\| \frac{\mu_n}{c_n}\,\overline{\partial} e_k \right\|_{L^{2p/(2+p)}(\mathbb{R}^2)} + \left\| \mu_n \overline{(\partial e_k) v_n} \right\|_{L^{2p/(2+p)}(\mathbb{R}^2)} \\ & + \left\| \mu_n \overline{e_k \partial v_n} \right\|_{L^{2p/(2+p)}(\mathbb{R}^2)}. \end{aligned} \tag{3.18}$$

The first right-hand term in (3.18) clearly goes to zero as $n \to \infty$ since $c_n \to \infty$. The function in the second right-hand term is supported in $\mathbb{D}$ owing to the factor μ_n and therefore also converges to zero since $v_n \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^2)$ by hypothesis. The function in the third term is again supported in $\mathbb{D}$ and, using a version of the Caccioppoli inequality adapted to the v_n's (see Lemma 3.2.7 below), we have $\|\partial v_n\|_{L^p(\mathbb{D})} \lesssim \|v_n\|_{L^p(2\mathbb{D})} + \mathcal{O}(c_n^{-1})$, which shows that the third right-hand term in (3.18) also goes to zero as $n \to \infty$. Thus, $v_n \to 0$ in $L^p(\mathbb{R}^2)$. Applying Lemma 3.2.7 to the compactly supported function $\overline{\partial} v_n$ shows that, also $\left\|\overline{\partial} v_n\right\|_{L^p} \to 0$ as $n \to \infty$, contradicting the fact that $\|v_n\|_{W^{1,p}(\mathbb{R}^2)} = 1$ for all n. From this contradiction we conclude that $\{v_n\}$ has a nonzero limit, again assuming that $c_n \to \infty$.

Next, we show that the limit function v is a weak solution of the equation $\overline{\partial} v = \mu \overline{\partial(e_\xi v)}$. For $\varphi \in C_0^\infty(\mathbb{R}^2)$ we compute from (3.17)

$$(\varphi, v_n) = c_n^{-1}(\varphi, \mu_n \overline{\partial} e_\xi) + (\varphi, \mu_n \overline{\partial(e_\xi v_n)}).$$

where $(f, g) = \displaystyle\int fg$. It is easy to see that the first right-hand term vanishes as $n \to \infty$. In the second term,

$$\begin{aligned} (\varphi, \mu_n \overline{\partial(e_\xi v_n)}) &= (\overline{\partial}(e_{-\xi})\mu\varphi, \overline{v}_n) + (e_\xi \mu \varphi, \overline{\partial v_n}) + (\varphi(\mu_n - \mu), \overline{\partial(e_k v_n)}) \\ &\to (\overline{\partial}(e_{-\xi})\mu\varphi, \overline{v}) + (e_\xi \mu\varphi, \overline{\partial v}) \text{ as } n \to \infty \end{aligned}$$

[2] The embedding $W^{1,p}(\mathbb{R}^2) \hookrightarrow C^{0,1-2/p}(\overline{\Omega})$ is compact, which one can use to show strong convergence in $L^p_{\text{loc}}(\mathbb{R}^2)$.

since $\|v_n\|_{W^{1,p}} = 1$ and $v_n \to v$ in L^p_{loc}. It follows that v is a weak solution of $\overline{\partial} v = \mu \overline{\partial (e_\xi v)}$ with $\|v\|_{W^{1,p}} \leq 1$. Thus, assuming that $\|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)}$ is not bounded, we have constructed a nonzero solution $v \in W^{1,p}(\mathbb{R}^2)$ of the equation $\overline{\partial} v = \mu \overline{\partial (e_\xi v)}$. However, this violates the uniqueness of the normalized CGO solution for Beltrami coefficient μ proved in [8, Theorem 4.2], a contradiction. We conclude that $\|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)}$ is bounded uniformly in n. □

To complete the proof of Lemma 3.2.6, we need to establish the a priori bounds on the sequence v_n constructed in its proof. To do so, we will need the *a priori* estimate for solutions of the Beltrami equation from Theorem 2.4.1.

Lemma 3.2.7. *Suppose that v_n is a sequence of functions as constructed in the proof of Lemma 3.2.6. Then, the estimate*

$$\|\partial v_n\|_{L^p(\mathbb{D})} \lesssim_\xi c_n^{-1} + \|v\|_{L^p(2\mathbb{D})}$$

where the implied constants are independent of n.

Proof. By construction, the function $f = e^{i\xi z}(c_n v_n + 1)$ satisfies the Beltrami equation

$$\overline{\partial} f = \mu_n \overline{\partial f}$$

and hence satisfies the distortion inequality. Thus by Theorem 2.4.1, for a compactly supported smooth function η we can write

$$\|\eta \partial f\|_p \leq C_{p,k} \|f \nabla \eta\|_p \tag{3.19}$$

Note that by the triangle inequality, we have

$$\begin{aligned} \|\eta \partial f\|_p = \left\|\eta \partial (e^{i\xi z}(c_n v_n + 1))\right\|_p \geq c_n \left\|\eta e^{i\xi z} \partial v_n\right\|_p \\ - |\xi| \left\|\eta e^{i\xi z}(c_n v_n + 1)\right\|_p \end{aligned} \tag{3.20}$$

which enables us to write

$$\begin{aligned} c_n \left\|\eta e^{i\xi z} \partial v_n\right\|_p \leq \left\|\eta \partial (e^{i\xi z}(c_n v_n + 1))\right\|_p \\ + |\xi| \left\|\eta e^{i\xi z}\right\|_p + c_n |\xi| \left\|\eta e^{i\xi z} v_n\right\|_p \end{aligned} \tag{3.21}$$

Next, we apply (3.19) to obtain

$$\begin{aligned} c_n \left\|\eta e^{i\xi z} \partial v_n\right\|_p \lesssim c_n \left\|e^{i\xi z}(\nabla \eta) v_n\right\|_p + \left\|e^{i\xi z} \nabla \eta\right\|_p \\ + |\xi| \left\|\eta e^{i\xi z}\right\|_p + c_n |\xi| \left\|\eta e^{i\xi z} v_n\right\|_p . \end{aligned} \tag{3.22}$$

Thus we conclude

$$\begin{aligned} \left\|\eta e^{i\xi z} \partial v_n\right\|_p \lesssim \left\|e^{i\xi z}(\nabla \eta) v_n\right\|_p + |\xi| \left\|\eta e^{i\xi z} v_n\right\|_p \\ + \frac{1}{c_n} \left(\left\|e^{i\xi z} \nabla \eta\right\|_p + |\xi| \left\|\eta e^{i\xi z}\right\|_p \right) \end{aligned} \tag{3.23}$$

To obtain the desired estimate, we choose η supported on the disk of radius 2 so that $\eta = e^{-i\xi z}$ in $\mathbb{D}$. □

We can now give the proof of Proposition 3.2.5 and thereby complete the proof of Theorem 3.0.2.

Proof of Proposition 3.2.5. By Lemma 3.2.6, the sequences $\{M_{\pm\mu_n} - 1\}$ for either choice of sign are bounded in $W^{1,p}(\mathbb{R}^2)$. We will take a single choice of sign, the $+$ sign, and write M_n for M_{μ_n} and M for M_μ from now on. The sequence $\{M_n - 1\}$ has a weak limit point in $W^{1,p}(\mathbb{R}^2)$ which we denote by $M^\sharp - 1$. By the Rellich-Kondrachov theorem, $M_n - 1$ converges in $L^p_{\text{loc}}(\mathbb{R}^2)$ to $M^\sharp - 1$. We wish to show that

$$\begin{aligned} \overline{\partial} M^\sharp &= \mu \overline{\partial \left(e_\xi M^\sharp \right)}, \\ M^\sharp - 1 &\in W^{1,p}(\mathbb{R}^2) \end{aligned} \tag{3.24}$$

since we can then conclude that $M^\sharp - 1$ is nonzero (as the PDE does not admit the solution $M^\sharp = 1$) and that $M^\sharp = M$ since the PDE is uniquely solvable for $M^\sharp - 1 \in W^{1,p}(\mathbb{R}^2)$.

From $\overline{\partial} M_n = \mu_n \overline{\partial \left(e_\xi M_n \right)}$ we conclude that for any $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} - \left(\overline{\partial} \varphi, M_n \right) &= \left(\varphi, \mu_n \overline{\partial \left(e_\xi M_n \right)} \right) \\ &= \left(\varphi, \mu \overline{\partial \left(e_\xi M_n \right)} \right) + \left((\mu_n - \mu), \overline{\partial \left(e_\xi M_n \right)} \right) \end{aligned}$$

The second right-hand term vanishes as $n \to \infty$ by dominated convergence since $\mu_n - \mu$ is supported in $\mathbb{D}$ while $\{\partial(e_\xi M_n)\}$ is uniformly bounded in $L^p(\mathbb{R}^2)$. Weak convergence of derivatives allows us to conclude that (3.24) holds. □

3.3 Summary

In this chapter, we examined the process of obtaining the scattering transform of a piecewise-continuous conductivity from its Dirichlet-to-Neumann map. The shortcut implemented in the numerical experiments in [9] used Nachman's integral equations to represent the scattering transform. Prior to our work, this procedure assumed a priori that the conductivity was Hölder continuous. We have shown that the smoothness assumption on the conductivity σ is not needed, provided σ is 1 in a neighborhood of the boundary (a statement made more precisely in (3.1)). In order to establish that the shortcut works, a type of continuity result in the map $\sigma \mapsto \mathbf{t}$ was established in the Astala-Päivärinta framework. Thus in the presence of infinite-precision data, the scattering transform of a piecewise-continuous conductivity can be found using the shortcut method presented in [9].

To address the second motivating question in [9], we need to first understand what sorts of conductivities correspond to a scattering transform that is compactly supported. Then we study the problem of CGO solutions for this type of conductivity and show that they exhibit a similar subexponential growth.

4.1 Decay of the Conductivity

The Fourier transform maps smooth functions to decaying functions and vice-versa. Earlier, we mentioned that the scattering transform can be thought of as a nonlinear analogue of the Fourier transform. Thus there is no reason to expect the Beltrami coefficient corresponding to a $\tau \in C_0^\infty(\mathbb{C})$ to be compactly supported. However, as we will show, the corresponding Beltrami coefficient exhibits large-z decay on the order of $\mathcal{O}\left(|z|^{-2}\right)$.

Theorem 4.1.1. *Suppose $\tau \in C_0^\infty(\mathbb{C})$ satisfies $\tau(0) = 0$ and $|\tau(\xi)| \leq 1$, then τ corresponds to a continuous Beltrami coefficient μ that satisfies*

$$\|\mu\|_\infty \leq k < 1 \text{ and } \mu(z) \sim \mathcal{O}\left(\frac{1}{|z|^2}\right) \text{ as } z \to \infty \tag{4.1}$$

Consequently, $\mu \in L^r(\mathbb{C})$ for all $1 < r \leq \infty$. Moreover, $\partial\mu \in L^r(\mathbb{C})$ for all $1 < r \leq \infty$.

Remark 4.1.2. *Note that $\mu = (1 - \sigma)/(1 + \sigma)$ where σ is reconstructed from τ. We will later extend the Astala-Päivärinta scattering transform to include the μ which construct such τ.*

The proof of this theorem is at the end of this section. First, we need to state a condition on the scattering transform $\mathbf{t}$ and thus on τ.

Lemma 4.1.3 (Lemma 5.1 of [36])**.** *For Nachman's scattering transform $\mathbf{t}(\xi)$, $\mathbf{t}(\xi) = \overline{\mathbf{t}(-\xi)}$. By the equivalence between $\mathbf{t}$ and τ,*

$$\bar{\xi}\tau(\xi) = -\xi\overline{\tau(-\xi)}$$

Davey-Stewartson System

To obtain information on the conductivity, we will study the connection between the Schrödinger problem and the linear system associated to the Davey-Stewartson II equation. We will use the notation of [38]. First we recall the direct problem:

$$\begin{aligned} \overline{\partial} m^1(z,\xi) &= u m^2(z,\xi), \\ (\partial + i\xi)\, m^2(z,\xi) &= \bar{u} m^1(z,\xi), \\ m^1(\,\cdot\,,\xi) - 1, m^2(\,\cdot\,,\xi) &\in L^4_z(\mathbb{R}^2). \end{aligned} \tag{4.2}$$

The scattering transform is computed from (4.2) via

$$\mathbf{s}(\xi) = -\frac{i}{\pi} \int e_\xi(z) \overline{u(z)} m^1(z,\xi)\, dz. \tag{4.3}$$

The functions m^1 and m^2 also obey the 'dual' problem

$$\begin{aligned} \overline{\partial}_\xi m^1(z,\xi) &= e_{-\xi}\mathbf{s}(\xi)\overline{m^2(z,\xi)}, \\ \overline{\partial}_\xi m^2(z,\xi) &= e_{-\xi}\mathbf{s}(\xi)\overline{m^1(z,\xi)}, \\ m^1(z,\,\cdot\,) - 1, m^2(z,\,\cdot\,) &\in L^4_\xi(\mathbb{R}^2). \end{aligned} \tag{4.4}$$

We can construct the solution to Nachman's problem using the functions m^1 and m^2. Cross-differentiating the equations (4.2) we see that $m = m^1 + m^2$ obeys

$$\begin{aligned} \overline{\partial}\,(\partial + i\xi)\, m &= \left(\partial u + |u|^2\right) m, \\ m(\,\cdot\,,\xi) - 1 &\in L^4_z(\mathbb{R}^2) \end{aligned} \tag{4.5}$$

which is the form for normalized CGO solutions of the zero-energy Schrödinger equation with potential

$$q = 4\left(\partial u + |u|^2\right),$$

provided u obeys the compatibility condition

$$\partial u = \overline{\partial u}.$$

The function m also obeys the dual equation

$$\begin{cases} \overline{\partial}_\xi m(z,\xi) = e_{-\xi}(z)\mathbf{s}(\xi)\overline{m(z,\xi)}, \\ m(z,\,\cdot\,) - 1 \in L^4_\xi(\mathbb{R}^2). \end{cases} \tag{4.6}$$

Comparing (4.6) and (1.12), we see that by taking $\mathbf{s}(\xi) = \tau(\xi)$ we recover the solution to Nachman's problem.

