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HILBERT BASES, DESCENT STATISTICS, AND COMBINATORIAL SEMIGROUP ALGEBRAS

McCabe J. Olsen

University of Kentucky, mccabe.olsen.math@gmail.com

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McCabe J. Olsen, Student

Dr. Benjamin J. Braun, Major Professor

Dr. Peter D. Hislop, Director of Graduate Studies

HILBERT BASES, DESCENT STATISTICS, AND COMBINATORIAL SEMIGROUP
ALGEBRAS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
McCabe James Olsen
Lexington, Kentucky

Director: Dr. Benjamin J. Braun, Professor of Mathematics
Lexington, Kentucky

2018

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ABSTRACT OF DISSERTATION

HILBERT BASES, DESCENT STATISTICS, AND COMBINATORIAL SEMIGROUP ALGEBRAS

The broad topic of this dissertation is the study of algebraic structure arising from polyhedral geometric objects. There are three distinct topics covered over three main chapters. However, each of these topics are further linked by a connection to the Eulerian polynomials.

Chapter 2 studies Euler-Mahonian identities arising from both the symmetric group and generalized permutation groups. Specifically, we study the algebraic structure of unit cube semigroup algebra using Gröbner basis methods to acquire these identities. Moreover, this serves as a bridge between previous methods involving polyhedral geometry and triangulations with descent bases methods arising in representation theory.

In Chapter 3, the aim is to characterize Hilbert basis elements of certain \mathbf{s} -lecture hall cones. In particular, the main focus is the classification of the Hilbert bases for the $1 \bmod k$ cones and the ℓ -sequence cones, both of which generalize a previous known result. Additionally, there is much broader characterization of Hilbert bases in dimension ≤ 4 for \mathbf{u} -generated Gorenstein lecture hall cones.

Finally, Chapter 4 focuses on certain algebraic and geometric properties of \mathbf{s} -lecture hall polytopes. This consists of partial classification results for the Gorenstein property, the integer-decomposition property, and the existence of regular, unimodular triangulations.

KEYWORDS: Ehrhart theory, lecture hall partitions, descent statistics, major index statistics, Gröbner basis, Hilbert basis

Author's signature: McCabe James Olsen

Date: April 26, 2018

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By
McCabe James Olsen

Director of Dissertation: Benjamin J. Braun

Director of Graduate Studies: Peter D. Hislop

Date: April 26, 2018

For Rachel

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Chapter 1 Introduction and Background

1.1 Introduction

This dissertation consists of four main sections. Chapter 1 is a collection of preliminaries and background information; it contains no original research or results. Chapter 2 is an investigation of certain bivariate generating functions, called *Euler-Mahonian identities* which arise from statistics on permutations and generalized permutations. These results are essentially those in the paper “Euler-Mahonian statistics and descent bases for semigroup algebras” [13], appearing in the *European Journal of Combinatorics*, which is joint with Benjamin Braun. Chapter 3 gives characterization of the Hilbert basis elements of certain *s-lecture hall cones*, namely the 1 mod k cones and the ℓ -sequence cones, as well as provides more general classifications in lower dimensions for the Gorenstein lecture hall cones. The results in this chapter are those found in the paper “Hilbert bases and lecture hall partitions” [35], which will appear in *The Ramanujan Journal*. Chapter 4 studies various algebraic and geometric properties of *s-lecture hall polytopes*, specifically providing partial results for the Gorenstein property and the integer-decomposition property. These results are those contained in the paper “Gorenstein properties and integer decomposition properties of lecture hall polytopes” [30], which is joint work with Takayuki Hibi and Akiyoshi Tsuchiya that will appear in *Moscow Mathematical Journal*.

At first glance, it may appear that these individual topics are somewhat disconnected. Specifically, Chapter 2, dealing with permutation statistics, seems to be quite distinct from Chapters 3 and 4 which deal with lecture hall partitions. However, these topics are linked two ways. The first is that all of the topics rely on the algebraic study of polyhedral objects. The second, and more surprising connection, is the central focus of the *Eulerian polynomial* throughout. This polynomial arises in Chapter 2 by considering a bivariate generalization and in both Chapters 3 and 4 by the role of a different family of generalizations known as *s-Eulerian polynomials* in studying lecture hall polytopes and lecture hall cones.

Each chapter to follow will begin with a brief description of the problem and background, as well as provide relevant terminology and notation. Subsequently, the material of Chapter 1 will occasionally be reintroduced in later chapters, though the notation will remain consistent throughout.

For the results which have already been published in peer-reviewed journals, all necessary permissions to reuse the work within this dissertation have been obtained.

We begin the discussion of background material with the symmetric group and permutation statistics.

1.2 The symmetric group and permutation statistics

Let $[n] = \{1, 2, \dots, n\}$ denote the n -set. The *symmetric group on n elements*, which we denote S_n , consists of all permutations of $[n]$ with the group operation of composition. We will refer to elements of S_n using *one-line notation*, that is give $\pi \in S_n$, we write $\pi = \pi_1 \pi_2 \dots \pi_n$ where π is permutation which maps $i \mapsto \pi_i$ for each $i \in [n]$. To emphasis the

group operation, given $\pi, \sigma \in S_n$ with $\pi = \pi_1\pi_2 \dots \pi_n$ and $\sigma = \sigma_1\sigma_2 \dots \sigma_n$, the permutation $\sigma \circ \pi = \sigma_{\pi_1}\sigma_{\pi_2} \dots \sigma_{\pi_n}$.

Example 1.2.1. Consider $\pi, \sigma \in S_5$ where $\pi = 15234$ and $\sigma = 41253$. The permutation π maps the elements of $[5]$ as follows:

$$1 \xrightarrow{\pi} 1, \quad 2 \xrightarrow{\pi} 5, \quad 3 \xrightarrow{\pi} 2, \quad 4 \xrightarrow{\pi} 3, \quad 5 \xrightarrow{\pi} 4.$$

The permutation σ maps the element of $[5]$ as follows:

$$1 \xrightarrow{\sigma} 4, \quad 2 \xrightarrow{\sigma} 1, \quad 3 \xrightarrow{\sigma} 2, \quad 4 \xrightarrow{\sigma} 5, \quad 5 \xrightarrow{\sigma} 3.$$

The permutation given by the composition $\sigma \circ \pi = \sigma_1\sigma_5\sigma_2\sigma_3\sigma_4 = 43125$, which we can see as

$$1 \xrightarrow{\pi} 1 \xrightarrow{\sigma} 4, \quad 2 \xrightarrow{\pi} 5 \xrightarrow{\sigma} 3, \quad 3 \xrightarrow{\pi} 2 \xrightarrow{\sigma} 1, \quad 4 \xrightarrow{\pi} 3 \xrightarrow{\sigma} 2, \quad 5 \xrightarrow{\pi} 4 \xrightarrow{\sigma} 5.$$

We are interested in the study of permutation statistics. A *permutation statistic* is a function $f : S_n \rightarrow \mathbb{Z}_{\geq 0}$ which encodes combinatorial data of the permutation. There are many examples of permutation statistics (c.f. [49, Chapter 1]), but we will be concerned primarily with statistics are related to descents in permutations. Given a permutation $\pi \in S_n$, the *descent set* of π is

$$\text{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\} \subseteq [n - 1],$$

and we say that i is a *descent* on π . The *descent statistic* is defined $\text{des}(\pi) = \#\text{Des}(\pi)$. An additional permutation statistic related to the descents of a permutation is the *major index*¹, denoted $\text{maj}(\pi)$, which is defined to be

$$\text{maj}(\pi) = \sum_{i \in \text{Des}(\pi)} i.$$

To demonstrate these statistics, please see the Example 1.2.2 and Table 1.1.

Example 1.2.2. Let $\pi = 6125374 \in S_7$. Note that $\text{Des}(\pi) = \{1, 4, 6\}$, as $6 > 1 < 2 < 5 > 3 < 7 > 4$. Subsequently, we have $\text{des}(\pi) = 3$ and $\text{maj}(\pi) = 1 + 4 + 6 = 11$.

It is easy to see that $|S_n| = n!$. However, it is often of interest and use to refine the enumeration of permutations by way of permutation statistics. One example of stratifying elements of S_n is using the descent statistic via the Eulerian polynomial. The n -th *Eulerian polynomial* $A_n(t)$ is given as follows

$$A_n(t) = \sum_{\pi \in S_n} t^{\text{des}(\pi)}. \tag{1.1}$$

The polynomial satisfies the generating function

$$\sum_{k \geq 0} (k + 1)^n t^k = \frac{A_n(t)}{(1 - t)^{n+1}}, \tag{1.2}$$

which was first studied by Euler [22].

¹This statistic is named for Major Percy A. MacMahon, who initiated its study. In the original work of MacMahon (e.g. [33]), the statistic is referred to as the “greater index.”

Table 1.1: Descent and major index statistics for S_3

π	$\text{Des}(\pi)$	$\text{des}(\pi)$	$\text{maj}(\pi)$
123	\emptyset	0	0
132	$\{2\}$	1	2
213	$\{1\}$	1	1
231	$\{2\}$	1	2
312	$\{1\}$	1	1
321	$\{1, 2\}$	2	3

Example 1.2.3. Let $n = 3$. By the data given in Table 1.1, we can see that $A_3(t) = 1 + 4t + t^2$ and subsequently, we have the identity

$$\sum_{k \geq 0} (k+1)^3 t^k = \frac{1 + 4t + t^2}{(1-t)^3}.$$

We say that a permutation statistic $\text{stat}_1 : S_n \rightarrow \mathbb{Z}_{\geq 0}$ is *Eulerian* if it has the same distribution over S_n the descent statistic, that is if

$$\sum_{\pi \in S_n} t^{\text{stat}_1(\pi)} = \sum_{\pi \in S_n} t^{\text{des}(\pi)}.$$

Likewise, we say that a statistic $\text{stat}_2 : S_n \rightarrow \mathbb{Z}_{\geq 0}$ is a *Mahonian* statistic if

$$\sum_{\pi \in S_n} t^{\text{stat}_2(\pi)} = \sum_{\pi \in S_n} t^{\text{maj}(\pi)}.$$

Furthermore, it is occasionally of interest to consider joint distributions of statistics. We say that a pair of statistics $(\text{stat}_1, \text{stat}_2)$ are *Euler-Mahonian statistics* if they satisfy the same joint distribution as the descent statistic and major index, that is if

$$\sum_{\pi \in S_n} t^{\text{stat}_1(\pi)} q^{\text{stat}_2(\pi)} = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}.$$

We call the polynomial given on the right-hand side the *Euler-Mahonian distribution polynomial*. In a similar manner to the classical Eulerian polynomial, this polynomial satisfies a generating function identity. The identity, in the given form, is due to Carlitz [16], though with some effort one can derive it from the works of MacMahon [33, Volume 2, Chapter IV, §462].

Theorem 1.2.4 (Carlitz, [16]). *For all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1-tq^j)}$$

where $[k+1]_q = 1 + q + q^2 + \dots + q^k$.

Example 1.2.5. Let $n = 3$. Using Table 1.1, we find that the Euler-Mahonian distribution polynomial is $1 + 2tq + 2tq^2 + t^2q^3$ and we have the generating function identity

$$\sum_{k \geq 0} [k+1]_q^3 t^k = \frac{1 + 2tq + 2tq^2 + t^2q^3}{(1-t)(1-tq)(1-tq^2)(1-tq^3)}.$$

1.3 Gröbner basis theory

Let $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ denote a polynomial ring in n variables over the complex numbers. Let $I \subset S$ be an ideal. A common question which arises is the ideal membership problem. That is, given a polynomial $f \in S$, is $f \in I$? Though a simple question to state, it is not always an easy question to answer; one tool for approaching these questions is using the theory of Gröbner bases.

Example 1.3.1. Let $S = \mathbb{C}[w, x, y, z]$, let $I = \langle xy - w^2, yw - x^2 \rangle$, and let $f = x^3 - w^3$ and $g = xy^2 - wz^2$. It is easy to see that $f \in I$ as

$$x^3 - w^3 = w(xy - w^2) - x(yw - x^2) \in \langle xy - w^2, yw - x^2 \rangle.$$

However, it is not as obvious that $g \notin I$.

In order to define a Gröbner basis it is necessary to introduce the notion of a *term order* for monomials. For the sake of notation, if we have $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, we denote $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Definition 1.3.2 (c.f. [17, Section 2.2]). A *term ordering* $<_{\text{mon}}$ on S is any relation on the set of monomials \mathbf{x}^α , with $\alpha \in \mathbb{Z}_{\geq 0}^n$, which satisfies

- (i) $<_{\text{mon}}$ is a total ordering on $\mathbb{Z}_{\geq 0}^n$.
- (ii) If $\beta <_{\text{mon}} \alpha$ and $\gamma \in \mathbb{Z}_{\geq 0}^n$, then $\beta + \gamma <_{\text{mon}} \alpha + \gamma$.
- (iii) $<_{\text{mon}}$ is a well-ordering on $\mathbb{Z}_{\geq 0}^n$.

We say that $\mathbf{x}^\beta <_{\text{mon}} \mathbf{x}^\alpha$ if $\beta <_{\text{mon}} \alpha$.

There are many examples of term orderings on S , which can often result in starkly different orderings for given monomials. Two such examples of orderings are *lexicographic ordering* (*lex*) and *graded reverse lexicographic ordering* (*grevlex*).

Definition 1.3.3 (Lexicographic Ordering). Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ with $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$. We say $\beta <_{\text{lex}} \alpha$ if the leftmost nonzero entry in the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We write $\mathbf{x}^\beta <_{\text{lex}} \mathbf{x}^\alpha$ if $\beta <_{\text{lex}} \alpha$.

Definition 1.3.4 (Graded reverse lexicographic ordering). Let $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$. We say that $\beta <_{\text{grevlex}} \alpha$ if

$$|\beta| = \sum_{i=1}^n \beta_i < |\alpha| = \sum_{i=1}^n \alpha_i,$$

or $|\alpha| = |\beta|$ and the rightmost nonzero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative. We write $\mathbf{x}^\beta <_{\text{grevlex}} \mathbf{x}^\alpha$ if $\beta <_{\text{grevlex}} \alpha$.

Given a nonzero polynomial $f = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha} \in S$ and some monomial order $<_{\text{mon}}$, we say that the *multidegree* of f is

$$\text{multideg}(f) = \max_{<_{\text{mon}}} \{ \alpha \in \mathbb{Z}_{\geq 0}^n : c_{\alpha} \neq 0 \},$$

the *leading coefficient* of f is

$$\text{LC}(f) = c_{\text{multideg}(f)} \in \mathbb{C},$$

the *leading monomial* of f is

$$\text{LM}(f) = \mathbf{x}^{\text{multideg}(f)},$$

and the *leading term* of f is

$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

Example 1.3.5. Consider $f = 3x_1^2x_2x_5 + 2x_2^3x_4 - x_4^5 \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$. If we consider the lexicographic ordering, we have that $x_4^5 <_{\text{lex}} x_2^3x_4 <_{\text{lex}} x_1^2x_2x_5$ because the exponent vectors $(0, 0, 0, 5, 0) <_{\text{lex}} (0, 3, 0, 1, 0) <_{\text{lex}} (2, 1, 0, 0, 1)$. Thus $\text{multideg}(f) = (2, 1, 0, 0, 1)$, $\text{LC}(f) = 3$, $\text{LM}(f) = x_1^2x_2x_5$, and $\text{LT}(f) = 3x_1^2x_2x_5$.

Alternatively, the graded lexicographic ordering gives $x_1^2x_2x_5 <_{\text{grevlex}} x_2^3x_4 <_{\text{grevlex}} x_4^5$ because we have $(2, 1, 0, 0, 1) <_{\text{grevlex}} (0, 3, 0, 1, 0) <_{\text{grevlex}} (0, 0, 0, 5, 0)$. Then we have $\text{multideg}(f) = (0, 0, 0, 5, 0)$, $\text{LC}(f) = -1$, $\text{LM}(f) = x_4^5$, and $\text{LT}(f) = -x_4^5$.

Given an ideal $I \subset S$, we let $\text{LT}(I)$ denote the set of leading terms of I , that is

$$\text{LT}(I) = \{c\mathbf{x}^\alpha : \text{there exists } f \in I \text{ with } \text{LT}(f) = c\mathbf{x}^\alpha\}.$$

The *leading term ideal* $\langle \text{LT}(I) \rangle$ is ideal generated by $\text{LT}(I)$. We can now formally define the notion of a *Gröbner basis* for I .

Definition 1.3.6. Fix a term order $<_{\text{mon}}$. A finite set $G = \{g_1, \dots, g_t\} \subset I$ is a *Gröbner basis* if

$$\langle \text{LT}(g_1), \dots, \text{LT}(g_t) \rangle = \text{LT}(I).$$

In the interest of computing Gröbner bases with a given term order on S , it is first necessary to describe a division algorithm for S . If $S = \mathbb{C}[x]$, then this division algorithm is the usual polynomial division from elementary algebra. However, for $S = \mathbb{C}[x_1, \dots, x_n]$ when $n \geq 2$, the division algorithm is more complicated and is dependent on a choice of term order.

Definition 1.3.7. Fix a term order $<_{\text{mon}}$ on $\mathbb{Z}_{\geq 0}^n$ and let $F = (f_1, \dots, f_s)$ be an ordered s -tuple of polynomials in S . Then every $f \in S$ can be written as

$$f = a_1f_1 + \dots + a_sf_s + r,$$

where $a_i, r \in S$, and either $r = 0$ or r is a linear combination of monomials, none of which is divisible by any of $\text{LT}(f_1), \dots, \text{LT}(f_s)$. We will write \bar{f}^F to denote the remained on division of f by the ordered S -tuple F .

Example 1.3.8. Let $S = \mathbb{C}[x, y, z]$. Consider the polynomial $f = x^4y + y^2z^2 + xyz + z^3$ which is written in decreasing order according to the grevlex term order and we will divide by the ordered polynomial tuple $F = (f_1, f_2) = (xy + z^2, xz - y)$. We first consider division of f by f_1 . Note that $\text{LT}(f) = x^4y$ which is divisible by $\text{LT}(f_1) = xy$. So, we now have

$$f = x^3(xy + z^2) + g_1$$

where $g_1 = -x^3z^2 + y^2z^2 + xyz + z^3$ which is written in decreasing order. We now consider division of g_1 by f_1 . Note that $\text{LT}(f_1)$ does not divide $\text{LT}(g_1) = -x^3z^2$ and $\text{LT}(f_1)$ does not divide the second greatest term of g_1 which is y^2z^2 either. However, $\text{LT}(f_1)$ does divide xyz , so we have

$$f = x^3(xy + z^2) + z(xy + z^2) + g_2,$$

where $g_2 = -x^3z^2 + y^2z^2$. The leading term $\text{LT}(f_1)$ does not divide any monomial in g_2 , so we are done with this division. Now, we consider g_2 divided by f_2 . Note that $\text{LT}(f_2) = xz$ divides $\text{LT}(g_2)$, so we have

$$f = x^3(xy + z^2) + z(xy + z^2) - x^2z(xz - y) + g_3,$$

where $g_3 = -x^2yz + y^2z^2$. Notice that $\text{LT}(f_2)$ does not divide any monomial in g_3 , so we are done and have

$$f = (x^3 + z)f_1 - x^2zf_2 + (-x^2yz + y^2z^2)$$

and thus

$$\bar{f}^F = -x^2yz + y^2z^2.$$

Given two polynomials $f, g \in S$, the S -polynomial of f and g is the combination

$$S(f, g) = \frac{\text{LCM}(\text{LM}(f), \text{LM}(g))}{\text{LT}(f)} \cdot f - \frac{\text{LCM}(\text{LM}(f), \text{LM}(g))}{\text{LT}(g)} \cdot g.$$

We now have necessary terminology to state *Buchberger's Algorithm* for computing a Gröbner basis.

Algorithm 1.3.9 (Buchberger's Algorithm, c.f. [17]). Fix a term order $<_{\text{mon}}$. Let $I = \langle f_1, \dots, f_s \rangle \subset S$ with $I \neq \{0\}$. Then a Gröbner basis for I can be constructed in a finite number of steps by the following algorithm:

Input: $F = (f_1, \dots, f_s)$

Output: a Gröbner basis $G = (g_1, \dots, g_t)$ for I

$G := F$

REPEAT

$G' := G$

FOR each pair $\{p, q\}$, $p \neq q$ in G' **DO**

$S := \overline{S(p, q)}^{G'}$

IF $S \neq 0$, **THEN** $G = G' \cup \{S\}$

UNTIL $G = G'$

Example 1.3.10. Let $I = \langle xy - z^2, x^2 - yz \rangle \subset \mathbb{C}[x, y, z]$. Following Algorithm 1.3.9, we initially compute one S -polynomial

$$\begin{aligned} \overline{S(xy - z^2, x^2 - yz)}^{(xy - z^2, x^2 - yz)} &= \overline{\frac{x^2y}{xy}(xy - z^2) - \frac{x^2y}{x^2}(x^2 - yz)}^{(xy - z^2, x^2 - yz)} \\ &= \overline{y^2z - xz^2}^{(xy - z^2, x^2 - yz)} \\ &= y^2z - xz^2, \end{aligned}$$

as the resulting polynomial cannot be divided by either polynomial in the ideal. So, we add it to the Gröbner basis $G = (xy - z^2, x^2 - yz, y^2z - xz^2)$. We now need to consider two additional S -polynomials:

$$\begin{aligned} \overline{S(xy - z^2, y^2z - xz^2)}^G &= \overline{\frac{xy^2z}{xy}(xy - z^2) - \frac{xy^2z}{y^2z}(y^2z - xz^2)}^G \\ &= \overline{z^2(x^2 - yz)}^G \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \overline{S(x^2 - yz, y^2z - xz^2)}^G &= \overline{\frac{x^2y^2z}{y^2z}(y^2z - xz^2) - \frac{x^2y^2z}{x^2}(x^2 - yz)}^G \\ &= \overline{x^3z^2 - y^3z^2}^G \\ &= \overline{xz^2(x^2 - yz) - yz(y^2z - xz^2)} \\ &= 0. \end{aligned}$$

Thus, we have a complete Gröbner basis.

1.4 Convex geometry and Ehrhart theory

In this section, we briefly review the theory of lattice point enumeration in convex rational polyhedra. To provide motivation for why the enumeration of lattice points is of interest, we state a classical theorem of Georg Alexander Pick, which describes a surprising phenomenon for convex lattice polygons which is known as Pick's Theorem.

Theorem 1.4.1 (Pick's Theorem [38]). *Let $\mathcal{P} \subset \mathbb{R}^2$ be a lattice polygon, that is a polygon with integer points for vertices. Let A denote the area of \mathcal{P} , let B denote the number of lattice (integer) points on the boundary of \mathcal{P} , and let I denote the number of lattice points in the interior of \mathcal{P} . Then*

$$A = I + \frac{1}{2}B - 1.$$

Example 1.4.2. Let P be the lattice polygon given in Figure 1.1. Notice that there are 10 boundary lattice points given in red and 28 interior lattice points given in blue. Applying Theorem 1.4.1 gives the area

$$A = 28 + \frac{1}{2}(10) - 1 = 32$$

which is the area obtained from using basic geometry.

We will be interested in two types of discrete geometric objects, namely convex rational polytopes and convex rational polyhedral cones. A *convex rational polytope* $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of finitely many rational points in \mathbb{R}^n , that is

$$\mathcal{P} = \text{conv}\{v_1, \dots, v_s : v_i \in \mathbb{Q}^n\}.$$

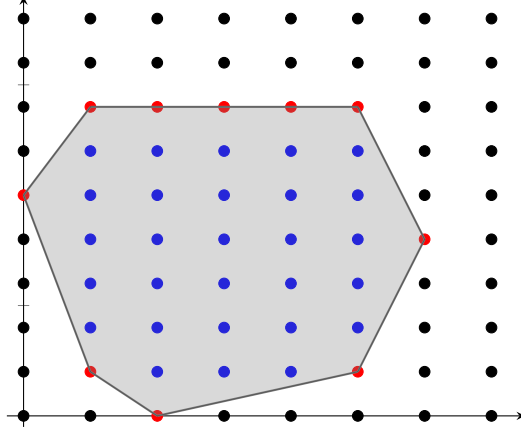


Figure 1.1: A convex lattice polygon

This description of the polytope is known as the *vertex description* or the \mathcal{V} -*description*. Alternatively, a convex rational polytope \mathcal{P} can be described as the intersection of finitely many rational halfspaces. This description of the polytope is the *halfspace description* or the \mathcal{H} -*description*. Any polytope \mathcal{P} has both a \mathcal{V} -description and an \mathcal{H} -description, though this is nontrivial (see e.g. [52]). We say that \mathcal{P} is a *lattice polytope* if it is the convex hull of finitely many lattice points

$$\mathcal{P} = \text{conv}\{v_1, \dots, v_s : v_i \in \mathbb{Z}^n\},$$

or alternatively if \mathcal{P} is the intersection of finitely many integral halfspaces.

Example 1.4.3. Consider the unit cube $[0, 1]^3 \subset \mathbb{R}^3$. This polytope has \mathcal{V} -description

$$[0, 1]^3 = \text{conv} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

where the vertices are expressed as column vectors of \mathbb{R}^3 . This polytope also has \mathcal{H} -description given by

$$0 \leq x, y, z \leq 1.$$

A *convex rational polyhedral cone* C in \mathbb{R}^n is the solution set to a finite collection of linear inequalities $A\mathbf{x} \geq 0$ for some matrix A with rational entries, or equivalently for some elements $w_1, w_2, \dots, w_j \in \mathbb{Q}^n$,

$$C = \text{span}_{\mathbb{R}_{\geq 0}} \{w_1, w_2, \dots, w_j\}.$$

The elements w_i are known as *ray generators*. We say that C is *pointed* if it does not contain a linear subspace of \mathbb{R}^n . The lattice points $C \cap \mathbb{R}^n$ form a semigroup which is known to have a unique minimal additive generating set called a *Hilbert basis* (c.f. [34]).

Example 1.4.4. Consider the polyhedral cone

$$C = \text{span}_{\mathbb{R}_{\geq 0}} \{(0, 1), (4, 7)\} \subset \mathbb{R}^2$$

which is illustrated in Figure 1.2. The Hilbert basis of C contains 3 elements: $\{(0, 1), (4, 7), (1, 2)\}$.

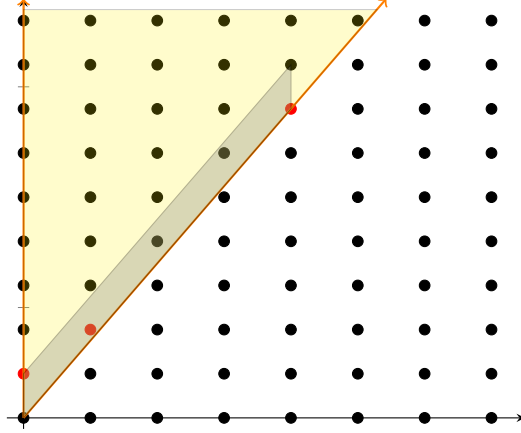


Figure 1.2: A pointed, rational cone in \mathbb{R}^2 with ray generators $(0, 1)$ and $(4, 7)$. The Hilbert basis is the set $\{(0, 1), (4, 7), (1, 2)\}$.

Let $\mathcal{P} \subset \mathbb{R}^n$ be a d -dimensional rational polytope ($d \leq n$). For $t \in \mathbb{Z}_{\geq 0}$, the *lattice point enumerator* $i(\mathcal{P}, t)$ gives the number of lattice points in $t\mathcal{P} = \{t\alpha : \alpha \in \mathcal{P}\}$ the t th dilation of \mathcal{P} . That is,

$$i(\mathcal{P}, t) = \#(t\mathcal{P} \cap \mathbb{Z}^n), \quad t \in \mathbb{Z}_{\geq 0}.$$

When $t = 0$, $t\mathcal{P} = \{0\}$ and $i(\mathcal{P}, 0) = 1$. If \mathcal{P} is a lattice polytope, it is a well-known theorem of Ehrhart [19] that $i(\mathcal{P}, t)$ is a polynomial in the variable t of degree d . Moreover, if \mathcal{P} is rational, but not lattice, it is known that $i(\mathcal{P}, t)$ is a *quasi-polynomial* in the variable t of degree d [19]. Subsequently, $i(\mathcal{P}, t)$ is called the *Ehrhart polynomial* of \mathcal{P} or *Ehrhart quasi-polynomial* of \mathcal{P} respectively.

Suppose that \mathcal{P} is lattice. The *Ehrhart series* for \mathcal{P} , $\text{Ehr}_{\mathcal{P}}(z)$, is the rational generating function

$$\text{Ehr}_{\mathcal{P}}(z) = \sum_{t \geq 0} i(\mathcal{P}, t)z^t = \frac{h^*(\mathcal{P}, z)}{(1 - z)^{d+1}}$$

where $h^*(\mathcal{P}, z) = h_0^* + h_1^*z + h_2^*z^2 + \cdots + h_d^*z^d$ is the h^* -polynomial of \mathcal{P} and $h^*(\mathcal{P}) = (h_0^*, h_1^*, h_2^*, \dots, h_d^*)$ the h^* -vector of \mathcal{P} . The h^* -polynomial (h^* -vector) is endowed with the following properties:

- $h_0^* = 1$, $h_1^* = i(\mathcal{P}, 1) - (d + 1)$, and $h_d^* = \#(\mathcal{P} \setminus \partial\mathcal{P} \cap \mathbb{Z}^d)$;
- $h_i^* \geq 0$ for all $0 \leq i \leq d$ ([47]);
- If $h_d^* \neq 0$, then $h_1^* \leq h_i^*$ for each $0 \leq i \leq d - 1$ ([28]).

