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The Partition Lattice in Many Guises

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THE PARTITION LATTICE IN MANY GUISES

ABSTRACT OF DISSERTATION

THE PARTITION LATTICE IN MANY GUISES

This dissertation is divided into four chapters. In Chapter 2 the $\mathfrak{S}_{n-1}$-equivariant homology groups of upper order ideals in the partition lattice are computed. The homology groups of these filters are written in terms of border strip Specht modules as well as in terms of links in an associated complex in the lattice of compositions. The classification is used in Section 2.15 to compute the $\mathfrak{S}_{n-1}$-equivariant homology of many well-studied subcomplexes of the partition lattice, including the d-divisible partition lattice and the Frobenius complex. In Chapter 3 the box polynomial $B_{m,n}(x)$ is defined in terms of all integer partitions λ that fit in an m by n box. The real roots of the box polynomial are completely characterized, and an asymptotically tight bound on the norms of the complex roots is also given. An equivalent definition of the box polynomial is given via applications of the finite difference operator Δ to the monomial x^{m+n}. The box polynomials are also used to find identities counting set partitions with all even or odd blocks, respectively. Chapter 4 extends results from Chapter 3 to give combinatorial proofs for the ordinary generating function for set partitions with all even or all odd block sizes, respectively. This is achieved by looking at a multivariable generating function analog of the Stirling numbers of the second kind using restricted growth words. Chapter 5 introduces a colored variant of the ordered partition lattice, denoted Q_n^α, as well an associated complex known as the α-colored permutahedron, whose face poset is Q_n^α. Connections between the Eulerian polynomials and Stirling numbers of the second kind are developed via the fibers of a map from Q_n^α to the symmetric group $\mathfrak{S}_n$.

KEYWORDS: partition lattice, filter, Stirling numbers, box polynomial

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Date: _____ May 2, 2017_____

THE PARTITION LATTICE IN MANY GUISES

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Date: May 2, 2017

Dedicated to my advisor, Richard Ehrenborg.

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Lastly, I would like to thank my friends and family for helping me through graduate school. No need to name names–you know who you are.

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1.1   Introduction

> Well, the beginning, that is
> dead and buried.
>
> ―――――――――――――――
> *Celia*
> *As You Like It, I.2.241*

This thesis is comprised of four main sections. Chapter 1 collects prerequisites. As Celia opines in the opening quote for this chapter, nothing in this preliminary chapter is my own work. Chapter 2 examines the topology and representation theory of a large class of filters in the partition lattice Π_n. In particular, the homology groups of filters in the partition lattice are determined explicitly in terms of an associated complex Δ living in the poset of compositions of n. A major tool used in Chapter 2 will be to convert topological data from the partition lattice Π_n to the *ordered* partition lattice Q_n using *Quillen's Fiber Lemma*, which will be discussed further in Section 1.7. In particular, the first main result is Theorem 2.6.3, where the homology groups of a large class of filters in the ordered set partition poset Q_n are determined. The topological information of Theorem 2.6.3 is transferred to Π_n in Theorem 2.11.5 using Quillen's Fiber Lemma.

Chapter 3 explores the box polynomials, a polynomial defined in terms of all integer partitions λ that fit in the m by n box. Many algebraic and combinatorial properties of these polynomials are established, including a characterization of their real roots, as well as asymptotics of their complex roots in Section 3.3. Additionally, connections between the box polynomials and set partitions are pursued in Section 3.2.

Chapter 4 extends results from Chapter 3 to give combinatorial proofs for the ordinary generating function for set partitions with all even or all odd block sizes, respectively. As explained in the introduction to Chapter 4, this is achieved by looking at a multivariable generating function analog of the Stirling numbers of the second kind, i.e., a generalization of the ordinary generating function in Equation (1.2.3), via *restricted growth words*.

Lastly, Chapter 5 introduces a colored generalization of the ordered partition lattice Q_n as well as a colored generalization of the permutahedron. In particular, we allow each block of the ordered set partition to have one of α colors, and impose a new cover relation where adjacent blocks of the ordered set partition can only be merged if they have the same color. Unfortunately, as the genesis of an idea is often sacrificed in the editing of a mathematical paper, it is worth noting here that all of Chapter 5 began with Exercise 33 of Chapter 1 of Stanley's treatise [36].

While at first glance these chapters seem unrelated, the common theme that binds them is the partition lattice Π_n in its many guises, explaining the title of this dissertation. In Chapters 2 and 4 the partition lattice is the main actor, while in Chapter 5 the partition lattice appears in its ordered form, Q_n. In Chapter 3 integer partitions take the spotlight.

Each subsequent chapter will begin with a brief introduction of the history of the problem, as well as an introduction to the relevant mathematical notation and terminology. That being

said, the material of Chapter 1 will sometimes be re-introduced in other chapters. Lastly, to emulate a great early American mathematical paper by Sylvester [40], each chapter begins with a Shakespearean quote related to the content therein.

We begin our chapter with preliminaries on the Stirling numbers of the second kind.

1.2 Stirling Numbers

Much of this thesis is devoted to the combinatorics of the partition lattice, Π_n. For this reason it is only fitting that we begin our narrative with a history of set partitions.

Let $[n] = \{1, 2, \ldots, n\}$ denote the n-set. A *partition* of $[n]$ is a disjoint union of subsets $B_1, B_2, \ldots, B_k$ such that $\cup_{i=1}^{k} B_i = [n]$. According to Knuth, [44], set partitions were first studied systematically by the Japanese around 1500. In a popular parlour game of the time, *genji-ko*, five unknown incense were burned and players were asked to identify which of the scents were the same, and which were different. The number of outcomes of this game is 52, the number of set partitions of the five set [5], or in modern parlance, the fifth *Bell number* $B_5 = 52$. Diagrams were soon developed to model the 52 set partitions of [5]; see [44, pg. 25].

For a set partition $\pi = \{B_1, \ldots, B_m\}$ of $[n]$ we call each B_i a *block* of π. We denote a break in blocks with a vertical bar |. As an example, we write the set partition $\{\{1, 2, 4\}, \{3, 5\}\}$ of [5] as 124|35. We write set partitions with elements ordered least to greatest from left to right within a block. We then order the blocks by their least element from left to right.

The *Stirling Number of the second kind*, $S(n, k)$, named after Scottish mathematician James Stirling, enumerates set partitions of $[n]$ into k blocks. The Stirling numbers satisfy the recursion

$$S(n + 1, k) = k \cdot S(n, k) + S(n, k - 1), \tag{1.2.1}$$

which is easily seen by keeping track of which block the element $n + 1$ joins. Additionally, the Stirling numbers have initial conditions $S(0, 0) = 1$ and $S(n, 0) = 0$ for $n \geq 1$.

We will now record some closed forms and generating functions for the Stirling numbers which will be used in Chapters 3 and 4.

To obtain a closed form for $S(n, k)$ we count surjections from $[n]$ to $[k]$ for $n \geq k$ in two ways. A surjection from $[n]$ to $[k]$ determines a set partition of $[n]$ into k parts, thus all surjections are counted by $k! \cdot S(n, k)$. By inclusion/exclusion the number of surjections is counted by $\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$, yielding

$$S(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n. \tag{1.2.2}$$

The ordinary generating function for $S(n, k)$ is given by

$$\sum_{n \geq k} S(n, k)x^n = \frac{x^k}{(1 - x)(1 - 2x)\cdots(1 - kx)}, \tag{1.2.3}$$

see [36, Eq.1.94(c)]. Equation (1.2.3) is proven in sub-section 4.2 using *restricted growth words*; see [7]. By equating coefficients in Equation (1.2.3), it follows that

$$S(n + k, k) = h_k(1, 2, \ldots, n), \tag{1.2.4}$$

where h_k is the degree k homogeneous symmetric function.

The exponential generating function for $S(n,k)$ is given by

$$\sum_{n \geq k} S(n,k)\frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}. \tag{1.2.5}$$

Equation (1.2.5) is proven with the composition principle of exponential generating functions.

1.3 Posets

A *partially ordered set* P, colloquially referred to as a poset, is a set endowed with a binary relation $\leq$ satisfying the following:

- (reflexive) For each $p \in P$ we have $p \leq p$.

- (antisymmetric) If $p \leq q$ and $q \leq p$ then $p = q$.

- (transitive) If $p \leq q$ and $q \leq r$ then $p \leq r$.

Noticeably lacking from the definition of a partially ordered set is that any two elements be related by $\leq$. If two elements p and q are not related via $\leq$, we say that p and q are *incomparable* elements. As an example, a and b of Figure 1.1 are incomparable. For more poset terminology, see Chapter 3 of [36].

Example 1.3.1 (The Boolean algebra)**.** Consider the collection of subsets of the n-set $[n]$ ordered by inclusion. This forms a poset called the *Boolean algebra*, B_n. Note that $\{1,2,3\}$ and $\{2,3,4\}$ are incomparable elements in B_4 as neither is a subset of the other.

A *chain* in a poset P is a collection of elements that are totally ordered. The size of a chain is defined to be the number of elements in the chain.

If $p \leq q$ and there is no element $r \neq q$ such that $p \leq r \leq q$ we say that q *covers* p. We represent posets graphically with the *Hasse diagram*. The Hasse diagram of a poset P is a graph with a node for each element of P and with edges given by the covering relations in P.

Given two posets P and Q we define the *direct product* poset $P \times Q$ on the collection of pairs (p,q), for $p \in P$ and $q \in Q$, with order relation given by $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$.

In a poset P we let the *meet* of two elements p and q be the collection of maximal mutual lower bounds of p and q. Equivalently, we define the *join* of p and q in P to be the collection of minimal mutual upper bounds of p and q.

A *lattice* is a poset P such that any two elements p and q have a unique meet and join. In a lattice we let $a \wedge b$ denote the meet of a and b, and we let $a \vee b$ denote the join of a and b. Figure 1.1 shows the Hasse digram of a non-lattice. Notice that both c and d are meets for a and b, and therefore a and b do not have a unique meet.

Let P and Q be posets. A function $f : P \longrightarrow Q$ is a *poset* map if $r \leq_P s$ implies $f(r) \leq_Q f(s)$. Posets P and Q are isomorphic as posets if there is an invertible poset map $f : P \longrightarrow Q$ whose inverse f^{-1} is also a poset map. We indicate poset isomorphism in the usual way, $P \cong Q$.
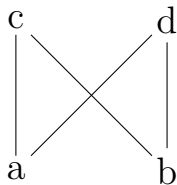
Figure 1.1: A non-lattice.

We will now introduce an important poset invariant, the *Möbius function*.

Let P be a locally finite poset, that is, a poset where each interval $[p, q]$ is finite. Let I be the collection of all intervals $[p, q]$ in P. Let $I(P)$ denote the collection of maps from $I \longrightarrow \mathbb{C}$. For ease of notation, for $f \in I(P)$ applied to the interval $[p, q]$ we write $f(p, q)$ rather than $f([p, q])$.

We now give $I(P)$ an algebra structure, called the *incidence algebra* of P. Let addition and ring subtraction in $I(P)$ be defined pointwise, and define the algebra multiplication to be given by convolution. Specifically, if $f, g \in I(P)$ then $f * g$ applied to the interval $[p, q]$ is given by:

$$(f * g)(p, q) = \sum_{p \leq s \leq q} f(p, s)g(s, q).$$

It can be shown that the identity for the incidence algebra is given by the function that takes value one on trivial intervals $[p, p]$ and is zero on intervals $[p, q]$ for $p < q$. We denote this function by δ.

The zeta function $\zeta \in I(P)$ is identically 1 on all intervals. The *Möbius function* is the inverse to the zeta function, that is, $\mu * \zeta = \zeta * \mu = \delta$. The Möbius function is defined recursively via:

- $\mu(p, p) = 1$

- $\mu(p, q) = -\sum_{p \leq s < q} \mu(p, s)$

While the Möbius function is an element of the incidence algebra and thus is a complex-valued function on the set of intervals in P, if the poset P is bounded, meaning P has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$, we write $\mu(P)$ to indicate the value of the Möbius function on the interval $P = [\hat{0}, \hat{1}]$.

The Möbius function is clearly a poset invariant, i.e. if $P \cong Q$ then $\mu(P) = \mu(Q)$, and is therefore an interesting number to keep track of. Using the definition of the Möbius function to compute $\mu(P)$ can be tedious, thus we now record two theorems that are helpful for computing $\mu(P)$ and are used throughout this dissertation.

Theorem 1.3.2 (Weisner's Theorem). *Let P be a lattice. Let $a \in P$ such that $a \neq \hat{1}$. Then the following holds:*

$$\sum_{x : x \wedge a = \hat{0}} \mu([x, \hat{1}]) = 0.$$

4

$$123$$

12|3   13|2   1|23

1|2|3
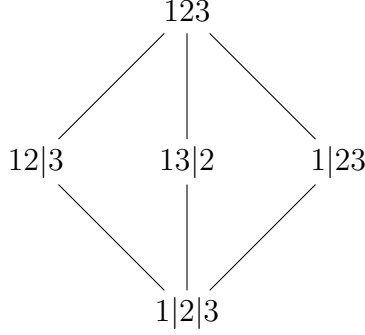
Figure 1.2: The Partition Lattice Π_3

We also record Philip Hall's Theorem, which relates the Euler characteristic of the order complex of $\overline{P} := P - \{\hat{0}, \hat{1}\}$ to the Möbius function of P. For the definition of the order complex of P, see Section 1.7. Furthermore, recall that for Δ a simplicial complex, the *reduced Euler characteristic* of Δ, denoted $\widetilde{\chi}$, is given by the alternating sum of its face numbers $\widetilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i(\Delta)$.

Theorem 1.3.3 (Philip Hall's Theorem). *Let P be a bounded poset. Let $\overline{P} := P - \{\hat{0}, \hat{1}\}$. Then its Möbius function is given by $\mu(P) = \chi(\triangle(\overline{P}))$.*

1.4 The Partition Lattice and Integer Partitions

The collection of set partitions of $[n]$, denoted Π_n, is called the *partition lattice*. The cover relation in Π_n is given by $\sigma \prec \tau$ if exactly two blocks of σ were merged to form τ. With this cover relation Π_n is graded with minimal element $1|2|\cdots|n$ and maximal element $123\cdots n$.

We indicate that a set partition σ has k blocks by writing $|\sigma| = k$. A set partition $\sigma \in \Pi_n$ has rank k if $|\sigma| = n - k$, and thus the number of elements of rank k in Π_n is counted by the Stirling number $S(n, n - k)$. The maximal element $12 \cdots n$ of Π_n has one block, and therefore the rank of Π_n is $n - 1$.

For $\sigma = B_1|B_2|\cdots|B_k$ in Π_n, the upper order ideal $\{\tau \in \Pi_n | \tau \geq \sigma\}$ is isomorphic to Π_k, since we can think of the k blocks of σ as the elements $\{1, 2, \ldots, k\}$ which we must merge to move up in Π_n. Analogously, the lower order ideal $\{\tau \in \Pi_n | \tau \leq \sigma\}$ is isomorphic to $\Pi_{|B_1|} \times \Pi_{|B_2|} \times \cdots \times \Pi_{|B_k|}$, since to move down from σ in Π_n we must break up the blocks of σ independently.

The Möbius function $\mu(\Pi_n)$ can be computed with Weisner's Theorem 1.3.2. Note that since $\hat{0} \wedge a = \hat{0}$, we can rewrite Weisner's Theorem as

$$\mu([\hat{0}, \hat{1}]) = - \sum_{x : x \wedge a = \hat{0}, x \neq \hat{0}} \mu([x, \hat{1}]). \tag{1.4.1}$$

Let a be the coatom $12 \cdots (n - 1)|n$ in Π_n. The collection of non-zero set partitions τ such that $\tau \wedge a = \hat{0}$ consists of atoms with unique doubleton block $\{i, n\}$ for $1 \leq i \leq n - 1$. As
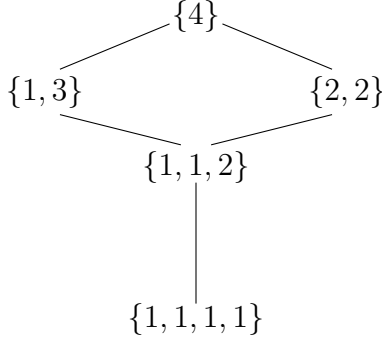
Figure 1.3: I_4, the poset of integer partitions of 4.

there are $n-1$ such atoms, and as each interval $[\tau, \hat{1}] \cong \Pi_{n-1}$, by Weisner's Theorem (1.4.1) we have that $\mu(\Pi_n) = -(n-1) \cdot \mu(\Pi_{n-1})$. Therefore, by induction,

$$\mu(\Pi_n) = (-1)^{n-1}(n-1)!. \tag{1.4.2}$$

A large portion of this thesis is devoted to sub-posets of the partition lattice Π_n. To sort set partitions by block size, we introduce the notion of *type*.

Let I_n be the poset of integer partitions of n. The elements of I_n are multisets of positive integers $\{i_1, i_2, \ldots, i_k\}$ such that $i_1 + i_2 + \cdots + i_k = n$. The cover relation in I_n is given by adding of (not necessarily adjacent) parts, that is, $\{i_1, i_2, \ldots, i_k\} \prec \{i_1 + i_2, \ldots, i_k\}$. Note that the minimal element of I_n is $\{1, 1, \ldots, 1\}$ and maximal element is $\{n\}$. As an example, see Figure 1.3.

For a set partition $\sigma \in \Pi_n$ we define the *type* of σ, denoted type(σ), as the integer partition of n given by the cardinality of the blocks of σ. As an example, type$(14|25|3) = \{1, 2, 2\} \in I_5$. It is worth noting that type $: \Pi_n \longrightarrow I_n$ is a poset map.

With the type of a set partition defined, we can create sublattices of Π_n by type. As an example, the d-divisible partition lattice, denoted Π_n^d, is the collection of all set partitions in Π_n whose type has all parts divisible by d. In other words, the d-divisible partition lattice consists of all set partitions with block sizes divisible by d.

1.5 Ordered Set Partitions and Compositions

In this section we consider the ordered equivalent of Section 1.4.

An *ordered set partition* of the set $[n]$ is a disjoint union of the set $[n]$ into blocks where the order between the blocks matters. To distinguish from ordinary set partitions, we write ordered set partitions in parentheses with commas indicating a break in blocks. For example, $(13, 245)$ and $(245, 13)$ are distinct ordered set partitions of $[5]$.

The collection of ordered set partitions of n, denoted Q_n, has a poset structure with cover relation given by the merging of adjacent blocks. We record the cover relation for Q_n here:

$$(B_1, B_2, B_3, \ldots, B_k) \prec (B_1, B_2, \ldots, B_i \cup B_{i+1}, \ldots, B_k) \tag{1.5.1}$$
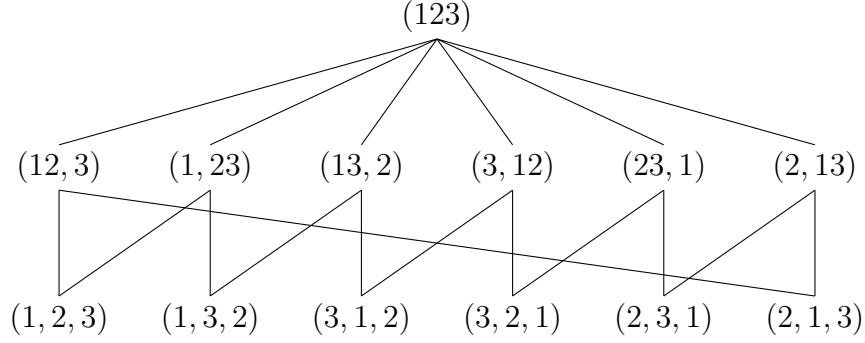
Figure 1.4: The Ordered Partition Lattice Q_3

Figure 1.4 shows the Hasse diagram for the ordered partition lattice Q_3. Notice Q_n does not have a unique minimal element, but rather has $n!$ minimal elements, one for each permutation of $[n]$. This shows that Q_n is not a lattice.

To make Q_n into a lattice we artificially adjoin a minimal element $\hat{0}$. The poset $Q_n \cup \{\hat{0}\}$ is now a lattice and we can compute $\mu(Q_n \cup \{\hat{0}\})$ using Philip Hall's formula, Theorem 1.3.3, since $Q_n \cup \{\hat{0}\}$ is the face lattice of the permutahedron P_n. We postpone this calculation until Chapter 5.

Additionally, using Figure 1.4 as an example, we see that intervals in the poset Q_n are isomorphic to Boolean algebras.

In analogy to unordered set partitions, we again need to define the type of an ordered set partition. For this notion we need the poset of compositions of n, denoted $\mathrm{Comp}(n)$.

A *composition* of n is an ordered list $(i_1, i_2, \ldots, i_k)$ of positive integers such that $i_1 + i_2 + \cdots + i_k = n$. We denote the collection of compositions of n by $\mathrm{Comp}(n)$ and we endow $\mathrm{Comp}(n)$ with a poset structure with cover relation given by adding of adjacent parts.

To reflect that the order amongst the blocks in Q_n matters, we say that the *type* of an ordered set partition $\sigma \in Q_n$ is the composition of n given by the cardinality of its blocks in order. For example, type$((12, 345)) = (2, 3) \in \mathrm{Comp}(5)$. Continuing the analogy from Section 1.4, the map type $: Q_n \longrightarrow \mathrm{Comp}(n)$ is also a poset map.

The poset of compositions of n is well-behaved. In fact, there is a poset isomorphism $\mathrm{Comp}(n) \cong B_{n-1}$, where B_{n-1} is the Boolean algebra on $n-1$ elements. The isomorphism is given by mapping the composition $(c_1, c_2, \ldots, c_k)$ to the subset of $[n-1]$ given by the partial sum of the first $k-1$ parts of $\vec{c}$, that is, $\vec{c}$ corresponds to $\{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1}\}$.

Lastly, we mention there is a well-behaved map from $f : Q_n \longrightarrow \Pi_n$, known as the *forgetful* map. The map f is defined as you might expect. Let $(A_1, A_2, \ldots, A_k) \in Q_n$, then f is given by:

$$f((A_1, A_2, \ldots, A_k)) = A_1 | A_2 | \ldots | A_k \qquad (1.5.2)$$

The forgetful map will be a crucial tool used to help characterize the topology of sub-posets of the partition lattice Π_n in Chapter 2.

1.6   Permutations and Descents

The symmetric group $\mathfrak{S}_n$ is the collection of permutations of the n-set $[n]$ under composition. We will refer to permutations in one line notation. For $\alpha \in \mathfrak{S}_n$ we write $\alpha_1 \alpha_2 \ldots \alpha_n$ where $\alpha_i = \alpha(i)$. As an example, $231 \in \mathfrak{S}_3$ is the map given by $1 \mapsto 2$, $2 \mapsto 3$, and $3 \mapsto 1$.

A *descent* of a permutation $\alpha \in \mathfrak{S}_n$ is an index $1 \leq i \leq n$ such that $\alpha_i > \alpha_{i+1}$. The *descent set* of α, denoted $\mathrm{DES}(\alpha)$, is the set of indices $\{i_1, i_2, \ldots, i_k\}$ where α has descents. Using the isomorphism $B_{n-1} \cong \mathrm{Comp}(n)$ of Section 1.5, we can equivalently define the *descent composition* of α to be the composition $\vec{c} = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$. We call the parts of the descent composition $\vec{c}$ the *runs* of α since they correspond to increasing runs in the one line notation of α.

For $\alpha = 231$ the descent set of α is $\mathrm{DES}(231) = \{2\}$ and the descent composition of $\alpha = 231$ is $\vec{c} = (2, 1)$.

Let $S = \{i_1, \ldots, i_k\}$ be a subset of $[n-1]$. We define $\alpha(S)$ to be the number of permutations σ in $\mathfrak{S}_n$ such that $\mathrm{DES}(\sigma) \subseteq S$. Suppose S corresponds to the composition $\vec{c} = (c_1, \ldots, c_{k+1})$ in $\mathrm{Comp}(n)$. Choosing a permutation $\sigma \in \mathfrak{S}_n$ with possible descents in the set S is equivalent to first choosing a subset of size c_1 from $[n]$ and writing it in increasing order to form the first run of σ. From the remaining $n - c_1$ integers choose a subset of size c_2 and write in increasing order to form the second run of σ, and so on. The permutation σ created can only possibly have descents coming from the set $S = \{i_1, i_2, \ldots, i_k\}$. Therefore, $\alpha(S)$ is given by the multinomial coefficient:

$$\alpha(S) = \binom{n}{c_1, c_2, \ldots, c_{k+1}}. \tag{1.6.1}$$

For notational convenience, we sometimes write the multinomial coefficient in Equation (1.6.1) as $\binom{n}{\vec{c}}$.

Analogously, define $\beta_n(S)$ to be the number of permutations $\sigma \in \mathfrak{S}_n$ such that $\mathrm{DES}(\sigma) = S$. Since $\alpha(S) = \sum_{T \subseteq S} \beta(S)$, by inclusion/exclusion we obtain:

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S/T|} \alpha(T). \tag{1.6.2}$$

The notation $\alpha(S)$ and $\beta(S)$ comes from the *rank selection* literature; see [36, Cor.3.1,3.2].

1.7   Poset Topology

Sections 1.2 through 1.5 have laid much of the poset-theoretic groundwork to be used throughout this dissertation. We now discuss topological considerations and how we attribute topological properties to posets. For further details see Wachs' overview article [43].

Recall an abstract simplicial complex Δ is a collection of sets that is closed under subsets, that is, if $\sigma \in \Delta$ and if $\tau \subseteq \sigma$ then we must have that $\tau \in \Delta$. The *dimension* of a face $\sigma \in \Delta$ is given by $|\sigma| - 1$. We call the elements $\sigma \in \Delta$ of dimension 0 the *vertices* of Δ.

For an abstract simplicial complex Δ there is a closely related topological space called the *geometric realization* of Δ, denoted $|\Delta|$, that allows us to attribute topological properties to abstract simplicial complexes just as the order complex allows us to give topological
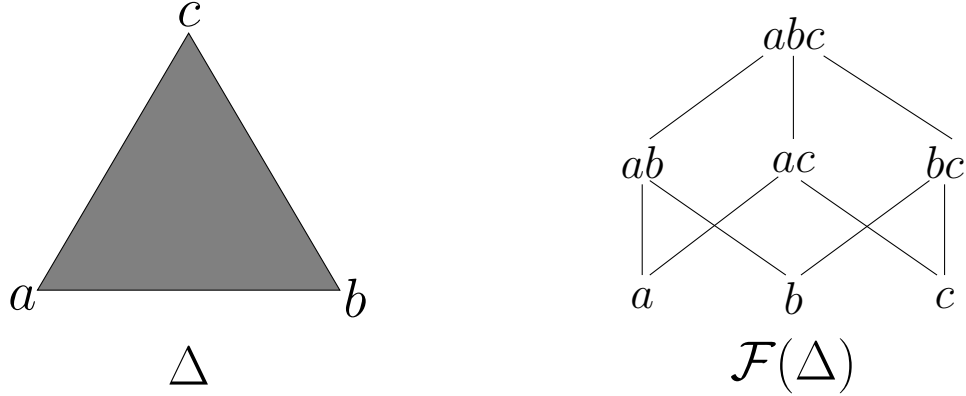
Figure 1.5: The 2-dimensional simplex $\triangle$ and its face poset $\mathbb{B}_3/\{\hat{0}\}$.

properties to posets. Except for exceptional cases which we will avoid, we treat $\triangle$ and its geometric realization $|\triangle|$ as interchangeable. For a complete discussion of the geometric realization, see [43].

We now introduce the *order complex* of a poset P.

Definition 1.7.1. Let P be a poset. The *order complex* of P, denoted $\triangle(P)$, is the abstract simplicial complex whose i-dimensional faces are the chains of size $i + 1$ in P.

Definition 1.7.1 is best illustrated with an example, see Figure 1.7. Note that the elements of P, which are chains of size 1, correspond to the vertices of $\triangle(P)$, which are 0-dimensional faces. The order complex allows us to give topological attributes to posets. Throughout this dissertation, as well as in the topological combinatorics literature, any topological characteristic of P is the corresponding notion applied to $\triangle(P)$.

A poset P that has a minimal element $\hat{0}$ or a maximal element $\hat{1}$ will have a contractible order complex $\triangle(P)$, and therefore we often remove $\hat{1}$ or $\hat{0}$ from P to ensure that $\triangle(P)$ has non-trivial topology. We let $\overline{P}$ denote $P - \{\hat{0}, \hat{1}\}$.

Much of topological combinatorics is concerned with classifying the homotopy type of a poset P. Techniques for determining the homotopy type of P include discrete Morse matchings and *EL*-labelings. Both of these techniques rely on labeling the edges of the Hasse diagram of P while avoiding certain forbidden structures.

We now introduce discrete Morse theory. For a complete introduction see Forman [15]. The goal of discrete Morse theory is to characterize the homotopy type of an abstract simplicial complex $\triangle$. In order to do this we introduce the *face poset* of $\triangle$ and discuss a special partial matching on the Hasse diagram of the face poset.

Definition 1.7.2. Let $\triangle$ be an abstract simplicial complex. The *face poset* of $\triangle$, denoted $\mathcal{F}(\triangle)$, is the poset of *non-empty* simplices of $\triangle$ ordered by set inclusion.

As an example, let $\triangle$ be the two-dimensional simplex in $\mathbb{R}^2$. Then $\mathcal{F}(\triangle)$ is the Boolean algebra $\mathbb{B}_3 - \{\hat{0}\}$, as is illustrated in Figure 1.5.

For a simplicial complex Γ we have the following relationship between Γ and its face poset $\mathcal{F}(\Gamma)$:

Proposition 1.7.1. *Let Γ be a simplicial complex. The order complex of the face lattice $\mathcal{F}(\Gamma)$ is the barycentric subdivision of Γ. In particular, there is a homotopy equivalence $\Gamma \simeq \triangle(\mathcal{F}(\Gamma))$.*

Given two simplicial complexes Δ and Γ with distinct vertex sets we define the *join* of Δ and Γ, denoted $\Delta * \Gamma$, to be the abstract simplicial complex given by

$$\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \, \gamma \in \Gamma\}.$$

Note that if $\dim(\Delta) = m$ and $\dim(\Gamma) = n$, then $\dim(\Delta * \Gamma) = (m + 1) + (n + 1) - 1 = m + n + 1$.

Definition 1.7.3. A *partial matching M* of a poset P is a collection of edges from the Hasse diagram of P such that each element of P is in at most one edge of the matching M. In other words, if the Hasse diagram of P is thought of as a graph, then a partial matching on P is simply a graph matching.

Consider the edges of the Hasse diagram of P as initially oriented down, meaning if $a \prec b$ in P we think of the edge as pointing down from b to a in the Hasse diagram. Now, suppose P has a partial matching M, à la Definition 1.7.3. Orient the edges from M upward. If the newly oriented Hasse diagram has no directed cycles, then we say that the matching M of P is a *discrete Morse matching*. An unmatched element of P is called a *critical* element.

We now demonstrate the topological implications of a discrete Morse matching with a theorem from Forman [15].

Theorem 1.7.2 (Forman). *Let Δ be an abstract simplicial complex. Suppose there is a discrete Morse matching on the face poset $\mathcal{F}(\Delta)$. Additionally, suppose there are c_i critical elements of $\mathcal{F}(P)$ of dimension $i \geq 0$, where the dimension of an element of $\mathcal{F}(\Delta)$ refers to the dimension of the corresponding simplex in Δ. Then Δ is homotopy equivalent to a CW complex with c_i cells of dimension i.*

In general, a discrete Morse matching with a small number of critical cells is desirable as then all of the topological information of Δ is stored in a smaller, more computationally manageable complex.

Example 1.7.3. Figure 1.6 shows a simplicial complex Δ, its face poset $\mathcal{F}(\Delta)$, and a discrete Morse matching on $\mathcal{F}(\Delta)$ with critical cells c and bd marked in red. Our Morse matching has one critical cell of dimension 0 and one critical cell of dimension 1. Theorem 1.7.2 gives that Δ is homotopy equivalent to a complex with one zero cell and one cell of dimension one, which of course is the one-dimensional sphere S^1.

We now record a theorem that gives a sufficient condition for a discrete Morse matching. We will then note the homotopy type of the truncated partition lattice $\overline{\Pi}_n$ coming from a discrete Morse matching.
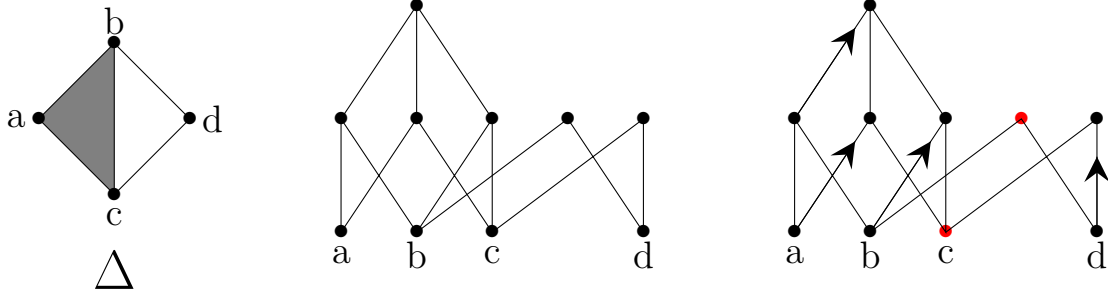
Figure 1.6: A simplicial complex Δ, its face poset $\mathcal{F}(\Delta)$, and a discrete Morse matching with critical cells c and bd.