Expansions

Consider the system

$$\begin{cases} \overline{\partial} m^1 = u m^2, \\ (\partial + i\xi)\, m^2 = \overline{u} m^1, \\ m^1(\,\cdot\,,\xi) - 1,\ m^2(\,\cdot\,,\xi) \in L^4(\mathbb{R}^2) \end{cases} \tag{4.7}$$

for $u \in \mathscr{S}(\mathbb{R}^2)$. For such u it is not difficult to show that $m^1(\,\cdot\,,\xi)$ and $m^2(\,\cdot\,,\xi)$ belong to $L^\infty(\mathbb{R}^2)$ for each ξ with bounds independent of ξ[1]. We can then use the identity

$$\int \frac{1}{z-\zeta} f(\zeta)\, d\zeta = \sum_{j=0}^{N} \frac{1}{z^{j+1}} \int \zeta^j f(\zeta)\, d\zeta + \frac{1}{z^{N+1}} \int \frac{\zeta^{N+1}}{z-\zeta} f(\zeta)\, d\zeta$$

[1]Much of the careful analysis of the Davey-Stewartson-II system was done by Sung in [45, 46, 47].

to show that

$$m^1(z,\xi) = 1 + \frac{1}{\pi z}\int u(\zeta) m^2(\zeta,\xi)\, d\zeta + \frac{1}{\pi z^2}\int \zeta u(\zeta) m^2(\zeta,\xi)\, d\zeta \qquad (4.8)$$
$$+ \frac{1}{\pi z^2}\int \frac{\zeta^2 u(\zeta)}{z-\zeta} m^2(\zeta,\xi)\, d\zeta$$

$$m^2(z,\xi) = \frac{e_{-\xi}(z)}{\pi \overline{z}}\int e_\xi(\zeta)\overline{u(\zeta)} m^1(\zeta,\xi)\, d\zeta + \frac{e_{-\xi}(z)}{\pi \overline{z}^2}\int e_\xi(\zeta)\overline{\zeta}\overline{u(\zeta)} m^1(\zeta,\xi)\, d\zeta \qquad (4.9)$$
$$+ \frac{e_{-\xi}(z)}{\pi \overline{z}^2}\int e_\xi(\zeta)\frac{\overline{\zeta}^2\overline{u(\zeta)}}{\overline{\zeta}-\overline{z}} m^1(\zeta,\xi)\, d\zeta$$

where we've only expanded to order $|z|^{-2}$. To identify the coefficients in these asymptotic expansions, we exploit the fact that m^1 and m^2 also obey the dual equations

$$\begin{cases} \overline{\partial}_\xi m^1 = e_{-\xi}\mathbf{s}\overline{m^2}, \\ \overline{\partial}_\xi m^2 = e_{-\xi}\mathbf{s}\overline{m^1}, \\ m^1(z,\,\cdot\,) - 1,\, m^2(z,\,\cdot\,) \in L^2_\xi(\mathbb{R}^2). \end{cases} \qquad (4.10)$$

where $\mathbf{s}$, the scattering transform of u, was defined in (4.3) (this immediately identifies the first nontrivial coefficient in the large-z expansion (4.9) of m^2). It is clear from (4.8)–(4.9) that m^1 and m^2 admit large-z asymptotic expansions of the form

$$m^1(z,\xi) \sim 1 + \sum_{j\geq 1} \frac{\alpha_j(\xi)}{z^j},$$
$$m^2(z,\xi) \sim e_{-\xi}(z)\sum_{j\geq 1} \frac{\beta_j(\xi)}{\overline{z}^j}.$$

Substituting these expansions into (4.10), we can find recurrence relations for α_j and β_j:

$$\overline{\partial}_\xi \alpha_j = \mathbf{s}\overline{\beta_j}$$
$$\beta_1 = i\mathbf{s}$$
$$\overline{\partial}_\xi \beta_j + i\beta_{j+1} = \mathbf{s}\overline{\alpha_j}$$

where the first relation comes from the first equation in (4.10), and the remaining relations come from the second equation of (4.10). Hence $\beta_1 = -i\mathbf{s}$ and $\alpha_1 = \partial_\xi^{-1}(|s|^2)$. Using these facts in (4.8) and (4.9), we conclude that

$$m^1(z,\xi) = 1 + \frac{1}{iz}\partial_\xi^{-1}\left(|\mathbf{s}(\,\cdot\,)|^2\right)(\xi) + \frac{1}{\pi z^2}\int \zeta u(\zeta) m^2(\zeta,\xi)\, d\zeta \qquad (4.11)$$
$$+ \frac{1}{\pi z^2}\int \frac{\zeta u(\zeta)}{z-\zeta} m^2(\zeta,\xi)\, d\zeta$$

$$m^2(z,\xi) = \frac{ie_{-\xi}(z)}{\overline{z}}\mathbf{s}(\xi) + \frac{e_{-\xi}(z)}{\pi \overline{z}^2}\int e_\xi(z)\overline{\zeta}\overline{u(\zeta)} m^1(\zeta,\xi)\, d\zeta \qquad (4.12)$$
$$+ \frac{e_{-\xi}(z)}{\pi \overline{z}^2}\int e_\xi(z)\frac{\overline{\zeta}\overline{u(\zeta)}}{\overline{\zeta}-\overline{z}} m^1(\zeta,\xi)\, d\zeta$$

Proof of Theorem 4.1.1. In [15], Brown and Uhlmann make a change of dependent variable

$$q = -\frac{1}{2}\partial \log \sigma \tag{4.13}$$

that transforms the divergence form equation $\nabla \cdot (\sigma \nabla u) = 0$ into the linear system (4.2) with scattering map (4.3) and transform $\mathbf{s}(\xi)$ (which we have shown is the same as $\tau(\xi)$. Sung showed that this scattering map takes $\mathscr{S}(\mathbb{R}^2)$ to itself [46, Theorem 4.4]. If $\mathbf{s} \in \mathscr{S}(\mathbb{R}^2)$, we can conclude that $q = -\frac{1}{2}\partial \log \sigma \in \mathscr{S}(\mathbb{R}^2)$. This implies that $-2\partial^{-1} q$ is certainly a bounded function. Therefore,

$$\sigma = e^{-2\partial^{-1} q} \tag{4.14}$$

is a smooth non-negative function. Thus if $\mathbf{s} \in \mathscr{S}(\mathbb{R}^2)$, the quantity $\mu = (1-\sigma)/(1+\sigma)$ is a smooth, bounded function with $\|\mu\|_\infty \leq k < 1$.

Together with the formula $m(z,\xi) = m^1(z,\xi) + m^2(z,\xi)$ for the solution of (4.6) and the reconstruction formula

$$\sigma(z) = \lim_{\xi \to 0} m(z,\xi) \tag{4.15}$$

we examine the large-z asymptotics of the conductivity σ. Because

$$m(z,\xi)^2 - 1 = (m(z,\xi) - 1)^2 + 2(m(z,\xi) - 1) \tag{4.16}$$

we need to examine the asymptotics of $m(z,\xi)$ as $\xi \to 0$.
Using the fact that $\tau(\xi) = \mathbf{s}(\xi)$, assume now that $\mathbf{s}$ satisfies the conditions

$$\mathbf{s}(0) = 0, \quad \overline{\mathbf{s}(\xi)} = -\frac{\bar{\xi}}{\xi}\mathbf{s}(-\xi). \tag{4.17}$$

Then $|\mathbf{s}(\xi)|^2$ is an even function so that $\partial_\xi^{-1}\left(|\mathbf{s}(\,\cdot\,)|^2\right)(0) = 0$.

We then conclude from (4.11) and (4.12) that

$$m^1(z,0) = 1 + \frac{1}{\pi z^2}\int \zeta u(\zeta) m^2(\zeta,0)\,d\zeta + \frac{1}{\pi z^2}\int \frac{\zeta u(\zeta)}{z-\zeta} m^2(\zeta,0)\,d\zeta \tag{4.18}$$

$$m^2(z,0) = \frac{1}{\pi \bar{z}^2}\int \bar{\zeta}\overline{u(\zeta)} m^1(\zeta,0)\,d\zeta + \frac{1}{\pi \bar{z}^2}\int \frac{\bar{\zeta}\overline{u(\zeta)}}{\bar{\zeta}-\bar{z}} m^1(\zeta,0)\,d\zeta \tag{4.19}$$

Since $m(z,\xi) - 1 = m^1(z,\xi) - 1 + m^2(z,\xi)$ we see that

$$m(z,0) - 1 \sim \mathcal{O}\left(\frac{1}{|z|^2}\right) \text{ as } z \to \infty. \tag{4.20}$$

Thus $\sigma(z) - 1 \sim \mathcal{O}\left(|z|^{-2}\right)$ and $\mu(z) \sim \mathcal{O}\left(|z|^{-2}\right)$ as $z \to \infty$.

For the derivative, note

$$\partial\mu = \frac{-2\partial\sigma}{(1+\sigma)^2}.$$

We see that an estimate on $\partial\sigma$ via $\overline{\partial} m^1(z,0)$ and $\partial m^2(z,0)$ is sufficient to conclude $\partial\mu \in L^r(\mathbb{C})$ for $r \in (1,\infty)$. Since m^1 and m^2 are bounded, it follows from (4.7) that, also, $\overline{\partial} m^1$ and $(\partial_z + i\xi)m^2$ are bounded functions of rapid decrease. It follows that the first derivatives $(\partial m^1)(z,0)$, $\left(\overline{\partial} m^1\right)(z,0)$, $(\partial m^2)(z,0)$, and $\left(\overline{\partial} m^2\right)(z,0)$ all belong to $L^r(\mathbb{R}^2)$ for any $r \in (1,\infty)$ by the boundedness of the Beurling transform on L^r for such r.

□

4.2 Existence of CGO Solutions

In [8], Astala and Päivärinta show the existence and uniqueness of complex geometric optics (CGO) solutions to the Beltrami equation

$$\overline{\partial} f = \mu \overline{\partial f}$$

where the Beltrami coefficient $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \le k < 1$ and $\text{supp}(\mu) \subset \mathbb{D}$. We extend their theorem to a broader class of μ. Our approach draws heavily upon the techniques used by Astala and Päivärinta in [8]. At key points where the authors use the compact support assumption, we find an alternative strategy to replace the compact support with the decay condition.

Theorem 4.2.1. *Suppose that μ is a real-valued measurable function with $\|\mu\|_\infty \le k < 1$ and that $\mu \in L^r(\mathbb{C})$ for all r with $1 < r < \infty$. Let $2 < p < 1 + k^{-1}$. Then for each $\xi \in \mathbb{C}$ there exists a unique solution $f \in W^{1,p}_{loc}(\mathbb{C})$ to the equation*

$$\overline{\partial} f = \mu \overline{\partial f} \tag{4.21}$$

where f can be written as $f(z,\xi) = e^{i\xi z} M(z,\xi)$ and

$$M(\cdot,\xi) - 1 \in W^{1,p}(\mathbb{C}) \tag{4.22}$$

Some Useful Propositions

We will study an integral operator built from the solid Cauchy transform and the Beurling transform. Let

$$\nu \in \bigcap_{1<q<\infty} L^q(\mathbb{C}) \quad \alpha \in \bigcap_{1<q<\infty} L^q(\mathbb{C}) \cap L^\infty(\mathbb{C}) \tag{4.23}$$

where $\|\nu\|_\infty \le k < 1$ and suppose p is in the interval $1 + k < 2 < p < 1 + k^{-1}$. In analogy with Proposition 4.1 of [8], we define an operator K

$$Kg = P(I - \nu \bar{S})^{-1}(\alpha \bar{g}) \tag{4.24}$$

We will prove some properties of this operator that will be useful in proving our main theorem.

Proposition 4.2.2. *Suppose ν, and α are as described in equation* (4.23) *where $\|\nu\|_\infty \le k < 1$ and $1 + k < q < 2 < p < 1 + k^{-1}$. Then the following hold.*

1. $K : L^p(\mathbb{C}) \to L^p(\mathbb{C})$

2. $(I - K)$ *is injective on* $L^p(\mathbb{C})$

The proof of statement (2) is due to Astala-Päivärinta. In order to prove this and subsequent statements, we will make use of a few useful facts, which are consequences of Hölder's inequality.

Lemma 4.2.3. *Suppose* $g \in L^p \cap L^q$ *for* $1 < q < 2 < p < \infty$ *and* $\|g\|_\infty \le k$. *Then the following hold.*

$$\|g\|_2 \le k^{(2-q)/2} \|g\|_q^{q/2} \tag{4.25}$$

$$\|gh\|_{2p/(p+2)} \le \|g\|_2 \|h\|_p \tag{4.26}$$

Proof of Proposition 4.2.2. In the first place, since $\|\nu\|_\infty \le k < 1$ and $1 + k < q < 2 < p < 1 + k^{-1}$, we know by Lemma 2.5.3 that $(I - \nu\bar{S})$ is invertible on $L^p(\mathbb{C})$. Now we can write

$$\begin{aligned} Kg &= P(I - \nu\bar{S})^{-1}(\alpha\bar{g}) \\ &= P\left[\alpha\bar{g} - \nu\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right] \end{aligned} \tag{4.27}$$

Employing estimate (2.14), we have

$$\|Kg\|_p \lesssim_p \|\alpha\bar{g}\|_{2p/(p+2)} + \left\|\nu\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right\|_{2p/(p+2)}$$

Our assumptions on α and ν let us use estimate (4.26) to obtain (in combination with other estimates on the Beurling operator and the resolvent):

$$\|Kg\|_p \lesssim_p \|\alpha\|_2 \|g\|_p + \|\nu\|_2 \|\bar{S}\|_{L^p \to L^p} \left\|(I - \nu\bar{S})^{-1}\right\|_{L^p \to L^p} \|\alpha\|_\infty \|g\|_p \tag{4.28}$$

We now show $I - K$ is injective. Suppose $g \in L^p$ such that $g = Kg$. This implies that we can write

$$\begin{aligned} \bar{\partial}g &= (I - \nu\bar{S})^{-1}(\alpha\bar{g}) \\ \bar{\partial}g - \nu\overline{S(\bar{\partial}g)} &= \alpha\bar{g} \\ \bar{\partial}g - \nu\overline{\partial g} &= \alpha\bar{g} \end{aligned}$$

Then g satisfies the differential inequality

$$|\bar{\partial}g| \le k|\partial g| + |\alpha||g|$$

Moreover, we claim that since $g = Kg$, g must vanish at ∞. Using the same decomposition as in (4.27) and our estimate (4.26), we can conclude that $\alpha\bar{g}$ and $\nu\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})$ are in $L^{2p/(p+2)}(\mathbb{C})$. Since α and ν are both bounded, we can also conclude $\alpha\bar{g}$ and $\nu\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})$ are in $L^p(\mathbb{C})$. Since,

$$\frac{2p}{p+2} < 2 < p,$$

Lemma 2.5.2 implies $g(z) = Kg(z) \to 0$ as $|z| \to \infty$. Therefore, by Theorem 2.4.2, we conclude that $g \equiv 0$ and hence $(I - K)$ is injective. □

In the following section, we will want to solve an integral equation of the form $(I-K)g=h$ in L^p. We will use the Fredholm alternative to accomplish this, but we need to establish the compactness of the operator K. The key distinction from the case of Proposition 4.1 of [8] is that we no longer assume that α and ν have support in $\mathbb{D}$. Instead, we impose an integrability condition which equates to having decay at infinity.

Proposition 4.2.4. *Suppose ν, and α are as described in equation (4.23) where $\|\nu\|_\infty \leq k < 1$ and $1+k < q < 2 < p < 1+k^{-1}$. Then the operator K defined in (4.24) is compact on $L^p(\mathbb{C})$. Consequently, $(I-K)$ is invertible on $L^p(\mathbb{C})$.*

Proof. Let $\chi_n(z)$ be the characteristic function of the ball of radius n centered at the origin. Take $\alpha_n = \chi_n\alpha$ and $\nu_n = \chi_n\nu$. We can now construct a sequence of operators $\{K_n\}$ on $L^p(\mathbb{C})$ by

$$K_n g = P(I-\nu_n\bar{S})^{-1}(\alpha_n\bar{g}). \tag{4.29}$$

Then by Proposition 4.1 of [8], the K_n are all compact. We will show $K_n \to K$ in operator norm, and since the set of compact operators is norm-closed, we can conclude that K is compact.
It suffices to show that $\|(K_n-K)g\|_p \to 0$ as $n\to\infty$ uniformly in $g\in L^p(\mathbb{C})$ with $\|g\|_p = 1$. Note that

$$\begin{aligned}(K-K_n)g &= P\Big[(I-\nu_n\bar{S})^{-1}[(\alpha_n-\alpha)\bar{g}] \\ &\quad + \Big[(I-\nu_n\bar{S})^{-1} - (I-\nu\bar{S})^{-1}\Big](\alpha\bar{g})\Big]\end{aligned} \tag{4.30}$$

and by the second resolvent formula, we can rewrite this as

$$\begin{aligned}(K-K_n)g &= \underbrace{P\Big[(I-\nu_n\bar{S})^{-1}[(\alpha_n-\alpha)\bar{g}]\Big]}_{I} \\ &\quad + \underbrace{P\Big[(I-\nu_n\bar{S})^{-1}(\nu_n-\nu)\bar{S}(I-\nu\bar{S})^{-1}(\alpha\bar{g})\Big]}_{II}\end{aligned} \tag{4.31}$$

We employ a decomposition in the spirit of equation (4.27).

$$\begin{aligned}\|I\|_p &= \Big\|P\Big[[(\alpha_n-\alpha)\bar{g}] - \nu_n\bar{S}(I-\nu_n\bar{S})^{-1}[(\alpha_n-\alpha)\bar{g}]\Big]\Big\|_p \\ &\lesssim_p \|(\alpha_n-\alpha)\bar{g}\|_{2p/(p+2)} + \Big\|\nu_n\bar{S}(I-\nu_n\bar{S})^{-1}[(\alpha_n-\alpha)\bar{g}]\Big\|_{2p/(p+2)} \\ &\lesssim_p \|\alpha_n-\alpha\|_2\|g\|_p + \|\nu_n\|_p\big\|\bar{S}\big\|_{L^2\to L^2}\Big\|(I-\nu_n\bar{S})^{-1}\Big\|_{L^2\to L^2}\|(\alpha_n-\alpha)\bar{g}\|_2 \\ &\lesssim_p \|\alpha_n-\alpha\|_2\|g\|_p + \frac{1}{1-k}\|\nu\|_p\|\alpha_n-\alpha\|_{2p/(p-2)}\|g\|_p \\ &\to 0 \text{ as } n\to\infty\end{aligned}$$

