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A REPRESENTATION THEOREM FOR MATERIAL TENSORS OF TEXTURED THIN SHEETS WITH WEAK PLANAR ANISOTROPY

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A REPRESENTATION THEOREM FOR MATERIAL TENSORS OF TEXTURED THIN SHEETS WITH WEAK PLANAR ANISOTROPY

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy in the
College of Arts and Sciences at the
University of Kentucky

By
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Lexington, Kentucky

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ABSTRACT OF DISSERTATION

A REPRESENTATION THEOREM FOR MATERIAL TENSORS OF TEXTURED THIN SHEETS WITH WEAK PLANAR ANISOTROPY

Herein we consider material tensors that pertain to thin sheets or thin films, which we model as two-dimensional objects. We assume that the thin sheet in question carries a crystallographic texture characterized by an orientation distribution function defined on the rotation group SO(3), which is almost transversely-isotropic about the sheet normal so that mechanical and physical properties of the thin sheet have weak planar-anisotropy. We present a procedure by which a special orthonormal basis can be determined in each tensor subspace invariant under the action of the orthogonal group O(2). We call members of such special bases irreducible basis tensors under O(2). For the class of thin sheets in question, we derive a representation formula in which each tensor in any given tensor subspace Z is written as the sum of a transversely-isotropic term and a linear combination of orthonormal irreducible basis tensors in Z, where the coefficients are given explicitly in terms of texture coefficients and undetermined material parameters. In addition to the general representation formula, we present also the specialized form for subspaces of tensor products of second-order symmetric tensors, a type commonly found in mechanics of materials.

KEYWORDS: Polycrystals, Texture, Almost transversely-isotropic sheets, Material tensors, Irreducible tensor basis, Representation formula

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Chapter 1 Introduction

Many materials are polycrystalline aggregates of tiny crystallites or grains of various sizes and shapes separated by grain boundaries. In theories on physical properties of polycrystals, as a first approximation, all effects of grain boundaries are ignored and the constituent crystallites of the polycrystal are taken as parts of perfect single crystals, the crystal lattices of which have different orientations in space. As each crystallite is anisotropic, the macroscopic physical properties of the polycrystal will likewise be anisotropic unless the orientations of its constituent crystallites are completely random. Manufacturing processes (e.g., annealing and hot/cold rolling in the case of sheet metals), however, usually impart the constituent crystallites of polycrystalline products with preferred orientations, which are called crystallographic texture in materials science and are quantitatively characterized by orientation distribution functions (ODFs) defined on the rotation group SO(3) [1, 3, 4, 5, 9, 11, 20, 21].

Material properties are often described by tensors of various types. Material tensors pertaining to polycrystalline aggregates should manifest the influence of crystallographic texture on material properties. Many papers which study the effects of texture on various material properties have been published. But, until the work of Man and Huang [14], all these papers were restricted to some specific classes of tensors (e.g. second-order tensors that describe thermal conductivity, optical refractive index and electrical conductivity, the fourth-order elasticity tensor, the sixth-order acoustoelastic tensor, etc.) and were restricted to some specific texture and crystal symmetries (e.g., orthorhombic aggregates of cubic or hexagonal crystallites were mostly studied). Man and Huang [14] derived a representation theorem by which any material tensor of a weakly-textured polycrystal can be expressed as a linear combination of an orthonormal set of irreducible basis tensors, with the components given explicitly in terms of texture coefficients and a set of undetermined material parameters. In their paper they provide a procedure by which the irreducible basis tensors can be constructed explicitly. The representation theorem of Man and Huang, however, is valid only for "weakly-textured polycrystals", which means that they are almost isotropic in the following sense: All the texture coefficients of the polycrystalline aggregate c^l_{mn} with $l \geq 1$ (i.e., except c^0_{00}, which pertains to the isotropic term; see (4.1)) are sufficiently small that material tensors which characterize its physical properties can be taken as linear in texture coefficients with $l \geq 1$.

Manufacturing processes such as hot/cold rolling and annealing, however, often produce sheet metals which show high normal plastic anisotropy and low planar plastic anisotropy, as reflected in high average r-value and low Δr-value, respectively (see, e.g., [6, 7]). In other words, the material is almost transversely-isotropic about the sheet normal but cannot be taken as almost isotropic. In terms of texture coefficients with $l \geq 1$, all c^l_{mn} with $m \neq 0$ are small, while $c^l_{0n} = 0$ for odd l and there is no restriction on c^l_{0n} for even l; see the discussion in Chapter 4 until (4.15). The objective of the present research is to extend the representation theorem of Man and Huang to cover this rather common situation for thin sheets and thin films.

In this dissertation we model the homogeneous thin sheet or film in question as a two-dimensional object that lies in a Euclidean plane E^2 in the three-dimensional physical space, and we consider material tensors that pertain to its in-plane properties. On the other hand, we allow flipping the sheet over to exchange its top and bottom faces as a legitimate symmetry operation. The material tensors are based on V, the translation space of E^2. The group of symmetry operations for V is the orthogonal group O(2). As the thin sheet consists of crystallites with orientations in three-dimensional space, we will keep using the ODF defined on the rotation group SO(3) to define its texture.

The plan of this dissertation is as follows. After presenting some preliminaries in Chapter 2, we discuss decomposition of a tensor space into its irreducible parts in Chapter 3. In particular we present a method to determine a special orthonormal basis in any invariant subspace of tensors which will be instrumental in our proof of the representation theorem in Chapter 4. For reasons to be given in Chapter 3, members of such special bases will be called irreducible tensor basis under the group O(2). To illustrate the procedure to generate the irreducible basis tensors in a given tensor space, we provide examples where the tensor space in question is a subspace of tensor products of second-order symmetric tensors, partly because this type of tensors is commonly found in mechanics of materials, and partly because we can use the Kelvin notation to simplify the presentation of the resulting irreducible basis tensors.

Chapter 4 is devoted to a derivation of the representation theorem we want to get. There we adopt the same physical assumption (4.17) as that of Man and Huang [14, equation (41)], which is suggested by the Principle of Material Frame-Indifference. We consider thin sheets (or films) that are almost transversely-isotropic about the sheet normal. For the class of thin sheets in question, we derive a representation formula in which each tensor in any given tensor subspace Z is written as the sum of a transversely-isotropic term and a linear combination of orthonormal irreducible basis tensors in Z, where the coefficients are given explicitly in terms of texture coefficients and undetermined material parameters. In addition to the general representation formula, we present also the specialized form for subspaces of tensor products of second-order symmetric tensors.

We end the dissertation with some concluding remarks in Chapter 5.

Chapter 2 Preliminaries

2.1 The groups SO(2) and O(2)

Let V be the translation space of the two-dimensional Euclidean space E^2. The vector space V is endowed with an inner product, which is usually called the dot product in physics. In what follows we denote the inner product of two vectors $\mathbf{a}, \mathbf{b} \in V$ by $\mathbf{a} \cdot \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b} \rangle$. Let Lin be the space of linear transformations on V. We choose an orthonormal basis in V by arbitrarily selecting two orthonormal vectors and calling them $\mathbf{e}_1, \mathbf{e}_2$. Henceforth unless stated otherwise we will always use $\{\mathbf{e}_i : i = 1, 2\}$ as the chosen basis in V. Under the chosen orthonormal basis each linear transformation on V can be represented by a 2×2 real matrix. Let the space of 2×2 real matrices be denoted by $M(2)$. We identify each linear transformation $\mathbf{A}$ on V by its representative matrix, which we denote by the same symbol $\mathbf{A}$.

A linear transformation $\mathbf{Q}$ is orthogonal if it preserves the inner product on V, i.e., $\langle \mathbf{Qa}, \mathbf{Qb} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$ for any two vectors $\mathbf{a}, \mathbf{b} \in V$. Let $\mathbf{I}$ be the identity matrix in $M(2)$ and $\mathbf{Q}^T$ denote the transpose of $\mathbf{Q}$. It follows immediately from the definition that an orthogonal transformation $\mathbf{Q}$ satisfies the condition $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, which implies $\det(\mathbf{Q})^2 = 1$ or $\det \mathbf{Q} = \pm 1$. It is easy to verify that the orthogonal transformations on V constitute a group, which we call O(2), and that the orthogonal transformations $\mathbf{R}$ which satisfy $\det \mathbf{R} = 1$ constitute a subgroup of O(2), which we denote by SO(2). To determine the general form of the matrices in SO(2) and O(2) under the basis $\{\mathbf{e}_i : i = 1, 2\}$, we first find the general form of the matrices in SO(2). Let a, b, c and d be the four real entries of a 2×2 matrix $\mathbf{A}$ in SO(2) as follows:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \det(\mathbf{A}) = ad - cb = 1 \tag{2.1}$$

Since $\mathbf{A}$ is orthogonal, we have $\mathbf{A}^{-1} = \mathbf{A}^T$, which implies

$$d = a \quad \text{and} \quad c = -b. \tag{2.2}$$

From the condition $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, we see that

$$a^2 + b^2 = 1. \tag{2.3}$$

Thus the real numbers a and b should satisfy the condition that

$$-1 \leq a, b \leq 1. \tag{2.4}$$

Without loss of generality, we introduce a real parameter φ such that

$$a = \cos\varphi, \quad b = -\sin\varphi, \tag{2.5}$$

in which case a and b both satisfy conditions (2.3) and (2.4). Hence the general form for a real and orthogonal 2×2 matrix $\mathbf{R}(\varphi)$ with determinant 1 can be written as

$$\mathbf{R}(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad \text{for } \varphi \in (-\pi, \pi]. \tag{2.6}$$

The orthogonal group in two dimensions O(2) is defined by the set of real and orthogonal 2×2 matrices with determinant ± 1. Following similar reasoning, we find that we may express the general form of a real and orthogonal 2×2 matrix $\widetilde{\mathbf{R}}(\varphi)$ with determinant -1 as

$$\widetilde{\mathbf{R}}(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix}, \quad \text{for } \varphi \in (-\pi, \pi], \tag{2.7}$$

which can also be written as

$$\widetilde{\mathbf{R}}(\varphi) = \mathbf{M}_x\mathbf{R}(\varphi), \quad \mathbf{M}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

Thus the matrix group O(2) can be generated by

$$\mathbf{R}(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad \mathbf{M}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.9}$$

where $\varphi \in (-\pi, \pi]$. Note that each $\mathbf{R}(\varphi)$ is a rotation and $\mathbf{M}_x$ is the reflection across the x-axis (i.e., the axis defined by $\mathbf{e}_1$). By direct computations it is easy to show that

$$\mathbf{M}_x\mathbf{M}_x = \mathbf{I}, \quad \mathbf{M}_x\mathbf{R}(\varphi) = \mathbf{R}(-\varphi)\mathbf{M}_x, \quad \text{and} \quad \mathbf{R}(\alpha)\mathbf{R}(\beta) = \mathbf{R}(\alpha+\beta), \tag{2.10}$$

which imply

$$\mathbf{M}_x\mathbf{R}(\varphi)\mathbf{M}_x^{-1} = \mathbf{R}(-\varphi), \tag{2.11}$$

$$\begin{aligned} \mathbf{R}(\beta)(\mathbf{M}_x\mathbf{R}(\varphi))\mathbf{R}(-\beta) &= \mathbf{M}_x\mathbf{R}(-\beta)\mathbf{R}(\varphi-\beta) \\ &= \mathbf{M}_x\mathbf{R}(\varphi - 2\beta) \quad \text{for all angles } \varphi \text{ and } \beta. \end{aligned} \tag{2.12}$$

Equation (2.12) can be written as

$$\mathbf{R}(\beta)\widetilde{\mathbf{R}}(\varphi)\mathbf{R}(-\beta) = \widetilde{\mathbf{R}}(\varphi - 2\beta) \quad \text{for all angles } \varphi \text{ and } \beta. \tag{2.13}$$

2.2 The Haar Integral on O(2)

Our discussions in this section is based on the following theorem.

Theorem 2.1. [16, 23] *Let G be a compact topological group, and let $C(G)$ be the set of real-valued continuous functions on G. Then there exists a unique mapping $\mathcal{I} : G \to \mathbb{R}$, $f \mapsto \mathcal{I}(f) := \int_G f(g)\, dg$, called the Haar integral, which enjoys the following properties:*

(a) $\int_G (c_1 f_1 + c_2 f_2)\, dg = c_1 \int_G f_1\, dg + c_2 \int_G f_2\, dg$ *for each* $f_1, f_2 \in C(g)$ *and* $c_1, c_2 \in \mathbb{R}$*;*

(b) if f *is non-negative and not identically zero, then* $\int_G f\, dg > 0$*;*

(c) $\int_G f(hg)\, dg = \int_G f(gh)\, dg = \int_G f(g)\, dg$ *for each* $h \in G$*;*

(d) $\int_G 1\, dg = 1$.

The orthogonal group O(2) is a compact Lie group with two connected components (see, e.g., [10], pp. 195, 235):

$$\mathrm{O}^+(2) = \{\mathbf{A} \in \mathrm{O}(2) : \det \mathbf{A} = 1\}, \qquad \mathrm{O}^-(2) = \{\mathbf{A} \in \mathrm{O}(2) : \det \mathbf{A} = -1\}. \tag{2.14}$$

Topologically each connected component of the group O(2) can be identified with a unit circle $\{e^{i\varphi} : \varphi \in \mathbb{R}\}$ so that O(2) is the disjoint union of two unit circles. The Haar integral on O(2) is then given [25, p. 376] by

$$\begin{aligned}\int_{\mathrm{O}(2)} f(g)\, dg &= \int_{\mathrm{O}^+(2)} f(g)\, dg + \int_{\mathrm{O}^-(2)} f(g)\, dg \\ &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(\mathbf{R}(\varphi))\, d\varphi + \int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{R}(\varphi))\, d\varphi\right).\end{aligned} \tag{2.15}$$

The right-hand side of (2.15) obviously satisfies defining properties (a), (b), and (d) of the Haar integral. Let us check property (c). It suffices to restrict attention to the cases $h = \mathbf{R}(\beta)$ and $h = \mathbf{M}_x\mathbf{R}(\beta)$ for an arbitrarily given angle β. For $h = \mathbf{R}(\beta)$, we have

$$\begin{aligned}\int_{\mathrm{O}(2)} f(hg)\, dg &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(\mathbf{R}(\beta)\mathbf{R}(\varphi))\, d\varphi + \int_{-\pi}^{\pi} f(\mathbf{R}(\beta)\mathbf{M}_x\mathbf{R}(\varphi))\, d\varphi\right) \\ &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(\mathbf{R}(\beta + \varphi))\, d\varphi + \int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{R}(-\beta + \varphi))\, d\varphi\right) \\ &= \int_{\mathrm{O}(2)} f(g)\, dg,\end{aligned} \tag{2.16}$$

where we have appealed to (2.10). For $h = \mathbf{M}_x\mathbf{R}(\beta)$, there holds

$$\begin{aligned}\int_{\mathrm{O}(2)} f(hg)\,dg &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{R}(\beta)\mathbf{R}(\varphi))\,d\varphi + \int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{R}(\beta)\mathbf{M}_x\mathbf{R}(\varphi))\,d\varphi\right)\\ &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{R}(\beta+\varphi))\,d\varphi + \int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{M}_x\mathbf{R}(-\beta+\varphi))\,d\varphi\right)\\ &= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi} f(\mathbf{M}_x\mathbf{R}(\varphi))\,d\varphi + \int_{-\pi}^{\pi} f(\mathbf{R}(\varphi))\,d\varphi\right)\\ &= \int_{\mathrm{O}(2)} f(g)\,dg. \end{aligned} \tag{2.17}$$

The proof for $\int_{\mathrm{O}(2)} f(gh)\,dg = \int_{\mathrm{O}(2)} f(g)\,dg$ for each $h \in \mathrm{O}(2)$ is similar.

For complex-valued continuous functions $f(g) = u(g) + iv(g)$ on a compact group G, where $u, v \in C(G)$, we define

$$\int_G f(g)\,dg = \int_G u(g)\,dg + i\int_G v(g)\,dg. \tag{2.18}$$

2.3 Group Representations

2.3.1 Basics of Group Representations

Definition 2.3.1. *Let G be a group and let X be a complex linear space $\neq 0$. Consider a mapping T of G into the set of all linear operators carrying X into itself, written $g \to T(g)$, with the following properties:*

(1) $T(e) = 1$, where 1 is the identity operators in X;

(2) $T(g_1g_2) = T(g_1)T(g_2)$ for all $g_1,\ g_2 \in G$.

Then T is called a representation of G in the space X. The space X is called the representation space and the operators $T(g)$ representation operators.

If X is finite-dimensional and let $GL(X)$ be the space of non-singular linear transformations on X. Then a homomorphism

$$T : G \to GL(X)$$

is a representation of G on X.

A representation in a space X is called *irreducible* if X admits no subspace except for 0 and X itself that is invariant under all operators of the representation. A subspace $M \subseteq X$ is said to be invariant under a representation T if it is invariant under all operators $T(g)$ of this representation.