Theorem 1.7.4 (Kozlov;Patchwork Theorem, Theorem 11.10 [18]). *Let $\phi : P \longrightarrow Q$ be an order preserving poset map. Assume on each fiber $\phi^{-1}(q)$ we have an acyclic matching. Then the union of these matchings is an acyclic matching for P.*

As its name suggests, Theorem 1.7.4 allows us to build acyclic matchings up from smaller matchings. Using the Patchwork Theorem, Kozlov computes:

Theorem 1.7.5 (Kozlov;Theorem 11.18 [18]). $\triangle(\overline{\overline{\Pi}}_n) \simeq \bigvee_{(n-1)!} S^{n-3}$.

The wedge in Theorem 1.7.5 arises because the matching used to prove the theorem has only one critical cell of dimension 0, and therefore the resulting complex must be a wedge.

We see Theorem 1.7.5 illustrated in Figure 1.2 for $n = 3$. Notice if $\hat{0}$ and $\hat{1}$ are removed from Π_3, what remains is a wedge of two zero-dimensional spheres.

While we will not be using EL-labelings, it is worth noting that both Gessel [2] and Wachs [41] have EL-labelings of the truncated partition lattice $\overline{\overline{\Pi}}_n$ that recover Theorem 1.7.5.

We end the section with a brief discussion of *Quillen's Fiber Lemma*.

Suppose P and Q are posets with a poset map $f : P \longrightarrow Q$. Quillen's fiber lemma states that if the preimage of each principle lower order ideal in Q is contractible, then $\triangle(P)$ and $\triangle(Q)$ are homotopy equivalent.

Theorem 1.7.6 (Quillen Fiber Lemma). *Let $f : P \longrightarrow Q$ be a poset map such that for each $q \in Q$ the order complex $\triangle(f^{-1}(Q_{\leq q}))$ is contractible. Then $\triangle(P)$ and $\triangle(Q)$ are homotopy equivalent.*

Additionally, there is an equivariant version of Theorem 1.7.6, which we state in Section 1.8.

1.8 Simplicial Homology

We now review the rudiments of simplicial homology. For a more complete introduction, see [18].
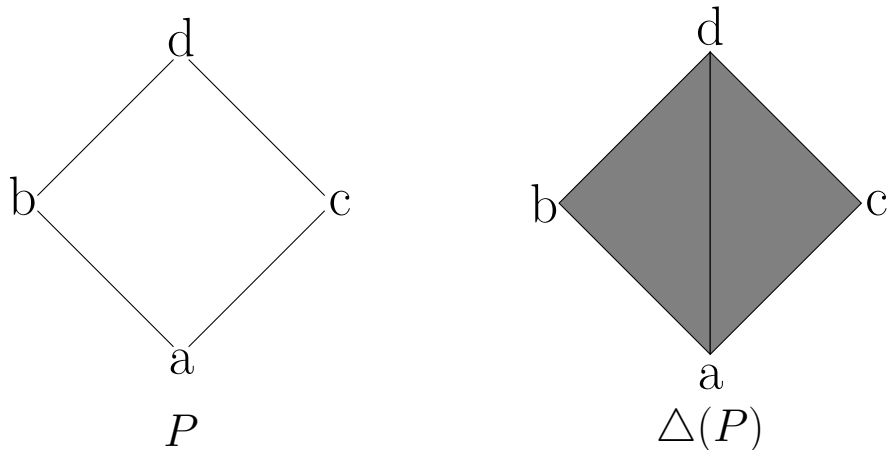
Figure 1.7: A poset P and its order complex $\triangle(P)$.

Let Δ be an abstract simplicial complex. The *dimension* of an element $\sigma \in \Delta$ is given by $\dim(\sigma) = |\sigma| - 1$. Equivalently, define the dimension of the complex Δ to be the maximum dimension over all faces $\sigma \in \Delta$.

Let Δ be an n-dimensional abstract simplicial complex. For each $1 \leq k \leq n$ let C_k be the complex vector space with basis given by the faces of Δ of dimension k, that is, $C_k = \langle \sigma : \sigma \in \Delta, \dim(\sigma) = k \rangle$. The vector space C_k is called the *k'th-chain space* of Δ. The elements of C_k are called chains.

Let $\sigma = \{a_0, a_1 \ldots, a_k\}$ be a k-dimensional face of Δ. We define a map $\partial_k : C_k \longrightarrow C_{k-1}$, called the *differential*, on the basis elements of C_k by:

$$\partial(\{a_1, \ldots, a_{k+1}\}) = \sum_{i=0}^{k} (-1)^i \{a_1, a_2, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_{k+1}\}. \tag{1.8.1}$$

Note that ∂_k is indeed a map into chains of one lower dimension since removing any element from $\sigma = \{a_0, \ldots, a_k\} \in C_k$ must still be in Δ as simplicial complexes are closed under inclusion. Moreover, we extend ∂_k to all chains linearly. We can now string together these differentials to obtain Equation (1.8.2).

$$0 \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} C_{k-2} \cdots C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 \tag{1.8.2}$$

It is routine to show that the composition $\partial_{i-1} \circ \partial_i = 0$, allowing us to define the *i'th homology group* of Δ.

Definition 1.8.1. Let Δ be a simplicial complex of dimension k. Using Equation (1.8.2) we define the i'th homology group of Δ by

$$H_i(\Delta) = \ker \partial_i / \operatorname{im} \partial_{i+1}.$$

12

$$V \xrightarrow{\cdot g} V$$
$$\downarrow \alpha \qquad \downarrow \alpha$$
$$W \xrightarrow{\cdot g} W$$

Figure 1.8: An isomorphism of representations.

The i'th-*reduced* homology, denoted $\widetilde{H}_i(\Delta)$, of the complex Δ agrees with Definition 1.8.1 for each $i \geq 1$, but at $i = 0$ the rank of $\widetilde{H}_0(\Delta)$ is 1 less than the rank of $H_0(\Delta)$.

If G is a group that acts on P such that the G action on P is a poset map for each $g \in G$, then this action of G on P descends to a representation of G on the homology groups $\widetilde{H}_i(\triangle(P))$. For a brief introduction to group representations, see Section 1.9.

In Chapter 2 we will have the symmetric group $\mathfrak{S}_n$ acting on the partition lattice by permutation. This action descends to a $\mathfrak{S}_n$–representation on the homology groups of the order complex $\widetilde{H}(\Pi_n)$. A tool that will help us compute these representations is Quillen's equivariant fiber lemma:

Theorem 1.8.1 (Equivariant Quillen Fiber Lemma)**.** *Let $f : P \longrightarrow Q$ be a G-poset map such that $\triangle(f^{-1}(Q_{\leq q}))$ is acyclic for each $q \in Q$. Then $\widetilde{H}_k(\triangle(P))$ and $\widetilde{H}_k(\triangle(Q))$ are isomorphic as G-representations for all k.*

1.9   Representations of the Symmetric Group

In this section we will give a brief introduction to the representations of finite groups. In particular, we will look at representations of the symmetric group on n elements, denoted by $\mathfrak{S}_n$. Much of what is discussed in this section can be found in Chapters one and two of Sagan [32]. The content of this section will be used, and expanded upon, in Sections 2.5 and 2.13.

Recall that the *General Linear Group* over a vector space V, denoted $\mathrm{GL}(V)$, is the group of invertible linear transformations from V to V under composition.

A *representation* of a group G is a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$ for V a vector space over the complex numbers $\mathbb{C}$. The dimension of ρ is given by $\dim(V)$. A representation of G on V lets G act on V by $g \cdot v := \rho(g)(v)$.

Two representations of G on V and W are isomorphic as representations if there is a linear isomorphism of vector spaces $\alpha : V \longrightarrow W$ such that the diagram of Figure 1.9 commutes for each $g \in G$.

A representation $\rho : G \longrightarrow \mathrm{GL}(V)$ is *irreducible* if the only subspaces of V fixed under the action of G are 0 and V. Otherwise, we say the representation is *reducible*.

We now give two examples of representations of an arbitrary group G and discuss whether or not they are irreducible.

Example 1.9.1. The *trivial* representation of G on a vector space V is given by $g \cdot v = v$ for all $v \in V$. As every element $v \in V$ is fixed under the trivial representation, the trivial representation is irreducible if and only if V is one dimensional.
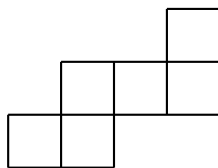
Figure 1.9: The border strip $B(\vec{c})$ associated with the composition $\vec{c} = (2, 3, 1)$.

Example 1.9.2. The *regular* representation of a group G is defined as follows. Let G be a finite group. Let $\mathbb{C}[G] = \langle e_g : g \in G \rangle$ be the complex vector space of dimension $|G|$ with basis elements given by group elements. Let G act on $\mathbb{C}[G]$ by $g \cdot \sum_i \alpha_i e_{g_i} = \sum_i \alpha_i e_{g \cdot g_i}$. Consider the one dimensional subspace H spanned by $\sum_G e_g$. The action of G fixes H, and thus the regular representation of G is irreducible if and only G is the trivial group on one element, or $G = \langle e \rangle$.

Suppose V and W are both G-representations. We form a G-representation on the direct sum $V \oplus W$ by $g \cdot (v, w) = (g \cdot v, g \cdot w)$.

Additionally, for G a finite group, any finite dimensional representation of G over a field of characteristic 0, such as $\mathbb{C}$, decomposes as a direct sum of irreducible representations.

Theorem 1.9.3 (Maschke's Theorem). *Let G be a finite group. Let ρ be a finite dimensional complex representation of G on V. Then V is isomorphic as a G-representation to the direct sum $\bigoplus_{i=1}^{m} V_i$, where each V_i is irreducible.*

Theorem 1.9.3 shows that studying representations of G can be accomplished by understanding the irreducible representations of G. Moreover, using character theory, see Sections 1.8 through 1.10 of [32], it can be shown that the number of irreducible representations of G, up to isomorphism of representations, is given by the number of conjugacy classes of G.

The conjugacy classes of the symmetric group $\mathfrak{S}_n$ are characterized by cycle type, and are therefore enumerated by the number of integer partitions of n. Each partition λ of n determines an irreducible representation of $\mathfrak{S}_n$ called the *Specht* module, denoted S^λ. The construction of the Specht module is discussed in Section 2.5.

The dimension of the Specht module S^λ is given by the number of standard Young tableaux of shape λ, often denoted f^λ. With character theory or with the beautiful RSK algorithm, see Section 3.1 of [32], one obtains the formula:

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n! \tag{1.9.1}$$

In Chapter 2 we will classify the $\mathfrak{S}_{n-1}$ action on the reduced homology groups of arbitrary filters in the partition lattice not in terms of Specht modules, but rather *border strip Specht modules*. We now discuss the construction of the border strip Specht module.

Let $\vec{c} = (c_1, \ldots, c_k)$ be a composition of n. A *border strip* of shape $\vec{c}$, denoted $B(\vec{c})$, is a skew shape whose ith row has length c_i and whose ith and $(i+1)$st columns overlap in precisely one position. See Figure 1.9. A filling of a border strip $B(\vec{c})$ is a way to place the

numbers $1, 2, \ldots, n$ in the shape $B(\vec{c})$. Once filled with the numbers $1, 2, \ldots, n$ we call the border strip a *border strip tableau*. The symmetric group acts on tableau t by relabeling. Two border strip tableaux of shape $\vec{c}$ are *row equivalent* if they have the same entries in each row. An equivalence class of tableaux under row equivalence is called a *tabloid*, and is denoted $[t]$. The symmetric group acts on tabloids by $\sigma \cdot [t] = [\sigma \cdot t]$.

We define the permutation module $M^{B(\vec{c})}$ to be the vector space with basis given by tabloids of shape $\vec{c}$. The dimension of $M^{B(\vec{c})}$ is the multinomial $\binom{n}{\vec{c}}$. For a tableau t, the collection of permutations that leave the columns of t fixed is called the *column stabilizer*, denoted $\mathfrak{S}_{\vec{c}}^{C}$.

Define the *polytabloid* to be an element in the permuation module $M^{B(\vec{c})}$ given by the alternating sum $e_t = \sum_{\gamma \in \mathfrak{S}_{\vec{c}}^{C}} (-1)^{\gamma} \cdot [\gamma \cdot t]$. The *border strip Specht module* is the span of the polytabloids in $M^{B(\vec{c})}$. It can be shown, see [32], that a basis for the Specht module $S^{B(\vec{c})}$ is given by e_t for t a *standard* tableau of shape $\vec{c}$, meaning the tableau increases from left to right in rows and increases down columns. The only way for t to be a standard tableau of shape $\vec{c}$ is if reading the filling of $B(\vec{c})$ from southwest to northeast gives a permutation with descent composition $\vec{c}$. Therefore, $\dim(S^{B(\vec{c})}) = \beta_n(\vec{c})$, where $\beta_n(\vec{c})$ is defined in Equation (1.6.2).

Unlike the Specht modules, the border strip Specht modules are in general reducible representations of $\mathfrak{S}_n$, but can be rectified to a direct sum of irreducible Specht modules using the sliding game *jeu de taquin*, see Appendix $A.1.2$ of [35].

Chapter 2 Filters in the Partition Lattice

> Like to a double cherry,
> seeming parted, But yet an
> union in partition
>
> ―――――――――――――――
>
> *Helena*
> *A Midsummer Night's Dream,*
> *III.2.1249-50*

2.1 Introduction

In Section 1.4 we introduced the partition lattice Π_n. In Section 1.7 we discussed topological properties of posets, and in particular, we saw that a discrete Morse matching can be used to characterize the homotopy type of $\triangle(\overline{\Pi}_n)$, see Theorem 1.7.5. In this chapter we will classify the topology and $\mathfrak{S}_{n-1}$ action on certain sub-posets of the partition lattice Π_n. Our main tool will be the equivariant version of Quillen's fiber lemma, see Theorem 1.8.1.

We begin with a history of work done on the d-divisible partition lattice, that is, the collection of partitions in Π_n having all block sizes divisible by d. After a discussion of the history of the d-divisible partition lattice we transition into our plan of attack for computing the homology of filters in the partition lattice.

Work on the d-divisible partition lattice began with Sylvester's physics dissertation [39], where he considered the even partition lattice, or equivalently, all set partitions with all even sized block. Sylvester computed the Möbius function of this lattice and showed that it equals, up to a sign, the tangent number, or equivalently the number of permutations in $\mathfrak{S}_n$ with descent set $\{2, 4, \ldots, n-2\}$. The name *tangent number* comes from the fact that the exponential generating function for *alternating* permutations, or permutations in $\mathfrak{S}_n$ with descents at all odd or all even positions, respectively, is given $\sec(x) + \tan(x)$, see [36, Proposition 1.6.1].

Stanley then introduced the d-divisible partition lattice. This is the collection of all set partitions with blocks having size divisible by d, denoted by Π_n^d. He showed that the Möbius function is, up to a sign, the number of permutations in the symmetric group $\mathfrak{S}_{n-1}$ with descent set $\{d, 2d, \ldots, n-d\}$; see [34].

Wachs [41] and Sagan [31] independently proved that the poset $\Pi_n^d \cup \{\hat{0}\}$ is *EL*-shellable, and thus the homotopy type of the complex $\triangle(\Pi_n^d - \{\hat{1}\})$ is a wedge of spheres of the same dimension. Additionally, Wachs gave explicit matrices for the action of $\mathfrak{S}_n$ on this homology.

Calderbank, Hanlon and Robinson [9] continued work on the order complex $\triangle(\Pi_n^d - \{\hat{1}\})$ by giving generating functions for the $\mathfrak{S}_n$ characters acting on the unique non-vanishing homology group of the d-divisible partition lattice, as well as the generating function for $\mathfrak{S}_n$ characters on partition lattice where all blocks have size 1 mod d. Additionally, Calderbank, Hanlon, and Robinson proved a conjecture of Stanley's stating that the restriction of the $\mathfrak{S}_n$

action on the top homology of the even partition lattice to $\mathfrak{S}_{n-1}$ is given by the border strip Specht module $B((d, d, \ldots, d, d-1))$.

Ehrenborg and Jung [12] further generalized the d-divisible partition lattice by defining a subposet $\Pi_{\vec{c}}^*$ of Π_n for a composition $\vec{c}$ of n. The subposet $\Pi_{\vec{c}}^*$ reduces to the d-divisible partition lattice when the composition $\vec{c}$ is given by $\vec{c} = (d, d, \ldots, d)$. Their work on the filter $\Pi_{\vec{c}}^*$ consists of three main results. First, they showed that the Möbius function of $\Pi_{\vec{c}}^* \cup \{\hat{0}\}$ equals, up to a given sign, the number of permutations in $\mathfrak{S}_n$ ending with the element n with descent composition $\vec{c}$. Second, they showed that the order complex $\triangle(\Pi_{\vec{c}}^* - \{\hat{1}\})$ is homotopy equivalent to a wedge of spheres of the same dimension. Lastly, if $\vec{c} = (c_1, c_2, \ldots, c_k)$, they proved that the action of $\mathfrak{S}_{n-1}$ on the top homology group of $\triangle(\Pi_{\vec{c}}^* - \{\hat{1}\})$ is given by the Specht module corresponding to the composition $(c_1, c_2, \ldots, c_k - 1)$.

In this chapter we continue this research program by considering a more general class of filters in the partition lattice. A *filter* F in a poset P is an upper order ideal, i.e., if $x \in F$ and $x \leq y$, then $y \in F$. Let Δ be a filter in the poset of compositions. Since the poset of compositions is isomorphic to a Boolean algebra, the filter Δ under the reverse order is a lower order ideal and hence can be viewed as the face poset of a simplicial complex. We define the associated filter Π_Δ^* in the partition lattice. This definition extends the definition of $\Pi_{\vec{c}}^*$. In fact, when Δ is a simplex generated by the composition $\vec{c}$ the two definitions agree.

Our main result is that we can determine all the reduced homology groups of the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ in terms of the reduced homology groups of links in Δ and in terms of Specht modules of border shapes; see Theorem 2.11.5. The proof proceeds by induction on the simplicial complex Δ and builds up the isomorphism of Theorem 2.6.3 using Mayer–Vietoris sequences. As our main tool, we use Quillen's fiber lemma to translate topological data from the filter Q_Δ^* to the filter Π_Δ^*. Our base case relies on a result of Ehrenborg and Jung describing the homology for the complex $Q_{\vec{c}}^*$.

We also present a proof of our main result, Theorem 2.6.3, using an equivariant poset fiber theorem of Björner, Wachs and Welker [4]. Even though this approach is concise, it does not yield an explicit construction of the isomorphism of Theorem 2.6.3. In particular, our hands-on approach using Mayer–Vietoris sequences yields a view of how the homology groups of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ are changing as the complex Δ is built up. Once again, a previous result of Ehrenborg and Jung is needed to apply the poset fiber theorem of Björner, Wachs and Welker.

Our main result yields explicit expressions for the reduced homology groups of the complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$, most notably when Δ is homeomorphic to a ball or to a sphere. The same holds when Δ is a shellable complex. We are able to describe the homotopy type of the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ using the homotopy fiber theorem of Björner, Wachs and Welker. Again, when Δ is homeomorphic to a ball or to a sphere, we obtain that Π_Δ^* is a wedge of spheres. We are also able to lift discrete Morse matchings from Δ and its links to form a discrete Morse matching on the filter of ordered set partitions Q_Δ^*.

In Sections 2.15 through 2.17 we give a plethora of examples of our results. We consider the case when our complex Δ is generated by a knapsack partition to obtain a previous result of Ehrenborg and Jung. In Section 2.16 we study the case when Λ is a semigroup of positive integers and we consider the filter of partitions whose block sizes belong to the semigroup Λ. When Λ is generated by the arithmetic progression $a, a+d, a+2d, \ldots$ we are able to describe the reduced homology groups of the associated filter in the partition lattice.

The particular case when d divides a was studied by Browdy [5], where the filter Λ consists of partitions whose block sizes are divisible by d and are greater than or equal to a. Finally, in Section 2.17 we study the filter corresponding to the semigroup generated by two relative prime integers. Here we are able to give explicit results for the top and bottom reduced homology groups.

Other previous work in this area is due to Björner and Wachs [3]. Additionally, Sundaram studied the subposet of the partition lattice defined by a set of forbidden block sizes using plethysm and the Hopf trace formula; see [37, 38].

We end the chapter by posing questions for further study.

2.2 Integer and set partitions

We define an integer partition λ to be a finite multiset of positive integers. Thus the multiset $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a partition of n if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Sometimes it will be necessary to consider the multiplicity of the elements of the partition λ. We then write

$$\lambda = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_p^{m_p}\},$$

where we tacitly assume that $\lambda_i \neq \lambda_j$ for two different indices $i \neq j$.

Let I_n be the set of all integer partitions of n. We make I_n into a partial order where the cover relation is adding two parts, that is, in terms of multisets

$$\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k\} \prec \{\lambda_1 + \lambda_2, \lambda_3, \ldots, \lambda_k\}.$$

Note that the composition $\{1, 1, \ldots, 1\}$ is the minimal element and $\{n\}$ is the maximal element in the partial order.

Let Π_n denote the set of all set partitions of $[n] = \{1, 2, \ldots, n\}$. Define a partial order on Π_n by merging blocks, that is,

$$\{B_1, B_2, B_3, \ldots, B_k\} \prec \{B_1 \cup B_2, B_3, \ldots, B_k\}.$$

The poset Π_n is in fact a lattice. Let $|\pi|$ denote the number of blocks of the partition π. Furthermore, for a set partition $\pi = \{B_1, B_2, \ldots, B_k\}$ define its type to be the integer partition of n given by the multiset $\mathrm{type}(\pi) = \{|B_1|, |B_2|, \ldots, |B_k|\}$.

The symmetric group $\mathfrak{S}_n$ acts on subsets of $[n]$ by relabeling the elements. Similarly, the symmetric group $\mathfrak{S}_n$ acts on the partition lattice by relabeling. Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a set partition. Then the action is given by $\alpha \cdot \pi = \{\alpha(B_1), \alpha(B_2), \ldots, \alpha(B_k)\}$. Finally, when we speak about the action of the symmetric group $\mathfrak{S}_{n-1}$, we view the group $\mathfrak{S}_{n-1}$ as the subgroup $\{\alpha \in \mathfrak{S}_n : \alpha_n = n\}$ of the symmetric group $\mathfrak{S}_n$.

2.3 Compositions and ordered set partitions

A composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n is an ordered list of positive integers such that $c_1 + c_2 + \cdots + c_k = n$. Let $\mathrm{Comp}(n)$ be the set of all compositions of n. Furthermore, introduce the cover relation given by adding adjacent entries, that is,

$$(c_1, \ldots, c_i, c_{i+1}, \ldots, c_k) \prec (c_1, \ldots, c_i + c_{i+1}, \ldots, c_k).$$

This makes $\mathrm{Comp}(n)$ into a poset, and it is isomorphic to the Boolean algebra on $n-1$ elements. Note that $(1, 1, \ldots, 1)$ and (n) are the minimal and maximal elements of $\mathrm{Comp}(n)$, respectively. Also, we define the type of a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ to be the integer partition $\mathrm{type}(\vec{c}) = \{c_1, c_2, \ldots, c_k\}$ of n. Furthermore, let $|\vec{c}|$ denote the number of parts of the composition $\vec{c}$.

We now review terminology from Section 1.6.

For a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n, the *multinomial coefficient* is given by

$$\binom{n}{\vec{c}} = \binom{n}{c_1, c_2, \ldots, c_k} = \frac{n!}{c_1! \cdot c_2! \cdots c_k!}.$$

For $\alpha \in \mathfrak{S}_n$, let the descent set of α, denoted by $\mathrm{Des}(\alpha)$, be the subset of $[n-1]$ given by $\mathrm{Des}(\alpha) = \{i \in [n-1] : \alpha(i) > \alpha(i+1)\}$. Throughout this chapter it will be more convenient to consider $\mathrm{Des}(\alpha)$ as a composition of n, that is, if $\mathrm{Des}(\alpha) = \{i_1 < i_2 < \cdots < i_k\}$, then we consider $\mathrm{Des}(\alpha)$ as a composition of n given by $\mathrm{Des}(\alpha) = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$. Note that the identity permutation $12 \cdots n$ has descent composition (n).

Let $\beta_n(\vec{c})$ be the number of permutations α in $\mathfrak{S}_n$ such that $\mathrm{Des}(\alpha) = \vec{c}$. Likewise, define $\beta_n^*(\vec{c})$ to be the number of permutations α in $\mathfrak{S}_n$ with descent composition $\vec{c}$ and $\alpha(n) = n$. Observe that

$$\binom{n-1}{c_1, \ldots, c_{k-1}, c_k - 1} = \sum_{\substack{\vec{d} \in \mathrm{Comp}(n) \\ \vec{c} \leq \vec{d}}} \beta_n^*(\vec{d}), \tag{2.3.1}$$

is the $\mathrm{Comp}(n)$ analog of Equation 1.6.2.

An *ordered set partition* $\sigma = (C_1, C_2, \ldots, C_p)$ of $[n]$ is a list of non-empty sets, or blocks, such that the set $\{C_1, C_2, \ldots, C_p\}$ is a partition of the set $[n]$, where the order of the blocks now matters. Furthermore, let $|\sigma|$ denote the number of blocks in the ordered set partition σ.

Let Q_n be the set of all ordered set partitions on the set $[n]$. Introduce a partial order on Q_n where the cover relation is joining adjacent blocks, that is,

$$(C_1, \ldots, C_i, C_{i+1}, \ldots, C_p) \prec (C_1, \ldots, C_i \cup C_{i+1}, \ldots, C_p).$$

Observe that the poset Q_n has the maximal element $([n])$, along with $n!$ minimal elements, namely the ordered set partitions $(\{\alpha_1\}, \{\alpha_2\}, \ldots, \{\alpha_n\})$, one for each permutation $\alpha \in \mathfrak{S}_n$. Moreover, every interval in Q_n is a Boolean algebra.

Define the *type* of an ordered set partition $\sigma = (C_1, C_2, \ldots, C_k)$ to be the composition of n given by listing the cardinalities of its blocks, that is, $\mathrm{type}(\sigma) = (|C_1|, |C_2|, \ldots, |C_k|)$.

Definition 2.3.1. For a permutation $\alpha \in \mathfrak{S}_n$ and a composition $\vec{d} = (d_1, d_2, \ldots, d_k)$ of n, let $\sigma(\alpha, \vec{d})$ denote the unique ordered set partition in Q_n of type $\vec{d}$ whose elements are given, in order, by the permutation α, that is,

$$\sigma(\alpha, \vec{d}) = (\{\alpha(1), \ldots, \alpha(d_1)\}, \{\alpha(d_1 + 1), \ldots, \alpha(d_2)\}, \ldots, \{\alpha(d_{k-1} + 1), \ldots, \alpha(n)\}).$$

Finally, the symmetric group $\mathfrak{S}_n$ acts on ordered set partitions by relabeling, that is

$$\alpha \cdot (C_1, C_2, \ldots, C_k) = (\alpha(C_1), \alpha(C_2), \ldots, \alpha(C_1)).$$

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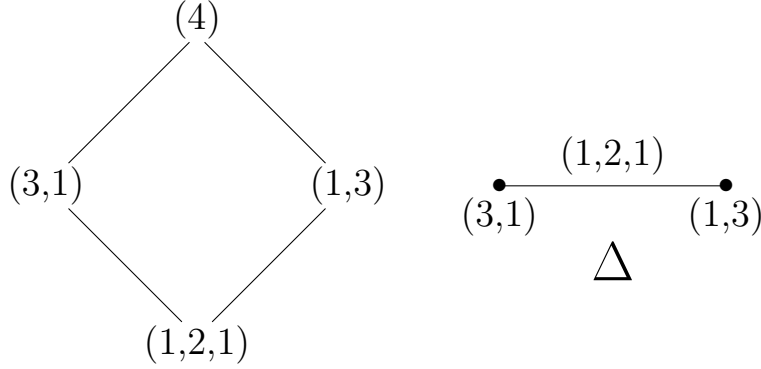
Figure 2.1: The filter $F = \langle (1, 2, 1) \rangle \subseteq \mathrm{Comp}(4)$ and its realization as a simplicial poset $\Delta := F^*$ in $\mathrm{Comp}(4)^*$.

2.4 Topological considerations

Recall a simplicial complex Δ is a finite collection of sets such that the empty set belongs to Δ and Δ is closed under inclusion. We will find it easier to view a simplicial complex as a partially ordered set Δ with additional conditions.

Definition 2.4.1. Let Δ be a poset such that

- Δ has a unique minimal element $\hat{0}$

- every interval $[\hat{0}, x]$ for $x \in \Delta$ is a Boolean algebra.

Then we say that Δ is a *simplicial poset*.

Notice that a poset P is simplicial if P is the face poset of a simplicial complex. Furthermore, note that the second condition in the definition of a simplicial poset makes the poset Δ ranked, since every saturated chain between the minimal element $\hat{0}$ and an element x has the same length. Thus the *dimension* of an element x is defined by its rank minus one, that is, $\dim(x) = \rho(x) - 1$. Note that this is in analogy to the dimension of a face σ in a simplicial complex Δ being given by $\dim(\sigma) = |\sigma| - 1$.

A filter in a poset P is an upper order ideal. Hence if F is a filter in P, then the dual filter F^* in P^* is now a lower order ideal. In particular, if $\Delta \subseteq \mathrm{Comp}(n)$ is a filter, since upper order ideals in $\mathrm{Comp}(n)$ are Boolean algebras, the dual of Δ is a simplicial poset in the dual space $\mathrm{Comp}(n)^*$, which has cover relation given by splitting rather than merging. To emphasize that we have dualized, we use $\leq^*$ to denote the order relation in the dualized $\mathrm{Comp}(n)$. See Figure 2.1 for an example.

Lastly, the *link* of a face F in a simplicial complex Δ is given by $\mathrm{lk}_F(\Delta) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$. However, working with the poset definition of a simplicial complex, we have the following equivalent definition of the link: $\mathrm{lk}_x(\Delta) = \{y \in \Delta : x \leq y\}$, that is, the link is the principle filter generated by the face x. One advantage of this definition is that we do not have to relabel the faces when considering the link.

From now on our simplicial complex Δ will be a filter in the composition lattice $\mathrm{Comp}(n)$, with the dual order $\leq^*$.

Let $C_k(\mathrm{Comp}(n))$ be the linear span of all compositions of n into $k+2$ parts. We obtain a chain complex by defining the boundary map as follows. Define the map $\partial_{k,j} : C_k(\mathrm{Comp}(n)) \longrightarrow C_{k-1}(\mathrm{Comp}(n))$ by

$$\partial_{k,j}(c_1, \ldots, c_j, c_{j+1}, \ldots, c_{k+2}) = (c_1, \ldots, c_j + c_{j+1}, \ldots, c_{k+2}).$$

Then the boundary map on $\mathrm{Comp}(n)$ is given by $\partial_k = \sum_{j=1}^{k+1}(-1)^{j-1} \cdot \partial_{k,j}$.

Consider the dual order on the set of ordered set partitions Q_n. For $\Delta \subseteq \mathrm{Comp}(n)$ a complex, let $Q_\Delta = \{\tau \in Q_n : \mathrm{type}(\tau) \in \Delta\}$. The filter Q_Δ is also a simplicial poset, so we refer to Q_Δ as a complex.

Define $C_k(Q_n)$ to be the linear span of all ordered set partitions of $[n]$ with $k+2$ blocks. The boundary map $\partial_k : C_k(Q_n) \longrightarrow C_{k-1}(Q_n)$ on Q_n is given by $\partial_k(\sigma(\alpha, \vec{d})) = \sigma(\alpha, \partial_k(\vec{d}))$, where $\partial_k(\vec{d})$ is the boundary map applied to the composition $\vec{d}$ in $C_k(\mathrm{Comp}(n))$, and where $\sigma(\alpha, \vec{c})$ is given in Definition 2.3.1. This boundary map is inherited by the subcomplex Q_Δ.

Finally, for simplicial complexes Δ and Γ in $\mathrm{Comp}(n)$ and $\mathrm{Comp}(m)$ respectively, their *join* is defined to be poset

$$\Delta * \Gamma = \{\vec{c} \circ \vec{d} : \vec{c} \in \Delta, \vec{d} \in \Gamma\},$$

where $\circ$ denote the concatenation of compositions. Note that the join $\Delta * \Gamma$ has the composition (n, m) as its minimal element. Furthermore, we have the following basic lemma on Morse matchings of joins of complexes.

Lemma 2.4.1. *Let Δ and Γ be two complexes in $\mathrm{Comp}(n)$ each having a discrete Morse matching. Let Δ^c and Γ^c be the sets of critical cells of Δ and Γ, respectively. Then the join $\Delta * \Gamma$ has a Morse matching where the critical cells are*

$$\{\vec{c} \circ \vec{d} : \vec{c} \in \Delta^c, \vec{d} \in \Gamma^c\}.$$

2.5 Border strips and Specht modules

A border strip B is a connected skew-shape which does not contain a two by two square. For each composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ there is a unique border strip such that the number of boxes in the ith row is given by c_i and every two adjacent rows overlap in one position. Denote this border strip by $B(\vec{c})$. See Figure 1.9 for an example.

In an analogous fashion, for a composition $\vec{c} \in \mathrm{Comp}(n)$ we define the border shape $A(\vec{c})$ to be the skew-shape whose ith row has length c_i such that the rows of $A(\vec{c})$ are non-overlapping.