Example 1.4.5. Figure 1.3 shows the first three dilates of $[0, 1]^3$. Notice that the first dilate contains 8 lattice points, the second dilate contains 27 lattice points ($8 + 19$), and third dilate contains 64 lattice points ($8 + 19 + 37$). These first three dilations are suggestive of a pattern which does in fact continue as

$$i([0, 1]^3, t) = (t + 1)^3.$$

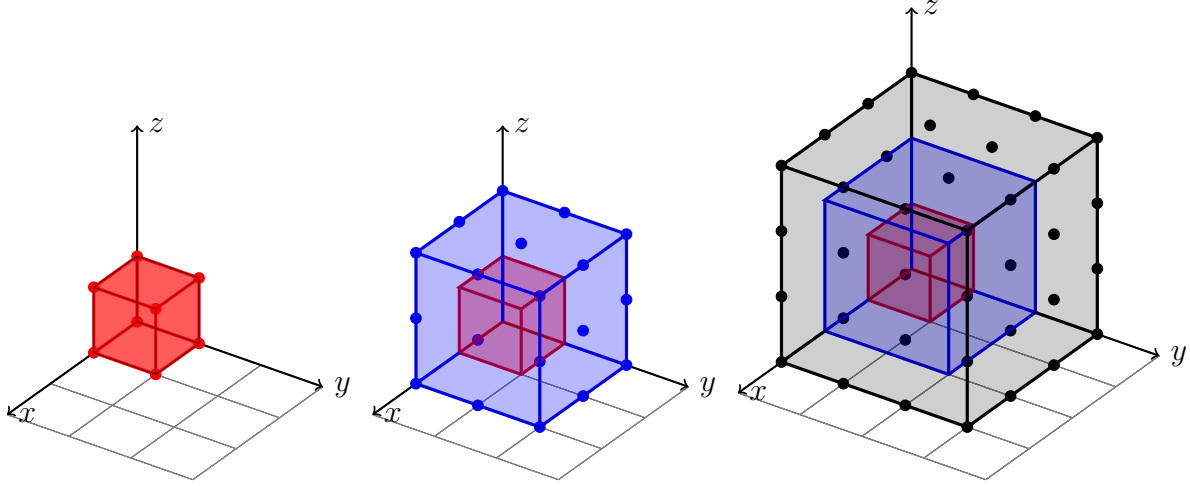


Figure 1.3: Dilations of $[0, 1]^3$ for $t = 1, 2, 3$

From previous computations in Example 1.2.3, we then know that Ehrhart series is

$$\text{Ehr}_{[0,1]^3}(z) = \sum_{t \geq 0} (t+1)^3 z^t = \frac{1 + 4z + z^2}{(1-z)^4}$$

and hence the h^* -vector is $h^*([0, 1]^3) = (1, 4, 1, 0)$. Moreover, one can see that in general for $[0, 1]^n$ for any $n \geq 1$,

$$i([0, 1]^n, t) = (t+1)^n$$

and therefore

$$\text{Ehr}_{[0,1]^n}(z) = \sum_{t \geq 0} (t+1)^n z^t = \frac{A_n(z)}{(1-z)^{n+1}},$$

where $A_n(z)$ is the Eulerian polynomial.

Ehrhart polynomials and Ehrhart series can additionally be identified as the *Hilbert function* and *Hilbert series* of an associated commutative algebra. For a polytope \mathcal{P} , consider the *cone over* \mathcal{P}

$$\text{cone}(\mathcal{P}) := \text{span}_{\mathbb{R}_{\geq 0}}\{(1, p) : p \in \mathcal{P}\}.$$

The *affine semigroup algebra* of \mathcal{P} over \mathbb{C} is

$$\mathbb{C}[\mathcal{P}] := \mathbb{C}[t^m \cdot \mathbf{x}^p : (m, p) \in \text{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}] \subset \mathbb{C}[t, x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}],$$

where $\mathbf{x}^p = x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$ when $(m, p) \in \text{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}$. Given that $\text{cone}(\mathcal{P})$ is a pointed, rational cone in \mathbb{R}^{n+1} , $\text{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}$ has a unique minimal generating, the Hilbert basis of the cone. Subsequently, the algebra $\mathbb{C}[\mathcal{P}]$ is a finitely generated, graded commutative algebra. Recall that given $\mathcal{A} = \bigoplus_{b \in \mathbb{Z}^n} \mathcal{A}_b$ a finitely generated, \mathbb{Z}^n -graded commutative \mathbb{C} -algebra. The *Hilbert series* of \mathcal{A} is

$$\text{Hilb}(\mathcal{A}; \mathbf{z}) = \sum_{b \in \mathbb{Z}^n} \dim_{\mathbb{C}}(\mathcal{A}_b) \cdot \mathbf{z}^b. \quad (1.3)$$

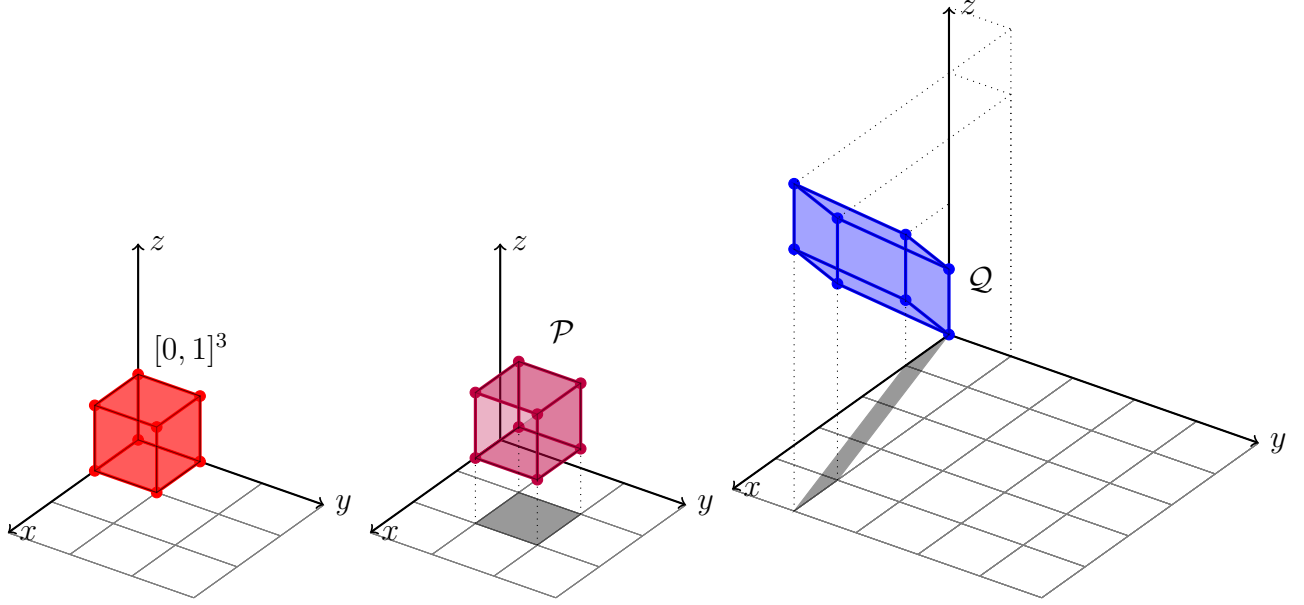


Figure 1.4: Unimodularly equivalent lattice polytopes to $[0, 1]^3$

For a polytope semigroup algebra $\mathbb{C}[\mathcal{P}]$, it is common to consider $\mathbb{C}[\mathcal{P}]$ as an \mathbb{N} -graded algebra where the grading is given by the z -degree. In this case, we have

$$\text{Hilb}(\mathbb{C}[\mathcal{P}]; z) = \sum_{m \geq 0} \#(m\mathcal{P} \cap \mathbb{Z}^n) \cdot z^m$$

which coincides with the *Ehrhart series* of \mathcal{P} (c.f [9, 34]).

Let $\mathbb{Z}^{n \times n}$ denote the set of $d \times d$ integer matrices. A matrix $A \in \mathbb{Z}^{n \times n}$ is *unimodular* if $\det(A) = \pm 1$. Given lattice polytopes $\mathcal{P} \subset \mathbb{R}^n$ and $\mathcal{Q} \subset \mathbb{R}^n$ of dimension d , we say that \mathcal{P} and \mathcal{Q} are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ and a vector $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathcal{Q} = f_U(\mathcal{P}) + \mathbf{w}$, where f_U is the linear transformation of \mathbb{R}^n defined by U , i.e., $f_U(\mathbf{v}) = U\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

Example 1.4.6. Consider the unit cube $[0, 1]^3$. Both the polytopes \mathcal{P} and \mathcal{Q} shown in Figure 1.4 are unimodularly equivalent to $[0, 1]^3$. First, it is easy to see that \mathcal{P} is simply a translation of $[0, 1]^3$, specifically by the vector $(1, 1, 1)^T$. To see \mathcal{Q} , note that

$$\mathcal{Q} = \text{conv} \begin{bmatrix} 0 & 0 & 1 & 4 & 1 & 4 & 5 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 4 & 1 & 3 & 5 & 4 \end{bmatrix} = f_U([0, 1]^3) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the linear transformation is given by the matrix

$$U = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

It is an elementary computation to check that $\det(U) = -1$.

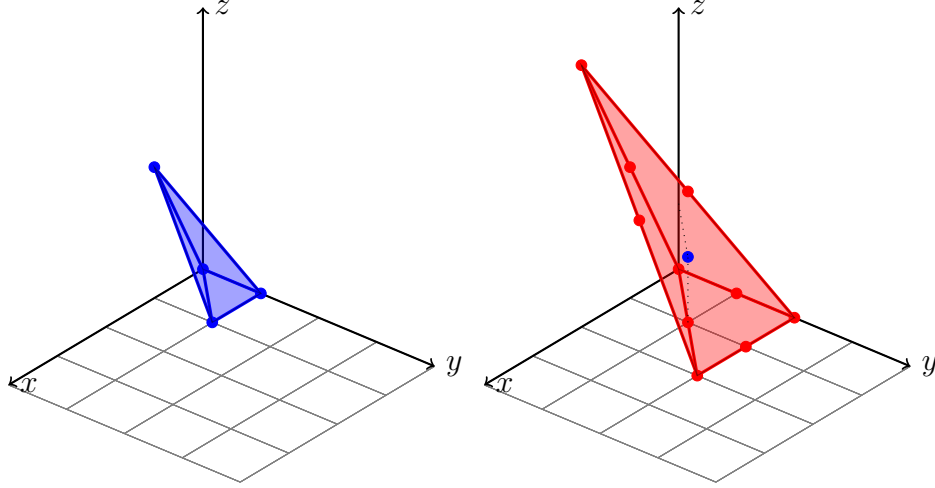


Figure 1.5: A non-IDP polytope: The Reeve Tetrahedron

We say that a lattice polytope \mathcal{P} is *Fano* if $(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^n = \{\mathbf{0}\}$. We say that \mathcal{P} is *reflexive* if it is Fano and its dual polytope

$$\mathcal{P}^\vee = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in \mathcal{P}\}$$

is a lattice polytope. Moreover, it follows from [27] that the following statements are equivalent:

- \mathcal{P} is unimodularly equivalent to some reflexive polytope;
- $h^*(\mathcal{P}, z)$ is of degree d and is symmetric, that is $h_i^* = h_{d-i}^*$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

We say that \mathcal{P} is *Gorenstein* if the affine semigroup algebra $\mathbb{C}[\mathcal{P}]$ is a Gorenstein algebra. However, it will be more convenient to use equivalent definitions. We say that \mathcal{P} is *Gorenstein of index c* where $c \in \mathbb{Z}_{>0}$ if $c\mathcal{P}$ is unimodularly equivalent to a reflexive polytope [18]. Equivalently, \mathcal{P} is Gorenstein if and only if $h^*(\mathcal{P}, z)$ is symmetric with $\deg(h^*(\mathcal{P}, z)) = d - c + 1$ ([46]).

Example 1.4.7. The polytope $[0, 1]^n$ is a Gorenstein polytope of index 2 for any n , which is evident as $h^*([0, 1]^n, z) = A_n(z)$ and the Eulerian polynomial $A_n(z)$ is a palindromic polynomial of degree $n - 1$.

We say that a lattice polytope \mathcal{P} has the *integer decomposition property* (or IDP for short) if for any $q \in t \cdot \mathcal{P} \cap \mathbb{Z}^n$, we can express $q = q_1 + q_2 + \dots + q_t$ where each $q_i \in \mathcal{P} \cap \mathbb{Z}^n$. If \mathcal{P} has the IDP, then we can more concisely describe $\mathbb{C}[\mathcal{P}]$. In particular, if $\mathcal{P} \cap \mathbb{Z}^n = \{p_1, p_2, \dots, p_k\}$ and \mathcal{P} satisfies the integer decomposition property, then

$$\mathbb{C}[\mathcal{P}] := \mathbb{C}[t \cdot \mathbf{x}^{p_i} : 1 \leq i \leq k] \subset \mathbb{C}[t, x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}].$$

Example 1.4.8. Consider the polytope

$$\mathcal{P} = \text{conv} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

This polytope \mathcal{P} is a *Reeve Tetrahedron* and it contains only its vertices as lattice points, as demonstrated in Figure 1.5. The second dilate of this polytope contains that lattice point $(1, 1, 1)$, which cannot be written as a combination of two of the vertices of \mathcal{P} . Thus, the polytope \mathcal{P} does not have the IDP.

1.5 Lecture hall partitions

In this section, we introduce the notion of a lecture hall partition, as well as related discrete geometric objects. First, recall that given some $d \in \mathbb{Z}_{\geq 1}$, a *partition* of d , is a sequence of numbers $a = (a_1, \dots, a_m) \in \mathbb{Z}_{\geq 1}^m$ with $a_1 \leq a_2 \leq \dots \leq a_m$ such that $a_1 + a_2 + \dots + a_m = d$. Given such a partition a , m is the number of parts of a .

We now define the notion of a lecture hall partition. Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$ be a sequence of n positive integers. The set of *\mathbf{s} -lecture hall partitions* is

$$L_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

Note that unlike the definition of a partition, lecture hall partitions can contain leading zeros. The conditions on each λ_i can be thought of as height constraints on the i th row of a lecture hall so that each student can see over the heads of students in the previous rows. The study of lecture hall partitions was initiated in two seminal papers by Bousquet-Mélou and Eriksson [10, 11] in 1997 and have since been vastly studied in various algebraic, discrete geometric, and number theoretic contexts (c.f. [40]).

Example 1.5.1. Consider the set $L_3^{(1,2,3)}$. The partition $(0, 4, 7)$ is an element of $L_3^{(1,2,3)}$ because

$$\frac{0}{1} < \frac{4}{2} < \frac{7}{3},$$

but the partition $(0, 5, 7)$ is not an element of $L_3^{(1,2,3)}$ because

$$\frac{0}{1} < \frac{5}{2} > \frac{7}{3}.$$

Remark 1.5.2. We should note that if \mathbf{s} is a weakly increasing sequence, the set $L_n^{(\mathbf{s})}$ can be thought of as refinement of the set of all partitions (of any integer) with at most n parts. Moreover, if \mathbf{s} is a constant sequence $\mathbf{s} = (a, a, \dots, a)$, for any $a \in \mathbb{Z}_{\geq 1}$, then $L_n^{(\mathbf{s})}$ is precisely the set of all partitions with at most n parts.

We can define a some statistics on the set of lecture hall partitions. Given \mathbf{s} , the set of *\mathbf{s} -inversion sequences* is the set

$$\mathbf{I}_n^{(\mathbf{s})} := \{ \mathbf{e} \in \mathbb{Z}^n : 0 \leq e_i < s_i \}.$$

Given $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$, the *ascent set* of \mathbf{e} is the set

$$\text{Asc}(\mathbf{e}) := \left\{ i \in \{0, 1, \dots, n-1\} : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\}$$

where by convention $s_0 = 1$ and $e_0 = 0$. Moreover, the *ascent number* of \mathbf{e} is $\text{asc}(\mathbf{e}) = |\text{Asc}(\mathbf{e})|$.

Table 1.2: Inversion sequences and ascent numbers for $\mathbf{s} = (1, 2, 3)$

\mathbf{e}	$\text{Asc}(\mathbf{e})$	$\text{asc}(\mathbf{e})$
$(0, 0, 0)$	\emptyset	0
$(0, 0, 1)$	$\{2\}$	1
$(0, 0, 2)$	$\{2\}$	1
$(0, 1, 0)$	$\{1\}$	1
$(0, 1, 1)$	$\{1\}$	1
$(0, 1, 2)$	$\{1, 2\}$	2

Example 1.5.3. For $\mathbf{s} = (1, 2, 3)$, the set of \mathbf{s} -inversion sequences is

$$\mathbf{I}_n^{(1,2,3)} = \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 1, 1), (0, 1, 2)\}.$$

The ascent sets and ascent numbers for these inversion sequences are given in Table 1.2.

Given the notion of a lecture hall partition, one can define three related discrete geometric objects of interest, namely lecture hall cones, lecture hall polytopes, and rational lecture hall partitions. First, given $\mathbf{s} \in \mathbb{Z}_{\geq 1}^n$, the \mathbf{s} -lecture hall cone $\mathcal{C}_n^{(\mathbf{s})}$ in the rational polyhedral cone defined by the following inequalities

$$\mathcal{C}_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

For a small example of such a lecture hall cone, refer to Figure 1.2, as this is precisely the cone $\mathcal{C}_2^{(4,7)}$.

Given $\mathbf{s} \in \mathbb{Z}_{\geq 1}^n$, the \mathbf{s} -lecture hall polytope $\mathbf{P}_n^{(\mathbf{s})}$ is the lattice polytope given by

$$\mathbf{P}_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1 \right\},$$

which is a bounded region of the $\mathcal{C}_n^{(\mathbf{s})}$. Alternatively, we note that $\mathbf{P}_n^{(\mathbf{s})}$ can be expressed via the convenient \mathcal{V} -description

$$\mathbf{P}_n^{(\mathbf{s})} := \text{conv} \{(0, \dots, 0), (0, \dots, 0, s_i, s_{i+1}, \dots, s_n) : 1 \leq i \leq n\}.$$

For some three dimensional examples corresponding to the sequences $(1, 2, 3)$ and $(15, 22, 13)$, see Figure 1.6.

One question of interest is the associated Ehrhart theory of \mathbf{s} -lecture hall polytopes. In general, the explicit form of the Ehrhart polynomials $i(\mathbf{P}_n^{(\mathbf{s})}, t)$ are unknown. However, explicit formulas for $h^*(\mathbf{P}_n^{(\mathbf{s})}, z)$ are known for arbitrary \mathbf{s} -sequences.

Theorem 1.5.4 (Savage–Schuster, [41, Theorem 5]). *The h^* polynomial for $\mathbf{P}_n^{(\mathbf{s})}$ is*

$$h^*(\mathbf{P}_n^{(\mathbf{s})}, z) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} z^{\text{asc}(\mathbf{e})}.$$

These polynomials are called the \mathbf{s} -Eulerian polynomials.

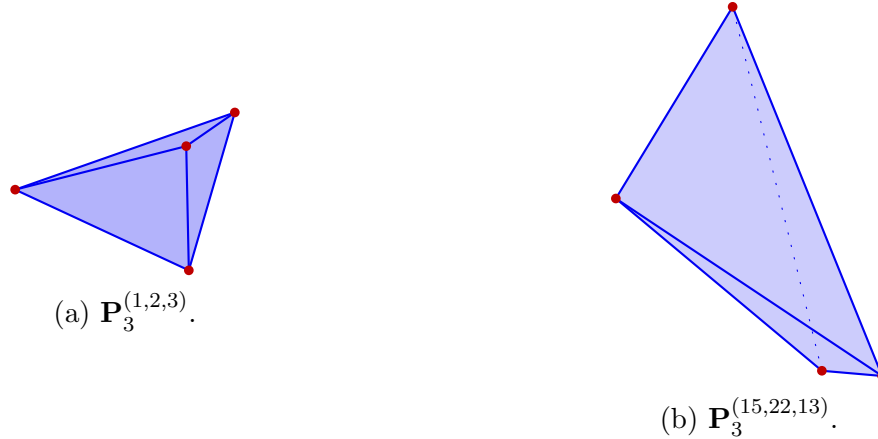


Figure 1.6: Two three-dimensional lecture hall polytopes (not drawn to scale)

Remark 1.5.5. The term \mathbf{s} -Eulerian polynomial comes from the following observation for the sequence $\mathbf{s} = (1, 2, \dots, n)$:

$$\sum_{\mathbf{e} \in \mathbf{1}_n^{(1,2,\dots,n)}} z^{\text{asc}(\mathbf{e})} = \sum_{\pi \in S_n} z^{\text{des}(\pi)} = A_n(z),$$

which is the usual Eulerian polynomial. For a concrete example of this observation, for $n = 3$ the permutation data from Table 1.1 and the inversion sequence data from Table 1.2 indicate that these polynomials must be the same in this case. Subsequently, these polynomials are a generalized family of Eulerian-like polynomials based on the parameter \mathbf{s} .

The \mathbf{s} -Eulerian polynomials have been the subject of much study (c.f. [40]). For particular \mathbf{s} , more explicit combinatorial formulas are known. Moreover, these polynomials are known to be real-rooted for arbitrary \mathbf{s} [42].

We define one additional polyhedral object associated to \mathbf{s} -lecture hall partitions. Given $\mathbf{s} \in \mathbb{Z}_{\geq 1}$, the *rational \mathbf{s} -lecture hall polytope* is the rational polytope given as follows

$$R_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq \frac{1}{s_n} \right\}.$$

Chapter 2 Euler-Mahonian statistics and descent bases for semigroup algebras

2.1 Introduction

The goal of this chapter is to study bivariate generating functions relating to certain pairs of statistics on permutations and generalized permutations. To do so, we will use algebraic methods on the affine semigroup algebra of a particular polytope, namely the unit cube $[0, 1]^n$. For additional details, please appeal to the introduction. In particular, see Section 1.2 for permutation details, see Section 1.3 for relevant Gröbner basis particulars, and see Section 1.4 for additional background on polytopes and Ehrhart theory.

Let $\mathcal{P} = [0, 1]^n$, which is known to satisfy the integer decomposition property. Let $R_n := \mathbb{C}[[0, 1]^n]$ denote the affine semigroup algebra of $[0, 1]^n$ which has the following description:

$$R_n = \mathbb{C}[t \cdot x_{a_1} \cdots x_{a_i} \mid A = \{a_1, \dots, a_i\} \subseteq [n]] \subset \mathbb{C}[t, x_1, x_2, \dots, x_n].$$

Alternatively, we can define R_n as the quotient of a polynomial ring by a toric ideal. Let T_n be a polynomial ring in 2^n variables, where each variable corresponds to a subset of $[n]$, thus

$$T_n := \mathbb{C}[z_A : A \subseteq [n]].$$

Define the toric ideal

$$I_n := \langle z_A z_B - z_{A \cap B} z_{A \cup B} \mid A \not\subseteq B \text{ and } B \not\subseteq A \rangle.$$

It is known that $R_n \cong T_n/I_n$. For background and details see [34]. This algebra also arises as the Hibi ring for the antichain on n elements, as the unit cube is the order polytope of the antichain (see e.g. [20, 25, 26] for additional details of Hibi rings). We will use R_n to denote T_n/I_n when it is convenient.

Let $\mathcal{A} = \bigoplus_{b \in \mathbb{Z}^n} \mathcal{A}_b$ be a finitely generated, \mathbb{Z}^n -graded commutative \mathbb{C} -algebra. The *Hilbert series* of \mathcal{A} is

$$\text{Hilb}(\mathcal{A}; \mathbf{z}) = \sum_{b \in \mathbb{Z}^n} \dim_{\mathbb{C}}(\mathcal{A}_b) \cdot \mathbf{z}^b. \quad (2.1)$$

For a polytope semigroup algebra $\mathbb{C}[\mathcal{P}]$, it is common to consider $\mathbb{C}[\mathcal{P}]$ as an \mathbb{N} -graded algebra where the grading is given by the t -degree. In this case, we have

$$\text{Hilb}(\mathbb{C}[\mathcal{P}]; t) = \sum_{m \geq 0} \#(m\mathcal{P} \cap \mathbb{Z}^n) \cdot t^m$$

which coincides with the *Ehrhart series* of \mathcal{P} . The reader is invited to consult [34] and [9] for background on Hilbert series and Ehrhart theory receptively. In the case of R_n , it is well-known that the Hilbert series with respect to the t -degree is $\sum_{k \geq 0} (k+1)^n t^k$, leading us to the topic of Euler-Mahonian identities.

The following bivariate generalization using the major index of the Eulerian polynomial identity 1.2 is usually attributed to Carlitz.

Theorem 2.1.1 (Carlitz, [16]). *For all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1 - tq^j)}$$

where $[k+1]_q = 1 + q + q^2 + \dots + q^k$.

In this form, this identity is due to Carlitz [16], though with some effort one can derive it from the works of MacMahon [33, Volume 2, Chapter IV, §462]. We will call this identity the *Euler-Mahonian identity*, which has arisen in a variety of contexts in recent years. Some such scenarios include lecture hall partition generating function identities [36], polyhedral-geometric studies of the semigroup algebra for cone($[0, 1]^n$) [5], Hilbert series related to a descent basis for the coinvariant algebra of S_n [2], 0-Hecke algebra actions on Stanley-Reisner rings [32], and quasisymmetric function identities [45].

Generalizing to colored permutation groups $\mathbb{Z}_r \wr S_n$, one can consider the *flag statistics* as well as the *negative statistics*, the latter of which we define in Section 2.2. These statistics were originally introduced for the hyperoctohedral group $B_n \cong \mathbb{Z}_2 \wr S_n$ [1] and generalized for $r \geq 2$ to $\mathbb{Z}_r \wr S_n$ [3, 4]. For these families of statistics, the following Euler-Mahonian identities exist.

Theorem 2.1.2 (Bagno, [3]). *Given any $r \geq 2$, for all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{ndes}(\rho, \epsilon)} q^{\text{nmajor}(\rho, \epsilon)}}{(1-t) \prod_{j=1}^n (1 - t^r q^{rj})}$$

Theorem 2.1.3 (Bagno-Biagioli, [4]). *Given any $r \geq 2$, for all $n \geq 1$,*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{fdes}(\rho, \epsilon)} q^{\text{fmajor}(\rho, \epsilon)}}{(1-t) \prod_{j=1}^n (1 - t^r q^{rj})}$$

The goal of this chapter is twofold. First, we produce a new algebraic interpretation of negative permutation statistics by considering $\mathbb{Z}_r \wr S_n$ -quotient algebras of R_n . To do so, we consider an ideal $\overline{\text{invar}(r, n)} \subset R_n$ which is generated by certain invariants of R_n under a $\mathbb{Z}_r \wr S_n$ -action, defined in detail in Section 2.3. We obtain the following theorem using Gröbner basis techniques.

Theorem 2.1.4 (see Theorem 2.4.1). *There exists a basis of $R_n / \overline{\text{invar}(r, n)}$ of the form $\{b_{(\sigma, X)}^r + \overline{\text{invar}(r, n)}\}$ with elements indexed by pairs (σ, X) that are in bijection with colored permutations $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$. Further, $b_{(\sigma, X)}^r$ encodes $\text{ndes}(\pi, \epsilon)$ and $\text{nmajor}(\pi, \epsilon)$. The bijective correspondence of $(\sigma, X) \leftrightarrow (\pi, \epsilon)$ is given in Remark 2.2.7.*

Our second goal is to consider a multigraded Hilbert series of R_n and the quotient $R_n / \overline{\text{invar}(r, n)}$. These computations allow us to recover the identities given by Theorem 2.1.1 and Theorem 2.1.2. These new proofs provide a new perspective on identities of this type.

Moreover, the new proof of Theorem 2.1.1 serves to expand connections between the commutative-algebraic and representation-theoretic methods [2] for the S_n -coinvariant algebra $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$, where $\mathcal{I}_n := \langle e_1, \dots, e_n \rangle$ with e_i denoting the i -th elementary symmetric function, and polyhedral-geometric methods for $\text{cone}([0, 1]^n)$ [5]. Additionally, we provide a short proof that this quotient algebra is isomorphic as a graded S_n -module to the S_n -coinvariant algebra $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$. We believe that these results, like those given in [5], support the idea that $\text{cone}([0, 1]^n)$ and its associated semigroup algebra are analogues of the polynomial ring in n variables that give rise to interesting and different structures and results in similar contexts.

2.2 Colored permutation groups and decent sets

The wreath product $\mathbb{Z}_r \wr S_n \cong (\mathbb{Z}_r)^n \rtimes S_n$ of a cyclic group of order r with S_n consists of pairs (π, ϵ) where $\pi \in S_n$ and $\epsilon \in \{\omega^0, \omega^1, \dots, \omega^{r-1}\}^n$ for $\omega := e^{2\pi i/r}$ a primitive r th root of unity. These groups are often called *colored permutation groups* and the elements are commonly referred to as *colored* or *indexed* permutations. We adopt the usual window notation, denoting the pair (π, ϵ) by $[\pi(1)^{\epsilon_1} \pi(2)^{\epsilon_2} \dots \pi(n)^{\epsilon_n}]$ where $\epsilon_j = \omega^{c_j}$. Additionally, we will use the notation j^{c_j} and (ω^{c_j}, j) to denote elements of $\{\omega^0, \omega^1, \dots, \omega^{r-1}\} \times [n]$.

Elements $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ can be identified as a permutation matrix for π where the 1 in position $(\pi(i), i)$ is replaced with ϵ_i . The algebraic structure of $\mathbb{Z}_r \wr S_n$ is described by matrix multiplication where entry-by-entry multiplication of the nonzero entries is given by the group operation of \mathbb{Z}_r . This means that given $(\pi, \epsilon), (\pi', \epsilon') \in \mathbb{Z}_r \wr S_n$

$$(\pi', \epsilon') \circ (\pi, \epsilon) = (\pi' \circ \pi, (\epsilon_1 \cdot \epsilon'_{\pi(1)}, \dots, \epsilon_n \cdot \epsilon'_{\pi(n)})),$$

or represented in window notation we have

$$[\pi'(1)^{\epsilon'_1} \dots \pi'(n)^{\epsilon'_n}] \circ [\pi(1)^{\epsilon_1} \dots \pi(n)^{\epsilon_n}] = [\pi' \circ \pi(1)^{\epsilon_1 + \epsilon'_{\pi(1)}} \dots \pi' \circ \pi(n)^{\epsilon_n + \epsilon'_{\pi(n)}}]$$

where the addition is modulo r . A more explicit understanding of these wreath products may be found in [3, 4, 5, 36].