Two representations T, S of a group G in spaces X and Y are called *equivalent* (written $T \sim S$) if there is a one-to-one linear operator A carrying X onto Y and satisfying the condition

$$AT(g) = S(g)A, \quad \text{for all } g \in G \tag{2.19}$$

It is possible that $Y = X$, and in this case we speak if the equivalence of representations in the same space. Condition (2.19) shows that $AT(g)x = S(g)Ax$ for all $x \in X$ and $g \in G$. That is , if A maps x into y (i.e., $Ax = y$), then A also maps $T(g)x$ into $S(g)y$ (i.e., $AT(g)x = S(g)y$). Condition (2.19) can also be written as follows:

$$T(g) = A^{-1}S(g)A, \quad \text{for all } g \in G \tag{2.20}$$

Theorem 2.2. *Representations S and T on X and Y are equivalent representations if and only if $n_S = n_T$ and under a proper choice of bases in X and Y, the matrix elements of the representatons S and T coincide.*

Definition 2.3.1. *A Hermitian inner product on a complex vector space V is a complex-valued bilinear form on V which is antilinear in the second slot, and is positive definite. That is, it satisfies the following properties, where $\overline{z}$ denotes the complex conjugate of z.*

1. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *and* $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$

2. $\langle ax, y\rangle = a\langle x, y\rangle$ *and* $\langle x, ay\rangle = \bar{a}\langle x, y\rangle$

3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$

4. $\langle x, x\rangle \geq 0$*, with equality only if* $x = 0$

A basic example in $\mathbb{C}^n$ is of the form $\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y_i}$

A linear space X equipped with a Hermitian inner product is called a pre-Hilbert space. A linear operator A on a pre-Hilbert space X is called unitary if A is a one-to-one mapping of X onto X and

$$\langle Ax, Ay\rangle = \langle x, y\rangle \quad \text{for all } x, y \in X. \tag{2.21}$$

A representation T of G in a pre-Hilbert space X is called unitary if all operators of the representation are unitary. Since operators of a representation in X are by definition one-to-one mappings X onto X, a represntation T is unitary if and only if

$$\langle T(g)x, T(g)y\rangle = \langle x, y\rangle, \quad \text{for all } g \in G,\ x,\ y \in X \tag{2.22}$$

Theorem 2.3. *Two unitary representations T and S of G in Euclidean spaces X and Y are equivalent if and only if there are orthonormal bases in X and Y with respect to which the matric of the representations T and S coincide.*

A representation T of group G in a space X is called *completely reducible* if T is the direct sum of irreducible representations of G, i.e.,

$$X = X_1 + \cdots + X_m$$

is a direct sum of subspaces X_k ($k = 1, \cdots, m$), each of which is invariant under G, and each restriction T_k of T on X_k ($k = 1, \cdots, m$) is an irreducible representation of G.

Theorem 2.4. *Every finite-dimensional unitary representation of a group G is completely reducible.*

A representation T of G in X is said to be (strongly) continuous if $g \to T(g)x$ is continuous on G for every $x \in X$. A set $\{T_i\}$ of representations of the group G is called a complete set of irreducible representations of G if the representations T_i are irreducible and are pairwise inequivalent, and every irreducible representation of G is equivalent to one of the representations T_i

Finally, we record a simple theorem that we shall use in the next subsection.

Theorem 2.5. *Every irreducible finite-dimensional representation of an abelian group is one-dimensional.*

2.3.2 Irreducible Representations of SO(2)

Consider the representation of SO(2) given by (2.6). This representation must be reducible because SO(2) is an abelian group. In order to determine a complete set of irreducible representations of SO(2), we begin by evaluating the eigenvalues of matrix (2.6) as follows:

$$\begin{aligned}
\det(\mathbf{R}(\varphi) - \lambda \mathbf{I}) &= \begin{pmatrix} \cos\varphi - \lambda & -\sin\varphi \\ \sin\varphi & \cos\varphi - \lambda \end{pmatrix} \\
&= (\cos\varphi - \lambda)^2 + \sin^2\varphi \\
&= \lambda^2 - 2\cos\varphi\lambda + 1 = 0.
\end{aligned} \tag{2.23}$$

By solving for λ, we have

$$\lambda_{-1} = \cos\varphi - i\sin\varphi = e^{-i\varphi}, \quad \lambda_1 = \cos\varphi + i\sin\varphi = e^{i\varphi} \tag{2.24}$$

as the two eigenvalues of matrix (2.6). And we can solve for the eigenvectors corresponding to the two eigenvalues of (2.6), which are

$$\mathbf{u}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{2.25}$$

Thus the two resulting non-equivalent one-dimensional irreducible representations are given by

$$\mathbf{D}^1(\varphi) = e^{i\varphi}, \quad \text{and} \quad \mathbf{D}^{-1}(\varphi) = e^{-i\varphi} \tag{2.26}$$

In what follows, for any rotation $\mathbf{R}(\varphi)$ in SO(2), we denote the representation $\mathbf{D}(\mathbf{R}(\varphi))$ by $\mathbf{D}(\varphi)$. An one-dimensional irreducible representation is a set of linear transformations $\{\mathbf{D}(\varphi) : \varphi \in (-\pi, \pi]\}$ of one-dimensional complex vector space X onto itself, *i,e.*: $\mathbf{D}(\varphi) : X \to X$. Since $\mathbf{D}(\varphi)$ as a function of φ is periodic with period 2π, for convenience we extend the domain of $\mathbf{D}(\cdot)$ from $(-\pi, \pi]$ to $(-\infty, \infty)$. When a vector $\mathbf{u}$ represents the only basis vector of X, then $\mathbf{D}(\varphi)$ is characterized by:

$$\mathbf{D}(\varphi)\mathbf{u} = c(\varphi)\mathbf{u}, \quad \text{where } c(\varphi) \text{ is a complex number} \tag{2.27}$$

And for any rotation $\mathbf{R}(\varphi)$ in SO(2), it is easy to show that

$$(\mathbf{R}(\varphi))^n = \mathbf{R}(n\varphi) \tag{2.28}$$

Similarly, the representation $\mathbf{D}(\varphi)$ also satisfies that

$$c(n\varphi)\mathbf{u} = \mathbf{D}(n\varphi)\mathbf{u} = (\mathbf{D}(\varphi)\mathbf{u})^n = (c(\varphi))^n\mathbf{u} \tag{2.29}$$

Since the identity operator $\mathbf{R}(0)$ is represented by the identity 1, thus $\mathbf{D}(0)\mathbf{u} = \mathbf{u}$, *i.e.*: $\mathbf{D}(0) = 1$. Consequently, we conclude that the irreducible representations of $\mathbf{R}(\varphi)$ in SO(2) are given by

$$\mathbf{D}^k(\varphi) = e^{-ik\varphi}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{2.30}$$

It is easy to show that each value of k gives an irreducible representation inequivalent to any irreducible representation of other values of k from Fourier-analysis.

Consequently, when we organize the basis of vector space X as $\{\varphi_0, \varphi_{-1}, \varphi_1, \varphi_{-2}, \varphi_2, \ldots\}$, then the matrix form of a representation $\mathbf{D}(\mathbf{R}(\varphi))$ takes the form as a diagonal matrix at the Fourier basis φ_m with diagonal elements $e^{-im\varphi}$, i.e.:

$$\mathbf{D}(\mathbf{R}(\varphi)) = \begin{pmatrix} \ddots & & & & & & \\ & e^{2i\varphi} & & & & & \\ & & e^{i\varphi} & & & & \\ & & & 1 & & & \\ & & & & e^{-i\varphi} & & \\ & & & & & e^{-2i\varphi} & \\ & & & & & & \ddots \end{pmatrix} \tag{2.31}$$

In terms of the one-dimensional standard irreducible representations, (2.31) can be written as

$$\mathbf{D}(\mathbf{R}(\varphi)) = \begin{pmatrix} \ddots & & & & & & \\ & \mathbf{D}^2(\varphi) & & & & & \\ & & \mathbf{D}^1(\varphi) & & & & \\ & & & \mathbf{D}^0(\varphi) & & & \\ & & & & \mathbf{D}^{-1}(\varphi) & & \\ & & & & & \mathbf{D}^{-2}(\varphi) & \\ & & & & & & \ddots \end{pmatrix} \tag{2.32}$$

2.4 Material Tensors

2.4.1 Tensors

Recall that V is the translation space of the two-dimensional Euclidean space. Let $V^r = V \times V \times V \cdots \times V$ (r copies), where $r \geq 2$. A mapping $\boldsymbol{H} : V^r \to \mathbb{R}$ is multilinear if it is linear with respect to each of its vector arguments, i.e.,

$$\boldsymbol{H}[\mathbf{v}_1, \cdots, \mathbf{v}_i + \alpha \mathbf{v}_i', \cdots, \mathbf{v}_r] = \boldsymbol{H}[\mathbf{v}_1, \ldots, \mathbf{v}_i, \cdots, \mathbf{v}_r] + \alpha \boldsymbol{H}[\mathbf{v}_1, \cdots, \mathbf{v}_i, \ldots, \mathbf{v}_r] \tag{2.33}$$

for each $\mathbf{v}_i$ $(1 \leq i \leq r)$, $\mathbf{v}_i' \in V$ and $\alpha \in \mathbb{R}$. In mathematics such multilinear mappings are called rth-order tensors. Let $\mathbf{u}_1, \cdots, \mathbf{u}_r$ be in V. The tensor product of $\mathbf{u}_1, \cdots, \mathbf{u}_r$ is the rth-order tensor $\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_r : V^r \to \mathbb{R}$ defined by

$$\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_r[\mathbf{v}_1, \cdots, \mathbf{v}_r] = (\mathbf{u}_1 \cdot \mathbf{v}_1) \cdots (\mathbf{u}_r \cdot \mathbf{v}_r) \quad \text{for each} \quad (\mathbf{v}_1, \cdots, \mathbf{v}_r) \in V^r. \tag{2.34}$$

We call tensor products of vectors simple tensors. Under the usual definition of addition and of scalar multiplication of mappings, the set of rth-order tensors clearly forms a linear space over $\mathbb{R}$, which we denote by $V^{\otimes r} := V \otimes V \otimes \cdots \otimes V$ (r copies) and call the space of rth-order tensors.

Let $\mathbf{e}_1$, $\mathbf{e}_2$ constitute a orthonormal basis in V. Every $\boldsymbol{H} \in V^{\otimes r}$ can be written in the form

$$\boldsymbol{H} = H_{i_1 i_2 \cdots i_r} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r}, \tag{2.35}$$

where the Einstein summation convention is in force, and

$$H_{i_1 i_2 \cdots i_r} = \boldsymbol{H}[\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_r}]. \tag{2.36}$$

We define an inner product $\langle \cdot, \cdot \rangle$ on $V^{\otimes r}$ by requiring that

$$\langle \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r, \mathbf{w}_1 \otimes \cdots \otimes \mathbf{w}_r \rangle = (\mathbf{u}_1 \cdot \mathbf{w}_1) \cdots (\mathbf{u}_r \cdot \mathbf{w}_r). \tag{2.37}$$

Clearly simple tensors of the form $\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r}$, where each suffix runs over the indices 1 and 2, constitute an orthonormal basis in $V^{\otimes r}$. Hence $\dim V^{\otimes r} = 2^r$. For

$$\boldsymbol{H} = H_{i_1 i_2 \cdots i_r} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r} \quad \textit{and} \quad \boldsymbol{K} = K_{i_1 i_2 \cdots i_r} \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_r}, \tag{2.38}$$

we have

$$\langle \boldsymbol{H}, \boldsymbol{K} \rangle = H_{i_1 i_2 \cdots i_r} K_{i_1 i_2 \cdots i_r}. \tag{2.39}$$

Each orthogonal linear transformation $\boldsymbol{Q}$ on V induces an orthogonal linear transformation $\boldsymbol{Q}^{\otimes r} : V^{\otimes r} \to V^{\otimes r}$ defined by

$$\boldsymbol{Q}^{\otimes r}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r) = \boldsymbol{Q}\mathbf{u}_1 \otimes \cdots \otimes \boldsymbol{Q}\mathbf{u}_r, \tag{2.40}$$

for all $\mathbf{u}_1, \cdots, \mathbf{u}_r \in V$.

In continuum physics, many attributes of material points are characterized by multilinear mappings. Let a physical attribute Π of a given point P be described by an rth-order tensor $\boldsymbol{H}$. When the material point P undergoes a rotation or a rotation followed by an inversion defined by $\boldsymbol{Q} \in \mathrm{O}(2)$, the multilinear mapping that characterizes its attribute Π changes from $\boldsymbol{H}$ to $\mathcal{T}_{\boldsymbol{Q}}\boldsymbol{H}$. We say that Π is characterized by a material tensor $\boldsymbol{H}$ if

$$\mathcal{T}_{\boldsymbol{Q}}\boldsymbol{H} = \boldsymbol{Q}^{\otimes r}\boldsymbol{H}. \tag{2.41}$$

2.4.2 Complexification of Tensor Space

Let $V_c = \{\mathbf{u} + i\mathbf{v} : \mathbf{u} \in V, \mathbf{v} \in V\}$ be its complexification. We equip V_c with the Hermitian product induced by the inner product in V for real vectors. Under the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$ in V, the Hermitian product of vectors $\mathbf{w} = \langle w_1, w_2\rangle$ and $\mathbf{v} = \langle z_1, z_2\rangle$ is given by $< \mathbf{w}, \mathbf{z} >= w_1\bar{z}_1 + w_2\bar{z}_2$. A rotation $\boldsymbol{Q}$ on V can be extended to a linear transformation on V_c, which we still denote by $\boldsymbol{Q}$, defined as follows:

$$\boldsymbol{Q}(\mathbf{u} + i\mathbf{v}) = \boldsymbol{Q}\mathbf{u} + i\boldsymbol{Q}\mathbf{v} \tag{2.42}$$

for each $\mathbf{u}$, $\mathbf{v}$ in V. Each rotation $\boldsymbol{Q} : V \mapsto V$ is orthogonal, and its extension on V_c is unitary.

Let $V_c^{\otimes r}$ be the complexification of the r-fold tensor product $V \otimes V \otimes \cdots \otimes V$ (r factors). Obvioulsy $V_c^{\otimes r} = (V_c)^r = V_c \otimes V_c \otimes \cdots \otimes V_c$ (r factors). We equip $V_c^{\otimes r}$ with the Hermitian product induced by the Hermitian product on V_c, which satisfies

$$\langle \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r, \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_r\rangle = \prod_{j=1}^{r}\langle \mathbf{u}_j, \mathbf{v}_j\rangle \tag{2.43}$$

for all $\mathbf{u}_1, \cdots, \mathbf{u}_r$ and $\mathbf{v}_1, \cdots, \mathbf{v}_r$ in V_c. For two tensors $\boldsymbol{H} = H_{i_1\cdots i_r}\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{ir}$ and $\boldsymbol{K} = K_{i_1\cdots i_r}\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{ir}$ in $V_c^{\otimes r}$, the Hermitian product is given by

$$\langle \boldsymbol{H}, \boldsymbol{K}\rangle = H_{i_1\cdots i_r}\overline{K_{i_1\cdots i_r}}, \tag{2.44}$$

and $\|\boldsymbol{H}\| := \sqrt{\langle \boldsymbol{H}, \boldsymbol{H} \rangle}$ defines the norm of tensor $\mathbf{H}$.

Each rotation $\boldsymbol{Q}$ on V induces a linear transformation $\boldsymbol{Q}^{\otimes r}$ on $V_c^{\otimes r}$ and $V_c^{\otimes r}$ defined by

$$\boldsymbol{Q}^{\otimes r}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r) = \mathbf{Q}\mathbf{u}_1 \otimes \cdots \otimes \boldsymbol{Q}\mathbf{u}_r \tag{2.45}$$

for all $\mathbf{u}_1, \cdots, \mathbf{u}_r$ in V_c. Note that $\boldsymbol{Q}^{\otimes r} : V_c^r \to V_c^r$ is a unitary transformation on V_c^r. The map $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes r}$ defines a continuous linear representation of the orthogonal group O(2) on V^r and V_c^r respectively. A subspace $Z \subset V^r$ is said to be invariant under the action of the rotation group if it remains invariant under $\boldsymbol{Q}^{\otimes r}$ for each $\boldsymbol{Q} \in \mathrm{O}(2)$. Obviously, if Z is an invariant subspace of V^r, then Z_c is an invariant subspace of V_c^r. In what follows, we will consider only with tensor spaces Z and their complexification Z_c which remain invariant under the action of O(2). And we will work with representations $\boldsymbol{Q} \mapsto \boldsymbol{Q}^{\otimes r} \mid Z_c$ on Z_c, which are unitary representations.

To specify the various types of tensors (or tensor spaces), we adopt a system of notation advocated by Jahn[8] and Sirotin[24] . In this notation, V^2 stands for the tensor product of $V \otimes V$. $[V^2]$ the space of symmetric second-order tensors. $[[V^2]^2]$ stands for the symmetric square of $[V^2]$, (i.e., the symmetrized tensor product of $[V^2]$ and $[V^2]$), $[[V^2]^3]$ the symmetric cube of $[V^2]$, $[V^2][[V^2]^2]$the tensor product of $[V^2]$ and $[[V^2]^2]$, . . ., etc.

When V is replaced by its complexification V_c, the same procedure of bulding the tensor space Z will result in its complexification Z_c. Therefore, V_c^2 represents the complexification of V^2, $[[V_c^2]^2]$ represents the complexification of $[[V^2]^2]$, . . ., etc.

2.5 The Kelvin Notation

In this section we will recast the Kelvin notation [15, 13] in three-dimensional linear elasticity to suit our present context that the basic vector space V is two-dimensional.