Let R_i be the interval $[c_1 + \cdots + c_{i-1} + 1, c_1 + \cdots + c_{i-1} + c_i]$. The row stabilizer of the border strip $B(\vec{c})$ is the subgroup $\mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \cdots \times \mathfrak{S}_{R_k}$ of the symmetric group $\mathfrak{S}_n$.

Since the poset $\mathrm{Comp}(n)$ of all compositions of n is a Boolean algebra, every composition has a complementary composition $\vec{c}^c$. To obtain the complement of a composition write

every part of the composition as a sum of 1s and then exchange the plus signs with the commas. Similarly, the column stabilizer is defined as the row stabilizer of the border strip of the complementary composition. More precisely, let $(d_1, d_2, \ldots, d_p)$ be the complementary composition $\vec{c}^c$ and let K_i be the interval $[d_1 + \cdots + d_{i-1} + 1, d_1 + \cdots + d_{i-1} + d_i]$. Then the column stabilizer is the subgroup $\mathfrak{S}_{\vec{c}}^C = \mathfrak{S}_{K_1} \times \mathfrak{S}_{K_2} \times \cdots \times \mathfrak{S}_{K_p}$. See Figure 1.9.

We now review some basic representation theory of the symmetric group. For a less terse introduction, see [32, Chapter 3]. A border strip tableau t of shape $\vec{c}$ is a filling of the border strip $B(\vec{c})$. In Figure 1.9 we look at the border strip tableau $B(\vec{c})$ for $\vec{c} = (2, 3, 1)$. Note that the row stabilizer is the group $\mathfrak{S}_{\vec{c}} = \mathfrak{S}_{[1,2]} \times \mathfrak{S}_{[3,5]} \times \mathfrak{S}_{[6,6]}$. Note that $\vec{c} = (1 + 1, 1 + 1 + 1, 1)$ so that the complementary composition is $\vec{c}^c = (1, 1 + 1, 1, 1 + 1) = (1, 2, 1, 2)$. Hence the column stabilizer is $\mathfrak{S}_{\vec{c}}^C = \mathfrak{S}_{[1,1]} \times \mathfrak{S}_{[2,3]} \times \mathfrak{S}_{[4,4]} \times \mathfrak{S}_{[5,6]}$.

A border strip tabloid, denoted $[t]$, is a border strip tableau under row equivalence. Define the permutation module, $M^{B(\vec{c})}$, to be the vector space with basis elements given by all tabloids of shape $B(\vec{c})$. A polytabloid for the border strip $\vec{c}$ is defined by the alternating sum $e_t = \sum_{\gamma \in \mathfrak{S}_{\vec{c}}^C} (-1)^\gamma \cdot [\gamma \cdot t]$, where $\mathfrak{S}_{\vec{c}}^C$ is the column stabilizer of the tableau t of shape $\vec{c}$. Lastly, the Specht module, denoted $S^{B(\vec{c})}$, is the subspace of $M^{B(\vec{c})}$ generated by polytabloids. The dimension of the Specht module $S^{B(\vec{c})}$ is given by the descent set statistics $\beta_n(\vec{c})$, while the dimension of the permutation module $M^{B(\vec{c})}$ is given by the multinomial coefficient $\binom{n}{\vec{c}}$.

We now define two operations on compositions. The motivation comes from the associated Specht and permutation modules. For a composition $\vec{c} = (c_1, \ldots, c_{k-1}, c_k)$ let $\vec{c} - 1$ denote the composition $(c_1, \ldots, c_{k-1}, c_k - 1)$ if $c_k \geq 2$, and otherwise let $\vec{c} - 1$ denote the empty composition. Similarly, let $\vec{c}/1$ denote the composition $(c_1, \ldots, c_{k-1}, c_k - 1)$ if $c_k \geq 2$, and otherwise let $\vec{c}/1$ denote the composition $(c_1, \ldots, c_{k-1})$. Note that $\vec{c}/1$ is always a composition of $n - 1$ if $\vec{c}$ is a composition of n.

For a composition $\vec{c}$ of n let $B^*(\vec{c})$ denote the border strip $B(\vec{c} - 1)$. In our results the associated Specht module is $S^{B^*(\vec{c})}$ and the group action is by $\mathfrak{S}_{n-1}$. We can think of this Specht module as a submodule of $S^{B(\vec{c})}$ spanned by all standard Young tableaux where the northeastern-most box is filled with n. Note that when the composition ends with the entry 1, there are no such standard Young tableaux, and hence $S^{B^*(\vec{c})}$ is the zero module.

If we require the tabloids that span the permutation module $M^{B(\vec{c})}$ to have the filling n in the northeastern-most box, we obtain the submodule $M^{B^\#(\vec{c})}$. With this in mind, we make the definition $B^\#(\vec{c}) = B(\vec{c}/1)$. Similarly, we define the shape $A^\#(\vec{c}) = A(\vec{c}/1)$.

Furthermore, the dimensions of the Specht module $S^{B^*(\vec{c})}$ and the permutation module $M^{B^\#(\vec{c})}$ are $\beta_n^*(\vec{c})$ and $\binom{n}{\vec{c}/1}$, respectively. Additionally, we have the decomposition

$$M^{B^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \leq \vec{d}} S^{B^*(\vec{d})},$$

which is the representation theoretic analogue of equation (2.3.1). Please see Lemma 2.13.3 for a proof.

2.6   The simplicial complex Q_Δ^*

We now define the simplicial complex Q_Δ^*, which will serve us as an important stepping stone to understanding the topology of a large class of filters in the partition lattice. We make the

transition from Q_Δ^* to the partition lattice using Quillen's Fiber Lemma, see Theorem 1.8.1. Note that by considering the reverse orders in $\mathrm{Comp}(n)$ and in Q_n we obtain two simplicial posets. Hence for Δ a non-empty filter in $\mathrm{Comp}(n)$, we view Δ as a simplicial complex under the reverse order $\leq^*$. See the discussion in Section 2.4.

Definition 2.6.1. Let Δ be a filter in $\mathrm{Comp}(n)$, that is, let Δ be a simplicial complex of compositions of n. Define Q_Δ^* to be all ordered set compositions whose type is in the complex Δ and whose last block contains the element n, that is,

$$Q_\Delta^* = \{\sigma = (C_1, C_2, \ldots, C_k) \in Q_n \ : \ \mathrm{type}(\sigma) \in \Delta, \ n \in C_k\}.$$

Note that we view Q_Δ^* as a simplicial complex. Our purpose is to study the reduced homology groups of this complex.

Recall that the link of a composition $\vec{c}$ in Δ is the filter

$$\mathrm{lk}_{\vec{c}}(\Delta) = \{\vec{d} \in \Delta \ : \ \vec{d} \leq^* \vec{c}\},$$

where $\leq^*$ is the reverse of the partial order of $\mathrm{Comp}(n)$. Since $\mathrm{lk}_{\vec{c}}(\Delta)$ is now a simplicial poset with minimal element $\vec{c}$, we have a dimension shift from Δ to $\mathrm{lk}_{\vec{c}}(\Delta)$ given by

$$\dim_{\mathrm{lk}_{\vec{c}}(\Delta)}(\vec{d}) = \dim_\Delta(\vec{d}) - |\vec{c}| + 1 \tag{2.6.1}$$

for $\vec{d} \in \mathrm{lk}_{\vec{c}}(\Delta)$.

Remark 2.6.1. The symmetric group $\mathfrak{S}_{n-1}$ acts on Q_Δ^* by permutation, whereas the action of $\mathfrak{S}_{n-1}$ on the complex Δ is the trivial action, meaning for any $\alpha \in \mathfrak{S}_{n-1}$ the action of α on a composition $\vec{d} \in \Delta$ is given by $\alpha \cdot \vec{d} = \vec{d}$. Therefore, as an $\mathfrak{S}_{n-1}$ module, the homology of $\mathrm{lk}_{\vec{c}}(\Delta)$ will be a direct sum of one dimension trivial representations of $\mathfrak{S}_{n-1}$.

Furthermore, the type map from Q_Δ^* to Δ respects the $\mathfrak{S}_{n-1}$ action, since the two ordered set partitions σ and $\alpha \cdot \sigma$ have the same type for $\sigma \in Q_n$ and $\alpha \in \mathfrak{S}_{n-1}$.

A special case of Q_Δ^* is when the simplicial complex Δ is a simplex, that is, Δ is generated by one composition $\vec{c}$. This case was studied by Ehrenborg and Jung in [12]. Their results are given below.

Theorem 2.6.2 (Ehrenborg–Jung)**.** *Let $\vec{c}$ be a composition of n into k parts. Then the complex $Q_{\vec{c}}^*$ is a wedge of $\beta_n^*(\vec{c})$ spheres of dimension $k-2$. Furthermore, the top homology group $\widetilde{H}_{k-2}(Q_{\vec{c}}^*)$ is isomorphic to the Specht module $S^{B^*(\vec{c})}$ as an $\mathfrak{S}_{n-1}$-module. This isomorphism $\phi : S^{B^*(\vec{c})} \longrightarrow \widetilde{H}_{k-2}(Q_{\vec{c}}^*)$ is given by*

$$\phi\left(e_t\right) = \sum_{\gamma \in \mathfrak{S}_{\vec{c}}^C} (-1)^\gamma \cdot \sigma(\alpha \cdot \gamma, \vec{c}),$$

where the permutation $\alpha \in \mathfrak{S}_n$ is obtained by reading the entries of the tabloid t from southwest to northeast and attaching the element n at the end.

Note that Ehrenborg and Jung formulated their result in terms of pointed set partitions. That is, our notation $Q^*_{\vec{c}}$ is their notation $\Delta_{\vec{d}}$, where $\vec{d} = (c_1, \ldots, c_{k-1}, c_k - 1)$. They allow the last entry of a composition to be zero and similarly the last entry of an ordered set partition to be empty. Moreover, our notation $\Pi^*_{\vec{c}}$ is in their notation $\Pi^{\bullet}_{\vec{d}}$.

We can now state the main result of this section.

Theorem 2.6.3. *Let Δ be a simplicial complex of compositions of n. Then the ith reduced homology group of the simplicial complex Q^*_Δ is given by*

$$\widetilde{H}_i(Q^*_\Delta) \cong \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i - |\vec{c}| + 1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

Furthermore, this isomorphism holds as $\mathfrak{S}_{n-1}$-modules.

We will prove Theorem 2.6.3 at the end of Section 2.9.

2.7 The homomorphism ϕ_i^Δ

In this section and the next two sections we present a proof of Theorem 2.6.3 using induction and Mayer–Vietoris sequences. The induction basis is when Δ is generated by a single composition $\vec{c}$ in $\mathrm{Comp}(n)$, and the proof of the basis follows from Theorem 2.6.2. The induction step is to assume that Theorem 2.6.3 holds for Δ, Γ, and the intersection $\Delta \cap \Gamma$, and to show that it also holds for the union $\Delta \cup \Gamma$. This step requires Mayer–Vietoris sequences. Finally, since any simplicial complex is a union of simplices, Theorem 2.6.3 holds for arbitrary simplicial complexes Δ in $\mathrm{Comp}(n)$.

We begin by defining the isomorphism of Theorem 2.6.3 explicitly. Section 2.8 covers the induction basis, whereas Section 2.9 covers the induction step.

Throughout this chapter we will let $i_{\vec{c}}$ denote the shift $i - |\vec{c}| + 1$.

Definition 2.7.1. Let $D_i^{\vec{c}}(\Delta)$ be the tensor product $C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$ where $C_j(\mathrm{lk}_{\vec{c}}(\Delta))$ is the jth chain group of the link $\mathrm{lk}_{\vec{c}}(\Delta)$. Let $D^{\vec{c}}(\Delta)$ be the chain complex whose ith chain group is $D_i^{\vec{c}}(\Delta)$ and whose boundary map is $\partial \otimes \mathrm{id}$. Lastly, let $D(\Delta)$ be the chain complex with ith chain group $\bigoplus_{\vec{c} \in \Delta} D_i^{\vec{c}}(\Delta)$ with the differential $\bigoplus_{\vec{c} \in \Delta} \partial \otimes \mathrm{id}$.

Definition 2.7.2. Define the chain complex $E^{\vec{c}}(\Delta)$ analogous to $D^{\vec{c}}(\Delta)$ of Definition 2.7.1 above by replacing the permutation module $M^{B^\#(\vec{c})}$ with the Specht module $S^{B^*(\vec{c})}$. We also have the corresponding chain complex $E(\Delta)$ with the same differential.

Lemma 2.7.1. *The homology of the chain complexes $D(\Delta)$ and $E(\Delta)$ are given by*

$$\widetilde{H}_i(D(\Delta)) \cong \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})},$$

$$\widetilde{H}_i(E(\Delta)) \cong \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

Proof. The homology of the chain complex $D^{\vec{c}}(\Delta)$ is given by $\ker(\partial_{i_{\vec{c}}} \otimes \mathrm{id}) / \mathrm{im}(\partial_{i_{\vec{c}}+1} \otimes \mathrm{id}) \cong (\ker(\partial_{i_{\vec{c}}}) \otimes M^{B^\#(\vec{c})}) / (\mathrm{im}(\partial_{i_{\vec{c}}+1}) \otimes M^{B^\#(\vec{c})}) \cong \ker(\partial_{i_{\vec{c}}}) / \mathrm{im}(\partial_{i_{\vec{c}}+1}) \otimes M^{B^\#(\vec{c})} \cong \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$. The analogous result holds for $E^{\vec{c}}(\Delta)$ and the lemma follows by direct summing. $\square$

For the rest of this section we let t denote a tabloid in the permutation module $M^{B^\#(\vec{c})}$ and $\alpha \in \mathfrak{S}_n$ is the permutation obtained by reading the entries of the tabloid t in increasing order from southwest to northeast and adjoining the element n at the end.

Definition 2.7.3. The $\mathfrak{S}_{n-1}$-action on $D_i^{\vec{c}}(\Delta)$ is given by $\tau \cdot (\vec{d} \otimes t) = \vec{d} \otimes (\tau \circ t)$, for $\tau \in \mathfrak{S}_{n-1}$ and $\vec{d} \otimes t$ a basis element of $D_i^{\vec{c}}(\Delta) = C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$.

Notice that Definition 2.7.3 states that $\mathfrak{S}_{n-1}$ acts on $D_i^{\vec{c}}(\Delta)$ by acting trivially on the chain group $C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta))$ and by relabeling on $M^{B^\#(\vec{c})}$.

Definition 2.7.4. For a simplicial complex Δ and a composition $\vec{c}$ in Δ define the map

$$\phi_i^{\Delta, \vec{c}} : C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})} \longrightarrow C_i(Q_\Delta^*),$$

on basis elements by $\phi_i^{\Delta, \vec{c}}(\vec{d} \otimes t) = \sigma(\alpha, \vec{d})$.

Since $\vec{d} \in C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta))$ is a basis element, we know that $\vec{d}$ is a simplex of $\mathrm{lk}_{\vec{c}}(\Delta)$ of dimension $i_{\vec{c}} = i - |\vec{c}| + 1$, and thus by the dimension shift in equation (2.6.1), we have that $|\vec{d}| = i + 2$, so that $\phi_i^{\Delta, \vec{c}}(\vec{d}) = \sigma(\pi, \vec{d})$ is an ordered partition of dimension i. Lastly, since tabloids in $M^{B^\#(\vec{c})}$ have n in the last block, we are guaranteed that $\phi_i^{\Delta, \vec{c}}(\vec{d}) \in C_i(Q_\Delta^*)$.

Lemma 2.7.2. *The map $\phi_i^{\Delta, \vec{c}} : C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})} \longrightarrow C_i(Q_\Delta^*)$ respects the $\mathfrak{S}_{n-1}$-action.*

Proof. Let $\tau \in \mathfrak{S}_{n-1}$ and $\vec{d} \otimes t$ be a basis element of $C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$. Then we have

$$\phi_i^{\Delta, \vec{c}}(\tau \cdot (\vec{d} \otimes t)) = \phi_i^{\Delta, \vec{c}}(\vec{d} \otimes (\tau \cdot t)) = \sigma(\tau \cdot \alpha, \vec{d}) = \tau \cdot \sigma(\alpha, \vec{d}) = \tau \cdot \phi_i^{\Delta, \vec{c}}(\vec{d} \otimes t). \qquad \square$$

Lemma 2.7.3. *The map $\phi_i^{\Delta, \vec{c}}$ is an equivariant chain map between the complexes $D^{\vec{c}}(\Delta)$ and $C_i(Q_\Delta^*)$. That is, the following diagram commutes:*

$$\begin{array}{ccc}
D_i^{\vec{c}}(\Delta) & \xrightarrow{\ \partial \otimes \mathrm{id}\ } & D_{i-1}^{\vec{c}}(\Delta) \\
\downarrow{\scriptstyle \phi_i^{\Delta,\vec{c}}} & & \downarrow{\scriptstyle \phi_{i-1}^{\Delta,\vec{c}}} \\
C_i(Q_\Delta^*) & \xrightarrow{\ \partial\ } & C_{i-1}(Q_\Delta^*)
\end{array}$$

Proof. Recall that the boundary map ∂ of $\mathrm{Comp}(n)$ as well as the boundary map ∂ of Q_Δ^* are given in Section 2.4. Let $\vec{d} \otimes t \in C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$. Tracing first right then down we obtain:

$$\phi_{i-1}^{\Delta, \vec{c}} \circ (\partial \otimes \mathrm{id})(\vec{d} \otimes t) = \phi_{i-1}^{\Delta, \vec{c}}(\partial(\vec{d}) \otimes t) = \sigma(\alpha, \partial(\vec{d})).$$

Next, we trace down then right to obtain the same result:

$$\partial \circ \phi_i^{\Delta, \vec{c}}(\vec{d} \otimes t) = \partial(\sigma(\alpha, \vec{d})) = \sigma(\alpha, \partial(\vec{d})).$$

The equivariance of $\phi_i^{\Delta, \vec{c}}$ is a consequence of Lemma 2.7.2. $\qquad \square$

Lemma 2.7.4. *The map $\phi_i^{\Delta, \vec{c}}$ induces a map*

$$\phi_i^{\Delta, \vec{c}} : \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})} \longrightarrow \widetilde{H}_i(Q_\Delta^*)$$

given by $\phi_i^\Delta(\overline{\vec{d}} \otimes t) = \overline{\sigma(\alpha, \vec{d})}$, for $\vec{d} \in C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta))$ a cycle.

Proof. Since $\phi_i^{\Delta,\vec{c}}$ is an equivariant chain map between the chain complexes $D^{\vec{c}}(\Delta)$ and $C_i(Q_\Delta^*)$ by Lemma 2.7.3, the result follows. $\qquad\square$

From now on, the use of the bar to indicate the quotient in passing from the chain space to the homology group will be suppressed for ease of notation.

Definition 2.7.5. Define the map ϕ_i^Δ from $D_i(\Delta) = \bigoplus_{\vec{c} \in \Delta} D_{i_{\vec{c}}}^{\vec{c}}(\Delta)$ to $C_i(Q_\Delta^*)$ by adding all the $\phi_i^{\Delta,\vec{c}}$ maps together, that is,

$$\phi_i^\Delta = \sum_{\vec{c} \in \Delta} \phi_i^{\Delta,\vec{c}}. \tag{2.7.1}$$

Observe that ϕ_i^Δ restricts to a map from $E_i(\Delta)$ to $C_i(Q_\Delta^*)$. Therefore ϕ_i^Δ also induces a map from $\widetilde{H}_i(E(\Delta)) = \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}$ to $\widetilde{H}_i(Q_\Delta^*)$ using Lemma 2.7.4.

2.8   The main theorem

We can now state the missing isomorphism of Theorem 2.6.3.

Definition 2.8.1. Let $K_i(\Delta) = \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}$. Notice this is the right-hand side of Theorem 2.6.3.

Remark 2.8.1. Note that Lemma 2.7.1 tells us that the homology of the complex $E(\Delta)$ is $K(\Delta)$. That is, $\widetilde{H}_i(E(\Delta)) \cong_{\mathfrak{S}_{n-1}} K_i(\Delta)$. In fact, we introduced the notation $K(\Delta)$ for brevity, in order to keep the upcoming commutative diagrams readable.

We now present the main theorem of the section.

Theorem 2.8.2. *Let Δ be a subcomplex of $\mathrm{Comp}(n)$. Then the map*

$$\phi_i^\Delta : K_i(\Delta) \longrightarrow \widetilde{H}_i(Q_\Delta^*)$$

is an $\mathfrak{S}_{n-1}$-equivariant isomorphism.

Notice that Lemma 2.7.3 says that equation (2.7.1) is a well defined map from the homology of $E(\Delta)$, which by Lemma 2.7.1 is $K_i(\Delta)$, to the homology $H_i(Q_\Delta^*)$.

Our proof of Theorem 2.6.3 is by induction on the simplicial complex Δ. The induction basis is when Δ is a simplex, including the case when Δ is the empty simplicial complex.

Proposition 2.8.3 (Base case for Theorem 2.6.3)**.** *Assume that Δ is a filter in $\mathrm{Comp}(n)$ generated by one composition, that is, Δ is a simplicial complex. Then Theorem 2.6.3 holds for Δ.*

Proof. Suppose that $\Delta \subseteq \mathrm{Comp}(n)$ is generated by a composition $\vec{d} = (d_1, d_2, \ldots, d_k)$. Theorem 2.6.2 states that Q_Δ^* only has reduced homology in dimension $k-2$. Additionally, Theorem 2.6.2 states that the action of $\mathfrak{S}_{n-1}$ on the top homology of Q_Δ^* is given by the border shape Specht module $S^{B^*(\vec{c})}$, that is, $\widetilde{H}_{k-2}(Q_\Delta) \cong_{\mathfrak{S}_n} S^{B^*(\vec{c})}$.

Next we show that $\phi_i^\Delta : K_i(\Delta) \longrightarrow \widetilde{H}_i(Q_\Delta^*)$ is an isomorphism for all i. When $i \neq k-2$ both sides are the trivial module, that is, $K_i(\Delta) = 0 = \widetilde{H}_i(Q_\Delta^*)$ and hence the map ϕ_i^Δ is

directly an isomorphism. Now assume that $i = k - 2$. Since all the links $\mathrm{lk}_{\vec{c}}(\Delta)$ for $\vec{c} <^* \vec{d}$ are contractible, we have

$$K_{k-2}(\Delta) = \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{k-2-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})} = \widetilde{H}_{-1}(\mathrm{lk}_{\vec{d}}(\Delta)) \otimes S^{B^*(\vec{d})}.$$

Notice that $\mathrm{lk}_{\vec{d}}(\Delta)$ consists only of the composition $\vec{d}$ itself, so that the (-1)-dimensional reduced homology group $\widetilde{H}_{-1}(\mathrm{lk}_{\vec{d}}(\Delta))$ is the homology of the chain space $C_{-1}(\mathrm{lk}_{\vec{d}}(\Delta))$, which is the one dimensional vector space with the generator $\vec{d}$. Therefore, the map ϕ_{k-2}^{Δ} : $\widetilde{H}_{-1}(\mathrm{lk}_{\vec{d}}(\Delta)) \otimes S^{B^*(\vec{d})} \longrightarrow \widetilde{H}_{k-2}(Q_{\Delta}^*)$ is given by

$$\vec{d} \otimes e_t = \vec{d} \otimes \left(\sum_{\gamma \in \mathfrak{S}_{\vec{d}}^C} (-1)^{\gamma} \cdot [\gamma \cdot t] \right) \longmapsto \sum_{\gamma \in \mathfrak{S}_{\vec{d}}^C} (-1)^{\gamma} \cdot \sigma(\alpha \cdot \gamma, \vec{d}).$$

But this is an isomorphism by Theorem 2.6.2. $\qquad\square$

As a direct corollary we have that Theorem 5.3.2 holds for the empty simplex $\{(n)\} \subseteq$ Comp(n).

Corollary 2.8.4. *Theorem 5.3.2 holds for the empty simplicial complex, that is, the simplicial complex consisting only of the composition (n).*

Proof. Apply Proposition 2.8.3 to the simplicial complex Δ generated by the composition (n) in Comp(n). $\qquad\square$

2.9 The induction step

As any simplicial complex is a union of smaller simplicial complexes, we prove that Theorem 5.3.2 holds for the complex $\Delta \cup \Gamma \subseteq$ Comp(n), assuming that Theorem 5.3.2 holds for Δ, Γ, as well as the intersection $\Delta \cap \Gamma$.

Lemma 2.9.1. *The following two identities hold for the link:* $\mathrm{lk}_{\vec{c}}(\Delta \cap \Gamma) = \mathrm{lk}_{\vec{c}}(\Delta) \cap \mathrm{lk}_{\vec{c}}(\Gamma)$ *and* $\mathrm{lk}_{\vec{c}}(\Delta \cup \Gamma) = \mathrm{lk}_{\vec{c}}(\Delta) \cup \mathrm{lk}_{\vec{c}}(\Gamma)$.

Lemma 2.9.2. *The following two identities hold for the ordered set partition poset:* $Q_{\Delta \cap \Gamma}^* = Q_{\Delta}^* \cap Q_{\Gamma}^*$ *and* $Q_{\Delta \cup \Gamma}^* = Q_{\Delta}^* \cup Q_{\Gamma}^*$.

The proofs of these two lemmas are straightforward.

Before we begin the proof of Theorem 5.3.2, let us remind ourselves of Definition 2.7.1. That is, for each composition $\vec{c}$ in Δ we have the chain complex $D^{\vec{c}}(\Delta)$ whose ith chain group is $D_i^{\vec{c}}(\Delta) = C_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes M^{B^\#(\vec{c})}$. Furthermore, $D(\Delta)$ is the chain complex obtained by direct summing $D^{\vec{c}}(\Delta)$ over all $\vec{c} \in \Delta$.

We now begin the proof of the induction step of Theorem 5.3.2.

Lemma 2.9.3. *For $\vec{c} \in \Delta \cap \Gamma$ the following diagram is commutative, and its rows are exact.*

$$0 \longrightarrow D_i^{\vec{c}}(\Delta \cap \Gamma) \longrightarrow D_i^{\vec{c}}(\Delta) \oplus D_i^{\vec{c}}(\Gamma) \longrightarrow D_i^{\vec{c}}(\Delta \cup \Gamma) \longrightarrow 0$$

$$\downarrow{\phi_i^{\Delta \cap \Gamma, \vec{c}}} \qquad \downarrow{\phi_i^{\Delta, \vec{c}} \oplus \phi_i^{\Gamma, \vec{c}}} \qquad \downarrow{\phi_i^{\Delta \cup \Gamma, \vec{c}}}$$

$$0 \longrightarrow C_i(Q_{\Delta \cap \Gamma}^*) \longrightarrow C_i(Q_\Delta^*) \oplus C_i(Q_\Gamma^*) \longrightarrow C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0$$

Proof. The horizontal maps in the above diagram are given by the construction of the Mayer–Vietoris sequence applied to $\mathrm{lk}_{\vec{c}}(\Delta \cup \Gamma) = \mathrm{lk}_{\vec{c}}(\Delta) \cup \mathrm{lk}_{\vec{c}}(\Gamma)$ in the top row, and $Q_{\Delta \cup \Gamma}^* = Q_\Delta^* \cup Q_\Gamma^*$ in the bottom row. The top horizontal maps have also been tensored with the identity map on the Specht modules. As the Specht module is free, both the top and bottom rows of the diagram remain exact.

We show commutativity of the left square, as the right square is analogous. Let $\vec{d} \otimes \alpha \in C_{i_{\vec{c}}}(\Delta \cap \Gamma) \otimes M^{B^\#(\vec{c})}$ be a basis element. First we trace right then down to obtain:

$$\vec{d} \otimes \alpha \longmapsto (\vec{d} \otimes \alpha) \oplus -(\vec{d} \otimes \alpha) \longmapsto \sigma(\alpha, \vec{d}) \oplus -\sigma(\alpha, \vec{d}).$$

We obtain the same result by first tracing down then right:

$$\vec{d} \otimes \alpha \longmapsto \sigma(\alpha, \vec{d}) \longmapsto \sigma(\alpha, \vec{d}) \oplus -\sigma(\alpha, \vec{d}). \qquad \square$$

Lemma 2.9.4. *For each $\vec{c} \in \Delta - \Gamma$, we have the commutative diagram with exact rows:*

$$0 \longrightarrow 0 \longrightarrow D_i^{\vec{c}}(\Delta) \xrightarrow{\mathrm{id}} D_i^{\vec{c}}(\Delta) \longrightarrow 0$$

$$\downarrow{0} \qquad \downarrow{\phi_i^\Delta \oplus 0} \qquad \downarrow{\phi_i^{\Delta \cup \Gamma}}$$

$$0 \longrightarrow C_i(Q_{\Delta \cap \Gamma}^*) \longrightarrow C_i(Q_\Delta^*) \oplus C_i(Q_\Gamma^*) \longrightarrow C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0$$

Proof. The left-hand square commutes trivially. We show the right-hand square commutes by first tracing right then down:

$$\vec{c} \otimes \pi \longmapsto \vec{c} \otimes \pi \longmapsto \sigma(\pi, \vec{c}).$$

Now we trace down then right:

$$\vec{c} \otimes \pi \longmapsto \sigma(\pi, \vec{c}) \oplus 0 \longmapsto \sigma(\pi, \vec{c}) + 0 = \sigma(\pi, \vec{c})$$

Exactness of the rows in the diagram follows from Lemma 2.9.3, as the bottom row has remained unchanged. $\qquad \square$

Notice that we can replace $\Delta - \Gamma$ in Lemma 2.9.4 with $\Gamma - \Delta$, which we will need in the following proof.

Lemma 2.9.5. *The following diagram is commutative, and its rows are exact.*

$$0 \longrightarrow D_i(\Delta \cap \Gamma) \longrightarrow D_i(\Delta) \oplus D_i(\Gamma) \longrightarrow D_i(\Delta \cup \Gamma) \longrightarrow 0$$

$$\downarrow{\phi_i^{\Delta \cap \Gamma}} \qquad \downarrow{\phi_i^\Delta \oplus \phi_i^\Gamma} \qquad \downarrow{\phi_i^{\Delta \cup \Gamma}}$$

$$0 \longrightarrow C_i(Q_{\Delta \cap \Gamma}^*) \longrightarrow C_i(Q_\Delta^*) \oplus C_i(Q_\Gamma^*) \longrightarrow C_i(Q_{\Delta \cup \Gamma}^*) \longrightarrow 0$$

Proof. The proof is by taking direct sums of the previous two short exact sequences. First, take the direct sum of the diagram in Lemma 2.9.3 for each $\vec{c} \in \Delta \cap \Gamma$. Next, take the resulting short exact sequence of chain complexes and take its direct sum with the diagram in Lemma 2.9.4 for each $\vec{c} \in \Delta - \Gamma$. Finally, switch Δ and Γ in Lemma 2.9.4 and direct sum the resulting diagram with the diagram from Lemma 2.9.4 for each $\vec{c} \in \Gamma - \Delta$. Observe that the second row of the diagram remains the same throughout this process. Also, note that the top row is exact as it is the direct sum of exact sequences. All together, this yields the desired commutative diagram. $\qquad\square$

Proposition 2.9.6. *The following diagram is commutative, and its rows are exact.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_i(\Delta \cap \Gamma) & \longrightarrow & E_i(\Delta) \oplus E_i(\Gamma) & \longrightarrow & E_i(\Delta \cup \Gamma) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \phi_i^{\Delta \cap \Gamma}} & & \downarrow{\scriptstyle \phi_i^{\Delta} \oplus \phi_i^{\Gamma}} & & \downarrow{\scriptstyle \phi_i^{\Delta \cup \Gamma}} & & \\
0 & \longrightarrow & C_i(Q^*_{\Delta \cap \Gamma}) & \longrightarrow & C_i(Q^*_\Delta) \oplus C_i(Q^*_\Gamma) & \longrightarrow & C_i(Q^*_{\Delta \cup \Gamma}) & \longrightarrow & 0
\end{array}
$$

Proof. Since $E_i(\Delta)$ is a subspace of $D_i(\Delta)$, it follows from Lemma 2.9.5 that the diagram is commutative. Furthermore, that the second row is exact also follows from this lemma. It remains to show that the first row is exact. However, this follows by the same reason that the first row of Lemma 2.9.5 is exact, but with the permutation module $M^{B^{\#}(\vec{c})}$ replaced with the Specht module $S^{B^*(\vec{c})}$. $\qquad\square$

Proposition 2.9.7 (Induction step of Theorem 5.3.2)**.** *Assume that Theorem 5.3.2 holds for the simplicial complexes Δ, Γ, and the intersection $\Delta \cap \Gamma$. Then Theorem 5.3.2 also holds for the union $\Delta \cup \Gamma$.*

Proof. Consider the diagram of short exact sequences of chain complexes given in Proposition 2.9.6. Use the zig-zag lemma to obtain the Mayer–Vietoris sequence:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_i(\Delta \cap \Gamma) & \longrightarrow & K_i(\Delta) \oplus K_i(\Gamma) & \longrightarrow & K_i(\Delta \cup \Gamma) & \longrightarrow & \cdots \\
& & \downarrow{\scriptstyle \phi_i^{\Delta \cap \Gamma}} & & \downarrow{\scriptstyle \phi_i^{\Delta} \oplus \phi_i^{\Gamma}} & & \downarrow{\scriptstyle \phi_i^{\Delta \cup \Gamma}} & & \\
\cdots & \longrightarrow & \widetilde{H}_i(Q^*_{\Delta \cap \Gamma}) & \longrightarrow & \widetilde{H}_i(Q^*_\Delta) \oplus \widetilde{H}_i(Q^*_\Gamma) & \longrightarrow & \widetilde{H}_i(Q^*_{\Delta \cup \Gamma}) & \longrightarrow & \cdots
\end{array}
$$

The assumption that Theorem 5.3.2 holds for the complexes $\Delta \cap \Gamma$, Δ, and Γ implies that $\phi_i^{\Delta \cap \Gamma}$ and $\phi_i^{\Delta} \oplus \phi_i^{\Gamma}$ are isomorphisms. The five-lemma now implies that $\phi_i^{\Delta \cup \Gamma}$ is also an isomorphism. Furthermore, $\phi_i^{\Delta \cup \Gamma}$ is an $\mathfrak{S}_{n-1}$-equivariant map by Lemma 2.7.2. $\qquad\square$

We have now proven Theorem 5.3.2, and hence Theorem 2.6.3, by induction. The base case was proven in Proposition 2.8.3, and the induction step was proven in Proposition 2.9.7.