To review one definition of descents for wreath products, we define a total order as follows. Given $j^{c_j}, k^{c_k} \in \{\omega^0, \omega^1, \dots, \omega^{r-1}\} \times [n]$ we say that $j^{c_j} < k^{c_k}$ if $c_j > c_k$ or if $c_j = c_k$ and $j < k$ hold.

Definition 2.2.1. Let $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$. The *type-A descent set* is defined to be

$$\text{Des}_A(\pi, \epsilon) := \{i \in [n-1] : \pi_i^{c_i} > \pi_{i+1}^{c_{i+1}}\}$$

and the *type-A descent statistic* is

$$\text{des}_A(\pi, \epsilon) := \#\text{Des}_A(\pi, \epsilon).$$

The *type-A major index* is

$$\text{major}_A(\pi, \epsilon) := \sum_{j \in \text{Des}_A(\pi, \epsilon)} j$$

Example 2.2.2. Let $(\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^2 3^0] \in \mathbb{Z}_4 \wr S_6$. Then $\text{Des}_A(\pi, \epsilon) = \{1, 2, 4\}$, $\text{des}_A(\pi, \epsilon) = 3$, and $\text{major}_A(\pi, \epsilon) = 7$.

We now review a different notion of descent statistics for $\mathbb{Z}_r \wr S_n$, namely the *negative* statistics.

Definition 2.2.3. For an element $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$, we define the *negative inverse multiset* as

$$\text{NNeg}(\pi, \epsilon) := \underbrace{\{i, i, \dots, i : i \in [n]\}}_{c_i \text{ times}}.$$

The *negative descent multiset* is

$$\text{NDes}(\pi, \epsilon) := \text{Des}_A(\pi, \epsilon) \cup \text{NNeg}((\pi, \epsilon)^{-1}).$$

The *negative descent statistic* is

$$\text{ndes}(\pi, \epsilon) := \#\text{NDes}(\pi, \epsilon).$$

The *negative major index* is

$$\text{nmajor}(\pi, \epsilon) := \sum_{i \in \text{NDes}(\pi, \epsilon)} i.$$

Example 2.2.4. If $(\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^2 3^0] \in \mathbb{Z}_4 \wr S_6$, then $(\pi, \epsilon)^{-1} = [4^0 1^3 6^0 3^1 5^2 2^1]$ and hence $\text{NNeg}((\pi, \epsilon)^{-1}) = \{2, 2, 2, 4, 5, 5, 6\}$. Further,

$$\text{NDes}(\pi, \epsilon) = \{1, 2, 4\} \cup \{2, 2, 2, 4, 5, 5, 6\} = \{1, 2, 2, 2, 2, 4, 4, 5, 5, 6\}$$

and thus $\text{ndes}(\pi, \epsilon) = 10$ and $\text{nmajor}(\pi, \epsilon) = 33$.

We will use the following representation for elements of $\mathbb{Z}_r \wr S_n$.

Definition 2.2.5. The *increasing elements* of $\mathbb{Z}_r \wr S_n$, denoted $\mathcal{I}_{r,n}$, is the subset of elements satisfying $\text{des}_A(\pi, \epsilon) = 0$.

It is a simple exercise to see that any element of $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ can be represented uniquely as

$$(\pi, \epsilon) = (\rho, \delta) \circ (\sigma, (1, 1, \dots, 1))$$

for some $\sigma \in S_n$ and $(\rho, \delta) \in \mathcal{I}_{r,n}$. Subsequently, we have that

$$\mathbb{Z}_r \wr S_n = \bigcup_{\sigma \in S_n} \mathcal{I}_{r,n} \sigma$$

where we use σ in place of $(\sigma, (1, 1, \dots, 1))$ for ease of notation.

We also have the following observation.

Proposition 2.2.6 (Beck–Braun, [5, Proposition 5.11]). *For $(\rho, \delta) \in \mathcal{I}_{r,n}$ and $\sigma \in S_n$,*

$$\text{NNeg}((\rho, \delta)\sigma)^{-1} = \text{NNeg}((\rho, h^*)^{-1}).$$

Further, each permutation $(\rho, h^) \in \mathcal{I}_{r,n}$ is uniquely determined by $\text{NNeg}((\rho, \delta)^{-1})$.*

Remark 2.2.7. We will often denote $(\pi, \epsilon) \in \mathbb{Z}_r \wr S_n$ by the pair (σ, X) where $\sigma \in S_n$ satisfies $(\rho, \delta)\sigma = (\pi, \epsilon)$ with $(\rho, \delta) \in \mathcal{I}_{r,n}$ and $X = \text{NNeg}((\pi, \epsilon)^{-1})$. This establishes a bijective correspondence between elements of $\mathbb{Z}_r \wr S_n$ and pairs (σ, X) with $\sigma \in S_n$ and X a multiset of elements of $[n]$ in which each element appears with multiplicity strictly less than r . For convenience of notation, we will write $(\sigma, X) \in \mathbb{Z}_r \wr S_n$ when this interpretation is preferred.

Example 2.2.8. Let $(\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^2 3^0] \in \mathbb{Z}_4 \wr S_6$ and consider $(\rho, \delta) = [4^3 6^3 5^2 2^1 1^0 3^0] \in \mathcal{I}_{4,6}$ and $\sigma = 421536 \in S_6$. Note that $(\pi, \epsilon) = (\rho, \delta) \circ (\sigma, (1, \dots, 1))$ as

$$[2^1 6^3 4^3 1^0 5^2 3^0] = [4^3 6^3 5^2 2^1 1^0 3^0] \circ [4^0 2^0 1^0 5^0 3^0 6^0].$$

Moreover, $(\rho, \delta)^{-1} = [5^0 4^3 6^0 1^1 3^2 2^1]$ and $\text{NNeg}((\rho, \delta)^{-1}) = \{2, 2, 2, 4, 5, 5, 6\}$. Therefore, $\text{NNeg}((\rho, \delta)^{-1}) = \text{NNeg}(((\rho, \delta)\sigma)^{-1}) = \text{NNeg}((\pi, \epsilon)^{-1})$. Thus, (π, ϵ) corresponds to the pair $(\sigma, X) = (421536, \{2, 2, 2, 4, 5, 5, 6\})$,

2.3 $\mathbb{Z}_r \wr S_n$ -quotient algebras of R_n and descent bases

For convenience, we will view $R_n \cong T_n/I_n$ as the quotient of a polynomial ring by the toric ideal I_n . First consider the S_n case. We define an S_n action on T_n given as $S_n \times T_n \rightarrow T_n$ defined on the variables by $(\pi, z_A) \mapsto z_{\pi(A)} = z_{\{\pi(a_1), \dots, \pi(a_k)\}}$ where $A = \{a_1, \dots, a_k\}$. Note that this action passes to $R_n \cong T_n/I_n$, where it corresponds to the usual action of S_n on $\{x_1, \dots, x_n\}$ of permutation of the variables because

$$z_{\{a_1, a_2, \dots, a_k\}} \mapsto x_{a_1} x_{a_2} \cdots x_{a_k}$$

and

$$z_{\{\pi(a_1), \dots, \pi(a_k)\}} \mapsto x_{\pi(a_1)} x_{\pi(a_2)} \cdots x_{\pi(a_k)}$$

which is the usual permutation of variables action of S_n . We consider the following ideal of elements which are invariant under this action:

$$\text{invar}(1, n) := \left\langle \hat{e}_k := \sum_{|A|=k} z_A \mid \text{for all } 0 \leq k \leq n \right\rangle.$$

The elements \hat{e}_k are the T_n -analogue to the usual elementary symmetric functions e_k in the polynomial ring on n variables. Notice that this ideal cannot be the full ideal of invariants for this action on T_n because there must be 2^n algebraically independent invariants [51, Proposition 2.1.1]. However, the generators are indeed invariant and this is the appropriate ideal for our purposes. We say the S_n quotient algebra of R_n is $R_n/\overline{\text{invar}(1, n)}$, where $\overline{\text{invar}(1, n)}$ is the image of $\text{invar}(1, n)$ in the quotient T_n/I_n . For convenience, we will consider the ring $T_n/J_{1,n}$ where $J_{1,n} := \overline{\text{invar}(1, n)} + I_n$, as it is a straightforward exercise in algebra to show that $T_n/J_{1,n} \cong R_n/\overline{\text{invar}(1, n)}$.

Next, we consider $\mathbb{Z}_r \wr S_n$ for $r \geq 2$. Consider the action $\mathbb{Z}_r \wr S_n \times T_n \rightarrow T_n$ defined on the variables by $((\pi, \epsilon), z_A) \mapsto (\prod_{i \in A} \epsilon_i) \cdot z_{\pi(A)} = (\prod_{i \in A} \epsilon_i) \cdot z_{\{\pi(a_1), \dots, \pi(a_k)\}}$ where $A = \{a_1, \dots, a_k\}$. We consider an ideal generated by invariant elements of this action:

$$\text{invar}(r, n) := \left\langle z_{\emptyset}, \hat{e}_{r,k} := \sum_{|A|=k} z_A^r \mid \text{for all } 1 \leq k \leq n \right\rangle.$$

This is consistent with the above in the $r = 1$ case. This ideal also does not contain all of the invariants of T_n under this action, but the ideal is the appropriate choice of invariant generators for our scenario. We say the $\mathbb{Z}_r \wr S_n$ quotient algebra of R_n is $R_n/\overline{\text{invar}}(r, n)$, and we will consider the ring $T_n/J_{r,n}$ where $J_{r,n} := \text{invar}(r, n) + I_n$, as we have $T_n/J_{r,n} \cong R_n/\overline{\text{invar}}(r, n)$.

Now, we will define descent bases for our quotients. First consider $T_n/J_{1,n}$. We wish to construct a basis based on the descent sets of S_n that is analogous to the Garsia-Stanton descent basis. The Garsia-Stanton descent basis is a basis for the S_n -coinvariant algebra $\mathbb{C}[x_1, \dots, x_n]/\mathcal{I}_n$ with coset representatives

$$a_\pi = \prod_{j \in \text{Des}(\pi)} x_{\pi(1)} \cdots x_{\pi(j)}$$

for all $\pi \in S_n$, where the ideal $\mathcal{I}_n = \langle e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n) \rangle$ is generated the the elementary symmetric functions $e_i(x_1, \dots, x_n) = \sum_{a_1 < \dots < a_i} x_{a_1} \cdots x_{a_i}$. Garsia and Stanton originally showed this was a basis in [23] using the theory of Stanley-Reisner rings. In [2], Adin, Brenti, and Roichman provide another proof of this result and use the basis heavily in their proof of the Euler-Mahonian identity for S_n . We introduce an analogue of the Garsia-Stanton basis for $T_n/J_{1,n}$, which is

$$\hat{a}_\pi := \prod_{j \in \text{Des}(\pi)} z_{\{\pi(1), \pi(2), \dots, \pi(j)\}}$$

for all $\pi \in S_n$.

Example 2.3.1. Let $\pi = 421536 \in S_6$. Since $\text{Des}(\pi) = \{1, 2, 4\}$, we have

$$a_\pi = z_{\{4\}} z_{\{2,4\}} z_{\{1,2,4,5\}}$$

Because of the correspondence given in Theorem 2.6.1, in this paper we will refer to the set $\{\hat{a}_\pi : \pi \in S_n\}$ as the *Garsia-Stanton basis*. Using Gröbner basis arguments in Section 2.4, we will show that this is indeed a basis for $T_n/J_{1,n}$.

We can generalize this to a basis of $T_n/J_{r,n}$ for $r \geq 2$.

Definition 2.3.2. The *negative descent basis* of $T_n/J_{r,n}$ consists of the elements

$$b_{(\sigma, X)}^r := \hat{a}_\sigma \cdot \prod_{j \in X} z_{\{\sigma(1), \sigma(2), \dots, \sigma(j)\}}$$

for all $\sigma \in S_n$ and X a multiset of $[n]$ where no element has multiplicity greater than $r - 1$.

Example 2.3.3. Let $(\sigma, X) = (421536, \{2, 2, 2, 4, 5, 5, 6\})$ corresponding to $(\pi, \epsilon) = [2^1 6^3 4^3 1^0 5^2 3^0] \in \mathbb{Z}_4 \wr S_6$. Then

$$\begin{aligned} b_{(\sigma, X)} &= z_{\{4\}} z_{\{2,4\}} z_{\{1,2,4,5\}} \cdot (z_{\{2,4\}})^3 \cdot (z_{\{1,2,4,5\}}) \cdot (z_{\{1,2,3,4,5\}})^2 \cdot (z_{\{1,2,3,4,5,6\}}) \\ &= z_{\{4\}} z_{\{2,4\}}^4 z_{\{1,2,4,5\}}^2 z_{\{1,2,3,4,5\}}^2 z_{\{1,2,3,4,5,6\}} \end{aligned}$$

We will show that this is a basis in Section 2.4. It follows from Remark 2.2.7 that if (σ, X) corresponds to $(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n$, then $\text{NNeg}((\rho, \epsilon)^{-1}) = X$ and $\text{Des}_A(\rho, \epsilon) = \text{Des}(\sigma)$. So, elements of this basis correspond to NDes sets of $\mathbb{Z}_r \wr S_n$, hence the name “negative descent basis.” It is important to observe that this is distinct from the basis developed by R. Adin, F. Brenti, and Y. Roichman [2] for the hyperoctohedral group $B_n \cong \mathbb{Z}_2 \wr S_n$, as their basis related to the *flag descent sets*.

2.4 Descent bases via Gröbner bases for $J_{r,n}$

Our goal in this section is to prove the following theorem by finding a Gröbner basis for the ideal $J_{r,n}$.

Theorem 2.4.1. *For $r \geq 2$, $\{b_{(\sigma, X)}^r : (\sigma, X) \in \mathbb{Z}_r \wr S_n\}$ is a basis of $T_n/J_{r,n}$. When $r = 1$, $\{\hat{a}_\pi : \pi \in S_n\}$ is a basis of $T_n/J_{1,n}$.*

Before proving this theorem, we briefly review Gröbner bases. For a detailed reference on the theory and computation of Gröbner bases, we invite the reader to consult [17, 21]. Consider the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$. Recall that a *term order* $<_{\text{mon}}$ on S is a relation on $\mathbb{Z}_{\geq 0}^n$ which is a total ordering, a well ordering, and satisfies the condition that if $\alpha <_{\text{mon}} \beta$ and $\gamma \in \mathbb{Z}_{\geq 0}^n$, then $\alpha + \gamma <_{\text{mon}} \beta + \gamma$. Given two monomials $m_1 = \prod_{i=1}^n x_i^{\alpha_i}$ and $m_2 = \prod_{i=1}^n x_i^{\beta_i}$, we say that $m_1 <_{\text{mon}} m_2$ if $(\alpha_1, \alpha_2, \dots, \alpha_n) <_{\text{mon}} (\beta_1, \beta_2, \dots, \beta_n)$. Given $f \in S$, the *leading monomial of f* , denoted $\text{LM}(f)$, is the largest monomial of f with respect to the term order $<_{\text{mon}}$. For notation, we will denote monomials as $\mathbf{x}^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ where $\alpha \in \mathbb{Z}_{\geq 0}^n$. The *leading term of f* , denoted $\text{LT}(f)$, is the leading monomial with its coefficient. Given an ideal $I \subset S$, $\text{LT}(I) = \{c\mathbf{x}^\alpha : \exists f \in I \text{ s.t. } \text{LT}(f) = c\mathbf{x}^\alpha\}$ and $\langle \text{LT}(I) \rangle$ is the ideal generated by elements of $\text{LT}(I)$, which we call the *leading term ideal* of I . A finite subset $G = \{g_1, \dots, g_t\}$ of an ideal I is called a *Gröbner basis* for I if

$$\langle \text{LT}(g_1), \dots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle.$$

Given a polynomial ideal $I \subset S$ and a fixed term order, we can algorithmically construct a Gröbner basis using a classical result known as *Buchberger’s Algorithm*. However, one can optimize this classical algorithm to be more efficient. Before stating an optimized version, we must introduce notation. Given two polynomials f, g , the *S-polynomial* of f and g is

$$S(f, g) = \frac{\mathbf{x}^\gamma}{\text{LT}(f)} \cdot f - \frac{\mathbf{x}^\gamma}{\text{LT}(g)} \cdot g$$

where $\mathbf{x}^\gamma = \text{lcm}(\text{LM}(f), \text{LM}(g))$. Given a polynomial f and an ordered s -tuple of polynomials $F = (f_1, \dots, f_s)$, let \bar{f}^F denote the remainder of f after division by each polynomial in F performed in order. The reader should consult [17] for a thorough discussion of multivariate polynomial division.

Algorithm 2.4.2 (Optimized Buchberger Algorithm). Let $I = \langle f_1, \dots, f_s \rangle \subset S$. Then a Gröbner basis for I can be constructed in a finite number of steps as follows:

Input: $F = (f_1, \dots, f_s)$

Output: G , a Gröbner basis for I

Initial state: $B := \{(i, j) : 1 \leq i < j \leq s\}$; $G := F$; $t := s$

WHILE $B \neq \emptyset$ **DO**

Select $(i, j) \in B$

IF $\text{lcm}(\text{LT}(f_i), \text{LT}(f_j)) \neq \text{LT}(f_i) \cdot \text{LT}(f_j)$, **AND** $\text{Criterion}(f_i, f_j, B)$ is false **THEN**

$\mathcal{S} := \overline{S(f_i, f_j)}^G$

IF $\mathcal{S} \neq 0$ **THEN**

$t := t+1$; $f_t := \mathcal{S}$; $G := G \cup \{f_t\}$; $B := B \cup \{(i, t) : 1 \leq i \leq t-1\}$

$B := B - \{(i, j)\}$,

where $\text{Criterion}(f_i, f_j, B)$ is true provided that there is some $k \notin \{i, j\}$ for which the pairs $[i, k]$ (i.e. (i, k) if $i < k$ or (k, i) if $k < i$) and $[j, k]$ are NOT in B and $\text{LT}(f_k)$ divides $\text{lcm}(\text{LT}(f_i), \text{LT}(f_j))$.

Our motivation to compute a Gröbner basis for the ideal $J_{r,n}$ is the following theorem attributed to Macaulay.

Theorem 2.4.3 (Macaulay, c.f.[21]). *Let $<_{\text{mon}}$ be a term order and let $I \subset S$ be an ideal. Then the monomials in S which do not belong to $\langle \text{LT}(I) \rangle$ form a \mathbb{C} -basis for S/I .*

Determining a Gröbner basis for $J_{r,n}$ yields a useful description of $\langle \text{LT}(J_{r,n}) \rangle$. Thus, Theorem 2.4.1 is an immediate consequence of Theorem 2.4.3, Proposition 2.4.4, and Theorem 2.4.10 below.

Proposition 2.4.4. *Fix $r \geq 1$ and $n \geq 1$. Consider the monomial ideal $N_{r,n}$ in T_n generated by the following elements:*

- z_\emptyset
- z_A^r , where $A = [k]$ for all $1 \leq k \leq n$
- z_A^{r+1} where $A \neq [k]$ for any $0 \leq k \leq n$
- $z_A z_B$ such that $A \not\subseteq B$ and $B \not\subseteq A$
- $z_A^r z_B$ where $A \neq [k]$ for any $0 \leq k \leq n$, such that $A \subset B$ and $\min(B \setminus A) > \max(A)$
- $z_A z_B^r$ where $B \neq [k]$ for any $0 \leq k \leq n$, such that $A \subset B$ and there is an ℓ with $[\ell] \not\subseteq A$, $[\ell] \subset B$, and $B \setminus A \subset [\ell]$
- $z_{A_1} z_{A_2}^r z_{A_3}$, where $A_2 \neq [k]$ for any $0 \leq k \leq n$, such that $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$

The monomials outside of this ideal are precisely the elements of the negative descent basis for $T_n/J_{r,n}$ (for $r = 1$, this is the Garsia-Stanton basis described above).

Proof. We will first show the argument for $r = 1$, the Garsia-Stanton basis, then we will generalize the argument for $r \geq 2$. Assume unless otherwise stated that elements of sets are written in ascending order, e.g. $A = \{a_1, a_2, \dots, a_\ell\}$ implies $a_1 < a_2 < \dots < a_\ell$. First, note that the following observations imply that every monomial \hat{a}_π is not divisible by any of the generators of $N_{1,n}$.

- z_\emptyset clearly cannot divide \hat{a}_π by construction.
- $z_{\{1,2,3,\dots,k\}}$ cannot divide \hat{a}_π , as this would imply that there is a descent at the position k , but there is no element smaller than k which has not already appeared.
- z_A^2 cannot divide \hat{a}_π , as by definition each set A which arises from $\text{Des}(\pi)$ must be unique.
- By definition, if $z_A z_B$ is a factor of \hat{a}_π , it implies that $A \subset B$ or vice versa.
- If $z_A z_B$ divides \hat{a}_π with $A \subset B$ such that $A = \{a_1, a_2, \dots, a_\ell\}$ and $B = A \cup \{b_1, \dots, b_k\}$, we must have that $b_1 < a_\ell$ else there is no descent possible at position ℓ .
- If $z_A z_B$ divides \hat{a}_π with $A \subset B$ such that $[\ell] \not\subset A$ and $[\ell] \subset B$, we must have some element $x \in B \setminus A$ such that $x \notin [\ell]$, else no descent could occur since $\pi(|B|) \in [\ell]$ and $[\ell] \subset \{\pi(1), \dots, \pi(|B|)\}$.
- If $z_{A_1} z_{A_2} z_{A_3}$ divides \hat{a}_π where $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$, then no descent could occur between set A_2 and A_3 , i.e. in position $\pi(|A_2|)$.

Suppose next that we have a monomial in $m \in T_n$, which is divisible by none of the generators of $N_{1,n}$. We claim that there exists some $\pi \in S_n$ such that $m = \hat{a}_\pi$; to prove this claim, first we write

$$m = z_{B_1} z_{B_2} \cdots z_{B_s}$$

where $B_1 \subset B_2 \subset \cdots \subset B_s$. We denote $B_1 = \{\beta_{1_1}, \dots, \beta_{1_{m_1}}\}$ and $B_i = B_{i-1} \cup \{\beta_{i_1}, \dots, \beta_{i_{m_i}}\}$ for all $1 < i \leq s$. Note that this union corresponds to the permutation

$$\pi = \beta_{1_1} \cdots \beta_{1_{m_1}} \beta_{2_1} \cdots \beta_{2_{m_2}} \cdots \beta_{s_1} \cdots \beta_{s_{m_s}} \gamma_1 \cdots \gamma_t$$

where $\gamma_1 < \gamma_2 < \cdots < \gamma_t$ are the elements which do not appear in any B_i set. Moreover, we have that $\beta_{i_{m_i}} > \beta_{i+1_1}$ and $\beta_{s_{m_s}} > \gamma_1$ and these will be the only such descents since m is not divisible by any of the generators of $N_{r,n}$. Hence, m is a Garsia-Stanton descent element \hat{a}_π . (This argument is similar to standard P -partition arguments [49, Lemma 3.15.3].)

Now suppose that $r \geq 2$. By a similar argument to that just given, $b_{(\pi,X)}^r$ is not divisible by a monomial from among the generators of $N_{r,n}$, since:

- z_\emptyset clearly cannot divide $b_{(\pi,X)}^r$ by construction.
- $z_{[k]}^r$ cannot appear in $b_{(\pi,X)}^r$, as, since the greatest possible multiplicity of any element in X is $r - 1$, this would imply that there is a position k descent in π when all smaller elements than $\pi(k)$ have already appeared in π .
- z_A^{r+1} for $A \neq [k]$ cannot appear in $b_{(\pi,X)}^r$ as we only obtain a single z_A from \hat{a}_π , and we can obtain at most $r - 1$ copies of z_A from the product over X . Note that if z_A^r appears in $b_{(\pi,X)}^r$, then one of the z_A terms must have come from the product indexed by $\text{Des}(\pi)$, and thus $|A| \in \text{Des}(\pi)$.
- By definition, $z_A z_B$ a factor of $b_{(\pi,X)}^r$ implies $A \subseteq B$ or vice-versa.

- If $z_A^r z_B$ appears in $b_{(\pi, X)}^r$ where $A \subset B$ with $A = \{a_1, a_2, \dots, a_\ell\}$ and $B = A \cup \{b_1, \dots, b_k\}$, it follows that $|A| \in \text{Des}(\pi)$, thus we must have that $b_1 < a_\ell$ else there is no descent occurring in π in position $|A|$.
- If $z_A z_B^r$ appears in $b_{(\pi, X)}^r$ where $A \subset B$ with $[\ell] \not\subset A$ and $[\ell] \subset B$, then $|B| \in \text{Des}(\pi)$. Hence, there must exist an element $x \in B \setminus A$ such that $x \notin [\ell]$, else no descent can occur.
- If $z_{A_1} z_{A_2}^r z_{A_3}$ appears in $b_{(\pi, X)}^r$ such that $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$, then no descent can occur between set A_2 and A_3 , i.e. in position $\pi(|A_2|)$, but the power of r on $z_{A_2}^r$ forces that there is such a descent. Hence this divisibility is not possible.

Suppose next that we have a monomial $m_r \in T_n$ that is divisible by none of the generators of $N_{r,n}$. We claim that there exists some $\pi \in S_n$ and X a multiset of $[n]$ with every element having multiplicity strictly less than r such that $m_r = b_{(\pi, X)}^r$. An example illustrating the following proof is given in Example 2.4.5. To prove this claim, first we write

$$m_r = z_{B_1}^{b_1} z_{B_2}^{b_2} \cdots z_{B_s}^{b_s}$$

where we have $B_1 \subset B_2 \subset \cdots \subset B_s$. Note that $b_i \leq r$ if $B_i \neq [k]$ and $b_i \leq r - 1$ if $B_i = [k]$. As in the previous case, inductively define $B_i = B_{i-1} \cup \{\beta_{i_1}, \dots, \beta_{i_{m_i}}\}$. Construct a new monomial

$$m'_r = \begin{cases} z_{B_1} z_{B_2} \cdots z_{B_s} & \text{if } B_s \neq [n] \\ z_{B_1} z_{B_2} \cdots z_{B_{s-1}} & \text{if } B_s = [n] \end{cases}$$

and the set

$$\tilde{X} = \begin{cases} \underbrace{\{c_1, \dots, c_1\}}_{b_1-1 \text{ times}}, \underbrace{\{c_2, \dots, c_2\}}_{b_2-1 \text{ times}}, \dots, \underbrace{\{c_s, \dots, c_s\}}_{b_s-1 \text{ times}} & \text{if } B_s \neq [n] \\ \underbrace{\{c_1, \dots, c_1\}}_{b_1-1 \text{ times}}, \underbrace{\{c_2, \dots, c_2\}}_{b_2-1 \text{ times}}, \dots, \underbrace{\{c_{s-1}, \dots, c_{s-1}\}}_{b_{s-1}-1 \text{ times}}, \underbrace{\{c_s, \dots, c_s\}}_{b_s \text{ times}} & \text{if } B_s = [n] \end{cases}$$

where $c_i = |B_i|$. We associate to m'_r the permutation

$$\pi = \beta_{1_1} \cdots \beta_{1_{m_1}} \beta_{2_1} \cdots \beta_{2_{m_2}} \cdots \beta_{s_1} \cdots \beta_{s_{m_s}} \gamma_1 \cdots \gamma_t$$

where $\gamma_1 < \gamma_2 < \cdots < \gamma_t$ are any elements which do not appear in any B_i set. Since the β -values within each B_i are increasing, the only possible descents occur between $\beta_{i_{m_i}}$ and β_{i+1_1} . If we have $\beta_{i_{m_i}} > \beta_{i+1_1}$, then we have a descent and we do nothing. (Note that the final three types of generators of $N_{r,n}$ force a descent to occur if b_i takes on a maximal value of r or $r - 1$, showing that all seven of the types of generators of $N_{r,n}$ are required for this argument to hold.) If we have

$$\beta_{i_{m_i}} < \beta_{i+1_1} \tag{2.2}$$

then there is no descent. Let m_{fail} be the product of z_{B_i} over all the i values such that (2.2) holds and define

$$\tilde{m}_r := m'_r / m_{fail}.$$

We have that $\hat{a}_\pi = \tilde{m}_r$ by our argument in the $r = 1$ case. Moreover, we set $X := \tilde{X} \cup \{c_i : \text{where } \beta_{i_{m_i}} < \beta_{i+1_1}\}$, where as before $c_i = |B_i|$. With this choice of permutation and multiset we obtain $m = b_{(\pi, X)}^r$. \square

Example 2.4.5. Let $r = 4$ and $n = 6$ and use the notation from the preceding proof. Consider the monomial

$$z_{\{4\}} z_{\{2,4\}}^4 z_{\{1,2,4,5\}}^2 z_{\{1,2,3,4,5\}}^2 z_{\{1,2,3,4,5,6\}}.$$

Thus, $B_2 = \{2, 4\}$, $B_5 = \{1, 2, 3, 4, 5, 6\}$, and so on. We have that $m'_4 = z_{\{4\}} z_{\{2,4\}} z_{\{1,2,4,5\}} z_{\{1,2,3,4,5\}}$ since $B_5 = \{1, 2, 3, 4, 5, 6\}$. We have that $\tilde{X} = \{2, 2, 2, 4, 5, 6\}$, where the 6 is included since $B_5 = \{1, 2, 3, 4, 5, 6\}$. In this case, the permutation $\pi = 421536$, and $m_{fail} = z_{\{1,2,3,4,5\}}$. Thus, we have that

$$\tilde{m}_4 = z_{\{4\}} z_{\{2,4\}} z_{\{1,2,4,5\}} = \hat{a}_\pi$$

and

$$X = \tilde{X} \cup \{5\} = \{2, 2, 2, 4, 5, 5, 6\}.$$

It is straightforward to check that

$$b_{(421536, \{2,2,2,4,5,5,6\})}^4 = z_{\{4\}} z_{\{2,4\}}^4 z_{\{1,2,4,5\}}^2 z_{\{1,2,3,4,5\}}^2 z_{\{1,2,3,4,5,6\}}$$

as desired. Note that here we have recovered the correspondence given in Example 2.3.3.