We used the fact that since the resolvent $(I - \nu_n \bar{S})^{-1}$ is constructed with a Neumann series dependent on $\|\nu_n\|_\infty$, we know that

$$\left\|(I - \nu_n \bar{S})^{-1}\right\|_{L^p \to L^p} \leq \frac{1}{1 - k \left\|\bar{S}\right\|_{L^p \to L^p}} \text{ for all } n \tag{4.32}$$

In particular, the constant is $\frac{1}{1-k}$ when $p = 2$ since the Beurling transform is an isometry on $L^2(\mathbb{C})$. Similarly, we can estimate

$$\begin{aligned}
\|II\|_p &= \left\| P\left[(\nu_n - \nu)\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g}) - \nu_n \bar{S}(I - \nu_n\bar{S})^{-1}(\nu_n - \nu)\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right]\right\|_p \\
&\lesssim_p \left\|(\nu_n - \nu)\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right\|_{2p/(p+2)} \\
&+ \left\|\nu_n \bar{S}(I - \nu_n\bar{S})^{-1}(\nu_n - \nu)\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right\|_{2p/(p+2)} \\
&\lesssim_p \|\nu_n - \nu\|_2 \left\|\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right\|_p \\
&+ \|\nu_n\|_p \left\|\bar{S}\right\|_{L^2 \to L^2} \left\|(I - \nu_n\bar{S})^{-1}\right\|_{L^2 \to L^2} \left\|(\nu_n - \nu)\bar{S}(I - \nu\bar{S})^{-1}(\alpha\bar{g})\right\|_2 \\
&\lesssim_p \|\nu_n - \nu\|_2 \left\|\bar{S}\right\|_{L^p \to L^p} \left\|(I - \nu\bar{S})^{-1}\right\|_{L^p \to L^p} \|\alpha\|_\infty \|g\|_p \\
&+ \frac{1}{1-k} \|\nu\|_p \|\nu_n - \nu\|_{2p/(p-2)} \left\|\bar{S}\right\|_{L^p \to L^p} \left\|(I - \nu\bar{S})^{-1}\right\|_{L^p \to L^p} \|\alpha\|_\infty \|g\|_p \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

This implies $\|(K_n - K)g\|_p \to 0$ as $n \to \infty$, and thus K is compact on $L^p(\mathbb{C})$. By Proposition 4.2.2 and the Fredholm alternative, the operator $(I - K)$ is invertible on $L^p(\mathbb{C})$. □

Proof of Theorem 4.2.1

In order to study this, we substitute the ansatz $f(z, \xi) = e^{i\xi z} M(z, \xi)$ into the equation (4.21). The function M satisfies

$$\begin{cases} \bar{\partial} M(z, \xi) = \mu(z)\overline{\partial(e_\xi(z) M(z, \xi))} \\ M(\cdot, \xi) - 1 \in W^{1,p}(\mathbb{C}) \end{cases} \tag{4.33}$$

where $e_\xi(z)$ is the phase function

$$e_\xi(z) = e^{i(\xi z + \bar{\xi}\bar{z})} \tag{4.34}$$

We first show that solving this PDE is equivalent to solving an integral equation.

Theorem 4.2.5. *Suppose μ is a real-valued measurable function with $\|\mu\|_\infty \leq k < 1$ and that $\mu \in L^r(\mathbb{C})$ for $1 < r < \infty$. Let $2 < p < 1 + k^{-1}$. The PDE (4.33) has a unique solution $M(\cdot, \xi) - 1 \in W^{1,p}(\mathbb{C})$ if and only if there exists a unique solution $M(\cdot, \xi) - 1 \in W^{1,p}(\mathbb{C})$ to the equation*

$$(I - K)(M - 1) = K(1) \tag{4.35}$$

where the operator K is defined in Proposition 4.2.2.

Proof. Suppose we have a solution to (4.33). Since $M-1 \in W^{1,p}(\mathbb{C})$ we can write

$$\overline{\partial}(M-1) = \mu\overline{\partial(e_\xi(M-1)+e_\xi)} \tag{4.36}$$

$$= \mu\overline{\partial}e_{-\xi}\overline{M-1} + \mu e_{-\xi}\overline{\partial(M-1)} + \mu\overline{\partial}e_{-\xi} \tag{4.37}$$

We now use the identity $\overline{\partial g} = \overline{\partial P\overline{\partial}g} = \bar{S}(\overline{\partial}g)$ to write

$$(I-\mu e_{-\xi}\bar{S})(\overline{\partial}(M-1)) = \mu\overline{\partial}e_{-\xi}\overline{(M-1)} + \mu\overline{\partial}e_{-\xi} \tag{4.38}$$

Write

$$\nu = \mu e_{-\xi} \quad \alpha = \mu\overline{\partial}e_{-\xi} \tag{4.39}$$

and note that the assumptions on μ imply that the operator $(I-\nu\bar{S})$ is invertible on $L^p(\mathbb{C})$ by Lemma 2.5.3. This allows us to write equation (4.38) as

$$\overline{\partial}(M-1) = (I-\nu\bar{S})^{-1}(\alpha(\overline{M-1})) + (I-\nu\bar{S})^{-1}(\alpha) \tag{4.40}$$

Applying the solid Cauchy transform to each side yields

$$(M-1) = P(I-\nu\bar{S})^{-1}(\alpha(\overline{M-1})) + P(I-\nu\bar{S})^{-1}(\alpha\bar{1}) \tag{4.41}$$

So, the integral equation can be rewritten as

$$(I-K)(M-1) = K(1) \tag{4.42}$$

Note that each of these steps is reversible. To show the integral equation is solvable, we consider the right-hand side of (4.42). We claim that the inhomogeneous term is in $L^p(\mathbb{C})$. Indeed, we can use a decomposition similar to equation (4.27).

$$\begin{aligned}
\|K(1)\|_p &= \left\|P\left[\alpha - \nu\bar{S}(I-\nu\bar{S})^{-1}(\alpha)\right]\right\|_p \\
&\lesssim_p \|\alpha\|_{2p/(p+2)} + \left\|\nu\bar{S}(I-\nu\bar{S})^{-1}(\alpha)\right\|_{2p/(p+2)} \\
&\lesssim_p \|\alpha\|_{2p/(p+2)} + \|\nu\|_2 \left\|\bar{S}\right\|_{L^p\to L^p} \left\|(I-\nu\bar{S})^{-1}\right\|_{L^p\to L^p} \|\alpha\|_p \\
&= \left\|\mu\overline{\partial}e_{-\xi}\right\|_{2p/(p+2)} + C_p\|\mu e_{-\xi}\|_2 \left\|\bar{S}\right\|_{L^p\to L^p} \left\|\mu\overline{\partial}e_{-\xi}\right\|_p
\end{aligned}$$

Thus a solution to the integral equation (4.42) also satisfies the PDE (4.33). □

We can now conclude the proof of Theorem 4.2.1 as follows. By Proposition 4.2.4, the equation (4.42) can be solved in $L^p(\mathbb{C})$, which allows us to write:

$$M-1 = (I-K)^{-1}(K(1))$$

It remains to show that $M-1 \in W^{1,p}(\mathbb{C})$. As seen in equation (4.41), $(M-1)$ is the sum of Cauchy transforms of L^p functions, which combined with the boundedness of the Beurling transform, implies that $\nabla(M-1) \in L^p(\mathbb{C})$. This completes the proof.

4.3 Principal Solutions for Linear Beltrami Equations

In their paper [8], Astala and Päivärinta showed that CGO solutions to the Beltrami equation can be written as

$$f(z,\xi) = e^{i\xi\varphi(z,\xi)}$$

where $\varphi(z,\xi)$ is a *principal solution* to the nonlinear Beltrami equation

$$\overline{\partial}\varphi(z,\xi) = -\frac{\bar{\xi}}{\xi}\mu(z)e_{-\xi}(\varphi(z,\xi))\overline{\partial\varphi(z,\xi)}. \tag{4.43}$$

The theory of quasiconformal maps uses the term principal solution to refer to solutions of the equation that have large z asymptotics:

$$\varphi(z,\xi) = z + \mathcal{O}\left(\frac{1}{z}\right).$$

This definition makes sense when μ is compactly supported, but for our class of μ, we will make the following weaker definition

Definition 4.3.1. *Suppose that $\|\mu\|_\infty \le k < 1$ and $\mu \in L^r(\mathbb{C})$ for all $1 < r < \infty$. Let $2 < p < 1 + k^{-1}$. We say that $\varphi \in W^{1,p}_{loc}(\mathbb{C})$ is a **principal solution** to (4.43) if $w(z) := \varphi(z) - z \in W^{1,p}(\mathbb{C})$ and $w(z) \to 0$ as $|z| \to \infty$.*

Remark 4.3.2. *The condition that $w(z) \to 0$ as $|z| \to \infty$ is redundant. The integrability properties of μ imply that the function $w(z)$ is the Cauchy transform of a function in $L^p \cap L^q$ for $1 < q < 2 < p < \infty$. Thus by Lemma 2.5.2, $w(z) \to 0$ as $|z| \to \infty$. We include this statement in the definition to support the connection to principal solutions of Beltrami equations with compactly supported coefficients.*

In this section we will show that this definition makes sense by examining existence and uniqueness of principal solutions for linear Beltrami equations. The next section will prove the existence and uniqueness of principal solutions to the nonlinear Beltrami equation (4.43).

As a "warm-up," we consider the linear Beltrami equation

$$\overline{\partial}f = \mu\overline{\partial f}. \tag{4.44}$$

where $\|\mu\|_\infty \le k < 1$ and $\mu \in L^r(\mathbb{C})$ for all $1 < r < \infty$, and let $1 + k < q < 2 < p < 1 + k^{-1}$. In analogy to Definition 4.3.1, we say that $f \in W^{1,p}_{\text{loc}}(\mathbb{C})$ is a *principal solution* to (4.44) if $f(z) = z + w(z)$ for $w \in W^{1,p}(\mathbb{C})$ and $w(z) \to 0$ as $|z| \to \infty$. The following discussion and proof was suggested by Kari Astala [3]. We made a small change in the uniqueness proof. We will prove:

Theorem 4.3.3. *Suppose that $\mu \in L^q(\mathbb{C})$ for all $1 < q < \infty$, and let $1 + k < q < 2 < p < 1 + k^{-1}$. There exists a unique principal solution f of (4.44).*

Proof. Writing $f = z + w$, we deduce from (4.44) that w solves the inhomogeneous Beltrami equation

$$\overline{\partial} w = \mu + \mu \overline{\partial w} \tag{4.45}$$

Using the operator (2.19), we rewrite (4.45) as

$$\overline{\partial} w = \mu + \mu \bar{S}(\overline{\partial} w)$$

For any $s \in [q, p]$, $\left\| \mu \bar{S} \right\|_{L^s \to L^s} < 1$ by Lemma 2.5.3, so $(I - \mu S)^{-1}$ exists as a bounded operator on $L^s(\mathbb{C})$ with norm bounded by a constant depending only on k. Thus we can solve for $\overline{\partial} w \in L^s(\mathbb{C})$ given by

$$\overline{\partial} w = (I - \mu \bar{S})^{-1} \mu$$

for any $s \in (q, p)$ by the hypothesis on μ. Note that this interval contains $p = 2$. We can then recover w using the Cauchy transform:

$$w(z) = P\left[(I - \mu \bar{S})^{-1} \mu \right](z) \tag{4.46}$$

which defines a bounded continuous function vanishing at infinity by Lemma 2.5.2.

We claim that w is unique. If not, suppose w' is another solution and let $v = w - w'$. Then

$$\overline{\partial} v = \mu \overline{\partial v}, \quad \overline{\partial} v \in L^p(\mathbb{C}).$$

From the boundedness of the Beurling operator it follows that, also $\partial v \in L^p(\mathbb{C})$. Applying [4, Theorem 14.4.8], we see that v is a constant, hence 0 by the vanishing conditions on w and w'. □

4.4 Principal Solutions for the Nonlinear Beltrami Equation

In this section we consider principal solutions in the sense of Definition 4.3.1 to (4.43). We assume that $\|\mu\|_\infty \leq k < 1$ and $\mu \in L^r(\mathbb{C})$ for $1 < r < \infty$. Let q and p be $1 + k < q < 2 < p < 1 + 1/k$. Then $\mu \in L^p \cap L^q$ where

$$\|f\|_{L^p \cap L^q} := \|f\|_p + \|f\|_q \,.$$

From this definition and Lemma 2.5.3, it follows that the operator $\bar{S}$ defined by (2.19) satisfies the estimate

$$\left\| \mu \bar{S} f \right\|_{L^p \cap L^q} \leq c \, \|f\|_{L^p \cap L^q} \tag{4.47}$$

for a number $0 < c < 1$ depending only on the pair (p, q).

We will prove:

Theorem 4.4.1. *Suppose that μ is a real-valued measurable function with $\|\mu\|_\infty \leq k < 1$, and that $\mu \in L^r(\mathbb{C})$ for $1 < r < \infty$. Let $1 + k < q < 2 < p < 1 + k^{-1}$. Then, for each $\xi \in \mathbb{C}$ there exists a unique principal solution φ to the nonlinear Beltrami equation* (4.43).

The proof is a direct consequence of Propositions 4.4.2 and 4.4.4 of what follows.

To prove existence we write $\varphi = z+w$ so that w obeys the inhomogeneous equation

$$\overline{\partial} w = -\mu_\xi e_{-\xi}(z+w) - \mu_\xi e_{-\xi}(z+w)\overline{\partial w}. \tag{4.48}$$

Write $w = Pf$ for a potential $f \in L^p(\mathbb{C}) \cap L^q(\mathbb{C})$ to be determined. For such a potential, w is bounded continuous function vanishing at infinity. The potential f satisfies

$$f = -\mu_\xi e_{-\xi}(z+Pf) - \mu_\xi e_{-\xi}(z+Pf)\bar{S}f. \tag{4.49}$$

where $\mu_\xi(z) = \frac{\bar{\xi}}{\xi}\mu(z)$.

Proposition 4.4.2. *Suppose that $\|\mu_\xi\|_\infty \leq k < 1$ and that $1+k < q < 2 < p < 1+k^{-1}$. Finally, suppose that $\mu \in L^r(\mathbb{C})$ for $1 < r < \infty$. Then, there exists a continuous solution $w = Pf$ to (4.48) for some $f \in L^p(\mathbb{C}) \cap L^q(\mathbb{C})$.*

Remark 4.4.3. *As a consequence of the representation $w = Pf$ and Lemma 2.5.2, w is a continuous function that vanishes at infinity, and clearly $\overline{\partial} w \in L^p(\mathbb{C})$. Moreover, it follows from (4.49) and Hölder's inequality that, in fact, $f \in L^s(\mathbb{C})$ for any $s \in (1,2)$. Choosing $s = (2p)/(2+p)$ gives $w \in L^p(\mathbb{C})$. Hence, $w \in W^{1,p}(\mathbb{C})$. The proof of Proposition 4.4.5 will include more details on these estimates.*

Proof of Proposition 4.4.2. We'll use an approach similar to the proof of Theorem 8.2.1 in [4]. To solve (4.49), we first consider the problem

$$\Phi = -\mu_\xi e_{-\xi}(z+Pf) - \mu_\xi e_{-\xi}(z+Pf)\bar{S}\Phi \tag{4.50}$$

to be solved for $\Phi \in L^p(\mathbb{C}) \cap L^q(\mathbb{C})$ for given $f \in L^p(\mathbb{C}) \cap L^q(\mathbb{C})$. This problem makes sense because $Pf \in L^\infty$, $\bar{S}$ maps $L^p \cap L^q$ to itself, and $e_{-\xi}(\,\cdot\,)$ is unimodular. Moreover, in the obvious iteration scheme to solve it, we easily estimate from (4.50) that

$$\|\Phi_{n+1} - \Phi_n\|_{L^p \cap L^q} \leq c\,\|\Phi_n - \Phi_{n-1}\|_{L^p \cap L^q}$$

where $c \in (0,1)$ is the constant from the estimate (4.47). This shows that (4.50) has a unique solution for each $f \in L^p \cap L^q$. The solution map $T : f \to \Phi$ is a nonlinear operator on $L^p \cap L^q$; a function f solves (4.49) if $f = Tf$. To show that the operator T has a fixed point, we will show that T preserves a sufficiently large closed ball B about the origin of $L^p \cap L^q$, that T is continuous, and that the image of B under T has compact closure. It will then follow from the Schauder fixed point theorem (Theorem 2.6.1) that T has a fixed point f, and hence that $w = Pf$ solves (4.48).