We choose a pair of orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in V and keep this choice throughout the discussion. Then the symmetric second-order tensors

$$\mathbf{f}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{f}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{f}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \tag{2.46}$$

constitute an orthonormal basis in $[V^2]$, which is a 3-dimensional space. We adopt the Einstein summation convention for repeated Latin indices, which run from 1 to 2, and for repeated Greek indices, which run from 1 to 3.

For a second-order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ in $[V^2]$, we write $\mathbf{A}$ in the Kelvin notation as a 3×1 column vector

$$\hat{\mathbf{A}} = [\hat{a}_1, \hat{a}_2, \hat{a}_3]^T, \tag{2.47}$$

with

$$\hat{a}_1 = \langle \mathbf{A}, \mathbf{f}_1 \rangle = A_{11}, \quad \hat{a}_2 = \langle \mathbf{A}, \mathbf{f}_2 \rangle = A_{22}, \quad \hat{a}_3 = \langle \mathbf{A}, \mathbf{f}_3 \rangle = \sqrt{2}A_{12}, \tag{2.48}$$

which represents under the basis (2.46) the relation

$$\mathbf{A} = \hat{a}_\alpha \hat{\mathbf{f}}_\alpha. \tag{2.49}$$

The inner product of tensors $\mathbf{A}$ and $\mathbf{B}$ in $[V^2]$ is given by

$$\langle \mathbf{A}, \mathbf{B} \rangle = A_{ij}B_{ij} = \hat{a}_\alpha \hat{b}_\alpha. \tag{2.50}$$

Under the Kelvin notation, tensors in $[[V^2]^2]$ are identified with symmetric linear transformations on the 3-dimensional vector space $[V^2]$. Let $\mathbf{C} = C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ be a fourth-order tensor in $[[V^2]^2]$. Following common practice, we shall use the same symbol $\mathbf{C}$ to denote that which results when we interpret $\mathbf{C} \in [[V^2]^2]$ as a symmetric linear transformation on $[V^2]$. In the Kevin notation the second-order tensor $\mathbf{C}$ on $[V^2]$ is represented under the basis (2.46) by the 3×3 symmetric matrix $\hat{\mathbf{C}} = [\hat{c}_{\alpha\beta}]$:

$$\mathbf{C} = \hat{c}_{\alpha\beta}\mathbf{f}_\alpha \otimes \mathbf{f}_\beta, \qquad \text{where } \hat{c}_{\alpha\beta} = \langle \mathbf{f}_\alpha, \mathbf{C}\mathbf{f}_\beta \rangle. \tag{2.51}$$

In terms of the tensor components C_{ijkl}, we have $\hat{c}_{11} = C_{1111}$, $\hat{c}_{13} = \sqrt{2}C_{1112}$, $\hat{c}_{33} = 2C_{1212}$, etc.

A rotation $\mathbf{R}$ on V induces an orthogonal transformation $\mathbf{R}^{\otimes r} := \mathbf{R} \otimes \cdots \otimes \mathbf{R}$ (r factors) on $V^{\otimes r}$ defined by

$$\mathbf{R}^{\otimes r}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_r) = \mathbf{R}\mathbf{u}_1 \otimes \cdots \otimes \mathbf{R}\mathbf{u}_r \tag{2.52}$$

for all $\mathbf{u}_1, \ldots, \mathbf{u}_r$ in V. To avoid confusion, we use the symbols $\mathbf{R}^{\otimes 2}$ and $\mathbf{R}^{\otimes 4}$ also denote the restrictions $\mathbf{R}^{\otimes 2}|[V^2]$ and $\mathbf{R}^{\otimes 4}|[[V^2]^2]$, respectively. Under Kelvin notation, $\mathbf{R}^{\otimes 2}|[V^2]$ is represented by a 3×3 matrix,

$$\mathbb{R} = [\mathbb{R}_{\alpha\beta}], \quad \text{where } \mathbb{R}_{\alpha\beta} = \langle \mathbf{f}_\alpha, \mathbf{R}^{\otimes 2}\mathbf{f}_\beta \rangle. \tag{2.53}$$

The matrix entries $\mathbb{R}_{\alpha\beta}$ can be computed explicitly in terms of the components R_{ij}, where $R_{ij} = \langle \mathbf{e}_i, \mathbf{R}\mathbf{e}_j \rangle$ of $\mathbf{R}$. The matrix $[\mathbb{R}_{\alpha\beta}]$, which represents the linear transformation $\mathbf{R}^{\otimes 2}|[V^2]$ under the basis (2.46), is orthogonal; it satisfies [15]

$$\mathbb{Q}\mathbb{Q}^T = \mathbb{Q}^T\mathbb{Q} = \mathbb{I}, \tag{2.54}$$

where $\mathbb{I}$ is the 3×3 identity matrix. Under the orthogonal transformation $\mathbf{R}^{\otimes 2}|[V^2]$, a linear transformation $\mathbf{L}$: $[V^2] \to [V^2]$ becomes the transformation

$$(\mathbf{R}^{\otimes 2}|[V^2])\mathbf{L}(\mathbf{R}^{\otimes 2}|[V^2])^T. \tag{2.55}$$

Given a fourth-order tensor $\mathbf{C}$ in $[[V^2]^2]$, the 3×3 matrix that represents the symmetric linear transformation $\mathbf{R}^{\otimes 4}\mathbf{C}$ on $[V^2]$ is given by $\mathbb{R}\hat{\mathbf{C}}\mathbb{R}^T$, which is interpreted as a product of 3×3 matrices. We write

$$\mathbf{Q}^{\otimes 4}\mathbf{C} = \mathbb{Q}\hat{\mathbf{C}}\mathbb{Q}^T. \tag{2.56}$$

Henceforth all 3×3 matrix representation of tensors in $[[V^2]^2]$ or in its complexification $[[V_c^2]^2]$ are given in the Kelvin notation. Moreover, except for places where we want to emphasize that we are referring to the 3×3 matrix, we will drop the superscript ˆ in $\hat{\mathbf{C}}$ and use the same symbol $\mathbf{C}$ to denote the fourth-order tensor in question as well as its 3×3 matrix representation.

Chapter 3 Decomposition of a Tensor into Its Irreducible Parts on O(2)

3.1 Irreducible Representations of O(2)

It is well known (see, e.g., [25], p. 376) that a complete set of irreducible unitary representations of O(2) is given by the mapping $\rho_k : \mathrm{O}(2) \to \mathrm{M}(2)$ where $k = 0, 0', 1, 2...$, such that for each $\varphi \in \mathbb{R}$,

$$\rho_k(\mathbf{R}(\varphi)) = \begin{pmatrix} \cos(k\varphi) & -\sin(k\varphi) \\ \sin(k\varphi) & \cos(k\varphi) \end{pmatrix}, \quad \rho_k(\mathbf{M}_x\mathbf{R}(\varphi)) = \begin{pmatrix} \cos(k\varphi) & -\sin(k\varphi) \\ -sin(k\varphi) & -\cos(k\varphi) \end{pmatrix}, \quad \text{for } k \geq 1 \tag{3.1}$$

$$\rho_0(\mathbf{R}(\varphi)) = 1, \quad \rho_0(\mathbf{M}_x\mathbf{R}(\varphi)) = 1; \tag{3.2}$$

$$\rho_{0'}(\mathbf{R}(\varphi)) = 1, \quad \rho_{0'}(\mathbf{M}_x\mathbf{R}(\varphi)) = -1; \tag{3.3}$$

For completeness, however, we will provide a proof of this fact.

Since $\rho_k(\mathbf{R}(\varphi)), \rho_k(\mathbf{M}_x\mathbf{R}(\varphi)) \in \mathrm{M}(2)$ are orthogonal matrices they satisfy

$$\langle \rho_k(\mathbf{R}(\varphi))\mathbf{u}, \rho_k(\mathbf{R}(\varphi))\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \tag{3.4}$$

$$\langle \rho_k(\mathbf{M}_x\mathbf{R}(\varphi))\mathbf{u}, \rho_k(\mathbf{M}_x\mathbf{R}(\varphi))\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \tag{3.5}$$

for any $\mathbf{u}$ and $\mathbf{v}$ in V^2 and $\langle \mathbf{u}, \mathbf{v} \rangle$ is the inner product in V^2. Thus the extension of ρ_k on V_c^2 is a unitary representation of O(2). It is obvious that the mappings ρ_k are continuous.

To show that ρ_k is irreducible, let χ_k be character of ρ_k given by what follows:

$$\chi_k(\mathbf{R}(\varphi)) = e^{ik\varphi} + e^{-ik\varphi}, \quad \chi_k(\mathbf{M}_x\mathbf{R}(\varphi)) = 0, \quad \text{for } k \geq 1; \tag{3.6}$$

$$\chi_0(\mathbf{R}(\varphi)) = 1, \quad \chi_0(\mathbf{M}_x\mathbf{R}(\varphi)) = 1; \tag{3.7}$$

$$\chi_{0'}(\mathbf{R}(\varphi)) = 1, \quad \chi_{0'}(\mathbf{M}_x\mathbf{R}(\varphi)) = -1. \tag{3.8}$$

Then we have

$$\begin{aligned}
\langle \chi_k, \chi_k \rangle &= \frac{1}{4\pi}\int_{-\pi}^{\pi} 2\cos(k\varphi)\cdot 2\cos(2k\varphi)d\varphi + \frac{1}{4\pi}\int_{-\pi}^{\pi} 0\cdot 0 d\varphi \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} 4\cos^2(k\varphi)d\varphi \\
&= 1 \qquad\qquad \text{for all } k \geq 1, \qquad (3.9)\\
\langle \chi_0, \chi_0 \rangle &= \frac{1}{4\pi}\int_{-\pi}^{\pi} 1\cdot 1 d\varphi + \frac{1}{4\pi}\int_{-\pi}^{\pi} 1\cdot 1 d\varphi \\
&= \frac{1}{4\pi}\cdot 2\pi + \frac{1}{4\pi}\cdot 2\pi \\
&= 1, \qquad (3.10)\\
\langle \chi_{0'}, \chi_{0'} \rangle &= \frac{1}{4\pi}\int_{-\pi}^{\pi} 1\cdot 1 d\varphi + \frac{1}{4\pi}\int_{-\pi}^{\pi} (-1)\cdot(-1) d\varphi \\
&= \frac{1}{4\pi}\cdot 2\pi + \frac{1}{4\pi}\cdot 2\pi \\
&= 1. \qquad (3.11)
\end{aligned}$$

Thus ρ_k is irreducible for all $k = 0, 0', 1, 2, \cdots$.

To prove completeness, we need to show that any irreducible continuous unitary representation ρ of O(2) is equivalent to one of the representations ρ_k. We will prove by contradiction after going over some preliminaries.

A function f defined on a group G is a central function if it satisfies

$$f(hgh^{-1}) = f(g), \quad \text{for each } g \text{ and } h \text{ in } G \qquad (3.12)$$

The character functions χ of linear representations on a group G are central functions. By (2.10)–(2.13) we observe that for the irreducible representation ρ of O(2),

$$\chi_\rho(\mathbf{R}(\varphi)) = \chi_\rho(\mathbf{M}_x\mathbf{R}(\varphi)\mathbf{M}_x^{-1}) = \chi_\rho(\mathbf{R}(-\varphi)), \qquad (3.13)$$

$$\chi_\rho(\mathbf{M}_x\mathbf{R}(\varphi)) = \chi_\rho(\mathbf{M}_x\mathbf{R}(\varphi - 2\beta)) \qquad (3.14)$$

for any angles φ and β. If we choose $\beta = \frac{\varphi}{2}$ in (3.14), then we obtain

$$\chi_\rho(\mathbf{M}_x\mathbf{R}(\varphi)) = \chi_\rho(\mathbf{M}_x\mathbf{R}(0)) = \chi_\rho(\mathbf{M}_x) \tag{3.15}$$

for any angle φ. From (3.13) and (3.15), we can see that χ_ρ is an even function of φ.

We now proceed to prove that the set of irreducible unitary representations ρ_k ($k = 0, 0', 1, 2, \cdots$) is complete. Let ρ be a continuous, irreducible unitary representation of O(2) that is not equivalent to any representation $\mathbb{D}^k$ ($k = 0, 0', 1, 2, \cdots$). Since O(2) is compact, ρ is finite-dimensional [26, p. 16].Let χ_ρ be the character of ρ, and let $\chi_\rho(\boldsymbol{M}_x) = c$. Then we have

$$\langle\chi_\rho, \chi_0\rangle = \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}\chi_\rho(\boldsymbol{R}(\alpha))\cdot 1\, d\alpha + \int_{-\pi}^{\pi} c\cdot 1\, d\alpha\right) = 0, \tag{3.16}$$

$$\langle\chi_\rho, \chi_{0'}\rangle = \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}\chi_\rho(\boldsymbol{R}(\alpha))\cdot 1\, d\alpha + \int_{-\pi}^{\pi} c\cdot (-1)\, d\alpha\right) = 0, \tag{3.17}$$

and for $k = 1, 2, \cdots$,

$$\langle\chi_\rho, \chi_k\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_\rho(\boldsymbol{R}(\alpha))\cos k\alpha\, d\alpha = 0. \tag{3.18}$$

Equations (3.16) and (3.17) imply

$$\int_{-\pi}^{\pi}\chi_\rho(\boldsymbol{R}(\alpha))\cdot 1\, d\alpha = 0, \qquad \text{and} \qquad c = 0. \tag{3.19}$$

Because $\chi_\rho(\boldsymbol{R}(\cdot))$ is continuous and even on $[-\pi, \pi]$, and because the set of functions $\{\cos k\alpha : k = 0, 1, 2, \cdots\}$ is complete in the subspace of even functions in $L^2[-\pi, \pi]$, equations (3.18) and $(3.19)_1$ dictate that $\chi_\rho(\boldsymbol{R}(\cdot)) = 0$ on $[-\pi, \pi]$. On the other hand, since ρ is irreducible and $\chi_\rho(\boldsymbol{M}_x\boldsymbol{R}(\alpha)) = 0$ for each α, we have

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}|\chi_\rho(\boldsymbol{R}(\alpha))|^2\, d\alpha = 1. \tag{3.20}$$

Hence the assumption that the irreducible representation ρ is not equivalent to any $\mathbb{D}^k$ ($k = 0, 0', 1, 2, \cdots$) leads to a contradiction.

Henceforth we denote any unitary representation of $O(2)$ that is equivalent to ρ_k by $\mathbb{D}^k$.

Remark 3.1. *Under the orthonormal basis* $\mathbf{u}_{\bar{1}}$ *and* $\mathbf{u}_1$ *in* V_c *as given by (2.25), we have*

$$\rho_k(\boldsymbol{R}(\varphi)) = \begin{pmatrix} e^{-ik\varphi} & 0 \\ 0 & e^{ik\varphi} \end{pmatrix}, \quad \rho_k(\boldsymbol{M}_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad for\ k \geqslant 1 \tag{3.21}$$

3.2 Decomposition of a Tensor into Its Irreducible Parts

If a subspace $Z \subset V^{\otimes r}$ is invariant under the action of orthogonal group O(2), then its complexification Z_c is an invariant subspace of $V_c^{\otimes r}$. Since each finite-dimensional unitary representation of a compact group is reducible, each tensor representation $\mathbf{Q} \mapsto \mathbf{Q}^{\otimes r} \mid Z_c$ of the orthogonal group O(2) can be decomposed as a direct sum of subrepresentations, each of which is equivalent to some $\mathbb{D}^k$:

$$Z_c = n_0\mathbb{D}^0 + n_{0'}\mathbb{D}^{0'} + n_1\mathbb{D}^1 + \cdots + n_r\mathbb{D}^r, \tag{3.22}$$

where n_k is the multiplicity of $\mathbb{D}^k$ in the decomposition. Explicit decomposition formulas for specific tensor spaces can be derived by various methods, and some examples are given in the literature (see, e.g., [2, Table 7]). Later we shall use several decomposition formulas when we present examples to illustrate our procedure to determine irreducible tensor bases for tensor spaces. These examples include the tensor spaces $[[V_c^2]^3]$ and $[[V_c^2]^2] \otimes [V_c^2]$. According to the convention given in Section 2.4.2, $[[V_c^2]^3]$ is the complexification of $[[V^2]^3]$, which stands for the symmetric cube of the space of symmetric second-order tensors. $[[V_c^2]^2] \otimes [V_c^2]$ is the complexification of $[[V^2]^2] \otimes [V^2]$, which is the tensor product of $[[V^2]^2]$ and $[V_c^2]$. The decompositions of these two spaces are given by the following formulas:

$$[[V_c^2]^3] = \mathbb{D}^6 + \mathbb{D}^4 + 2\mathbb{D}^2 + 2\mathbb{D}^0, \tag{3.23}$$

$$[[V_c^2]^2] \otimes [V_c^2] = \mathbb{D}^6 + 2\mathbb{D}^4 + 4\mathbb{D}^2 + 3\mathbb{D}^0 + \mathbb{D}^{0'}, \tag{3.24}$$

Here we include a proof of these two decomposition formulas by the method of characters.