Definition 2.4.6. Given two sets A and B such that $|A| = |B| = k$, we say that A is *lexicographically before* B if there exists $i \in A$ such that $i \notin B$ and given any $j \in B$ such that $j < i$ we have $j \in A$.

For example, the ordering of 3-subsets of the 5-set would be $1, 2, 3 < 1, 2, 4 < 1, 2, 5 < 1, 3, 4 < 1, 3, 5 < 1, 4, 5 < 2, 3, 4 < 2, 3, 5 < 2, 4, 5 < 3, 4, 5$. Our next step is to prove that the monomials listed in Proposition 2.4.4 arise as leading terms of $J_{r,n}$ when the following monomial term order is imposed on T_n .

Definition 2.4.7. Give the variables of T_n the linear order $z_A > z_B$ if $|A| < |B|$ or if $|A| = |B|$ and A is lexicographically before B . With respect to this ordering of variables, endow T_n with the graded reverse lexicographic (or *grevlex*) term order. In this setting, grevlex order is as follows. Let $(\alpha_A)_{A \subseteq [n]}$ and $(\beta_A)_{A \subseteq [n]}$ be vectors in $\mathbb{Z}_{\geq 0}^{\sum_{A \subseteq [n]} |A|}$ with entries totally ordered by setting the A -th coordinate to be larger than the B -th coordinate if and only if $z_A > z_B$. For two monomials in T_n , we have

$$\prod_{A \subseteq [n]} z_A^{\alpha_A} >_{grevlex} \prod_{A \subseteq [n]} z_A^{\beta_A}$$

if either (1) $\sum_{A \subseteq [n]} \alpha_A > \sum_{A \subseteq [n]} \beta_A$ or (2) $\sum_{A \subseteq [n]} \alpha_A = \sum_{A \subseteq [n]} \beta_A$ and in $(\alpha_A - \beta_A)_{A \subseteq [n]}$ the right most non-zero entry is negative.

Example 2.4.8. The variables in T_3 are ordered as follows:

$$z_\emptyset > z_{\{1\}} > z_{\{2\}} > z_{\{3\}} > z_{\{1,2\}} > z_{\{1,3\}} > z_{\{2,3\}} > z_{\{1,2,3\}}$$

We have that

$$z_{\{2\}}^4 > z_\emptyset z_{\{1\}}^2 z_{\{1,2\}}$$

since the exponent vectors for these monomials with respect to the linear order of the variables above are $(0, 0, 4, 0, 0, 0, 0, 0)$ and $(1, 2, 0, 0, 1, 0, 0, 0)$, hence we have

$$(0, 0, 4, 0, 0, 0, 0, 0) - (1, 2, 0, 0, 1, 0, 0, 0) = (-1, -2, 4, 0, -1, 0, 0, 0)$$

with negative right-most non-zero entry.

We will need the following definition for the proof of Theorem 2.4.10.

Definition 2.4.9. We call a pair of subsets A and B such that $A \not\subseteq B$ and $B \not\subseteq A$ a *Sperner 2-pair*.

Theorem 2.4.10. *There exists a Gröbner basis $G_{r,n}$ of $J_{r,n}$ for which the ideal generated by $\text{LT}(G_{r,n})$ is the ideal $N_{r,n}$ generated by terms of the form listed in Proposition 2.4.4.*

Prior to proving the general Gröbner basis result, it is useful to consider a small example. Take $J_{3,2} = \langle z_{\{1\}}z_{\{2\}} - z_{\emptyset}z_{[2]}, z_{\emptyset}, z_{\{1\}}^3 + z_{\{2\}}^3, z_{[2]}^3 \rangle$. From the list of desired leading terms given in Proposition 2.4.4, the only term not immediately accounted for is $z_{\{2\}}^3$. The only nontrivial S -polynomial to consider initially is

$$\begin{aligned} S(z_{\{1\}}z_{\{2\}} - z_{\emptyset}z_{[2]}, z_{\{1\}}^3 + z_{\{2\}}^3) &= \frac{z_{\{1\}}^3 z_{\{2\}}}{z_{\{1\}}z_{\{2\}}} \cdot (z_{\{1\}}z_{\{2\}} - z_{\emptyset}z_{[2]}) - \frac{z_{\{1\}}^3 z_{\{2\}}}{z_{\{1\}}^3} \cdot (z_{\{1\}}^3 + z_{\{2\}}^3) \\ &= -z_{\{2\}}^4 - z_{\emptyset}z_{\{1\}}^2 z_{\{1,2\}}. \end{aligned}$$

Under our term order, the leading term is $-z_{\{2\}}^4$, which is as desired. In order to show that no additional polynomials appear in the Gröbner basis, an exhaustive check of all other S -polynomials shows they reduce to 0. Alternatively, we can argue that no other terms will appear because we can compute that $\dim_{\mathbb{C}}(T_2/J_{3,2}) = 3^2 \cdot 2 = 18$ via a Hilbert series argument that is explicitly given by (2.3) in the proof below, thus no other leading terms can appear without contradicting this known dimension. In small examples, either argument will suffice. However, for arbitrary r and n , the latter argument is more efficient.

Proof of Theorem 2.4.10. Use the term order for T_n described above. Our proof will involve computing S -polynomials starting from the generators of $J_{r,n}$. To minimize the number of computations required, we first make a dimension argument showing that the number of monomials outside of the leading term ideal for $J_{r,n}$ is the number of elements of the negative descent basis. We then compute S -polynomials to produce elements with all of the leading terms listed in Proposition 2.4.4, which will complete the proof. We will compute the S -polynomials for arbitrary r , but we will make two dimension arguments, for $r = 1$ and $r \geq 2$.

Consider $r = 1$. It is a straightforward observation to notice that the number of elements of $p \in R_n$ such that $\deg(p) = t^k$ are precisely the lattice points at height k in the cone $([0, 1]^n)$ and the cardinality of these elements is $(k + 1)^n$. Combining this observation with [49, Proposition 1.4.4], we see that the Hilbert series of R_n is given by

$$\text{Hilb}(R_n; t) = \sum_{k \geq 0} (k + 1)^n t^k = \frac{A_n(t)}{(1 - t)^{n+1}}$$

where $A(n) = \sum_{\pi \in S_n} t^{\text{des}(\pi)}$ is the Eulerian polynomial. Let $\mathcal{C}_{1,n} := \mathbb{C}[\hat{e}_k + I_n | 0 \leq k \leq n]$, and note that the elements $\hat{e}_k + I_n$ are algebraically independent since they specialize in R_n (by setting $t = 1$) to the usual elementary symmetric functions; note that $\text{Hilb}(\mathcal{C}_{1,n}; t) = \frac{1}{(1-t)^{n+1}}$. Hochster's Theorem implies that R_n is Cohen-Macaulay [31], and since $\text{invar}(1, n)$ is an ideal generated by an algebraically independent system of parameters, we have

$$\text{Hilb}(T_n/J_{1,n}; t) = A_n(t)$$

by [26, Lemma 17.1]. The \mathbb{C} -dimension of $T_n/J_{1,n}$ is

$$\dim_{\mathbb{C}}(T_n/J_{1,n}) = \text{Hilb}(T_n/J_{1,n}; 1) = A(1) = n!,$$

which is the number of elements in the Garsia-Stanton descent basis, as desired.

Now, suppose that $r \geq 2$. Let $\mathcal{C}_{r,n} = \mathbb{C}[z_{\emptyset} + I_n, \hat{e}_{r,k} + I_n | 1 \leq k \leq n]$. Given that R_n is Cohen-Macaulay and that $\hat{e}_{r,k} + I_n$ and $z_{\emptyset} + I_n$ are algebraically independent, hence $\text{Hilb}(\mathcal{C}_{r,n}; t) = \frac{1}{(1-t)(1-t^r)^n}$, we have that

$$\text{Hilb}(R_n; t) = \sum_{k \geq 0} (k+1)^n t^k = \frac{B_{r,n}(t)}{(1-t)(1-t^r)^n}$$

where $B_{r,n}(t) = A_n(t) \cdot (1+t+\dots+t^{r-1})^n$ by our previous calculation for $r=1$. Thus,

$$\text{Hilb}(T_n/J_{r,n}; t) = A_n(t) \cdot (1+t+\dots+t^{r-1})^n$$

from which we can conclude that

$$\dim_{\mathbb{C}}(T_n/J_{r,n}) = \text{Hilb}(T_n/J_{r,n}; 1) = B_{r,n}(1) = r^n n!, \quad (2.3)$$

which is the number of elements in the negative descent basis, as desired.

Next, we move to S -polynomial calculations. Our goal is to compute S -polynomials until all the elements listed in Proposition 2.4.4 arise as leading terms; since at that point we will have reached the correct value of $\dim_{\mathbb{C}}(T_n/J_{r,n}) = \dim_{\mathbb{C}}(T_n/\text{LT}(J_{r,n}))$, we must have a Gröbner basis.

We begin by noting that some of our desired leading terms arise from the generators of $J_{r,n}$. First, $z_A z_B$ such that $A \not\subset B$ and $B \not\subset A$ where $A \neq [k] \neq B$ for any k are leading terms of I_n . The monomials z_{\emptyset} and z_A^r where $A = [k]$ for $k = 1, \dots, n$ are the leading terms of $\text{invar}(r, n)$. These account for the fourth, first, and second items listed in Proposition 2.4.4, respectively.

To obtain an element with the leading term z_A^{r+1} as given in the third bullet of Proposition 2.4.4, suppose that $|A| = k$ and consider the following S -polynomial:

$$\begin{aligned} & S(\hat{e}_{r,k}, z_{[k]} z_A - z_{[k] \cap A} z_{[k] \cup A}) \\ &= \frac{z_{[k]}^r z_A}{z_{[k]}^r} \left(z_{[k]}^r + z_{A_1}^r + z_{A_2}^r + \dots + z_A^r + \dots + z_{A_{\binom{n}{k}-1}}^r \right) \\ & \quad - \frac{z_{[k]}^r z_A}{z_{[k]} z_A} (z_{[k]} z_A - z_{[k] \cap A} z_{[k] \cup A}) \\ &= z_A \left(z_{A_1}^r + z_{A_2}^r + \dots + z_A^r + \dots + z_{A_{\binom{n}{k}-1}}^r \right) + z_{[k]}^{r-1} z_{[k] \cap A} z_{[k] \cup A} \end{aligned}$$

Note that the term order implies that

$$z_A z_{A_1}^r > z_A z_{A_2}^r > \dots > z_A z_A^r > \dots > z_A z_{A_{\binom{n}{k}-1}}^r > z_{[k]}^{r-1} z_{[k] \cap A} z_{[k] \cup A}$$

However, for each i where $A_i \neq A$, $z_A z_{A_i}$ is the leading term of a polynomial of $J_{r,n}$, and we use $z_A z_{A_i} - z_{A \cap A_i} z_{A \cup A_i} \in J_{r,n}$ to rewrite $z_A z_{A_i}^r$, yielding

$$S(\hat{e}_{r,k}, z_{[k]} z_A - z_{[k] \cap A} z_{[k] \cup A}) = z_A^{r+1} + \sum_j z_{A \cap A_j} z_{A_j}^{r-1} z_{A \cup A_j} \quad (2.4)$$

where the sum is over all j such that $|A_j| = k$, $A_j \neq A$, and $A \cap A_j \neq \emptyset$, since any terms involving z_\emptyset are elements of $J_{r,n}$. The observation that $|A| < |A \cup A_j|$ for all such j implies that z_A^{r+1} is the leading term of this polynomial, as desired.

Assume that we have added all prior S -polynomial calculations to the generators of $J_{r,n}$. To obtain terms of the form $z_A^r z_B$, where $A \subset B$ with $\max(A) < \min(B \setminus A)$ as listed in the fifth bullet of Proposition 2.4.4, let $|A| = k$. We compute the S -polynomial of $\hat{e}_{r,k}$ and the generator of I_n with leading term $z_{[k]} z_B$. Note that $z_{[k]} z_B$ is the leading term of a generator of I_n , since by assumption $A \neq [k]$ thus if $[k] \subset B$ this would violate the condition $\max(A) < \min(B \setminus A)$. We compute:

$$\begin{aligned} & S(\hat{e}_{r,k}, z_{[k]} z_B - z_{[k] \cap B} z_{[k] \cup B}) \\ &= \frac{z_{[k]}^r z_B}{z_{[k]}^r} \left(z_{[k]}^r + z_{A_1}^r + z_{A_2}^r + \cdots + z_A^r + \cdots + z_{A_{\binom{n}{k}-1}}^r \right) \\ & \quad - \frac{z_{[k]}^r z_B}{z_{[k]} z_B} (z_{[k]} z_B - z_{[k] \cap B} z_{[k] \cup B}) \\ &= z_B \left(z_{A_1}^r + z_{A_2}^r + \cdots + z_A^r + \cdots + z_{A_{\binom{n}{k}-1}}^r \right) + z_{[k]}^{r-1} z_{[k] \cap B} z_{[k] \cup B} \end{aligned}$$

We have the ordering

$$z_{A_1}^r z_B > z_{A_2}^r z_B > \cdots > z_A^r z_B > \cdots > z_{A_{\binom{n}{k}-1}}^r z_B > z_{[k]}^{r-1} z_{[k] \cap B} z_{[k] \cup B}.$$

Moreover, by the condition $\max(A) < \min(B \setminus A)$ and the use of lexicographic order on subsets, we know that $A_i \not\subset B$ for all i such that $z_{A_i}^r z_B > z_A^r z_B$. This is true because if $A_i \subset B$, then there exists some $j \in A_i$, $j \notin A$ so that $\max(A) < j$ and the condition that $|A| = |A_i|$ implies that there must exist some $s \in A$ such that $s \notin A_i$ and for all $t \in A_i$ such that $t < s$ we have $t \in A$, which would contradict $z_{A_i}^r z_B > z_A^r z_B$ by the definition of our variable ordering arising from the lexicographic ordering on subsets. The condition that $A_i \not\subset B$ implies that $z_{A_i}^r z_B$ is a leading term of a polynomial in I_n . Applying $z_{A_i} z_B - z_{A_i \cap B} z_{A_i \cup B} \in J_{r,n}$ to the term $z_{A_i}^r z_B$ will produce $z_{A_i \cap B} z_{A_i}^{r-1} z_{A_i \cup B} < z_A^r z_B$. Therefore, we will have

$$\begin{aligned} & S(\hat{e}_{r,k}, z_{[k]} z_B - z_{[k] \cap B} z_{[k] \cup B}) = \\ & z_A^r z_B + \sum_j z_{A_j}^r z_B + \sum_m z_{A_m \cap B} z_{A_m}^{r-1} z_{A_m \cup B} \end{aligned}$$

where the first sum is over all j so that $|A_j| = |A|$, $A_j \neq A$, and $A_j \subset B$, which implies that $z_A > z_{A_j}$ by condition $\max(A) < \min(B \setminus A)$. The second sum is over all m such that

$|A_m| = |A|$ where A_m and B are a Sperner 2-pair with $A_m \cap B \neq \emptyset$, as if the intersection was empty then the resulting term would be a multiple of z_\emptyset and hence an element of $J_{r,n}$. It follows from a simple cardinality argument that $z_{A_m \cup B} < z_B$, and thus $z_A^r z_B$ is a leading term in $J_{r,n}$.

Assume again that we have added all prior S -polynomial calculations to the generators of $J_{r,n}$. To obtain terms of the form $z_A z_B^r$ where there is an ℓ such that $[\ell] \not\subset A$, $[\ell] \subset B$ and $B \setminus A \subset [\ell]$, as listed in the sixth bullet of Proposition 2.4.4, let $|B| = k$. We compute the S -polynomial of $\hat{e}_{r,k}$ and the generator of I_n with leading term $z_A z_{[k]}$, which is a leading term since there exists an element $x \in [\ell] \subset [k]$ such that $x \notin A$ and there also exists $y = \max(A) = \max(B) \notin [k]$:

$$\begin{aligned} & S(\hat{e}_{r,k}, z_A z_{[k]} - z_{A \cap [k]} z_{A \cup [k]}) \\ &= \frac{z_A z_{[k]}^r}{z_{[k]}^r} \left(z_{[k]}^r + z_{B_1}^r + \cdots + z_B^r + \cdots + z_{B_{\binom{n}{k}-1}}^r \right) \\ &\quad - \frac{z_A z_{[k]}^r}{z_A z_{[k]}} (z_A z_{[k]} - z_{A \cap [k]} z_{A \cup [k]}) \\ &= z_A \left(z_{B_1}^r + \cdots + z_B^r + \cdots + z_{B_{\binom{n}{k}-1}}^r \right) + z_{[k]}^{r-1} z_{A \cap [k]} z_{A \cup [k]} \end{aligned}$$

which yields the term order of

$$z_A z_{B_1}^r > z_A z_{B_2}^r > \cdots > z_A z_B^r > \cdots > z_A z_{B_{\binom{n}{k}-1}}^r > z_{[k]}^{r-1} z_{A \cap [k]} z_{A \cup [k]}.$$

Note that $A \not\subset B_i$ for all i such that $z_{B_i} > z_B$. This is true because if $A \subset B_i$ for $B_i \neq B$, then given that $|B| = |B_i|$ we must have $z_B > z_{B_i}$ because $B \setminus A$ contains precisely the smallest elements not contained in A and thus $B_i \setminus A$ must contain at least one larger element meaning that B_i is lexicographically after B . The condition that $A \not\subset B_i$ for all i such that $z_{B_i} > z_B$ implies that $z_A z_{B_i}$ is the leading term of a polynomial in I_n . As in our previous cases, this leads to the calculation

$$\begin{aligned} & S(\hat{e}_{r,k}, z_A z_{[k]} - z_{A \cap [k]} z_{A \cup [k]}) = \\ & z_A z_B^r + \sum_j z_A z_{B_j}^r + \sum_m z_{A \cap B_m} z_{B_m}^{r-1} z_{A \cup B_m} \end{aligned}$$

where the first sum is over all j such that $|B_j| = |B|$, $B \neq B_j$, and $A \subset B_j$. The second sum is over all m such that A and B_m are a Sperner 2-pair with $A \cap B_m \neq \emptyset$. Notice that we know that $|B| = k$ and $B \neq [k]$ which says that there is at least some subset $\{j_1, \dots, j_t\} \subset B$ such that $j_i > k$ for all i and the defining condition $[\ell] \not\subset A$, $[\ell] \subset B$ and $B \setminus A \subset [\ell]$ implies that $j_i \in A$ for some i . Thus, $|A \cup [k]| > k = |B|$. Ergo, we have $z_A z_B^r$ as the leading term.

Our final case is to obtain the terms listed in the seventh bullet of Proposition 2.4.4, i.e. those of type $z_{A_1} z_{A_2}^r z_{A_3}$ where $A_1 \subset A_2 \subset A_3$ and $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ with $A_2 \neq [j]$ for all j . Assume that we have added all prior S -polynomials to the generators of $J_{r,n}$. We consider the S -polynomial for the elements $z_{A_2} z_{A_1 \cup (A_3 \setminus A_2)} - z_{A_1} z_{A_3}$ and the generator from (2.4) given by $z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}$ where $|C_j| = |A_2| = k$, $A_2 \neq C_j$, and $A_2 \cap C_j \neq \emptyset$. Let $B := A_1 \cup (A_3 \setminus A_2)$ for convenience of notation, and compute:

$$\begin{aligned}
& S \left(z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}, z_{A_2} z_B - z_{A_1} z_{A_3} \right) \\
&= \frac{z_{A_2}^{r+1} z_B}{z_{A_2}^{r+1}} \left(z_{A_2}^{r+1} + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} \right) \\
&\quad - \frac{z_{A_2}^{r+1} z_B}{z_{A_2} z_B} (z_{A_2} z_B - z_{A_1} z_{A_3}) \\
&= z_B \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} + z_{A_1} z_{A_2}^r z_{A_3} \\
&= z_{A_1 \cup (A_3 \setminus A_2)} \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j} + z_{A_1} z_{A_2}^r z_{A_3}
\end{aligned}$$

We now wish to show the $z_{A_1} z_{A_2}^r z_{A_3}$ is the leading term. Consider the terms involving C_j . There are three possible cases

1. $|A_2 \cup C_j| > |A_3|$
1. $|A_2 \cup C_j| < |A_3|$
1. $|A_2 \cup C_j| = |A_3|$

which we consider individually.

Case 1: If we have that $|A_2 \cup C_j| > |A_3|$, then we have $z_{A_1} z_{A_2}^r z_{A_3} > z_{A_2 \cap C_j} z_{A_1 \cup (A_3 \setminus A_2)} z_{C_j}^{r-1} z_{A_2 \cup C_j}$ immediately by the definition of graded reverse lexicographic order.

Case 2: Suppose that we have $|A_2 \cup C_j| < |A_3|$. Note that this implies that there exists $x \in A_3$ such that $x \notin A_2 \cup C_j$ and hence $x \in A_1 \cup (A_3 \setminus A_2)$. We also have $y \in A_2 \cup C_j$ such that $y \notin A_1 \cup (A_3 \setminus A_2)$. Hence, we have that $A_1 \cup (A_3 \setminus A_2)$ and $A_2 \cup C_j$ are a Sperner 2-pair. This implies that we can replace the monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$\begin{aligned}
& z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_1 \cup (A_3 \setminus A_2)) \cap (A_2 \cup C_j)} z_{(A_1 \cup (A_3 \setminus A_2)) \cup (A_2 \cup C_j)} \\
&= z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (C_j \cap (A_3 \setminus A_2))} z_{A_3 \cup C_j}
\end{aligned}$$

It is clear that $|A_3 \cup C_j| \geq |A_3|$. If the inequality is strict, then we are done. If $A_3 \cup C_j = A_3$, note that $C_j \subset A_3$ and that $C_j \cap (A_3 \setminus A_2) \neq \emptyset$ since $|C_j| = |A_2|$. We will now consider the variable $z_{A_1 \cup (C_j \cap (A_3 \setminus A_2))}$. We note that two subcases arise:

2.i. $A_1 \cup (C_j \cap (A_3 \setminus A_2)) = A_1 \cup C_j$ (equivalently $C_j \cap A_1 = C_j \cap A_2$)

2.ii. $A_1 \cup (C_j \cap (A_3 \setminus A_2))$ and $A_2 \cap C_j$ are a Sperner 2-pair.

Subcase 2.i: Note that $|A_1 \cup C_j| \geq |A_2|$ with equality occurring if $A_1 \cup C_j = C_j$. If the inequality is strict, we are done. If $A_1 \cup C_j = C_j$, then $|C_j| = |A_2|$, but since $C_j \cap A_1 = C_j \cap A_2$ and $C_j \cap (A_3 \setminus A_2) \neq \emptyset$, the condition $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ implies that A_2 is lexicographically before C_j , which is desired.

Subcase 2.ii: The existence of such a Sperner 2-pair allows us to replace the monomial through division by

$$\begin{aligned} & z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2))) \cap (A_2 \cap C_j)} z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2))) \cup (A_2 \cap C_j)} z_{C_j}^{r-1} z_{A_3} \\ &= z_{(A_1 \cup (C_j \cap (A_3 \setminus A_2))) \cap (A_2 \cap C_j)} z_{A_1 \cup C_j} z_{C_j}^{r-1} z_{A_3} \end{aligned}$$

Showing the desired outcome is now identical to the argument in Subcase 2.i.

Case 3: Suppose that we have $|A_2 \cup C_j| = |A_3|$. In this case, it is sufficient to consider the following three plausible sub-cases.

- 3.i.** $A_2 \cup C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair.
- 3.ii.** The subcase 3.i. is false, but $A_2 \cap C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair.
- 3.iii.** $A_2 \cap C_j$, $A_2 \cup C_j$, and $A_1 \cup (A_3 \setminus A_2)$ have no Sperner 2-pairs between them.

Subcase 3.i: Suppose we have that the sets $A_2 \cup C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair. This means that via division, we can replace the existing monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$\begin{aligned} & z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cup C_j) \cup (A_1 \cup (A_3 \setminus A_2))} \\ &= z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{A_3 \cup C_j} \end{aligned}$$

By virtue of the Sperner 2-pair assumptions, we have that there exists $x \in C_j$ such that $x \notin A_3$, which yields $|A_3 \cup C_j| > |A_3|$ and hence

$$z_{A_1} z_{A_2}^r z_{A_3} > z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{(A_2 \cup C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{A_3 \cup C_j}$$

and we are done.

Subcase 3.ii: Suppose that $A_2 \cap C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are a Sperner 2-pair, but that $A_2 \cup C_j$ and $A_1 \cup (A_3 \setminus A_2)$ are not. Then note that we have $A_1 \cup (A_3 \setminus A_2) \subset A_2 \cup C_j$, which implies that $A_3 \setminus A_2 \subset C_j$, and hence $A_2 \cup C_j = A_3$ by the cardinality assumption. Now, by the existence of the Sperner 2-pair, we can replace via division the existing monomial $z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_1 \cup (A_3 \setminus A_2)} z_{A_2 \cup C_j}$ with the monomial

$$z_{(A_2 \cap C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{C_j}^{r-1} z_{A_3}$$

Moreover, notice that $C_j \subseteq ((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))$. If the equality is strict, we have that $|A_2| < |((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))|$ and we are done. If we have equality, then we know $|A_2| = |((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))|$. By the assumption that $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$, this implies that A_2 is lexicographically before $((A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2)))$. Thus we will have

$$z_{A_1} z_{A_2}^r z_{A_3} > z_{(A_2 \cap C_j) \cap (A_1 \cup (A_3 \setminus A_2))} z_{(A_2 \cap C_j) \cup (A_1 \cup (A_3 \setminus A_2))} z_{C_j}^{r-1} z_{A_3}$$

which is as desired.

Subcase 3.iii: Suppose that the sets $A_2 \cap C_j$, $A_2 \cup C_j$, and $A_1 \cup (A_3 \setminus A_2)$ have no Sperner 2-pairs between them. This implies the following containment

$$A_2 \cap C_j \subset A_1 \cup (A_3 \setminus A_2) \subset A_2 \cup C_j = A_3$$

because $A_2 \cap C_j \subseteq A_1$ and $A_3 \subseteq A_2 \cup C_j$, which follows from the necessary containment and the fact that these sets have the same cardinality. These observations allow us to conclude that $C_j \subseteq A_1 \cup (A_3 \setminus A_2)$. If the containment is strict, we have that $|A_1 \cup (A_3 \setminus A_2)| > |A_2|$ and we are done. If equality holds, we have $|A_1 \cup (A_3 \setminus A_2)| = |A_2|$. However, the assumed condition that $\max(A_2 \setminus A_1) < \min(A_3 \setminus A_2)$ implies that A_2 is lexicographically before $A_1 \cup (A_3 \setminus A_2)$. Thus, we have that

$$z_{A_1} z_{A_2}^r z_{A_3} > z_{A_2 \cap C_j} z_{A_1 \cup (A_3 \setminus A_2)} z_{C_j}^{r-1} z_{A_3}$$

which is our desired result.

Given all of the above, we can conclude that

$$S \left(z_{A_2}^r + \sum_j z_{A_2 \cap C_j} z_{C_j}^{r-1} z_{A_2 \cup C_j}, z_{A_2} z_B - z_{A_1} z_{A_3} \right) = z_{A_1} z_{A_2}^r z_{A_3} + p_{A_1 A_2^r A_3}$$

where $p_{A_1 A_2^r A_3}$ is a polynomial with $\text{LT}(p_{A_1 A_2^r A_3}) < z_{A_1} z_{A_2}^r z_{A_3}$.

We have now shown that all of our desired leading terms appear through the optimized Buchberger Algorithm. Because of our previous dimension calculation for $T_n/J_{r,n}$, we know that no additional leading terms can result from further computations, thus we have a Gröbner basis. \square

We have thus established Theorem 2.4.1, as it follows immediately from Theorem 2.4.3, Proposition 2.4.4, and Theorem 2.4.10.

2.5 Combinatorial identities

We will now compute multigraded Hilbert series to prove Theorems 2.1.1 and 2.1.2. Recall from Section 1.4 that we can define a Hilbert series with respect to a \mathbb{Z}^m -grading for any $m \geq 1$ as in (1.3). We now define the \mathbb{Z}^2 -grading which arises from the the defined degree on variables $\deg(z_A) = tq^{|A|}$, where we note that $\deg(z_\emptyset) = t$. We denote this bivariate Hilbert series as $\text{Hilb}(A; t, q)$ for a graded module A of T_n . It is straightforward [5] to show that $\text{Hilb}(R_n; t, q) = \sum_{k \geq 0} [k+1]_q^n t^k$, which we assume for both of the following proofs. We will use the notation $\mathcal{C}_{r,n}$ introduced in the proof of Theorem 2.4.10.

Proof of Theorem 2.1.1. Given that R_n is Cohen-Macaulay and the elements of $\text{invar}(1, n)$ are an algebraically independent homogeneous system of parameters as argued in the proof of Theorem 2.4.10, we can express the Hilbert series in the form

$$\text{Hilb}(R_n; t, q) = \frac{\text{Hilb}(T_n/J_{1,n}; t, q)}{\prod_{j=0}^n (1 - tq^j)}.$$

This follows because it is an elementary exercise to compute that

$$\text{Hilb}(\mathcal{C}_{1,n}; t, q) = \frac{1}{(1-t)(1-tq) \cdots (1-tq^n)}.$$

To compute the numerator, we have

$$\text{Hilb}(T_n/J_{1,n}; t, q) = \sum_{\pi \in S_n} \deg(\hat{a}_\pi) = \sum_{\pi \in S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)}$$

by using the basis for $T_n/J_{1,n}$ given by Theorem 2.4.1. This completes the proof. \square

Proof of Theorem 2.1.2. Given that R_n is Cohen-Macaulay and $\text{invar}(r, n)$ is an algebraically independent homogenous system of parameters, we can express the Hilbert series as

$$\text{Hilb}(R_n; t, q) = \frac{\text{Hilb}(T_n/J_{r,n}; t, q)}{(1-t) \prod_{j=1}^n (1-t^r q^{rj})}.$$

This follows because, as in the previous proof, it is straightforward to show that

$$\text{Hilb}(\mathcal{C}_{r,n}; t, q) = \frac{1}{(1-t)(1-t^r q^r)(1-t^r q^{2r}) \cdots (1-t^r q^{rn})}.$$

Hence, we compute the numerator by employing the basis given in Theorem 2.4.1

$$\begin{aligned} \text{Hilb}(T_n/J_{r,n}; t, q) &= \sum_{(\pi, X) \in \mathbb{Z}_r \wr S_n} \deg(b_{(\pi, X)}^r) \\ &= \sum_{(\pi, X) \in \mathbb{Z}_r \wr S_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} t^{|X|} q^{\sum_{i \in X} i} \\ &= \sum_{(\rho, \epsilon) \in \mathbb{Z}_r \wr S_n} t^{\text{ndes}(\rho, \epsilon)} q^{\text{nmajor}(\rho, \epsilon)}, \end{aligned}$$

completing the proof. \square

2.6 Concluding Remarks

It is worth mentioning that when $r = 1$ there is a graded S_n -module isomorphism between $T_n/J_{1,n}$ and $\mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$.