2.10   Alternate Proof of Theorem 2.6.3

As mentioned in the introduction, we now give an alternate proof of Theorem 2.6.3 using a poset fiber theorem of Björner, Wachs and Welker [4].

Theorem 2.10.1. *Let Δ be a simplicial complex of compositions of n. Then the ith reduced homology group of the simplicial complex Q_Δ^* is given by*

$$\widetilde{H}_i(Q_\Delta^*) \cong \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

Furthermore, this isomorphism holds as $\mathfrak{S}_{n-1}$-modules.

Proof. Consider the two posets Δ and Q_Δ^* with the reverse order $\leq^*$ and the poset map

$$\mathrm{type} : Q_\Delta^* - \{([n])\} \longrightarrow \Delta - \{(n)\}.$$

Observe that the type map respects the action of the symmetric group $\mathfrak{S}_{n-1}$. Now the inverse image $\mathrm{type}^{-1}(\Delta_{\leq^* \vec{c}})$ is the filter $Q_{\vec{c}}^*$. Since $Q_{\vec{c}}^*$ only has reduced homology in dimension $|\vec{c}| - 2$ by Theorem 2.6.2, we have that the fiber $\triangle(\mathrm{type}^{-1}(\Delta_{\leq^* \vec{c}}))$ is $(|\vec{c}| - 3)$-acyclic, where $|\vec{c}| - 3$ is the length of the longest chain in $\mathrm{type}^{-1}(\Delta_{<^* \vec{c}})$. Hence Theorem 9.1 of [4] applies. Since $\mathfrak{S}_{n-1}$ acts trivially on Δ (see Remark 2.6.1), we have that the stabilizer $\mathrm{Stab}_{\mathfrak{S}_{n-1}}(\vec{c})$ is in fact the whole group $\mathfrak{S}_{n-1}$. Thus there is no representation to induce and we have that:

$$\widetilde{H}_i(Q_\Delta^*) \cong_{\mathfrak{S}_{n-1}} \widetilde{H}_i(\Delta) \oplus \bigoplus_{\vec{c} \in \Delta - \{(n)\}} \widetilde{H}_{|\vec{c}|-2}(\mathrm{type}^{-1}(\Delta_{\leq^* \vec{c}})) \otimes \widetilde{H}_{i-|\vec{c}|+1}((\Delta - \{(n)\})_{>^* \vec{c}})$$

$$\cong_{\mathfrak{S}_{n-1}} \widetilde{H}_i(\Delta) \oplus \bigoplus_{\vec{c} \in \Delta - \{(n)\}} S^{B^*(\vec{c})} \otimes \widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)),$$

where the first summand corresponds to $\vec{c} = (n)$ and the trivial representation $S^{B^*(n)}$, proving the result. $\qquad\square$

2.11 Filters in the set partition lattice

In Theorem 2.6.3 we characterized each homology group of Q_Δ^*, a subspace of ordered set partitions. We will now translate the topological data we have gathered on Q_Δ^* into data on the usual partition lattice Π_n.

Recall that Q_Δ^* is the collection of ordered set partitions, containing n in the last block, whose type is contained in the simplicial complex $\Delta \subseteq \mathrm{Comp}(n)$. Recall that the *forgetful* map of Equation (1.5.2) is given by forgetting the order between blocks, that is, $f((C_1, C_2, \ldots, C_k)) = \{C_1, C_2, \ldots, C_k\}$.

Definition 2.11.1. Let $\Pi_\Delta^* \subseteq \Pi_n$ be the image of Q_Δ^* under the forgetful map f.

Lemma 2.11.1. *Suppose that F is a filter in the integer partition lattice. Let Δ_F be the filter of compositions given by $\{\vec{c} \in \mathrm{Comp}(n) : \mathrm{type}(\vec{c}) \in F\}$. Then the associated filter $\Pi_{\Delta_F}^*$ in the partition lattice is given by $\{\pi \in \Pi_n : \mathrm{type}(\pi) \in F\}$.*

Proof. Choose $\pi \in \Pi_n$ such that $\mathrm{type}(\pi) \in F$, with $\pi = \{B_1, B_2, \ldots, B_k\}$ where we suppose $n \in B_k$. The ordered set partition $\tau = (B_1, B_2, \ldots, B_k)$ is an element of $Q_{\Delta_n}^*$, since $\mathrm{type}(\tau) = \mathrm{type}(\pi) \in F$. Hence π is in the image of the forgetful map f. The other direction is clear. $\qquad\square$

Remark 2.11.2. In general, taking the image of a filter $\Delta \subseteq \mathrm{Comp}(n)$ under the map type does not define a filter in the integer partition lattice I_n. For example, consider the simplex Δ in $\mathrm{Comp}(6)$ generated by $(3, 2, 1)$. Note that $\mathrm{type}(\Delta)$ consists of the four partitions $\{\{3, 2, 1\}, \{3 + 2, 1\}, \{3, 2 + 1\}, \{3 + 2 + 1\}\} = \{\{3, 2, 1\}, \{5, 1\}, \{3, 3\}, \{6\}\}$. But this is not a filter in I_6 since it does not contain the partition $\{4, 2\}$.

Lemma 2.11.3. *The forgetful map* $f : Q_\Delta^* \longrightarrow \Pi_\Delta^*$ *respects the* $\mathfrak{S}_{n-1}$-*action.*

Proof. Let $\alpha \in \mathfrak{S}_{n-1}$ and $\sigma = (C_1, \ldots, C_k) \in Q_\Delta^*$. Then we have that

$$f(\alpha \cdot \sigma) = f((\alpha(C_1), \ldots, \alpha(C_k))) = \{\alpha(C_1), \ldots, \alpha(C_k)\} = \alpha \cdot f(\sigma). \qquad \square$$

The $\mathfrak{S}_{n-1}$ action on Π_Δ^* extends to the chains in the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$.

For a statement of the equivariant version of the Quillen Fiber Lemma, please see [43, Theorem 5.2.2].

Proposition 2.11.4. *The forgetful map* $f : Q_\Delta^* - \{\hat{1}\} \longrightarrow \Pi_\Delta^* - \{\hat{1}\} = P$ *satisfies the condition of Quillen's Equivariant Fiber Lemma, see Theorem 1.8.1. In particular, for a partition* $\pi = \{B_1, B_2, \ldots, B_k\}$ *in* P, *the order complex* $\triangle(f^{-1}(P_{\geq \pi}))$ *is the barycentric subdivision of a cone, and is therefore contractible and acyclic.*

Proof. Let B_k be the block of the partition π that contains the element n. Note that because every ordered partition in Q_Δ^* must have the element n in its last block, we must have that each ordered set partition in the fiber $f^{-1}(\pi)$ has the set B_k as its last block. Furthermore, the last block of each ordered set partition in $f^{-1}(P_{\geq \pi})$ contains the block B_k.

We claim that $f^{-1}(P_{\geq \pi})$ is a cone with apex $([n] - B_k, B_k)$. Let $\sigma \in f^{-1}(P_{\geq \pi})$ be the ordered set partition $\sigma = (C_1, \ldots, C_{p-1}, C_p)$. Note that the number of blocks of σ, p, is greater than or equal to 2 as we have removed the maximal element $\hat{1}$ from Q_Δ^*. If $C_p = B_k$ then the face σ contains the vertex $([n] - B_k, B_k)$. If $C_p \supsetneq B_k$ then both σ and the vertex $([n] - B_k, B_k)$ are contained in the face $(C_1, \ldots, C_{p-1}, C_p - B_k, B_k)$ in $f^{-1}(P_{\geq \pi})$. Hence $f^{-1}(P_{\geq \pi})$ is the face poset of a cone with vertex $([n] - B_k, B_k)$, and therefore $\triangle(f^{-1}(P_{\geq \pi}))$ is the barycentric subdivision of a cone, and hence contractible and acyclic, by Proposition 1.7.1. $\qquad \square$

Combining Proposition 2.11.4 with Theorem 2.6.3, we have the following result for the homology of the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$.

Theorem 2.11.5. *The ith reduced homology group of the order complex of* $\Pi_\Delta^* - \{\hat{1}\}$ *is given as an* $\mathfrak{S}_{n-1}$-*module as*

$$\widetilde{H}_i(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i - |\vec{c}| + 1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}.$$

Remark 2.11.6. Suppose that $\mathrm{lk}_{\vec{c}}(\Delta)$ has reduced homology in dimension j. Then Theorem 2.11.5 tells us that this reduced homology contributes to dimension $j + |\vec{c}| - 1$ of the reduced homology of the order complex of $\Pi_\Delta^* - \{\hat{1}\}$.

We end the section with a discussion of Morse matchings in the link $\mathrm{lk}_{\vec{c}}(\Delta)$. Assume that the link $\mathrm{lk}_{\vec{c}}(\Delta)$ has a discrete Morse matching with critical cell $\vec{d}$, which also contributes to the reduced homology of $\mathrm{lk}_{\vec{c}}(\Delta)$. For instance, this case occurs if $\vec{d}$ is a facet. Similarly,

$\vec{d}$ will contribute to the reduced homology of $\mathrm{lk}_{\vec{c}}(\Delta)$ if $\vec{d}$ is a homology facet of a shelling. In either case, the critical cell $\vec{d}$ contributes to the reduced homology of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ in dimension $\dim_{\mathrm{lk}_{\vec{c}}(\Delta)}(\vec{d}) + |\vec{c}| - 1 = \dim_\Delta(\vec{d}) = |\vec{d}| - 2$, by equation (2.6.1). Note that this dimension is independent of the composition $\vec{c}$.

2.12 Consequences of the main result

As the title of this section suggests, we will now derive results from Theorem 2.11.5 using topological data from Δ.

Theorem 2.12.1. *Assume that Δ is homeomorphic to a k-dimensional manifold with or without boundary. Then the reduced homology of the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is given by*

$$\widetilde{H}_i(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \widetilde{H}_i(\Delta) \otimes 1_{\mathfrak{S}_{n-1}} \quad \text{for } i < k,$$

and the top dimensional homology is given by

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \widetilde{H}_k(\Delta) \otimes 1_{\mathfrak{S}_{n-1}} \oplus \bigoplus_{\vec{c} \in \mathrm{Int}(\Delta)} S^{B^*(\vec{c})},$$

where $1_{\mathfrak{S}_{n-1}}$ is the trivial representation of $\mathfrak{S}_{n-1}$, and the direct sum is over the interior faces of the manifold Δ. Moreover, these isomorphisms hold as $\mathfrak{S}_{n-1}$-modules.

Proof. Since Δ is homeomorphic to a k-dimensional manifold, we may apply the comment preceding Proposition 3.8.9 of [36], which states that for any $\vec{c} \in \Delta$, where $\vec{c}$ is not the empty composition (n), we have that $\mathrm{lk}_{\vec{c}}(\Delta)$ has the homology groups of a sphere of dimension $k - |\vec{c}| + 1$ if $\vec{c}$ is on the interior of Δ, or the homology groups of a ball of dimension $k - |\vec{c}| + 1$ if $\vec{c}$ is on the boundary. Hence if $\vec{c}$ is on the boundary of Δ it does not contribute to the reduced homology of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$. If instead $\vec{c}$ is in the interior of Δ then it will contribute to the reduced homology group of dimension $(k - |\vec{c}| + 1) + |\vec{c}| - 1 = k$, the top homology of the complex, by Remark 2.11.6. Finally, observe that the composition (n) contributes to all homology groups of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ when Δ has homology, and that the Specht module $S^{B^*(n)}$ is the trivial representation $1_{\mathfrak{S}_{n-1}}$. $\qquad\square$

We now give two immediate corollaries of Theorem 2.12.1, namely when Δ is a sphere or a ball.

Corollary 2.12.2. *Suppose that Δ is homeomorphic to a sphere of dimension k. Then the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ only has homology in dimension k, which is given by:*

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} S^{B^*(\vec{c})}.$$

Corollary 2.12.3. *Suppose that Δ is homeomorphic to a ball of dimension k. Then the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ only has homology in dimension k, which is given by:*

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \mathrm{Int}(\Delta)} S^{B^*(\vec{c})}.$$

Next we obtain a result about $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ when Δ is shellable.

Proposition 2.12.4. *Suppose that Δ is a shellable complex of dimension k. Then the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ only has reduced homology in dimension k, which is given by:*

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \widetilde{\beta}_{k-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}.$$

Proof. Note that the face $\vec{c}$ has dimension $|\vec{c}| - 2$. Hence the link $\mathrm{lk}_{\vec{c}}(\Delta)$ has dimension $k - \dim(\vec{c}) - 1 = k - |\vec{c}| + 1$, by Equation (2.6.1). Since the link is shellable, all of its reduced homology is in dimension $k - |\vec{c}| + 1$ and this contributes only to the reduced homology of dimension k of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ by Remark 2.11.6. $\qquad\square$

2.13 The representation ring

The *representation ring* $R(G)$ of a group G is the free abelian group with generators given by representations V of G, modulo the subgroup generated by $V + W - V \oplus W$. Elements of the representation ring are called virtual representations, because summands can have negative coefficients. For finite groups, complete reducibility implies $R(G)$ is just the free abelian group generated by the irreducible representations V of G.

Remark 2.13.1. Suppose G acts trivially on the space V. Then $V \otimes W \cong_G \dim(V) \cdot W$ in the representation ring $R(G)$.

Proof. Since G acts trivially on V we know that $V \cong_G \mathbb{C}^{\dim(V)}$. Thus, $V \otimes W \cong_G \mathbb{C}^{\dim(V)} \otimes W \cong_G \dim(V) \cdot W$. $\qquad\square$

In the representation ring we can compute the alternating sum of the homology groups of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$, which we do in the following proposition. This can be seen as $\mathfrak{S}_{n-1}$-analogue of the reduced Euler characteristic.

Proposition 2.13.2. *As virtual $\mathfrak{S}_{n-1}$-representations we have that*

$$\bigoplus_{i \geq -1}(-1)^i \cdot \widetilde{H}_i(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong \bigoplus_{\vec{c} \in \Delta}(-1)^{|\vec{c}|-1} \cdot \widetilde{\chi}(\mathrm{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}.$$

Proof. We begin the proof by applying alternating sums to both sides of Theorem 2.11.5.

$$\bigoplus_{i \geq -1}(-1)^i \cdot \widetilde{H}_i(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong \bigoplus_{i \geq -1}(-1)^i \cdot \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}$$

$$\cong \bigoplus_{\vec{c} \in \Delta} \bigoplus_{i \geq -1}(-1)^i \cdot \widetilde{\beta}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}$$

$$\cong \bigoplus_{\vec{c} \in \Delta}(-1)^{|\vec{c}|-1} \cdot \bigoplus_{j \geq -1}(-1)^j \cdot \widetilde{\beta}_j(\mathrm{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})}$$

$$\cong \bigoplus_{\vec{c} \in \Delta}(-1)^{|\vec{c}|-1} \cdot \widetilde{\chi}(\mathrm{lk}_{\vec{c}}(\Delta)) \cdot S^{B^*(\vec{c})},$$

where the second step is by Remark 2.13.1, since $\mathfrak{S}_{n-1}$ acts trivially on $\widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta))$. Furthermore, in the last step we used that the alternating sum of the Betti numbers is the reduced Euler characteristic. $\qquad\square$

The next lemma is straightforward to prove using jeu-de-taquin; see [32] or [35, A.1.2].

Lemma 2.13.3. *The permutation module $M^{B^\#(\vec{c})}$ is equal to the direct sum over all border strip Specht modules $S^{B^*(\vec{d})}$ for $\vec{d} \leq^* \vec{c}$. That is,*

$$M^{B^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d} \leq^* \vec{c}} S^{B^*(\vec{d})}.$$

Proof. Recall that the border strip of shape $A^\#(\vec{c})$ was defined in Section 2.5.

We have the isomorphism $S^{A^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} M^{A^\#(\vec{c})}$ because the rows of the shape $A(\vec{c}/1)$ are non-overlapping, thus polytabloids of shape $A(\vec{c}/1)$ are tabloids of shape $A(\vec{c}/1)$. Additionally, we have $M^{A^\#(\vec{c})} \cong_{\mathfrak{S}_{n-1}} M^{B^\#(\vec{c})}$, since tabloids are defined as row equivalence classes of tableaux and $A(\vec{c}/1)$ and $B(\vec{c}/1)$ have the same rows. Combining these two $\mathfrak{S}_{n-1}$-isomorphisms yields $M^{B^\#(\vec{c})} \cong S^{A^\#(\vec{c})}$.

Now consider the $k-1$ empty boxes situated to the left of every row in the Specht module defined by the shape $A^\#(\vec{c})$, but above the last box of the previous row. For each of these boxes perform a jeu-de-taquin slide into this box.

For each slide, there are two alternatives. If the slide is horizontal, it moves the upper row one step to the left such that the two rows overlap in one position. If the slide is vertical then every entry in the lower row moves one step up.

After performing all the $k-1$ slides the result is a border shape of shape $B^\#(\vec{c})$, where the composition $\vec{c}$ is less than or equal to the composition $\vec{d}$ in the dual order. $\qquad \square$

Proposition 2.13.2 can also be proved using the Hopf trace formula; see [43, Theorem 2.3.9].

Second proof of Proposition 2.13.2. Recall that $\widetilde{H}_i(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong \widetilde{H}_i(Q_\Delta^*)$. By applying the Hopf trace formula we have that

$$\bigoplus_{i \geq -1} (-1)^i \cdot \widetilde{H}_i(Q_\Delta^*) \cong \bigoplus_{i \geq -1} (-1)^i \cdot C_i(Q_\Delta^*)$$

$$\cong \bigoplus_{\vec{d} \in \Delta} (-1)^{|\vec{d}|} \cdot M^{B^\#(\vec{d})}$$

$$\cong \bigoplus_{\vec{d} \in \Delta} (-1)^{|\vec{d}|} \cdot \bigoplus_{\vec{c} \leq^* \vec{d}} S^{B^*(\vec{c})}$$

$$\cong \bigoplus_{\vec{c} \in \Delta} \sum_{\substack{\vec{d} \geq^* \vec{c} \\ \vec{d} \in \Delta}} (-1)^{|\vec{d}|} \cdot S^{B^*(\vec{c})}$$

$$\cong \bigoplus_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \sum_{\substack{\vec{d} \geq^* \vec{c} \\ \vec{d} \in \Delta}} (-1)^{|\vec{d}|-|\vec{c}|-1} \cdot S^{B^*(\vec{c})}.$$

Notice that in the second isomorphism we have used that the chain space $C_i(Q_\Delta^*)$ has basis given by all ordered set partitions into $i+2$ parts with type in Δ. This is equivalent to the direct sum over all permutation modules $M^{B^\#(\vec{d})}$ where $\vec{d} \in \Delta$ is a composition of n into $i+2$ parts. The remaining step is to observe that the inner sum of the last line is given by the reduced Euler characteristic $\widetilde{\chi}(\mathrm{lk}_{\vec{c}}(\Delta))$. $\qquad \square$

We observe that in the case when the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ has all its reduced homology concentrated in one dimension, the proof of Proposition 2.13.2 using the Hopf trace formula gives a shorter proof of our main result, Theorem 2.11.5.

Lastly, by taking dimension on both sides of Proposition 2.13.2 we obtain the reduced Euler characteristic of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$.

Corollary 2.13.4. *The reduced Euler characteristic of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is given by*

$$\widetilde{\chi}(\triangle(\Pi_\Delta^* - \{\hat{1}\})) = \sum_{\vec{c} \in \Delta} (-1)^{|\vec{c}|-1} \cdot \widetilde{\chi}(\mathrm{lk}_{\vec{c}}(\Delta)) \cdot \beta_n^*(\vec{c}).$$

Note that this corollary extends Theorem 3.1 from [13].

2.14 The homotopy type of Π_Δ^*

We turn our attention to the homotopy type of the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$. By combining the poset fiber theorems of Quillen [29] and Björner, Wachs and Welker [4] we obtain the next result. Recall that $*$ denotes the (free) join of complexes.

Theorem 2.14.1. *The order complex of $\Pi_\Delta^* - \{\hat{1}\}$ is homotopy equivalent to the complex of ordered set partitions Q_Δ^*, that is, $\triangle(\Pi_\Delta^* - \{\hat{1}\}) \simeq Q_\Delta^*$. Furthermore, the following homotopy equivalence holds*

$$Q_\Delta^* \simeq \triangle(\Delta - \{(n)\}) \vee \{Q_{\vec{c}}^* * \mathrm{lk}_{\vec{c}}(\Delta) : \vec{c} \in \Delta - \{(n)\}\},$$

where $\vee$ denotes identifying each vertex $\vec{c}$ in $\triangle(\Delta - \{(n)\})$ with any vertex in $Q_{\vec{c}}^*$. In the case when the complex Δ is connected then the homotopy equivalence simplifies to

$$Q_\Delta^* \simeq \bigvee_{\vec{c} \in \Delta} Q_{\vec{c}}^* * \mathrm{lk}_{\vec{c}}(\Delta).$$

Proof. The first homotopy equivalence follows by applying Quillen's fiber lemma to the forgetful map f, which yields $\triangle(\Pi_\Delta^* - \{\hat{1}\}) \simeq \triangle(Q_\Delta^* - \{\hat{1}\}) \cong Q_\Delta^*$, since $\triangle(Q_\Delta^* - \{\hat{1}\})$ is the barycentric subdivision of Q_Δ^*.

The second homotopy equivalence, in both cases, follows by Theorem 1.1 in [4], with the same reasoning as in the proof of Theorem 2.6.3. Furthermore, when $\vec{c} = (n)$ then the complex $Q_{\vec{c}}^*$ is the empty complex, which is the identity for the join. $\square$

Corollary 2.14.2. *Let Δ be a connected simplicial complex. Assume furthermore, that each link (including Δ) $\mathrm{lk}_{\vec{c}}(\Delta)$ is a wedge of spheres. Then the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is also a wedge of spheres. Furthermore, the number of i-dimensional spheres is given by the sum*

$$\sum_{\vec{c} \in \Delta} \beta_n^*(\vec{c}) \cdot \widetilde{\beta}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)), \tag{2.14.1}$$

where $\widetilde{\beta}_j$ denotes the jth reduced Betti number.

Next we have the homotopy versions of Corollaries 2.12.2 and 2.12.3. To prove the next two corollaries, we are again using the comment preceding Proposition 3.8.9 of [36] to determine the reduced Betti numbers of the links.

Corollary 2.14.3. *Suppose that Δ is homeomorphic to a sphere of dimension k. Then the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is a wedge of k-dimensional spheres and the number of spheres is given by the sum:*

$$\sum_{\vec{c} \in \Delta} \beta_n^*(\vec{c}).$$

Corollary 2.14.4. *Suppose that Δ is homeomorphic to a ball of dimension k. Then the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is a wedge of k-dimensional spheres and the number of spheres is given by the sum:*

$$\sum_{\vec{c} \in \mathrm{Int}(\Delta)} \beta_n^*(\vec{c}).$$

We end this section with a discussion of how we can lift discrete Morse matchings from the links of Δ to the complex of order set partitions Q_Δ^*.

Definition 2.14.1. For an ordered set partition $\sigma = (C_1, C_2, \ldots, C_k)$ of n, where $C_i = \{c_{i,1} < c_{i,2} < \cdots < c_{i,j_i}\}$ and $|C_i| = j_i$, define the permutation $\mathrm{perm}(\sigma) \in \mathfrak{S}_n$ to be the elements of the blocks written out in the order of the blocks, that is,

$$\mathrm{perm}(\sigma) = c_{1,1}, c_{1,2}, \ldots, c_{1,j_1}, c_{2,1}, c_{2,2}, \ldots, c_{k,j_k}.$$

Define the descent set of an ordered set partition σ to be $\mathrm{Des}(\sigma) = \mathrm{Des}(\mathrm{perm}(\sigma))$. Observe that the descent composition of an ordered set partition is an order preserving map from the poset of ordered set partitions Q_n to the poset of compositions Comp_n, that is, $\mathrm{Des} : Q_n^* \longrightarrow \mathrm{Comp}(n)$ is a poset map.

Lemma 2.14.5. *Let Δ be a filter in the composition poset $\mathrm{Comp}(n)$. For the order preserving map $\mathrm{Des} : Q_\Delta^* \longrightarrow \Delta$ the poset fiber $\mathrm{Des}^{-1}(\vec{c})$ is the (poset) direct sum of $\beta_n^*(\vec{c})$ copies of the poset $\mathrm{lk}_{\vec{c}}(\Delta) = \{\vec{d} \in \Delta \,:\, \vec{d} \leq^* \vec{c}\}$.*

Proof. Let σ be an ordered set partition and assume that the ith block C_i is the disjoint union of the two non-empty sets X and Y such that $\max(X) < \min(Y)$. Observe now that the two ordered set partitions σ and $(\ldots, C_{i-1}, X, Y, C_{i+1}, \ldots)$ have the same descent composition, since there is no descent between blocks X and Y.

Let $\vec{c}$ be a composition in the filter Δ. For any ordered set partition τ in the fiber $\mathrm{Des}^{-1}(\vec{c})$ we know that τ has descent composition $\vec{c}$, that is, $\mathrm{Des}(\tau) = \vec{c}$. As τ can only have descents between blocks, we know the minimal elements of $\mathrm{Des}^{-1}(\vec{c})$ have the form $\sigma(\alpha, \vec{c})$, for $\alpha \in \mathfrak{S}_n$ satisfying $\mathrm{Des}(\alpha) = \vec{c}$ and $\alpha_n = n$. To remain in the same fiber as these minimal elements, we are free to break blocks as in previous paragraph, hence

$$\mathrm{Des}^{-1}(\vec{c}) = \{\sigma(\alpha, \vec{d}) \,:\, \vec{c} \leq^* \vec{d}, \, \vec{d} \in \Delta, \, \mathrm{Des}(\alpha) = \vec{c}, \, \alpha_n = n\}.$$

Notice that the poset $\mathrm{lk}_{\vec{c}}(\Delta)$ is isomorphic to the poset $\{\sigma(\alpha, \vec{d}) : \vec{d} \leq \vec{c}, \, \vec{d} \in \Delta\}$ for a fixed permutation $\alpha \in \mathfrak{S}_n$ satisfying $\mathrm{Des}(\alpha) = \vec{c}$ and $\alpha_n = n$. Finally, for a composition $\vec{e} \in \mathrm{lk}_{\vec{c}}(\Delta)$ and a permutation $\beta \in \mathfrak{S}_n$ different from α such that $\mathrm{Des}(\beta) = \vec{c}$ and $\beta_n = n$, consider the two ordered set partitions $\sigma(\beta, \vec{e})$ and $\sigma(\alpha, \vec{d})$, where $\vec{d} \in \mathrm{lk}_{\vec{c}}(\Delta)$. By examining the first increasing run in the permutations α and β where their elements differ, we conclude that the

two ordered set partitions $\sigma(\beta, \vec{e})$ and $\sigma(\alpha, \vec{d})$ are incomparable. Thus the fiber $\mathrm{Des}^{-1}(\vec{c})$ is a direct sum of copies of the poset $\mathrm{lk}_{\vec{c}}(\Delta)$, one for each permutation α in $\mathfrak{S}_n$ satisfying $\mathrm{Des}(\alpha) = \vec{c}$ and $\alpha(n) = n$. $\qquad\square$

Theorem 2.14.6. *Let Δ be a simplicial complex of compositions such that every link $\mathrm{lk}_{\vec{c}}(\Delta)$ has a Morse matching where the critical cells are facets of the link $\mathrm{lk}_{\vec{c}}(\Delta)$. Then the simplicial complex Q_Δ^* has a Morse matching, where the number of i-dimensional critical cells is given by equation* (2.14.1).

Proof. Apply the Patchwork Theorem [18, Theorem 11.10] to the poset map $\mathrm{Des} : Q_\Delta^* \longrightarrow \Delta$. By Lemma 2.14.5, each fiber is a direct sum of links of Δ, each of which has a Morse matching, where each critical cell is a facet. Hence Q_Δ^* is homotopy equivalent to a wedge of spheres, and thus the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is also a wedge of spheres. The number of i-dimensional critical cells of Q_Δ^* in the fiber $\mathrm{Des}^{-1}(\vec{c})$ is the number of critical cells of dimension $i - |\vec{c}| + 1$ in the link $\mathrm{lk}_{\vec{c}}(\Delta)$ times the number of copies of the link, that is $\beta_n^*(\vec{c})$. By summing over all compositions $\vec{c}$ in Δ the result follows. $\qquad\square$

Now suppose that Δ is a shellable complex, in the broader non-pure sense. Then each link in Δ is also shellable, and thus for each link there exists a discrete Morse matching whose critical cells are facets of the link; see Chapter 12 of [18].

Corollary 2.14.7. *If Δ is a non-pure shellable complex then Theorem 2.14.6 applies and the simplicial complex Q_Δ^* has a Morse matching where the number of i-dimensional critical cells is given by equation* (2.14.1).

Proof. This follows directly from two observations: (i) a non-pure shellable complex has a Morse matching with all critical cells being facets (ii) each link of a non-pure shellable complex is non-pure shellable. See Section 12.1 in [18]. $\qquad\square$

2.15 Examples

In this section we use Theorem 2.11.5 and its consequences from Section 2.12 to derive results about various filters Π_Δ^*.

Example 2.15.1. Let $\vec{d}$ be a composition of n into $k+2$ parts and let Δ be the simplex generated by $\vec{d}$. Since the simplex is homeomorphic to the k dimensional ball, by Corollary 2.12.3 we have that the kth reduced homology group is given by

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} S^{B^*(\vec{d})},$$

since the only face of Δ in the interior of Δ is the facet $\vec{d}$. This example illustrates Theorems 5.3 and 7.4 in [12]. Moreover, this is the base case of the authors' proof of Theorem 2.6.3 using the Mayer–Vietoris sequence.

Example 2.15.2. Let $\vec{d}$ be a composition into $k+3$ parts and let Δ be the boundary of the simplex generated by $\vec{d}$, that is, Δ is homeomorphic to a k-dimensional sphere. Then Δ is

shellable and the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is a wedge of k-dimensional spheres. Now by Corollary 2.12.2 we have that the kth reduced homology group is given by

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} <^* \vec{d}} S^{B^*(\vec{c})} \cong_{\mathfrak{S}_{n-1}} M^{B^\#(\vec{d})}/S^{B^*(\vec{d})}.$$

Note that we have used Lemma 2.13.3 to express the permutation module $M^{B^\#(\vec{d})}$ as a direct sum of Specht modules.

Example 2.15.3. Let $\vec{d}$ be a composition of n into $k + r$ parts, where $r \geq 1$. Let Δ be the k-skeleton of the simplex generated by the composition $\vec{d}$. Note that Δ is shellable, so by Corollary 2.14.7 the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is a wedge of k-dimensional spheres. By Proposition 2.12.4 we have the following calculation in the representation ring:

$$\widetilde{H}_k(\triangle(\Pi_\Delta^* - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\substack{\vec{c} < \vec{d} \\ |\vec{c}| \leq k+2}} \binom{k + r - |\vec{c}| - 1}{k - |\vec{c}| + 2} \cdot S^{B^*(\vec{c})}, \tag{2.15.1}$$

where we have used that $\widetilde{\beta}_{k-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) = (-1)^{k-|\vec{c}|+1} \cdot \widetilde{\chi}(\mathrm{lk}_{\vec{c}}(\Delta))$ since $\mathrm{lk}_{\vec{c}}(\Delta)$ is shellable. Lastly, we also used a basic identity on the alternating sum of binomial coefficients, which arises in computing the Euler characteristic of the link.

Example 2.15.4 (The d-divisible partition lattice with minimal elements removed). Let n be a multiple of d. Consider the boundary of the simplex generated by the composition $(d, d, \ldots, d)$ of n. Then Δ is a $(n/d - 3)$-dimensional simplicial complex, and Π_Δ^* is the d-divisible partition lattice without its minimal elements. By applying Example 2.15.2 we obtain that $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is a wedge of $(n/d - 3)$-dimensional spheres and the reduced homology group is given by $\widetilde{H}_{n/d-3}(\Pi_\Delta^* - \{\hat{1}\}) \cong_{\mathfrak{S}_{n-1}} M^{B^\#(d,\ldots,d,d)}/S^{B^*(d,d,\ldots,d)}$.