First, using (4.50), we have the pointwise estimate

$$|\Phi(z)| \leq |\mu_\xi| + |\mu_\xi|\,\left|\bar{S}\Phi\right|$$

from which it follows that

$$\|\Phi\|_{L^p \cap L^q} \leq \frac{1}{1-c}\,\|\mu_\xi\|_{L^p \cap L^q}\,.$$

We now choose B to be the ball of radius $(2/(1-c))\left\|\mu_\xi\right\|_{L^p\cap L^q}$ about 0. Clearly T maps B into itself. To show that T is continuous and $T(B)$ is compact, it suffices to show that if $\varphi_n \to \varphi_0$ weakly in $L^p \cap L^q$, if $T\varphi_n = \Phi_n$, and if $T\varphi_0 = \Phi_0$, then $\Phi_n \to \Phi_0$ strongly in $L^p \cap L^q$.

By the compactness of the operator P as a map from $L^p \cap L^q$ to $C(\{|z| \leq R\})$ [4, Theorem 4.3.14], we have $\sup_{|z|\leq R} |(P\varphi_n)(z) - (P\varphi_0)(z)| \to 0$ as $n \to \infty$ for each fixed $R > 0$. Using (4.50) for the respective pairs (φ_n, Φ_n) and (φ_0, Φ_0), we see that

$$\begin{aligned}\Phi_n(z) - \Phi_0(z) = &\mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right] \\ &+ \mu_\xi e_{-\xi}(z + P\varphi_n)\left[\bar{S}\Phi_n - \bar{S}\Phi_0\right] \\ &+ \mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right]\bar{S}\Phi_0\end{aligned} \tag{4.51}$$

Letting

$$\begin{aligned}E_n = &\mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right] \\ &+ \mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right]\bar{S}\Phi_0\end{aligned} \tag{4.52}$$

we deduce from (4.51) that

$$(1-c)\left\|\Phi_n - \Phi_0\right\|_{L^p\cap L^q} \leq \left\|E_n\right\|_{L^p\cap L^q}$$

so it suffices to show that $\left\|E_n\right\|_{L^p\cap L^q} \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. To estimate the first right-hand term in (4.52), we first choose R so large that

$$\left\|\chi_{>R}\mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right]\right\|_{L^p\cap L^q} \leq 2\left\|\chi_{>R}\mu_\xi\right\|_{L^p\cap L^q} < \varepsilon/4,$$

where $\chi_{>R}$ is the characteristic function of the set $\{z : |z| > R\}$ and, in what follows, $\chi_{\leq R} = 1 - \chi_{>R}$. We then use the uniform convergence of $P\varphi_n$ to $P\varphi_0$ on compact subsets of $\mathbb{C}$ to choose n so large that

$$\left\|\chi_{\leq R}\mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right]\right\|_{L^p\cap L^q} < \varepsilon/4.$$

To bound the second right-hand term in (4.52), we similarly choose R so large that

$$2\left\|\chi_{>R}\mu_\xi\bar{S}\Phi_0\right\|_{L^p\cap L^q} < \varepsilon/4.$$

Next we estimate

$$\begin{aligned}&\left\|\chi_{\leq R}\mu_\xi\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right]\bar{S}\Phi_0\right\|_{L^p\cap L^q} \\ &\qquad \leq \left\|\mu\right\|_\infty \left\|\chi_{\leq R}\left[e_{-\xi}(z + P\varphi_n) - e_{-\xi}(z + P\varphi_0)\right]\right\|_\infty \left\|\bar{S}\Phi_0\right\|_{L^p\cap L^q}\end{aligned}$$

and choose n so large that this term is also $< \varepsilon/4$. This shows that $\left\|E_n\right\|_{L^p\cap L^q} < \varepsilon$ for n sufficiently large, so that $\Phi_n \to \Phi_0$ in $L^p \cap L^q$.

It now follows that there is at least one $f \in L^p \cap L^q$ that solves (4.49), and hence at least one $w = Pf$ that solves (4.48). □

Next, we show that the principal solution is unique. Our uniqueness proof borrows ideas from the proof of [4, Theorems 8.5.1 and 8.5.3]. The hypothesis $w_i \in W^{1,p}(\mathbb{C})$ in the uniqueness theorem is justified by Remark 4.4.3.

Proposition 4.4.4. *Let $\mu \in L^p \cap L^q$ where $\|\mu\|_\infty \leq k < 1$ and $1+k < q < 2 < p < 1+1/k$. Suppose that w_1 and w_2 are solutions of (4.48) with $w_i \in W^{1,p}(\mathbb{C})$, and $w_i(z) \to 0$ as $|z| \to \infty$. Then $w_1 = w_2$.*

Proof. Let $v = w_1 - w_2$. It follows from (4.48) that

$$\overline{\partial} v = \alpha(z)\partial v + F(z)v \tag{4.53}$$

where

$$\begin{aligned}\alpha(z) &= -\mu_\xi e_{-\xi}(z+w_1)\frac{\overline{\partial v}}{\partial v},\\ F(z) &= F_1 + F_2,\end{aligned}$$

where

$$\begin{aligned}F_1 &= -\mu_\xi e_{-\xi}(z)\left[\frac{e_{-\xi}(w_1)-e_{-\xi}(w_2)}{w_1-w_2}\right],\\ F_2 &= -\mu_\xi e_{-\xi}(z)\overline{\partial w_2}\left[\frac{e_{-\xi}(w_1)-e_{-\xi}(w_2)}{w_1-w_2}\right].\end{aligned}$$

Note that $\|\alpha\|_\infty \leq k < 1$. For real numbers p and q, we note that

$$\begin{aligned}\left|e^{ip}-e^{iq}\right| &= \left|\int_0^1 ie^{tp+(1-t)q}(p-q)\,dt\right|\\ &\leq |p-q|\end{aligned}$$

For the real numbers $p = 2\Re(\xi w_1(z))$ and $q = 2\Re(\xi w_2(z))$, we can conclude

$$\left|e^{2i\Re(\xi w_1(z))} - e^{2i\Re(\xi w_2(z))}\right| \leq 2\left|\Re(\xi w_1(z) - \xi w_2(z))\right| \leq 2|\xi||w_1(z)-w_2(z)|. \tag{4.54}$$

Therefore,

$$\left|\frac{e_{-\xi}(w_1)-e_{-\xi}(w_2)}{w_1-w_2}\right| \leq 2|\xi| \tag{4.55}$$

This estimate enables us to show $F \in L^p \cap L^q$. This is obviously true of F_1, and we may estimate

$$\begin{aligned}\|F_2\|_p &\leq 2|\xi|\,\|\mu_\xi\|_\infty\,\|w_2\|_{W^{1,p}},\\ \|F_2\|_q &\leq 2|\xi|\,\|\mu_\xi\|_{qp/(p-q)}\,\|w_2\|_{W^{1,p}}\end{aligned}$$

We will introduce a change of variable to remove the Fv from equation (4.53). Suppose we could find a bounded function θ so that $v = e^\theta g$. Then substituting into (4.53) gives us

$$\overline{\partial}\theta e^\theta + e^\theta\overline{\partial} g = \alpha\partial\theta e^\theta g + \alpha e^\theta \partial g = Fe^\theta g \tag{4.56}$$

Our goal is to reduce (4.56) to $\overline{\partial} g = \alpha\partial g$. This implies that θ must satisfy the following in $L^p(\mathbb{C})$

$$\overline{\partial}\theta = \alpha\theta + F. \tag{4.57}$$

Given α and F, define $\theta = P(I - \alpha S)^{-1}F$. The resolvent exists on $L^s(\mathbb{C})$ for $s \in (1+k, 1+k^{-1})$. Since $F \in L^p \cap L^q$, Lemma 2.5.2 implies that θ is a bounded function that vanishes at ∞. As constructed, $\overline{\partial}\theta \in L^p(\mathbb{C})$ and θ satisfies (4.57) in $L^p(\mathbb{C})$.

Since $v \in L^p(\mathbb{C})$ and θ is a bounded function, $g \in L^p(\mathbb{C})$ and g satisfies $\overline{\partial}g = \alpha\partial g$. In addition, we can write $g = e^{-\theta}v$ and conclude

$$\overline{\partial}g = -\overline{\partial}\theta e^{-\theta}v + e^{-\theta}\overline{\partial}v \tag{4.58}$$

The function $v \in W^{1,p}(\mathbb{C})$. By Morrey's inequality, v is a bounded function, and $\overline{\partial}v \in L^p(\mathbb{C})$. Thus $\overline{\partial}g \in L^p(\mathbb{C})$ and $g \in W^{1,p}(\mathbb{C})$. So, g is a solution to the homogeneous Beltrami equation

$$\begin{cases} \overline{\partial}g = \alpha\partial g \\ g \in W^{1,p}(\mathbb{C}) \end{cases} \tag{4.59}$$

We can now apply a generalized Liouville theorem (e.g., Theorem 2.4.2) to conclude that g is a constant. Since w_1 and w_2 vanish at infinity, we conclude that $g = 0$, hence $v = 0$ and hence $w_1 = w_2$.

□

Now that we have established the existence and uniqueness of the principal solution φ to (4.43), we will show that the function $\varphi(z) - z$ satisfies some estimates uniformly in the parameter ξ.

Proposition 4.4.5. *Let $\varphi(z,\xi) = z + g(z)$ be the solution to the nonlinear Beltrami equation* (4.43) *from Theorem 4.4.1. Let $1 + k < q \leq 2 < p < 1 + k^{-1}$. Then*

$$\sup_{\xi\in\mathbb{C}} \|g\|_p < \infty \tag{4.60}$$

$$\sup_{\xi\in\mathbb{C}} \left\|\overline{\partial}g\right\|_{L^p\cap L^q} < \infty \tag{4.61}$$

Proof. We write $g(z) = (Pf)(z)$ where from Proposition 4.4.2, $f \in L^p \cap L^q$. Recall the equation that f satisfies, (4.49)

$$f = -\mu_\xi e_{-\xi}(z + Pf) - \mu_\xi e_{-\xi}(z + Pf)\bar{S}f \tag{4.62}$$

The properties of μ imply that f can be represented as a Neumann series

$$f = -\sum_{n=0}^{\infty} \left(\mu_\xi e_{-\xi}(z + Pf)\bar{S}\right)^n \left(\mu_\xi e_{-\xi}(z + Pf)\right) \tag{4.63}$$

Because Pf is a bounded function and $e_{-\xi}(\cdot)$ is unimodular (as is the $\frac{\bar{\xi}}{\xi}$ term in μ_ξ), we can estimate

$$\sup_{\xi\in\mathbb{C}} \|f\|_{L^p\cap L^q} \leq C_{p,q,k} \|\mu\|_{L^p\cap L^q} \tag{4.64}$$

using the estimate (4.47) and the geometric series. The representation $g = Pf$ gives (4.61). To get a uniform bound on $\|g\|_p$, we need to show that $f \in L^s(\mathbb{C})$ for any $s \in (1,2)$ where the estimate is uniform in ξ. The particular selection $s = \frac{2p}{2+p}$ will give the uniform bound on $\|g\|_p$. Let $s \in (1,2)$ and suppose $2 < p < 1 + k^{-1}$. Then we can use (4.62) to estimate

$$\|f\|_s \leq \|\mu\|_s + \left\|\mu \bar{S} f\right\|_s \leq \|\mu\|_s + \|\mu\|_{\frac{sp}{p-s}} \left\|\bar{S}\right\|_{L^p \to L^p} \|f\|_p$$

□

4.5 Solution to a Nonlinear Equation for $\partial\varphi$

In the following section, we will prove some key properties of the principal solution to the nonlinear Beltrami equation

$$\begin{cases} \overline{\partial}\varphi = -\frac{\bar{\xi}}{\xi}\mu(z)e_\xi(\varphi(z))\overline{\partial\varphi} \\ \varphi(z,\xi) \to z \quad \text{as } |z| \to \infty \end{cases} . \tag{4.65}$$

We collect the coefficients in the Beltrami equation as

$$\tilde{\mu}(z) = -\frac{\bar{\xi}}{\xi}e_{-\xi}(\varphi(z))\mu(z) \tag{4.66}$$

In order to establish the properties of φ, we must first examine the following nonlinear equation

$$\overline{\partial}\omega = (\partial\tilde{\mu} + \partial\bar{\omega}\tilde{\mu})e^{\bar{\omega}-\omega} \tag{4.67}$$

and show that it has a solution $\omega \in L^q(\mathbb{C})$ for some $q > 2$. This equation comes from the substitution $\partial\varphi = e^\omega$. For more details on its derivation, see Lemma 4.6.1 in the next section.

A Priori Estimates

We first prove an estimate on one of the terms in equation (4.67).

Lemma 4.5.1. *For each $1 + k < p < 1 + k^{-1}$ and each fixed $\xi \in \mathbb{C}$, the coefficient $\tilde{\mu}$ satisfies*

$$\|\partial\tilde{\mu}\|_p < \infty \tag{4.68}$$

Proof. We first calculate

$$\partial\tilde{\mu}(z) = -\frac{\bar{\xi}}{\xi}e^{-i(\xi\varphi+\bar{\xi}\bar{\varphi})}(\xi\partial\varphi + \bar{\xi}\partial\bar{\varphi})(-i)\mu - \frac{\bar{\xi}}{\xi}e_{-\xi}(\varphi(z))\partial\mu(z)$$

This implies

$$\begin{aligned} \|\partial\tilde{\mu}\|_p &\leq \left\|\bar{\xi}(\partial\varphi - 1)\mu\right\|_p + \left\|\bar{\xi}\mu\right\|_p + \left\|\xi(\overline{\partial}\varphi)\mu\right\|_p + \|\partial\mu\|_p && (4.69) \\ &\leq |\xi|k\,\|\partial\varphi - 1\|_p + |\xi|\,\|\mu\|_p + |\xi|k\left\|\overline{\partial}\varphi\right\|_p + \|\partial\mu\|_p && (4.70) \end{aligned}$$

The function $\overline{\partial}\varphi$ can be represented as a Neumann series. In particular,

$$\overline{\partial}\varphi = (I - \tilde{\mu}\bar{S})^{-1}(\tilde{\mu}) \tag{4.71}$$

Since $|\tilde{\mu}(z)| \leq k < 1$, the Neumann series converges in $L^p(\mathbb{C})$.[2] Thus, we know $\overline{\partial}\varphi \in L^p(\mathbb{C})$ (and consequently $(\partial\varphi - 1) \in L^p(\mathbb{C})$. Thus the first and third terms are bounded. Because μ satisfies the conclusion of Theorem 4.1.1, the other terms are bounded as well. Therefore $\|\partial\tilde{\mu}\|_p$ is bounded for each fixed ξ. □

We will also need this useful fact about compact operators between L^p-spaces.