Theorem 2.6.1. *The map $\varphi : T_n/J_{1,n} \rightarrow \mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$ defined by algebraically extending $z_A + J_{1,n} \mapsto \prod_{i \in A} x_i + \mathcal{I}_n$ is an S_n -isomorphism.*

Proof. Consider $T_n/J_{1,n}$ under the q -grading used in the multigrading for Section 2.5, i.e. $\deg(z_A) = |A|$. Let $\mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$ be graded by total degree. It is clear that φ respects grading, by definition. Moreover, it is clear that φ is an algebra isomorphism, since

$$\begin{aligned} \varphi(z_A + J_{1,n}) \cdot \varphi(z_B + J_{1,n}) &= \left(\prod_{i \in A} x_i + \mathcal{I}_n \right) \cdot \left(\prod_{j \in B} x_j + \mathcal{I}_n \right) \\ &= \left(\prod_{i \in A} x_i \right) \cdot \left(\prod_{j \in B} x_j \right) + \mathcal{I}_n \\ &= \varphi(z_A z_B + J_{1,n}) \end{aligned}$$

which implies $\varphi(\hat{a}_\pi + J_{1,n}) = a_\pi + \mathcal{I}_n$ for all $\pi \in S_n$.

Now we show that the action is preserved. Consider $z_A + J_{1,n}$ and $\sigma \in S_n$, and observe that

$$\begin{aligned}
 \sigma \circ \varphi(z_A + J_{1,n}) &= \sigma \left(\prod_{i \in A} x_i \right) + \mathcal{I}_n \\
 &= \prod_{i \in A} x_{\sigma(i)} + \mathcal{I}_n \\
 &= \prod_{i \in \sigma(A)} x_i + \mathcal{I}_n \\
 &= \varphi(z_{\sigma(A)} + J_{1,n}) \\
 &= \varphi \circ \sigma(z_A + J_{1,n}).
 \end{aligned}$$

□

It would be interesting to determine if the representation-theoretic results of Adin, Brenti, and Roichman [2] are easier to establish in the context of $T_n/J_{1,n}$ rather than $\mathbb{C}[x_1, x_2, \dots, x_n]/\mathcal{I}_n$.

Chapter 3 Hilbert bases and lecture hall partitions

3.1 Introduction

In this chapter, we seek to understand the additive generating sets of lecture hall partitions by way of polyhedral geometry. This chapter is largely self-contained; many of the necessary concepts and definitions are contained within this chapter, including several which have been defined in previous sections. However, for a more thorough review, we refer the reader to Section 1.5 and Section 1.4 for background on lecture hall partitions and polyhedral geometry respectively.

Recall that given a sequence of positive integers $\mathbf{s} \in \mathbb{Z}_{\geq 1}^n$, the \mathbf{s} -lecture hall partitions to be the set

$$L_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

In the case when \mathbf{s} is weakly (or strictly) increasing, \mathbf{s} -lecture hall partitions are a refinement of the set of all partitions. Bousquet-Mélou and Eriksson first introduced the notion of \mathbf{s} -lecture hall partitions in two seminal papers [10, 11], and since then these objects have been vastly studied in various contexts. See the excellent survey of Savage [40] for an overview of much of this work.

One question which remains open in general is the following:

Question 3.1.1. Can we determine the minimal additive generating set for $L_n^{(\mathbf{s})}$ for an arbitrary \mathbf{s} ? Are there nontrivial bounds on the cardinality of this set?

While this is in general a difficult question to answer, one method for answering is to employ tools from polyhedral geometry. Given a sequence $\mathbf{s} = (s_1, \dots, s_n)$, we can define the \mathbf{s} -lecture hall cone to be the rational, pointed, simplicial polyhedral cone given by

$$\mathcal{C}_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

For a rational, pointed cone $C \in \mathbb{R}^n$, the *Hilbert basis* of C is the minimal additive generating set of $C \cap \mathbb{Z}^n$. Noting that $\mathcal{C}_n^{(\mathbf{s})} \cap \mathbb{Z}^n = L_n^{(\mathbf{s})}$, we can now reformulate Question 3.1.1 in terms of polyhedral geometry.

Question 3.1.2. Can we determine the Hilbert basis of $\mathcal{C}_n^{(\mathbf{s})}$ for arbitrary \mathbf{s} ? Can we give nontrivial bounds on the cardinality of this set?

At first glance, this seems like a potentially fruitful question. Beck, Braun, Köppe, Savage, and Zafeirakopoulos [7] show that the elements of the Hilbert basis of $\mathcal{C}_n^{(\mathbf{s})}$ for $\mathbf{s} = (1, 2, \dots, n)$ are naturally indexed by subsets $A \subseteq [n-1]$. Moreover, these elements are all of degree 1 with respect to a particular grading of $\mathcal{C}_n^{(\mathbf{s})}$. This motivates looking for a general form for arbitrary \mathbf{s} .

Unfortunately, it is unlikely that there is a general structure for the Hilbert bases of \mathbf{s} -lecture hall cones and it is almost a certainty that no nontrivial bounds on the cardinality

exist. This can be seen in the simplest case $n = 2$. Let $\mathbf{s} = (s_1, s_2)$ and notice that we have upper and lower bounds, namely $s_1 + 1$ forms an upper bound given by enumerating lattice points in the fundamental parallelepiped of $\mathcal{C}_n^{(\mathbf{s})}$ and 3 is a lower bound provided $s_1 \geq 2$ (2 is the lower bound if $s_1 = 1$). These bounds are in fact sharp, as the sequence $\mathbf{s} = (s_1, k \cdot s_1 + 1)$ for any $k \in \mathbb{Z}_{\geq 1}$ gives a cone whose Hilbert basis has cardinality $s_1 + 1$, whereas the cone for the sequence $\mathbf{s} = (s_1, k \cdot s_1 - 1)$ for any $k \in \mathbb{Z}_{\geq 1}$ has a Hilbert basis of cardinality 3.

Subsequently, in order to obtain meaningful results, we must place some additional restrictions. Motivated by recent work on lecture hall cones [6, 7], we restrict to the case of *\mathbf{u} -generated Gorenstein \mathbf{s} -lecture hall cones* (Defined in Section 3.2). We pose the following question.

Question 3.1.3. Can we determine the Hilbert basis of $\mathcal{C}_n^{(\mathbf{s})}$ where \mathbf{s} is an arbitrary \mathbf{u} -generated Gorenstein sequence? Can we give the cardinality of the set of Hilbert basis elements, or find nontrivial bounds to this set?

In this chapter, we make progress towards answering Question 3.1.3. Section 3.2 is devoted to providing necessary definitions and terminology. In Sections 3.3 and 3.4, we consider well-studied families of sequences, namely the 1 mod k sequences and the ℓ -sequences, which both specialize to the sequence $\mathbf{s} = (1, 2, \dots, n)$. In Sections 3.5, 3.6, and 3.7, we provide a characterization for the Hilbert bases of \mathbf{u} -generated Gorenstein \mathbf{s} -lecture hall cones in \mathbb{R}^n for $n \leq 4$, noting that the complexity of the Hilbert bases grows rapidly as the dimension increases. We conclude in Section 3.8 by providing some direction for future work in the context of commutative algebra, particularly the study of toric ideals and free resolutions.

3.2 Preliminaries

We recall a few definitions from polyhedral geometry. A *polyhedral cone* C in \mathbb{R}^n is the solution set to a finite collection of linear inequalities $Ax \geq 0$ for some real matrix A , or equivalently for some elements $w_1, w_2, \dots, w_j \in \mathbb{R}^n$,

$$C = \text{span}_{\mathbb{R}_{\geq 0}} \{w_1, w_2, \dots, w_j\}.$$

The elements w_i are called *ray generators*. The cone C is said to be *rational* if the matrix A contains rational entries (equivalently if each $w_i \in \mathbb{Q}^n$), it is said to be *simplicial* if it is defined by n independent inequalities (equivalently if $j = n$ and $\{w_i\}_{i=1}^n$ are linearly independent), and it is said to be *pointed* if it does not contain a linear subspace of \mathbb{R}^n . Let C° denote the interior of C .

Given any pointed rational cone $C \subset \mathbb{R}^n$, a *proper grading* of C is a function $g : C \cap \mathbb{Z}^n \rightarrow \mathbb{Z}_{\geq 0}^r$, for some r , satisfying (i) $g(\lambda + \mu) = g(\lambda) + g(\mu)$; (ii) $g(\lambda) = 0$ implies $\lambda = 0$; and (iii) for any $v \in \mathbb{N}^r$, $g^{-1}(v)$ is finite. Moreover, the integer points $C \cap \mathbb{Z}^n$ form a semigroup. Semigroups of this type have unique minimal generating sets known as the *Hilbert basis* of C . Additionally, pointed rational cones give rise to a semigroup algebra structure $\mathbb{C}[C] := \mathbb{C}[C \cap \mathbb{Z}^n]$. For background and details see [9, 34].

We say that a pointed, rational cone $C \subset \mathbb{R}^n$ is *Gorenstein* if there exists a point $\mathbf{c} \in C^\circ$ such that $C^\circ \cap \mathbb{Z}^n = \mathbf{c} + (C \cap \mathbb{Z}^n)$. This point is known as the *Gorenstein point* of C . Due to theorems of Stanley [46], this notion of Gorenstein is equivalent to the commutative algebra

notion of Gorenstein, as C is Gorenstein if and only if the algebra $\mathbb{C}[C]$ is Gorenstein. For reference and commutative algebra details, see [15, 48].

It will also be useful to recall several definitions for convex polytopes and Ehrhart Theory. Let $\mathcal{P} \subset \mathbb{R}^n$ be a n -dimensional convex polytope with vertex set $\{v_1, v_2, \dots, v_d\}$. We say \mathcal{P} is a *lattice polytope* if $v_i \in \mathbb{Z}^n$ for each i . Likewise, we say that \mathcal{P} is a *rational polytope* if $v_i \in \mathbb{Q}^n$ for each i . The *lattice point enumerator* of \mathcal{P} is the function

$$i(\mathcal{P}, t) = \#(t \cdot \mathcal{P} \cap \mathbb{Z}^n)$$

where $t \cdot \mathcal{P} = \{t \cdot \alpha : \alpha \in \mathcal{P}\}$ is the t th dilate of \mathcal{P} with $t \in \mathbb{Z}_{\geq 0}$. By theorems of Ehrhart [19], if \mathcal{P} is lattice, $i(\mathcal{P}, t)$ is a polynomial in the variable t of degree n and if \mathcal{P} is rational, $i(\mathcal{P}, t)$ is a quasipolynomial in the variable t of degree d . Subsequently, we will call $i(\mathcal{P}, t)$ the *Ehrhart polynomial* of \mathcal{P} or the *Ehrhart quasipolynomial* of \mathcal{P} in each respective case.

Given a sequence $\mathbf{s} = (s_1, \dots, s_n)$, the \mathbf{s} -*lecture hall cone* is the rational, pointed, simplicial polyhedral cone defined as follows

$$\mathcal{C}_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

Alternatively, one may consider a ray generator description with integral generators

$$\mathcal{C}_n^{(\mathbf{s})} = \text{span}_{\mathbb{R}_{\geq 0}} \{(0, \dots, 0, 1), (0, \dots, 0, s_i, s_{i+1}, \dots, s_{n-1}, s_n) : 1 \leq i \leq n-1\}.$$

It is easy to see that $\mathcal{C}_n^{(\mathbf{s})} \cap \mathbb{Z}^n = L_n^{(\mathbf{s})}$. There are many choices for properly grading the $L_n^{(\mathbf{s})}$, though three useful notions are as follows:

- $\lambda \mapsto (\lambda_1, \lambda_2, \dots, \lambda_n)$;
- $\lambda \mapsto \lambda_n$;
- $\lambda \mapsto (\lambda_n - \lambda_{n-1})$.

In a similar manner, we can define the \mathbf{s} -*lecture hall polytope* to be as follows.

$$\mathbf{P}_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$

A related geometric structure is the *rational \mathbf{s} -lecture hall polytope*, which is defined similarly:

$$R_n^{(\mathbf{s})} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq \frac{1}{s_n} \right\}.$$

Remark 3.2.1. For a given lecture hall cone $\mathcal{C}_n^{(\mathbf{s})}$, we may assume that $\gcd(s_1, s_2, \dots, s_n) = 1$. If we have $\gcd(s_1, s_2, \dots, s_n) = m > 1$, we could consider the sequence $\mathbf{t} = (t_1, \dots, t_n)$ defined by $t_i = s_i/m$ and notice that it is clear by definition that $\mathcal{C}_n^{(\mathbf{s})} = \mathcal{C}_n^{(\mathbf{t})}$. However, when considering the lecture hall polytope $\mathbf{P}_n^{(\mathbf{s})}$ or the rational lecture hall polytope $R_n^{(\mathbf{s})}$, it is not permissible to make this assumption, as $\mathbf{P}_n^{(\mathbf{s})} \not\cong \mathbf{P}_n^{(\mathbf{t})}$ and $R_n^{(\mathbf{s})} \not\cong R_n^{(\mathbf{t})}$. In fact, we have $\mathbf{P}_n^{(\mathbf{s})} = m \cdot \mathbf{P}_n^{(\mathbf{t})}$ and $R_n^{(\mathbf{s})} = m \cdot R_n^{(\mathbf{t})}$.

There has been much study of these three polyhedral geometric objects (see e.g. [6, 7, ?, 36, 37, 40, 43]). In particular, a characterization of which \mathbf{s} -sequences yield Gorenstein cones was implicitly given by Bousquet-Mélou and Eriksson in [11] and explicitly stated by Beck, Braun, Köppe, Savage, and Zafeirakopoulos in [6] as follows:

Theorem 3.2.2 (Beck et al [6, Corollary 2.6], Bousquet-Mélou, Eriksson [11, Proposition 5.4]). *For a positive integer sequence \mathbf{s} , the \mathbf{s} -lecture hall cone $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if there exists some $\mathbf{c} \in \mathbb{Z}^n$ satisfying*

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j+1})$$

for $j > 1$, with $c_1 = 1$.

Moreover, in the case of \mathbf{s} -sequences where $\gcd(s_i, s_{i+1}) = 1$ holds for all i , we have a refinement to this theorem. We say that \mathbf{s} is \mathbf{u} -generated by a sequence \mathbf{u} of positive integers if $s_2 = u_1 s_1 - 1$ and $s_{i+1} = u_i s_i - s_{i-1}$ for $i > 1$.

Theorem 3.2.3 (Beck et al [6, Theorem 2.8], Bousquet-Mélou, Eriksson [11, Proposition 5.5]). *Let $\mathbf{s} = (s_1, \dots, s_n)$ be a sequence of positive integers such that $\gcd(s_i, s_{i+1}) = 1$ for $1 \leq i < n$. Then $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if \mathbf{s} is \mathbf{u} -generated by some sequence $\mathbf{u} = (u_1, u_2, \dots, u_{n-1})$ of positive integers. When such a sequence exists, the Gorenstein point \mathbf{c} for $\mathcal{C}_n^{(\mathbf{s})}$ is defined by $c_1 = 1$, $c_2 = u_1$, and for $2 \leq i < n$, $c_{i+1} = u_i c_i - c_{i-1}$.*

It is a natural question to consider the Hilbert basis of a given polyhedral cone. While the question of characterizing the Hilbert bases for $\mathcal{C}_n^{(\mathbf{s})}$ given arbitrary \mathbf{s} is intractible, a natural redirection is to restrict to the case of \mathbf{u} -generated Gorenstein \mathbf{s} -sequences. To provide further motivation, Beck, Braun, Köppe, Savage, and Zafeirakopoulos in [7] give an explicit description of the Hilbert basis in the case of $\mathbf{s} = (1, 2, \dots, n)$, which is \mathbf{u} -generated by $\mathbf{u} = (3, 2, 2 \dots, 2)$. The Hilbert basis is given as follows.

Theorem 3.2.4 (Beck et al [7, Theorem 5.1]). *For each $A = \{a_1 < a_2 < \dots < a_k\} \subseteq [n-1]$, define the element v_A to be*

$$v_A = (0, \dots, 0, a_1, a_2, \dots, a_l, a_k + 1).$$

The Hilbert basis for $L_n^{(1,2,\dots,n)}$ is

$$\mathcal{H}_n^{(1,2,\dots,n)} := \{v_A : A \subseteq [n-1]\}.$$

As a corollary, the semigroup algebra $\mathbb{C}[\mathcal{C}_n^{(1,2,\dots,n)}]$ is generated entirely by elements in degree 1 with respect to the grading given by $\lambda \mapsto (\lambda_n - \lambda_{n-1})$.

3.3 The 1 mod k sequences

For any $k \in \mathbb{Z}_{\geq 1}$, we can define the 1 mod k sequence to be

$$\mathbf{s} = (1, k + 1, 2k + 1, \dots, (n - 1)k + 1).$$

For convenience of notation, let $L_{k,n} := L_n^{(s)}$, let $\mathcal{C}_{k,n} := \mathcal{C}_n^{(s)}$, and let $\mathbf{P}_{k,n} := \mathbf{P}_n^{(s)}$. This sequence is \mathbf{u} -generated by $\mathbf{u} = (k+2, 2, 2, \dots, 2)$, and hence Gorenstein. Note that if $k = 1$, we obtain the sequence $\mathbf{s} = (1, 2, \dots, n)$. This generalization has been well studied, most notably by Savage and Viswanathan [43] using a discrete geometric point of view. We now give a concise description for the Hilbert basis of $\mathcal{C}_{k,n}$.

Theorem 3.3.1. *For all $k \geq 1$, the Hilbert basis $\mathcal{H}_{k,n}$ of $L_{k,n}$ consists of the following elements:*

- The element $v_A := (0, 0, \dots, 0, a_1, a_2, \dots, a_k, a_k + 1)$ for each $A \subseteq [n-2]$ where $A = \{a_1 < a_2 < \dots < a_k\}$.
- Element $w \in L_{k,n}$, where $w_{n-1} = (n-2)k + 1$ and $w_n = (n-1)k + 1$.

Proof. The Hilbert basis for the case of $k = 1$ is known by Theorem 3.2.4 and the description can be translated to be written in this language with ease. Subsequently, we will prove the result assuming $k \geq 2$.

First we claim that v_A are all possible elements of degree one with respect to the grading given by $\deg(\lambda) = \lambda_n - \lambda_{n-1}$. Let $a = (a_1, a_2, \dots, a_{n-1}, a_n) \in L_{k,n}$ such that $a_n - a_{n-1} = 1$ and $a_{n-1} < n-1$. We can see that $a = v_A$ for some set A for the following reasons:

- (i) For each $1 \leq i \leq n-1$, $a_{n-1} < n-1$ implies that $a_{n-i} < n-i$ because we have the inequalities

$$\frac{n-i}{k(n-i)+1} \leq \frac{n-i+1}{k(n-i+1)+1} \leq \dots \leq \frac{n-3}{k(n-3)+1} \leq \frac{n-2}{k(n-2)+1}$$

but we also clearly have

$$\frac{n-i+1}{k(n-i)+1} \not\leq \frac{n-i+1}{k(n-i+1)+1};$$

- (ii) We must have $a_i < a_{i+1}$ for all $i \leq n-1$ as the inequalities

$$\frac{a_{i+1}-1}{k(i-1)+1} < \frac{a_{i+1}}{ki+1}$$

is equivalent to $a_{i+1} \leq i$, but we also clearly have

$$\frac{a_{i+1}}{k(i-1)+1} \not\leq \frac{a_{i+1}}{ki+1}.$$

Hence, we have $a = (0, \dots, 0, a_j, a_{j+1}, \dots, a_{n-1}, a_{n-1} + 1)$ which means $a = v_A$ for the set $A = \{a_j < a_{j+1} < \dots < a_{n-1}\} \subset [n-2]$. Now suppose that $a \in L_{k,n}$ and suppose that $a_{n-1} = j \geq n-1$. Notice that $a_n \geq j+2$, because if we suppose that $a_n = j+1$, then we arrive at a contradiction as

$$\frac{j}{k(n-2)+1} \leq \frac{j+1}{k(n-1)+1}$$

holds if and only if $j < (n - 1)$, which violates hypothesis. Therefore, a must be of degree 2 or higher.

Second, note that $w \in L_{k,n}$, with $w_{n-1} = (n - 2)k + 1$ and $w_n = (n - 1)k + 1$ cannot be written as a combination of elements of the type v_A . This follows from a grading argument as w has degree k . If we consider $a = \sum_{i=1}^k v_{A_i}$, it is clear that $a_{n-1} \leq k(n - 2) < k(n - 2) + 1 = w_{n-1}$ and we have the result.

Now, suppose that $a \in L_{k,n}$. There are three possible cases:

1. $a_{n-1} < k(n - 2) + 1$ and $a_n < k(n - 1) + 1$;
2. $a_{n-1} < k(n - 2) + 1$ and $a_n \geq k(n - 1) + 1$;
3. $a_{n-1} \geq k(n - 2) + 1$ and $a_n \geq k(n - 1) + 1$.

Case 1: Suppose that $a_{n-1} < k(n - 2) + 1$ and $a_n < k(n - 1) + 1$. Given that $s_1 = 1$, this condition forces $a_1 = 0$, because

$$a_1 \leq \frac{a_n}{k(n - 1) + 1} < 1$$

and likewise for all $2 \leq i \leq n - 2$ we have $a_i < k(i - 1) + 1$ because

$$\frac{a_i}{k(i - 1) + 1} < 1.$$

Moreover, we note that for all such i , we have

$$\frac{a_i}{k(i - 1) + 1} < \frac{a_{i+1}}{ki + 1}$$

because equality would force

$$a_{i+1} = a_i + k \cdot \frac{a_i}{k(i - 1) + 1}$$

which cannot be an integer by our previous observation and that $\gcd(k, k(i - 1) + 1) = 1$. Let j be the largest index such that $a_j < j - 1$. We now write $a = b + c$ where

$$b = (0, a_2, \dots, a_j, j, j + 1, \dots, n - 1)$$

and

$$c = (0, 0, \dots, 0, a_{j+1} - j, a_{j+2} - (j + 1), \dots, a_n - (n - 1)).$$

It is clear that $b = v_A$ for some $A \subseteq [n - 2]$. To show that $c \in L_{k,n}$, notice that for all $i \geq j$ we have that

$$\frac{a_i - i + 1}{k(i - 1) + 1} \leq \frac{a_{i+1} - 1}{ki + 1}$$

is equivalent to

$$a_i(ki + 1) + 1 \leq a_{i+1}(k(i - 1) + 1)$$

which is equivalent to

$$\frac{a_i}{k(i - 1) + 1} < \frac{a_{i+1}}{ki + 1}$$

and thus we have the desired result. So by induction, a of this form can be written as the sum of elements of the type v_A .

Case 2: Suppose that $a_{n-1} < k(n-2) + 1$ and $a_n \geq k(n-1) + 1$. We claim that $a - v_\emptyset = a - (0, 0, \dots, 0, 1) \in L_{k,n}$. If we have $a_n > k(n-1) + 1$, this is immediate. So, suppose that $a_n = k(n-1) + 1$, then

$$\frac{a_{n-1}}{k(n-2) + 1} \leq \frac{k(n-2)}{k(n-2) + 1} < \frac{k(n-1)}{k(n-1) + 1} = \frac{a_n - 1}{k(n-1) + 1}$$

holds because $k > 0$. Thus, for a of this form we can reduce to Case 1.

Case 3: Suppose that $a_{n-1} \geq k(n-2) + 1$ and $a_n \geq k(n-1) + 1$. Let j be the largest index such that $a_j < k(j-1) + 1$. We write $a = b + c$, where

$$b = (a_1, a_2, \dots, a_j, kj + 1, k(j+1) + 1, \dots, k(n-2) + 1, k(n-1) + 1)$$

and

$$c = (0, \dots, 0, a_{j+1} - (kj + 1), a_{j+2} - (k(j+1) + 1), \dots, a_{n-1} - (k(n-2) + 1), a_n - (k(n-1) + 1)).$$

It is clear that $b \in L_{k,n}$ with $b_{n-1} = k(n-2) + 1$ and $b_n = k(n-1) + 1$, which is an element of our proposed Hilbert basis. Moreover, because for all $i \geq j$ we have $a_i \geq k(i-1) + 1$ by assumption, it is immediate that $c \in L_{k,n}$. Thus, by induction, this case will reduce to either Case 1 or Case 2 showing the result. \square

In addition to the description of the Hilbert basis, we can also give the cardinality of the Hilbert basis by using Ehrhart theoretic methods.

Corollary 3.3.2. *The cardinality of these Hilbert bases are*

$$|\mathcal{H}_{k,n}| = \frac{(k+1)^{n-2} + (k-1)}{k} + 2^{n-2}$$

Proof. Given that we have an element v_A for all $A \subseteq [n-2]$, this yields 2^{n-2} elements. To enumerate the remaining Hilbert Basis elements, note that there is a clear bijection between $w \in L_{k,n}$ with $w_{n-1} = (n-2)k + 1$ and $w_n = (n-1)k + 1$, and elements $w' \in L_{k,n-2}$ such that $w'_{n-2} \leq (n-3)k + 1$. However, for any such w' , one can identify w' as a lattice point in the polytope $\mathbf{P}_{k,n-2}$. Savage and Viswanathan [43, Theorem 2] prove that the Ehrhart Polynomial of $\mathbf{P}_{k,n}$ is given by

$$i(P_{k,n}, t) = (-1)^t \sum_{p=0}^t \binom{\frac{1}{k} - 1}{t-p} \binom{-1/k}{p} (kp + 1)^n.$$

Evaluating at $i(P_{k,n-2}, t)$ at $t = 1$ yields

$$i(P_{k,n-2}, 1) = (-1) \binom{\frac{1}{k} - 1}{1} + (-1) \binom{-1/k}{0} (k+1)^{n-2} = \frac{(k+1)^{n-2} + (k-1)}{k}.$$

Thus, the proof is complete.¹ \square

¹In discussions with Carla Savage following the publication of the results in this chapter [39], this result can be obtained via an elementary enumeration argument of lattice points without appealing to the unwieldy and mysterious Ehrhart polynomial formula.

3.4 The ℓ -sequences

For any $\ell \in \mathbb{Z}_{\geq 2}$, the ℓ -sequence is the sequence $\mathbf{s} = (s_1, s_2, \dots, s_n)$ recursively defined as follows: $s_{i+1} = \ell s_i - s_{i-1}$ with $s_0 = 0$ and $s_1 = 1$. For convenience of notation let $L_n^\ell := L_n^{(\mathbf{s})}$, $\mathcal{C}_n^\ell := \mathcal{C}_n^{(\mathbf{s})}$, and $R_n^\ell := R_n^{(\mathbf{s})}$. Note that it is easy to see that any ℓ -sequence is strictly increasing. Moreover, we have that ℓ sequences are \mathbf{u} -generated by the sequence $\mathbf{u} = (\ell + 1, \ell, \ell, \dots, \ell)$ and hence \mathcal{C}_n^ℓ is Gorenstein. If we let $\ell = 2$, we reduce to the known case of $\mathbf{s} = (1, 2, \dots, n)$. The ℓ -sequences have appeared from a number theoretic point of view by way of the ℓ -lecture hall theorem and ℓ -Euler Theorems studied in [11] and [44]. We now give an explicit description of the Hilbert basis for any ℓ -sequence lecture hall cone.