Setting $d = 1$ in the last example shows that the action of $\mathfrak{S}_{n-1}$ on the reduced homology group of $\triangle(\Pi_n - \{\hat{0}, \hat{1}\})$ is $M^{B^\#(1,\ldots,1,1)} = M^{B(1,\ldots,1)}$, which is the regular representation of $\mathfrak{S}_{n-1}$.

Example 2.15.5 (The truncated d-divisible partition lattice). To generalize Example 2.15.4 and specialize Example 2.15.3, let $n = (k + r) \cdot d$ and consider the k-skeleton of the simplex generated by the composition $(d, d, \ldots, d)$ of n. Here Π_Δ^* consists of all set partitions in the d-divisible partition lattice with at most $k+2$ parts. Directly we have that the order complex $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is a wedge of k-dimensional spheres and its k-dimensional reduced homology is given by equation (2.15.1).

Examples 2.15.4 and 2.15.5 are both rank selected subposets of the d-divisible partition lattice, which is (pure) shellable, and thus by Stanley [33], each of these complexes are (pure) shellable.

Example 2.15.6. An integer partition $\lambda = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_p^{m_p}\}$ of the non-negative integer n is called *knapsack* if all the sums $\sum_{i=1}^p e_i \cdot \lambda_i$, where $0 \leq e_i \leq m_i$, are distinct. In other words, λ is knapsack if

$$\left| \left\{ \sum_{i=1}^p j_i \cdot \lambda_i : 0 \leq j_i \leq m_i \right\} \right| = \prod_{i=1}^p (m_i + 1).$$

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For a knapsack partition λ into $k-1$ parts of $n-m$, where $m < n$, define $\Delta_{\lambda,m}$ to be the simplicial complex $\Delta_{\lambda,m}$ which has the facets $(c_1, c_2, \ldots, c_{k-1}, c_k)$ where $\mathrm{type}(c_1, c_2, \ldots, c_{k-1}) = \lambda$ and the last part c_k is m. Then the complex $\Delta_{\lambda,m}$ is homeomorphic to a $(k-2)$-dimensional ball; see the proof of Theorem 4.4 in [13]. Applying Corollary 2.12.3, we obtain the following result:

$$\widetilde{H}_{k-2}(\triangle(\Pi^*_{\Delta_{\lambda,m}} - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \mathrm{Int}(\Delta_{\lambda,m})} S^{B^*(\vec{c})}.$$

Furthermore, the set of interior faces of $\Delta_{\lambda,m}$ is given by compositions $\vec{c}$ in $\Delta_{\lambda,m}$ such that when each part of $\vec{c}$ is written as a sum of parts of λ, those parts are distinct. This example is Theorem 10.3 in [12]. Moreover, $\Delta_{\lambda,m}$ is shellable, so Theorem 2.14.6 yields a Morse matching of $Q^*_{\Delta_{\lambda,m}}$; see Theorem 8.2 of [12].

2.16 The Frobenius complex

We now consider a different class of examples stemming from [11]. Let Λ be a semigroup of positive integers, that is, a subset of the positive integers which is closed under addition. Let Δ_n be the collection of all compositions of n whose parts belong to Λ, that is,

$$\Delta_n = \{(c_1, \ldots, c_k) \in \mathrm{Comp}(n) \; : \; c_1, \ldots, c_k \in \Lambda\}.$$

Since Λ is closed under addition, we obtain that Δ_n is a filter in the poset of compositions $\mathrm{Comp}(n)$ and hence we view it as a simplicial complex. This complex is known as the Frobenius complex; see [11]. Using Lemma 2.11.1 the associated filter in the partition lattice is given by

$$\Pi_n^\Lambda = \{\{B_1, \ldots, B_k\} \in \Pi_n \; : \; |B_1|, \ldots, |B_k| \in \Lambda\}.$$

Let Ψ_n be the generating function

$$\Psi_n = \sum_{i \geq -1} \widetilde{\beta}_i(\Delta_n) \cdot t^{i+1}.$$

Observe that for a composition $\vec{c}$ in Δ_n we have that the link $\mathrm{lk}_{\vec{c}}(\Delta_n)$ is given by the join

$$\mathrm{lk}_{\vec{c}}(\Delta_n) = \Delta_{c_1} * \Delta_{c_2} * \cdots * \Delta_{c_k}. \tag{2.16.1}$$

Hence we can apply the Künneth theorem to obtain that the ith reduced Betti number of the link is given by

$$\widetilde{\beta}_i(\mathrm{lk}_{\vec{c}}(\Delta_n)) = [t^{i+1}]\Psi_{c_1} \cdot \Psi_{c_2} \cdots \Psi_{c_k}.$$

Hence using Theorem 2.11.5, the ith reduced Betti number of the order complex $\triangle(\Pi_n^\Lambda - \{\hat{1}\})$ is given in the representation ring of $\mathfrak{S}_{n-1}$ by

$$\widetilde{H}_i(\triangle(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \sum_{\vec{c}} [t^{i_{\vec{c}}+1}]\Psi_{c_1} \cdot \Psi_{c_2} \cdots \Psi_{c_k} \cdot S^{B^*(\vec{c})},$$

where the sum is over all compositions $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n.

A more explicit approach is possible when the complex Δ_n has a discrete Morse matching. By combining equation (2.16.1), Lemma 2.4.1, and a Morse matching from [11], we create a Morse matching on every link. We will see this method in the remainder of this section.

We continue by studying one concrete example. Let a and d be two positive integers. Let Λ be the semigroup generated by the arithmetic progression

$$\Lambda = \langle a, a + d, a + 2d, \ldots \rangle.$$

Since for $j \geq a$ we have that $a + j \cdot d = d \cdot a + a + (j - a) \cdot d$, the semigroup is generated by the finite arithmetic progression

$$\Lambda = \langle a, a + d, a + 2d, \ldots, a + (a - 1)d \rangle.$$

Clark and Ehrenborg proved that the Frobenius complex Δ_n is a wedge of spheres of different dimensions; see [11, Theorem 5.1]. Observe that their result is formulated in terms of sets, instead of compositions. However, the two notions are equivalent via the natural bijection given by sending a composition $(c_1, c_2, \ldots, c_k)$ of n to the subset $\{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\}$ of the set $[n-1]$. To state their result, let A be the set $\{a + d, a + 2d, \ldots, a + (a - 1) \cdot d\}$.

Proposition 2.16.1. *For n in the semigroup Λ, there is a discrete Morse matching on the Frobenius complex Δ_n such that the critical cells are compositions $\vec{c} = (c_1, \ldots, c_k)$ characterized by*

(i) *All but the last entry of the composition belongs to the set A, that is, $c_1, \ldots, c_{k-1} \in A$.*

(ii) *The last entry c_k belongs to $\{a\} \cup A$.*

Furthermore, all the critical cells are facets.

Proof. When a and d are relative prime, that is, $\gcd(a, d) = 1$, this result is Lemma 5.10 in [11]. When a and d are not relative prime, the result follows by scaling down the three parameters a, d and n by $a' = a/\gcd(a, d)$, $d' = d/\gcd(a, d)$ and $n' = n/\gcd(a, d)$. Now the result applies the semigroup $\Lambda' = \langle a', a' + d', a' + 2d', \ldots \rangle$ and its associated Frobenius complex $\Delta'_{n'}$. However, this complex is isomorphic to Δ_n by sending the composition $\vec{c} = (c_1, \ldots, c_k)$ in $\Delta'_{n'}$ to the composition $\gcd(a, d) \cdot \vec{c} = (\gcd(a, d) \cdot c_1, \ldots, \gcd(a, d) \cdot c_k)$ in Δ_n. $\square$

Corollary 2.16.2. *The order complex $\triangle(\Pi_n^\Lambda - \{\hat{1}\})$ is a wedge of spheres.*

Proof. Since Δ_n has a discrete Morse matching where each critical cell is a facet, Δ_n is homotopy equivalent to a wedge of spheres. Furthermore, by equation (2.16.1) we know that every link of Δ_n is a wedge of spheres. Finally, by Corollary 2.14.2 we obtain the result. $\square$

Next we need to extend Lemma 2.13.3 to collect Specht modules together. We call the sum $c_1 + c_2 + \cdots + c_j$ an *initial sum* of a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ for $1 \leq j \leq k$.

Definition 2.16.1. For an interval $[\vec{d}, \vec{b}]$ in the lattice of compositions $\mathrm{Comp}(n)$ let $B^*(\vec{d}, \vec{b})$ be the skew-shape where the row lengths are given by $d_1, d_2, \ldots, d_{r-1}, d_r - 1$ and if the initial sum $d_1 + \cdots + d_j$ is equal to an initial sum of the composition $\vec{b} - 1$, then jth row and the $(j + 1)$st row overlap in one column. All other rows of $B^*(\vec{d}, \vec{b})$ are non–overlapping.

Table 2.1: The reduced homology groups of the order complex $\triangle(\Pi_n^{\langle 3,5,7 \rangle} - \{\hat{1}\})$ for the even cases $n = 8, 10, 12$ and 14.

n	$\widetilde{H}_0$	$\widetilde{H}_2$
8	(border strip) $\oplus$ (border strip)	0
10	(border strip) $\oplus$ (border strip) $\oplus$ (border strip)	0
12	(border strip) $\oplus$ (border strip)	(border strip)
14	(border strip)	(border strip) $\oplus$ (border strip) $\oplus$ (border strip) $\oplus$ (border strip)

As an example, if $\vec{d} = (2, 5, 4, 1, 3, 2)$ and $\vec{b} = (2 + 5 + 4, 1 + 3, 2)$, then $B^*(\vec{d}, \vec{b})$ is the border strip with row lengths $2, 5, 4, 1, 3$ and $2 - 1 = 1$ which overlaps between the rows of length 4 and 1 and the rows of length 3 and 1. Note that $B^*(\vec{d}, (n)) = A^*(\vec{d})$.

The proof of the next lemma is the same as the proof of Lemma 2.13.3, that is, it uses jeu-de-taquin moves where two adjacent rows do not overlap.

Lemma 2.16.3. *Let $\vec{b}$ and $\vec{d}$ be two compositions in $\mathrm{Comp}(n)$ such that $\vec{d} \leq \vec{b}$. Then the Specht module $S^{B^*(\vec{d}, \vec{b})}$ is given by the direct sum*

$$S^{B^*(\vec{d}, \vec{b})} \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d} \leq \vec{c} \leq \vec{b}} S^{B^*(\vec{c})}.$$

In order to state the main result for the semigroup $\Lambda = \langle a, a + d, a + 2d, \ldots \rangle$ and the associated filter in the partition lattice, we need one last definition.

Definition 2.16.2. For a composition $\vec{d}$ of n with entries in the set $\{a\} \cup A$, let $\vec{b}(\vec{d})$ be the composition greater than or equal to $\vec{d}$ obtained by adding runs of entries of $\vec{d}$ together where each run ends with the entry a.

As an example, for $a = 3$, $d = 2$ we have $A = \{5, 7\}$. Hence for the composition $\vec{d} = (5, 3, 7, 5, 3, 3, 7, 5)$ we obtain $\vec{b}(\vec{d}) = (5 + 3, 7 + 5 + 3, 3, 7 + 5) = (8, 15, 3, 12)$.

Remark 2.16.4. Observe that the skew-shape $S^{B^*(\vec{d}, \vec{b}(\vec{d}))}$ has the row lengths $d_1, \ldots, d_{r-1}, d_r - 1$ and satisfies the condition that $d_i = a$ if and only if there is overlap between ith and $(i+1)$st rows. See Definition 2.16.1.

Theorem 2.16.5. *Let a and d be two positive integers and let Π_n^Λ be the filter in the partition lattice Π_n where each partition π consists of blocks whose cardinalities belong to the*

semigroup Λ *generated by the arithmetic progression* $a, a + d, \ldots, a + (a - 1) \cdot d$. *Then the* *i*th *reduced homology group of the order complex* $\triangle(\Pi_n^\Lambda - \{\hat{1}\})$ *is given by the direct sum*

$$\widetilde{H}_i(\triangle(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d}} S^{B^*(\vec{d}, \vec{b}(\vec{d}))},$$

where the sum is over all compositions $\vec{d}$ *into* $i + 2$ *parts such that every entry belongs to the set* $\{a\} \cup A = \{a, a + d, a + 2 \cdot d, a + (a - 1) \cdot d\}$.

Proof. Let $\vec{c}$ be a composition in the complex Δ_n. Using the Morse matching given by Proposition 2.16.1 and Lemma 2.4.1, we obtain that a critical cell $\vec{d}$ in the link $\mathrm{lk}_{\vec{c}}(\Delta_n) = \Delta_{c_1} * \Delta_{c_2} * \cdots * \Delta_{c_k}$ is a composition $\vec{d} \leq \vec{c}$ where the entries of $\vec{d}$ belong to the set $\{a\} \cup A$. Furthermore, in the run of entries of $\vec{d}$ that sums to the entry c_i of the composition $\vec{c}$, only the last entry of the run is allowed to be equal to a. Using Theorem 2.11.5 we have

$$\widetilde{H}_i(\triangle(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta_n} \widetilde{H}_{i - |\vec{c}| + 1}(\mathrm{lk}_{\vec{c}}(\Delta_n)) \otimes S^{B^*(\vec{c})}.$$

$$\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta_n} \bigoplus_{\vec{d}} S^{B^*(\vec{c})},$$

where the inner sum consists of compositions $\vec{d}$ satisfying the above conditions and with $|\vec{d}| = i + 2$. By changing the order of summation we obtain

$$\widetilde{H}_i(\triangle(\Pi_n^\Lambda - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{d}} \bigoplus_{\vec{c}} S^{B^*(\vec{c})},$$

where the outer direct sum is over all compositions $\vec{d}$ of n into $i + 2$ parts where each part is in the set $\{a\} \cup A$ and the inner direct sum is over all compositions $\vec{c}$ greater than $\vec{d}$, obtained by adding runs of entries of $\vec{d}$ where an entry equal to a can only be at the end of a run. The inner direct sum is hence given by the Specht module $S^{B^*(\vec{d}, \vec{b}(\vec{d}))}$ by Remark 2.16.4 and Lemma 2.16.3, and therefore the result follows. $\square$

Corollary 2.16.6. *The order complex* $\triangle(\Pi_n^\Lambda - \{\hat{1}\})$ *only has non-vanishing reduced homology in dimension* i *when* $n \equiv (i + 2) \cdot a \bmod d$ *for* $d \geq 2$.

Proof. Since all entries in the set $\{a\} \cup A$ are congruent to a modulo d, we have $n = \sum_{j=1}^{i+2} d_j \equiv (i + 2) \cdot a \bmod d$. $\square$

In Tables 2.1 and 2.2 we have explicitly calculated the reduced homology groups for the order complex $\triangle(\Pi_n^{\langle 3,5,7 \rangle} - \{\hat{1}\})$ for $8 \leq n \leq 15$, that is, when $a = 3$ and $d = 2$. Instead of writing out the notation $S^{B^*(\vec{d}, \vec{b})}$ for the Specht modules we have drawn the associated border shapes. Observe that when a row has three boxes, there is overlap with the row above. For $a = 3$ and $d = 2$ Corollary 2.16.6 implies that the order complex only has non-vanishing homology in dimensions of the same parity as n.

Table 2.2: The reduced homology groups of the order complex $\triangle(\Pi_n^{\langle 3,5,7\rangle} - \{\hat{1}\})$ for the odd cases $n = 9$, 11, 13 and 15.

n	$\widetilde{H}_1$	$\widetilde{H}_3$
9		0
11		0
13		0
		0
15		

Example 2.16.7. When the integer d divides the integer a, the homology groups of Π_n^Λ have been studied. In this case, the filter Π_n^Λ consists of all partitions where the block sizes are divisible by d and the block sizes are greater than or equal to a. This filter was studied by Browdy [5], and our Theorem 2.16.5 reduces to her result; see Corollary 5.3.3 in [5].

Example 2.16.8. The previous example is particularly nice when $d = 1$. Then the semigroup Λ is given by $\Lambda = \{n \in \mathbb{P} \ : \ n \geq a\}$ and the filter Π_n^Λ consists of all partitions where $1, 2, \ldots, a - 1$ are forbidden block sizes. In this case it follows by Billera and Meyers [1] that Δ_n is non-pure shellable. Additionally, Björner and Wachs [3] gave an EL-labeling of $\Pi_n^\Lambda \cup \{\hat{0}\}$. This order complex was also considered by Sundaram in Example 4.4 in [37].

Finally, we note that Wachs [42] has plethystic recurrences for the $\mathfrak{S}_n$ representations on the homology groups of the filters in Examples 2.16.7 and 2.16.8.

2.17 The partition filter $\Pi_n^{\langle a,b\rangle}$

Let a and b be two relatively prime integers greater than 1. Let $\Pi_n^{\langle a,b\rangle}$ be the filter in Π_n generated by all partitions whose block sizes are all a or b. As an example, $\Pi_n^{\langle 2,3\rangle}$ consists of all partitions in Π_n with no singleton blocks. The corresponding complex Δ_n in $\mathrm{Comp}(n)$ consists of all compositions of n whose parts are contained in the set $\langle a, b\rangle = \{i \cdot a + j \cdot b \ : \ a, b \in \mathbb{N}\}$. When $a = 2$ and $b = 3$ the complex Δ_n is known as the complex of sparse sets; see [11, 19].

Following Theorem 4.1 in [11], we define the set $A = \{n \in \mathbb{P} : n \equiv 0, a, b \text{ or } a+b \bmod ab\}$

and the function $h : A \longrightarrow \mathbb{Z}_{\geq -1}$ as follows:

$$h(n) = \begin{cases} \frac{2n}{ab} - 2 & \text{if } n \equiv 0 \bmod ab, \\ \frac{2(n-a)}{ab} - 1 & \text{if } n \equiv a \bmod ab, \\ \frac{2(n-b)}{ab} - 1 & \text{if } n \equiv b \bmod ab, \\ \frac{2(n-a-b)}{ab} & \text{if } n \equiv a + b \bmod ab. \end{cases} \tag{2.17.1}$$

Then Theorem 4.1 in [11] states that Δ_n is either homotopy equivalent to a sphere or is contractible, according to

$$\Delta_n \simeq \begin{cases} S^{h(n)} & \text{if } n \in A, \\ \text{point} & \text{otherwise.} \end{cases}$$

Using equation (2.16.1) we see that if $\vec{c} \in \Delta_n$ has any part not in A, then $\mathrm{lk}_{\vec{c}}(\Delta_n)$ is contractible. If each part of $\vec{c}$ is in A, then $\mathrm{lk}_{\vec{c}}(\Delta_n) \simeq S^{h(c_1)} * \cdots * S^{h(c_k)} = S^{h(\vec{c})}$, where we define $h(\vec{c}) = k - 1 + \sum_{j=1}^{k} h(c_j)$ for compositions $\vec{c}$ with all parts in A, since the join of an n-dimensional sphere and an m-dimensional sphere is an $(n + m + 1)$-dimensional sphere. Note that $h(\vec{c})$ is undefined for all other compositions.

For a composition $\vec{c} = (c_1, \ldots, c_k)$ of n with all of its parts in A, let $\dim(\vec{c})$ denote the dimension of the reduced homology of $\triangle(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$ to which the composition $\vec{c}$ contributes. That is, $\dim(\vec{c})$ is given by

$$\dim(\vec{c}) = h(\vec{c}) + k - 1 = \sum_{i=1}^{k} h(c_i) + 2k - 2. \tag{2.17.2}$$

We can apply Theorem 2.11.5 to obtain

Theorem 2.17.1. *Let $2 \leq a < b$ with $\gcd(a, b) = 1$. Then the ith reduced homology group of $\triangle(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$ is given by the direct sum of Specht modules $\bigoplus_{\vec{c} \in F_i} S^{B^*(\vec{c})}$, where F_i is the collection of compositions $\vec{c}$ of n where all the parts are in the set A with $\dim(\vec{c}) = i$.*

Proof. We directly have

$$\widetilde{H}_i(\Delta(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i_{\vec{c}}}(\mathrm{lk}_{\vec{c}}(\Delta_n)) \otimes S^{B^*(\vec{c})}$$

$$\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in F_i} \widetilde{H}_{i_{\vec{c}}}(S^{h(\vec{c})}) \otimes S^{B^*(\vec{c})}$$

$$\cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in F_i} S^{B^*(\vec{c})}. \qquad \square$$

We now describe the top and bottom reduced homology of the order complex $\Delta(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$. We begin with the top homology.

Proposition 2.17.2. *Let $2 \leq a < b$ with $\gcd(a, b) = 1$. Let r be the unique integer such that $0 \leq r < a$ and $n \equiv rb \bmod a$. Then the top homology of $\triangle(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$, which occurs in dimension $(n - r(b - a))/a - 2$, is given by the direct sum of Specht modules $\bigoplus_{\vec{c} \in R} S^{B^*(\vec{c})}$, where R is the collection of compositions $\vec{c}$ of n where exactly r of the parts are equal to b or $a + b$, and the remaining parts are all equal to a.*

Proof. We present two procedures that will change a composition $\vec{c}$ into another composition $\vec{c}'$ such that the dimension of contribution from $\vec{c}'$ is greater than the contribution of $\vec{c}$, that is, $\dim(\vec{c}) < \dim(\vec{c}')$. The compositions which we cannot improve with this procedure are those described in the statement of the proposition.

We now describe the first replacement procedure. If the composition $\vec{c}$ has a part of the form

(i) jab, replace it with jb a's,

(ii) $jab + a$, replace it with $(jb + 1)$ a's,

(iii) $jab + b$, replace it with jb a's and one b,

(iv) $jab + a + b$, replace it with $(jb + 1)$ a's and one b,

to obtain a new composition $\vec{c}'$. We claim that $\dim(\vec{c}') - \dim(\vec{c}) = (b - a) \cdot j$. We check the computation in the case (iv), the other three cases are similar. The difference $\dim(\vec{c}') - \dim(\vec{c})$ only depends on the parts affected and the number of them. Hence

$$\dim(\vec{c}') - \dim(\vec{c}) = [(jb + 1) \cdot h(a) + h(b) + 2(jb + 2)] - [h(jab + a + b) + 2]$$
$$= [jb + 2] - [2j + 2] = (b - a) \cdot j > 0,$$

using that $h(a) = h(b) = -1$ and $h(jab + a + b) = 2j$. Hence this procedure increases the dimension.

Iterating this procedure we obtain a new composition with all the parts of the form a, b and $a + b$.

The second replacement procedure is as follows. Assume that there are a parts of the composition $\vec{c}$ that are different from a. Assume that p of these parts are equal to $a + b$, and hence $a - p$ of them are equal to b. Replace these a parts with $b + p$ parts equal to a to obtain a new composition $\vec{c}'$.

$$\dim(\vec{c}') - \dim(\vec{c}) = [(b + p) \cdot h(a) + 2(b + p)] - [p \cdot h(a + b) + (a - p) \cdot h(b) + 2a]$$
$$= [b + p] - [a + p] = b - a > 0.$$

Hence the new composition $\vec{c}'$ contributes to a homology of dimension $b - a > 0$ greater than the composition $\vec{c}$ does.

Iterating the last procedure, we are left with a composition $\vec{c}$ where the number of parts different from a is at most $a - 1$. By considering the equation $c_1 + \cdots + c_k = n$ modulo a, we obtain the number of parts different from a is given by the integer r from the statement of the proposition. Additionally, switching between one part of $a + b$ and the two parts a and b does not change the dimension of the contribution of the composition. Finally, we compute the contribution of the composition $(\underbrace{a, \ldots, a}_{(n-br)/a}, \underbrace{b, \ldots, b}_{r})$ to obtain the desired dimension. $\qquad \square$

Corollary 2.17.3. *Let $2 \leq a < b$ with $\gcd(a, b) = 1$. Assume that n is divisible by a. Then the top homology of $\triangle(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$, which occurs in dimension $n/a - 2$, is the Specht module $S^{B^*(a,a,\ldots,a)}$.*

Proof. When a divides n, then the integer r of Proposition 2.17.2 is 0. Thus the only contribution to reduced homology in dimension $n/a - 2$ is given by $(a, a, \ldots, a)$. $\qquad\square$

We now turn our attention to the bottom reduced homology.

Proposition 2.17.4. *Let $3 \leq a < b$ with $\gcd(a, b) = 1$. Let r and s be the two unique integers such that*

$$n \equiv rb \bmod a, \quad 0 \leq r < a, \quad n \equiv sa \bmod b \quad \text{and} \quad 0 \leq s < b.$$

Then the bottom reduced homology of $\triangle(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$ occurs in dimension $2 \cdot \frac{n-sa-rb}{ab} + r + s - 2$, and is given by the direct sum of Specht modules $S^{B^*(\vec{c})}$ over all compositions $\vec{c}$ such that the number of parts of $\vec{c}$ of the form $j \cdot ab + a$ and $j \cdot ab + a + b$ is s and the number of parts of the form $j \cdot ab + b$ and $j \cdot ab + a + b$ is r.

Proof. Just as in Proposition 2.17.2, we will define replacement procedures, where our goal now is to decrease the dimension of the homology that our composition contributes to, rather than increase it, as was the case in Proposition 2.17.2.

The first procedure takes b parts of the composition $\vec{c}$ of the form $jab + a$ and $jab + a + b$ and subtracts a from each of these b parts, and adjoins a new part ab. Notice that the resulting new composition $\vec{c}'$ remains a composition of n. Observe that $h(jab) = h(jab + a) - 1$, $h(jab + b) = h(jab + a + b) - 1$, and $h(ab) = 0$. Hence the dimension $\vec{c}'$ contributes to is $\dim(\vec{c}') = \sum_{i=1}^{k+1} h(c'_i) + 2(k+1) - 2 = \sum_{i=1}^{k} h(c_i) - b + 2(k+1) - 2 = \dim(\vec{c}) - b + 2 < \dim(\vec{c})$.

There is one small caveat. In the procedure, replacing a part a with 0 we obtain a weak composition, that is, we can introduce zero entries. Note the natural extension of the function h satisfies $h(0) = -2$. Assume that $\vec{c}'$ has a zero entry, say in its last entry, and let $\vec{c}''$ be the (weak) composition with this last entry removed. Then we have that $\dim(\vec{c}') = \sum_{i=1}^{k+1} h(c'_i) + 2(k+1) - 2 = \sum_{i=1}^{k} h(c''_i) + 2k - 2 = \dim(\vec{c}'')$. Thus zero entries can be removed without changing the dimension.

The second procedure is symmetric to the first in the two parameters a and b. That is, it takes a parts of the composition $\vec{c}$ of the form $jab + b$ and $jab + a + b$ and subtracts b from each of these a parts and adjoins a new part ab. Now we have $\dim(\vec{c}') = \dim(\vec{c}) - a + 2 < \dim(\vec{c})$, using the fact that $a \geq 3$.

Iterating these two procedures we obtain a composition which has at most $b - 1$ parts of the form $jab + a$ and $jab + a + b$, and at most $a - 1$ parts of the form $jab + b$ and $jab + a + b$. Hence this composition satisfies the condition of the statement of the proposition. Finally, one has to observe that all such composition contribute to the same dimension. $\qquad\square$

Corollary 2.17.5. *Assuming $3 \leq a < b$, $\gcd(a, b) = 1$ and that n is divisible by ab. Then the bottom reduced homology of the order complex $\triangle(\Pi_n^{\langle a,b \rangle} - \{\hat{1}\})$ is given by the permutation module $M^{B^\#(ab,\ldots,ab,ab)} = M^{B(ab,\ldots,ab,ab-1)}$.*

Proof. Now we have $r = s = 0$. Hence the compositions only have parts of the form $j \cdot ab$. The result follows from Lemma 2.13.3. $\qquad\square$

We end with a complete description in the case when $a = 2$.

Proposition 2.17.6. *Let b be odd and greater than or equal to 3. Then the ith reduced homology of $\triangle(\Pi_n^{\langle 2,b \rangle} - \{\hat{1}\})$ is given by the direct sum of Specht modules $S^{B^*(\vec{c})}$ over all compositions $\vec{c}$ with all parts congruent to 0 or 2 mod b, where exactly $(b(i+2)-n)/(b-2)$ entries of $\vec{c}$ are congruent to 2 modulo b. The bottom reduced homology occurs in dimension $\lceil n/b \rceil - 2$. Furthermore, when b divides n the bottom reduced homology is given by the permutation module $M^{B^\#(b,\ldots,b,b)} = M^{B(b,\ldots,b,b-1)}$.*

Proof. Since $a = 2$ the expression for $h(n)$ in equation (2.17.1) reduces to $h(n) = \lceil n/b \rceil - 2$ and the set A reduces to $\{n \in \mathbb{P} : n \equiv 0, 2 \bmod b\}$. Let $\vec{c}$ be a composition of n into k parts, where each part belongs to the set A. Furthermore, assume that $\vec{c}$ has s entries congruent to 2 modulo b. The contribution of $\vec{c}$ to the reduced homology of $\triangle(\Pi_n^{\langle 2,b \rangle} - \{\hat{1}\})$, given by equation (2.17.2), is in dimension

$$\dim(\vec{c}) = \sum_{i=1}^{k} h(c_i) + 2k - 2 = \sum_{i=1}^{k} \left\lceil \frac{c_i}{b} \right\rceil - 2 = \frac{\sum_{i=1}^{k} c_i + s \cdot (b-2)}{b} - 2 = \frac{n + s \cdot (b-2)}{b} - 2.$$

Solving for s in this equation yields the desired expression.

For real numbers x and y we have the inequality $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$. Hence we obtain the lower bound on the dimension of the homology: $\dim(\vec{c}) = \sum_{i=1}^{k} \left\lceil \frac{c_i}{b} \right\rceil - 2 \geq \left\lceil \frac{n}{b} \right\rceil - 2$. When b divides n the only way to obtain equality in the previous inequality is when all the parts of the composition are divisible by b. Hence the bottom reduced homology group is the direct sum over all compositions $\vec{c}$ of n where each part is divisible by b, that is, $(b, b, \ldots, b) \geq^* \vec{c}$. Hence we obtain the permutation module $M^{B^\#(b,\ldots,b,b)} = M^{B(b,\ldots,b,b-1)}$ by Lemma 2.13.3. □

2.18 Concluding remarks

With Theorem 2.11.5 we have been able to classify the action of $\mathfrak{S}_{n-1}$ on the top homology of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ for any complex $\Delta \subseteq \text{Comp}(n)$. In the case when $\triangle(\Pi_\Delta^* - \{\hat{1}\})$ is shellable, is there an EL-labeling of $\Pi_\Delta^* \cup \{\hat{0}\}$ that realizes this shelling order?

Is there a way we can classify the $\mathfrak{S}_n$-action on the homology groups of $\triangle(\Pi_\Delta^* - \{\hat{1}\})$, rather than the $\mathfrak{S}_{n-1}$-action? Browdy described the matrices representing the action of $\mathfrak{S}_n$ on the cohomology groups of the filter with block sizes belonging to the arithmetic progression $k \cdot d, (k+1) \cdot d, \ldots$; see [5, Section 5.4].

The partition lattice is naturally associated with the symmetric group, that is, the Coxeter group of type A. Miller [20] has extended the results about the filter $\Pi_{\vec{c}}^*$ to other root systems. Hence it is natural to ask if our results for the filter Π_Δ^* can be extended to other root systems.

Is there a non-pure shelling of the Frobenius complex generated by a and b? Alternatively, is there a Morse matching for this Frobenius complex such that all the critical cells are facets? While we do have this property for Λ defined by an arithmetic progression as in Section 2.16, unfortunately the general matching given in [11] does not have this property.

Lastly, all of our results are based upon Δ being a filter in the composition lattice $\text{Comp}(n)$. What if we remove the filter constraint? That is, let Ω be an arbitrary collection of compositions of n not containing the extreme composition (n). Define Q_Ω^* to be all ordered set partitions $\sigma = (C_1, C_2, \ldots, C_k)$ such that $\text{type}(\sigma) \in \Omega$ and containing n in the

last block C_k. Let Π_Ω be the image of Q_Ω^* under the forgetful map f. What can be said about the homology groups and the homotopy type of the order complex $\triangle(\Pi_\Omega)$? We need to understand the topology of the links $\mathrm{lk}_{\vec{c}}(\Omega)$, even though these links are not simplicial complexes.

Chapter 3 Box Polynomials

3.1 Box polynomials

In Chapter 3 we examine a one variable polynomial $B_{m,n}(x)$ defined using all integer partitions that fit in an m by n grid. The intersection between Chapter 2 and Chapter 3 is of course set partitions. In Section 3.2 we will see that the box polynomial $B_{m,n}$ evaluated at $-n/2$ enumerates set partitions of $m + n$ into n blocks of odd size.

   We begin in Section 3.1 by introducing the polynomials of our study. We proceed by giving alternate forms of the box polynomials and using these to prove properties about their real and complex roots.

   Lastly, for coherence reasons I mention that we were led to study these polynomials by way of the *excedance matrix*, see Section 3.4. If this chapter were presented in chronological order it would begin with Section 3.4.

   We now give the definition of the box polynomial $B_{m,n}(x)$.

Definition 3.1.1. The *box polynomial* is defined by the sum

$$B_{m,n}(x) = \sum_{\lambda \subseteq m \times n} \prod_{i=1}^{m} (x + \lambda_i),$$

where the sum is over all partitions $\lambda = (0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq n)$, that is, all partitions with m parts and each part at most n.

Example 3.1.1. Let $m = n = 2$. Using the table in Table 3.1 we see that $B_{2,2}(x) = 6x^2 + 12x + 7$.