Let the tensor representation ρ of O(2) be equivalent to the symmetric square of $\mathbb{D}^1$. Then $\rho(\mathbf{R}(\varphi))$ and $\rho(\mathbf{M}_x\mathbf{R}(\varphi))$ have eigenvalues

$$\lambda_1 = e^{i2\varphi}, \quad \lambda_2 = e^{-i2\varphi}, \quad \lambda_3 = 1, \tag{3.25}$$

and

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 1 \tag{3.26}$$

respectively. Then we have

$$\begin{aligned}\chi(\rho^{\otimes 3}(\mathbf{R}(\varphi))) &= \sum_{i\le j\le k} \lambda_i\lambda_j\lambda_k \\ &= \lambda_1^3 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_1\lambda_2\lambda_3 + \lambda_2^3 + \lambda_2^2\lambda_1 + \lambda_2^2\lambda_3 + \lambda_3^3 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2 \\ &= e^{i6\varphi} + e^{-i6\varphi} + e^{i4\varphi} + e^{-i4\varphi} + 2e^{i2\varphi} + 2e^{-i2\varphi} + 2, \qquad (3.27)\\ \chi(\rho^{\otimes 3}(\mathbf{M}_x\mathbf{R}(\varphi))) &= 2, \qquad (3.28)\end{aligned}$$

which prove the decomposition formula of $[[V_c^2]^3]$.

Using (3.25) and (3.26) with similar calculations, we can also derive the following formulas that pertain to the decomposition of $[[V_c^2]^2]$, i.e.,

$$\begin{aligned}\chi(\rho^{\otimes 2}(\boldsymbol{R}(\varphi))) &= \sum_{i\leq j} \lambda_i\lambda_j \\ &= \lambda_1^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2 \\ &= e^{i4\varphi} + e^{-i4\varphi} + e^{i2\varphi} + e^{-i2\varphi} + 2, \end{aligned} \tag{3.29}$$

$$\chi(\rho^{\otimes 2}(\mathbf{M}_x\mathbf{R}(\varphi))) = 2. \tag{3.30}$$

Now we proceed to prove formula (3.24) for $[[V_c^2]^2] \otimes [V_c^2]$. Let ρ be the tensor representation of $[[V_c^2]^2] \otimes [V_c^2]$. Then by (3.25), (3.26), (3.29), and (3.30), we have

$$\begin{aligned}\chi(\rho(\boldsymbol{R}(\varphi))) &= (e^{i4\varphi} + e^{-i4\varphi} + e^{i2\varphi} + e^{-i2\varphi} + 2)(e^{i2\varphi} + e^{-i2\varphi} + 1) \\ &= e^{i6\varphi} + e^{-i6\varphi} + 2e^{i4\varphi} + 2e^{-i4\varphi} + 4e^{i2\varphi} + 4e^{-i2\varphi} + 4, \end{aligned} \tag{3.31}$$

$$\chi(\rho(M_x\boldsymbol{R}(\varphi))) = (1 - 1 + 1 - 1 + 2)(1 - 1 + 1) = 2. \tag{3.32}$$

Hence formula (3.24) is proved.

3.3 Determination of Irreducible Tensor Bases

Existence of a special basis with some far-reaching properties in a tensor subspace on which the tensor representation $\mathbf{Q} \mapsto \mathbf{Q}^{\otimes r}$ is equivalent to some $\mathbb{D}^k$ is indicated by the following general considerations.

Lemma 3.2. *Let Z be a two-dimensional subspace of $V^{\otimes r}$, and let Z_c be its complexification. If $\boldsymbol{H} \in Z_c$, then $\overline{\boldsymbol{H}} \in Z_c$.*

Proof. Let $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ constitute a basis in Z. Then $\boldsymbol{H} = \alpha_1\boldsymbol{E}_1 + \alpha_2\boldsymbol{E}_2$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$. Since $\overline{\boldsymbol{E}_i} = \boldsymbol{E}_i$ for $i = 1, 2$, we have $\overline{\boldsymbol{H}} = \overline{\alpha_1}\boldsymbol{E}_1 + \overline{\alpha_2}\boldsymbol{E}_2 \in Z_c$. □

Proposition 3.3. *Let Z be a two-dimensional subspace of $V^{\otimes r}$ invariant under the action of* O(2)*, i.e., $\boldsymbol{Q}^{\otimes r}Z \subset Z$ for each $\boldsymbol{Q}$ in* O(2)*, and let Z_c be the complexification of Z. Let $\rho : \mathrm{O}(2) \to \mathrm{GL}(Z_c)$ be a representation of* O(2) *equivalent to $\mathbb{D}^k$, with $k \geq 1$. Then there exists an orthonormal basis $\boldsymbol{H}_{\bar{k}}$, $\boldsymbol{H}_k$ in Z_c such that*

$$\boldsymbol{R}(\varphi)^{\otimes r}\boldsymbol{H}_{\bar{k}} = e^{-ik\varphi}\boldsymbol{H}_{\bar{k}}, \quad \boldsymbol{R}(\varphi)^{\otimes r}\boldsymbol{H}_k = e^{ik\varphi}\boldsymbol{H}_k, \tag{3.33}$$

$$\boldsymbol{M}_x^{\otimes r}\boldsymbol{H}_{\bar{k}} = \boldsymbol{H}_k, \quad \boldsymbol{M}_x^{\otimes r}\boldsymbol{H}_k = \boldsymbol{H}_{\bar{k}}, \quad \boldsymbol{H}_{\bar{k}} = \overline{\boldsymbol{H}_k}. \tag{3.34}$$

Proof. By the hypotheses given in the lemma, and by Remark 3.1, there exists an orthonomoral basis $\boldsymbol{X}_{\bar{k}}$, $\boldsymbol{X}_k$ under which the matrix representation of of $\rho(\boldsymbol{R}(\varphi)) = \boldsymbol{R}^{\otimes r}(\varphi)$ and $\rho(\boldsymbol{M}_x) = \boldsymbol{M}_x^{\otimes r}$ are given by

$$\boldsymbol{R}^{\otimes r}(\varphi) = \begin{pmatrix} e^{-ik\varphi} & 0 \\ 0 & e^{ik\varphi} \end{pmatrix}, \quad \boldsymbol{M}_x(\varphi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad for\ k \geq 1. \tag{3.35}$$

It follows immediately that

$$\boldsymbol{R}(\varphi)^{\otimes r}\boldsymbol{X}_{\bar{k}} = e^{-ik\varphi}\boldsymbol{X}_{\bar{k}}, \quad \boldsymbol{R}(\varphi)^{\otimes r}\boldsymbol{X}_k = e^{ik\varphi}\boldsymbol{X}_k, \quad for\ all\ \varphi, \tag{3.36}$$

$$\boldsymbol{M}_x^{\otimes r}\boldsymbol{X}_{\bar{k}} = \boldsymbol{X}_k, \quad \boldsymbol{M}_x^{\otimes r}\boldsymbol{X}_k = \boldsymbol{X}_{\bar{k}}. \tag{3.37}$$

Taking the complex conjugate of both sides of above equation, we obtain

$$\boldsymbol{R}(\varphi)^{\otimes r}\overline{\boldsymbol{X}_k} = e^{-ik\varphi}\overline{\boldsymbol{X}_k}. \tag{3.38}$$

Since both $\overline{\boldsymbol{X}_k}$ and $\boldsymbol{X}_{\bar{k}}$ are unit vectors of the same one-dimensional subspace invariant under $\boldsymbol{R}^{\otimes r}$, there exists a $c \in \mathbf{C}$ with $|c| = 1$ such that

$$\overline{\boldsymbol{X}_k} = c\boldsymbol{X}_{\bar{k}}. \tag{3.39}$$

It follows from (3.38) and above equation that

$$\boldsymbol{M}_x^{\otimes r}\overline{\boldsymbol{X}_k} = c\boldsymbol{M}_x^{\otimes r}\boldsymbol{X}_{\bar{k}} = c\boldsymbol{X}_k. \tag{3.40}$$

Let $\boldsymbol{X}_k = X^k_{i_1\cdots i_r}\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_r}$ and $X^k_{i_1\cdots i_r} = a_{i_1\cdots i_r} + ib_{i_1\cdots i_r}$. Fpr those components $X^k_{i_1\cdots i_r}$ with an even (including zero) number of "2" in the subscripts $i_1,\cdots,i_r$, the above equation dictates that

$$a_{i_1\cdots i_r} - ib_{i_1\cdots i_r} = c(a_{i_1\cdots i_r} + ib_{i_1\cdots i_r}), \tag{3.41}$$

which implies

$$c = 1, \quad b = 0, \quad or \quad c = -1, \quad a = 0. \tag{3.42}$$

For those components $X^k_{i_1\cdots i_r}$ with an odd number of "2" in the subscripts $i_1,\cdots,i_r$, the corresponding requirment is that

$$-a_{i_1\cdots i_r} + ib_{i_1\cdots i_r} = c(a_{i_1\cdots i_r} + ib_{i_1\cdots i_r}), \tag{3.43}$$

which implies

$$c = 1, \quad a = 0, \quad or \quad c = -1, \quad b = 0. \tag{3.44}$$

Hence there are two possibilities:

- Case 1: $c = 1$. All components $X^k_{i_1\cdots i_r}$ with an even number of "2" in the subscripts $i_1\cdots i_r$ are real, whereas those with an odd number of "2" are imaginary.

- Case 2: $c = -1$. All components $X^k_{i_1\cdots i_r}$ with an even number of "2" in the subscripts $i_1\cdots i_r$ are imaginary, whereas those with an odd number of "2" are odd.

Under Case 1, let $\boldsymbol{H}_k := \boldsymbol{X}_k$ and $\boldsymbol{H}_{\bar{k}} := \overline{\boldsymbol{X}_k}$. Then $\boldsymbol{H}_k$ and $\boldsymbol{H}_{\bar{k}}$ clearly constitute an orthonormal basis that satisfies (3.33). Under Case 2, let $\boldsymbol{H}_k := i\boldsymbol{X}_k$ and $\boldsymbol{H}_{\bar{k}} := i\overline{\boldsymbol{X}_{\bar{k}}}$. By (3.39) and

$c = -1$, we have

$$\boldsymbol{H}_{\bar{k}} = -i\overline{\boldsymbol{X}_k} = \overline{\boldsymbol{H}_k}. \tag{3.45}$$

By the definition of $\boldsymbol{H}_k$, $\boldsymbol{H}_{\bar{k}}$, and by the properties of $\boldsymbol{X}_k$, $\boldsymbol{X}_{\bar{k}}$, we observe that $\boldsymbol{H}_k$ and $\boldsymbol{H}_{\bar{k}}$ constitute an orthonormal basis in Z_c. By (3.38), we have

$$\boldsymbol{R}(\varphi)^{\otimes r}\boldsymbol{H}_{\bar{k}} = e^{-ik\varphi}\boldsymbol{H}_{\bar{k}}, \quad \boldsymbol{M}_x^{\otimes r}\boldsymbol{H}_{\bar{k}} = \boldsymbol{H}_k, \quad \boldsymbol{M}_x^{\otimes r}\boldsymbol{H}_k = \boldsymbol{H}_{\bar{k}}. \tag{3.46}$$

□

We will develop a procedure to determine explicitly a special orthonormal basis for any invariant subspace Z_c of $V_c^{\otimes r}$ under $O(2)$, for which the basis tensors enjoy properties similar to those displayed in (3.33) and (3.34). We call such a special orthonormal basis an irreducible tensor basis. In what follows we use $\mathbb{D}^k$ and $\mathbf{D}^k$ to denote the irreducible representations of O(2) and SO(2), respectively. Suppose the decomposition of Z_c under $O(2)$ is

$$\begin{aligned} Z_c &= n_0\mathbb{D}^0 + n_{0'}\mathbb{D}^{0'} + n_1\mathbb{D}^1 + \cdots + n_r\mathbb{D}^r, \\ &= n_0\mathbb{D}^0 + n_{0'}\mathbb{D}^{0'} + \sum_{k=1}^{r} n_k\mathbb{D}^k \\ &= n_0\mathbb{D}^0 + n_{0'}\mathbb{D}^{0'} + \sum_{k=1}^{r} \mathbb{X}_k \end{aligned} \tag{3.47}$$

where $\mathbb{X}_k = n_k\mathbb{D}^k$, for $n \geq 1$.

Under SO(2), each subspace that transforms as $\mathbb{D}^0$ or $\mathbb{D}^{0'}$ remains invariant. Each $\mathbb{D}^k$, which is of dimension 2, splits into a direct sum of two subspaces that transform as $\mathbf{D}^k$ and $\mathbf{D}^{-k}$, respectively. Hence the decomposition of Z_c under SO(2) can be expressed as:

$$Z_c = (n_0 + n_{0'})\mathbf{D}^0 + n_1(\mathbf{D}^1 + \mathbf{D}^{-1}) + \ldots + n_r(\mathbf{D}^r + \mathbf{D}^{-r}). \tag{3.48}$$

Since the invariant subspaces $\mathbb{X}_k$ in the decomposition are mutually orthogonal, we can examine each $\mathbb{X}_k$, respectively. Let $\mathbf{A}$ be a tensor in $\mathbb{X}_k$. It can be decomposed as a direct sum:

$$\mathbf{A} = \sum_{j=1}^{r} \mathbf{A}_j, \tag{3.49}$$

where $\mathbf{A}_j$ is some tensor in $\mathbb{X}_j$. Let $\mathfrak{I}^k : Z_c \to Z_c$ be defined by

$$\mathfrak{I}^k(\mathbf{A}) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{R}^{\otimes r} \mathbf{A} e^{-ik\varphi} d\varphi. \tag{3.50}$$

Here we claim that $\mathfrak{I}^k$ is a projection operator onto $\mathbb{X}_k$ for all $k \geq 1$. (An operator T is a projection if $T^2 = T$.) Indeed we have

$$\begin{aligned}
\mathfrak{I}^k(\mathbf{A}) &= \frac{1}{2\pi} \int_0^{2\pi} \mathbf{R}^{\otimes r} \mathbf{A} e^{-ik\varphi} d\varphi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \mathbf{R}^{\otimes r} \sum_{j=1}^{r} \mathbf{A}_j e^{-ik\varphi} d\varphi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{r} \mathbf{R}^{\otimes r} \mathbf{A}_j e^{-ik\varphi} d\varphi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{r} e^{ij\varphi} \mathbf{A}_j e^{-ik\varphi} d\varphi \\
&= \delta_{kj} \mathbf{A}_j \\
&= \mathbf{A}_k,
\end{aligned} \tag{3.51}$$

and

$$\mathfrak{I}^k(\mathfrak{I}^k(\mathbf{A})) = \mathfrak{I}^k(\mathbf{A}_k) = \mathbf{A}_k. \tag{3.52}$$

Thus $\mathfrak{I}^k$ is a projection from Z_c onto $\mathbb{X}_k$ and we can see that $\mathfrak{I}^k$ is surjective.

To start with, we determine an explicit basis for $\mathbb{X}_k$ under O(2) for $k \geq 1$. We let $\mathfrak{I}^k$ run over each member of any basis in Z in an arbitrarily chosen linear order. There will be a first basis tensor, which we call $\mathbf{A}_1^k$, for which $\mathfrak{I}^k(\mathbf{A}_1^k) \neq \mathbf{0}$. Then $\mathfrak{I}^k(\mathbf{A}_1^k)$ and $\overline{\mathfrak{I}^k(\mathbf{A}_1^k)}$

constitute an irreducible tensor basis in the invariant subspace $\mathbb{X}_k$ under O(2). If $n_k = 1$, then we are done with $\mathbb{X}_k$ for this step.

If $n_k > 1$, we continue to let $\mathfrak{I}^k$ to run over the selected basis tensors in Z in the order after $\mathbf{A}_1^k$. There will be a second basis tensor, which we call $\mathbf{A}_2^k$, for which $\mathfrak{I}^k(\mathbf{A}_2^k) \neq \mathbf{0}$ and $\mathfrak{I}^k(\mathbf{A}_2^k)$ is not a multiple of $\mathfrak{I}^k(\mathbf{A}_1^k)$. Then we get $\mathfrak{I}^k(\mathbf{A}_2^k)$ as a second tensor basis of $\mathbb{X}_k$. For $i = 1, 2$, let $\mathbb{X}_k^{(i)}$ be the span of $\mathfrak{I}^k(\mathbf{A}_i^k)$ and $\overline{\mathfrak{I}^k(\mathbf{A}_i^k)}$. If $\mathbb{X}_k = \mathbb{X}_k^{(1)} + \mathbb{X}_k^{(2)}$, then our work with $\mathbb{X}_k$ is done for this step.

If $n_k > 2$, we continue to let $\mathfrak{I}^k$ to run over the selected basis tensors in Z in the order after $\mathbf{A}_1^k$ and $\mathbf{A}_2^k$. There will be a third basis tensor, which we call $\mathbf{A}_3^k$, for which $\mathfrak{I}^k(\mathbf{A}_3^k) \neq \mathbf{0}$ and $\mathfrak{I}^k(\mathbf{A}_3^k)$ is not a linear combination of $\mathfrak{I}^k(\mathbf{A}_1^k)$ and $\mathfrak{I}^k(\mathbf{A}_2^k)$. Let let $\mathbb{X}_k^{(3)}$ be the span of $\mathfrak{I}^k(\mathbf{A}_3^k)$ and $\overline{\mathfrak{I}^k(\mathbf{A}_3^k)}$. If $\mathbb{X}_k = \mathbb{X}_k^{(1)} + \mathbb{X}_k^{(2)} + \mathbb{X}_k^{(3)}$, then our work with $\mathbb{X}_k$ is done for this step.

If $n_k > 3$, we repeat the procedure until we find for each $1 \leq s \leq n_k$ an irreducible tensor basis $\mathfrak{I}^k(\mathbf{A}_s^k)$ for $\mathbb{X}_k^{(s)}$ of $\mathbb{X}_k$, and $\mathbb{X}_k = \sum_{s=1}^{n_k} \mathbb{X}_k^{(s)}$.