Theorem 3.4.1. *Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be an ℓ -sequence for some $\ell \geq 2$. The Hilbert basis \mathcal{H}_n^ℓ can be described as follows*

$$\mathcal{H}_n^\ell = \bigcup_{i=0}^n \{ \lambda \in L_n^\ell : \lambda_{n-1} = s_i, \lambda_n = s_{i+1} \}$$

Proof. Note that the Hilbert basis for $\ell = 2$ is given by Theorem 3.2.4, and can be translated into this notation with ease. We will use the convention that $s_i = 0$ if $i \leq 0$. We claim first that there are no redundancies in this set. First note that $w \in L_n^\ell$ with $w_{n-1} = s_2 = \ell$ and $w_n = s_3 = \ell^2 - 1$ cannot be written as a combination of smaller elements of the proposed Hilbert basis. This is true because it would imply $w = \ell \cdot v' + c \cdot u$ where $v_{n-1} = 1$, $v_n = \ell$, $u_{n-1} = 0$, and $u_n = 1$, but this is contradiction as $w_n = \ell^2 + c$ for some positive integer c . Now, suppose that for some $i \geq 3$ there exists $w \in L_n^\ell$ such that $w_{n-1} = s_i$ and $w_n = s_{i+1}$ with $w = \sum v_j$ where each v_j is an element of the proposed Hilbert basis as well. This would imply that

$$s_i = \sum_{k=1}^m a_k \cdot s_k$$

where $a_k \in \mathbb{Z}_{\geq 0}$ and $m < i$ and that

$$s_{i+1} = \sum_{k=1}^m a_k \cdot s_{k+1}$$

must also hold. However, since we have that $s_{i+1} = \ell \cdot s_i - s_{i-1}$ combining these two gives us that

$$s_{i-1} = \sum_{k=1}^m a_k \cdot s_{k-1}.$$

We can now use this equality along with $s_i = \ell \cdot s_{i-1} - s_{i-2}$ to deduce that

$$s_{i-2} = \sum_{k=1}^m a_k \cdot s_{k-2}.$$

In fact, we can continue this iteration so that

$$s_{i-j} = \sum_{k=1}^m a_k \cdot s_{k-j}.$$

In the case of $j = i - 2$, we have that

$$s_2 = \sum_{k=1}^m a_k \cdot s_{k-i+2} = a_{i-1} \cdot s_1 = a_{i-1}$$

which implies that $m = i - 1$ and that $a_m = a_{i-1} = \ell$ as $s_2 = \ell$. However, this implies that

$$s_i = \ell \cdot s_{i-1} + \sum_{k=1}^{i-2} a_k \cdot s_k$$

with $a_k \in \mathbb{Z}_{\geq 0}$, which is a contradiction to $s_i = \ell \cdot s_{i-1} - s_{i-2}$. Thus, we have no redundancy.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) \in L_n^\ell$. First note that if $\lambda_{n-1} \geq s_i$, then $\lambda_n \geq s_{i+1}$. Notice that the inequality

$$\frac{s_i}{s_{n-1}} < \frac{s_{i+1}}{s_n}$$

is equivalent to

$$s_i s_n < s_{i+1} s_{n-1}$$

and making the substitutions $s_n = \ell s_{n-1} - s_{n-2}$ and $s_{i+1} = \ell s_i - s_{i-1}$ and simplifying leads to the new equivalent statement

$$s_{i-1} s_{n-1} < s_i s_{n-3}.$$

Repeating this process similarly shows that the above inequalities are equivalent to

$$s_{i-j} s_{n-j} < s_{i-j+1} s_{n-j-1}$$

for any $1 \leq j \leq i$. So, if $j = i$, note that $s_0 = 0$, $s_1 = 1$, and we have that $0 < s_{n-j-1}$ which is necessarily true. Thus we have the desired inequality. Moreover the inequality

$$\frac{s_i}{s_{n-1}} \leq \frac{s_{i+1} - 1}{s_n}$$

is equivalent to

$$s_i s_n \leq s_{i+1} s_{n-1} - s_{n-1}.$$

Making similar reductions as above for any $1 \leq j \leq i$ this is equivalent to one of the following:

$$\begin{cases} s_{i-j} s_{n-j} \leq s_{i-j+1} s_{n-j-1} - s_{n-1}, & \text{if } j \text{ is even,} \\ s_{i-j} s_{n-j} + s_{n-1} \leq s_{i-j+1} s_{n-j-1}, & \text{if } j \text{ is odd.} \end{cases}$$

If we consider $j = i$ we have that either of the preceding is equivalent to

$$s_{n-1} \leq s_{n-j-1}$$

which is a contradiction because \mathbf{s} is a strictly increasing sequence for any ℓ . Ergo, we have that $\lambda_{n-1} \geq s_i$ implies $\lambda_n \geq s_{i+1}$.

Consider λ_{n-1} . If $\lambda_{n-1} \geq s_{n-1}$, we have $\lambda_n \geq s_n$. Let j be the smallest integer such that $\lambda_j \geq s_j$. Notice that $(\lambda_1, \dots, \lambda_{j-1}, s_j, \dots, s_{n-1}, s_n) \in L_n^\ell$ and moreover $\lambda - (\lambda_1, \dots, \lambda_{j-1}, s_j, \dots, s_{n-1}, s_n) \in L_n^\ell$ follows immediately.

Now suppose that $s_i \leq \lambda_{n-1} < s_{i+1}$. Notice, since $s_1 = 1$, that we can write the element $\lambda_{n-1} = k \cdot s_i + \sum_{a_p \in A} s_{a_p}$ where $1 \leq k < \ell$, A is a multiset of elements of $[i-1]$ of cardinality $r < \infty$, and each a_p is chosen to be as large as possible. Then we have that $\lambda_n > k \cdot s_{i+1} + \sum_{a_p \in A} s_{a_p+1}$. This is an elementary exercise akin to the previous proof that $\lambda_{n-1} \geq s_i$ implies $\lambda_n > s_{i+1}$. To see that $\lambda - (\lambda_1, \dots, \lambda_{n-1}, s_i, s_{i+1}) \in L_n^\ell$, first suppose that we write $\lambda_{n-1} = k \cdot s_i + \sum_{t=0}^{i-1} b_t \cdot s_t$ where $b_t \in \mathbb{Z}_{\geq 0}$ are the multiplicities of the elements of the multiset described above. Now, we have

$$\frac{\left(k \cdot s_i + \sum_{t=0}^{i-1} b_t \cdot s_t\right) - s_i}{s_{n-1}} \leq \frac{\left(k \cdot s_{i+1} + \sum_{t=0}^{i-1} b_t \cdot s_{t+1}\right) - s_{i+1}}{s_n} \leq \frac{\lambda_n - s_{i+1}}{s_n}$$

The second equality is immediate by previous observation and the first inequality is equivalent to

$$(k-1)s_i s_n + s_n \cdot \sum_{t=0}^{i-1} b_t \cdot s_t \leq (k-1)s_{i+1} s_{n-1} + s_{n-1} \cdot \sum_{t=0}^{i-1} b_t \cdot s_{t+1}.$$

By expanding using $s_n = \ell \cdot s_{n-1} - s_{n-2}$ on the right hand side and $s_{i+1} = \ell \cdot s_i - s_{i-1}$ and $s_{t+1} = \ell \cdot s_t - s_{t-1}$ on the left hand side, we have after simplification

$$(k-1)s_{n-2}s_i + \sum_{t=0}^{i-1} b_t \cdot s_t \geq (k-1)s_{n-1}s_{i-1} + \sum_{t=0}^{i-1} b_t \cdot s_{t-1}$$

In a similar manner to the above, we can do this repeatedly to arrive at the equivalent statement

$$(k-1)s_{n-j}s_{i-j} + \sum_{t=0}^{i-1} b_t \cdot s_{t-j} \leq (k-1)s_{n-j-1}s_{i-j+1} + \sum_{t=0}^{i-1} b_t \cdot s_{t-j+1}$$

for any $0 \leq j \leq i$. When $j = i$, we have

$$0 = (k-1)s_{n-i}s_0 \leq (k-1)s_{n-i-1}s_1 = (k-1)s_{n-i-1}$$

which is necessarily true. Therefore, by induction, we have a complete Hilbert basis. \square

We now provide a method for computing the cardinality of the Hilbert basis for any ℓ -sequence. Though not given by an explicit algebraic expression, this formula gives a nice combinatorial interpretation for the cardinality of the Hilbert basis elements of ℓ -sequences.

Corollary 3.4.2. *The cardinality of the Hilbert basis \mathcal{H}_n^ℓ is given by the following:*

$$|\mathcal{H}_n^\ell| = 2 + \sum_{j=1}^{n-2} i(R_{n-2}^\ell, s_j)$$

where $i(R_{n-2}^\ell, t)$ denotes the Ehrhart quasipolynomial of the rational lecture hall polytope R_{n-2}^ℓ .²

Proof. Suppose that $\lambda \in L_n^\ell$ such that $\lambda_{n-1} = s_{i+1}$ and $\lambda_n = s_{i+2}$ for some $1 \leq i \leq n-2$. This implies that $\lambda_{n-2} \leq s_i$ by applying arguments used in the proof of Theorem 3.7.1. Therefore, we can bijectively associate λ with a lattice point λ' in the s_i th dilate of the rational lecture hall polytope R_{n-2}^ℓ , so $\lambda' \in (s_i \cdot R_{n-2}^\ell \cap \mathbb{Z}^{n-2})$. Therefore, all such Hilbert basis elements are enumerated by $i(R_{n-2}^\ell, s_i)$. All Hilbert basis elements are counted in this way with the exception of two, namely $(0, \dots, 0, 0, 1)$ and $(0, \dots, 0, 1, \ell)$, as $s_1 = 1$ and $s_2 = \ell$. Thus, we have the desired. \square

As an aside, note that the $(n-2)$ th summand actually gives $i(R_{n-2}^\ell, s_{n-2}) = i(\mathbf{P}_{n-2}^\ell, 1)$. This means that some of the Hilbert basis elements correspond to lattice points in the integral lecture hall polytope \mathbf{P}_{n-2}^ℓ , which one may have suspected from the results in the $1 \pmod k$ cones. This phenomenon occurs in later cases as well.

3.5 Two dimensional Gorenstein sequences

We begin our low dimensional characterization for the Hilbert bases of \mathbf{u} -generated Gorenstein lecture cones by considering the two dimensional case. Notice that when $n = 2$, Remark 3.2.1 implies that there is no distinction between Gorenstein and \mathbf{u} -generated Gorenstein. Applying Theorems 3.2.2 and 3.2.3 provides the following description for the Gorenstein condition.

Lemma 3.5.1. *Suppose that $\mathbf{s} = (s_1, s_2)$ such that $\mathcal{C}_2^{(\mathbf{s})}$ is Gorenstein. Then $\mathbf{s} = (s_1, ks_1 - 1)$ for $k \geq 1$.*

Using this description, we will now classify the Hilbert bases for all two dimensional Gorenstein lecture hall cones as follows.

Theorem 3.5.2. *Let $\mathcal{C}_2^{(\mathbf{s})}$ be a Gorenstein lecture hall cone with $\mathbf{s} = (s, ks - 1)$ for some $k \geq 1$. The Hilbert basis of $\mathcal{C}_2^{(\mathbf{s})}$ is $\mathcal{H}_2^{(\mathbf{s})} = \{(0, 1), (s, ks - 1), (1, k)\}$.*

Proof. Let $(a, b) \in L_2^{(\mathbf{s})}$. First, suppose that $a \geq s$ and note that this immediately implies that $b \geq ks - 1$. We have that $(a, b) - (s, ks - 1) \in L_2^{(\mathbf{s})}$ because

$$\frac{a - s}{s} \leq \frac{b - (ks - 1)}{ks - 1}$$

follows directly from

$$\frac{a}{s} \leq \frac{b}{ks - 1}$$

and that $a \geq s$ and $b \geq ks - 1$.

²In conversations with Carla Savage following the publication of the results in this chapter [39], it appears that one can create an enumeration argument using only Ehrhart polynomial evaluations of regular \mathbf{s} -lecture hall polytopes from the ℓ -sequences. This makes an eventual explicit algebraic expression more attainable.

Now suppose that $1 \leq a \leq s - 1$. If $a \geq 1$, then $b \geq k$ because $\frac{1}{s} < \frac{k}{ks-1}$, but $\frac{1}{s} > \frac{k-1}{ks-1}$. Observe that

$$\frac{a}{s} < \frac{b}{ks-1}$$

must hold, because equality implies that $b = ak - \frac{a}{s}$ which by the assumption $1 \leq a \leq s - 1$ cannot be an integer. Now, we claim that $(a, b) - (1, k) \in L_2^{(s)}$ as

$$\frac{a-1}{s} \leq \frac{b-k}{ks-1}$$

is equivalent to

$$a(ks-1) < bs$$

which is equivalent to our observation above.

Finally, note that if $a = 0$ and $b \geq 1$, $(a, b) - (0, 1) \in L_2^{(s)}$ is immediate. Therefore, we have shown that any \mathbf{s} -lecture hall partition can be written as a sum of these elements, which completes the proof. \square

We note that when $n = 2$, the Gorenstein condition ensures that the Hilbert basis is of the smallest possible cardinality, $|\mathcal{H}_2^{(s)}| = 3$ when $s_1 \geq 2$ and $|\mathcal{H}_2^{(s)}| = 2$ if $s_1 = 1$. This further motivates the restriction to \mathbf{u} -generated Gorenstein cones.

3.6 Three dimensional \mathbf{u} -generated Gorenstein sequences

We continue our low dimensional characterization for \mathbf{u} -generated Gorenstein lecture hall cones by considering the three dimensional case. When $n = 3$, a direct application of Theorem 3.2.3 yields the following description.

Lemma 3.6.1. *Suppose that $\mathbf{s} = (s_1, s_2, s_3)$ such that $\mathcal{C}_3^{(s)}$ is Gorenstein with $\gcd(s_i, s_{i+1}) = 1$ for all i . Then $\mathbf{s} = (s, ks - 1, \ell(ks - 1) - s)$ for integers $s \geq 1$, $k \geq 1$ and $\ell \geq 1$.*

Using the above, we now completely characterize the Hilbert bases for all \mathbf{u} -generated Gorenstein lecture hall cones for $n = 3$.

Theorem 3.6.2. *Suppose that $\mathbf{s} = (s, ks - 1, \ell(ks - 1) - s)$. Then*

- *If $s \geq 2$, then the Hilbert basis is*

$$\mathcal{H}_3^{(s)} = \{(0, 0, 1), (0, 1, \ell), (0, k, \ell k - 1), (1, k, \ell k - 1), (j, ks - 1, \ell(ks - 1) - s) \mid 0 \leq j \leq s\}.$$

- *If $s = 1$, then the Hilbert basis is*

$$\mathcal{H}_3^{(s)} = \{(0, 0, 1), (0, 1, \ell), (0, k - 1, \ell(k - 1) - 1), (1, k - 1, \ell(k - 1) - 1)\}.$$

Proof. First, we will suppose that $s \geq 2$. To begin, we claim that the proposed Hilbert basis has no redundancy. To show this, it sufficient to show that $(j, ks - 1, \ell(ks - 1) - s)$ cannot be written as a sum of other proposed elements. Suppose this is possible, then there exist positive integers α, β , and γ such that $\alpha(k) + \beta = ks - 1$. This has solutions $\alpha = s - i$ and

$\beta = ki - 1$ for $1 \leq i < s$. However, we must also have $\alpha(\ell k - 1) + \beta(\ell) + \gamma = \ell(ks - 1) - s$ and evaluating at the above solution implies that $\gamma = -i$. This is a contradiction.

Let $(a, b, c) \in L_3^{(s)}$. First note that if $a \geq s$, this implies that $b \geq ks - 1$ and $c \geq \ell(ks - 1) - s$. It is clear then that $(a, b, c) - (s, ks - 1, \ell(ks - 1) - s) \in L_3^{(s)}$. If $0 \leq a < s$ and $b \geq ks - 1$, then it follows that $c \geq \ell(ks - 1) - s$ and $(a, b, c) - (a, ks - 1, \ell(ks - 1) - s) \in L_3^{(s)}$.

Next suppose that $1 \leq a < s$ and $b < ks - 1$. Notice that $a \geq 1$ implies that $b \geq k$ and $c \geq \ell k - 1$ because $\frac{1}{s} < \frac{k}{ks-1} < \frac{\ell k - 1}{\ell(ks-1)-s}$ but $\frac{1}{s} > \frac{k-1}{ks-1}$ and $\frac{k}{ks-1} > \frac{\ell k - 2}{\ell(ks-1)-s}$. Additionally, we can see that the inequalities must be strict

$$\frac{a}{s} < \frac{b}{ks-1} < \frac{c}{\ell(ks-1)-s}.$$

This follows because equality of the first and second fractions implies that $b = ak - \frac{a}{s}$ which is not an integer by the assumption $1 \leq a < s - 1$ and equality of second and third fractions implies that $c = b\ell - \frac{bs}{ks-1}$ which is not an integer by the assumption $b < ks - 1$. Now, we claim that $(a, b, c) - (1, k, \ell k - 1) \in K_3^{(s)}$ as

$$\frac{a-1}{s} \leq \frac{b-k}{ks-1} \leq \frac{c-\ell k+1}{\ell(ks-1)-s}$$

is equivalent to

$$a(ks-1) + 1 \leq bs \quad \text{and} \quad b(\ell(ks-1)-s) \leq c(ks-1) - 1$$

or

$$a(ks-1) < bs \quad \text{and} \quad b(\ell(ks-1)-s) < c(ks-1)$$

which is equivalent to the strict inequalities shown above.

Now suppose that $a = 0$. If $b \geq k$, we have $(0, b, c) - (0, k, \ell k - 1) \in L_3^{(s)}$ immediately by the previous argument. So, suppose that $1 \leq b < k$, and notice that this implies that $c \geq b\ell$ as $\frac{b}{ks-1} < \frac{b\ell}{\ell(ks-1)-s}$. However, we also have $\frac{b}{ks-1} > \frac{b\ell-1}{\ell(ks-1)-s}$ as this is equivalent to $sb < ks - 1$ which follows from $b \leq k - 1$. We now claim that $(0, b, c) - (0, 1, \ell) \in L_3^{(s)}$ as we have the following inequalities

$$\frac{b-1}{ks-1} \leq \frac{b\ell-\ell}{\ell(ks-1)-s} \leq \frac{c-\ell}{\ell(ks-1)-s}.$$

The second inequality is immediate by $c \geq b\ell$ and the first inequality is equivalent to $b \geq 1$.

Thus, by induction, any element of $L_3^{(s)}$ can be written as a sum of these elements and we have the Hilbert basis.

Now, we suppose that $s = 1$. It is clear that there is no redundancy in the proposed Hilbert basis. Note that we must have $k \geq 2$. Let $(a, b, c) \in L_3^{(s)}$. Consider b . If $b \geq k - 1$, then $c \geq \ell(k - 1) - 1$. If $a \geq 1$, then $(a, b, c) - (1, k - 1, \ell(k - 1) - 1) \in L_3^{(s)}$ is immediate. If $a = 0$, then $(a, b, c) - (0, k - 1, \ell(k - 1) - 1) \in L_3^{(s)}$ is also immediate. Now, if $1 \leq b < k - 1$, note that $a = 0$ and $c \geq b\ell$, which follows from the same argument given in the previous case. Moreover, we also have $(a, b, c) - (0, 1, \ell) \in L_3^{(s)}$ immediately from work of the previous case. Thus, by induction, we have the Hilbert basis. \square

We note that in this case, the cardinality of the Hilbert basis is directly dependent on the starting value s_1 , with $|\mathcal{H}_3^{(s)}| = s_1 + 5$ when $s_1 \geq 2$ and $|\mathcal{H}_3^{(s)}| = 4$ when $s_1 = 1$.

3.7 Four dimensional \mathbf{u} -generated Gorenstein sequences

We conclude our low dimensional characterization of \mathbf{u} -generated Gorenstein lecture hall cones in the case of four dimensions. We have the following description for the Hilbert bases.

Theorem 3.7.1. *Suppose that $\mathbf{s} = (s_1, s_2, s_3, s_4)$ is \mathbf{u} -generated by $\mathbf{u} = (u_1, u_2, u_3)$ such that $\mathcal{C}_4^{(\mathbf{s})}$ is a Gorenstein lecture hall cone. Recall that $\mathbf{c} = (c_1, c_2, c_3, c_4)$ is the Gorenstein point of $\mathcal{C}_4^{(\mathbf{s})}$, with $c_1 = 1$, $c_2 = u_1$, and $c_{i+1} = u_i c_i - c_{i-1}$ for $i \geq 2$. Then*

(a) *If $s_1 = 1$ and $u_1 = 2$ and the Hilbert basis is*

$$\mathcal{H}_4^{(\mathbf{s})} = \{(0, 0, 0, 1), (0, 0, 1, u_3), (0, 0, s_3, s_4), (0, 1, s_3, s_4), (1, 1, s_3, s_4)\}.$$

(b) *If $s_1 = 1$ and $u_1 \geq 3$ and the Hilbert basis is*

$$\mathcal{H}_4^{(\mathbf{s})} = \left\{ \begin{array}{l} (0, j, s_3, s_4) \text{ for all } 0 \leq j \leq s_2 \\ (1, s_2, s_3, s_4), (0, 0, 0, 1), (0, 0, 1, u_3), (0, 0, u_2, u_2 u_3 - 1), (0, 1, u_2, u_2 u_3 - 1) \end{array} \right\}.$$

(c) *If $s_1 = 2$ and $u_1 = 1$, then the Hilbert basis is*

$$\mathcal{H}_4^{(\mathbf{s})} = \{(2, 1, s_3, s_4), (1, 1, s_3, s_4), (0, 1, s_3, s_4), (0, 0, s_3, s_4), (0, 0, 1, u_3), (0, 0, 0, 1)\}.$$

(d) *If $s_1 \geq 3$ and $u_1 = 1$, then the Hilbert basis is*

$$\mathcal{H}_4^{(\mathbf{s})} = \left\{ \begin{array}{l} \lambda \in L_4^{(\mathbf{s})} \text{ with } \lambda_3 = s_3 \text{ and } \lambda_4 = s_4 \\ (0, 0, 0, 1), (0, 0, 1, u_3), (0, 0, c_3, c_4), (0, 1, c_3, c_4), (1, 1, c_3, c_4) \end{array} \right\}.$$

(e) *If $s_1 \geq 2$ and $u_1 \geq 2$, then the Hilbert basis is*

$$\mathcal{H}_4^{(\mathbf{s})} = \left\{ \begin{array}{l} \lambda \in L_4^{(\mathbf{s})} \text{ with } \lambda_3 = s_3 \text{ and } \lambda_4 = s_4 \\ (0, j, c_3, c_4) \text{ for all } 0 \leq j \leq c_2 \\ (c_1, c_2, c_3, c_4), (0, 0, 0, 1), (0, 0, 1, u_3), (0, 0, u_2, u_2 u_3 - 1), (0, 1, u_2, u_2 u_3 - 1) \end{array} \right\}.$$

Proof. For each of the cases, we will consider $\mathcal{C}_4^{(\mathbf{s})}$ with respect to the grading defined by $\lambda \mapsto (\lambda_4 - \lambda_3)$. The first two cases (a) and (b) can be reduced to the three dimensional case and hence follow directly from the proof of Theorem 3.6.2.

Case (c): First note that to have a valid sequence $u_2 \geq 3$ and we have that $s_2 = 1$, $s_3 = u_2 - 2$, and $s_4 = u_3(u_2 - 2) - 1$. It is clear that there are no redundancies among the elements of the proposed Hilbert basis.

We will now show that an arbitrary element of $L_4^{(\mathbf{s})}$ can be written as a sum of elements of this basis by induction. Let $\lambda \in L_4^{(\mathbf{s})}$ and consider λ_3 . If $\lambda_3 \geq s_3$, we then consider $\lambda_2 = 0$ or $\lambda_2 \geq 1$. If $\lambda_2 = 0$, it is clear that $\lambda - (0, 0, s_3, s_4) \in L_4^{(\mathbf{s})}$. If we have that $\lambda_2 \geq 1$, we then consider $\lambda_1 = 0$, $\lambda_1 = 1$, or $\lambda_1 \geq 2$. We can see that if $\lambda_1 = 0$, then $\lambda - (0, 1, s_3, s_4) \in L_4^{(\mathbf{s})}$

and if $\lambda_1 = 1$, then $\lambda - (1, 1, s_3, s_4) \in L_4^{(s)}$. If $\lambda_1 \geq 2$, note that $\lambda_2 \geq \lceil \frac{\lambda_1}{2} \rceil$ which means that $\lambda - (2, 1, s_3, s_4) \in L_4^{(s)}$ follows from

$$\frac{\lambda_1 - 2}{2} \leq \left\lceil \frac{\lambda_1}{2} \right\rceil - 1 \leq \lambda_2 - 1$$

which is clearly true.

Now, suppose that $1 \leq \lambda_3 < s_3$. This implies that $\lambda_1 = \lambda_2 = 0$ and $\lambda_4 \geq u_3 \lambda_3$. The first two are trivial and the latter follows because

$$\frac{\lambda_3}{u_2 - 2} < \frac{u_3 \lambda_3}{u_3(u_2 - 2) - 1}$$

holds, but the inequality

$$\frac{\lambda_3}{u_2 - 2} \leq \frac{u_3 \lambda_3 - 1}{u_3(u_2 - 2) - 1}$$

is equivalent to $\lambda_3 \geq s_3$ which is a contradiction. Moreover, we have that $\lambda - (0, 0, 1, u_3) \in L_4^{(s)}$ because

$$\frac{\lambda_3 - 1}{u_2 - 2} \leq \frac{u_3 \lambda_3 - u_3}{u_3(u_2 - 2) - 1} \leq \frac{\lambda_4 - u_3}{u_3(u_2 - 2) - 1}$$

is immediate from the previous observations. Therefore, by induction, we have a complete Hilbert basis.

Case (d): First, we claim that this set contains no redundancy. Note that no element $\lambda \in L_4^{(s)}$ with $\lambda_3 = c_3$ and $\lambda_4 = c_4$ can be written as a combination of smaller elements. Given that $c_3 = u_2 - 1$ and $c_4 = u_3 u_2 - u_3 - 1$, this would imply that $c_4 = (u_2 - 1)u_3 + b$ where $b \in \mathbb{Z}_{\geq 0}$ which is impossible. Suppose that $w \in L_4^{(s)}$ such that $w_3 = s_3$, $w_4 = s_4$ and there are additional elements of the proposed Hilbert basis such that $\sum_{i=1}^d v_i = w$. Note that this would imply there are integers $m, n, p \in \mathbb{Z}_{\geq 0}$, where $m \leq s_1 - 2$ so that we have $s_3 = m \cdot c_3 + n = m \cdot u_2 - m + n$ and $s_4 = m \cdot c_4 + n \cdot u_3 + p$. However, we also have $s_4 = u_3 s_3 - s_1 + 1$. Combining and simplification yields the result $s_1 = m - p - 1$, which is a contradiction to $m \leq s_1 - 2$, and hence we have no such sum.

We will now show that an arbitrary element of $L_4^{(s)}$ can be written as a sum of elements of this basis by induction. Suppose that $\lambda \in L_4^{(s)}$ and consider λ_3 . If we have $\lambda_3 \geq s_3$, then consider λ_2 and λ_1 . One of three cases will hold (i) $\lambda < s_2$, (ii) $\lambda_2 \geq s_2$ with $\lambda_1 < s_1$, or (iii) $\lambda_1 \geq s_1$. For (i), it is clear that $\lambda - (\lambda_1, \lambda_2, s_3, s_4) \in L_4^{(s)}$, for (ii) it is clear that $\lambda - (\lambda_1, s_2, s_3, s_4) \in L_4^{(s)}$ and for (iii) it is clear that $\lambda - (s_1, s_2, s_3, s_4) \in L_4^{(s)}$ all of which are valid lecture hall partitions.

Now, suppose that $c_3 \leq \lambda_3 < s_3$. We can then write $\lambda_3 = \alpha \cdot c_3 + \beta$ where either $1 \leq \alpha < s_1 - 2$ and $0 \leq \beta \leq c_3 - 1 = u_2 - 2$ or $\alpha = s_1 - 2$ and $0 \leq \beta \leq u_2 - 3$, because $s_3 = (s_1 - 2)c_3 + (u_2 - 2)$. We then claim that $\lambda_4 \geq \alpha \cdot c_4 + \beta \cdot u_3$. This follows because

$$\frac{\alpha \cdot c_3 + \beta}{s_3} < \frac{\alpha \cdot c_4 + \beta \cdot u_3}{s_4}$$

reduces using $c_3 = u_2 - 1$, $c_4 = u_3 \cdot (u_2 - 1) - 1$, $s_3 = u_2 s_1 - u_2 - s_1$, and $s_4 = u_3 u_2 s_1 - u_3 s_1 - u_2 u_3 - s_1 + 1$ to the inequality

$$-\alpha + \beta(1 - s_1) < 0$$

which is obviously true. However, the inequality

$$\frac{\alpha \cdot c_3 + \beta}{s_3} \leq \frac{\alpha \cdot c_4 + \beta \cdot u_3 - 1}{s_4}$$

reduces in the same manner to

$$\alpha + \beta(s_1 - 1) \geq u_2 s_1 - s_1 - u_2$$

which is contradiction because of the conditions 1 that $\alpha \leq s_1 - 3$ and $\beta \leq u_2 - 2$ or $\alpha = s_1 - 2$ and $\beta \leq u_2 - 3$.

Now consider λ_2 . Suppose that the $1 \leq \lambda_2 < s_1 - 1 = s_2$, then $\lambda_3 \geq c_3 \lambda_2$. This follows because

$$\frac{\lambda_2}{s_2} < \frac{\lambda_2(u_2 - 1)}{s_3}$$

is equivalent to $\lambda_2 > 0$, but the inequality

$$\frac{\lambda_2}{s_2} \leq \frac{\lambda_2(u_2 - 1) - 1}{s_3}$$

cannot hold because it reduces to $s_1 - 1 \leq \lambda_2$, which is a contradiction.

If $\lambda_2 \geq 1$ we have

$$\frac{\lambda_2 - 1}{s_2} \leq \frac{\lambda_2 \cdot c_3 - c_3}{s_3} \leq \frac{\lambda_3 - c_3}{s_3} = \frac{\alpha \cdot c_3 + \beta - c_3}{s_3} \leq \frac{\alpha \cdot c_4 + \beta \cdot u_3 - c_4}{s_4} \leq \frac{\lambda_4 - c_4}{s_4}.$$

Each of these inequalities follows directly from previous observations. It now holds that if $\lambda_1 = 0$, we have that $\lambda - (0, 0, c_3, c_4) \in L_4^{(s)}$ in the case $\lambda_2 = 0$ and $\lambda - (0, 1, c_3, c_4) \in L_4^{(s)}$ provided $\lambda_2 > 0$.

Moreover, if $1 \leq \lambda_2 < s_1 - 1$ and $\lambda_1 \geq 1$, we have that $\frac{\lambda_1}{s_1} < \frac{\lambda_2}{s_1 - 1}$ as equality creates a contradiction. Hence, we get the equivalent inequality

$$\frac{\lambda_1 - 1}{s_1} \leq \frac{\lambda_2 - 1}{s_1 - 1}$$

which means that $\lambda - (1, 1, c_3, c_4) \in L_4^{(s)}$.