   Another way to express the box polynomials is in terms of the complete symmetric function h_m. It follows from Definition 3.1.1 that $B_{m,n}(x) = h_m(x, x+1, \ldots, x+n)$. Directly from this expression we have that the box polynomial evaluated at $x = 0$ is given by the Stirling number of the second kind, that is,

$$B_{m,n}(0) = S(m + n, n), \tag{3.1.1}$$
$$B_{m,n}(1) = S(m + n + 1, n + 1). \tag{3.1.2}$$

Furthermore, using the complete symmetric function also yields the generating function

$$\sum_{m \geq 0} B_{m,n}(x) \cdot t^m = \frac{1}{(1 - x \cdot t) \cdot (1 - (x+1) \cdot t) \cdots (1 - (x+n) \cdot t)}. \tag{3.1.3}$$

Table 3.1: The box polynomial $B_{2,2}(x)$. The table lists all partitions λ that fit in the 2×2 box, upper left justified.

λ	$\prod_{i=1}^{m}(x + \lambda_i)$
$\emptyset$	x^2
▢	$x^2 + x$
▢▢ (vertical)	$x^2 + 2x + 1$
▢▢ (horizontal)	$x^2 + 2x$
▢▢▢	$x^2 + 3x + 2$
▢▢▢▢	$x^2 + 4x + 4$
	$B_{2,2}(x) = 6x^2 + 12x + 7$

Note that the box polynomial $B_{m,n}(x)$ has degree m and the sum defining this polynomial has $\binom{m+n}{m}$ terms, since each partition λ fitting in the m by n box can be specified uniquely by a lattice walk from $(0,0)$ to (m,n) with east and north steps. Just as the binomial coefficients satisfy the Pascal recursion we have the following recursion for the box polynomials.

Proposition 3.1.2. *The box polynomial $B_{m,n}(x)$ satisfies the recursion*

$$B_{m,n}(x) = x \cdot B_{m-1,n}(x) + B_{m,n-1}(x + 1),$$

with initial conditions $B_{m,0}(x) = x^m$ and $B_{0,n}(x) = 1$.

Proof. The initial conditions are straightforward to verify. The recursion follows the same reasoning as the Pascal recursion. Either the first part λ_1 of the partition λ is 0 or greater than or equal to 1. In the first case we have $\lambda = (0) \circ \mu$ where μ is a partition contained in a $(m-1) \times n$ box and $\circ$ denotes concatenation. Here we have $\prod_{i=1}^{m}(x + \lambda_i) = x \cdot \prod_{i=1}^{m-1}(x + \mu_i)$. Summing over all μ yields $x \cdot B_{m-1,n}(x)$. In the second case, $\lambda = (\nu_1 + 1, \nu_2 + 1, \ldots, \nu_m + 1)$ where ν is contained in a $m \times (n-1)$ box. Now $\prod_{i=1}^{m}(x + \lambda_i) = \prod_{i=1}^{m-1}(x + 1 + \nu_i)$ and summing over all ν yields $B_{m,n-1}(x + 1)$. $\square$

We continue by giving a different derivation of the box polynomials, namely as the image of the forward difference operator. We begin by defining the relevant polynomial operators. Let E be the shift operator given by $E(p)(x) = p(x + 1)$. Let Δ be the forward difference operator defined by $\Delta = E - \mathrm{Id}$, so, $\Delta(p(x)) = p(x + 1) - p(x)$. Note that the difference operator is shift invariant, that is, $\Delta \circ E = E \circ \Delta$. Finally, let $\underline{x}$ be the operator given by $\underline{x}(p(x)) = x \cdot p(x)$, that is, it multiplies by the variable x.

Lemma 3.1.3. *For all non-negative integers n, $\Delta^n \circ \underline{x} = \underline{x} \circ \Delta^n + n \cdot E \circ \Delta^{n-1}$.*

Proof. When $n = 0$ there is nothing to prove. Begin by observing that

$$\Delta(x \cdot p(x)) = (x+1) \cdot p(x+1) - x \cdot p(x) = x \cdot \big(p(x+1) - p(x)\big) + p(x+1) = \underline{x} \circ \Delta(p(x)) + E(p(x)),$$

and hence the identity holds for $n = 1$. The general case follows by induction using the case $n = 1$ in the induction step. $\qquad\square$

We now give the operator interpretation of the box polynomials.

Theorem 3.1.4. *The box polynomial $B_{m,n}(x)$ satisfies $B_{m,n}(x) = \Delta^n(x^{m+n})/n!$.*

Proof. The proof is by induction on m and n. The base case $m = 0$ or $n = 0$ is straight-forward. Using the recursion of Proposition 3.1.2 and the induction hypothesis we have that

$$\begin{aligned}
B_{m,n}(x) &= x \cdot B_{m-1,n}(x) + B_{m,n-1}(x+1) \\
&= \underline{x}(B_{m-1,n}(x)) + E(B_{m,n-1}(x)) \\
&= 1/n! \cdot \underline{x} \circ \Delta^n(x^{m-1+n}) + 1/(n-1)! \cdot E \circ \Delta^{n-1}(x^{m+n-1}) \\
&= 1/n! \cdot (\underline{x} \circ \Delta^n + n \cdot E \circ \Delta^{n-1})(x^{m+n-1}) \\
&= 1/n! \cdot \Delta^n \circ \underline{x}(x^{m+n-1}),
\end{aligned}$$

where the last step is Lemma 3.1.3, completing the induction. $\qquad\square$

A different and direct proof of Theorem 3.1.4 is as follows.

Second proof of Theorem 3.1.4. Using the relation $\Delta \underline{x} = \underline{x}\Delta + E$ each occurence of Δ in $\Delta^n \underline{x}^{m+n}$ can be moved to the right until it either cancels an $\underline{x}$ and the pair becomes a shift operator E, or it reaches all the way to the right. Because $\Delta(1)$ is zero, the last case vanishes. Since the order of the n $\underline{x}$'s that become the shift operator E does not matter, we divide by $n!$ on both sides and obtain that

$$1/n! \cdot \Delta^n \underline{x}^{m+n}(1) = \sum_{p_0+p_1+\cdots+p_n=m} \underline{x}^{p_0} E \underline{x}^{p_1} E \cdots E \underline{x}^{p_n}(1).$$

Let λ be the partition which has p_i parts equal to i. Then the term $\underline{x}^{p_0} E \underline{x}^{p_1} E \cdots E \underline{x}^{p_n}(1)$ is indeed the product $\prod_{j=1}^m (x + \lambda_i)$ and the result follows by observing that the condition $p_0 + p_1 + \cdots + p_n = m$ is equivalent to the partition λ satisfying $\lambda \subseteq m \times n$. $\qquad\square$

By the binomial theorem applied to $\Delta = E - I$, the nth power of the difference operator Δ^n is given by $\sum_{r=0}^n (-1)^{n-r} \cdot \binom{n}{r} \cdot E^r$. Therefore, the box polynomial is given by:

$$B_{m,n}(x) = \frac{1}{n!} \cdot \sum_{r=0}^n (-1)^{n-r} \cdot \binom{n}{r} \cdot (x+r)^{m+n}. \tag{3.1.4}$$

Lemma 3.1.5. *The derivative of the box polynomial $B_{m,n}(x)$ satisfies*

$$\frac{d}{dx} B_{m,n}(x) = (m+n) \cdot B_{m-1,n}(x).$$

Proof. The derivative operator $\frac{d}{dx}$ commutes with the difference operator Δ. Therefore,

$$\frac{d}{dx}B_{m,n}(x) = \frac{d}{dx}\frac{1}{n!} \cdot \Delta^n(x^{m+n}) = (m+n) \cdot \frac{1}{n!} \cdot \Delta^n(x^{(m-1)+n}) = (m+n) \cdot B_{m-1,n}(x). \quad \square$$

Alternatively, Definition 3.1.1 of the box polynomials can be used to prove Lemma 3.1.5. Let λ be a partition contained in the $m \times n$ box. Removing one arbitrary entry yields a partition μ in a $(m-1) \times n$ box. Let us write this relationship as $\lambda \sim \mu$. Note that given $\mu \subseteq (m-1) \times n$ there are $m+n$ possible partitions λ such that $\lambda \sim \mu$, as there are $m+n$ possible entries to insert in μ. Now apply the product rule to Definition 3.1.1 and change the order of summation:

$$\frac{d}{dx}B_{m,n}(x) = \sum_{\lambda \subseteq m \times n} \frac{d}{dx} \prod_{i=1}^{m}(x + \lambda_i)$$

$$= \sum_{\lambda \subseteq m \times n} \sum_{\lambda \sim \mu} \prod_{i=1}^{m-1}(x + \mu_i)$$

$$= \sum_{\mu \subseteq (m-1) \times n} \sum_{\lambda \sim \mu} \prod_{i=1}^{m-1}(x + \mu_i)$$

$$= (m+n) \cdot B_{m-1,n}(x).$$

Proposition 3.1.6. *The box polynomials satisfy $B_{m,n}(-n-x) = (-1)^m \cdot B_{m,n}(x)$. That is, $B_{m,n}(x)$ is even about $x = -n/2$ when m is even, and $B_{m,n}(x)$ is odd about $x = -n/2$ when m is odd.*

Proof. This result follows from equation (3.1.4) by substituting r to $n-r$ and pulling out the sign $(-1)^m$. Alternatively, it follows from the definition of the box polynomials by changing the partition λ to the complementary partition $\lambda^* = (n - \lambda_m, n - \lambda_{m-1}, \ldots, n - \lambda_1)$ and observing that the product $\prod_{j=1}^{m}(-n - x + \lambda_j)$ is given by $(-1)^m \cdot \prod_{j=1}^{m}(x + \lambda_j^*)$. $\quad \square$

We now use equation (3.1.4) to give yet another form for the box polynomials.

Corollary 3.1.7. *The box polynomial $B_{m,n}(x)$ has a closed form given by*

$$B_{m,n}(x) = \sum_{j=0}^{m} \binom{m+n}{j} \cdot S(m+n-j, n) \cdot x^j.$$

Proof. By equation (3.1.4) we have that the box polynomial is given by:

$$B_{m,n}(x) = \frac{1}{n!} \cdot \sum_{r=0}^{n} (-1)^{n-r} \cdot \binom{n}{r} \cdot (x+r)^{m+n}$$

$$= \frac{1}{n!} \cdot \sum_{r=0}^{n} (-1)^{n-r} \cdot \binom{n}{r} \cdot \sum_{j=0}^{m+n} \binom{m+n}{j} \cdot r^{m+n-j} \cdot x^j$$

$$= \sum_{j=0}^{m+n} \binom{m+n}{j} \cdot \frac{1}{n!} \cdot \sum_{r=0}^{n} (-1)^{n-r} \cdot \binom{n}{r} \cdot r^{m+n-j} \cdot x^j$$

$$= \sum_{j=0}^{m+n} \binom{m+n}{j} \cdot S(m+n-j, n) \cdot x^j,$$

where in the last step we used a classical identity for the Stirling numbers, see (1.2.2). Note that $S(m+n-j, n)$ is zero when $j > m$ and hence the upper bound of the sum is m. $\square$

Lemma 3.1.8. *For non-negative integers m, n_1 and n_2 we have the identity*

$$B_{m,n_1+n_2+1}(x) = \sum_{k=0}^{m} B_{k,n_1}(x) \cdot B_{m-k,n_2}(x + n_1 + 1).$$

Proof. Any partition $\lambda \subseteq m \times (n_1 + n_2 + 1)$ can be written uniquely as the concatenation of the two partitions μ and $\nu + n_1 + 1$ where $\mu \subseteq k \times n_1$, $\nu \subseteq (m-k) \times n_2$, and $\nu + n_1 + 1$ shifts every entry of ν with $n_1 + 1$. By summing over all possibilities the identity follows. $\square$

Lemma 3.1.9. *For m and n non-negative integers we have:*

$$B_{m,n-1}(-n) = (-1)^m \cdot S(m+n, n).$$

Proof. We can extend the sum over all partitions $\lambda \subseteq m \times (n-1)$ to the sum $\lambda \subseteq m \times n$ by noticing that all new terms are in fact 0. That is,

$$B_{m,n-1}(-n) = \sum_{\lambda \subseteq m \times (n-1)} \prod_{i=1}^{m} (\lambda_i - n) = \sum_{\lambda \subseteq m \times n} \prod_{i=1}^{m} (\lambda_i - n) = B_{m,n}(-n) = (-1)^m \cdot B_{m,n}(0).$$

The last step is Proposition 3.1.6 and then apply Equation (3.1.1). $\square$

3.2 Connection with set partitions

Earlier we observed that the box polynomial $B_{m,n}(x)$ evaluated at $x = 0$ and $x = 1$ yields the Stirling numbers of the second kind, which enumerates set partitions. In this section we consider other evaluations of the box polynomial that also enumerate various flavors of set partitions.

Proposition 3.2.1. *For $n \geq 2$, the box polynomial evaluated at $x = -1$, $B_{m,n}(-1)$, enumerates the number of the set partitions of the set $\{1, 2, \ldots, m+n\}$ into n blocks such that no block contains two consecutive integers and the elements 1 and $m+n$ do not belong to the same block.*

Proof. Consider the set $\{1, 2, \ldots, m+n\}$ as the congruence classes modulo $m+n$, that is, $\mathbb{Z}_{m+n}$. In other words, the element $m+n$ is followed by 1. Let A be a subset of $\{1, 2, \ldots, m+n\}$. Then the number of set partitions π of $[m+n]$ such that if i belongs to A then i and $i+1$ belong to the same block of π is given by $S(m+n-|A|, n)$, since we can first choose a set partition of $\mathbb{Z}_{m+n} - A$ and then insert $i \in A$ into the same block as $i+1$. Hence by inclusion-exclusion the desired number of set partitions is given by

$$\sum_{A \subseteq \mathbb{Z}_{m+n}} (-1)^{|A|} \cdot S(m+n-|A|, n) = \sum_{j=0}^{m+n} \binom{m+n}{j} \cdot (-1)^j \cdot S(m+n-j, n). \qquad (3.2.1)$$

Observe that when the variable j exceeds m, the associated term vanishes. Now the result follows by Corollary 3.1.7. $\qquad \square$

It is immediately clear that the box polynomial evaluated at minus one, aka the right side of Equation (3.2.1), should have an interpretation via inclusion/exclusion. While it is not difficult to prove an identity with inclusion/exclusion if you know what are you are trying to count, it can be difficult to figure out what it is you are counting. While I knew Proposition 3.2.1 should have an inclusion/exclusion proof, the idea to count set partitions avoiding i and $i+1$ in the same block came as follows.

Let $x = -1$ in Equation (3.1.3) to get a generating function for $B_{m,n}(-1)$. This generating function has been studied, namely as a Monthly problem posed by Knuth entitled "Partitions of a circular set", [17]. After matching up generating functions, the proof of Proposition 3.2.1 became clear.

The next three propositions will rely on the notion of *restricted growth words*. For a complete introduction, please see Section 4.2.

Proposition 3.2.2. *Let r be a positive integer. Then the box polynomial evaluated at $x = r$, $B_{m,n}(r)$, is the number of set partitions of $m+n+r$ into $n+r$ blocks such that the elements 1 through r all belongs to different blocks.*

Proof. Set $x = r$ in the generating function in Equation (3.1.3) to obtain

$$\sum_{m \geq 0} B_{m,n}(r) \cdot t^m = \frac{1}{(1 - r \cdot t) \cdot (1 - (r+1) \cdot t) \cdots (1 - (r+n) \cdot t)}. \qquad (3.2.2)$$

Observe that this is generating function of the number of restricted growth words of the form

$$w = 12 \cdots r \cdot u_r \cdot (r+1) \cdot u_{r+1} \cdots (r+n) \cdot u_{r+n}, \qquad (3.2.3)$$

where u_i is a word in the letters 1 through i, as is shown in Section 4.2. Furthermore, restricted growth words w, of the form shown in Equation 3.2.3, with the sum of the lengths $l(u_r) + l(u_{r+1}) + \cdots + l(u_{r+n})$ equaling m, are in bijection with set partitions of $m+n+r$ into $n+r$ blocks such that $1, 2, \ldots, r$ all belong to separate blocks. The result now follows by equating coefficients. $\qquad \square$

Proposition 3.2.2 is illustrated in Example 3.2.3 below.

Example 3.2.3. Let $m = 1, n = 2$, and $r = 2$. We evaluate the box polynomial $B_{1,2}(x)$ at 2.

Using Definition 3.1.1, $B_{1,2}(x) = (x + 0) + (x + 1) + (x + 2) = 3x + 3$, since the partitions λ fitting in the 1×2 box are the partitions $\emptyset, (1), (2)$.

Proposition 3.2.2 states that $B_{1,2}(2)$ counts sets partitions of $m + n + r = 1 + 2 + 2 = 5$ into $n + r = 2 + 2 = 4$ parts such that 1 and 2 are in separate blocks. There are $\binom{5}{2} = 10$ set partitions of 5 into 4 blocks, since each such set partition has a unique doubleton block. Since 1 and 2 need to be in different blocks, there are $10 - 1 = 9$ such set partitions. Lastly, notice that $B_{1,2}(2) = 3(2) + 3 = 9$.

Proposition 3.2.4. *Let r be a positive integer. Then the box polynomial evaluated at $x = r$, $B_{m,n}(r)$, is given by the sum*

$$B_{m,n}(r) = \sum_{i=0}^{r-1} s(r, r - i) \cdot S(m + n + r - i, r + n),$$

where $s(r, i)$ denotes the (signed) Stirling number of the first kind.

Proof. By Equation (3.2.2) we have that

$$\sum_{m \geq 0} B_{m,n}(r) \cdot t^m = \frac{(1 - t) \cdot (1 - 2 \cdot t) \cdots (1 - (r - 1) \cdot t)}{(1 - t) \cdot (1 - 2 \cdot t) \cdots (1 - (r + n) \cdot t)} \tag{3.2.4}$$

$$= \left(\sum_{i=0}^{r-1} s(r, r - i) \cdot t^i \right) \cdot \frac{1}{(1 - t) \cdot (1 - 2 \cdot t) \cdots (1 - (r + n) \cdot t)} \tag{3.2.5}$$

In Equation (3.2.5) we have used that $p(t) = (t - 1) \ldots (t - (r - 1)) = \sum_{k=0}^{r-1} s(r, k) t^k$, see [36, Proposition 1.3.7]. Therefore, the polynomial with reversed coefficients, or $t^{n-1} p(1/t)$, is given by

$$\sum_{i=0}^{r-1} s(r, r - i) \cdot t^i = (1 - t) \cdot (1 - 2 \cdot t) \cdots (1 - (r - 1) \cdot t).$$

Lastly, notice that the right side of Equation (3.2.5) is the generating function for Stirling numbers of the second kind, see Equation (1.2.3).

The coefficient of t^m follows by multiplying these two generating functions. $\qquad \square$

Proposition 3.2.5. *Let r be a positive integer such that $n \geq 2r$. In this case, the box polynomial $B_{m,n}(x)$ evaluated at the integer $-r$ enumerates set partitions into $n - r$ parts such that the minimal element of the block B_i is congruent to $i \mod 2$ for $1 \leq i \leq r$.*

Proof. Once again, set $x = -r$ in Equation (3.1.3) to obtain:

$$\sum_{m \geq 0} B_{m,n}(-r) \cdot t^m = \frac{1}{(1 + r \cdot t) \cdot (1 + (r - 1) \cdot t) \cdots (1 + t) \cdot (1 - t) \cdots (1 - (n - r) \cdot t)}. \tag{3.2.6}$$

As $n \geq 2r$, we have that $n - r \geq r$. This means we can pair terms in Equation (3.2.6) to obtain:

$$\sum_{m \geq 0} B_{m,n}(-r)t^m = \frac{1}{(1 - t^2) \cdot (1 - (2t)^2) \cdots (1 - (rt)^2) \cdot (1 - (r+1) \cdot t) \cdots (1 - (n-r) \cdot t)}.$$
(3.2.7)

The right side of Equation (3.2.7) is the generating function for restricted growth words of the form

$$w = 1u_1 2u_2 \cdots r u_r (r+1) u_{r+1} \cdots (n-r) u_{n-r},$$

such that the lengths of the u_i are even for $1 \leq i \leq r$.

Since the lengths of u_i are even for $1 \leq i \leq r$, this means that $l(1u_1 2u_2 \ldots iu_i)$ has the same parity as i. In other words, the minimal element of the $(i+1)$'st block of the partition is even if i is even and odd if i is odd, for $1 \leq i \leq r$.

If the lengths of u_j sum to m, that is $l(u_1) + \cdots + l(u_{n-r}) = m$, then the right side of Equation (3.2.7) is the generating function for set partitions of $m+n-r$ into $n-r$ parts such that the minimal element of the ith block B_i has the same parity as i, for $1 \leq i \leq r+1$. $\quad\square$

Example 3.2.6. Let $m = 1$, $n = 3$, and $r = 1$. As $3 \geq 2 \cdot 1$, Proposition 3.2.5 states that $B_{1,3}(-1)$ counts set partitions of $m + n - r = 3$ into $n - r = 2$ blocks such that the minimal element of the first block is odd and the minimal element of the second block is even. The minimal element of the first block is always 1, which is odd, thus we only need that the minimal element of the second block is even. There are two such partitions, $1|23$ and $13|2$. Lastly, $B_{1,3}(x) = 4x + 6$ and $B_{1,3}(-1) = 2$.

Remark 3.2.7. Let's look at the box polynomial $B_{m,n}(-1)$ for n at least 2. Combining Proposition 3.2.1 and Proposition 3.2.2, there is a bijection between set partitions of $m + n$ into n blocks such that i and $i+1$ are in different blocks, including 1 and n, and set partitions of $m + n - 1$ into $n - 1$ blocks such that the smallest element of the second block is even. A combinatorial proof of this bijection can be found in the solution to [36, Problem 108].

Proposition 3.2.8. *Let n be at least two. The sum*

$$\sum_{j=2}^{m+n} B_{m+n-j,j}(-1)$$

counts set partitions of $m + n$ with no singleton blocks.

Proof. Using Proposition 3.2.1 at $r = -1$, the box polynomial $B_{m,n}(-1)$ counts set partitions of $m + n$ into n blocks such that i and $i + 1$ are in different blocks, including 1 and $m + n$. Therefore, $B_{m+n-j,j}(-1)$ counts set partitions of $m+n-j+j = m+n$ into j blocks avoiding i and $i + 1$ being in the same block. The result now follows by summing over all block sizes and with [36, Problem 108 part (b)]. $\quad\square$

Example 3.2.9. Let $m = n = 2$. Proposition 3.2.8 says that

$$B_{2,2}(-1) + B_{1,3}(-1) + B_{0,4}(-1)$$

counts set partitions of [4] with no singleton blocks. The above sum, easily computed by hand with Definition 3.1.1, is 4. On the other hand, the four set partitions of [4] with no singleton blocks are 12|34, 13|24, 14|23, and 1234.

We now transition into a connection between the box polynomials and chromatic polynomials of graphs. Using the chromatic polynomial of the cycle on n-vertices a new proof of Proposition 3.2.1 is given in Proposition 3.2.11. Additionally, an enumeration of the set partitions of $[n]$ avoiding i and $i+1$ in the same block for $1 \leq i \leq n-1$, in terms of the box polynomial, is given via the chromatic polynomial of the path graph in Proposition 3.2.12.

Proposition 3.2.10. *Let G be a graph on vertex set $[n]$. Then the number of set partitions of $[n]$ into k blocks such that adjacent vertices of G are in different blocks is given by*

$$\frac{1}{k!} \cdot \Delta^k(\chi(G; x))|_{x=0},$$

where $\chi(G; t)$ is the chromatic polynomial of the graph G.

Proof. Observe that the chromatic polynomial $\chi(G; t)$ enumerates ordered set partitions into t (possibly empty) blocks where the blocks are independent sets of G. The latter partitions are ordered since any collection of t independent sets of G can be colored in $t!$ ways, and blocks can be empty as a coloring of G with k colors may not use all k colors. By inclusion-exclusion, the number of ordered sets partitions into k blocks where the blocks are non-empty independent sets is given by the alternating sum $\sum_{i=0}^{k}(-1)^{k-i} \cdot \binom{k}{i} \cdot \chi(G; i)$, since $\binom{k}{i} \cdot \chi(G; k-i)$ counts set partitions of n into k parts with blocks forming independent sets of G with at least i empty blocks.

The result follows by removing the order between the blocks, that is, dividing by $k!$. Finally, express the result in terms of the forward difference operator Δ applied k times. $\square$

Note that the empty graph, or the graph on n-vertices with no edges, has chromatic polynomial t^n. Combining this with Proposition 3.2.10 reproduces Equation (1.2.2):

$$S(n, k) = 1/k! \cdot \Delta^k(x^n)|_{x=0} = 1/k! \cdot \sum_{i=0}^{k}(-1)^{k-i} \cdot \binom{k}{i} \cdot i^n. \qquad (3.2.8)$$

Furthermore, since the chromatic polynomial of a cycle of length n is given by $\chi(C_n; t) = (t-1)^n + (-1)^n \cdot (t-1)$, we have the following consequence.

Corollary 3.2.11. *The number of set partitions of $[n]$ into $k \geq 2$ blocks such that the elements i and $i+1$ are in different blocks, including 1 and n, is given by the box polynomial $B_{n-k,k}(x)$ evaluated at $x = -1$.*

Proof. By Proposition 3.2.10 the sought after enumeration is given by:

$$\frac{1}{k!} \cdot \Delta^k\left((x-1)^n + (-1)^n \cdot (x-1)\right)|_{x=0} = \frac{1}{k!} \cdot \Delta^k\left((x-1)^n\right)|_{x=0}$$

$$= \frac{1}{k!} \cdot \Delta^k\left(x^n\right)|_{x=-1},$$

which is the box polynomial $B_{n-k,k}(x)$ evaluated at $x = -1$ by Theorem 3.1.4. Note that we have used that $\Delta^k((-1)^n \cdot (x-1)) = 0$ since $k \geq 2$ which is greater than the degree of $(x-1)$. $\square$

Note that Corollary 3.2.11 can also be shown by letting $m = n - k$ and $n = k$ in Proposition 3.2.1.

We now record one more proposition using a known chromatic polynomial, namely the chromatic polynomial for the path on n vertices, P_n.

Proposition 3.2.12. *The box polynomial $B_{n-k,k-1}(x)$ evaluated at $x = 0$, or equivalently the Stirling number $S(n-1, k-1)$, counts set partitions of n into k parts such that i and $i + 1$ are in different blocks (not including 1 and n).*

Proof. Apply Proposition 3.2.10 to the path graph, P_n, with chromatic polynomial given by $\chi(P_n; x) = x(x-1)^{n-1}$. Notice that $\chi(P_n, x) = \underline{x}(x-1)^{n-1}$, where $\underline{x}$ is the multiplication by x operator.

Since $\Delta^k \underline{x} = \underline{x}\Delta^k + k \cdot \Delta^{k-1}E$, for E the shift operator $E(p(x)) = p(x+1)$, by Proposition 3.2.10 we have:

$$\frac{1}{k!} \cdot \Delta^k(\underline{x}(x-1)^{n-1}) = \frac{1}{k!} \cdot \left(\underline{x}\Delta^k(x-1)^{n-1}|_{x=0} + k \cdot \Delta^{k-1}E((x-1)^{n-1})|_{x=0}\right)$$

$$= \frac{1}{(k-1)!}\Delta^{k-1}(x^{n-1})|_{x=0},$$

where $\frac{1}{(k-1)!}\Delta^{k-1}(x^{n-1})|_{x=0} = B_{n-k,k-1}(0)$, or equivalently the Stirling number $S(n-1, k-1)$, by Equation (3.1.1). $\square$

Remark 3.2.13. By summing over all k, Proposition 3.2.12 gives that the collection of all set partitions of $[n]$ such that i and $i + 1$ are not in the same block for $1 \leq i \leq n - 1$ is $\sum_{k=2}^{n} S(n-1, k-1) = B(n-1)$, the $n - 1$st Bell number. Once again, this reproduces the bijection of [36, Exercise 108 part (a)].

Proposition 3.2.14. *The expression $2^m \cdot B_{m,n}(-n/2)$ is the number of set partitions of a set of cardinality $m + n$ into n blocks of odd size, denoted by $T_{m+n,n}$.*

Proof. Using (3.1.4) evaluated at $x = -n/2$ yields

$$2^m \cdot B_{m,n}(-n/2) = \frac{1}{2^n \cdot n!} \cdot \sum_{r=0}^{n} (-1)^{n-r} \cdot \binom{n}{r} \cdot (2r - n)^{m+n}. \tag{3.2.9}$$

The exponential generating function for partitions with n blocks with odd cardinalities is $\sinh(x)^n/n! = (e^x - e^{-x})^n/(2^n \cdot n!)$. Using the binomial theorem and considering the coefficient of $x^{m+n}/(m+n)!$ yields the right hand side of Equation (3.2.9). $\square$

From Proposition 3.1.6 it follows that when m is odd the box polynomial $B_{m,n}$ has $-n/2$ as a root of odd multiplicity. This also follows from Proposition 3.2.14 since there are no such set partitions when m is odd. However, when m is even and greater than or equal to

2 there are at least $\binom{m+n}{n-1}$ such partitions, namely consider the set partitions with $n-1$ singleton blocks and one block of size $m+1$. Hence $-n/2$ is not a root of the box polynomial when $m \geq 2$ is even. Finally, returning to the case when m is odd, we know that the root $-n/2$ does not have multiplicity greater than 1, since this would imply by Lemma 3.1.5 that its derivative $B_{m-1,n}(-n/2)$ has a root at $-n/2$, contradicting the fact that $m-1$ is even.

We now return to discuss the number of set partitions where all the blocks have odd cardinality. We begin to express this number in terms of Stirling numbers of the second kind.

Corollary 3.2.15. *For n even, the ordinary generating function for the numbers $T_{m+n,n}$ is given by*

$$\sum_{m \geq 0} T_{m+n,n} \cdot t^m = \frac{1}{(1 - 2^2 \cdot t^2) \cdot (1 - 4^2 \cdot t^2) \cdots (1 - n^2 \cdot t^2)}.$$

Proof. By Proposition 3.2.14 and the generating function in Equation (3.1.3) we have that

$$\sum_{m \geq 0} T_{m+n,n} \cdot t^m = \sum_{m \geq 0} B_{m,n}(-n/2) \cdot (2t)^m$$

$$= \frac{1}{(1 + n/2 \cdot 2t) \cdot (1 + (n/2 - 1) \cdot 2t) \cdots (1 - (n/2 - 1) \cdot 2t) \cdot (1 - n/2 \cdot 2t)}.$$

The last step is combine factors using $(1 + k \cdot t) \cdot (1 - k \cdot t) = 1 - k^2 \cdot t^2$. □

By equating coefficients of t^m of Corollary 3.2.15. we have an immediate corollary.

Corollary 3.2.16. *For m and n both even, the number $T_{m+n,n}$ is given by the complete symmetric function*

$$T_{m+n,n} = h_{m/2}(2^2, 4^2, \ldots, n^2).$$

We will reprove this result in Chapter 4 using restricted growth words and integer walks. We now look at another consequence of Equation (3.1.3).

Corollary 3.2.17. *Let n be an even integer. Then the number of set partitions of a set of cardinality $m + n$ into n blocks of odd size is given by the following convolution of Stirling numbers of the second kind*

$$T_{m+n,n} = 2^m \cdot \sum_{k=0}^{m} (-1)^k \cdot S(k + n/2, n/2) \cdot S(m - k + n/2, n/2).$$

Proof. We factor the generating function for $2^{-m} \cdot T_{m+n,n}$ as

$$\sum_{m \geq 0} T_{m+n,n} \cdot (t/2)^m = \frac{1}{(1 - t^2) \cdot (1 - 2^2 \cdot t^2) \cdots (1 - (n/2)^2 \cdot t^2)}$$

$$= \frac{1}{(1 + t) \cdot (1 + 2 \cdot t) \cdots (1 + n/2 \cdot t)} \cdot \frac{1}{(1 - t) \cdot (1 - 2 \cdot t) \cdots (1 - n/2 \cdot t)}.$$

The second factor is the generating function for the Stirling numbers $S(m + n/2, n/2)$, see equation (1.2.3). The first factor is the generating function for $(-1)^m \cdot S(m+n/2, n/2)$. The result follows since the product of generating functions corresponds to the convolution of the coefficients. $\qquad\square$

A second proof is using Proposition 3.2.14 and Lemma 3.1.8 with $n = (n/2-1)+n/2+1$,

$$
\begin{aligned}
T_{m+n,n} &= 2^m \cdot B_{m,n}(-n/2) \\
&= 2^m \cdot \sum_{k=0}^{m} B_{k,n/2-1}(-n/2) \cdot B_{m-k,n/2}(0) \\
&= 2^m \cdot \sum_{k=0}^{m} (-1)^k \cdot S(k + n/2, n/2) \cdot S(m - k + n/2, n/2),
\end{aligned}
$$

where the last step is equation (3.1.1) and Lemma 3.1.9.

For a different approach to these set partitions using restricted growth words, see Chapter 4.

3.3 Bounds on the roots

We now discuss the location of the roots of the box polynomial $B_{m,n}(x)$.