Now we show that the basis tensors constructed by the procedure above satisfy (3.33) and (3.34). In our examples to illustrate the procedure, we will take advantage of the Kelvin notation for tensor spaces based on $[V^2]$. Hence we shall phrase our argument in the context of those spaces. In fact there is no loss in generality, because the argument for the general case of tensor spaces based on V is entirely similar.

Let $\mathbb{M}_x : [V^2] \to [V^2]$ be the restriction of $\mathbf{M}_x^{\otimes 2}$ on $[V^2]$. Under the Kelvin notation (see

Section 2.5) and the basis $\mathbf{f}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2$ and $\mathbf{f}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, we have

$$\begin{aligned}
\mathbb{M}_x \mathbf{f}_1 &= \boldsymbol{M}\mathbf{e}_1 \otimes \boldsymbol{M}\mathbf{e}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 = \mathbf{f}_1 \\
\mathbb{M}_x \mathbf{f}_2 &= \boldsymbol{M}\mathbf{e}_2 \otimes \boldsymbol{M}\mathbf{e}_2 = -\mathbf{e}_2 \otimes (-\mathbf{e}_2) = \mathbf{f}_2 \\
\mathbb{M}_x \mathbf{f}_3 &= \frac{1}{\sqrt{2}}(\boldsymbol{M}\mathbf{e}_1 \otimes \boldsymbol{M}\mathbf{e}_2 + \boldsymbol{M}\mathbf{e}_2 \otimes \boldsymbol{M}\mathbf{e}_1) = -\mathbf{f}_3.
\end{aligned} \tag{3.53}$$

Hence $\mathbb{M}_x$ is represented by the matrix

$$\mathbb{M}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.54}$$

For a non-trivial tensor $\mathbf{A}' \in Z \subset [V^2]^{\otimes r}$, let $\mathbf{A} = \mathbf{A}' + \mathbb{M}_x^{\otimes r}\mathbf{A}'$, where $\mathbb{M}_x^{\otimes r}$ is the linear transformation on $[V^2]^{\otimes r}$ induced by $\mathbf{M}_x$ on V. Since Z is an invariant subspace under the action of O(2), $\mathbf{A} \in Z$ and all components of $\mathbf{A}$ are real. Moreover, it is easy to show that $\mathbb{M}_x^{\otimes r}\mathbf{A} = \mathbf{A}$. Indeed we have

$$\begin{aligned}
\mathbb{M}_x^{\otimes r}\mathbf{A} &= \mathbb{M}_x^{\otimes r}\mathbf{A}' + \mathbb{M}_x^{\otimes r}\mathbb{M}_x^{\otimes r}\mathbf{A}' \\
&= \mathbb{M}_x^{\otimes r}\mathbf{A}' + \mathbf{A}' = \mathbf{A}.
\end{aligned} \tag{3.55}$$

Let $\mathbb{R}(\varphi)$ be the linear transformation on $[V^2]$ induced by the rotation $\mathbf{R}(\varphi)$ on V. Under the Kelvin notation and the basis $\mathbf{f}_i$ $(i = 1, 2, 3)$, $\mathbb{R}(\varphi)$ is represented by the matrix

$$\mathbb{R}(\varphi) = \begin{pmatrix} \cos^2\varphi & \sin^2\varphi & -\sqrt{2}\cos\varphi\sin\varphi \\ \sin^2\varphi & \cos^2\varphi & \sqrt{2}\cos\varphi\sin\varphi \\ \sqrt{2}\cos\varphi\sin\varphi & -\sqrt{2}\cos\varphi\sin\varphi & \cos^2\varphi - \sin^2\varphi \end{pmatrix}. \tag{3.56}$$

Let

$$\boldsymbol{X}_k = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\theta)^{\otimes r}\mathbf{A}e^{-ik\theta}d\theta. \tag{3.57}$$

Since

$$\begin{aligned}
\boldsymbol{X}_{\bar{k}} &= \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\theta)^{\otimes r}\mathbf{A}e^{-i\bar{k}\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\theta)^{\otimes r}\mathbf{A}e^{ik\theta}d\theta \\
&= \overline{\boldsymbol{X}_k},
\end{aligned} \tag{3.58}$$

we have proved that $\mathbf{X}_k$ and $\mathbf{X}_{\bar{k}}$ satisfy the last equation in (3.34).

Note that

$$\begin{aligned}
\mathbb{R}(\varphi)^{\otimes r}\boldsymbol{X}_k &= \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes r}\mathbb{R}(\theta)^{\otimes r}\mathbf{A}e^{-ik\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi+\theta)^{\otimes r}\mathbf{A}e^{-ik\theta}d\theta.
\end{aligned} \tag{3.59}$$

Let $\psi = \varphi + \theta$. Then we have

$$\begin{aligned}
\mathbb{R}(\varphi)^{\otimes r}\boldsymbol{X}_k &= \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\psi)^{\otimes r}\mathbf{A}e^{-ik(\psi-\varphi)}d\psi \\
&= e^{ik\varphi}\frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\psi)^{\otimes r}\mathbf{A}e^{-ik\psi}d\psi \\
&= e^{ik\varphi}\boldsymbol{X}_k.
\end{aligned} \tag{3.60}$$

Similarly, it is easy to show that $\mathbb{R}(\varphi)^{\otimes r}\boldsymbol{H}_{\bar{k}} = e^{-ik\varphi}\boldsymbol{H}_{\bar{k}}$.

When we apply the operator $\mathbb{M}_x^{\otimes r}$ on the tensor $\mathbf{A} \in Z$, we notice that the components $A_{i_1 \ldots i_r}$ of tensor $\mathbf{A}$ with an odd number of "3" in the subscripts $i_1, \ldots, i_r$ will change the sign. However by choosing $\mathbf{A}$ following the procedure above satisfying $\mathbb{M}_x\mathbf{A} = \mathbf{A}$, we guarantee that the components $A_{i_1 \ldots i_r}$ with an odd number of "3" in the subscripts $i_1, \ldots, i_r$ have to be zero. Thus we have

$$
\begin{aligned}
\mathbb{M}_x^{\otimes r}\boldsymbol{X}_k &= \frac{1}{2\pi}\int_0^{2\pi}\mathbb{M}_x^{\otimes r}\mathbb{R}(\theta)^{\otimes r}\mathbf{A}e^{-ik\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathbb{R}(-\theta)^{\otimes r}\mathbb{M}_x^{\otimes r}\mathbf{A}e^{-ik\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathbb{R}(-\theta)^{\otimes r}\mathbf{A}e^{-ik\theta}d\theta. \qquad (3.61)
\end{aligned}
$$

Let $\psi = -\theta$. It follows that

$$
\begin{aligned}
\mathbb{M}_x^{\otimes r}\boldsymbol{X}_k &= \frac{1}{2\pi}\int_0^{-2\pi}\mathbb{R}(\psi)^{\otimes r}\mathbf{A}e^{ik\psi}(-d\psi) \\
&= \boldsymbol{X}_{\bar{k}} \qquad (3.62)
\end{aligned}
$$

which implies that $\mathbb{M}_x^{\otimes r}\boldsymbol{H}_k = \boldsymbol{H}_{\bar{k}}$.

A set of orthonormal irreducible basis tensors in Z_c can be constructed from the $\mathbf{X}_k$'s through the Gram-Schmidt procedure (cf. [14]).

Suppose a set of irreducible basis tensors $\boldsymbol{B}_{k,s} = \mathfrak{I}^k(\mathbf{A}_s^k)$ $(1 \le s \le n_k, 0 \le k \le r)$ has been determined by the procedure described above. We apply the Gram-Schmidt procedure to obtain the orthonormal irreducible basis tensors $\boldsymbol{H}_{k,s}$ as follows:

$$
\begin{aligned}
&\boldsymbol{H}_{k,1} = \frac{\boldsymbol{B}_{k,1}}{\parallel \boldsymbol{B}_{k,1} \parallel}, \quad \boldsymbol{H}_{k,2} = \frac{\boldsymbol{B}_{k,2} - \langle \boldsymbol{B}_{k,2}, \boldsymbol{H}_{k,1}\rangle \boldsymbol{H}_{k,1}}{\parallel \boldsymbol{B}_{k,2} - \langle \boldsymbol{B}_{k,2}, \boldsymbol{H}_{k,1}\rangle \boldsymbol{H}_{k,1} \parallel}, \\
&\boldsymbol{H}_{k,s+1} = \frac{\boldsymbol{B}_{k,s+1} - \sum_{j=1}^{s}\langle \boldsymbol{B}_{k,s+1}, \boldsymbol{H}_{k,j}\rangle \boldsymbol{H}_{k,j}}{\parallel \boldsymbol{B}_{k,s+1} - \sum_{j=1}^{s}\langle \boldsymbol{B}_{k,s+1}, \boldsymbol{H}_{k,j}\rangle \boldsymbol{H}_{k,j} \parallel} \quad \text{for } 1 \le s \le n_k - 1. \qquad (3.63)
\end{aligned}
$$

3.4 Examples of Calculating Irreducible tensor bases under O(2)

First we consider irreducible tensor basis for $[V_c^2]$, which is the space of symmetric second-order tensors. An orthonormal irreducible tensor basis for $[V_c^2]$ is given by the following formulas:

$$\boldsymbol{H}_2 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad \boldsymbol{H}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{H}_{\bar{2}} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{3.64}$$

It can be easily checked that

$$\mathbf{R}(\varphi)^{\otimes 2}\boldsymbol{H}_2 = \frac{1}{2}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = e^{2i\varphi}\boldsymbol{H}_2. \tag{3.65}$$

for any angle φ. Similarly, it is easy to verify by direct computations that $\mathbf{H}_2$, $\mathbf{H}_0$, and $\mathbf{H}_{\bar{2}}$ satisfy the other equations in (3.33) and (3.34).

From now on, we denote $[V_c^2]$ by $\mathbb{V}$ for convenience. Now we consider irreducible tensor basis for $[\mathbb{V}^2]$, i.e $[[V_c^2]^2]$. Under the Kelvin notation and the basis $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ in $[\mathbb{V}^2]$, we have, for any angle φ,

$$\mathbf{R}(\varphi)^{\otimes 4}\mathbf{A} = \mathbb{R}(\varphi)\hat{\mathbf{A}}\mathbb{R}(\varphi)^T, \tag{3.66}$$

where $\mathbb{R}(\varphi)$ is given by (3.56) and $\hat{\mathbf{A}}$ is the 3×3 matrix that represent $\mathbf{A}$ under the Kelvin notation. For $\hat{\mathbf{A}}$ with $A_{11} = 1$ and all other components zero, we have

$$\mathbb{Q}\hat{\mathbf{A}}\mathbb{Q}^T = \begin{pmatrix} \cos^4\varphi & \cos^2\varphi\sin^2\varphi & \sqrt{2}\cos^3\varphi\sin\varphi \\ \cos^2\varphi\sin^2\varphi & \sin^4\varphi & \cos^2\varphi\sin^2\varphi \\ \sqrt{2}\cos^3\varphi\sin\varphi & \sqrt{2}\cos\varphi\sin^3\varphi & 2\cos^2\varphi\sin^2\varphi \end{pmatrix} \tag{3.67}$$

An orthonormal irreducible tensor basis for $[\mathbb{V}^2]$ is given by the following formulas:

$$\boldsymbol{H}_4 = \frac{1}{4}\begin{pmatrix} 1 & -1 & -\sqrt{2}i \\ -1 & 1 & -\sqrt{2}i \\ \sqrt{2}i & -\sqrt{2}i & -2 \end{pmatrix}, \quad \boldsymbol{H}_2 = \frac{1}{4}\begin{pmatrix} 2 & 0 & \sqrt{2}i \\ 0 & -2 & \sqrt{2}i \\ \sqrt{2}i & \sqrt{2}i & 0 \end{pmatrix},$$

$$\boldsymbol{H}_{0,1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{H}_{0,2} = \frac{1}{2\sqrt{6}}\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \boldsymbol{H}_{\bar{2}} = \overline{H_2}, \quad \boldsymbol{H}_{\bar{4}} = \overline{H_4}. \tag{3.68}$$

It is easy to verify by direct computation that the basis tensors above satisfy (3.33) and (3.34).

In what follows we adopt the Kelvin notation for tensors in $[\mathbb{V}^3]$, where $\mathbb{V}$ is taken as a three-dimensional space above. The components of a sixth-order tensor $\mathbf{B}$ in this space is displayed in array form as

$$\boldsymbol{B} = \begin{pmatrix} B_{111} & B_{112} & B_{113} & B_{211} & B_{212} & B_{213} & B_{311} & B_{312} & B_{313} \\ B_{121} & B_{122} & B_{123} & B_{221} & B_{222} & B_{223} & B_{321} & B_{322} & B_{323} \\ B_{131} & B_{132} & B_{133} & B_{231} & B_{232} & B_{233} & B_{331} & B_{332} & B_{333} \end{pmatrix}, \tag{3.69}$$

where the components B_{ijk} are totally symmetric with respect to its indices. There are ten independent components, which we take as B_{111}, B_{112}, B_{113}, B_{122}, B_{123}, B_{133}, B_{222}, B_{223}, B_{233} and B_{333}.

From (3.23), we know that the decomposition formula of $[\mathbb{V}^3]$ is given by $[\mathbb{V}^3] = \mathbb{D}^6 + \mathbb{D}^4 + 2\mathbb{D}^2 + 2\mathbb{D}^0$. The tensors $\boldsymbol{H}_6$ and $\boldsymbol{H}_4$ are obtained by normalizing the tensors

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}e^{-i6\varphi}d\varphi, \quad \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}e^{-i4\varphi}d\varphi, \tag{3.70}$$

respectively, with $A_{111} = 1$ and $A_{ijk} = 0$ otherwise:

$$\boldsymbol{H}_6 = \frac{1}{2\sqrt{15}} \begin{pmatrix} 1 & -1 & \sqrt{2}i & -1 & 1 & -\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & -2 \\ -1 & 1 & -\sqrt{2}i & 1 & -1 & \sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & 2 \\ \sqrt{2}i & -\sqrt{2}i & -2 & -\sqrt{2}i & \sqrt{2}i & 2 & -2 & 2 & -2\sqrt{2}i \end{pmatrix}, \tag{3.71}$$

$$\mathbf{H}_4 = \frac{1}{4\sqrt{6}} \begin{pmatrix} 3 & -1 & 2\sqrt{2}i & -1 & -1 & 0 & 2\sqrt{2}i & 0 & -2 \\ -1 & -1 & 0 & -1 & 3 & -2\sqrt{2}i & 0 & -2\sqrt{2}i & -2 \\ 2\sqrt{2}i & 0 & -2 & 0 & -2\sqrt{2}i & -2 & -2 & -2 & 0 \end{pmatrix}. \tag{3.72}$$

Up to a constant real factor, the tensors $\boldsymbol{B}_{0,1}$ and $\boldsymbol{B}_{2,1}$ are given by the integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}d\varphi, \quad \frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}e^{-i2\varphi}d\varphi \tag{3.73}$$

respectively, with $A_{111} = 1$ and $A_{ijk} = 0$ otherwise. And the tensors $\boldsymbol{B}_{0,2}$ and $\boldsymbol{B}_{2,2}$ are given by the same integrals respectively, with $A_{112} = A_{121} = A_{211} = 1$ and $A_{ijk} = 0$ otherwise:

$$\boldsymbol{B}_{0,1} = \begin{pmatrix} 5 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 \end{pmatrix} \tag{3.74}$$

$$\boldsymbol{H}_{2,1} = \begin{pmatrix} 15 & 1 & 5\sqrt{2}i & 1 & -1 & 3\sqrt{2}i & 5\sqrt{2}i & 3\sqrt{2}i & 2 \\ 1 & -1 & 3\sqrt{2}i & -1 & -15 & 5\sqrt{2}i & 3\sqrt{2}i & 5\sqrt{2}i & -2 \\ 5\sqrt{2}i & 3\sqrt{2}i & 2 & 3\sqrt{2}i & 5\sqrt{2}i & -2 & 2 & -2 & 6\sqrt{2}i \end{pmatrix}, \tag{3.75}$$

$$\boldsymbol{B}_{0,2} = \begin{pmatrix} 3 & 7 & 0 & 7 & 7 & 0 & 0 & 0 & -2 \\ 7 & 7 & 0 & 7 & 3 & 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 & 0 & -2 & -2 & -2 & 0 \end{pmatrix}, \tag{3.76}$$

$$\boldsymbol{B}_{2,2} = \begin{pmatrix} 3 & 13 & \sqrt{2}i & 1 & -13 & 7\sqrt{2}i & \sqrt{2}i & 7\sqrt{2}i & -6 \\ 13 & -1 & 7\sqrt{2}i & -13 & -3 & \sqrt{2}i & 7\sqrt{2}i & \sqrt{2}i & 6 \\ \sqrt{2}i & 7\sqrt{2}i & -6 & 7\sqrt{2}i & \sqrt{2}i & 6 & -6 & 6 & -18\sqrt{2}i \end{pmatrix}. \tag{3.77}$$