Now, suppose that $1 \leq \lambda_3 < c_3$. Notice that this implies that $\lambda_1 = \lambda_2 = 0$ and $\lambda_4 \geq u_3 \lambda_3$. The first two inequalities are immediate, and the latter inequality follows from the fact that

$$\frac{\lambda_3}{s_3} < \frac{\lambda_3 \cdot u_3}{s_4}$$

is equivalent to $s_1 > 1$, which is true by assumption, but the inequality

$$\frac{\lambda_3}{s_3} \leq \frac{\lambda_3 \cdot u_3 - 1}{s_4}$$

using the observation $c_3 = u_2 - 1$, reduces to

$$u_2 s_1 - u_2 - s_1 \leq \lambda_3 (s_1 - 1) \leq (u_2 - 2)(s_1 - 1) = u_2 s_1 - u_2 - 2s_1 + 2$$

which contradicts the assumption that $s_1 \geq 3$. Subsequently, we have $\lambda - (0, 0, 1, u_3) \in L_4^{(s)}$ by the above observation and applying the arguments used in case (c). Thus, by induction, we have a complete Hilbert basis in this case.

Case (e): We verify that the proposed set contains no redundancy. It is clear that the elements $(0, 0, 0, 1)$, $(0, 0, 1, u_3)$, $(0, 0, u_2, u_2 u_3 - 1)$, and $(0, 0, u_2, u_2 u_3 - 1)$ cannot be written as a combination of one another. So suppose first that an element $(0, j, c_3, c_4) = \sum_{i=1}^e v_i$ where the v_i are elements of smaller degree. This would imply that there exist $a, b, d \in \mathbb{Z}_{\geq 0}$ such that $a \cdot u_2 + b = c_3$ and $a \cdot (u_2 u_3 - 1) + b \cdot u_3 + d = c_4$, with the restriction that $a \leq u_1 - 1$ because $c_3 = u_1 u_2 - 1$. However, we also have $c_4 = u_3 c_3 - c_2 = u_3 c_3 - u_1$, which means that we have that $a - d = u_1$, which contradicts $a \leq u_1 - 1$. Thus, these elements cannot be written as a sum of elements of lower degree.

Now, suppose that $w \in L_4^{(s)}$ with $w_3 = s_3$ and $w_4 = s_4$. If there was some collection of elements lower degree in the proposed basis such that $w = \sum_{i=1}^e v_i$, this would imply there would be integers $m, n, p, q \in \mathbb{Z}_{\geq 0}$ with $m \leq s_1 - 1$, $n \leq u_1 - 2$ such that $s_3 = m \cdot c_3 + n \cdot u_2 + p$, $s_4 = m \cdot c_4 + n(u_2 u_3 - 1) + p \cdot u_3 + q$, but also $s_4 = u_3 s_3 - s_2$. When we combine and simplify, we have that $s_2 = m \cdot c_2 + n - q$. However, this implies that $u_1 s_1 - 1 \leq (u_1 - 2)(s_1 - 1) - q$, which implies that $u_1 + s_1 \leq 3 - q$ which would be a contradiction to $s_1 \geq 2$ and $u_1 \geq 2$. Hence, there are no redundancies in the proposed Hilbert basis.

We will now show that an arbitrary element of $L_4^{(s)}$ can be written as a sum of elements of this basis by induction. Let $\lambda \in L_4^{(s)}$ and consider λ_3 . If $\lambda_3 \geq s_3$, we can construct an element of $w \in L_4^{(s)}$ such that $w_3 = s_3$ and $w_4 = s_4$ so that $\lambda - w \in L_4^{(s)}$ by following analogous construction to the previous cases.

If $c_3 \lambda < s_3$, note that then we can consider λ_1 . If $\lambda_1 = 0$, we have that $\lambda - (0, \lambda_2, c_3, c_4) \in L_4^{(s)}$ provided that $\lambda_2 \leq c_2$ or $\lambda - (0, c_2, c_3, c_4) \in L_4^{(s)}$ in the case $\lambda_2 > c_2$. If $\lambda_1 \geq 1$, we have $\lambda - (c_1, c_2, c_3, c_4) \in L_4^{(s)}$. Each of these statements follow identically from the arguments for the $c_3 \leq \lambda_3 < s_3$ made in case (d).

Now, suppose that $u_2 \leq \lambda_3 < c_3 = u_2 u_1 - 1$. We can write $\lambda_3 = \alpha \cdot u_2 + \beta$ where either $1 \leq \alpha \leq u_1 - 2$ and $0 \leq \beta \leq u_2 - 1$ or $\alpha = u_1 - 1$ and $0 \leq \beta \leq u_2 - 2$, as $c_3 = (u_1 - 1)u_2 + (u_2 - 1)$. Note that this implies that $\lambda_4 \geq \alpha(u_2 u_3 - 1) + \beta \cdot u_3$ because the inequality

$$\frac{\alpha \cdot u_2 + \beta}{s_3} < \frac{\alpha(u_2 u_3 - 1) + \beta \cdot u_3}{s_4}$$

reduces to $0 < \alpha \cdot s_1 + \beta(u_1 s_1 - 1)$ which is true by assumption. Additionally, the inequality

$$\frac{\alpha \cdot u_2 + \beta}{s_3} \leq \frac{\alpha(u_2 u_3 - 1) + \beta \cdot u_3 - 1}{s_4}$$

reduces to

$$u_2 u_1 s_1 - u_2 - s_1 \leq \alpha s_1 + \beta(u_1 s_1 - 1)$$

due to the assumptions on α and β implies that either

$$u_2 u_1 s_1 - u_2 - s_1 \leq (u_1 - 2)s_1 + (u_2 - 1)(u_1 s_1 - 1) = (u_2 u_1 s_1 - u_2 - s_1) - s_1 + 1$$

which contradicts $s_1 \geq 2$, or it implies

$$u_2 u_1 s_1 - u_2 - s_1 \leq (u_1 - 1)s_1 + (u_2 - 2)(u_1 s_1 - 1) = (u_2 u_1 s_1 - u_2 - s_1) - u_1 s_1 + 2$$

which contradicts $s_1 \geq 2$ and $u_1 \geq 2$. Moreover, note that this implies that $\lambda_2 < u_1$, which implies that $\lambda_1 = 0$. Additionally, we have that $1 \leq \lambda_2 < u_1$ implies that $\lambda_3 \geq \lambda_2 u_2$ which follows because the inequality

$$\frac{\lambda_2}{s_2} = \frac{\lambda_2}{s_1 u_1 - 1} < \frac{\lambda_2 u_2}{u_2 u_1 s_1 - u_2 - s_1} = \frac{\lambda_2 u_2}{s_3}$$

is immediate, but the inequality

$$\frac{\lambda_2}{s_1 u_1 - 1} < \frac{\lambda_2 u_2 - 1}{u_2 u_1 s_1 - u_2 - s_1}$$

reduces to $u_1 s_1 - 1 \leq \lambda_2 s_1$ which contradicts $\lambda_2 < u_1$. Therefore, we get the inequalities

$$\frac{\lambda_2 - 1}{s_2} \leq \frac{\lambda_2 u_2 - u_2}{s_3} \leq \frac{\lambda_3 - u_2}{s_3}$$

and

$$\frac{\lambda_3 - u_2}{s_3} = \frac{(\alpha - 1) \cdot u_2 + \beta}{s_3} \leq \frac{(\alpha - 1) \cdot (u_2 u_3 - 1) + \beta u_3}{s_4} \leq \frac{\lambda_4 - (u_2 u_3 - 1)}{s_4}.$$

Thus, if we have $\lambda - (0, 1, u_2, u_2 u_3 - 1) \in L_4^{(s)}$ when $\lambda_2 \geq 1$ and $\lambda - (0, 0, u_2, u_2 u_3 - 1) \in L_4^{(s)}$ when $\lambda_2 = 0$.

If $1 \leq \lambda_3 < u_2$, we get $\lambda - (0, 0, 1, u_3) \in L_4^{(s)}$ by repeating analogous arguments to previous cases (see case (c)). Thus, by induction, we have a complete Hilbert basis. \square

Given the explicit Hilbert basis in the case of $n = 4$, it is additionally of interest to consider the cardinalities of the set in each case. The following is computation of the these cardinalities.

Corollary 3.7.2. *For each case of Theorem 3.7.1, the cardinality of the Hilbert basis is as follows:*

(a) If $s_1 = 1$ and $u_1 = 2$, then $|\mathcal{H}_4^{(s)}| = 5$.

(b) If $s_1 = 1$ and $u_1 \geq 3$, then $|\mathcal{H}_4^{(s)}| = s_2 + 6$.

(c) If $s_1 = 2$ and $u_1 = 1$, then $|\mathcal{H}_4^{(s)}| = 6$.

(d) If $s_1 \geq 3$ and $u_1 = 1$, then $|\mathcal{H}_4^{(s)}| = \frac{(s_1 + 1)(s_1 - 2)}{2} + 5$.

(e) If $s_1 \geq 2$ and $u_1 \geq 2$, then $|\mathcal{H}_4^{(s)}| = \frac{u_1(s_1(s_1 + 1))}{2} + u_1^2 + 6$.

Proof. The cases of (a), (b), and (c) are immediate. Consider case (d). It is necessary to enumerate the number of $\lambda \in L_4^{(s)}$ such that $\lambda_3 = s_3$ and $\lambda_4 = s_4$. We should notice that this equivalent to determining the number of lattice points in a lecture hall polytope, namely $\mathbf{P}_2^{(s_1, s_2)}$. Given that this is a lattice triangle, this is an easy task. Note that $s_2 = s_1 - 1$ and the vertices of $\mathbf{P}_2^{(s_1, s_2)}$ are $(0, 0)$, $(0, s_1 - 1)$, and $(s_1, s_1 - 1)$. Recall Pick's theorem says that if \mathcal{P} is a lattice polygon with area A , I interior lattice points, and B boundary lattice points, then

$$A = I + \frac{B}{2} - 1$$

must hold (see [9] for details and proof). We can see that there are $2s_1$ lattice points on the boundary of $\mathbf{P}_2^{(s_1, s_2)}$ as the hypotenuse contains only the vertices as lattice points by $\gcd(s_1, s_2) = 1$. Moreover, since the area is $\frac{s_1(s_1-1)}{2}$, we get that there are $\frac{s_1^2-3s_1-2}{2}$ interior lattice points. Adding the interior points, the boundary points, and the additional five Hilbert basis elements gives

$$\frac{s_1^2 - 3s_1 - 2}{2} + 2s_1 + 5 = \frac{(s_1 + 1)(s_1 - 2)}{2} + 5.$$

To show (e), we apply similar methods. We must enumerate the lattice points of $\mathbf{P}_2^{(s_1, s_2)}$, where $s_2 = u_1 s_1 - 1$, which has vertices $(0, 0)$, $(0, u_1 s_1 - 1)$, and $(s_1, u_1 s_1 - 1)$. We find that there are $s_1(u_1 + 1)$ boundary points, again noting that the hypotenuse contains only the two vertices. Applying Pick's theorem, yields that there are $\frac{u_1 s_1^2 - s_1(u_1 + 2) + 2}{2}$ interior points. Hence, we have that $\mathbf{P}_2^{(s_1, s_2)}$ contains $\frac{u_1(s_1(s_1+1))}{2} + 1$ lattice points. Additionally, elements of the form $(0, j, c_3, c_4)$ account for $c_2 + 1 = (u_1^2 - 1) + 1 = u_1^2$ elements, and there are 5 additional described elements. This gives the cardinality desired. \square

3.8 Concluding remarks and future directions

It is possible that one could consider continuing the low dimensional characterization to $n = 5$ or greater dimensions. However, there are two observations, which discourage this pursuit. First, as noted by the case of $n = 4$, as dimension increases so does the complexity and variation of the Hilbert basis. Experimental evidence using Normaliz [14] indicates that there would be many more cases to consider in the case of $n = 5$ and this will likely shroud the significance of knowing the Hilbert bases. Secondly, cardinality arguments are unlikely to exist in general for greater dimension. The cardinality of the Hilbert basis is controlled in large part by the first $n - 2$ terms of the \mathbf{s} -sequence. In particular, it appears that to obtain the cardinality of the Hilbert basis, one must always compute the number of lattice points in $\mathbf{P}_{n-2}^{(s)}$. In the case of $n = 4$, $n - 2 = 2$ and Pick's theorem makes this possible, but there is no analogue to Pick's theorem for dimension ≥ 3 , which makes the task much more difficult.

There are a number of different directions for future research in this vein. To begin, one could consider the computation of Hilbert bases for more families of \mathbf{s} -lecture hall cones.

One particular family of well studied sequences which fall under the umbrella of \mathbf{u} -generated Gorenstein sequence are the (k, ℓ) -sequences (see [40, Section 5] for definition and importance). However, it is certainly possible that some sequences yield lecture hall cones with combinatorially interesting Hilbert bases which are not \mathbf{u} -generated Gorenstein, or even Gorenstein. It may be interesting to consider certain sequences of this type (e.g. the Fibonacci sequence).

In addition, knowing the Hilbert bases for \mathbf{s} -lecture hall cones opens the door for a number of questions of an algebraic flavor. It is well-known that $\mathbb{C}[\mathcal{C}_n^{(\mathbf{s})}] \cong \mathbb{C}[x_1, \dots, x_d]/I_{\mathbf{s}}$ where $d = |\mathcal{H}_n^{(\mathbf{s})}|$ and $I_{\mathbf{s}}$ is a toric ideal. It would be of interest to compute these toric ideals in certain cases to determine if these ideals admit algebraically or combinatorially interesting Gröbner bases under certain term orders, as well as other algebraic or algebro-geometric properties. Moreover, one could consider free resolutions of $I_{\mathbf{s}}$ to determine the multigraded Betti numbers of the $\mathcal{C}_n^{(\mathbf{s})}$, either using algebraic or combinatorial methods (see [50]). These are unknown even in the case of $\mathbf{s} = (1, 2, \dots, n)$.

Chapter 4 Gorenstein properties and integer decomposition properties of lecture hall polytopes

4.1 Introduction

This chapter investigates algebraic and geometric properties of \mathbf{s} -lecture hall polytopes. For pertinent details on Ehrhart theory, consult Section 1.4; for background on lecture hall polytopes, see Section 1.5.

Recall that given $\mathbf{s} \in \mathbb{Z}_{\geq 1}^n$, the \mathbf{s} -lecture hall polytope is

$$\mathbf{P}_n^{(\mathbf{s})} := \left\{ \mathbf{x} \in \mathbb{R}^d : 0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \dots \leq \frac{x_n}{s_n} \leq 1 \right\}$$

which alternatively has the vertex representation as the column vectors of the matrix

$$\begin{bmatrix} 0 & s_n & s_n & s_n & \cdots & s_n \\ 0 & 0 & s_{n-1} & s_{n-1} & \cdots & s_{n-1} \\ 0 & 0 & 0 & s_{n-2} & \cdots & s_{n-2} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & s_1 \end{bmatrix}$$

where x_n is given by the first row and so on with x_1 given by the last row. It should be noted that there is a easy unimodular equivalence $\mathbf{P}_n^{(\mathbf{s})} \cong \mathbf{P}_n^{(s_n, \dots, s_2, s_1)}$.

For a given $\mathbf{s} \in \mathbb{Z}_{> 1}^n$, recall the \mathbf{s} -inversion sequences by the set $\mathbf{I}_n^{(\mathbf{s})} := \{\mathbf{e} \in \mathbb{Z}^d : 0 \leq e_i < s_i\}$. Given $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$, we define the *ascent set* of \mathbf{e} by

$$\text{Asc}(\mathbf{e}) := \left\{ i : 0 \leq i < n \text{ and } \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\}$$

with the convention that $e_0 = 1$ and $s_0 = 1$. Let $\text{asc}(\mathbf{e}) := |\text{Asc } \mathbf{e}|$. The following result of the h^* -polynomials of \mathbf{s} -lecture hall polytopes for arbitrary \mathbf{s} :

$$h^*(\mathbf{P}_n^{(\mathbf{s})}, z) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} z^{\text{asc}(\mathbf{e})}. \quad (4.1)$$

Moreover, these polynomials are real-rooted and hence unimodal.

The theory of lecture hall polytopes and lecture hall partitions is extensive [40] and many questions have been answered. Some particular motivating work includes the thorough study of Gorenstein properties for \mathbf{s} -lecture hall cones [6]. These results imply Ehrhart theoretic properties of the *rational \mathbf{s} -lecture hall polytopes* $R_{(\mathbf{s})}^n$, but do not imply the same properties for $\mathbf{P}_{(\mathbf{s})}^n$. Additionally, the existence of a unimodular triangulation for the \mathbf{s} -lecture hall cone of $\mathbf{s} = (1, 2, \dots, n)$ was recently shown [7]. However, showing the existence or nonexistence of a unimodular triangulation of $\mathbf{P}_{(\mathbf{s})}^n$ for most \mathbf{s} is still an open question. This motivates the following unanswered questions:

- For what \mathbf{s} is $\mathbf{P}_n^{(\mathbf{s})}$ Fano, reflexive, or Gorenstein?
- For what \mathbf{s} does $\mathbf{P}_n^{(\mathbf{s})}$ satisfies the integer decomposition property?
- If $\mathbf{P}_n^{(\mathbf{s})}$ satisfies the integer decomposition property, for what conditions will it admit a unimodular triangulation?

In this paper, we answer these questions for particular large classes of \mathbf{s} as progress towards a complete characterization. First we consider $\mathbf{P}_n^{(\mathbf{s})}$ when \mathbf{s} is a monotonic sequence. We will show necessary and sufficient conditions for Fano and reflexive in the case when \mathbf{s} is a sequence with $0 \leq s_{i+1} - s_i \leq 1$ for all $0 \leq i \leq n - 1$ (or equivalently $0 \leq s_i - s_{i-1} \leq 1$ for all $0 \leq i \leq n - 1$), the case when \mathbf{s} is a strictly monotonic sequence, and the case when \mathbf{s} is constant then strictly increasing. In the two latter cases, we can also provided necessary and sufficient conditions for when $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein. We continue to show that $\mathbf{P}_n^{(\mathbf{s})}$ satisfies the integer decomposition property for all monotonic \mathbf{s} and show that in some special cases, we can prove that $\mathbf{P}_n^{(\mathbf{s})}$ admits a unimodular triangulation, which is a stronger condition. Furthermore, if we have two lecture hall polytopes $\mathbf{P}_n^{(\mathbf{s})}$ and $\mathbf{P}_d^{(\mathbf{t})}$ which are Gorenstein and/or satisfies the integer decomposition property, we can construct a $(n+d+1)$ -dimensional lecture hall polytope with the respective property.

4.2 Fano, Reflexive, and Gorenstein

Suppose that \mathbf{s} is a monotonic sequence. We give necessary and sufficient conditions for when $\mathbf{P}_n^{(\mathbf{s})}$ is Fano or reflexive in the special cases of \mathbf{s} a strictly increasing sequence and \mathbf{s} a sequence which increases by at most one. In the case of strictly increasing, we can also find necessary and sufficient conditions for when $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein.

Remark 4.2.1. All of the results in this section can be rephrased in the obvious way for when \mathbf{s} is decreasing. This follows from the observation $\mathbf{P}_n^{(s_1, s_2, \dots, s_n)} \cong \mathbf{P}_n^{(s_n, s_{n-1}, \dots, s_1)}$.

Strictly increasing \mathbf{s} -sequences

Suppose that $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ for all $i \in \{1, 2, \dots, n-1\}$. We have the following necessary and sufficient conditions for when $\mathbf{P}_n^{(\mathbf{s})}$ is translation equivalent to a Fano polytope.

Theorem 4.2.2. *Suppose \mathbf{s} is a sequence of strictly increasing positive integers. Then $\mathbf{P}_n^{(\mathbf{s})}$ is translation equivalent to a Fano polytope if and only if $s_1 = 2$ and $s_{i+1} \leq 2s_i$ for all $1 \leq i \leq d - 1$. Moreover, if $\mathbf{P}_n^{(\mathbf{s})}$ is Fano, the unique interior point of $\mathbf{P}_n^{(\mathbf{s})}$ is $(s_n - 1, s_{n-1} - 1, \dots, s_2 - 1, s_1 - 1)^T$.*

Proof. Suppose that \mathbf{s} is a sequence with the property that $s_1 = 2$ and $s_{i+1} \leq 2s_i$. We will show that this implies that $\mathbf{P}_n^{(\mathbf{s})}$ is Fano. It is sufficient to show that $\mathbf{I}_n^{(\mathbf{s})}$ has exactly 1 inversion sequence \mathbf{e} such that $\text{asc}(\mathbf{e}) =$, as this implies that $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = 1$ by 4.1. If we let $\mathbf{e} = (s_1 - 1, s_2 - 1, s_3 - 1, \dots, s_d - 1)$, we should note that $\text{asc}(\mathbf{e}) = n$ because

$$\frac{s_i - 1}{s_i} < \frac{s_{i+1} - 1}{s_{i+1}}$$

follows from the fact that $-s_{i+1} < -s_i$ which is true by assumption. To claim that this is the only such inversion sequence note that

$$\frac{s_i - 1}{s_i} < \frac{s_{i+1} - 2}{s_{i+1}}$$

is never true for any i because this would imply that $-s_{i+1} < -2s_i$ which is false by assumption. Moreover, in order for \mathbf{e} to have an ascent in position 1, we need $e_1 = 1 = s_1 - 1$, so it follows that there is a single inversion sequence of this type. Hence, Additionally, we should note that because we have

$$0 < \frac{s_1 - 1}{s_1} < \frac{s_2 - 1}{s_2} < \dots < \frac{s_n - 1}{s_n} < 1$$

it follows that the point $(s_n - 1, s_{n-1} - 1, \dots, s_2 - 1, s_1 - 1)^T$ does not lie on a supporting hyperplane and is hence the unique interior point of $\mathbf{P}_n^{(\mathbf{s})}$.

Now, suppose that \mathbf{s} is not of the prescribed form. We will show that $\mathbf{P}_n^{(\mathbf{s})}$ is not Fano. There are three possible cases:

- (i) $s_1 = 1$;
- (ii) $s_1 \geq 3$;
- (iii) $s_1 = 2$ and $s_{i+1} > 2s_i$ for some $1 \leq i \leq n - 1$.

Each of these cases preclude $\mathbf{P}_n^{(\mathbf{s})}$ from being Fano.

For (i), if $s_1 = 1$, it is clear from the vertex description of the polytope that $\mathbf{P}_n^{(\mathbf{s})} \cong \text{Pyr}(\mathbf{P}_{n-1}^{(s_2, s_3, \dots, s_n)})$ and hence $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = 0$.

For (ii), if $s_1 \geq 3$, it is easy to see that $\mathbf{P}_n^{(3, 4, \dots, n+2)} \subseteq \mathbf{P}_d^{(\mathbf{s})}$. We can see that $h_n^*(\mathbf{P}_n^{(3, 4, \dots, n+2)}) \geq 2$ because both the inversion sequences $\mathbf{e} = (1, 2, \dots, n)$ and $\mathbf{e}' = (2, 3, \dots, n+1)$ have the property $\text{asc}(\mathbf{e}) = \text{asc}(\mathbf{e}') = d$. So, $\mathbf{P}_n^{(3, 4, \dots, n+2)}$ has at least 2 interior points, which must also be interior points of $\mathbf{P}_n^{(\mathbf{s})}$, meaning it is not Fano.

For (iii), if we have $s_1 = 2$ but that there exists at least one $1 \leq i \leq n - 1$ such that $s_{i+1} > 2s_i$. If there exist multiple such i , choose the smallest. We can see that $\mathbf{P}_n^{(\mathbf{t})} \subseteq \mathbf{P}_n^{(\mathbf{s})}$, where $\mathbf{t} = (s_1, \dots, s_i, 2s_i + 1, 2s_i + 2, \dots, 2s_i + (n - i + 1))$. If we consider this smaller polytope, we can again ascertain that $h_n^*(\mathbf{P}_n^{(\mathbf{t})}) \geq 2$. Note that $\mathbf{e} = (s_1 - 1, \dots, s_i - 1, 2s_i, 2s_i + 1, \dots, 2s_i + (n - i))$ has $\text{asc}(\mathbf{e}) = n$ as

$$\frac{s_i - 1}{s_i} < \frac{2s_i}{2s_i + 1}$$

follows from $-s_i - 1 < 0$ and the other inequalities follow from previous arguments. However, $\mathbf{e}' = (s_1 - 1, \dots, s_i - 1, 2s_i - 1, 2s_i, \dots, 2s_i + (n - i - 1))$ also has the property $\text{asc}(\mathbf{e}') = n$ because

$$\frac{s_i - 1}{s_i} < \frac{2s_i - 1}{2s_i + 1}$$

is follows from $-1 < 0$ and

$$\frac{2s_i + k}{2s_i + k + 2} < \frac{2s_i + k + 1}{2s_i + k + 3}$$

follows from $0 < 4s_i + 2k + 6$. Hence, $\mathbf{P}_n^{(t)}$, and therefore $\mathbf{P}_n^{(s)}$, has at least two interior points, and is not Fano. \square

We can go further to provide necessary and sufficient conditions for when $\mathbf{P}_n^{(s)}$ is translation equivalent to a reflexive polytope.

Theorem 4.2.3. *Suppose that \mathbf{s} is a sequence of strictly increasing positive integers such that $\mathbf{P}_n(\mathbf{s})$ is Fano. Then $\mathbf{P}_n^{(s)}$ is reflexive (up to translation) if and only if for each $0 \leq i \leq n-1$, $k_i = s_{i+1} - s_i$ has the property $k_i | s_i$ and $k_i | s_{i+1}$.*

Proof. If $\mathbf{P}_n^{(s)}$ is Fano, by Theorem 4.2.2 we know that the interior point is $(s_n - 1, s_{n-1} - 1, \dots, s_2 - 1, s_1 - 1)^T$. If we translate $\mathbf{P}_n^{(s)}$ such that the interior point is the origin, the resulting polytope has vertices given by the columns of

$$\begin{bmatrix} 1 - s_n & 1 & 1 & 1 & \cdots & 1 \\ 1 - s_{n-1} & 1 - s_{n-1} & 1 & 1 & \cdots & 1 \\ 1 - s_{n-2} & 1 - s_{n-2} & 1 - s_{n-2} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{bmatrix}$$

This polytope has \mathcal{H} -representation

- $x_n \leq 1$
- $s_{i+1}x_i - s_i x_{i+1} \leq s_{i+1} - s_i$ for all $1 \leq i \leq n-1$
- $-x_1 \leq 1$

using the convention of x_n given by the first row and so on with x_1 given by the last row, as it is clear that each vertex satisfies n equations with equality and 1 with strict inequality.

It follows then that $(\mathbf{P}_n^{(s)})^\vee$ is a lattice polytope if and only if $k_i | s_i$ and $k_i | s_{i+1}$ where $k_i = s_{i+1} - s_i$. \square

We have the following corollary.

Corollary 4.2.4. *Suppose \mathbf{s} is a sequence of strictly increasing positive integers. Then $\mathbf{P}_n^{(s)}$ is Gorenstein of index 2 if and only if $\mathbf{s} = (\frac{t_1}{2}, \frac{t_2}{2}, \dots, \frac{t_n}{2})$ where $\mathbf{t} = (t_1, \dots, t_n)$ is a sequence such that $\mathbf{P}_n^{(t)}$ is reflexive. Moreover, there is no sequence \mathbf{s} of strictly increasing positive integers such that $\mathbf{P}_n^{(s)}$ is Gorenstein of index ≥ 3 .*

Proof. This follows immediately from the observation that $r\mathbf{P}_n^{(s)} = \mathbf{P}_n^{(rs_1, rs_2, \dots, rs_n)}$ and the condition that $s_1 = 2$ when $\mathbf{P}_n^{(s)}$ is reflexive. \square

Constant then strictly increasing \mathbf{s} -sequences

Suppose that we have a sequence of positive integers $\mathbf{s} = (s_1, s_2, \dots, s_i, s_{i+1}, \dots, s_n)$ such that $s_1 = s_2 = \dots = s_i$ and $s_j < s_{j+1}$ for all $j \geq i$. We will give necessary and sufficient conditions for when $\mathbf{P}_n^{(\mathbf{s})}$ is translation equivalent to a Fano polytope for such sequences.

Theorem 4.2.5. *Suppose that \mathbf{s} is a sequence such that $s_1 = \dots = s_i$ for some $1 \leq i \leq n$, and $s_j < s_{j+1}$ for all $i \leq j \leq n - 1$. The polytope $\mathbf{P}_n^{(\mathbf{s})}$ is translation equivalent to a Fano polytope if and only if $s_1 = \dots = s_i = i + 1$ and for all $j \geq i$, $s_{j+1} \leq 2s_j$. Moreover, the unique interior point is $(s_n - 1, \dots, s_{i+1} - 1, i, i - 1, \dots, 2, 1)^T$.*

Proof. Suppose that \mathbf{s} is a sequence of this form such that $s_1 = \dots = s_i = i + 1$ and $s_{j+1} \leq 2s_j$ for all $j \geq i$. We will show that $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = 1$ by showing that there is a unique inversion sequence \mathbf{e} such that $\text{asc}(\mathbf{e}) = d$. Let $\mathbf{e} = (1, 2, \dots, i, s_{i+1} - 1, s_{i+2} - 1, \dots, s_n - 1)$. It is clear that this sequence has d ascents, as $\frac{c}{i+1} < \frac{c+1}{i+1}$ for all $1 \leq c \leq i$,

$$\frac{s_j - 1}{s_j} < \frac{s_{j+1} - 1}{s_{j+1}}$$

for all $j > i$ because $s_j < s_{j+1}$, and

$$\frac{i}{i+1} < \frac{s_{i+1} - 1}{s_{i+1}} = 1 - \frac{1}{s_{i+1}}$$

because $s_{i+1} > i + 1$. To claim that this is the unique such inversion sequence, note that the only way to obtain an ascent each of the first i positions is have the sequence begin $1, 2, \dots, i$. From previous work, we know that

$$\frac{s_j - 1}{s_j} < \frac{s_{j+1} - 2}{s_{j+1}}$$

cannot hold by the assumption $s_{j+1} \leq 2s_j$ for all $j \geq i$. This ensures that no other such inversion sequence with n ascents exists. Thus, we have $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = 1$ so the polytope is Fano. Additionally, because we have

$$0 < \frac{1}{i+1} < \dots < \frac{i}{i+1} < \frac{s_{i+1} - 1}{s_{i+1}} < \dots < \frac{s_n - 1}{s_n} < 1$$

the point $(s_n - 1, \dots, s_{i+1} - 1, i, i - 1, \dots, 2, 1)^T$ is in $\mathbf{P}_n^{(\mathbf{s})}$ and cannot lie on any supporting hyperplane and is hence the unique interior point.