Theorem 3.3.1. *All roots of the box polynomial $B_{m,n}(x)$ have real part $-n/2$.*

Proof. If the polynomial $p(x)$ has roots all with real part a, then the polynomial $\Delta(p(x))$ has roots with all real parts $a - 1/2$. This statement is due to Pólya [27], who stated it as an exercise which was solved by Obreschkoff [25]. (For a more general statement, see Lemma 9.13 in [28].) Applying this result n times to the polynomial x^{m+n} yields the result; see Theorem 3.1.4. $\qquad\square$

Example 3.3.2. When $n = 1$ the roots of the box polynomial $B_{m,1}(x) = (x+1)^{m+1} - x^{m+1}$ are given by

$$
-\frac{1}{2} + i \cdot \frac{1}{2} \cdot \frac{\sin\left(\frac{2\pi \cdot j}{m+1}\right)}{\cos\left(\frac{2\pi \cdot j}{m+1}\right) - 1}
$$

for $1 \leq j \leq m$. Note that the largest imaginary part is about $(m + 1)/2\pi$.

Example 3.3.3. When $n = 2$ the real roots of the box polynomial $B_{m,2}(x)$ are of the form $-1 + i \cdot u$ where $u = \frac{v}{\sqrt{1-v^2}}$ and v is a root of the equation $T_{m+2}(v) = v^{m+2}$, where the T_{m+2} is the Chebyshev polynomial of the first kind.

Example 3.3.4. For small values of m the imaginary part of the roots of the box polynomial $B_{m,n}(x)$ are listed in Table 3.2.

Theorem 3.3.5. *The imaginary parts of the roots of the box polynomial $B_{m,n}(x)$ are bounded above by mn/π and below by $-mn/\pi$.*

Table 3.2: The imaginary parts of the roots of $B_{m,n}$ for small m.

m	
1	0
2	$\pm\sqrt{n/12}$
3	$0, \pm\sqrt{n/4}$
4	$\pm\sqrt{\frac{30n\pm\sqrt{150n^2+30n}}{120}}$
5	$0, \pm\sqrt{\frac{10n\pm\sqrt{5n^2+3n}}{24}}$

Proof. Assume that $x = -n/2 + i \cdot y$ where $y \geq mn/\pi$. For $0 \leq \lambda_j \leq n$ we have that the real part of $x + \lambda_j$ lies in in the closed interval $[-n/2, n/2]$. Hence the argument of $x + \lambda_j$ is bounded by

$$\pi/2 - \frac{\pi/2}{m} < \pi/2 - \arctan\left(\frac{n/2}{mn/\pi}\right) \leq \arg(x + \lambda_j) \leq \pi/2 + \arctan\left(\frac{n/2}{mn/\pi}\right) < \pi/2 + \frac{\pi/2}{m},$$

where we used the inequality $\arctan(\theta) < \theta$ for θ positive. Thus the argument of the product $\prod_{j=1}^{m}(x + \lambda_j)$ is bounded by

$$(m - 1) \cdot \pi/2 < \arg\left(\prod_{j=1}^{m}(x + \lambda_j)\right) < (m + 1) \cdot \pi/2.$$

Hence for all partitions λ the products $\prod_{j=1}^{m}(x + \lambda_j)$ all lie in the same open half plane. Therefore their sum, which is the box polynomial $B_{m,n}(x)$, also lies in this open half plane. Thus $B_{m,n}(x)$ is non-zero, proving the upper bound. The lower bound follows by complex conjugation. $\square$

A different bound is obtained as follows.

Theorem 3.3.6. *All the roots z_j of the box polynomial $B_{m,n}(x)$ lie in the annulus with inner radius $n/2$ and outer radius $S(m + n, n) \cdot (2/n)^{m-1} \cdot \binom{m+n}{n}^{-1}$, that is,*

$$n/2 \leq |z_j| \leq \frac{S(m + n, n)}{(\frac{n}{2})^{m-1} \cdot \binom{m+n}{n}}.$$

Proof. The inner radius follows since all roots have real part $-n/2$ by Theorem 3.3.1. Let $z_1, z_2, \ldots, z_m$ be the roots of the box polynomial $B_{m,n}(x)$. Then we know that the product $(-1)^m \cdot z_1 z_2 \cdots z_m$ is the ratio of the constant term $S(m + n, n)$ over the leading term $\binom{m+n}{m}$, that is, $S(m + n, n) \cdot \binom{m+n}{m}^{-1}$. We obtain the upper bound as follows:

$$|z_j| = \prod_{k \neq j} |z_k|^{-1} \cdot S(m + n, n) \cdot \binom{m + n}{m}^{-1} \leq (2/n)^{m-1} \cdot S(m + n, n) \cdot \binom{m + n}{m}^{-1}. \quad \square$$

Proposition 3.3.7. *The inner and outer radii of the annulus in Theorem 3.3.6 are asymptotically equivalent as n tends to infinity, that is,*

$$\frac{S(m+n,n)}{(\frac{n}{2})^{m-1} \cdot \binom{m+n}{n}} \sim n/2.$$

Proof. Note that the Stirling number of the second kind $S(m+n,n)$ is given by

$$S(m+n,n) = \sum_{\substack{\lambda_1,\ldots,\lambda_k \geq 2 \\ \sum_{i=1}^{k} \lambda_i = k+m}} \binom{n+m}{\sum_{i=1}^{k} \lambda_i} \cdot p(\lambda_1,\ldots,\lambda_k),$$

where $\lambda_1,\ldots,\lambda_k$ are the cardinalities of the non-singleton blocks and $p(\lambda_1,\ldots,\lambda_k)$ does not depend on n. As a polynomial in n, the only term in this expression with maximal degree corresponds to $\lambda_1 = \cdots = \lambda_m = 2$. This corresponds to counting set partitions into m pairs and $n-m$ singleton blocks, of which there are $\binom{n+m}{2m} \cdot (2m-1)!!$. Hence the Stirling number and the leading terms are asymptotically equivalent, that is,

$$S(m+n,n) \sim \binom{n+m}{2m} \cdot (2m-1)!! \sim \frac{n^{2m} \cdot (2m-1)!!}{(2m)!} = \frac{n^{2m}}{2^m \cdot m!} \sim (n/2)^m \cdot \binom{n+m}{m},$$

where we used $m! \cdot \binom{n+c}{m} \sim n^m$ twice. The last statement is equivalent to the proposition. $\square$

Conjecture 3.3.8. *The imaginary part of the roots of the box polynomial $B_{m,n}(x)$ is bounded by $O(m \cdot \sqrt{n})$.*

3.4 The excedance matrix

The *excedance algebra* is defined as the quotient

$$\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle / (\mathbf{ba} - \mathbf{ab} - \mathbf{a} - \mathbf{b}). \tag{3.4.1}$$

It was introduced by Clark and Ehrenborg [11] and motivated by Ehrenborg and Steingrímsson's study of the excedance set statistic in [14]. For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_{n+1}$ in the symmetric group $\mathfrak{S}_{n+1}$ define its *excedance word* $u = u_1 u_2 \cdots u_n$ by $u_j = \mathbf{b}$ if $\pi_j > j$ and $u_j = \mathbf{a}$ otherwise. In other words, the letter $\mathbf{b}$ encodes where the excedances occur in the permutation. Let the bracket $[u]$ denote the number of permutations in the symmetric group with excedance word u. The bracket is the excedance set statistic and it satisfies the recursion $[u \cdot \mathbf{ba} \cdot v] = [u \cdot \mathbf{ab} \cdot v] + [u \cdot \mathbf{a} \cdot v] + [u \cdot \mathbf{b} \cdot v]$; see [14, Proposition 2.1]. This recursion is the motivation for the excedance algebra. Also note that we have the initial conditions that $[\mathbf{a} \cdot u] = [u \cdot \mathbf{b}] = [u]$ and $[1] = 1$.

Consider the polynomial $E(m,n)$ which is the sum of all $\mathbf{ab}$-words with exactly m $\mathbf{a}$'s and n $\mathbf{b}$'s. For instance, $E(2,2)$ is given by $\mathbf{aabb} + \mathbf{abab} + \mathbf{abba} + \mathbf{baab} + \mathbf{baba} + \mathbf{bbaa}$. After the quotient of equation (3.4.1) every element in the excedance algebra can be expressed in the standard basis $\{\mathbf{a}^i \mathbf{b}^j\}_{i,j \geq 0}$. Let $c_{i,j}^{m,n}$ be the coefficient of $\mathbf{a}^i \mathbf{b}^j$ in the expansion of $E(m,n)$, that is,

$$E(m,n) = \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} c_{i,j}^{m,n} \cdot \mathbf{a}^i \cdot \mathbf{b}^j.$$

Similarly for any polynomial u in the excedance algebra, define the coefficients $c_{i,j}(u)$ by

$$u = \sum_{0 \leq i,j} c_{i,j}(u) \cdot \mathbf{a}^i \cdot \mathbf{b}^j.$$

Let the excedance matrix $M(m,n)$ be the $(m+1) \times (n+1)$ matrix whose (i,j) entry is $c_{i,j}^{m,n}$ with rows and columns indexed from 0 to m, respectively, 0 to n. For instance, $M(2,2)$ is the matrix

$$M(2,2) = \begin{pmatrix} 0 & 4 & 7 \\ 4 & 14 & 12 \\ 7 & 12 & 6 \end{pmatrix},$$

since we have the expansion $E(2,2) = 6 \cdot \mathbf{aabb} + 12 \cdot \mathbf{aab} + 7 \cdot \mathbf{aa} + 12 \cdot \mathbf{abb} + 14 \cdot \mathbf{ab} + 4 \cdot \mathbf{a} + 7 \cdot \mathbf{bb} + 4 \cdot \mathbf{b}$. By symmetry we know that $M(n,m)$ is the transpose of $M(m,n)$, that is, $c_{j,i}^{n,m} = c_{i,j}^{m,n}$.

Proposition 3.4.1. *The sum over all entries of the excedance matrix $M(m,n)$ is the Eulerian number $A(m+n+1, n+1)$.*

Proof. Note that the bracket $u \longmapsto [u]$ is a linear functional on the excedance algebra. Hence the bracket $[E(m,n)]$ enumerates the number of permutations in the symmetric group $\mathfrak{S}_{m+n+1}$ with n excedances, that is, $A(m+n+1, n+1)$. By expanding $E(m,n)$ into the standard basis we have $[E(m,n)] = \sum_{i,j} c_{i,j}^{m,n} \cdot [\mathbf{a}^i \mathbf{b}^j]$, which is the sum of all the matrix entries since $[\mathbf{a}^i \mathbf{b}^j] = 1$. $\qquad \square$

We apply Lemma 2.6 of [11] to the sum of monomials $E(m,n)$ to obtain the following result.

Lemma 3.4.2. *The alternating sums of the southwest to northeast diagonals in the excedance matrix satisfy $\sum_{i+j=k} (-1)^i \cdot c_{i,j}^{m,n} = 0$ for $k < m+n$. Furthermore, the last entry is given by $c_{m,n}^{m,n} = \binom{m+n}{m}$.*

Proof. Lemma 2.6 of [11] states that if u is an **ab**-word with m **a**'s and n **b**'s, then $\sum_{i+j=k} (-1)^i \cdot c_{i,j}(u) = \delta_{m+n,0}$. Summing this results over all such monomials yields the result. $\qquad \square$

We now prove a recursion for the entries of the excedance matrix.

Proposition 3.4.3. *The entries of the excedance matrix $M(m,n)$ satisfy*

$$c_{i,j}^{m,n} = c_{i,j-1}^{m,n-1} + \sum_{k=j}^{n} \binom{k}{j} \cdot c_{i-1,k}^{m-1,n} + \sum_{k=j}^{n} \binom{k}{j-1} \cdot c_{i,k}^{m-1,n}.$$

Proof. One way to obtain the coefficient of $\mathbf{a}^i \mathbf{b}^j$, or the entry $c_{i,j}^{m,n}$ of $M(m,n)$, is by post-multiplying monomials of the form $\mathbf{a}^i \mathbf{b}^{j-1}$ by $\mathbf{b}$, which yields the first term $c_{i,j-1}^{m,n-1}$ of the proposition.

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Note that a monomial of the form $\mathbf{a}^{i-1}\mathbf{b}^k$ can yield $\mathbf{a}^i\mathbf{b}^j$ for $k \geq j$ by post-multiplication by $\mathbf{a}$. As the $\mathbf{a}$ moves past each of the k $\mathbf{b}$'s at the end of $\mathbf{a}^{i-1}\mathbf{b}^k$, we choose $k - j$ of the $\mathbf{ba}$ pairs to become $\mathbf{a}$, and all other pairs become $\mathbf{ab}$. This eliminates $k - j$ copies of $\mathbf{b}$ and no copies of $\mathbf{a}$, leaving one term of the form $\mathbf{a}^i\mathbf{b}^j$, yielding the middle sum $\sum_{k=j}^{n} \binom{k}{j} \cdot c_{i-1,k}^{m-1,n}$.

Finally, we can obtain $\mathbf{a}^i\mathbf{b}^j$ by post-multiplying a monomial of the form $\mathbf{a}^i\mathbf{b}^k$ by $\mathbf{a}$, for $k \geq j$. Note that the power of $\mathbf{a}$ is the same in $\mathbf{a}^i\mathbf{b}^j$ and $\mathbf{a}^i\mathbf{b}^k$, so, as we are post multiplying by $\mathbf{a}$, we need to eliminate one copy of $\mathbf{a}$ and $k - j$ copies of $\mathbf{b}$ as we move the $\mathbf{a}$ past the k copies of $\mathbf{b}$. Eliminating the copy of $\mathbf{a}$ must be the last step, so we choose one of the first j $\mathbf{b}$'s from the left to become an $\mathbf{a}$. Suppose we choose the l'th $\mathbf{b}$ to become an $\mathbf{a}$. Of the remaining $k - l$ $\mathbf{b}$'s to the right of the l'th $\mathbf{b}$, choose $k - j$ of them to become $\mathbf{a}$'s. This yields the coefficient $\sum_{l=1}^{j} \binom{k-l}{k-j} = \binom{k}{j-1}$ in the final sum of the proposition. $\square$

We now make certain entries of the excedance matrix $M(m, n)$ explicit.

Corollary 3.4.4. *The two entries $c_{1,0}^{m,n}$ and $c_{0,1}^{m,n}$ of the excedance matrix $M(m, n)$ are given by the Eulerian number $A(m + n - 1, n)$.*

Proof. For a polynomial v in the excedance algebra, observe that when expanding $v \cdot \mathbf{b}$ into the standard basis, there is no $\mathbf{a}$ term, that is, $c_{1,0}(v \cdot \mathbf{b}) = 0$. If we further assume that v has no constant term, we obtain $c_{1,0}(\mathbf{a} \cdot v) = 0$. Finally, Corollary 2.5 in [11] states that $c_{1,0}(\mathbf{b} \cdot v \cdot \mathbf{a}) = [v]$. (Note that their indexes are reversed, that is, our $c_{i,j}(u)$ is their $c_{m-i,n-j}(u)$.) Using the identity

$$E(m, n) = \mathbf{a} \cdot E(m - 2, n) \cdot \mathbf{a} + \mathbf{a} \cdot E(m - 1, n - 1) \cdot \mathbf{b}$$
$$+ \mathbf{b} \cdot E(m - 1, n - 1) \cdot \mathbf{a} + \mathbf{b} \cdot E(m, n - 2) \cdot \mathbf{b},$$

and applying the linear functional $u \longmapsto c_{1,0}(u)$ we obtain

$$c_{1,0}(E(m, n)) = c_{1,0}(\mathbf{b} \cdot E(m - 1, n - 1) \cdot \mathbf{a}) = [E(m - 1, n - 1)].$$

This last expression enumerates the number of permutations in the symmetric group $\mathfrak{S}_{m+n-1}$ with $n - 1$ excedances. Finally, Lemma 3.4.2 implies $c_{1,0}^{m,n} = c_{0,1}^{m,n}$. $\square$

We now come to the connection between the excedance matrix and the box polynomials.

Proposition 3.4.5. *The box polynomial $B_{m,n}(x)$ is given by $\sum_{j=0}^{m} c_{j,n}^{m,n} \cdot x^j$.*

Proof. Since we are only interested in the last column of the excedance matrix, we are only interested in terms with n $\mathbf{b}$'s. In other words, when replacing $\mathbf{ba}$ by $\mathbf{ab} + \mathbf{a} + \mathbf{b}$ we can directly throw out the term $\mathbf{a}$. That is, we replace the relation with $\mathbf{ba} = \mathbf{ab} + \mathbf{b} = (\mathbf{a} + 1) \cdot \mathbf{b}$. Iterating this relation yields

$$E(m, n) = \sum_{p_0 + p_1 + \cdots + p_n = m} \mathbf{a}^{p_0} \cdot \mathbf{b} \cdot \mathbf{a}^{p_1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot \mathbf{a}^{p_n}$$

$$= \sum_{p_0 + p_1 + \cdots + p_n = m} \mathbf{a}^{p_0} \cdot (\mathbf{a} + 1)^{p_1} \cdots (\mathbf{a} + n)^{p_n} \cdot \mathbf{b}^n.$$

Now by applying the linear functional $L(\mathbf{a}^i\mathbf{b}^n) = x^i$ we have $L(E(m, n)) = B_{m,n}(x)$ by Definition 3.1.1. $\square$

By Corollary 3.1.7 we directly have

Corollary 3.4.6. *The entries in the last column of the excedance matrix $M(m,n)$ are given by $c_{j,n}^{m,n} = \binom{m+n}{j} \cdot S(m+n-j,n)$, while the entries in the last row are given by $c_{m,j}^{m,n} = \binom{m+n}{j} \cdot S(m+n-j,m)$.*

For instance, the $(m-1,n-1)$ entry of the excedance matrix is given by

$$
\begin{aligned}
c_{m-1,n-1}^{m,n} &= c_{m-2,n}^{m,n} + c_{m,n-2}^{m,n} \\
&= \binom{m+n}{m-2} \cdot S(n+2,n) + \binom{m+n}{n-2} \cdot S(m+2,m) \\
&= \binom{m+n}{m} \cdot \frac{m \cdot (m-1)}{(n+2) \cdot (n+1)} \cdot \left(3 \cdot \binom{n+2}{4} + \binom{n+2}{3} \right) \\
&\quad + \binom{m+n}{m} \cdot \frac{n \cdot (n-1)}{(m+2) \cdot (m+1)} \cdot \left(3 \cdot \binom{m+2}{4} + \binom{m+2}{3} \right) \\
&= \binom{m+n}{m} \cdot m \cdot n \cdot \frac{3mn - m - n}{12}
\end{aligned}
$$

3.5 Concluding remarks

Is there a way to prove that the Eulerian numbers are unimodal using the excedance set statistic? One possible approach is as follows. Let $\mathcal{E}(m,n)$ be the set of all **ab**-monomials with m **a**'s and n **b**'s. Is there an injective function $\varphi : \mathcal{E}(m,n) \longrightarrow \mathcal{E}(m+1,n-1)$ for all $m < n$ such that $[u] \leq [\varphi(u)]$? If such a function φ exists, the unimodality of the Eulerian numbers follows by summing over all monomials u in $\mathcal{E}(m,n)$.

A candidate function $\varphi(u)$ is defined by factoring u as $v \cdot w$, where v has exactly one more **a** than **b**'s. Then let $\varphi(v \cdot w) = \overline{v}^* \cdot w$, where $*$ reverses the word and the bar exchanges **a**'s and **b**'s. This function works for small length words, however at length 22 there is a counterexample, namely:

$$u = \mathbf{b}^5\mathbf{ababa}^5\mathbf{bababa}^2 \cdot \mathbf{a},$$
$$\varphi(u) = \mathbf{b}^2\mathbf{ababab}^5\mathbf{ababa}^5 \cdot \mathbf{a},$$

and $[u] = 150803880738467413$ which is greater than $[\varphi(u)] = 150373062932169969$.

Chapter 4 Set Partitions into Even and Odd Parts

> You're an odd man; give even
> or give none.

Cressida
Troilus and Cressida,
IV.5.2642

4.1   Introduction

In this chapter we examine the ordinary generating function for set partitions of an $m + n$ set into n blocks of odd or even cardinality, respectfully. This work spawned from work on the box polynomials in Section 3.2. In particular, Corollary 3.2.15 and Corollary 3.2.16 are proven combinatorially using *restricted growth words*, introduced in Section 4.2, and lattice path arguments.

As this section deals largely with the Stirling numbers of the second kind, we begin with a brief review of Section 1.2.

The Stirling numbers of the second kind $S(n, k)$ enumerate partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k blocks. They satisfy the ordinary generating function identity of Equation (1.2.3), which we recall in Equation (4.1.1) below:

$$\sum_{n \geq k} S(n, k) \cdot t^{n-k} = \frac{1}{(1 - t) \cdot (1 - 2t) \cdots (1 - kt)}. \tag{4.1.1}$$

Recall that the complete symmetric function $h_m(x_1, x_2, \ldots, x_k)$ satisfies the generating function identity

$$\sum_{m \geq 0} h_m(x_1, x_2, \ldots, x_k) \cdot t^m = \frac{1}{(1 - x_1 t) \cdot (1 - x_2 t) \cdots (1 - x_k t)}.$$

The expression $S(n, k) = h_{n-k}(1, 2, \ldots, k)$ for the Stirling numbers of the second kind follows directly. For a reference on Stirling numbers see [36, Section 1.9].

Let $T_{n,k}$ and $U_{n,k}$ denote the number of set partitions of the set $[n]$ into k blocks where each block has odd, respectively even, cardinality. These numbers have been well-studied in the literature. The classical approach is via their exponential generating functions $\sinh(t)^k / k!$ and $(\cosh(t) - 1)^k / k!$ or via a more bijective route; see [10, 30], [6], respectively. We study the ordinary generating functions of these numbers using restricted growth words and multivariate generating functions.

We will use the natural bijection between partitions and restricted growth words. Our first step is to generalize (4.1.1) to a multivariate generating function. Next, by picking up the terms where all the powers are even/odd, we obtain expressions for the ordinary generating function of partitions with each block size being odd, respectively even. By viewing these

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expressions as sums over walks on the integers, we give explicit product expressions for them. Here we use homogeneous bivariate generating functions, making the proofs of the essential identities straightforward. We end with a few open questions.

4.2 Restricted growth words

A *restricted growth word*, which we abbreviate as RG-word, is a word $u = u_1 u_2 \cdots u_n$ with the entries in the positive integers such that $u_j \leq \max(0, u_1, u_2, \ldots, u_{j-1}) + 1$ for all $1 \leq j \leq n$. The notion of RG-words was introduced by Milne; see [21, 22, 23]. More recently, they appear in the papers [7, 8].

Let $RG(n, k)$ denote the set of all RG-words of length n with largest entry k. The set $RG(n, k)$ is in bijection with the set partitions of the set $\{1, 2, \ldots, n\}$ into k blocks. Namely, given an RG-word $u = u_1 u_2 \cdots u_n$, construct a partition by letting elements i and j be in the same block if $u_i = u_j$. Hence the cardinality of $RG(n, k)$ is given by the Stirling number of the second kind $S(n, k)$. We compute the RG-word for a partition $\pi \in \Pi_9$ in Example 4.2.1 below.

Example 4.2.1. Consider the set partition $\pi \in \Pi_9$ given by $\pi = 135|2|468|79$. Elements within each block are ordered least to greatest, while the blocks are ordered in increasing order of the smallest element in each block from left to right. The RG word of π is given by $u(\pi) = 121313434$. Notice that $u_4 = 3$ since 4 is in block 3 of π. Also, note that $u(\pi) \in RG(9, 4)$.

Definition 4.2.1. For an RG-word $u = u_1 u_2 \cdots u_n$ in $RG(n, k)$, let x_u be the monomial $x_1^{c_1} \cdots x_k^{c_k}$, where for all i, c_i is one less than the number of times the letter i appears in u.

In particular, Definition 4.2.1 implies that if $u = u_1 u_2 \cdots u_n$ then the total degree of x_u is $n - k$. We compute an example of a monomial x_u in Example 4.2.2 below.

Example 4.2.2. Consider π of Example 4.2.1. We computed $u(\pi) = 121313434$, and thus $x_{u(\pi)} = x_1^2 x_2^0 x_3^2 x_4^1$.

We begin by generalizing equation (4.1.1) to a multivariate version.

Theorem 4.2.3. *For a non-negative integer k the sum of the monomial of an RG-word over all RG-words with largest entry k is given by*

$$\sum_{n \geq k} \sum_{u \in RG(n,k)} x_u = \frac{1}{(1 - x_1) \cdot (1 - x_1 - x_2) \cdots (1 - x_1 - x_2 - \cdots - x_k)}.$$

Proof. Every RG-word u has a unique factorization $u = 1 \cdot w_1 \cdot 2 \cdot w_2 \cdots k \cdot w_k$, where w_i is a word with entries 1 through i. For any word w, let $x(w)$ be the monomial where the power of x_i is the number of times i appears in w. Note that x_u is given by the product $x(w_1) \cdot x(w_2) \cdots x(w_k)$. The result now follows by the sum

$$\sum_{w_i} x(w_i) = \frac{1}{1 - x_1 - x_2 - \cdots - x_i},$$

where w_i ranges over all words with entries 1 through i. $\qquad\square$

Note that by setting $x_i = q^i$ we obtain a q-analogue of equation (4.1.1) which is due to Gould [16]; see also Theorem 4.1 in [7].

Let $RG^{\text{odd}}(n, k)$ denote the set of RG-words u in which each letter occurs an odd number of times and let $RG^{\text{even}}(n, k)$ denote the set of RG-words u in where each letter occurs an even number of times. By the bijection between RG-words and set partitions we have that $|RG^{\text{odd}}(n, k)| = T_{n,k}$ and $|RG^{\text{even}}(n, k)| = U_{n,k}$.

Theorem 4.2.4. *The multivariate generating functions for $RG^{odd}(n, k)$ and $RG^{even}(n, k)$ are given by*

$$\sum_{n \geq k} \sum_{u \in RG^{odd}(n,k)} x_u = \frac{1}{2^k} \cdot \sum_{\vec{c}} F(c_1 x_1, c_2 x_2, \ldots, c_k x_k), \tag{4.2.1}$$

$$\sum_{n \geq k} \sum_{u \in RG^{even}(n,k)} x_u = \frac{1}{2^k} \cdot \sum_{\vec{c}} c_1 \cdot c_2 \cdots c_k \cdot F(c_1 x_1, c_2 x_2, \ldots, c_k x_k), \tag{4.2.2}$$

where the sums are over all vectors $\vec{c} = (c_1, c_2, \ldots, c_k) \in \{-1, 1\}^k$ and $F(x_1, x_2, \ldots, x_k)$ is the generating function in Theorem 4.2.3.

Proof. This result follows from the fact that the RG-words in $RG^{\text{odd}}(n, k)$ have monomials with all even powers, and the words in $RG^{\text{even}}(n, k)$ have monomials with all odd powers.

Let's concentrate on the first sum. Suppose $x_u = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$ is a monomial in $F(x_1, x_2, \ldots, x_k)$, the generating function of Theorem 4.2.3. Additionally, let $O_u = \{i : \alpha_i \text{ is odd}\}$. For a composition $\vec{c}$, let $O_{\vec{c}}$ denoted the collection $\{i : c_i = -1\}$. The monomial x_u has coefficient -1 in the sum $F(c_1 x_1, \ldots, c_k x_k)$ for $O_{\vec{c}} \subseteq O_u$ and $|O_{\vec{c}}|$ odd. In the same way, the monomial x_u will have coefficient $+1$ in the sum $F(c_1 x_1, \ldots, c_k x_k)$ for $O_{\vec{c}} \subseteq O_u$ and $|O_{\vec{c}}|$ even. As there is a bijection between subsets of O_u of even and odd size, the overall contribution of the monomial x_u in Equation (4.2.1) is 0 if $O_u \neq \emptyset$. If x_u has all even powers, then x_u has coefficient $+1$ in $F(c_1 x_1, \ldots, c_k x_k)$ for each composition $\vec{c} \in \{-1, 1\}^k$, and thus it contributes x_u to Equation (4.2.1) since we divide by 2^k, the number of such compositions $\vec{c}$. Therefore, Equation (4.2.1) counts monomials with all even exponents.

Analogously, Equation 4.2.2 counts monomials with all odd exponents. Mimic the argument of the previous paragraph, but let E_u denote the collection of indices i such that x_i has an even power in x_u. Now the only monomials that contribute to Equation (4.2.2) have all odd exponents. $\qquad \square$

4.3 Generating functions

Let $W_k(a)$ be the set of all one-dimensional walks of length k starting at a taking steps either -1 or 1. That is, $W_k(a) = \{(a_0, a_1, \ldots, a_k) \in \mathbb{Z}^{k+1} : a_0 = a, a_i - a_{i-1} \in \{-1, 1\}\}$. Define the generating functions $G_k(s, t)$ and $G_k^{\pm}(s, t)$ over the set of walks beginning at 0 and of length k by the sums

$$G_k(s, t) = \frac{1}{2^k} \cdot \sum_{\vec{a} \in W_k(0)} \frac{1}{(s - a_0 t) \cdot (s - a_1 t) \cdots (s - a_k t)} \tag{4.3.1}$$

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$$G_k^\pm(s,t) = \frac{1}{2^k} \cdot \sum_{\vec{a} \in W_k(0)} \frac{(-1)^{(k-a_k)/2}}{(s - a_0 t) \cdot (s - a_1 t) \cdots (s - a_k t)} \tag{4.3.2}$$

Proposition 4.3.1. *The generating functions $G_k(s,t)$ and $G_k^\pm(s,t)$ satisfy the recursions*

$$G_{k+1}(s,t) = \frac{G_k(s-t,t) + G_k(s+t,t)}{2s},$$

$$G_{k+1}^\pm(s,t) = \frac{G_k^\pm(s-t,t) - G_k^\pm(s+t,t)}{2s},$$

with the initial condition $G_0(s,t) = G_0^\pm(s,t) = 1/s$.

Proof. Observe that the substitution $s \longmapsto s - j \cdot t$ translates the sequence $(a_0, a_1, \ldots, a_k)$ j steps up, that is,

$$G_k(s - j \cdot t, t) = \frac{1}{2^k} \cdot \sum_{\vec{a} \in W_k(j)} \frac{1}{(s - a_0 t) \cdot (s - a_1 t) \cdots (s - a_k t)}. \tag{4.3.3}$$

Each walk $\vec{a} \in W_{k+1}(0)$ has $a_0 = 0$ and $a_1 = 1$ or $a_1 = -1$. Therefore, we split the summands of Equation 4.3.1 according to $a_1 = 1$ or $a_1 = -1$. The last $k + 1$ coordinates of each $\vec{a} \in W_{k+1}(0)$ is an element of $W_k(1)$ or $W_k(-1)$, and thus by Equation (4.3.3) we have that

$$G_{k+1}(s,t) = \frac{G_k(s-t,t) + G_k(s+t,t)}{2s},$$

where we divide by s since each summand of $G_{k+1}(s,t)$ has a factor or $1/s$ that the summands of $G_k(s - t, t)$ and $G_k(s + t, t)$ both lack. Additionally, we divide by 2 since each $\vec{a} \in W_k(1)$ and each $\vec{a} \in W_k(-1)$ has length k.

The same proof applies to $G_k^\pm(s,t)$ by considering the difference

$$G_k^\pm(s - t, t) - G_k^\pm(s + t, t).$$

$\square$

Proposition 4.3.2. *The generating functions $G_k(s,t)$ and $G_k^\pm(s,t)$ are given by the products*

$$G_k(s,t) = \prod_{\substack{i=-k \\ i \equiv k \bmod 2}}^{k} (s - i \cdot t)^{-1}, \tag{4.3.4}$$

$$G_k^\pm(s,t) = (2k - 1)!! \cdot t^k \cdot \prod_{i=-k}^{k} (s - i \cdot t)^{-1}. \tag{4.3.5}$$

Proof. Let $g_k(s,t)$ be the right-hand side of equation (4.3.4). We would like to prove that $G_k(s,t)$ and $g_k(s,t)$ are equal. Observe first that $G_0(s,t) = 1/s = g_0(s,t)$. Next observe that

$$\frac{1}{(s-(k+1)t)} + \frac{1}{(s+(k+1)t)} = \frac{2s}{(s-(k+1)t)(s+(k+1)t)}.$$

Multiply both sides of the latter equality by $g_{k-1}(s,t)$, yielding $g_k(s-t,t) + g_k(s+t,t) = 2 \cdot s \cdot g_{k+1}(s,t)$. This shows that $g_k(s,t)$ satisfies the same recurrence relations as $G_k(s,t)$.

Let $g_k^{\pm}(s,t)$ be the right-hand side of equation (4.3.5). We have that $G_0^{\pm}(s,t) = 1/s = g_0^{\pm}(s,t)$. Now consider the difference

$$\frac{1}{(s-(k+1)t)(s-kt)} - \frac{1}{(s+(k+1)t)(s+kt)} = \frac{2s \cdot (2k+1)t}{(s-(k+1)t)(s-kt)(s+kt)(s+(k+1)t)}.$$

Multiply both sides by $(2k-1)!! \cdot t^k \cdot \prod_{i=-k+1}^{k-1}(s-i\cdot t)^{-1}$. This yields the recursion $g_k^{\pm}(s-t,t) - g_k^{\pm}(s+t,t) = 2 \cdot s \cdot g_{k+1}^{\pm}(s,t)$. $\qquad\square$

Combining these results yields the following generating functions.