Lemma 4.5.2. *Let $\{T_n\} \in \mathcal{L}(L^p, L^p)$ be a strongly convergent sequence (with strong limit $T \in \mathcal{L}(L^p, L^p)$)and $S \in \mathcal{L}(L^p, L^q)$ be a compact operator. Then the sequence $\{ST_n\} \in \mathcal{L}(L^p, L^q)$ converges in norm to ST.*

Proof. From Proposition 2.6.3, there exists a sequence of finite-rank operators $\{S_m\}$ such that $S_m \xrightarrow{n} S$ in $\mathcal{L}(L^p, L^q)$. For any m, consider the operators $S_m(T_n - T)$. These operators are of finite rank, and because the T_n converge strongly to T, we can conclude

$$\|S_m(T_n - T)\|_{\mathcal{L}(L^p,L^q)} \to 0 \text{ as } n \to \infty. \tag{4.72}$$

Note that because $T_n \xrightarrow{s} T$, the principle of uniform boundedness implies

$$\sup_n \|T_n - T\|_{\mathcal{L}(L^p,L^p)} \leq C$$

Let $\varepsilon > 0$ be given. Choose m so that $\|S - S_m\|_{\mathcal{L}(L^p,L^q)} C < \frac{\varepsilon}{2}$. For that m, choose N so that $n \geq N$ implies $\|S_m(T_n - T)\|_{\mathcal{L}(L^p,L^q)} < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned} \|S(T_n - T)\|_{\mathcal{L}(L^p,L^q)} &\leq \|S - S_m\|_{\mathcal{L}(L^p,L^q)} \|T_n - T\|_{\mathcal{L}(L^p,L^p)} + \|S_m(T_n - T)\|_{\mathcal{L}(L^p,L^q)} \\ &< \|S - S_m\|_{\mathcal{L}(L^p,L^q)} C + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

□

Proof of Existence

The goal is to use the Schauder fixed point theorem to show that (4.67) has a solution. This highly nonlinear equation is saved by the fact that the nonlinearities appear in unimodular factors.

Let $q > 2$ be so that $1 + k < \dfrac{2q}{2+q} < 1 + k^{-1}$. Define the operator T by

$$Tf = P(I - \tilde{\mu}e^{\bar{f}-f}C)^{-1}(\partial\tilde{\mu}e^{\bar{f}-f}) \tag{4.73}$$

[2]For more details on this convergence (and its uniformity in the parameter ξ, take a look at the proof of Proposition 4.4.5.

where C denotes the conjugation operator $(Cg)(z) = \overline{g(z)}$. Then we first note that $T : L^q(\mathbb{C}) \to L^q(\mathbb{C})$, since

$$\begin{aligned} \|Tf\|_q &\leq C_q \left\| (I - \tilde{\mu} e^{\bar{f}-f} C)^{-1} (\partial \tilde{\mu} e^{\bar{f}-f}) \right\|_{\frac{2q}{2+q}} \\ &\leq C_q \left\| (I - \tilde{\mu} e^{\bar{f}-f} C)^{-1} \right\|_{op} \left\| \partial \tilde{\mu} e^{\bar{f}-f} \right\|_{\frac{2q}{2+q}} \end{aligned}$$

where $\|\cdot\|_{op}$ denotes the operator norm from $L^{\frac{2q}{2+q}}(\mathbb{C}) \to L^{\frac{2q}{2+q}}(\mathbb{C})$. Since C is an isometry and $\left|\tilde{\mu} e^{\bar{f}-f}\right| \leq k < 1$ for all $z \in \mathbb{C}$ and uniformly for $f \in L^q(\mathbb{C})$, we can conclude

$$\left\| (I - \tilde{\mu} e^{\bar{f}-f} C)^{-1} \right\|_{op} \leq \frac{1}{1-k}$$

Thus we can bound $\|Tf\|_q$ by

$$\|Tf\|_q \leq \frac{C_q}{1-k} \left\| \partial \tilde{\mu} e^{\bar{f}-f} \right\|_{\frac{2q}{2+q}} = \frac{C_q}{1-k} \|\partial \tilde{\mu}\|_{\frac{2q}{2+q}} \tag{4.74}$$

which is finite by Lemma 4.5.1.
Next, define

$$B = \left\{ f \in L^q(\mathbb{C}) : \|f\|_q \leq \frac{C_q}{1-k} \|\partial \tilde{\mu}\|_{\frac{2q}{2+q}} \right\} \tag{4.75}$$

Then our calculation in equation (4.74) indicates $T(B) \subset B$. It now remains to show that $T(B)$ satisfies the requirements of the Kolmogorov-Riesz Theorem (Theorem 2.6.4). First, we show that $Tf(z)$ exhibits uniform decay in L^q-norm. We write

$$\begin{aligned} \chi_{|z|>4R}(z) Tf(z) &= \int \frac{\chi_{|z|>4R}(z)}{z-w} (I - \tilde{\mu} e^{\bar{f}-f} C)^{-1} (\partial \tilde{\mu} e^{\bar{f}-f})(w) \, dw \\ &= \int \frac{\chi_{|z|>4R}(z)}{z-w} \left[\chi_{|w|<R}(w) + \chi_{|w|\geq R}(w) \right] \underbrace{(I - \tilde{\mu} e^{\bar{f}-f} C)^{-1} (\partial \tilde{\mu} e^{\bar{f}-f})(w)}_{g(w)} \, dw \end{aligned}$$

We want to estimate $\left\| \chi_{|z|>4R} Tf \right\|_q$ for large R.

$$\left\| \chi_{|z|>4R} Tf \right\|_q \leq \underbrace{\left\| \chi_{|z|>4R} P(\chi_{|w|<R} g) \right\|_q}_{I} + \underbrace{\left\| \chi_{|z|>4R} P(\chi_{|w|>R} g) \right\|_q}_{II}$$

Choose $\alpha \in (0,1)$ so that $1 + k < \frac{2q}{2+(1-\alpha)q} < 1 + \frac{1}{k}$. Then we can write

$$\left\| \chi_{|z|>4R} P(\chi_{|w|<R} g) \right\|_q \leq \frac{1}{R^\alpha} \left\| \int \frac{1}{|z-w|^{1-\alpha}} g(w) \, dw \right\|_q$$

Next, apply the Hardy-Littlewood-Sobolev inequality (Theorem 2.5.1) to obtain

$$\begin{aligned} \left\| \chi_{|z|>4R} P(\chi_{|w|<R} g) \right\|_q &\leq \frac{1}{R^\alpha} C_{q,\alpha} \|g\|_{\frac{2q}{2+(1-\alpha)q}} \\ &\leq \frac{1}{R^\alpha} \frac{C_{q,\alpha}}{1-k} \|\partial \tilde{\mu}\|_{\frac{2q}{2+(1-\alpha)q}} \end{aligned}$$

Then for the second term recall that C denotes the conjugation operator. This means that we can estimate:

$$\begin{aligned}\left|\sum_{n=0}^{\infty}(\tilde{\mu}e^{\bar{f}-f}C)^n(\partial\tilde{\mu}e^{\bar{f}-f})\right| &\leq \sum_{n=0}^{\infty}|(\tilde{\mu}e^{\bar{f}-f}C)^n(\partial\tilde{\mu}e^{\bar{f}-f})| \\ &\leq \sum_{n=0}^{\infty}k^n|\partial\tilde{\mu}(w)| = \frac{1}{1-k}|\partial\tilde{\mu}(w)|\end{aligned}$$

Thus term II can be bounded by

$$\begin{aligned}\left\|\chi_{|z|>4R}P(\chi_{|w|\geq R}g)\right\|_q &\leq \left\|\int\frac{1}{|z-w|}\chi_{|w|\geq R}(w)\sum_{n=0}^{\infty}(\tilde{\mu}e^{\bar{f}-f}C)^n(\partial\tilde{\mu}e^{\bar{f}-f})(w)\ dw\right\|_q \\ &\leq \frac{1}{1-k}\left\|P(\chi_{|w|>R}|\partial\tilde{\mu}|)\right\|_q \\ &\leq \frac{C_q}{1-k}\left\|\chi_{|w|\geq R}|\partial\tilde{\mu}|\right\|_{\frac{2q}{q+2}}\end{aligned}$$

Thus we have a decay estimate

$$\left\|\chi_{|z|>4R}(z)Tf(z)\right\|_q \leq \frac{1}{R^\alpha}\frac{C_{q,\alpha}}{1-k}\left\|\partial\tilde{\mu}\right\|_{\frac{2q}{2+(1-\alpha)q}} + \frac{C_q}{1-k}\left\|\chi_{|w|\geq R}\partial\tilde{\mu}\right\|_{\frac{2q}{q+2}}.$$

We can choose R based on the decay of the $L^{\frac{2q}{2+q}}(\mathbb{C})$ norm of $\partial\tilde{\mu}$ and the constants in the first term. Thus this decay estimate is uniform for $f \in B$ satisfying the first requirement of Theorem 2.6.4.
Let M_h be the translation operator $M_hf(z) = f(z+h)$. We want to estimate

$$\|(M_h - I)(Tf)\|_q$$

for small h uniformly for $f \in B$. Let $\varepsilon > 0$ be given. Using the uniform decay estimate proven above, we choose R large enough so that

$$2\left\|\chi_{|z|>R}Tf\right\|_q < \frac{\varepsilon}{2} \tag{4.76}$$

uniformly for $f \in B$. We write

$$\|(M_h - I)(Tf)\|_q \leq \left\|\chi_{|z|<R}(M_h - I)(Tf)\right\|_q + 2\left\|\chi_{|z|>R}Tf\right\|_q \tag{4.77}$$

and we write

$$Tf = P(I - \tilde{\mu}e^{\bar{f}-f}C)^{-1}(\partial\tilde{\mu}e^{\bar{f}-f}) = Pg$$

Since translation commutes with P, we can write

$$\left\|\chi_{|z|<R}(M_h - I)(Tf)\right\|_q = \left\|\chi_{|z|<R}P(M_h - I)g\right\|_q$$

Choose $1 + k < s < 2$ such that $q < s^* = \frac{2s}{2-s}$. By Lemma 4.5.1, $g \in L^s(\mathbb{C})$ and the norm is uniform for $f \in B$. Note that $\{(M_h - I)\}$ is a strongly convergent sequence with limit 0 in $\mathcal{L}(L^s, L^s)$. By standard arguments (see, e.g. Theorem 4.3.14 in [4]), the operator $(\chi_{|z|<R}P) : L^s \to L^r$ is compact for all $1 \leq r < s^*$. In particular,

$$(\chi_{|z|<R}P) : L^s \to L^q \text{ is compact.} \tag{4.78}$$

By Lemma 4.5.2, we can conclude that $\{(\chi_{|z|<R}P)(M_h - I)\}$ converges in norm to 0 as $h \to 0$ in $\mathcal{L}(L^s, L^q)$. We can apply the convergence to

$$\begin{aligned} \left\|(\chi_{|z|<R}P)(M_h - I)g\right\|_q &\leq \left\|(\chi_{|z|<R}P)(M_h - I)\right\|_{\mathcal{L}(L^s,L^q)} \|g\|_s \\ &\leq \frac{1}{1-k} \left\|(\chi_{|z|<R}P)(M_h - I)\right\|_{\mathcal{L}(L^s,L^q)} \|\partial\tilde{\mu}\|_s \end{aligned} \tag{4.79}$$

Hence there exists a $\delta > 0$ such that for $|h| < \delta$, $\left\|(\chi_{|z|<R}P)(M_h - I)g\right\|_q < \frac{\varepsilon}{2}$ uniformly in g (and hence f).

Thus for $|h| < \delta$, we conclude

$$\begin{aligned} \|(M_h - I)(Tf)\|_q &\leq \left\|\chi_{|z|<R}(M_h - I)(Tf)\right\|_q + 2\left\|\chi_{|z|>R}Tf\right\|_q \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and the estimate is uniform for $f \in B$. Thus we have shown that $T(B) \subset B$ is precompact. The Schauder fixed point theorem (Theorem 2.6.1) shows that there is a fixed point $\omega \in B$ so that $\omega = T\omega$.

4.6 Properties of Principal Solutions

In the case of compactly supported Beltrami coefficients, the theory of quasiregular mappings allows one to deduce that a principal solution to a Beltrami equation is in fact a global homeomorphism. We seek to show that the solution φ to (4.43) is also a global homeomorphism. Its inverse will prove key in establishing the behavior of the solution as the parameter $\xi \to \infty$.

φ is a Global Homeomorphism

The process of showing that φ is a homeomorphism begins with showing that its Jacobian determinant does not vanish for any $z \in \mathbb{C}$.

Lemma 4.6.1. *Let φ be the solution to* (4.43) *for a fixed $\xi \in \mathbb{C}$. Then there exists a constant $G_\xi > 0$ such that for any $z \in \mathbb{C}$, the Jacobian of φ at z, denoted $J(z, \varphi)$ satisfies.*

$$J(z, \varphi) \geq G_\xi > 0$$

Proof. We first calculate

$$J(z, \varphi) = |\partial\varphi|^2 - |\overline{\partial}\varphi|^2$$

As φ solves a $\overline{\partial}$-problem with a pointwise bound on $\tilde{\mu}$, we can estimate:

$$J(z, \varphi) = (1 - |\tilde{\mu}|^2)|\partial\varphi|^2 \geq (1 - k^2)|\partial\varphi|^2 \tag{4.80}$$

We want to guarantee that $\partial\varphi$ is never 0. We claim that there is a bounded function ω such that $\partial\varphi = e^{\omega}$. Substituting this form into the Beltrami equation yields $\overline{\partial}\varphi = \tilde{\mu}e^{\bar{\omega}}$. Then by the compatibility condition for derivatives $\overline{\partial}(\partial\varphi) = \partial(\overline{\partial}\varphi)$ can be calculated as

$$\overline{\partial}\omega = (\partial\tilde{\mu} + \partial\bar{\omega}\tilde{\mu})e^{\bar{\omega}-\omega} \tag{4.81}$$

The equation (4.81) is the same nonlinear equation studied in the previous section (4.67). Thus our work in the previous section implies $\omega \in L^q(\mathbb{C})$ for some $q > 2$. We now want to show it is actually bounded. Equation (4.81) allows us to estimate

$$(1 - |\tilde{\mu}|)|\overline{\partial}\omega| \leq |\partial\tilde{\mu}|$$

Thus

$$|\overline{\partial}\omega| \leq \frac{1}{1-k}|\partial\tilde{\mu}| \tag{4.82}$$

Lemma 4.5.1 implies that $|\overline{\partial}\omega| \in L^p \cap L^q$ for $1 + k < q < 2 < p < 1 + k^{-1}$. Thus by Lemma 2.5.2, ω is a bounded function. □

Remark 4.6.2. *The representation $\partial\varphi = e^{\omega}$ has some important consequences for the continuity of the derivatives of φ. First note that Lemma 2.5.2 implies that $\omega(z)$ is continuous and $\omega(z) \to 0$ as $z \to \infty$. This implies that $\partial\varphi$ is continuous and*

$$\lim_{z\to\infty} \partial\varphi(z) = 1 \tag{4.83}$$

To borrow the notation of (4.66), $\overline{\partial}\varphi = \tilde{\mu}\overline{\partial\varphi}$. Theorem 4.1.1 claims that μ is continuous. The function $\varphi - z \in W^{1,p}(\mathbb{C})$ for $p > 2$, which implies by Morrey's inequality that φ is continuous. Thus $\tilde{\mu}$ is continuous and $|\tilde{\mu}(z)| \sim \mathcal{O}\left(|z|^{-2}\right)$ as $z \to \infty$. These facts come together to give

$$\lim_{z\to\infty} \overline{\partial}\varphi(z) = 0 \tag{4.84}$$

Theorem 4.6.3. *For each fixed $\xi \in \mathbb{C}$, the solution $\varphi(\cdot, \xi)$ is a global homeomorphism of $\widehat{\mathbb{C}}$.*

Proof. We begin by defining $\varphi(\infty, \xi) = \infty$ and thus extend φ to $\widehat{\mathbb{C}}$. We first show that the Jacobian of φ is positive in a neighborhood of ∞. This is equivalent to showing that the map

$$F(w) = \frac{1}{\varphi\left(\frac{1}{w}\right)}$$

has a positive Jacobian at $w = 0$.
By the chain rule, we calculate

$$\begin{aligned} \overline{\partial}F(w) &= \frac{1}{\varphi\left(\frac{1}{w}\right)^2}\overline{\partial}\varphi\left(\frac{1}{w}\right)\frac{1}{\bar{w}^2} \\ \partial F(w) &= \frac{1}{\varphi\left(\frac{1}{w}\right)^2}\partial\varphi\left(\frac{1}{w}\right)\frac{1}{w^2} \end{aligned}$$