We proceed to obtain an orthonormal basis by the Gram-Schmidt procedure described by Man and Huang [14]. First, we observe that

$$\frac{1}{4}(\boldsymbol{B}_{2,1} - \boldsymbol{B}_{2,2}) = \begin{pmatrix} 3 & -3 & \sqrt{2}i & -3 & 3 & -\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & 2 \\ -3 & 3 & -\sqrt{2}i & 3 & -3 & \sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & -2 \\ \sqrt{2}i & -\sqrt{2}i & 2 & -\sqrt{2}i & \sqrt{2}i & -2 & 2 & -2 & 6\sqrt{2}i \end{pmatrix}. \tag{3.78}$$

Then we take

$$\boldsymbol{H}_{2,1} = \frac{1}{8\sqrt{3}} \begin{pmatrix} 3 & -3 & \sqrt{2}i & -3 & 3 & -\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & 2 \\ -3 & 3 & -\sqrt{2}i & 3 & -3 & \sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & -2 \\ \sqrt{2}i & -\sqrt{2}i & 2 & -\sqrt{2}i & \sqrt{2}i & -2 & 2 & -2 & 6\sqrt{2}i \end{pmatrix} \tag{3.79}$$

and

$$\begin{aligned} \boldsymbol{H}_{2,2} &= \frac{\boldsymbol{B}_{2,1} - \langle \boldsymbol{B}_{2,1}, \boldsymbol{H}_{2,1} \rangle \boldsymbol{H}_{2,1}}{\|\boldsymbol{B}_{2,1} - \langle \boldsymbol{B}_{2,1}, \boldsymbol{H}_{2,1} \rangle \boldsymbol{H}_{2,1}\|} \\ &= \frac{1}{4\sqrt{3}} \begin{pmatrix} 3 & 1 & \sqrt{2}i & 1 & -1 & \sqrt{2}i & \sqrt{2}i & \sqrt{2}i & 0 \\ 1 & -1 & \sqrt{2}i & -1 & -3 & \sqrt{2}i & \sqrt{2}i & \sqrt{2}i & 0 \\ \sqrt{2}i & \sqrt{2}i & 0 & \sqrt{2}i & \sqrt{2}i & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.80}$$

Similarly, note that

$$\frac{1}{8}(\boldsymbol{B}_{0,1} + \boldsymbol{B}_{0,2}) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.81}$$

We choose

$$\boldsymbol{H}_{0,1} = \frac{1}{2\sqrt{2}} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.82}$$

and

$$\begin{aligned} \boldsymbol{H}_{0,2} &= \frac{\boldsymbol{B}_{0,1} - \langle \boldsymbol{B}_{0,1}, \boldsymbol{H}_{0,1} \rangle \boldsymbol{H}_{0,1}}{\|\boldsymbol{B}_{0,1} - \langle \boldsymbol{B}_{0,1}, \boldsymbol{H}_{0,1} \rangle \boldsymbol{H}_{0,1}\|} \\ &= \frac{1}{4\sqrt{3}} \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 2 \\ -1 & -1 & 0 & -1 & 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 \end{pmatrix}. \end{aligned} \tag{3.83}$$

The independent components of the irreducible basis tensors given above are displayed in the following table:

Tensor $\boldsymbol{B}$	B_{111}	B_{112}	B_{113}	B_{122}	B_{123}	B_{133}	B_{222}	B_{223}	B_{233}	B_{333}
$2\sqrt{15} \times \boldsymbol{H}_6$	1	−1	$\sqrt{2}i$	1	$-\sqrt{2}i$	−2	−1	$\sqrt{2}i$	2	$-2\sqrt{2}i$
$4\sqrt{6} \times \boldsymbol{H}_4$	3	−1	$2\sqrt{2}i$	−1	0	−2	3	$-2\sqrt{2}i$	−2	0
$8\sqrt{3} \times \boldsymbol{H}_{2,1}$	3	−3	$\sqrt{2}i$	3	$-\sqrt{2}i$	2	−3	$\sqrt{2}i$	−2	$6\sqrt{2}i$
$4\sqrt{3} \times \boldsymbol{H}_{2,2}$	3	1	$\sqrt{2}i$	−1	$\sqrt{2}i$	0	−3	$\sqrt{2}i$	0	0
$2\sqrt{2} \times \boldsymbol{H}_{0,1}$	1	1	0	1	0	0	1	0	0	0
$4\sqrt{3} \times \boldsymbol{H}_{0,2}$	3	−1	0	−1	0	2	3	0	2	0

Next we consider irreducible tensor basis for $\mathbb{V} \otimes [\mathbb{V}^2]$. Again we adopt the Kelvin notation for tensors in $\mathbb{V} \otimes [\mathbb{V}^2]$, where $\mathbb{V}$ is taken as a three-dimensional space with basis $\mathbf{f}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2$, and $\mathbf{f}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$. The components of a sixth-order tensor $\boldsymbol{B}$ in this space is displayed in array form as

$$\boldsymbol{B} = \begin{pmatrix} B_{111} & B_{112} & B_{113} & B_{211} & B_{212} & B_{213} & B_{311} & B_{312} & B_{313} \\ B_{121} & B_{122} & B_{123} & B_{221} & B_{222} & B_{223} & B_{321} & B_{322} & B_{323} \\ B_{131} & B_{132} & B_{133} & B_{231} & B_{232} & B_{233} & B_{331} & B_{332} & B_{333} \end{pmatrix}, \tag{3.84}$$

where $B_{ijk} = B_{ikj}$.

The integral

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}e^{-i4\varphi}d\varphi, \text{where } A_{112} = A_{121} = -1, \text{ and } A_{ijk} = 0 \text{ otherwise,}$$

delivers after normalization the tensor

$$\boldsymbol{H}_{4,1} = \frac{1}{4\sqrt{2}}\begin{pmatrix} 1 & -1 & \sqrt{2}i & 1 & -1 & \sqrt{2}i & 0 & 0 & 0 \\ -1 & 1 & -\sqrt{2}i & -1 & 1 & -\sqrt{2}i & 0 & 0 & 0 \\ \sqrt{2}i & -\sqrt{2}i & -2 & \sqrt{2}i & -\sqrt{2}i & -2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.85}$$

We take $\boldsymbol{B}_{4,2} = 4\sqrt{6} \times \boldsymbol{H}_4$, where $\boldsymbol{H}_4$ is given by (3.72). It turns out that $\langle \boldsymbol{B}_{4,2}, \boldsymbol{H}_{4,1}\rangle = 0$. Hence we take $\boldsymbol{H}_{4,2} = \boldsymbol{H}_4$.

With reference to (3.79) and (3.80), here we take

$$\boldsymbol{B}_{2,1} = \begin{pmatrix} 3 & -3 & \sqrt{2}i & -3 & 3 & -\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & 2 \\ -3 & 3 & -\sqrt{2}i & 3 & -3 & \sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & -2 \\ \sqrt{2}i & -\sqrt{2}i & 2 & -\sqrt{2}i & \sqrt{2}i & -2 & 2 & -2 & 6\sqrt{2}i \end{pmatrix}, \tag{3.86}$$

$$\boldsymbol{B}_{2,2} = \begin{pmatrix} 3 & 1 & \sqrt{2}i & 1 & -1 & \sqrt{2}i & \sqrt{2}i & \sqrt{2}i & 0 \\ 1 & -1 & \sqrt{2}i & -1 & -3 & \sqrt{2}i & \sqrt{2}i & \sqrt{2}i & 0 \\ \sqrt{2}i & \sqrt{2}i & 0 & \sqrt{2}i & \sqrt{2}i & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.87}$$

The integral

$$\frac{1}{2\pi}\int_0^{2\pi}\mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}e^{-i2\varphi}d\varphi,$$

where (i) $A_{112} = A_{121} = 1$ and $A_{ijk} = 0$ otherwise, and (ii) $A_{113} = A_{131} = -i$ and $A_{ijk} = 0$ otherwise, delivers up to a constant factor the tensors

$$\boldsymbol{B}_{2,3} = \begin{pmatrix} 1 & 7 & -\sqrt{2}i & -1 & -7 & \sqrt{2}i & 3\sqrt{2}i & 5\sqrt{2}i & -2 \\ 7 & 1 & \sqrt{2}i & -7 & -1 & -\sqrt{2}i & 5\sqrt{2}i & 3\sqrt{2}i & 2 \\ -\sqrt{2}i & \sqrt{2}i & -2 & \sqrt{2}i & -\sqrt{2}i & 2 & -2 & 2 & -6\sqrt{2}i \end{pmatrix}, \quad (3.88)$$

$$\boldsymbol{B}_{2,4} = \begin{pmatrix} 5 & -1 & 3\sqrt{2}i & 3 & 1 & \sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & 2 \\ -1 & -3 & \sqrt{2}i & 1 & -5 & 3\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & -2 \\ 3\sqrt{2}i & \sqrt{2}i & -2 & \sqrt{2}i & 3\sqrt{2}i & 2 & 2 & -2 & 2\sqrt{2}i \end{pmatrix}. \quad (3.89)$$

respectively.

We choose $\boldsymbol{H}_{2,1}$ and $\boldsymbol{H}_{2,2}$ to be the tensors given by

$$2\sqrt{3}\times\boldsymbol{H}_{2,1} = \frac{1}{4}(\boldsymbol{B}_{2,1} + 2\boldsymbol{B}_{2,2} - \boldsymbol{B}_{2,4}) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & \sqrt{2}i & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & \sqrt{2}i & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & \sqrt{2}i \end{pmatrix}, \quad (3.90)$$

and

$$4\sqrt{2}\times\boldsymbol{H}_{2,2} = \frac{1}{4}(4\boldsymbol{B}_{2,2} - \boldsymbol{B}_{2,1} - \boldsymbol{B}_{2,3}) = \begin{pmatrix} 2 & 0 & \sqrt{2}i & 2 & 0 & \sqrt{2}i & 0 & 0 & 0 \\ 0 & -2 & \sqrt{2}i & 0 & -2 & \sqrt{2}i & 0 & 0 & 0 \\ \sqrt{2}i & \sqrt{2}i & 0 & \sqrt{2}i & \sqrt{2}i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.91)$$

respectively. Clearly $\langle \boldsymbol{H}_{2,1}, \boldsymbol{H}_{2,2} \rangle = 0$. Let

$$\boldsymbol{C} = \frac{1}{4}\left(\boldsymbol{B}_{2,3} + \boldsymbol{B}_{2,4} - 2\boldsymbol{B}_{2,2}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & \sqrt{2}i & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & \sqrt{2}i & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -2\sqrt{2}i \end{pmatrix}. \quad (3.92)$$

Since $\langle \boldsymbol{C}, \boldsymbol{H}_{2,2} \rangle = 0$, we take

$$\begin{aligned} \boldsymbol{H}_{2,3} &= \frac{\boldsymbol{C} - \langle \boldsymbol{C}, \boldsymbol{H}_{2,1} \rangle \boldsymbol{H}_{2,1}}{\|\boldsymbol{C} - \langle \boldsymbol{C}, \boldsymbol{H}_{2,1} \rangle \boldsymbol{H}_{2,1}\|} \\ &= \frac{1}{4\sqrt{6}} \begin{pmatrix} 1 & 3 & 0 & -1 & -3 & 0 & \sqrt{2}i & 3\sqrt{2}i & 0 \\ 3 & 1 & 0 & -3 & -1 & 0 & 3\sqrt{2}i & \sqrt{2}i & 0 \\ 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & -2\sqrt{2}i \end{pmatrix}. \end{aligned} \quad (3.93)$$

Let

$$\boldsymbol{D} = \frac{1}{4}\left(\boldsymbol{B}_{2,4} - \boldsymbol{B}_{2,3}\right) = \begin{pmatrix} 1 & -2 & \sqrt{2}i & 1 & 2 & 0 & -\sqrt{2}i & -\sqrt{2}i & 1 \\ -2 & -1 & 0 & 2 & -1 & \sqrt{2}i & -\sqrt{2}i & -\sqrt{2}i & -1 \\ \sqrt{2}i & 0 & 0 & 0 & \sqrt{2}i & 0 & 1 & -1 & 2\sqrt{2}i \end{pmatrix}. \quad (3.94)$$

Since $\langle \boldsymbol{D}, \boldsymbol{H}_{2,1} \rangle = 0$, we take

$$\begin{aligned} \boldsymbol{H}_{2,4} &= \frac{\boldsymbol{D} - \langle \boldsymbol{D}, \boldsymbol{H}_{2,2} \rangle \boldsymbol{H}_{2,2} - \langle \boldsymbol{D}, \boldsymbol{H}_{2,3} \rangle \boldsymbol{H}_{2,3}}{\|\boldsymbol{D} - \langle \boldsymbol{D}, \boldsymbol{H}_{2,2} \rangle \boldsymbol{H}_{2,2} - \langle \boldsymbol{D}, \boldsymbol{H}_{2,3} \rangle \boldsymbol{H}_{2,3}\|} \\ &= \frac{1}{8} \begin{pmatrix} 1 & -1 & \sqrt{2}i & -1 & 1 & -\sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & 2 \\ -1 & 1 & -\sqrt{2}i & 1 & -1 & \sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & -2 \\ \sqrt{2}i & -\sqrt{2}i & -2 & -\sqrt{2}i & \sqrt{2}i & 2 & 2 & -2 & 2\sqrt{2}i \end{pmatrix}. \end{aligned} \quad (3.95)$$

The integral

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3}\mathbf{A}\,d\varphi, \text{ where } A_{112} = A_{121} = 1, \text{ and } A_{ijk} = 0 \text{ otherwise,}$$

delivers up to a constant real factor the tensor

$$\boldsymbol{B}_{0,3} = \begin{pmatrix} 1 & 3 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.96}$$

We take

$$\begin{aligned} \boldsymbol{H}_{0,3} &= \frac{\boldsymbol{B}_{0,3} - \langle \boldsymbol{B}_{0,3}, \boldsymbol{H}_{0,1} \rangle \boldsymbol{H}_{0,1} - \langle \boldsymbol{B}_{0,3}, \boldsymbol{H}_{0,2} \rangle \boldsymbol{H}_{0,2}}{\|\boldsymbol{B}_{0,3} - \langle \boldsymbol{B}_{0,3}, \boldsymbol{H}_{0,1} \rangle \boldsymbol{H}_{0,1} - \langle \boldsymbol{B}_{0,3}, \boldsymbol{H}_{0,2} \rangle \boldsymbol{H}_{0,2}\|} \\ &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & -2 & 1 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.97}$$

The integral

$$\frac{1}{2\pi} \int_0^{2\pi} \mathbb{R}(\varphi)^{\otimes 3} \mathbf{A} d\varphi, \text{ where } A_{113} = A_{131} = 1, \text{ and } A_{ijk} = 0 \text{ otherwise,}$$

delivers after normalization the tensor

$$\boldsymbol{H}_{0'} = \frac{1}{4} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.98}$$

Chapter 4 Representation Theorem for Almost Transversely-Isotropic Sheets

4.1 Characterization of Almost Transversely-Isotropic Texture

In materials science, macrotexture is usually described by the orientation distribution function (ODF) w defined on the rotation group SO(3). Generally the ODF can be expanded as an infinite series in terms of the Wigner D-functions D^l_{mn} [3, 11, 21, 27]

$$w(\boldsymbol{R}(\psi,\theta,\phi)) = \frac{1}{8\pi^2} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l} c^l_{mn} D^l_{mn}(\boldsymbol{R}(\psi,\theta,\phi)), \tag{4.1}$$

where the three-dimensional rotations are parametrized by the Euler angles (ψ,θ,ϕ), c^l_{mn} are expansion coefficients called texture coefficients, and the ODF is normalized so that its integral over the rotation group SO(3) is equal to 1, i.e.,

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} w(\psi,\theta,\phi)\sin\theta\, d\psi d\theta d\phi = 1. \tag{4.2}$$

Because w is real-valued, the texture coefficients and the Wigner D-functions satisfy the constraints that

$$c^l_{mn} = (-1)^{m+n}\overline{c^l_{\bar{m}\bar{n}}}, \tag{4.3}$$

$$D^l_{mn}(\boldsymbol{R}) = (-1)^{m+n}\overline{D^l_{\bar{m}\bar{n}}(\boldsymbol{R})}. \tag{4.4}$$

When the crystallites of the polycrystal have no preferred orientations (in other words, the polycrystal is isotropic), we have

$$w := w_{\text{iso}} = \frac{1}{8\pi^2}. \tag{4.5}$$

Let us choose a Cartesian coordinate system defined by an orthonormal triad $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ so that the homogeneous thin sheet in question lies in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. The texture of the thin sheet is said to be transversely-isotropic about the sheet normal if the ODF w remains unchanged after the sheet undergoes a rotation $\boldsymbol{R}(\mathbf{e}_3, \varphi)$, i.e., a rotation about $\mathbf{e}_3$ by an angle φ, or $\boldsymbol{R}(\mathbf{e}_2, \pi)$ that flips $\mathbf{e}_3$ to $-\mathbf{e}_3$. We proceed to prove that w is transversely-isotropic about the sheet normal (i.e., the axis defined by $\mathbf{e}_3$) if it is of the form

$$w_{\text{tiso}} = \frac{1}{8\pi^2} + \sum_{l=1}^{\infty} \sum_{n=-l}^{l} c^l_{0n} D^l_{0n}(\boldsymbol{R}), \tag{4.6}$$

$$c^l_{0n} = 0 \qquad \text{for odd } l. \tag{4.7}$$

In order to prove (4.6) and (4.7), we first introduce some transformation formulae [11]. Let w and $\mathcal{T}_{\boldsymbol{Q}} w$ be the ODF of a polycrystal before and after the sample undergoes a rotation $\boldsymbol{Q}$. Then it is clear that $\mathcal{T}_{\boldsymbol{Q}} w$ and w are related by

$$\mathcal{T}_{\boldsymbol{Q}} w(\boldsymbol{R}) = w(\boldsymbol{Q}^T \boldsymbol{R}) \tag{4.8}$$

for each rotation $\boldsymbol{R}$. For rotations $\boldsymbol{Q}$ in the symmetry group of the macrotexture, we have

$$\mathcal{T}_{\boldsymbol{Q}} w(\boldsymbol{R}) = w(\boldsymbol{R}) \qquad \text{for each rotation } \boldsymbol{R}. \tag{4.9}$$