Theorem 4.3.3. *For a non-negative integer k the ordinary generating function for the number of RG-words where each entry occurs an odd or even number of times is $G_k(1,t)$ or $G_k^{\pm}(1,t)$, respectively. That is,*

$$\sum_{n\geq k} T_{n,k} \cdot t^{n-k} = \prod_{\substack{i=-k \\ i \equiv k \bmod 2}}^{k} (1-i\cdot t)^{-1},$$

$$\sum_{n\geq k} U_{n,k} \cdot t^{n-k} = (2k-1)!! \cdot t^k \cdot \prod_{i=-k}^{k} (1-i\cdot t)^{-1}.$$

Proof. In Theorem 4.2.4, set $x_1 = \cdots = x_k = t$. Recall the monomial x_u has exponents $x_i^{\alpha_i}$ where α_i is one fewer than the number of times i appears in the RG-word u. Hence the substitution $x_1 = \cdots = x_k = t$ turns x_u into t^{n-k}. Therefore, the left-hand side of Equation (4.2.1) and Equation (4.2.2) become the generating functions for the cardinality of $RG^{\mathrm{odd}}(n,k)$, respectively $RG^{\mathrm{even}}(n,k)$.

Next, under the substitution $x_1 = \cdots = x_k = t$ the right-hand side of Equation (4.2.1) becomes

$$\frac{1}{2^k} \sum_{\vec{c}} F(c_1 t, c_2 t, \ldots, c_k t)$$

$$= \frac{1}{2^k} \sum_{\vec{c} \in \{-1,1\}^k} \frac{1}{1-c_1 t} \cdots \frac{1}{1-c_1 t - c_2 t - \cdots - c_k t}.$$

Note that the sum above is $G_k(1,t)$. Therefore we have

$$\sum_{n\geq k} T_{n,k} \cdot t^{n-k} = \prod_{\substack{i=-k \\ i \equiv k \bmod 2}}^{k} (1-i\cdot t)^{-1}$$

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by Proposition 4.3.2. In the signed case we use that the sign $c_1 \cdots c_k$ is given by $(-1)^{(k-a_k)/2}$. $\square$

When k is even the generating function for $T_{n,k}$ is given by

$$\sum_{n \geq k} T_{n,k} \cdot t^{n-k} = \frac{1}{(1 - 2^2 \cdot t^2) \cdot (1 - 4^2 \cdot t^2) \cdots (1 - k^2 \cdot t^2)}.$$

Similarly, for k odd we have

$$\sum_{n \geq k} T_{n,k} \cdot t^{n-k} = \frac{1}{(1 - 1^2 \cdot t^2) \cdot (1 - 3^2 \cdot t^2) \cdots (1 - k^2 \cdot t^2)}.$$

The generating function for $U_{n,k}$ is given by

$$\sum_{n \geq k} U_{n,k} \cdot t^{n-k} = \frac{(2k-1)!! \cdot t^k}{(1 - 1^2 \cdot t^2) \cdot (1 - 2^2 \cdot t^2) \cdots (1 - k^2 \cdot t^2)}.$$

We now obtain the following expressions in terms of the complete symmetric function.

Corollary 4.3.4. *The number of RG-words with odd, respectively even, number of each entry is given by*

$$T_{n,k} = \begin{cases} h_{\frac{n-k}{2}}(2^2, 4^2, \ldots, k^2) & k \text{ even,} \\ h_{\frac{n-k}{2}}(1^2, 3^2, \ldots, k^2) & k \text{ odd,} \end{cases}$$
$$U_{n,k} = (2k-1)!! \cdot h_{\frac{n}{2}-k}(1^2, 2^2, \ldots, k^2).$$

Using the recurrence $h_m(x_1, \ldots, x_k) = x_k \cdot h_{m-1}(x_1, \ldots, x_k) + h_m(x_1, \ldots, x_{k-1})$, this corollary yields the classical recurrences for $T_{n,k}$ and $U_{n,k}$.

4.4 Concluding remarks

Is there a bijective proof of Corollary 4.3.4? Is there a multivariate refinement of Theorem 4.3.3? For instance, is there a q-analogue of this theorem?

For more information on the poset and topological structure of partitions with all blocks odd/even, see [9, 34, 41].

Chapter 5 Alpha Colored Partition Lattice and Fiber Theorems

5.1 Introduction

This chapter grew out of exercise 33 of Chapter 1 of Stanley's text [36]. The problem is to prove that if $A_n(x)$ is the classical Eulerian polynomial, then $1/2 \cdot A_n(2)$ counts the number of ordered set partitions of n, or $|Q_n|$ in the notation of this thesis. My proof of this exercise involved examining the fibers of a map from Q_n to $\mathfrak{S}_n$, which yielded a generalization, Theorem 5.3.2, by coloring blocks in Q_n in a particular manner. This chapter builds a coherent narrative around Theorem 5.3.2 by creating a colored analog of the ordered partition lattice Q_n.

We begin with a brief history of the Eulerian polynomials and a discussion of the layout of this chapter.

The Eulerian polynomials have a long and rich history in combinatorics. Euler first defined the Eulerian polynomials as the numerator for the generating function of the n'th powers, that is, the degree n polynomial satisfying $A_n(x) = (1-x)^{n+1} \sum_{k \geq 0} k^n x^k$. Equivalently, the Eulerian polynomials can be defined as a sum over descents in the symmetric group $\mathfrak{S}_n$, namely as $A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{1+d(\pi)}$, for $d(\pi)$ the number of descents of π. Yet another way to arrive at the Eulerian polynomials is as the h-polynomial of the permutahedron P_n.

In this chapter we define the α-colored ordered partition lattice, denoted Q_n^α, for α a positive integer. In Section 5.2 we introduce the poset structure of Q_n^α and give an alternate combinatorial interpretation of the poset. We then proceed in Section 5.3 to prove the α-colored analog of the fiber theorem from Chapter 1 exercise 33 of Stanley [35]. Moreover, just as the face poset of the permutahedron is the ordered set partition lattice Q_n, we construct a polytopal complex called the α-colored permutahedron with face poset Q_n^α.

5.2 Preliminaries

We begin with a discussion of the lattice of compositions of n, denoted $\mathrm{Comp}(n)$, as well as a discussion of the ordered partition lattice, Q_n. Throughout this chapter, we denote the n-set by $[n] = \{1, 2, \ldots, n\}$.

Definition 5.2.1 ($\mathrm{Comp}(n)$). Let $\mathrm{Comp}(n)$ denote the poset of ordered integer partitions of n into non-negative parts, with cover relation given by adding adjacent parts. The minimum and maximum elements of $\mathrm{Comp}(n)$ are $\hat{0} = (1, 1, \ldots, 1)$ and $\hat{1} = (n)$ respectively. For a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ we refer to c_i as the i'th part of $\vec{c}$.

Let Q_n denote the poset of ordered set partitions on n objects. In contrast to the usual partition lattice Π_n, order among the blocks in Q_n matters. The cover relation in Q_n is given by the merging of adjacent blocks.

The *type* of an ordered set partition $\tau = (B_1, B_2, \ldots, B_k)$ in Q_n is defined to be the composition of n given by the cardinality of the blocks of τ in order, or $\mathrm{type}(\tau) = (|B_1|, |B_2|, \ldots, |B_k|)$ in $\mathrm{Comp}(n)$.

Definition 5.2.2 (α-colored partition lattice)**.** Let α be a positive integer. Let Q_n^α be the collection of ordered set partitions where each block has one of α colors, with the **last block a fixed color**. The cover relationship in Q_n^α is given by the merging of adjacent blocks of the same color.

We still let the type of $\tau \in Q_n^\alpha$ to be the composition of n given by the cardinality of the blocks of τ in order, forgetting about the colors.

Remark 5.2.1. There is a clear bijection between elements of Q_n^α and ordered set partitions where we color the breaks between the blocks of the partition, namely by coloring bars between blocks the color of the block to its left and forgetting the color of the last block.

Example 5.2.2. Let $n = 5$ and $\alpha = 2$. Instead of having two colors, we will let our blocks be hatted or bald, and force our last block to be hatted. Four distinct elements of Q_5^2 are $\hat{1}\hat{2}\hat{3}|\hat{4}\hat{5}$, $123|\hat{4}\hat{5}$, $\hat{4}\hat{5}|\hat{1}\hat{2}\hat{3}$ and $45|\hat{1}\hat{2}\hat{3}$. Alternatively, using the interpretation of Remark 5.2.1 the elements of Q_n^α can be thought of as having colored bars $|$ and $\hat{|}$, yielding the respective elements $123\hat{|}45, 123|45, 45\hat{|}123$, and $45|123$. The dictionary between the two interpretations is given below:

$$\hat{1}\hat{2}\hat{3}|\hat{4}\hat{5} \longleftrightarrow 123\hat{|}45$$
$$123|\hat{4}\hat{5} \longleftrightarrow 123|45$$
$$\hat{4}\hat{5}|\hat{1}\hat{2}\hat{3} \longleftrightarrow 45\hat{|}123$$
$$45|\hat{1}\hat{2}\hat{3} \longleftrightarrow 45|123.$$

While Q_n^α is primarily introduced as a means to develop the α-colored permutahedron P_n^α, we remark that just as ordered set partitions are an important tool in the computation of the composition of ordinary generating functions, the poset Q_n^α is an indexing poset for the n-fold composition of ordinary generating functions. This suggests another interpretation of Q_n^α.

When $\alpha = 1$, Q_n^1 is the usual ordered partition lattice, which we think of as lists of sets. When $\alpha = 2$, with "colors" $|$ and $\hat{|}$ as in Example 5.2.2, then Q_n^2 can be thought of as lists of lists of sets, where $\hat{|}$ denotes a comma in an outer list and $|$ denotes a comma in an inner list. Continuing in this fashion, we can think of Q_n^3 as lists of lists of lists of sets, and so on. The use of the terminology lists of lists of sets comes from Motzkin's paper [24]. This correspondence is demonstrated in the following example.

Example 5.2.3. Let $n = 5$ and $\alpha = 2$, with bars $|$ and $\hat{|}$. Then:

$$1\hat{|}23|45 \longleftrightarrow \{1, \{23, 45\}\}$$
$$45|1\hat{|}23 \longleftrightarrow \{\{45, 1\}, 23\}$$
$$45|1|23 \longleftrightarrow \{\{45, 1, 23\}\}$$

Proposition 5.2.4. *The exponential generating function for the cardinality of the α-colored ordered partition lattice is given by*

$$\sum_{n \geq 1} \frac{|Q_n^\alpha|}{n!} x^n = \frac{e^x - 1}{1 - \alpha(e^x - 1)}.$$

Proof. We use the composition principle of exponential generating functions. The α-colored ordered partition lattice can be described as a composition of two structures on the set $[n]$, namely, an inner non-empty structure given by $e^x - 1$, and an outer α-colored permutation structure with generating function given by $x/(1 - \alpha \cdot x)$. $\square$

5.3 α-colored ordered set partitions, Eulerian polynomials, and the permutahedron.

In this section we demonstrate the close relationship between the Eulerian polynomials and the permutahedron. By means of Theorem 5.3.2, we show that the Eulerian polynomial computes the Euler characteristic of the permutahedron. The key ingredient to the proof is that the face lattice of the permutahedron P_n is the ordered set partition lattice Q_n. In Section 5.4 we will mirror the analogy between the permutahedron and the Eulerian polynomial with a new polytopal complex which we call the α-colored permutahedron,.

We now proceed with the definition of descents in the symmetric group $\mathfrak{S}_n$ and of the permutahedron.

This paragraph is a brief restatement of the content of Section 1.6. For a permutation $\pi \in \mathfrak{S}_n$, the descent set of π is given by $D(\pi) = \{i \in [n-1] : \pi(i) > \pi(i+1)\}$. We let $d(\pi)$ be the number of descents of π, or $d(\pi) = |D(\pi)|$. It will often be more advantageous to think of the descent set of π as a composition of n in the usual way, and thus we define:

Definition 5.3.1. Let the descent set of π in the symmetric group $\mathfrak{S}_n$ be given by the set $\{i_1, i_2, \ldots, i_k\}$. We convert this descent set into the *descent composition* of π by $D(\pi) = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$.

We now define the Eulerian polynomial, as in Chapter 1 of [36].

Definition 5.3.2. $A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{1+d(\pi)}$ is the *Eulerian polynomial*.

Note that $\deg(A_n(x)) = n$ since the permutation $n(n-1)(n-1)\cdots 21$ has descents at all $n-1$ possible positions, thus it contributes $x^{(n-1)+1} = x^n$ to $A_n(x)$.

Recall that the *permutahedron*, P_n, is the $(n-1)$-dimensional polytope obtained by taking the convex hull of the permutations of $\mathfrak{S}_n$ in $\mathbb{R}^n$. The d-dimensional faces of P_n are in one to one correspondence with ordered set partitions of n into $n - d$ parts. For example, the vertices of P_n are given by permutations in $\mathfrak{S}_n$, which are in bijection with ordered set partitions of n into $n = n - 0$ parts. We note that the interior of P_n, of dimension $n - 1$, must then be in bijection with ordered set partitions of n into $n - (n-1) = 1$ part. In other words, P_n is contractible.

In parallel to Proposition 5.4.5 to come in Section 5.4, we now demonstrate that the Euler characteristic of the permutahedron P_n can be computed with the Eulerian polynomial $A_n(x)$.

We do this by counting the elements of the α-colored ordered partition lattice Q_n^α with the Eulerian polynomial in Theorem 5.3.2. Recall that the *Stirling number of the second kind,* $S(n,k)$, is the number of partitions of n into k non-zero blocks.

In Corollary 5.3.1 we will use that the face lattice of the permutahedron P_n is the dual of the ordered set partition lattice $Q_n \cup \{\hat{0}\}$. With this fact, we note here that the Möbius function $\mu(Q_n \cup \{\hat{0}\})$ is easily retrieved from equation (5.3.1) using Philip Hall's Theorem 1.3.3.

Corollary 5.3.1. *The Euler characteristic of P_n can be computed from the Eulerian polynomial $A_n(x)$.*

Proof. Given that the face lattice of the permutahedron is the dual of the ordered set partition lattice yields

$$\chi(P_n) = \sum_{k=0}^{n-1}(-1)^k f_k(P_n) = \sum_{k=0}^{n-1}(-1)^k S(n,n-k)\cdot(n-k)! = 1, \qquad (5.3.1)$$

since P_n is contractible. Note by reindexing we have that

$$\chi(P_n) = \sum_{k=0}^{n}(-1)^k S(n,k)k!$$

for n even and when n is odd we raise $(-1)^{k-1}$ rather than k. Also, we use upper bound n since $S(n,0) = 0$.

Using Theorem 5.3.2 we have that:

$$\sum_{k=0}^{n} S(n,k)k!\alpha^{k-1} = (\alpha+1)^n/\alpha \cdot A_n(\alpha/(\alpha+1))$$

$$= (\alpha+1)^n/\alpha \cdot \sum_{\pi\in\mathfrak{S}_n}(\alpha/(\alpha+1))^{1+d(\pi)}$$

$$= (\alpha+1)/\alpha \cdot L + \frac{(\alpha+1)^n}{\alpha} \cdot \frac{\alpha^n}{(\alpha+1)^n}$$

$$= (\alpha+1)/\alpha \cdot L + \alpha^{n-1},$$

where L is all summands of $A_n(\alpha/(\alpha+1))$ except for the term corresponding to the reverse identity permutation, that is, the permutation with descents at all possible $n-1$ positions. Lastly, let $\alpha = -1$ to obtain $\sum_{k=0}^{n} S(n,k)k!(-1)^{k-1} = (-1)^{n-1}$. The latter sum is the Euler characteristic of P_n when n is odd, and when n is even the Euler characteristic of P_n is obtained by multiplying both sides of the latter sum by -1. In both the even and odd case, the Euler characteristic is 1. $\qquad \square$

We now proceed to the main theorem of this section. The idea of the proof of Theorem 5.3.2 is to examine the fibers of a map from the α-colored partition lattice to the symmetric group $\mathfrak{S}_n$.

Theorem 5.3.2. *The following identity holds between the Eulerian polynomial and the Stirling numbers of the second kind:*

$$\frac{(\alpha+1)^n}{\alpha} A_n\left(\frac{\alpha}{\alpha+1}\right) = \sum_{k=0}^{n} S(n,k)k!\alpha^{k-1}.$$

Proof. We construct a map $P : Q_n^\alpha \longrightarrow \mathfrak{S}_n$ given by writing out the elements of each block of an α-colored ordered set partition in increasing order, then considering this string as a permutation in one line notation in $\mathfrak{S}_n$. We will think of elements in Q_n^α as having bars with one of α colors, as discussed in Remark 5.2.1.

Let $\pi \in \mathfrak{S}_n$ such that $d(\pi) = k$. Any α-colored ordered set partition in the fiber $P^{-1}(\pi)$ must have α-colored breaks at the descents of π, but otherwise it is free to have α-colored breaks at any position. Therefore, between ascents of π we can place a bar $|$ with one of α colors, or we can not place a bar, giving us $(\alpha + 1)$ choices for building an element of the fiber $P^{-1}(\pi)$ at each ascent of π. At the descents of π we must have a bar $|$ that can have one of α colors, so we have $\alpha^{d(\pi)}$ choices in total for the color combinations of these bars.

Putting it all together, if the descent runs of π have sizes $d_1, d_2, \ldots, d_{d(\pi)+1}$, then we have that

$$\begin{aligned}
|P^{-1}(\pi)| &= (\alpha+1)^{d_1-1}(\alpha+1)^{d_2-1}\cdots(\alpha+1)^{d_{d(\pi)+1}-1}\alpha^{d(\pi)} \\
&= (\alpha+1)^{d_1+d_2+\cdots+d_{k+1}-(k+1)}\alpha^{d(\pi)} \\
&= (\alpha+1)^{n-k-1}\alpha^{d(\pi)} \\
&= (\alpha+1)^{n-d(\pi)-1}\alpha^{d(\pi)}.
\end{aligned}$$

Since P is surjective we have that the union of the fibers is Q_n^α, thus:

$$\begin{aligned}
|Q_n^\alpha| &= \sum_{\pi\in\mathfrak{S}_n} |P^{-1}(\pi)| \\
&= \sum_{\pi\in\mathfrak{S}_n} (\alpha+1)^{n-d(\pi)-1}\alpha^{d(\pi)} \\
&= (\alpha+1)^n(1/\alpha) \sum_{\pi\in\mathfrak{S}_n} (\alpha+1)^{-d(\pi)-1}\alpha^{d(\pi)+1} \\
&= (\alpha+1)^n(1/\alpha) \sum_{\pi\in\mathfrak{S}_n} (\alpha/(\alpha+1))^{1+d(\pi)} \\
&= (\alpha+1)^n(1/\alpha) A_n(\alpha/(\alpha+1)).
\end{aligned}$$

Finally, we note that the cardinality of Q_n^α is given by $|Q_n^\alpha| = \sum_{k=0}^{n} S(n,k)k!\alpha^{k-1}$, since the number of α-colored ordered set partitions into k blocks is counted by $k! \cdot S(n,k)$ times α^{k-1}, with the latter term accounting for the α possible colors of the $k-1$ breaks. $\qquad\square$

Note that Theorem 5.3.2 recovers Theorem 5.3 of [26], as Petersen [26] defines the Eulerian polynomial as $A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{d(\pi)}$. Moreover, when Theorem 5.3.2 is viewed as a polynomial in α, each side of the statement of the theorem can be interpreted as the generating function for the faces of the braid arrangement $\mathcal{H}(n)$ per Theorem 5.3 of [26].

Corollary 5.3.3. *The Eulerian polynomial A_n evaluated at $1/2$ times 2^n equals the number of ordered set partitions of n, that is, $2^n \cdot A_n(1/2) = |Q_n^1| = |Q_n|$.*

The result follows by letting $\alpha = 1$ in Theorem 5.3.2 and by the symmetry of the Eulerian polynomials. Additionally, Corollary 5.3.3 recaptures Exercise 33 of Chapter 1 of [36].

We now give a non-topological consequence of Theorem 5.3.2.

Recall Euler's generating function definition for the Eulerian polynomials:

$$\sum_{k \geq 0} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}. \tag{5.3.2}$$

Using Theorem 5.3.2 and Equation (5.3.2) we obtain the following corollary.

Corollary 5.3.4. *For $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > -1/2$ the following holds:*

$$\sum_{k=0}^{\infty} k^n \left(\frac{\alpha}{\alpha+1}\right)^k = (\alpha+1) \cdot \alpha \sum_{k=0}^{n} S(n,k) k! \alpha^{k-1}.$$

The condition $\mathrm{Re}(\alpha) > -1/2$ is needed since the power series on the left side of Equation 5.3.2 has radius of convergence 1 by the ratio test.

5.4 The construction of P_n^α

When $\alpha = 1$, Q_n^α is the usual ordered partition lattice, where the number of elements in Q_n of rank i count the $(n-i)$-dimensional faces of the permutahedron, P_n. In this section we define an analogous polytopal complex P_n^α whose $(n-i)$-dimensional faces are counted by the elements of rank i in Q_n^α. We call P_n^α the α-*colored permutahedron*, see Definition 5.4.1.

Over the course of this section we will see that P_n^α has many similarities to the usual permutahedron P_n. Namely, the face lattice of P_n^α is Q_n^α, while the face lattice of P_n is Q_n. Furthermore, Proposition 5.4.5 shows that the Euler characteristic of P_n^α can be computed with the Eulerian polynomial $A_n(x)$ in a similar fashion to Corollary 5.3.1. Lastly, the components of P_n^α are products of permutohedra, and therefore P_n^α is a union of contractible components, just as P_n is contractible.
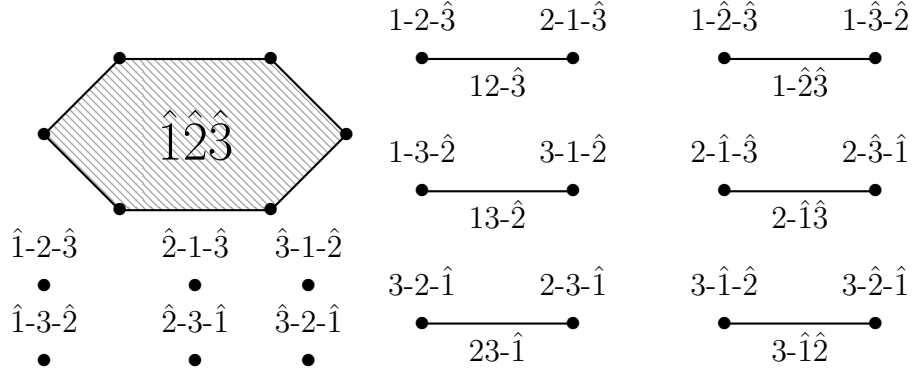
Keeping in mind that the face lattice of P_n^α should be Q_n^α, we reverse engineer the construction of P_n^α by describing the facets of the polytopal complex–each of which will determine a unique connected component of the complex.

Let $\vec{c} = (c_1, c_2, \ldots, c_k)$ be a composition of n. Recall the *multinomial coefficient* $\binom{n}{\vec{c}} = \binom{n}{c_1, c_2, \ldots, c_k}$. Notice that there are $(\alpha-1)^{k-1} \cdot \binom{n}{\vec{c}}$ elements of Q_n^α of type $\vec{c}$ with no adjacent blocks of the same color. This is because each α-colored ordered set partition of type $\vec{c}$ has blocks sizes given by c_i, and since we want no adjacent blocks of the same color, we have $\alpha - 1$ choices for the penultimate block B_{k-1}, since it can not match the fixed color of the last block. Coloring blocks from right to left in this fashion gives us that there are $(\alpha-1)^{k-1} \cdot \binom{n}{\vec{c}}$ elements of Q_n^α of type $\vec{c}$ with no adjacent blocks of the same color.

For each of these alternating color ordered partitions we define a facet $P_{|c_1|} \times P_{|c_2|} \times \cdots \times P_{|c_k|}$ of P_n^α. Since our goal is to define a polytopal complex P_n^α with face lattice Q_n^α, defining facets of P_n^α in this manner makes sense. This is because an α-colored ordered set partition

Figure 5.1: P_3^2 with "colors" bald and hatted.

$$\text{1-2-}\hat{3} \qquad \text{2-1-}\hat{3} \qquad \text{1-}\hat{2}\text{-}\hat{3} \qquad \text{1-}\hat{3}\text{-}\hat{2}$$
$$\text{12-}\hat{3} \qquad\qquad \text{1-}\hat{2}\hat{3}$$

$$\hat{1}\hat{2}\hat{3}$$

$$\text{1-3-}\hat{2} \qquad \text{3-1-}\hat{2} \qquad \text{2-}\hat{1}\text{-}\hat{3} \qquad \text{2-}\hat{3}\text{-}\hat{1}$$
$$\text{13-}\hat{2} \qquad\qquad \text{2-}\hat{1}\hat{3}$$

$$\hat{1}\text{-2-}\hat{3} \qquad \hat{2}\text{-1-}\hat{3} \qquad \hat{3}\text{-1-}\hat{2}$$

$$\text{3-2-}\hat{1} \qquad \text{2-3-}\hat{1} \qquad \text{3-}\hat{1}\text{-}\hat{2} \qquad \text{3-}\hat{2}\text{-}\hat{1}$$

$$\hat{1}\text{-3-}\hat{2} \qquad \hat{2}\text{-3-}\hat{1} \qquad \hat{3}\text{-2-}\hat{1}$$
$$\text{23-}\hat{1} \qquad\qquad \text{3-}\hat{1}\hat{2}$$

τ of type $\vec{c}$ with alternating colors is not covered by any element in Q_n^α per Definition 5.2.2, and thus should correspond to a facet of P_n^α.

Moreover, any element $\tau' \in Q_n^\alpha$ with $\tau' \leq \tau$ can be formed by splitting blocks of τ and by flipping blocks of τ of the same color. This splitting and flipping can be done independently, and since the faces of the usual permutahedron P_n are enumerated by the ordered partition lattice Q_n, the face lattice of the component $P_{|c_1|} \times P_{|c_2|} \times \cdots \times P_{|c_k|}$ will be the lower order ideal generated by its defining alternating color ordered set partition τ of type $\vec{c}$ in Q_n^α. Since our components are disjoint, this construction will yield $f(P_n^\alpha) = Q_n^\alpha$, and gives the following definition.

Definition 5.4.1 (α-colored permutahedron)**.** Let P_n^α be the polytopal complex with $(\alpha - 1)^{k-1} \cdot \binom{n}{\vec{c}}$ disjoint facets $P_{|c_1|} \times P_{|c_2|} \times \cdots \times P_{|c_k|}$ for each composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n. Each of these facets is labeled by a unique α-colored ordered partition, τ, of type $\vec{c}$ with no adjacent blocks of the same color.

The lower dimensional faces of codimension i in the facet labeled by τ are labeled by α-colored ordered set partitions in the lower order ideal generated by τ in Q_n^α into $|\tau| + i$ parts. By virtue of construction, we have that the face lattice of P_n^α is given by Q_n^α, that is $f(P_n^\alpha) = Q_n^\alpha$.

We now look at an example.

Example 5.4.1. Figure 5.1 shows P_3^2. Per Definition 5.4.1, P_3^2 has facets labeled by 2-colored ordered set partitions with alternating colors, which we mark as bald blocks and hatted blocks. Instead of a fixed last color, we force the last block of each 2-colored ordered set partition to be hatted. We now compute the facets of P_3^2 using Lemma 5.4.2. The compositions of three are $(1, 1, 1), (1, 2), (2, 1)$ and (3). Since $\binom{3}{1,1,1} = 6$, $\binom{3}{2,1} = 3$, $\binom{3}{1,2} = 3$ and $\binom{3}{3} = 1$, there are 6 facets of type $(1, 1, 1)$, 3 facets of type $(2, 1)$, 3 facets of type $(1, 2)$, and 1 facet of type (3). Each of these facets must be alternating in color with last color hatted.

We now list relevant topological properties of P_n^α.

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Lemma 5.4.2. *The number of components of P_n^α is given by*

$$\sum_{\vec{c} \in \mathrm{Comp}\,(n)} (\alpha - 1)^{|\vec{c}|-1} \cdot \binom{n}{\vec{c}}. \tag{5.4.1}$$

Recall $|\vec{c}|$ denotes the number of parts of $\vec{c}$.

The above sum enumerates all elements of Q_n^α with adjacent blocks having different colors. Since each of these partitions determine a component, the result follows.

Corollary 5.4.3. *The number of connected components of P_n^α is the total number of faces in $P_n^{\alpha-1}$.*

We can view equation (5.4.1) as summing over all $(\alpha - 1)$-colored ordered set partitions of type $\vec{c}$, and the result follows from Lemma 5.4.2.

Corollary 5.4.4. *The Euler characteristic $\chi(P_n^\alpha)$ can be counted in two ways as*

$$\sum_{k=0}^{n} (-1)^k S(n, n-k) \cdot (n-k)! \cdot \alpha^{n-k-1} = \sum_{\vec{c} \in \mathrm{Comp}\,(n)} (\alpha - 1)^{|\vec{c}|-1} \cdot \binom{n}{\vec{c}}. \tag{5.4.2}$$

Since P_n^α is a union of contractible components, Lemma 5.4.2 counts the Euler characteristic of P_n^α. We can also count the Euler characteristic by the alternating sum of the face numbers of Q_n^α, which is the left hand side of (5.4.2).

Note that letting $\alpha = 1$ in Equation (5.4.2) recaptures that the usual permutahedron is contractible, and thus has Euler characteristic 1, since composition $\vec{c}$ on the right side of Equation (5.4.2) vanishes except for $\vec{c} = (n)$. For the composition of n into one part we have $(1 - 1)^{|\vec{c}|-1} \cdot \binom{n}{n} = 0^0 \cdot 1 = 1$.

Lastly, to complete the analogy between P_n^α and the usual permutahedron P_n, we show that the Eulerian polynomial can also be used to compute the Euler characteristic $\chi(P_n^\alpha)$, just as we showed the Eulerian polynomial can be used to compute $\chi(P_n)$ in Corollary 5.3.1.

Proposition 5.4.5. *The Euler characteristic of P_n^α is given by*

$$\chi(P_n^\alpha) = \frac{(\alpha - 1)^n}{\alpha} A_n \left(\frac{\alpha}{\alpha - 1} \right).$$

Proof. This proof will mimic the proof of Theorem 5.3.2.

Consider the restriction of the map $P : Q_n^\alpha \longrightarrow \mathfrak{S}_n$ to partitions alternating in color. We denote this map P_A. Since partitions alternating in colors are the facets of P_n^α, and as each facet determines a unique connected component, the sum $\sum_{\pi \in \mathfrak{S}_n} |P_A^{-1}(\pi)|$ will give us our desired Euler characteristic.

For a fixed $\pi \in \mathfrak{S}_n$, the size of the fiber $P_A^{-1}(\pi)$ is given by

$$|P_A^{-1}(\pi)| = \sum_{\vec{d} \leq D(\pi)} (\alpha - 1)^{|\vec{d}|-1},$$

as we are allowed to add breaks between descents of π while maintaining alternating colors. Therefore,

$$\chi(P_n^\alpha) = \sum_{\pi \in \mathfrak{S}_n} |P_A^{-1}(\pi)|$$

$$= \sum_{\pi \in \mathfrak{S}_n} \sum_{\vec{d} \leq D(\pi)} (\alpha - 1)^{|\vec{d}|-1}$$

$$= (\alpha - 1)^{n-1} \sum_{\pi \in \mathfrak{S}_n} \sum_{\vec{d} \leq D(\pi)} (1/(\alpha-1))^{n-|\vec{d}|}.$$

Notice that the last sum $\sum_{\vec{d} \leq D(\pi)} (\alpha - 1)^{n-|\vec{d}|}$ is the rank generating function for the lower order ideal generated by $D(\pi) = \vec{c} = (c_1, c_2, \ldots, c_k)$ in $\mathrm{Comp}(n)$, evaluated at $\alpha - 1$. As lower order ideals in $\mathrm{Comp}(n)$ are products of Boolean algebras, we may express this inner sum as a product of rank generating functions for corresponding Boolean algebras:

$$\chi(P_n^\alpha) = (\alpha - 1)^{n-1} \sum_{\pi \in \mathfrak{S}_n} \sum_{\vec{d} \leq D(\pi)} (1/(\alpha-1))^{n-|\vec{d}|}$$

$$= (\alpha - 1)^{n-1} \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{k} F_{B_{c_i}}(1/(\alpha-1))$$

$$= (\alpha - 1)^{n-1} \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{k} (1 + 1/(\alpha-1))^{c_i-1}$$

$$= (\alpha - 1)^{n-1} \sum_{\pi \in \mathfrak{S}_n} \left(\frac{\alpha}{\alpha-1}\right)^{n-k}$$

$$= \frac{(\alpha-1)^n}{\alpha} A_n\left(\frac{\alpha}{\alpha-1}\right).$$

Observe we have used that the rank generating function of the Boolean Algebra B_n is given by $F_{B_n}(x) = (1+x)^{n-1}$. While some steps at the end of the calculation have been omitted, the reader may see Theorem 5.3.2 for similar reasoning. $\square$

A different proof of Proposition 5.4.5 uses the symmetry of the Eulerian polynomials. Namely, by Corollary 5.4.3, the Euler characteristic of P_n^α is given by the number of faces in $P_n^{\alpha-1}$, which has face poset $Q_n^{\alpha-1}$. Theorem 5.3.2 says that $|Q_n^{\alpha-1}| = \frac{\alpha^n}{\alpha-1} A_n(\frac{\alpha-1}{\alpha})$. Now use that $A_n(x) = \frac{1}{x^{n+1}} A_n(1/x)$, which is the symmetry of $A_n(x)$, to obtain Proposition 5.4.5.

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