Now we can calculate the Jacobian of F as

$$J(w,F) = \frac{1}{\left|\varphi\left(\frac{1}{w}\right)\right|^4} \frac{1}{|w|^4} \left(\left|\partial\varphi\left(\frac{1}{w}\right)\right|^2 - \left|\overline{\partial}\varphi\left(\frac{1}{w}\right)\right|^2 \right) \tag{4.85}$$

Note first that $\varphi(z) = z + g(z)$ where g is a bounded function. This implies

$$\lim_{w\to 0} w\varphi\left(\frac{1}{w}\right) = \lim_{w\to 0} w\left(\frac{1}{w} + g\left(\frac{1}{w}\right)\right) = 1$$

since g is bounded. Remark 4.6.2 on the representation $\partial\varphi = e^{\omega}$ gives the result

$$\lim_{w\to 0} \overline{\partial}\varphi\left(\frac{1}{w}\right) = 0, \quad \lim_{w\to 0} \partial\varphi\left(\frac{1}{w}\right) = 1$$

Thus the Jacobian at $w = 0$ is

$$J(0,F) = \lim_{w\to 0} J(w,F) = 1.$$

By Lemma 4.6.1 and the inverse function theorem, φ is a local homeomorphism for each point in $\widehat{\mathbb{C}}$. Because $\widehat{\mathbb{C}}$ is compact, we can cover it by finitely many of these neighborhoods. Let us now consider the set

$$N_i = \{w \in \widehat{\mathbb{C}} : w \text{ has } i \text{ preimages under } \varphi\}$$

In the first place, note that each N_i is a union of finite intersections of open neighborhoods from the cover and thus each N_i is open. On the other hand, for any fixed i,

$$N_i^c = \bigcup_{j\neq i} N_j,$$

and hence N_i is closed for each i. So, each of the N_i is either all of $\widehat{\mathbb{C}}$ or empty. The asymptotic condition at ∞ and the injectivity at ∞ implies that $N_1 = \widehat{\mathbb{C}}$ and $N_i = \emptyset$ for $i \geq 2$. Thus φ is in fact a global homeomorphism. □

A Beltrami Equation for φ^{-1}

In order to study the behavior of the solutions to the nonlinear Beltrami equation as $\xi \to \infty$, it is more useful to consider the function $\psi = \varphi^{-1}$. Here we include a derivation of the Beltrami equation for ψ. Because φ and ψ are inverses they satisfy the composition

$$(\psi \circ \varphi)(z) = z$$

We now take the $\overline{\partial}$ derivative of both sides of the equation. Using the chain rule, this gives us

$$\partial\psi(\varphi(z)) \cdot \overline{\partial}\varphi(z) + \overline{\partial}\psi(\varphi(z)) \cdot \overline{\partial\varphi(z)} = 0$$

Now we use the fact that φ satisfies (4.43).

$$\partial\psi(\varphi(z)) \left(-\frac{\bar{\xi}}{\xi}\mu(z)e_{-\xi}(\varphi(z))\overline{\partial\varphi(z)} \right) + \overline{\partial}\psi(\varphi(z)) \cdot \overline{\partial\varphi(z)} = 0$$

We can move the first term to the right-hand side and since $\partial\varphi(z) \neq 0$ for all z (see Lemma 4.6.1) we can divide $\overline{\partial\varphi}$ out. This leaves

$$\overline{\partial}\psi(\varphi(z)) = \frac{\bar{\xi}}{\xi} e_{-\xi}(\varphi(z))\mu(z)\partial\psi(\varphi(z))$$

We now make the change of variables $w = \varphi(z)$ to arrive at the desired equation

$$\overline{\partial}\psi(w) = \frac{\bar{\xi}}{\xi} e_{-\xi}(w)\mu(\psi(w))\partial\psi(w) \tag{4.86}$$

Theorem 4.6.4. *For each fixed $\xi \in \mathbb{C}$ and every $2 < p < 1 + k^{-1}$, there exists a unique solution $\psi(z, \xi)$ to the equation*

$$\begin{cases} \overline{\partial}\psi = \frac{\bar{\xi}}{\xi}\mu(\psi(z))e_{-\xi}\partial\psi \\ \psi(\cdot, \xi) - z \in W^{1,p}(\mathbb{C}) \end{cases}$$

Moreover,

$$\sup_{\xi\in\mathbb{C}} \|\psi - z\|_{W^{1,p}(\mathbb{C})} < \infty \tag{4.87}$$

Proof. The existence and uniqueness of the solution φ to (4.43) yields a unique function $\psi = \varphi^{-1}$ that satisfies the equation. It remains to show that $\psi(w, \xi) - w \in W^{1,p}(\mathbb{C})$. First, we show that $\psi(w, \xi) - w \in L^p(\mathbb{C})$ and that the L^p-norm is uniform in ξ. Next, we will show that for a fixed ξ, $\overline{\partial}\psi \in L^p(\mathbb{C})$, followed by a refined estimate that is uniform in ξ.
We first write $\varphi(z) = z + g(z)$ where $g \in W^{1,p}(\mathbb{C})$ for each of the p's in the range. If we change variables $z = \psi(w)$ we can write φ as

$$\varphi(\psi(w)) = \psi(w) + g(\psi(w))$$

Since φ and ψ are inverses, we can rearrange this and write

$$\psi(w) = w - g(\psi(w))$$

We first show that the function $g \circ \psi \in L^p(\mathbb{C})$ by using a change of variable

$$\begin{aligned} \int |g(\psi(w))|^p \, dw &= \int |g(z)|^p J(z, \varphi) \, dz \\ &\leq \int |g(z)|^p (1 - |\tilde{\mu}|^2)|1 + \partial g|^2 \, dz \\ &\leq \int |g(z)|^p + 2|g(z)|^{p/2}|g(z)|^{p/2}|\partial g| + |g(z)|^p|\partial g|^2 \, dz \\ &\leq \|g\|_p^p + 2\|g\|_\infty^{p/2}\|g\|_p^{p/2}\|\partial g\|_2 + \|g\|_\infty^p \|\partial g\|_2^2 \end{aligned} \tag{4.88}$$

Proposition 4.4.5 implies $\|g\|_{W^{1,p}}$ is uniform in ξ. It follows by Morrey's inequality, $g \in C^{0,\alpha}(\mathbb{C})$ for some $\alpha > 0$ with $\|g\|_\infty$ uniform in ξ. Proposition 4.4.5 also implies that $\|\partial g\|_2$ is uniform in ξ.

We now want to work with $\overline{\partial}\psi(w) = \overline{\partial}(\psi(w) - w)$. To that end, recall that the Jacobian matrix of φ, denoted Jac(φ) is the matrix

$$\mathrm{Jac}(\varphi)(z) = \begin{bmatrix} \partial\varphi & \overline{\partial}\varphi \\ \partial\bar{\varphi} & \overline{\partial\varphi} \end{bmatrix} \tag{4.89}$$

Since $\psi = \varphi^{-1}$, we know $\mathrm{Jac}(\psi)(z) = [\mathrm{Jac}(\varphi)]^{-1}(z)$. Thus,

$$\mathrm{Jac}(\psi)(w) = \begin{bmatrix} \partial\psi & \overline{\partial}\psi \\ \partial\bar{\psi} & \overline{\partial\psi} \end{bmatrix} = \frac{1}{J(\psi(w), \varphi)} \begin{bmatrix} \overline{\partial\varphi(\psi(w))} & -\overline{\partial}\varphi(\psi(w)) \\ -\partial\overline{\varphi(\psi(w))} & \partial\varphi(\psi(w)) \end{bmatrix} \tag{4.90}$$

So we can write

$$\int |\overline{\partial}\psi(w)|^p \; dw = \int \left| \frac{1}{J(\psi(w), \varphi)} (-\overline{\partial}\varphi(\psi(w))) \right|^p \; dw$$

We now employ a change of variable $w = \varphi(z)$ and recall that $J(z, \varphi)$ is a positive real number to conclude

$$\begin{aligned} \int \left| \frac{1}{J(\psi(w), \varphi)} (-\overline{\partial}\varphi(\psi(w))) \right|^p \; dw &= \int \frac{1}{J(z, \varphi)^p} |\overline{\partial}\varphi(z)|^p J(z, \varphi) \; dz \\ &= \int \frac{1}{J(z, \varphi)^{p-1}} |\overline{\partial}\varphi(z)|^p \; dz \end{aligned}$$

For a fixed ξ, let G_ξ be the (uniform in z) lower bound on $J(z, \varphi)$ from Lemma 4.6.1. We use the lower bound to conclude

$$\left\|\overline{\partial}\psi\right\|_p^p \leq \frac{1}{G_\xi^{p-1}} \left\|\overline{\partial}\varphi\right\|_p^p$$

By the L^p-boundedness of the Beurling transform,

$$\|\partial(\psi - w)\|_p \leq C_p \left\|\overline{\partial}(\psi - w)\right\|_p = C_p \left\|\overline{\partial}\psi\right\|_p \tag{4.91}$$

Thus we can conclude that for a fixed ξ, $\psi(w) = w + h(w)$ with $h \in W^{1,p}(\mathbb{C})$ for $2 < p < 1 + k^{-1}$.

The the range of p allows equation (4.86) to be rewritten as a Neumann series. Specifically,

$$\overline{\partial}\psi = \sum_{n=0}^{\infty} \left(\frac{\bar{\xi}}{\xi} (\mu \circ \psi) e_{-\xi} S \right)^n \left(\frac{\bar{\xi}}{\xi} (\mu \circ \psi) e_{-\xi} \right)$$

Using the result of Lemma 4.6.6 below, we can estimate the L^p-norm of $\overline{\partial}\psi$ in terms of the function $(\mu \circ \psi)$.

$$\begin{aligned} \left\|\overline{\partial}\psi\right\|_p &= \left\| \sum_{n=0}^{\infty} \left(\frac{\bar{\xi}}{\xi} (\mu \circ \psi) e_{-\xi} S \right)^n \left(\frac{\bar{\xi}}{\xi} (\mu \circ \psi) e_{-\xi} \right) \right\|_p \\ &\leq C_{p,k} \|\mu \circ \psi\|_p \end{aligned} \tag{4.92}$$

By Lemma 4.6.6, we know

$$\sup_{\xi\in\mathbb{C}} \|\mu\circ\psi\|_p < \infty.$$

Equations (4.88), (4.91), and (4.92) together imply

$$\sup_{\xi\in\mathbb{C}} \|\psi - z\|_{W^{1,p}(\mathbb{C})} < \infty$$

□

Remark 4.6.5. *The Neumann series representation for $\overline{\partial}\psi$ and the estimate (4.92) actually hold for $1+k < p < 1+k^{-1}$. The Neumann series converges in $L^p(\mathbb{C})$, and its operator norm is independent of ξ.*

Lemma 4.6.6. *Let $\psi(z,\xi)$ be the solution to equation (4.86) and suppose $\varphi(z,\xi) = z + g(z)$ be the solution to (4.43). Then for every $r \in (1,\infty)$,*

$$\sup_{\xi\in\mathbb{C}} \|\mu\circ\psi\|_r < \infty \tag{4.93}$$

Proof. We introduce a change of variable $w = \varphi(z)$.

$$\begin{aligned}
\int |\mu(\psi(w))|^r \, dw &= \int |\mu(z)|^r J(z,\varphi)\, dz \\
&= \int |\mu(z)|^r (1-|\tilde{\mu}|^2)|\partial\varphi|^2 \, dz \\
&\leq \int |\mu(z)|^r |1+\partial g(z)|^2 \, dz \\
&\leq \int |\mu(z)|^r + 2|\mu(z)|^r|\partial g| + |\mu(z)|^r|\partial g|^2 \, dz \\
&\leq \|\mu\|_r^r + 2\,\|\mu\|_{2r}^r \,\|\partial g\|_2 + k^r \,\|\partial g\|_2^2
\end{aligned}$$

Proposition 4.4.5 and the boundedness of the Beurling transform give $\sup_\xi \|\partial g\|_2 < \infty$ so the estimate is uniform in ξ. □

4.7 Subexponential Growth

The next piece of the puzzle that Astala and Päivärinta solved in [8] was the phenomenon of *subexponential growth.* The solutions to the nonlinear Beltrami equation (4.43) have a prescribed asymptotic behavior as $z \to \infty$, but in order to study the inverse problem, we need to study the asymptotic behavior as $\xi \to \infty$.

Theorem 4.7.1. *Suppose that μ is a real-valued measurable function with $\|\mu\|_\infty \leq k < 1$, $\mu \in L^r(\mathbb{C})$, and $\partial\mu \in L^r(\mathbb{C})$ for all r with $1 < r < \infty$. Suppose $2 < p < 1+k^{-1}$. Let φ be the solution to equation (4.43). Then*

$$\varphi(z,\xi) \to z \tag{4.94}$$

uniformly in $z \in \mathbb{C}$ as $\xi \to \infty$.

Remark 4.7.2. *The function* $\varphi(z,\xi)$ *comes from the CGO solution* f *to the Beltrami equation* $\overline{\partial}f = \mu\overline{\partial f}$, *where*

$$f(z,\xi) = e^{i\xi\varphi(z,\xi)}.$$

The asymptotic condition as $z \to \infty$ *means that* $f \sim e^{i\xi z}$, *and Theorem 4.7.1 implies the same asymptotic condition holds as* $\xi \to \infty$.

We will actually prove the theorem for $\psi = \varphi^{-1}$ because the nonlinearity moves from the phase to a composition with μ (see equation (4.86)). There are several key steps that will lead to a proof of Theorem 4.7.1. Our analysis will be an adaptation of the work of [8] as presented in [4]. They show the behavior for an associated linear problem, and, to get the behavior of the nonlinear equation, these authors use a normal families argument. Replacing the compact support with an integrability condition means that we can make estimates uniform in ξ (such as (4.87) and Lemma 4.6.6). This allows us to adapt Astala and Päivärinta's techniques for the linear Beltrami equation to the nonlinear case.

Preliminaries

Lemma 4.7.3. *Let* ψ *be the solution to equation* (4.86) *and suppose that* μ *has a large-z asymptotic expansion* $\mu \sim \mathcal{O}\left(|z|^{-2}\right)$. *Then*

$$|\mu(\psi(z))| \sim \mathcal{O}\left(\frac{1}{|z|^2}\right) \text{ as } z \to \infty$$

and the expansion is uniform in ξ.

Proof. First, write $\psi(z,\xi) = z + h(z,\xi)$. Then from Theorem 4.6.4 and Sobolev embedding, $|h(z,\xi)| \leq M$ uniformly in ξ. This implies

$$(|z| - M) \leq |z| - |h(z,\xi)| \leq |z + h(z,\xi)|$$

Therefore, for large enough z,

$$|\mu(\psi(z))| \lesssim \frac{1}{|z + h(z)|^2} \lesssim \frac{1}{(|z| - M)^2}$$

□

We will need the following result of Astala-Päivärinta for the next step in the argument.

Lemma 4.7.4 (Lemma 18.6.2 in [4])**.** *Let* $\varepsilon > 0$ *is given. Suppose also that* $\|\mu_\lambda\|_\infty \leq k < 1$ *and* μ_λ *has compact support. Let*

$$f_n = \mu_\lambda S_n \mu_\lambda S_{n-1} \mu_\lambda \cdots \mu_\lambda S_1 \mu_\lambda$$

where $S_j : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ *are Fourier multiplier operators, each with a unimodular symbol. Then there is a number* $R_n = R_n(k,\varepsilon)$ *depending only on* k, n, *and* ε, *such that*

$$|\widehat{f_n}(\eta)| < \varepsilon \text{ for } |\eta| > R_n.$$

The lemma as stated in [4] is for functions μ_λ which are supported in $\mathbb{D}$. The following scaling lemma allows us to apply the result to μ_λ which are supported in the ball $B(0,R) = \{z : |z| \leq R\}$. For $z \in B(0,R)$, let $z = Rx$ for some $x \in \mathbb{D}$. For a function f supported in $B(0,R)$, we denote

$$f_R(x) = f(Rx) \tag{4.95}$$

and note that f_R is supported in $\mathbb{D}$.