Now, suppose that \mathbf{s} does not have the desired properties. We will show that $\mathbf{P}_n^{(\mathbf{s})}$ is not Fano. There are 3 possibilities:

- (i) $s_1 = \dots = s_i \leq i$;
- (ii) $s_1 = \dots = s_i \geq i + 2$;
- (iii) $s_1 = \dots = s_i = i + 1$, but there exists some $j \geq i$ such that $2s_j < s_{j+1}$

Each of these cases preclude $\mathbf{P}_n^{(\mathbf{s})}$ from being Fano.

For (i), note that it is impossible for there to be an ascent in each of the first i positions. Hence, we have $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = 0$.

For (ii), notice that $\mathbf{P}_n^{(i+2, \dots, i+2, i+3, i+4, \dots, n+2)} \subset \mathbf{P}_n^{(\mathbf{s})}$. If we consider inversion sequences in $\mathbf{I}_n^{(i+2, \dots, i+2, i+3, i+4, \dots, n+2)}$, we have that both $\mathbf{e} = (1, 2, \dots, i, i+1, i+2, \dots, n)$ $\mathbf{e}' = (2, 3, \dots, i+1, i+2, i+3, \dots, n+1)$ have the property $\text{asc}(\mathbf{e}) = \text{asc}(\mathbf{e}') = n$ and hence $h_n^*(\mathbf{P}_n^{(i+2, \dots, i+2, i+3, i+4, \dots, n+2)}) \geq 2$, which implies it has at least two interior points, which are also interior points of $\mathbf{P}_n^{(\mathbf{s})}$.

For (iii), note that $\mathbf{P}_n^{(2, 3, \dots, i+1, s_{i+1}, \dots, s_n)} \subset \mathbf{P}_n^{(\mathbf{s})}$. By the proof of Theorem 4.2.2, we know that $h_n^*(\mathbf{P}_n^{(2, 3, \dots, i+1, s_{i+1}, \dots, s_n)}) \geq 2$, which implies that $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) \geq 2$. \square

Now that we have a complete characterization of when $\mathbf{P}_n^{(\mathbf{s})}$ is Fano for \mathbf{s} of this type, we can now give necessary and sufficient conditions for when they are reflexive.

Theorem 4.2.6. *Suppose that \mathbf{s} is a sequence such that $s_1 = \dots = s_i$ for some $1 \leq i \leq n$, and $s_j < s_{j+1}$ for all $i \leq j \leq n-1$ and suppose that $\mathbf{P}_n^{(\mathbf{s})}$ is Fano. Then $\mathbf{P}_n^{(\mathbf{s})}$ is reflexive if and only if for all $i \leq j \leq n-1$ we have $k_j | s_j$ and $k_j | s_{j+1}$ where $k_j = s_{j+1} - s_j$.*

Proof. By Theorem 4.2.5, we know that the interior point is $(s_n - 1, \dots, s_{i+1} - 1, i, i - 1, \dots, 2, 1)^T$. If we translate $\mathbf{P}_n^{(\mathbf{s})}$ so the interior point is the origin, the resulting polytope has vertices given as the columns of

$$\begin{bmatrix} 1 - s_n & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 - s_{n-1} & 1 - s_{n-1} & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 - s_{n-2} & 1 - s_{n-2} & 1 - s_{n-2} & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 - s_{i+1} & 1 - s_{i+1} & 1 - s_{i+1} & \cdots & 1 - s_{i+1} & 1 & 1 & \cdots & 1 & 1 \\ -i & -i & -i & \cdots & -i & -i & 1 & \cdots & 1 & 1 \\ 1 - i & 1 - i & 1 - i & \cdots & 1 - i & 1 - i & 1 - i & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & i - 1 \end{bmatrix}.$$

This polytope has \mathcal{H} -representation

- $-x_1 \leq 1$;
- $x_n \leq 1$;
- $x_{j-1} - x_j \leq 1$ for all $2 \leq j \leq i$;
- $s_{j+1}x_j - s_jx_{j+1} \leq s_{j+1} - s_j$ for all $i \leq j \leq n-1$.

using the convention that x_n is given by the first row and so on with x_1 given by the last row. It is easy to see that each column of the matrix satisfies precisely n equations with equality and 1 with strict inequality validating the \mathcal{H} -representation. It follows then that the dual polytope $(\mathbf{P}_n^{(\mathbf{s})})^\vee$ is a lattice polytope exactly when $k_j | s_j$ and $k_j | s_{j+1}$ where $k_j = s_{j+1} - s_j$ for $i \leq j \leq n-1$. \square

We can additionally give a description of Gorenstein lecture hall polytopes where \mathbf{s} is of this form.

Corollary 4.2.7. *Suppose that \mathbf{s} is a sequence such that $s_1 = \dots = s_i$ for some $1 \leq i \leq n$, and $s_j < s_{j+1}$ for all $i \leq j \leq n - 1$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein of index $k \in \mathbb{Z}_{>0}$ if and only if there exists a sequence $\mathbf{t} = (t_1, \dots, t_n)$ such that $t_j = ks_j$ for all j (which implies that $t_1 = \dots = t_i$ and $t_j < t_{j+1}$ for $j \geq i$) and $\mathbf{P}_n^{(\mathbf{t})}$ is reflexive.*

Proof. This is immediate with the observation that $k\mathbf{P}_n^{(\mathbf{s})} = \mathbf{P}_n^{(\mathbf{t})}$ and applying the conditions given in Theorem 4.2.6. \square

\mathbf{s} -sequences increasing by at most 1

We now consider an additional subclass of \mathbf{s} -sequences. Suppose the $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ and $0 \leq s_{i+1} - s_i \leq 1$ for all $1 \leq i \leq n - 1$. We have the following characterizations for when $\mathbf{P}_n^{(\mathbf{s})}$ is Fano and reflexive.

Theorem 4.2.8. *Suppose that $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ and $0 \leq s_{i+1} - s_i \leq 1$ for all $1 \leq i \leq n - 1$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is translation equivalent to a Fano polytope if and only if $s_n = n + 1$. Moreover, the unique interior point is $(n, n - 1, \dots, 2, 1)^T$.*

Proof. Suppose that $s_n = n + 1$. We will show that there is a unique $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$ such that $\text{asc}(\mathbf{e}) = n$. It is clear that the sequence $\mathbf{e} = (1, 2, \dots, n)$ satisfies this property, as both $\frac{i}{k} < \frac{i+1}{k}$ and $\frac{i}{k} < \frac{i+1}{k+1}$ are true which implies $\frac{i}{s_i} < \frac{i+1}{s_{i+1}}$. Moreover, to have maximum ascents, we must have $e_i < e_{i+1}$, which means that if $e_n \leq n - 1$, $e_1 = 0$ implying that there is no ascent in the first position. Thus, the sequence $\mathbf{e} = (1, 2, \dots, n)$ is the only inversion sequence with n ascents, giving $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = 1$. It also follows that the unique interior point of $\mathbf{P}_n^{(\mathbf{s})}$ is $(n, n - 1, \dots, 2, 1)^T$, as

$$0 < \frac{1}{s_1} < \frac{2}{s_2} < \dots < \frac{n}{s_n} < 1$$

implies that the point is in $\mathbf{P}_n^{(\mathbf{s})}$ and not on any supporting hyperplane.

Now, note that if $s_n \geq n + 2$, both the inversion sequences $(1, 2, 3, \dots, n)$ and $(2, 3, \dots, n + 1)$ has n ascents. Thus, $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) \geq 2$ in this case.

If we have that $s_n \leq n$, it follows that $\mathbf{P}_n^{(\mathbf{s})} \subseteq \mathbf{P}_n^{(\mathbf{t})}$ where $\mathbf{t} = (n, n, \dots, n)$. Since it is clear that for $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{t})}$ we have $i \in \text{Asc}(\mathbf{e})$ if and only if $e_{i-1} < e_i$ and since $e_i \in \{0, 1, \dots, n - 1\}$ there is no sequence with $\text{asc}(\mathbf{e}) = n$. Thus, we have $h_n^*(\mathbf{P}_n^{(\mathbf{s})}) = h_n^*(\mathbf{P}_n^{(\mathbf{t})}) = 0$. \square

Theorem 4.2.9. *Suppose that $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ and $0 \leq s_{i+1} - s_i \leq 1$ for all $1 \leq i \leq n - 1$. and suppose that $\mathbf{P}_n^{(\mathbf{s})}$ is Fano. Then $\mathbf{P}_n^{(\mathbf{s})}$ is reflexive if and only if $k_i | s_i$ and $k_i | s_{i+1}$ where $k_i = (i + 1)s_i - is_{i+1}$.*

Proof. By Theorem 4.2.8, the interior point of $\mathbf{P}_n^{(\mathbf{s})}$ is $(n, n-1, \dots, 2, 1)^T$. So, if we translate the polytope such that the origin is the interior point, we have the polytope with vertices

$$\begin{bmatrix} -n & 1 & 1 & 1 & \cdots & 1 \\ 1-n & 1-n & s_{n-1}-n+1 & s_{n-1}-n+1 & \cdots & s_{n-1}-n+1 \\ 2-n & 2-n & 2-n & s_{n-2}-n+2 & \cdots & s_{n-2}-n+2 \\ \vdots & \vdots & \vdots & & \ddots & \cdots \\ -1 & -1 & -1 & \cdots & -1 & s_1-1 \end{bmatrix}$$

which, using the convention of x_n given by the first row and so on with x_1 given by the last row, has the \mathcal{H} -representation

- $x_n \leq 1$
- $s_{i+1}x_i - s_i x_{i+1} \leq (i+1)s_i - i s_{i+1}$ for all $1 \leq i \leq n-1$
- $-x_1 \leq 1$

as it is not hard to see that each vertex satisfies n equations with equality and 1 equation with strict inequality. It is now clear that $(\mathbf{P}_n^{(\mathbf{s})})^\vee$ is a lattice polytope if and only if $k_i | s_i$ and $k_i | s_{i+1}$ for $k_i = (i+1)s_i - i s_{i+1}$. \square

4.3 Integer decomposition property and triangulations

We say \mathcal{P} satisfies the *integer-decomposition property (IDP)* if for all $\mathbf{z} \in k\mathcal{P} \cap \mathbb{Z}^n$ there exists $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in \mathcal{P} \cap \mathbb{Z}^n$ such that

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = \mathbf{z}.$$

If \mathcal{P} satisfies then integer decomposition property, we say that \mathcal{P} has the IDP. For \mathbf{s} -lecture hall polytopes where \mathbf{s} is monotonic sequence, we have the following theorem.

Theorem 4.3.1. *Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be a monotone sequence of positive integers. Then the polytope $\mathbf{P}_n^{(\mathbf{s})}$ has the IDP.*

Proof. Without loss of generality, suppose that \mathbf{s} is increasing. We will show that given $k \geq 2$, for any $\mathbf{x} \in k\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^d$, there exists some $\mathbf{y} \in \mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ such that $(\mathbf{x} - \mathbf{y}) \in (k-1)\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$. Note that this is sufficient, because this result allows integral closure to follow from induction on k .

First note that $k\mathbf{P}_n^{(\mathbf{s})} = \mathbf{P}_n^{(ks_1, ks_2, \dots, ks_n)}$, which is clear by definition. Let $\mathbf{x} = (x_n, x_{n-1}, \dots, x_1)^T \in k\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$, so we have that \mathbf{x} satisfies

$$0 \leq \frac{x_1}{ks_1} \leq \frac{x_2}{ks_2} \leq \cdots \leq \frac{x_n}{ks_n} \leq 1.$$

Note that since \mathbf{s} is increasing, given any $C \in \mathbb{Z}_{>0}$ by the above we must have that $x_i \leq Cs_i$ implies that $x_{i-1} \leq Cs_{i-1}$ and likewise $x_i > Cs_i$ implies $x_{i+1} > Cs_{i+1}$. So, let $1 \leq j \leq n$ be the minimum index such that $x_j > (k-1)s_j$. Then we let

$$\mathbf{y} = (x_n - (k-1)s_n, \dots, x_j - (k-1)s_j, 0, \dots, 0)^T$$

with $\mathbf{y} = \mathbf{0}$ if there is no such j .

We know that the lattice point is in $\mathbf{P}_n^{(\mathbf{s})}$ because for any $j \leq i < n$ we have

$$\frac{x_i - (k-1)s_i}{s_i} \leq \frac{x_{i+1} - (k-1)s_{i+1}}{s_{i+1}}$$

is equivalent to $\frac{x_i}{ks_i} \leq \frac{x_{i+1}}{ks_{i+1}}$ and $0 < x_i - (k-1)s_i \leq s_i$ by construction.

It is left to verify that $(\mathbf{x} - \mathbf{y}) = ((k-1)s_n, \dots, (k-1)s_j, x_{j-1}, \dots, x_1)^T \in \mathbf{P}_n^{((k-1)s_1, \dots, (k-1)s_n)} \cap \mathbb{Z}^n$. However, this is immediate, because $\frac{x_i}{(k-1)s_i} \leq \frac{x_{i+1}}{(k-1)s_{i+1}}$ is equivalent to $\frac{x_i}{ks_i} \leq \frac{x_{i+1}}{ks_{i+1}}$ and it is clear that since $x_{j-1} \leq (k-1)s_{j-1}$ by assumption that

$$\frac{x_{j-1}}{(k-1)s_{j-1}} \leq \frac{(k-1)s_j}{(k-1)s_j} = 1.$$

Thus, we have the $\mathbf{P}_n^{(\mathbf{s})}$ has the IDP. □

Recall that a *triangulation* of a lattice polytope \mathcal{P} is a subdivision of \mathcal{P} into n -dimensional simplices. We say that a triangulation is *unimodular* if each simplex h^* of the triangulation is unimodularly equivalent to the standard n -simplex or equivalently, each simplex has smallest possible normalized volume $\text{Vol}(h^*) = 1$. One should note that a polytope \mathcal{P} possessing a unimodular triangulation means that \mathcal{P} can be covered by IDP polytopes which implies that \mathcal{P} has the IDP. We will show the existence for a unimodular triangulation of $\mathbf{P}_n^{(\mathbf{s})}$ provided that for all $1 \leq i \leq n-1$, $s_{i+1} = n_i s_i$ where $n_i \in \mathbb{Z}_{>0}$.

First, we define chimney polytopes. Given a polytope $\mathcal{P} \subset \mathbb{R}^n$ and two integral linear functionals ℓ and u such that $\ell \leq u$, then the *chimney polytope* associated to \mathcal{P} , ℓ , and u is

$$\text{Chim}(\mathcal{P}, \ell, u) := \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{x} \in \mathcal{P}, \ell(\mathbf{x}) \leq y \leq u(\mathbf{x})\}.$$

For chimney polytopes we have the following theorem regarding triangulations.

Lemma 4.3.2 ([24, Theorem 2.8]). *If \mathcal{P} admits a unimodular triangulation, then so does $\text{Chim}(\mathcal{P}, \ell, u)$.*

With this in mind, we can now state and prove a theorem for $\mathbf{P}_n^{(\mathbf{s})}$ where \mathbf{s} is increasing of a particular form.

Theorem 4.3.3. *Let \mathbf{s} be an increasing sequence of positive integers such that $s_{i+1} = k_i s_i$ for some $k_i \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq n-1$. Then $\mathbf{P}_n^{(\mathbf{s})}$ admits a unimodular triangulation.*

Proof. Note that if \mathbf{s} has the property $s_n = k_{n-1} s_{n-1}$ for some $k_{n-1} \in \mathbb{Z}_{>0}$, we can express $\mathbf{P}_n^{(\mathbf{s})}$ as a chimney polytope, namely

$$\mathbf{P}_n^{(\mathbf{s})} \cong \text{Chim}(\mathbf{P}_{n-1}^{(s_1, \dots, s_{n-1})}, k_{n-1} x_{n-1}, s_n)$$

where \mathbf{s}_n is constant function of value s_n . It is easy to see this isomorphism as all of the supporting hyperplanes for $\text{Chim}(\mathbf{P}_{n-1}^{(s_1, \dots, s_{n-1})}, k_{n-1} x_{n-1}, s_n)$ are those of $\mathbf{P}_{n-1}^{(s_1, \dots, s_{n-1})}$ with the

addition of $x_n \leq s_n$ and $k_{n-1}x_{n-1} \leq x_n$. However, these hyperplanes are precisely the supporting hyperplanes of $\mathbf{P}_n^{(\mathbf{s})}$.

Now, note that any 1 dimensional lecture hall polytope trivially has a unimodular triangulation. So, if \mathbf{s} has the property that $s_{i+1} = k_i s_i$ for a positive integer k_i for each i , then applying Theorem 4.3.2 to this inductive chimney polytope construction of $\mathbf{P}_n^{(\mathbf{s})}$ yields the existence of a unimodular triangulation. \square

Remark 4.3.4. We should note that Theorem 4.3.3 implies that $\mathbf{P}_n d^{(\mathbf{s})}$ where \mathbf{s} has the property $s_{i+1} = \frac{s_i}{k_i}$ for some positive integer k_i for all i also admits a unimodular triangulation.

4.4 Constructing new examples

In this section, we construct new Gorenstein and IDP lecture hall polytopes. We will do this by identifying an \mathbf{s} -lecture hall polytope as the free sum of two smaller lecture hall polytopes which are Gorenstein and/or IDP.

Recall that given two lattice polytopes $\mathcal{P} \subset \mathbb{R}^{n_{\mathcal{P}}}$ and $\mathcal{Q} \subset \mathbb{R}^{n_{\mathcal{Q}}}$ such that $0_{n_{\mathcal{P}}} \in \mathcal{P}$ and $0_{n_{\mathcal{Q}}} \in \mathcal{Q}$, the *free sum* of \mathcal{P} and \mathcal{Q} is the $(n_{\mathcal{P}} + n_{\mathcal{Q}})$ -dimensional polytope given by $\mathcal{P} \oplus \mathcal{Q} = \text{conv}\{(0_{\mathcal{P}} \times \mathcal{Q}) \cup (\mathcal{P} \times 0_{\mathcal{Q}})\}$. We can view lecture hall polytopes as free sum of smaller lecture hall polytopes.

Proposition 4.4.1. *For integer sequences $\mathbf{s} = (s_1, \dots, s_n)$ and $\mathbf{t} = (t_1, \dots, t_d)$, we have $\mathbf{P}_{n+d}^{(\mathbf{s}, \mathbf{t})} \cong \mathbf{P}_n^{(\mathbf{s})} \oplus \mathbf{P}_d^{(\tilde{\mathbf{t}})}$, where $(\mathbf{s}, \mathbf{t}) = (s_1, \dots, s_n, t_1, \dots, t_d)$ and $\tilde{\mathbf{t}} = (t_d, t_{d-1}, \dots, t_1)$.*

Proof. Translate by the vector $(t_d, \dots, t_2, t_1, 0, 0, \dots, 0)^T$. \square

The following generalization of Braun's formula gives us conditions on the h^* -polynomial of a free sum of two polytopes.

Lemma 4.4.2 ([8, Theorem 1.4]). *Let $\mathcal{P} \subset \mathbb{R}^n$ and $\mathcal{Q} \subset \mathbb{R}^d$ be integral convex polytopes each containing its respective origin. Then $h^*(\mathcal{P} \oplus \mathcal{Q}, \lambda) = h^*(\mathcal{P}, \lambda)h^*(\mathcal{Q}, \lambda)$ holds if and only if either \mathcal{P} or \mathcal{Q} satisfies that the equation of each facet is of the form $\sum_{i=1}^f a_i x_i = b$ where a_i is an integer, $b \in \{0, 1\}$, and $f \in \{n, d\}$.*

We can now give a construction for larger lecture hall polytopes which must be Gorenstein.

Theorem 4.4.3. *Suppose that $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{t} = (t_1, t_2, \dots, t_d)$ are integer sequences such that $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein of index k and $\mathbf{P}_d^{(\mathbf{t})}$ is Gorenstein of index ℓ . Then $\mathbf{P}_{n+d+1}^{(\mathbf{s}, \mathbf{t})}$ is Gorenstein of index $k + \ell$.*

Proof. Note that by Proposition 4.4.1, we have that $\mathbf{P}_{n+d+1}^{(\mathbf{s}, \mathbf{t})} \cong \mathbf{P}_{n+1}^{(\mathbf{s}, 1)} \oplus \mathbf{P}_d^{(\tilde{\mathbf{t}})}$. By the \mathcal{H} -representation, we know that $\mathbf{P}_{n+1}^{(\mathbf{s}, 1)}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{d+1} a_i x_i = b$ where a_i is an integer, $b \in \{0, 1\}$. Moreover, from the \mathcal{V} -representation it is clear that $\mathbf{P}_{n+1}^{(\mathbf{s}, 1)} \cong \text{Pyr}(\mathbf{P}_n^{(\mathbf{s})})$, so it has the same h^* -vector and is thus Gorenstein. By Lemma 4.4.2, we then know that $h^*(\mathbf{P}_{n+d+1}^{(\mathbf{s}, \mathbf{t})}, \lambda) = h^*(\mathbf{P}_{n+1}^{(\mathbf{s}, 1)}, \lambda)h^*(\mathbf{P}_d^{(\tilde{\mathbf{t}})}, \lambda)$ because $\mathbf{P}_d^{(\tilde{\mathbf{t}})} \cong \mathbf{P}_d^{(\mathbf{t})}$.

Therefore, $h^*(\mathbf{P}_{n+d+1}^{(s,1,t)}, \lambda)$ is symmetric polynomial of degree $(n+d+1) - (k+\ell) + 1$ and we have the desired. \square

Additionally, necessary and sufficient conditions for the integral closure of a free sum of two polytopes are known. These are given in the following theorem.

Lemma 4.4.4 ([29, Theorem 0.1]). *Let $\mathcal{P} \subset \mathbb{R}^n$ and $\mathcal{Q} \subset \mathbb{R}^d$ be integral convex polytopes each containing its respective origin. Suppose that \mathcal{P} and \mathcal{Q} satisfy $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^n) = \mathbb{Z}^n$, $\mathbb{Z}(\mathcal{Q} \cap \mathbb{Z}^d) = \mathbb{Z}^d$, and*

$$(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{n+d} = \mu(\mathcal{P} \cap \mathbb{Z}^n) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^d)$$

where μ and ν are the canonical injections defined $\mu : \mathbb{R}^n \rightarrow \mathbb{R}^{n+d}$ by $\alpha \mapsto (\alpha, 0_d)$ and $\nu : \mathbb{R}^d \rightarrow \mathbb{R}^{n+d}$ by $\beta \mapsto (0_n, \beta)$. Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ has the IDP if and only if the following two conditions hold:

- each of \mathcal{P} and \mathcal{Q} has the IDP;
- either \mathcal{P} or \mathcal{Q} has the property that the equation of each facet is of the form $\sum_{i=1}^f a_i x_i = b$ where a_i is an integer, $b \in \{0, 1\}$, and $f \in \{n, d\}$.

We can now give a construction for larger IDP lecture hall polytopes.

Theorem 4.4.5. *Suppose that $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{t} = (t_1, t_2, \dots, t_d)$ are integer sequences such that $\mathbf{P}_n^{(\mathbf{s})}$ and $\mathbf{P}_d^{(\mathbf{t})}$ are IDP. Then $\mathbf{P}_{n+d+1}^{(\mathbf{s},1,\mathbf{t})}$ has the IDP.*

Proof. Note that for any 2 lecture hall polytopes $\mathbf{P}_n^{(\mathbf{s})}$ and $\mathbf{P}_d^{(\mathbf{t})}$, we have $\mathbb{Z}(\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n) = \mathbb{Z}^n$ and $\mathbb{Z}(\mathbf{P}_d^{(\mathbf{t})} \cap \mathbb{Z}^d) = \mathbb{Z}^d$ follow immediately.

Now, by Proposition 4.4.1, we have that $\mathbf{P}_{n+d+1}^{(\mathbf{s},1,\mathbf{t})} \cong \mathbf{P}_{n+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_d^{(\tilde{\mathbf{t}})}$. By the \mathcal{H} -representation, we know that $\mathbf{P}_{n+1}^{(\mathbf{s},1)}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{n+1} a_i x_i = b$ where a_i is an integer, $b \in \{0, 1\}$. To see that

$$(\mathbf{P}_{n+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_d^{(\tilde{\mathbf{t}})}) \cap \mathbb{Z}^{n+1+d} = \mu(\mathbf{P}_{n+1}^{(\mathbf{s},1)} \cap \mathbb{Z}^{n+1}) \cup \nu(\mathbf{P}_d^{(\tilde{\mathbf{t}})} \cap \mathbb{Z}^d)$$

holds, note that the right side is clearly contained in the left side. If we consider an element \mathbf{x} such that

$$\mathbf{x} \in (\mathbf{P}_{n+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_d^{(\tilde{\mathbf{t}})}) \cap \mathbb{Z}^{n+1+d} \setminus \left(\mu(\mathbf{P}_{n+1}^{(\mathbf{s},1)} \cap \mathbb{Z}^{n+1}) \cup \nu(\mathbf{P}_d^{(\tilde{\mathbf{t}})} \cap \mathbb{Z}^d) \right),$$

we have that $\sum_{i=1}^{n+d+1} c_i v_i = 1$ where c_i is constant and v_i is the i th vertex. However, we also must have that $x_{n+1} = 1$, which implies that $\sum_{i=1}^{n+1} c_i = 1$ from the definition of the free sum. So this implies that $\mathbf{x} \in \mu(\mathbf{P}_{n+1}^{(\mathbf{s},1)} \cap \mathbb{Z}^{n+1})$ which is a contradiction. The result now follows from Lemma 4.4.4. \square

4.5 Concluding Remarks

While we have been able to ascertain many previously unknown properties of lecture hall polytopes, full characterizations of all of these properties remain elusive. We conclude with two conjectures.

Conjecture 4.5.1. *For any $\mathbf{s} = (s_1, \dots, s_n)$, $\mathbf{P}_n^{(\mathbf{s})}$ has the IDP.*

For many randomly generated \mathbf{s} , we have found $\mathbf{P}_n^{(\mathbf{s})}$ to be IDP and we have been unable to find an example of a non IDP lecture hall polytope. Additionally, the convenient description of dilates of lecture hall polytopes, namely $c\mathbf{P}_n^{(\mathbf{s})} = \mathbf{P}_n^{(cs_1, cs_2, \dots, cs_n)}$, suggests that one may be able to generalize our arguments for monotone sequences to arbitrary \mathbf{s} .¹

Conjecture 4.5.2. *For any $\mathbf{s} = (s_1, \dots, s_n)$, $\mathbf{P}_n^{(\mathbf{s})}$ admits a unimodular triangulation.*

We have not come across any examples of lecture hall polytopes which do not admit a unimodular triangulation. However, using Gröbner bases has not proved fruitful given that though a variable ordering and monomial ordering which yield a quadratic squarefree Gröbner basis seem to always exist, it is not always the same ordering. A positive answer to this conjecture would resolve Conjecture 4.5.1 as well. Moreover, a counterexample, or a positive partial result such as the monotone case would be of great interest.

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¹This has been shown to be the case by Brändén and Solus [12]. In fact, the proof is essentially identical to the proof given for monotone sequences.

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VITA

MCCABE JAMES OLSEN

ACADEMIC POSITIONS

The Ohio State University, Columbus, Ohio

Arnold Ross Assistant Professor (accepted)

Beginning August 2018

EDUCATION

University of Kentucky, Lexington, KY

Ph.D. Mathematics

(Expected) May 2018

Advisor: Benjamin Braun

M.A. Mathematics

May 2015

Mercer University, Macon, GA

B.S. Mathematics

May 2013

B.A. Political Science

Cum Laude

AWARDS & FELLOWSHIPS

External Awards:

- AMS JMM Graduate Student Travel Grant Jan. 2018
- AMS Graduate Student Travel Grant Sept. 2017
- NSF EAPSI Fellowship Summer 2016

Internal Awards:

- College of Arts and Sciences Outstanding Teaching Assistant Award,
University of Kentucky Apr. 2018
- Wimberly C. Royster Outstanding Teaching Assistant Award,
University of Kentucky Apr. 2017
- Max Steckler Fellowship, University of Kentucky Aug. 2016–May 2017
- Riley Plymale Award for Mathematics, Mercer University Apr. 2013
- Adel Sherwood Tuition Scholarship, Mercer University Aug. 2009– May 2013

PUBLICATIONS

Publications are listed in reverse submission order.

- Florian Kohl and McCabe Olsen. Level algebras and \mathbf{s} -lecture hall polytopes. Submitted to *J. Combin. Theory Ser. A*. <https://arxiv.org/abs/1710.10892>
- McCabe Olsen. Hilbert bases and lecture hall partitions, *Ramanujan J.* To appear. <https://arxiv.org/abs/1703.02006>
- Dustin Hedmark, Cyrus Hettle, and McCabe Olsen. Flag descents and Eulerian polynomials for wreath product quotients. Submitted to *INTEGERS*. <https://arxiv.org/abs/1611.06259>
- Takayuki Hibi, McCabe Olsen, and Akiyoshi Tsuchiya. Gorenstein properties and integral decomposition properties of lecture hall polytopes. *Mosc. Math. J.* To appear. <http://arxiv.org/abs/1608.03934>
- Takayuki Hibi, McCabe Olsen, and Akiyoshi Tsuchiya. Self dual reflexive simplices with Eulerian polynomials. *Graphs Combin.* 33: 1401–1404, 2017. <http://arxiv.org/abs/1607.04871>
- Benjamin Braun and McCabe Olsen. Euler-Mahonian statistics and descent bases for semigroup algebras. *European J. Combin.*, 69:237–254, 2018. <http://arxiv.org/abs/1606.03007>
- Patrick Bahls, Elizabeth Bailey, and McCabe Olsen. New families of graphs whose independence polynomials have only real zeros. *Australas. J. Combin.* 60: 128–135, 2014.
- Julie Beier and McCabe Olsen. A not-so-simple Lie bracket expansion, (with J. Beier) *Involve*, 7, no. 5: 647-655, 2014.