For notational convenience, let us put $\tilde{w} = \mathcal{T}_{\boldsymbol{Q}}$. Let $\tilde{c}^l_{mn}$ and c^l_{mn} be the texture coefficients pertaining to $\tilde{w}$ and w, respectively. By (4.1) and (4.9), we have

$$\begin{aligned}
\tilde{w}(\boldsymbol{R}) &= w(\boldsymbol{Q}^T\boldsymbol{R}) \\
&= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l} c^l_{mn} D^l_{mn}(\boldsymbol{Q}^T\boldsymbol{R}) \\
&= \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l} c^l_{mn} (\sum_{p=-l}^{l} D^l_{mp}(\boldsymbol{Q}^T) D^l_{pn}(\boldsymbol{R})) \\
&= \sum_{l=1}^{\infty}\sum_{p=-l}^{l}\sum_{n=-l}^{l} (\sum_{m=-l}^{l} c^l_{mn} D^l_{mp}(\boldsymbol{Q}^{-1})) D^l_{pn}(\boldsymbol{R}) \qquad (4.10)
\end{aligned}$$

Thus we obtain

$$\tilde{c}^l_{pn} = \sum_{m=-l}^{l} c^l_{mn} D^l_{mp}(\boldsymbol{Q}^{-1})), \qquad (4.11)$$

which by renaming indices is equivalent to

$$\tilde{c}^l_{mn} = \sum_{s=-l}^{l} c^l_{sn} D^l_{sm}(\boldsymbol{Q}^{-1})). \qquad (4.12)$$

If the polycrystal is transversely isotropic about z-axis, the elements of G_{tex}, the group of texture symmetry, are $\mathbf{I}$, $\boldsymbol{R}(\mathbf{e}_2, \pi)$, and $\boldsymbol{R}(\mathbf{e}_3, \phi)$ for $0 \leq \phi < 2\pi$. Transversely isotropy dictates that $\tilde{c}^l_{mn} = c^l_{mn}$ for $\boldsymbol{Q}^{-1}$ given by the Euler angle $(0, 0, \phi)$ and $(0, \pi, 0)$ and $(0, 0, \phi)$ for $0 \leq \phi < 2\pi$, respectively.

For $\boldsymbol{Q}^{-1}$ given by $(0, 0, \phi)$, we have

$$\begin{aligned}
c^l_{mn} &= \sum_{s=-l}^{l} c^l_{sn} D^l_{sm}((0, 0, \phi)) \\
&= \sum_{s=-l}^{l} c^l_{sn} d^l_{sm}(0) e^{-im\phi} \\
&= c^l_{mn} e^{-im\phi}, \qquad (4.13)
\end{aligned}$$

which holds for any angle ϕ. Hence we obtain $c^l_{mn} = 0$ if $m \neq 0$.

For $\boldsymbol{Q}^{-1}$ given by $(0, \pi, 0)$, we have

$$
\begin{aligned}
c^l_{mn} &= \sum_{s=-l}^{l} c^l_{sn} D^l_{sm}((0,\pi,0)) \\
&= \sum_{s=-l}^{l} c^l_{sn} d^l_{sm}(\pi) \\
&= \sum_{s=-l}^{l} c^l_{sn} (-1)^{l+s} d^l_{s\bar{m}}(0) \\
&= (-1)^{l-m} c^l_{\bar{m}n}. \qquad (4.14)
\end{aligned}
$$

When $m = 0$, we obtain

$$
c^l_{0n} = (-1)^l c^l_{0n} = \begin{cases} 0 & \text{for odd } l \\ c^l_{0n} & \text{for even } l \end{cases}. \qquad (4.15)
$$

Hence we have proved (4.6)and (4.7) for w_{tiso}.

The texture of a homogeneous thin sheet is said to be almost transversely-isotropic about the sheet normal if under a Cartesian coordinate system with the sheet normal being the z-axis, the ODF that defines the texture is of the form

$$
w = w_{\text{tiso}} + \sum_{l=1}^{\infty} \sum_{m \in J_l \setminus \{0\}} \sum_{n \in J_l} c^l_{mn} D^l_{mn}, \qquad (J_l = \{-l, \cdots, -1, 0, 1, \cdots, l\}) \qquad (4.16)
$$

where w_{tiso} is given by (4.6) and all texture coefficients c^l_{mn} with $l \geq 1$ and $m \neq 0$ are small in the sense that only terms linear in these coefficients need to be considered as far as their effects on material tensors are concerned.

4.2 Representation Theorem

In their paper [14], Man and Huang consider material tensors $\boldsymbol{H}$ in three-dimensional space which are smooth functions of the ODF w in an SO(3)-invariant neighborhood $\mathcal{N}$ of w_{iso} in

SO(3). Their main physical assumption is the constraint

$$\boldsymbol{H}(\mathcal{T}_{\boldsymbol{Q}} w)[\boldsymbol{Q}\mathbf{v}_1, \ldots, \boldsymbol{Q}\mathbf{v}_r] = \boldsymbol{H}(w)[\mathbf{v}_1, \ldots, \mathbf{v}_r] \tag{4.17}$$

for each $\boldsymbol{Q} \in \mathrm{SO}(3)$, each $w \in \mathcal{N}$, and any $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V^{(3)}$, where $V^{(3)}$ denotes the translation space of the three-dimensional Euclidean space. Constraint (4.17) is suggested [11, 19] by the general principles of material frame-indifference and isotropy of space. Paroni [18] proved this constraint in the context of stochastic homogenization by making the assumption that the orientation field of the crystallites in the polycrystalline aggregate be statistically independent.

In this dissertation we consider material tensors based on the two-dimensional translation space V of the sheet plane. To make use of constraint (4.17), we must restrict $\boldsymbol{Q}$ to $\boldsymbol{R}(\mathbf{e}_3, \varphi)$ and $\boldsymbol{R}(\mathbf{e}_1, \pi)$, which correspond to $\mathbf{R}(\varphi)$ and $\mathbf{M}_x$, respectively, in our theory based on O(2), so that vectors in V are transformed into vectors in V.

In what follows we assume that the thin sheet in question carries a texture which is almost transversely-isotropic about the sheet normal and is characterized by a specific ODF of the type given by (4.16). We restrict attention to values of the r-th order tensor $\boldsymbol{H}[\mathbf{v}_1, \ldots, \mathbf{v}_r]$, where $\mathbf{v}$ are in V (or in $V^{(3)}$ with zero 3-component). We assume that $\boldsymbol{H}(\cdot)$ is smooth in a neighborhood of w_{tiso}. Let $D\boldsymbol{H}(w_{\mathrm{tiso}})$ denote the Frechet derivative of $\boldsymbol{H}$ at w_{tiso}. We consider the case that the physical property characterized by $\boldsymbol{H}$ in the given polycrystalline aggregate is adequate to be replaced by its approximation at w_{tiso}, i.e.,

$$\boldsymbol{H}(w) = \boldsymbol{H}(w_{\mathrm{tiso}}) + D\boldsymbol{H}(w_{\mathrm{tiso}})[w - w_{\mathrm{tiso}}]. \tag{4.18}$$

For the moment, let us not put any restriction on any restriction on the three dimensional rotation $\boldsymbol{Q}$ (i.e., allowing $\boldsymbol{Q}\mathbf{v}_i$ to have a non-zero 3-component). It then follows from the assumptions on $\boldsymbol{H}$ that for each w, we have

$$\boldsymbol{H}(w)[\mathbf{v}_1, \ldots, \mathbf{v}_r] = (\boldsymbol{H}(w_{\text{tiso}}) + D\boldsymbol{H}(w_{\text{tiso}})[w - w_{\text{tiso}}])[\mathbf{v}_1, \ldots, \mathbf{v}_r], \tag{4.19}$$

and

$$\boldsymbol{H}(\mathcal{T}_{\boldsymbol{Q}} w)[\boldsymbol{Q}\mathbf{v}_1, \ldots, \boldsymbol{Q}\mathbf{v}_r] = (\boldsymbol{H}(\mathcal{T}_{\boldsymbol{Q}} w_{\text{tiso}}) + D\boldsymbol{H}(\mathcal{T}_{\boldsymbol{Q}} w_{\text{tiso}}))[\mathcal{T}_{\boldsymbol{Q}}(w - w_{\text{tiso}})][\boldsymbol{Q}\mathbf{v}_1, \ldots, \boldsymbol{Q}\mathbf{v}_r]. \tag{4.20}$$

By the assumption (4.17), we have

$$(\boldsymbol{H}(w_{\text{tiso}})+D\boldsymbol{H}(w_{\text{tiso}})[w-w_{\text{tiso}}])[\mathbf{v}_1, \ldots, \mathbf{v}_r] = (\boldsymbol{H}(\mathcal{T}_{\boldsymbol{Q}} w_{\text{tiso}})+D\boldsymbol{H}(\mathcal{T}_{\boldsymbol{Q}} w_{\text{tiso}})[\mathcal{T}_{\boldsymbol{Q}}(w-w_{\text{tiso}})])[\boldsymbol{Q}\mathbf{v}_1, \ldots, \boldsymbol{Q}\mathbf{v}_r]. \tag{4.21}$$

Let

$$\mathcal{G} := \{\boldsymbol{Q} \in \text{SO}(3) : \mathcal{T}_{\boldsymbol{Q}} w_{\text{tiso}} = w_{\text{tiso}}\}. \tag{4.22}$$

Then by (4.21), $D\boldsymbol{H}(w_{\text{tiso}})[w - w_{\text{tiso}}]$ satisfies

$$(D\boldsymbol{H}(w_{\text{tiso}})[\mathcal{T}_{\boldsymbol{Q}}(w - w_{\text{tiso}})])[\boldsymbol{Q}\mathbf{v}_1, \ldots, \boldsymbol{Q}\mathbf{v}_r] = (D\boldsymbol{H}(w_{\text{tiso}})[w - w_{\text{tiso}}])[\mathbf{v}_1, \ldots, \mathbf{v}_r] \tag{4.23}$$

for each $\boldsymbol{Q} \in \mathcal{G}$ and any $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$. We will write $\boldsymbol{H}' := D\boldsymbol{H}(w_{\text{tiso}})$. Thus we have the equivalent form

$$\boldsymbol{Q}^{\otimes r}\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \boldsymbol{H}'(\mathcal{T}_{\boldsymbol{Q}} w_{\text{tiso}})[\mathcal{T}_{\boldsymbol{Q}}(w - w_{\text{tiso}})] \tag{4.24}$$

for each $\boldsymbol{Q} \in \mathcal{G}$ and w.

In what follows, let

$$\mathcal{H}_0 = \{f \in L^2(\text{SO}(3), \mathbb{C}) : f = \sum_{l=1}^{\infty} \sum_{m \in J_l \setminus \{0\}} \sum_{n \in J_l} c^l_{mn} D^l_{mn}, \tag{4.25}$$

$$c^l_{mn} \in \mathbb{C}, \quad J_l = \{-l, \cdots, -1, 0, 1, \cdots, l\}\}. \tag{4.26}$$

Note that by (4.16) we have $w - w_{\text{tiso}} \in \mathcal{H}_0$ for each ODF w.

Now we introduce a representation theorem as follows.

Theorem 4.1. *Let $\mathbb{V}$ be the space of symmetric second-order tensors which is also denoted by $[V^2]$. For $k \geq 1$, let Z be a two-dimensional subspace of $\mathbb{V}^{\otimes r}$ invariant under the action of* $\mathrm{O}(2)$. *Let Z_c be its complexification. Let $\mathbb{R}$ and $\mathbb{M}_x$ be defined as in Section 3.3. Suppose the representation $\mathbf{R}(\varphi) \mapsto \mathbb{R}(\varphi)^{\otimes r}|Z_c$, $\mathbf{M}_x\mathbf{R}(\varphi) \mapsto \mathbb{M}_x^{\otimes r}\mathbb{R}(\varphi)^{\otimes r}|Z_c$ is equivalent to the irreducible unitary representation $\mathbb{D}^k$. Let $\boldsymbol{H}' : \mathcal{H}_0 \to Z$ be linear and satisfy (4.24) for each $\boldsymbol{Q} \in \mathcal{G}$ and w. The assertion below is valid:*

There exists an orthonormal basis $\boldsymbol{H}_k$ and $\boldsymbol{H}_{\bar{k}}$ in Z_c such that

$$\begin{gathered} \boldsymbol{H}_{\bar{k}} = \overline{\boldsymbol{H}_k} \\ \mathbb{R}(\varphi)^{\otimes r}\boldsymbol{H}_k = e^{ik\varphi}\boldsymbol{H}_k, \quad \mathbb{R}(\varphi)^{\otimes r}\boldsymbol{H}_{\bar{k}} = e^{-ik\varphi}\boldsymbol{H}_{\bar{k}} \\ \mathbb{M}_x^{\otimes r}\boldsymbol{H}_k = \boldsymbol{H}_{\bar{k}}, \quad \mathbb{M}_x^{\otimes r}\boldsymbol{H}_{\bar{k}} = \boldsymbol{H}_k \end{gathered} \tag{4.27}$$

where $\bar{k} = -k$, and

$$\begin{aligned} \boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \frac{1}{2}\sum_{l=1}^{\infty}\sum_{n=-l}^{l} \Big(& c_{kn}^{l}\beta_k(w_{\text{tiso}})[D_{kn}^{l}(\cdot)]\boldsymbol{H}_k + c_{\bar{k}n}^{l}\beta_{\bar{k}}(w_{\text{tiso}})[D_{\bar{k}n}^{l}(\cdot)]\boldsymbol{H}_{\bar{k}} \\ & + (-1)^{l+k}c_{\bar{k}n}^{l}\beta_{\bar{k}}(w_{\text{tiso}})[D_{\bar{k}n}^{l}(\cdot)]\boldsymbol{H}_k + (-1)^{l+k}c_{kn}^{l}\beta_k(w_{\text{tiso}})[D_{kn}^{l}(\cdot)]\boldsymbol{H}_{\bar{k}} \Big), \end{aligned} \tag{4.28}$$

where $\beta_k(w_{\text{tiso}})$ and $\beta_{\bar{k}}(w_{\text{tiso}})$ are complex-valued linear functionals which satisfy

$$\overline{\beta_k(w_{\text{tiso}})[f]} = \beta_{\bar{k}}(w_{\text{tiso}})[\overline{f}] \qquad \text{for each } f \in L^2(SO(3), \mathbb{C}). \tag{4.29}$$

Proof. A procedure to construct orthonormal basis tensors $\boldsymbol{H}_k$ and $\boldsymbol{H}_{\bar{k}}$ in Z_c which satisfy (4.27) has already be given in Section 3.3.

Since $\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}]$ is in the two-dimensional invariant subspace Z_c under O(2), we can write

$$\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \beta_k(w_{\text{tiso}})[w - w_{\text{tiso}}]\boldsymbol{H}_k + \beta_{\bar{k}}(w_{\text{tiso}})[w - w_{\text{tiso}}]\boldsymbol{H}_{\bar{k}}, \tag{4.30}$$

for some complex-valued linear functionals $\beta_k, \beta_{\bar{k}}$ defined on $\mathcal{H}_0$. We proceed to prove that they satisfy

$$\overline{\beta_k(w_{\text{tiso}})[f]} = \beta_{\bar{k}}(w_{\text{tiso}})[\overline{f}] \qquad \text{for each } f \in \mathcal{H}_0. \tag{4.31}$$

To prove 4.31, we let $\mathcal{H}_0^c$ be the complexifications of $\mathcal{H}_0$. We extend the $\boldsymbol{H}' : \mathcal{H}_0 \to Z$ to the linear mapping from $\mathcal{H}_0^c$ to Z_c, which will still be denoted as $\boldsymbol{H}'$, defined by

$$\boldsymbol{H}'[g + ih] = \boldsymbol{H}'[g] + i\boldsymbol{H}'[h] \tag{4.32}$$

for each g and h in $\mathcal{H}_0$. $\boldsymbol{H}'$ still satisfies condition (4.24) after the extension.