Lemma 4.7.5. *Let $Q : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ be a Fourier multiplier operator with a unimodular symbol such that $\widehat{Qf}(p) = m(p)\hat{f}(p)$. Then*

$$(Qf)(Rx) = (Q_R f_R)(x)$$

where Q_R is the Fourier multiplier with symbol $\tilde{m}(p) = m\left(\frac{p}{R}\right)$, which is still unimodular.

Proof. We can write the Fourier multiplier operator using the symbol representation.

$$(Qf)(Rx) = \int e^{ip\cdot Rx} m(p)\hat{f}(p)\ dp$$

Make the substitution $\eta = Rp$. Then

$$(Qf)(Rx) = \int e^{i\eta\cdot x} m\left(\frac{\eta}{R}\right)\hat{f}\left(\frac{\eta}{R}\right)\frac{1}{R^2}\ d\eta$$

Next we compute

$$\begin{aligned}\hat{f}_R(p) &= \int e^{-ip\cdot x} f(Rx)\ dx \\ &= \int e^{-ip\cdot\left(\frac{y}{R}\right)} f(y)\frac{1}{R^2}\ dy \\ &= \hat{f}\left(\frac{p}{R}\right)\frac{1}{R^2}\end{aligned}$$

which allows us to conclude

$$(Qf)(Rx) = \int e^{i\eta\cdot x}\tilde{m}(\eta)\hat{f}_R(\eta)\ d\eta = (Q_R f_R)(x)$$

□

With this scaling lemma, Lemma 4.7.4 can be used when the function μ_λ has support in $B(0,R)$. Suppose ν has support in $B(0,R)$, and $\|\nu\|_\infty \leq k < 1$. Let S_i be the Fourier multiplier operators with unimodular symbol from the lemma. For $z \in B(0,R)$, write $z = Rx$ for $x \in \mathbb{D}$. Let

$$\begin{aligned}f_n(z) &= (\nu S_n \nu S_{n-1}\nu\cdots\nu S_1\nu)(z) \qquad (4.96)\\ &= (\nu S_n \nu S_{n-1}\nu\cdots\nu S_1\nu)(Rx)\end{aligned}$$

(4.97)

Let $\varepsilon > 0$ be given. We want to find a number $\tilde{R}$ so that for $|\eta| \geq \tilde{R}$, $|\widehat{f}(\eta)| < \varepsilon$. Define

$$g_n(x) = \left(\nu_R S_{nR} \nu_R S_{(n-1)R} \nu_R \cdots \nu_R S_{1R} \nu_R\right)(x)$$

Then Lemma 4.7.5 implies (using the notation (4.95))

$$g_n(x) = (f_n)_R(x) \tag{4.98}$$

The function ν_R is supported in $\mathbb{D}$ and the S_{jR} are Fourier multiplier operators with unimodular symbol. Lemma 4.7.4 implies there exists $R_{n,\varepsilon}$ such that

$$|\widehat{g}_n(p)| < \frac{\varepsilon}{R^2} \quad \text{for } |p| > R_{n,\varepsilon} \tag{4.99}$$

The scaling of the Fourier transform implies

$$\widehat{f}_n(\eta) = R^2 \mathcal{F}((f_n)_R)(R\eta) = R^2 \widehat{g}_n(R\eta). \tag{4.100}$$

Suppose $|\eta| > \frac{R_{n,\varepsilon}}{R}$. Then

$$|\widehat{f}_n(\eta)| = R^2 |\widehat{g}_n(R\eta)| < R^2 \left(\frac{\varepsilon}{R^2}\right) = \varepsilon \tag{4.101}$$

The next step in our analysis is to show that $\overline{\partial}\psi \to 0$ weakly in $L^p(\mathbb{C})$ as $\xi \to \infty$. We will eventually write $\psi(z) = z + P(\overline{\partial}\psi)(z)$. The weak convergence is powered by the uniform estimates on $\mu \circ \psi$ (Lemma 4.6.6) and transforming the phase $e_\xi(\cdot)$ into translation in Fourier space.

Lemma 4.7.6. *Let $\psi(z, \xi)$ be the solution to (4.86), and let $2 < p < 1 + \frac{1}{k}$. Then $\overline{\partial}\psi(\cdot, \xi)$ converges to 0 weakly in $L^p(\mathbb{C})$ as $\xi \to \infty$.*

Proof. Let f be a smooth test function whose Fourier transform $\widehat{f}$ has compact support, and let $\varepsilon > 0$ be given. We want to control

$$|\langle f, \overline{\partial}\psi \rangle|$$

as $\xi \to \infty$. First we write $\overline{\partial}\psi$ as a Neumann series

$$\overline{\partial}\psi = \sum_{n=0}^{\infty} \left(\frac{\bar{\xi}}{\xi}(\mu \circ \psi) e_{-\xi} S\right)^n \left(\frac{\bar{\xi}}{\xi}(\mu \circ \psi) e_{-\xi}\right).$$

Since the convergence of the Neumann series is uniform in ξ (see equation (4.92) and Lemma 4.6.6), we can choose N independent of ξ so that

$$\left\| \sum_{n \geq N}^{\infty} \left(\frac{\bar{\xi}}{\xi} e_{-\xi}(\mu \circ \psi) S\right)^n \left(\frac{\bar{\xi}}{\xi} e_{-\xi}(\mu \circ \psi)\right) \right\|_p < \frac{\varepsilon}{\|f\|_{p'}}$$

Then we can write

$$|\langle f, \overline{\partial}\psi \rangle| \leq \left| \left\langle f, \sum_{n=0}^{N} \left(\frac{\bar{\xi}}{\xi} e_{-\xi}(\mu \circ \psi) S\right)^n \left(\frac{\bar{\xi}}{\xi} e_{-\xi}(\mu \circ \psi)\right) \right\rangle \right| + \varepsilon \tag{4.102}$$

Consider one of the terms of the finite sum. Note that we can write

$$S(e_{-\xi}\varphi) = e_{-\xi}S_\xi\varphi$$

where $\widehat{S_\xi\varphi}(\eta) = m(\eta - \xi)\widehat{\varphi}$ where $m(\eta) = \frac{\eta}{\bar{\eta}}$. In other words, commuting the phase past the Beurling transform shifts the Fourier symbol (which remains unimodular). Consequently,

$$\begin{aligned}((\mu \circ \psi)e_{-\xi}S)^n\,((\mu \circ \psi)e_{-\xi}) &= e_{-(n+1)\xi}(\mu \circ \psi)S_{n\xi}(\mu \circ \psi)S_{(n-1)\xi}\cdots(\mu \circ \psi)S_\xi(\mu \circ \psi)\\ &= e_{-(n+1)\xi}G_n\end{aligned}$$

We reassociate the pairing

$$|\langle f, e_{-(n+1)\xi}G_n\rangle| = |\langle fe_{-(n+1)\xi}, G_n\rangle| \tag{4.103}$$

and examine G_n. Note that G_n is a multilinear function of $(\mu \circ \psi)$. Let χ_R denote the characteristic function of the set $B(0, R) = \{z \in \mathbb{C} : |z| \le R\}$. Choose R such that $\|(1 - \chi_R)(\mu \circ \psi)\|_{L^2 \cap L^\infty} < \varepsilon$. Note that due to Lemma 4.6.6 and Lemma 4.7.3, the choice of R is independent of ξ. We write $h = (\mu \circ \psi)\chi_R$. Define

$$H_n = hS_{n\xi}hS_{(n-1)\xi}h\cdots hS_\xi h$$

We can estimate

$$\begin{aligned}\|G_n - H_n\|_2 &\le \sum_{j=1}^{n+1}\left\|hS_{n\xi}h\cdots S_{j\xi}\left((\mu \circ \psi) - h\right)S_{(j-1)\xi}(\mu \circ \psi)\cdots(\mu \circ \psi)S_\xi(\mu \circ \psi)\right\|_2\\ &= \sum_{j=1}^{n} k^{n-1}\|\mu \circ \psi\|_2\,\|(1 - \chi_R)(\mu \circ \psi)\|_\infty + k^n\,\|(1 - \chi_R)(\mu \circ \psi)\|_2\\ &\le \frac{1}{1-k}\|\mu \circ \psi\|_2\,\|(1 - \chi_R)(\mu \circ \psi)\|_{L^2 \cap L^\infty}\\ &< \frac{\varepsilon}{1-k}\|\mu \circ \psi\|_2\end{aligned}$$

Therefore we will conclude

$$|\langle fe_{-(n+1)\xi}, G_n\rangle| \le |\langle fe_{-(n+1)\xi}, H_n\rangle| + \|G_n - H_n\|_2\,\|f\|_2 \tag{4.104}$$

Next, we use Plancherel's theorem to estimate[3]

$$|\langle fe_{-(n+1)\xi}, G_n\rangle| \le |\langle \mathcal{F}(fe_{-(n+1)\xi}), \widehat{H}_n\rangle| + \varepsilon \tag{4.105}$$

Applying our modified Lemma 4.7.4 to H_n, we find that there is a number $\tilde{R}$ depending only on k, ε, and R so that

$$|\widehat{H}_n(\eta)| < \varepsilon \text{ for } |\eta| \ge \tilde{R}$$

[3] The symbol $\mathcal{F}(\cdot)$ denotes the Fourier transform.

Note that the Fourier transform of $fe_{-(n+1)\xi}$ is $\widehat{f}(\eta-(n+1)\xi)$. We can select ξ large enough so that

$$\operatorname{supp}\{\widehat{f}(\eta-(n+1)\xi)\}\cap B(0,\tilde{R})=\emptyset \tag{4.106}$$

Hence, for sufficiently large ξ,

$$\left|\langle\widehat{f}(\eta-(n+1)\xi),\widehat{H}_n\rangle\right|<\varepsilon \tag{4.107}$$

Note that the choices made in the reductions and estimates are made independent of ξ and are used to bound finitely many terms in the Neumann series. Thus we can conclude $\overline{\partial}\psi$ converges to 0 weakly in $L^p(\mathbb{C})$ as $\xi\to\infty$. □

Proof of Theorem 4.7.1

We are now ready to prove the subexponential growth of CGO solutions for our class of Beltrami coefficients.

Convergence for Fixed z.

We can now use the weak convergence to show that $\psi(z,\xi)-z\to 0$ as $\xi\to\infty$ for a fixed z. Write

$$\psi(z,\xi)-z=P(\overline{\partial}\psi)=P(\chi_R\overline{\partial}\psi)+P((1-\chi_R)\overline{\partial}\psi) \tag{4.108}$$

First, estimate the second right-hand term of (4.108) uniformly in ξ by writing

$$\begin{aligned}P((1-\chi_R)\overline{\partial}\psi)&=P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}\partial\psi\right)\\&=P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}\right)+P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}S(\overline{\partial}\psi)\right)\end{aligned} \tag{4.109}$$

Suppose $1+k<q<2<p<1+\frac{1}{k}$. The first term vanishes as $R\to\infty$ uniformly in ξ due to Lemma 4.7.3 and dominated convergence. In the second term, equation (2.15) of Lemma 2.5.2 allows us to get a pointwise estimate via

$$\left|P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}S(\overline{\partial}\psi)\right)\right|\lesssim\left\|(1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}S(\overline{\partial}\psi)\right\|_{L^p\cap L^q}.$$

We use the facts that $\|(1-\chi_R)(\mu\circ\psi)\|_\infty\to 0$ as $R\to\infty$ while $\left\|S(\overline{\partial}\psi)\right\|_{L^p\cap L^q}\lesssim\left\|\overline{\partial}\psi\right\|_{L^p\cap L^q}$ is bounded uniformly in $\xi\in\mathbb{C}$ from Remark 4.6.5.

Secondly, we regard the first right-hand term of (4.108),

$$P(\chi_R\overline{\partial}\psi)(z)=\frac{1}{\pi}\int\frac{1}{z-\zeta}\chi_R(\zeta)(\overline{\partial}\psi)(\zeta)\,d\zeta,$$

as the dual pairing of $\overline{\partial}\psi\in L^p(\mathbb{C})$ with $(z-\zeta)^{-1}\chi_R(\zeta)\in L^{p'}(\mathbb{C})$ for some $p>2$. We use the weak convergence of $\overline{\partial}\psi(z,\xi)$ to 0 as $\xi\to\infty$ from Lemma 4.7.6 to conclude that $P(\chi_R\overline{\partial}\psi)(z)\to 0$ as $\xi\to\infty$.

Convergence Uniform in z

We want the establish that $\psi(z,\xi) \to z$ as $\xi \to \infty$ *uniformly* in z. The first step is to show that $|\psi(z,\xi) - z|$ is small for large z uniformly in ξ. This allows us to focus on establishing a uniform estimate in a large ball.

Lemma 4.7.7. *For all $\varepsilon > 0$, there exists $R > 0$ such that*

$$\sup_{\substack{|z|>R \\ \xi\in\mathbb{C}}} |\psi(z,\xi) - z| < \varepsilon \tag{4.110}$$

Proof. Let $\varepsilon > 0$ be given, and let $1 + k < q < 2 < p < 1 + k^{-1}$. Suppose $|z| > R$, where R will be determined later. As in (4.108), we write

$$\psi(z,\xi) - z = P(\overline{\partial}\psi) = P(\chi_R\overline{\partial}\psi) + P((1-\chi_R)\overline{\partial}\psi) \tag{4.111}$$

We use the same split in (4.109) to write

$$\psi(z,\xi) - z = \underbrace{P(\chi_R\overline{\partial}\psi)}_{I} + \underbrace{P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}\right)}_{II} \tag{4.112}$$
$$+ \underbrace{P\left((1-\chi_R)(\mu\circ\psi)e_{-\xi}S\overline{\partial}\psi\right)}_{III}$$

Let $0 < \alpha < \frac{2}{1+k} - 1 < 1$. Term I can be estimated by

$$\left|\int \frac{\chi_R(w)}{z-w}\overline{\partial}\psi(w)\ dw\right| \le \frac{1}{R^\alpha}\int \frac{1}{|z-w|^{1-\alpha}}|\overline{\partial}\psi(w)|\ dw \tag{4.113}$$

Since $1 + k < \frac{2}{1+\alpha} < 2$. We can use the Hardy-Littlewood-Sobolev inequality (Theorem 2.5.1) to get

$$\left|\int \frac{\chi_R(w)}{z-w}\overline{\partial}\psi(w)\ dw\right| \le \frac{1}{R^\alpha}\left\|\overline{\partial}\psi\right\|_{\frac{2}{1+\alpha}}. \tag{4.114}$$

Remark 4.6.5 implies that the estimate (4.114) is uniform in ξ. Thus we can choose $R_1 > 0$ independent of ξ such that for $|z| > 2R_1$,

$$\left|\int \frac{\chi_{R_1}(w)}{z-w}\overline{\partial}\psi(w)\ dw\right| \le \frac{1}{R_1^\alpha}\left\|\overline{\partial}\psi\right\|_{\frac{2}{1+\alpha}} < \frac{\varepsilon}{3}. \tag{4.115}$$

To estimate the term II, we are going to use equation (2.15) of Lemma 2.5.2 to obtain

$$\left|P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}\right)(z)\right| \lesssim_{p,q} \|(1-\chi_R)(\mu\circ\psi)\|_{L^p\cap L^q}. \tag{4.116}$$

By the uniform decay of $(\mu\circ\psi)$ in Lemma 4.7.3 and dominated convergence, there exists an $R_2 > 0$, independent of ξ, such that

$$\left|P\left((1-\chi_{R_2})\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}\right)(z)\right| \lesssim_{p,q} \|(1-\chi_{R_2})(\mu\circ\psi)\|_{L^p\cap L^q} < \frac{\varepsilon}{3} \tag{4.117}$$

For term III, we again use equation (2.15) of Lemma 2.5.2 to estimate

$$\begin{aligned} \left|P\left((1-\chi_R)(\mu\circ\psi)e_{-\xi}S\overline{\partial}\psi\right)(z)\right| &\lesssim_{p,q} \left\|(1-\chi_R)(\mu\circ\psi)S\overline{\partial}\psi\right\|_{L^p\cap L^q} \\ &\lesssim_{p,q} \left\|(1-\chi_R)(\mu\circ\psi)\right\|_\infty \left\|S\right\|_{op} \left\|\overline{\partial}\psi\right\|_{L^p\cap L^q} \end{aligned} \tag{4.118}$$

where $\|\cdot\|_{op}$ denotes the operator norm in $\mathcal{L}(L^p\cap L^q, L^p\cap L^q)$. Remark 4.6.5 implies that $\sup_{\xi\in\mathbb{C}} \left\|\overline{\partial}\psi\right\|_{L^p\cap L^q} < \infty$. The decay of $(\mu\circ\psi)$ is uniform in ξ (see Lemma 4.7.3). Thus there exists an $R_3 > 0$ independent of ξ such that

$$\begin{aligned} \left|P\left((1-\chi_{R_3})(\mu\circ\psi)e_{-\xi}S\overline{\partial}\psi\right)(z)\right| &\lesssim_{p,q} \left\|(1-\chi_{R_3})(\mu\circ\psi)\right\|_\infty \left\|S\right\|_{op} \left\|\overline{\partial}\psi\right\|_{L^p\cap L^q} \\ &< \frac{\varepsilon}{3} \end{aligned} \tag{4.119}$$

Let $R = \max\{R_1, R_2, R_3\}$, and suppose $|z| > R$. Then we can revisit equation (4.112) using the estimates (4.115), (4.117), and (4.119) to obtain

$$\begin{aligned} |\psi(z,\xi)-z| &\le \left|P(\chi_R\overline{\partial}\psi)\right| + \left|P\left((1-\chi_R)\frac{\bar{\xi}}{\xi}(\mu\circ\psi)e_{-\xi}\right)\right| \\ &+ \left|P\left((1-\chi_R)(\mu\circ\psi)e_{-\xi}S\overline{\partial}\psi\right)\right| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon \end{aligned} \tag{4.120}$$

□

Now that Lemma 4.7.7 establishes the result on the exterior of a ball, we will show $|\psi(z,\xi)-z| \to 0$ as $\xi\to\infty$ uniformly for z is a ball. We claim that the family $\{\psi(\cdot,\xi)\}$ is equicontinuous in z parameterized by ξ. We use the standard estimate (see, for example, [4, Theorem 4.3.13])

$$|(Pf)(z) - (Pf)(z')| \lesssim_p \|f\|_p |z-z'|^{1-2/p} \tag{4.121}$$

where $p > 2$ and $f = \overline{\partial}\psi$. The uniformity in ξ follows from the estimates in Theorem 4.6.4.

Let $\varepsilon > 0$ be given, and let R be the radius from Lemma 4.7.7. The equicontinuity of $\{\psi(\cdot,\xi)\}$ implies there is a $\delta > 0$ such that for every $z \in \overline{B(0,R)}$, $|z-z'| < \delta$ implies $|\psi(z,\xi)-\psi(z',\xi)| < \varepsilon$. The collection $\{B_\delta(z)\}_{z\in B_R}$ forms an open cover of $\overline{B(0,R)}$. Thus there are $\{z_j\}_{j=1}^N$ such that for any $z \in B(0,R)$, there is some z_j such that

$$|\psi(z,\xi)-z| \le |\psi(z_j,\xi)-z_j| + |\psi(z,\xi)-\psi(z_j,\xi)| + |z_j - z| \tag{4.122}$$

Therefore, we only need $|\psi(z_j,\xi)-z_j| \to 0$ as $\xi\to\infty$ for each z_j, $j = 1,\ldots,N$. This establishes the uniformity in z.

4.8 Summary

In this chapter, we began with the truncated scattering transform of a Beltrami coefficient $\mu \in L^\infty(\mathbb{D})$. In Section 1, we showed that in analogy with the Fourier transform, the truncated scattering transform corresponds to a Beltrami coefficient that is no longer compactly supported. Rather, it exhibits decay on the order of $\mathcal{O}\left(|z|^{-2}\right)$ as $z \to \infty$. The goal for the rest of the chapter was to recast the machinery of Astala-Päivärinta for this larger class of Beltrami coefficients. Along the way, we showed that the notion of principal solutions to Beltrami equations has an analogue in this new setting. The complex geometric optics (CGO) solutions to the Beltrami equation exist and can exhibit subexponential growth as the parameter $\xi \to \infty$.

Notation Index

The following is a list of symbols used in this dissertation.

$\mathbb{R}^n$	n-dimensional real Euclidean space
$\mathbb{C}$	field of complex numbers; we take $x \in \mathbb{R}^2$ as an element of $\mathbb{C}$
$\widehat{\mathbb{C}}$	the Riemann sphere (extended complex plane)
Ω	a bounded domain in $\mathbb{R}^n$
∂X	the boundary of a set X
$\mathbb{D}$	the unit disc in $\mathbb{C}$
dx	n-dimensional Lebesgue measure
dz	2-dimensional Lebesgue measure
ds	1-dimensional surface measure
$f\|_{\partial X}$	the restriction of a function $f : X \to \mathbb{C}$ to the domain ∂X.
$\nu(x)$	the outward unit normal at a point $x \in \partial X$
$L^r(\mathbb{R}^n)$	for $r \in (1, \infty)$, the Banach space of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ with $\\|f\\|_r := \left(\int \|f(x)\|^r \, dx\right)^{1/r} < \infty$
$L^\infty(\mathbb{R}^n)$	the Banach space of essentially bounded functions
$L^p \cap L^q$	the intersection of $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$
$W^{k,p}(\mathbb{C})$	the Sobolev space of distributions whose derivatives up to kth order are L^p-integrable.
$H^s(\partial\mathbb{D})$	the Sobolev space of functions on $\partial\mathbb{D}$ defined for non-integer values of s in Section 2.1
$C^{k,\alpha}(\mathbb{C})$	the space of k-times differentiable Hölder continuous functions with exponent α
$\mathscr{S}(\mathbb{R}^2)$	the Schwartz class of functions of rapid decrease
$C_0^\infty(\mathbb{C})$	the set of smooth compactly supported functions
$\mathcal{L}(X, Y)$	the Banach space of bounded operators from Banach space X to Banach space Y
∂f	the derivative of f with respect to the complex number z; $\partial f = \frac{1}{2}(\partial_x - i\partial_y) f$

$\overline{\partial} f$ the derivative of f with respect to the complex number $\overline{z}$; $\overline{\partial} f = \frac{1}{2}(\partial_x + i\partial_y) f$

$J(z, f)$ Jacobian determinant of $f : \mathbb{C} \to \mathbb{C}$ at z

$\mathrm{Jac}(f)(z)$ Jacobian matrix of $f : \mathbb{C} \to \mathbb{C}$ at z

σ conductivity function

Λ_σ Dirichlet-to-Neumann operator for σ, see (1.6)

$m(z, \xi)$ normalized CGO solution for the Schrödinger problem, see (1.9)

$\mathbf{t}(\xi)$ Nachman's scattering transform. See (1.10), (1.17)

$e_\xi(z)$ phase function, see (1.11)

$\mu(z)$ Beltrami coefficient corresponding to σ, see (1.24)

$f_\mu(z, \xi)$ CGO solution to the Beltrami equation, see Theorem 1.4.2

$M_\mu(z, \xi)$ normalized CGO solution to the Beltrami equation, see (1.25)

$\varphi(z, \xi)$ solution to the nonlinear beltrami equation (1.35) related to CGO solutions by (1.29)

$\tau(\xi)$ Astala-Päivärinta scattering transform, see (3.15)

P solid Cauchy transform, see (2.13)

S Beurling transform, see (2.17)

Bibliography

[1] Robert A. Adams. *Sobolev Spaces.* Academic Press, 1975.

[2] Giovanni Alessandrini. Stable determination of conductivity by boundary measurements. *Appl. Anal.*, 27(1-3):153–172, 1988.

[3] Kari Astala. Private communication.

[4] Kari Astala, Tadeusz Iwaniec, and Gaven Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane.* Princeton University Press, 2009.

[5] Kari Astala, Jennifer L. Mueller, Lassi Päivärinta, Allan Perämäki, and Samuli Siltanen. Direct electrical impedance tomography for nonsmooth conductivities. *Inverse Probl. Imaging*, 5(3):531–549, 2011.

[6] Kari Astala, Jennifer L. Mueller, Lassi Päivärinta, and Samuli Siltanen. Numerical computation of complex geometrical optics solutions to the conductivity equation. *Appl. Comput. Harmon. Anal.*, 29(1):2–17, 2010.

[7] Kari Astala and Lassi Päivärinta. A boundary integral equation for Calderón's inverse conductivity problem. *Collect. Math.*, (Vol. Extra):127–139, 2006.

[8] Kari Astala and Lassi Päivärinta. Calderón's inverse conductivity problem in the plane. *Ann. of Math. (2)*, 163(1):265–299, 2006.

[9] Kari Astala, Lassi Päivärinta, Juan Manuel Reyes, and Samuli Siltanen. Nonlinear Fourier analysis for discontinuous conductivities: computational results. *J. Comput. Phys.*, 276:74–91, 2014.

[10] Juan Antonio Barceló, Tomeu Barceló, and Alberto Ruiz. Stability of the inverse conductivity problem in the plane for less regular conductivities. *J. Differential Equations*, 173(2):231–270, 2001.

[11] Tomeu Barceló, Daniel Faraco, and Alberto Ruiz. Stability of Calderón inverse conductivity problem in the plane. *J. Math. Pures Appl. (9)*, 88(6):522–556, 2007.

[12] R. Beals and R. R. Coifman. Multidimensional inverse scatterings and nonlinear partial differential equations. In *Pseudodifferential operators and applications (Notre Dame, Ind., 1984)*, volume 43 of *Proc. Sympos. Pure Math.*, pages 45–70. Amer. Math. Soc., Providence, RI, 1985.

[13] Richard Beals and R. Coifman. The spectral problem for the Davey-Stewartson and Ishimori hierarchies. *Nonlinear evolution equations: Integrability and spectral methods*, pages 15–23, 1988.

[14] Marco Boiti, Jérôme Jp. Leon, Miguel Manna, and Flora Pempinelli. On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane. *Inverse Problems*, 3(1):25–36, 1987.

[15] Russell M. Brown and Gunther A. Uhlmann. Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Comm. Partial Differential Equations*, 22(5-6):1009–1027, 1997.

[16] Alberto-P. Calderón. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[17] Albert Clop, Daniel Faraco, and Alberto Ruiz. Stability of Calderón's inverse conductivity problem in the plane for discontinuous conductivities. *Inverse Probl. Imaging*, 4(1):49–91, 2010.

[18] Jeffery C. DiFranco and Kenneth T.-R. McLaughlin. A nonlinear Gibbs-type phenomenon for the defocusing nonlinear Schrödinger equation. *IMRP Int. Math. Res. Pap.*, (8):403–459, 2005.

[19] Toufic El Arwadi. Error estimates for reconstructed conductivities via the D-bar method. *Numer. Funct. Anal. Optim.*, 33(1):21–38, 2012.

[20] L. D. Faddeev. Growing solutions to Schrödinger's equation. *Doklady Akademii nauk SSSR*, 165(3):514, 1965.

[21] Daniel Faraco and Martí Prats. Characterization for stability in planar conductivities. *J. Differential Equations*, 264(9):5659–5712, 2018.

[22] Rupert L. Frank and Elliott H. Lieb. A new, rearrangement-free proof of the sharp Hardy-Littlewood-Sobolev inequality. In *Spectral theory, function spaces and inequalities*, volume 219 of *Oper. Theory Adv. Appl.*, pages 55–67. Birkhäuser/Springer Basel AG, Basel, 2012.

[23] Harald Hanche-Olsen and Helge Holden. The Kolmogorov-Riesz compactness theorem. *Expo. Math.*, 28(4):385–394, 2010.

[24] Harald Hanche-Olsen and Helge Holden. Addendum to "The Kolmogorov-Riesz compactness theorem" [Expo. Math. 28 (2010) 385–394] [MR2734454]. *Expo. Math.*, 34(2):243–245, 2016.

[25] Claudia N. L. Herrera, Miguel F. M. Vallejo, Jennifer L Mueller, and Raul G. Lima. Direct 2-d reconstructions of conductivity and permittivity from EIT data on a human chest. *IEEE transactions on medical imaging*, 34(1):267–274, 2015.

[26] Carlos Kenig and Mikko Salo. The Calderón problem with partial data on manifolds and applications. *Anal. PDE*, 6(8):2003–2048, 2013.

[27] Carlos Kenig and Mikko Salo. Recent progress in the Calderón problem with partial data. In *Inverse problems and applications*, volume 615 of *Contemp. Math.*, pages 193–222. Amer. Math. Soc., Providence, RI, 2014.

[28] Carlos E. Kenig, Johannes Sjöstrand, and Gunther Uhlmann. The Calderón problem with partial data. *Ann. of Math. (2)*, 165(2):567–591, 2007.

[29] Kim Knudsen, Matti Lassas, Jennifer L. Mueller, and Samuli Siltanen. D-bar method for electrical impedance tomography with discontinuous conductivities. *SIAM J. Appl. Math.*, 67(3):893–913, 2007.

[30] Kim Knudsen, Matti Lassas, Jennifer L Mueller, and Samuli Siltanen. Regularized d-bar method for the inverse conductivity problem. *Inverse Problems and Imaging*, 35(4):599, 2009.

[31] R. V. Kohn and M. Vogelius. Determining conductivity by boundary measurements. II. Interior results. *Comm. Pure Appl. Math.*, 38(5):643–667, 1985.

[32] Robert Kohn and Michael Vogelius. Determining conductivity by boundary measurements. *Comm. Pure Appl. Math.*, 37(3):289–298, 1984.

[33] Elliott H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)*, 118(2):349–374, 1983.

[34] Felipe Linares and Gustavo Ponce. *Introduction to nonlinear dispersive equations*. Universitext. Springer, New York, second edition, 2015.

[35] George Lytle, Peter Perry, and Samuli Siltanen. Nachman's reconstruction for the Calderon problem with discontinuous conductivities. *arXiv e-prints*, page arXiv:1809.09272, Sep 2018.

[36] Michael Music and Peter Perry. Global solutions for the zero-energy Novikov-Veselov equation by inverse scattering. *Nonlinearity*, 31(7):3413–3440, 2018.

[37] Adrian I. Nachman. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)*, 143(1):71–96, 1996.

[38] Adrian I. Nachman, Idan Regev, and Daniel I. Tataru. A Nonlinear Plancherel Theorem with Applications to Global Well-Posedness for the Defocusing Davey-Stewartson Equation and to the Inverse Boundary Value Problem of Calderon. *arXiv e-prints*, page arXiv:1708.04759, Aug 2017.

[39] R. G. Novikov. A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. *Funktsional. Anal. i Prilozhen.*, 22(4):11–22, 96, 1988.

[40] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[41] Raymond A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[42] Juliusz Schauder. Der fixpunktsatz in funktionalramen. *Studia Mathematica*, 2(1):171–180, 1930.

[43] Samuli Siltanen. *Electrical impedance tomography and Faddeev Green's functions*. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–Teknillinen Korkeakoulu (Helsinki) (Finland).

[44] Samuli Siltanen, Jennifer Mueller, and David Isaacson. An implementation of the reconstruction algorithm of A. Nachman for the 2D inverse conductivity problem. *Inverse Problems*, 16(3):681–699, 2000.

[45] Li-Yeng Sung. An inverse scattering transform for the Davey-Stewartson II equations. I. *J. Math. Anal. Appl.*, 183(1):121–154, 1994.

[46] Li-Yeng Sung. An inverse scattering transform for the Davey-Stewartson II equations. II. *J. Math. Anal. Appl.*, 183(2):289–325, 1994.

[47] Li-Yeng Sung. An inverse scattering transform for the Davey-Stewartson II equations. III. *J. Math. Anal. Appl.*, 183(3):477–494, 1994.

[48] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[49] Leo Tzou. The reflection principle and Calderón problems with partial data. *Math. Ann.*, 369(1-2):913–956, 2017.

[50] Henry Wilbraham. On a certain periodic function. *The Cambridge and Dublin Mathematical Journal*, 3:198–201, 1848.

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