Since $\overline{\boldsymbol{H}_k} = \boldsymbol{H}_{\bar{k}}$ and by the linearity of β_k and $\beta_{\bar{k}}$, we observe that

$$\begin{aligned}
\overline{\boldsymbol{H}'(w_{\text{tiso}})[g + ih]} &= \overline{\beta_k(w_{\text{tiso}})[g + ih]\boldsymbol{H}_k + \beta_{\bar{k}}(w_{\text{tiso}})[g + ih]\boldsymbol{H}_{\bar{k}}}, \\
&= \overline{\beta_k(w_{\text{tiso}})[g + ih]\boldsymbol{H}_k} + \overline{\beta_{\bar{k}}(w_{\text{tiso}})[g + ih]\boldsymbol{H}_{\bar{k}}} \\
&= \overline{\beta_k(w_{\text{tiso}})[g] + i\beta_k(w_{\text{tiso}})[h]}\boldsymbol{H}_{\bar{k}} + \overline{\beta_{\bar{k}}(w_{\text{tiso}})[g] + i\beta_{\bar{k}}(w_{\text{tiso}})[h]}\boldsymbol{H}_k \\
&= (\overline{\beta_k(w_{\text{tiso}})[g]} - i\overline{\beta_k(w_{\text{tiso}})[h]})\boldsymbol{H}_{\bar{k}} + (\overline{\beta_{\bar{k}}(w_{\text{tiso}})[g]} - i\overline{\beta_{\bar{k}}(w_{\text{tiso}})[h]})\boldsymbol{H}_k
\end{aligned} \tag{4.33}$$

On the other hand, we observe that

$$\begin{aligned}
\overline{\boldsymbol{H}'(w_{\text{tiso}})[g + ih]} &= \overline{\boldsymbol{H}'(w_{\text{tiso}})[g]} - i\overline{\boldsymbol{H}'(w_{\text{tiso}})[h]} \\
&= \boldsymbol{H}'(w_{\text{tiso}})[g] - i\boldsymbol{H}'(w_{\text{tiso}})[h] \\
&= \{\beta_k(w_{\text{tiso}})[g]\boldsymbol{H}_k + \beta_{\bar{k}}(w_{\text{tiso}})[h]\boldsymbol{H}_{\bar{k}}\} - i\{\beta_k(w_{\text{tiso}})[h]\boldsymbol{H}_k + \beta_{\bar{k}}(w_{\text{tiso}})[h]\boldsymbol{H}_{\bar{k}}\} \\
&= (\beta_k(w_{\text{tiso}})[g] - i\beta_k(w_{\text{tiso}})[h])\boldsymbol{H}_k + (\beta_{\bar{k}}(w_{\text{tiso}})[g] - i\beta_{\bar{k}}(w_{\text{tiso}})[h])\boldsymbol{H}_{\bar{k}}
\end{aligned} \tag{4.34}$$

Comparing the coefficients of $\boldsymbol{H}_k$ and $\boldsymbol{H}_{\bar{k}}$ respectively in (4.33) and (4.34), we have

$$\overline{\beta_{\bar{k}}(w_{\text{tiso}})[g+ih]} = \beta_k(w_{\text{tiso}})[g-ih], \quad \overline{\beta_k(w_{\text{tiso}})[g+ih]} = \beta_{\bar{k}}(w_{\text{tiso}})[g-ih], \tag{4.35}$$

i.e.,

$$\overline{\beta_{\bar{k}}[f]} = \beta_k[\overline{f}], \qquad \overline{\beta_k[f]} = \beta_{\bar{k}}[\overline{f}] \tag{4.36}$$

for each $f \in \mathcal{H}_0$.

From our basic assumption, there holds

$$\begin{aligned}\mathbf{Q}^{\otimes r}(\boldsymbol{H}'(w_{\text{tiso}})[w-w_{\text{tiso}}]) = \beta_k(\mathcal{T}_{\boldsymbol{Q}}w_{\text{tiso}})[\mathcal{T}_{\mathbb{Q}}(w-w_{\text{tiso}})]\boldsymbol{H}_k \\ + \beta_{\bar{k}}(\mathcal{T}_{\mathbb{Q}}w_{\text{tiso}})[\mathcal{T}_{\mathbb{Q}}(w-w_{\text{tiso}})]\boldsymbol{H}_{\bar{k}}.\end{aligned} \tag{4.37}$$

In what follows, assume that $\boldsymbol{Q} = \boldsymbol{R}(\boldsymbol{e}_2, \pi)$ or $\boldsymbol{Q} = \boldsymbol{R}(\boldsymbol{e}_3, \varphi)$, where φ is arbitrary. For these rotations, we have

$$\mathcal{T}_{\boldsymbol{Q}}w_{\text{tiso}} = w_{\text{tiso}}. \tag{4.38}$$

Substituting (4.38) and

$$\begin{aligned}\mathcal{T}_{\boldsymbol{Q}}(w(\boldsymbol{R}) - w_{\text{tiso}}) &= \sum_{l=1}^{\infty}\sum_{m\neq 0}\sum_{n=-l}^{l} c^l_{mn} D^l_{mn}(\boldsymbol{Q}^T\boldsymbol{R}) \\ &= \sum_{l=1}^{\infty}\sum_{m\neq 0}\sum_{n=-l}^{l} c^l_{mn}\left(\sum_{s=-l}^{l} D^l_{ms}(\boldsymbol{Q}^T)D^l_{sn}(\boldsymbol{R})\right)\end{aligned} \tag{4.39}$$

into equation (4.37) and multiplying both sides of the equation on the left by $(\boldsymbol{Q}^T)^{\otimes r}$, we obtain the following two cases:

Case 1: $\boldsymbol{Q} = \boldsymbol{R}(\boldsymbol{e}_3, \varphi)$. The Euler angles for $\boldsymbol{Q}$ is $(\psi, \theta, \phi) = (\varphi, 0, 0)$. Then $D^l_{ms}(\boldsymbol{Q}^T) =$

$e^{im\varphi} d^l_{ms}(0) = e^{im\varphi}\delta_{ms}$. We have

$$
\begin{aligned}
\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \sum_{l=1}^{\infty}\sum_{m\neq 0}\sum_{n,s=-l}^{l} \Big(& c^l_{mn} e^{im\varphi}\delta_{ms}\beta_k(w_{\text{tiso}})[D^l_{sn}(\cdot)]\,((\boldsymbol{R}(-\varphi))^{\otimes r}\boldsymbol{H}_k) \\
&+ c^l_{mn} e^{im\varphi}\delta_{ms}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{sn}(\cdot)]\,((\boldsymbol{R}(-\varphi))^{\otimes r}\boldsymbol{H}_{\bar{k}})\Big) \\
= \sum_{l=1}^{\infty}\sum_{m\neq 0}\sum_{n=-l}^{l} \Big(& c^l_{mn} e^{im\varphi}\beta_k(w_{\text{tiso}})[D^l_{mn}(\cdot)] e^{-ik\varphi}\boldsymbol{H}_k \\
&+ c^l_{mn} e^{im\varphi}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{mn}(\cdot)] e^{ik\varphi}\boldsymbol{H}_{\bar{k}}\Big). \qquad (4.40)
\end{aligned}
$$

Integrating both sides of (4.40) with respect to φ from 0 to 2π, we obtain

$$
\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \sum_{l=1}^{\infty}\sum_{n=-l}^{l}\Big(c^l_{kn}\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)]\boldsymbol{H}_k + c^l_{\bar{k}n}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\boldsymbol{H}_{\bar{k}}\Big). \qquad (4.41)
$$

Case 2: $\boldsymbol{Q} = \boldsymbol{R}(\boldsymbol{e}_2, \pi)\boldsymbol{R}(\boldsymbol{e}_3, \varphi)$. The Euler angles for $\boldsymbol{R}(\boldsymbol{e}_2, \pi)$ is $(\psi, \theta, \phi) = (0, \pi, 0)$. Note that $\boldsymbol{Q}^T = \boldsymbol{Q}$, and

$$
\begin{aligned}
D^l_{ms}(\boldsymbol{Q}^T) &= \sum_{p=-l}^{l} D^l_{mp}(\boldsymbol{R}(\boldsymbol{e}_2, \pi) D^l_{ps}(\boldsymbol{R}(\boldsymbol{e}_3, \varphi)) \\
&= \sum_{p=-l}^{l} d^l_{mp}(\pi) e^{-ip\varphi} d^l_{ps}(0) \\
&= \sum_{p=-l}^{l} (-1)^{l+m}\delta_{m\bar{p}}\, e^{-ip\varphi}\delta_{ps} \\
&= (-1)^{l+m} e^{-is\varphi}\delta_{m\bar{s}}. \qquad (4.42)
\end{aligned}
$$

We have

$$
\begin{aligned}
\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \sum_{l=1}^{\infty}\sum_{m\neq 0}\sum_{n,s=-l}^{l} \Big(& c^l_{mn}(-1)^{l+m} e^{-is\varphi}\delta_{m\bar{s}}\beta_k(w_{\text{tiso}})[D^l_{sn}(\cdot)]\Big((\mathbb{M}_y\boldsymbol{R}(\varphi))^{\otimes r}\boldsymbol{H}_k\Big) \\
&+ c^l_{mn}(-1)^{l+m} e^{-is\varphi}\delta_{m\bar{s}}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{sn}(\cdot)]\Big((\mathbb{M}_y\boldsymbol{R}(\varphi))^{\otimes r}\boldsymbol{H}_{\bar{k}}\Big)\Big) \\
= \sum_{l=1}^{\infty}\sum_{m\neq 0}\sum_{n=-l}^{l} \Big(& c^l_{mn}(-1)^{l+m} e^{im\varphi}\beta_k(w_{\text{tiso}})[D^l_{\bar{m}n}(\cdot)] e^{ik\varphi}\boldsymbol{H}_{\bar{k}} \\
&+ c^l_{mn}(-1)^{l+m} e^{im\varphi}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{m}n}(\cdot)] e^{-ik\varphi}\boldsymbol{H}_k\Big). \qquad (4.43)
\end{aligned}
$$

Integrating both sides of (4.43) with respect to φ from 0 to 2π, we obtain

$$\boldsymbol{H}'(w_{\text{tiso}})[w-w_{\text{tiso}}] = \sum_{l=1}^{\infty}\sum_{n=-l}^{l}\Big((-1)^{l+k}c^l_{\bar{k}n}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\boldsymbol{H}_k + \ (-1)^{l+k}c^l_{kn}\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)]\boldsymbol{H}_{\bar{k}}\Big). \tag{4.44}$$

Gathering (4.35) and (4.38), we can write the representation formula as

$$\begin{aligned}\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \frac{1}{2}\sum_{l=1}^{\infty}\sum_{n=-l}^{l}\Big(&c^l_{kn}\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)]\boldsymbol{H}_k + \ c^l_{\bar{k}n}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\boldsymbol{H}_{\bar{k}} \\ &+ (-1)^{l+k}c^l_{\bar{k}n}\beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\boldsymbol{H}_k + \ (-1)^{l+k}c^l_{kn}\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)]\boldsymbol{H}_{\bar{k}}\Big).\end{aligned} \tag{4.45}$$

□

If the texture of the sheet is monoclinic with $\boldsymbol{R}(\boldsymbol{e}_2, \pi) \in G_{\text{tex}}$, the group of texture symmetry, then the texture coefficients satisfy the requirement that

$$c^l_{mn} = (-1)^{l+m}c^l_{\bar{m}n}, \tag{4.46}$$

and representation formula (4.28) assumes the simpler form

$$\begin{aligned}\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] &= \frac{1}{2}\sum_{l=1}^{\infty}\sum_{n=-l}^{l}\Big(c^l_{kn}\left(\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)] + \beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\right)\boldsymbol{H}_k \\ &\qquad + \ c^l_{\bar{k}n}\left(\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)] + \beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\right)\boldsymbol{H}_{\bar{k}}\Big) \\ &= \sum_{l=1}^{\infty}\sum_{n=-l}^{l}\alpha_{kn}\left(c^l_{kn}\boldsymbol{H}_k + c^l_{\bar{k}n}\boldsymbol{H}_{\bar{k}}\right),\end{aligned} \tag{4.47}$$

where

$$\alpha_{kn} = \frac{1}{2}\left(\beta_k(w_{\text{tiso}})[D^l_{kn}(\cdot)] + \beta_{\bar{k}}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\right) \tag{4.48}$$

are material constants.

Applying the preceding theorem to each of the non-trivial irreducible two-dimensional subspaces of the decomposition

$$Z_c = n_0\mathbb{D}^0 + n_{0'}\mathbb{D}^{0'} + \sum_{k\geq 1} n_k\mathbb{D}^k,$$

we obtain the following representation theorem on $\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}]$.

Theorem 4.2. *Let $Z \subset \mathbb{V}^{\otimes r}$ be a subspace invariant under the action of* O(2)*. Let its complexification have the decomposition $Z_c = n_0\mathbb{D}^0 + n_{0'}\mathbb{D}^{0'} + \sum_{k\geq 1} n_k\mathbb{D}^k$. Let tensor function $\boldsymbol{H}' : \mathcal{H}_0 \to Z$ be linear and satisfy (4.24) for each rotation $\boldsymbol{Q} \in \mathcal{G}$ and* ODF *w. For each k in $J := \{j : n_j \geqslant 1, j \neq 0, 0'\}$ and $1 \leqslant s \leqslant n_k$, the assertion below is valid:*

There exists an orthonormal basis $\boldsymbol{H}_{k,s}$ in Z_c such that

$$\begin{aligned} \boldsymbol{H}_{\bar{k},s} &= \overline{\boldsymbol{H}_{k,s}} \\ \mathbb{R}(\varphi)^{\otimes r}\boldsymbol{H}_{k,s} = e^{ik\varphi}\boldsymbol{H}_{k,s}, &\quad \mathbb{R}(\varphi)^{\otimes r}\boldsymbol{H}_{\bar{k},s} = e^{-ik\varphi}\boldsymbol{H}_{\bar{k},s} \\ \mathbb{M}_x^{\otimes r}\boldsymbol{H}_{k,s} = \boldsymbol{H}_{\bar{k},s}, &\quad \mathbb{M}_x^{\otimes r}\boldsymbol{H}_{\bar{k},s} = \boldsymbol{H}_{k,s} \end{aligned} \tag{4.49}$$

where $\bar{k} = -k$, and

$$\begin{aligned} \boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \frac{1}{2}\sum_{k\in J}\sum_{s=1}^{n_k}\sum_{l=1}^{\infty}\sum_{n=-l}^{l}\Big(& c^l_{kn}\beta_{k,s}(w_{\text{tiso}})[D^l_{kn}(\cdot)]\boldsymbol{H}_{k,s} + c^l_{\bar{k}n}\beta_{\bar{k},s}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\boldsymbol{H}_{\bar{k},s} \\ &+ c^l_{kn}\beta_{\bar{k},s}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\boldsymbol{H}_{k,s} + c^l_{\bar{k}n}\beta_{k,s}(w_{\text{tiso}})[D^l_{kn}(\cdot)]\boldsymbol{H}_{\bar{k},s}\Big). \end{aligned} \tag{4.50}$$

If the texture of the sheet is monoclinic with $\boldsymbol{R}(\boldsymbol{e}_2, \pi) \in G_{\text{tex}}$, the group of texture symmetry, then the texture coefficients still satisfy the requirement (4.46), and representation formula (4.50) assumes the simpler form

$$\boldsymbol{H}'(w_{\text{tiso}})[w - w_{\text{tiso}}] = \sum_{k\in J}\sum_{s=1}^{n_k}\sum_{l=1}^{\infty}\sum_{n=-l}^{l}\alpha_{kn,s}\left(c^l_{kn}\boldsymbol{H}_{k,s} + c^l_{\bar{k}n}\boldsymbol{H}_{\bar{k},s}\right), \tag{4.51}$$

where

$$\alpha_{kn,s} = \frac{1}{2}\left(\beta_{k,s}(w_{\text{tiso}})[D^l_{kn}(\cdot)] + \beta_{\bar{k},s}(w_{\text{tiso}})[D^l_{\bar{k}n}(\cdot)]\right) \tag{4.52}$$

are material constants.

Theorem 4.1 and Theorem 4.2, while phrased in the context of Z_c being a subspace of $\mathbb{V}_c^{\otimes r} = [V_c^2]^{\otimes r}$ remain valid for general tensor subspaces of $V_c^{\otimes r}$, with obvious modifications such as replacing $\mathbb{R}(\varphi)$ and $\mathbb{M}_x$ by $\mathbf{R}(\varphi)$ and $\mathbf{M}_x$, respectively. The proofs are also essentially the same. If $Z_c \subset [V_c^2]^{\otimes r}$, then it is advantageous to use $\mathbb{V}$ rather than V as the base vector space because a tensor of order $2r$ in $V^{\otimes 2r}$ becomes a tensor of order r in $\mathbb{V}^{\otimes r}$ and the Kelvin notation can be used instead of the more cumbersome standard tensor notation.

Chapter 5 Closing Remarks

In this dissertation we have derived a representation formula for material tensors that pertain to textured thin sheets or thin films which carry weak planar anisotropy but possibly strong normal anisotropy. We model the texture of such sheets as almost transversely-isotropic about the sheet normal. The most glaring difference between our representation formula and that of Man and Huang [14] on almost isotropic polycrystals is that Man and Huang's formula is a finite sum which involves texture coefficients of order l no higher than the rank of the tensor in question, while the formula presented herein is an infinite series that involves texture coefficients of all orders $l \geq 1$. Nevertheless, the texture coefficients c^l_{mn} are themselves expansion coefficients of a convergent infinite series of Wigner D-functions. In some applications (e.g., plastic anisotropy of steel sheets), it has been reported that truncating the series expansion of the ODF at $l = 8$ or $l = 10$ would suffice.

A comparison of representation formula (4.50) and its counterpart (4.51) in the presence of monoclinic texture indicates that the representation formula will be significantly simplified in the presence of higher texture and crystal symmetries. For future work, specialized versions of the formula should be worked out for common cases such as orthorhombic aggregates of cubic or hexagonal crystallites. Such specialized representation formulas should be used in applications, e.g., to derive the angular dependence of the r-value in anisotropic plasticity and compare the findings with those available in [12] where the sheet metal in question is assumed to be almost isotropic.

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