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# Approximation of Solutions to the Mixed Dirichlet-Neumann Boundary Value Problem on Lipschitz Domains

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# APPROXIMATION OF SOLUTIONS TO THE MIXED DIRICHLET-NEUMANN BOUNDARY VALUE PROBLEM ON LIPSCHITZ DOMAINS

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DISSERTATION

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
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Lexington, Kentucky

Director: Dr. Russell Brown, Professor of Mathematics
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ABSTRACT OF DISSERTATION

APPROXIMATION OF SOLUTIONS TO THE MIXED DIRICHLET-NEUMANN BOUNDARY VALUE PROBLEM ON LIPSCHITZ DOMAINS

We show that solutions to the mixed problem on a Lipschitz domain $\Omega$ can be approximated in the Sobolev space $H^1(\Omega)$ by solutions to a family of related mixed Dirichlet-Robin boundary value problems which converge in $H^1(\Omega)$, and we give a rate of convergence. Further, we propose a method of solving the related problem using layer potentials.

KEYWORDS: Mixed problem, Boundary value problem, Lipschitz domain, Layer potentials, Approximation

Author's signature: Morgan Schreffler

Date: May 3, 2017

# APPROXIMATION OF SOLUTIONS TO THE MIXED DIRICHLET-NEUMANN BOUNDARY VALUE PROBLEM ON LIPSCHITZ DOMAINS

By
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Date: May 3, 2017

To Kate: The 284 to my 220.

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## 1.1 Statement and History of the Mixed Problem

Let $\Omega \subseteq \mathbf{R}^d$ be a bounded Lipschitz domain, $D \subseteq \partial\Omega$ a relatively open subset of $\partial\Omega$, $N = \partial\Omega \setminus \bar{D}$ the complementary open set in $\partial\Omega$, and $\Lambda = \bar{D} \cap \bar{N}$ the shared boundary of $D$ and $N$, and consider the mixed problem, also known as Zaremba's problem, with homogeneous Dirichlet data for the Laplacian:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } D \\ \frac{\partial u}{\partial \nu} = g & \text{on } N. \end{cases} \tag{MP}$$

For $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $-\Delta u := -\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}$ denotes the Laplacian and $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ denotes the normal derivative of $u$, where $\nu$ is the outward-pointing unit normal vector field on $\partial\Omega$. On a Lipschitz domain $\Omega$, $\nu$ is defined $\sigma$-a.e. on $\partial\Omega$, where $\sigma$ denotes $(d-1)$-dimensional surface measure.

In 1977, Dahlberg [7] studied the pure Dirichlet problem on Lipschitz domains, and in 1981 Jerison and Kenig [16] extended these results to more general elliptic operators and obtained regularity results. In 1979 Dahlberg [8] published a survey of his results up to that point pertaining to the pure Dirichlet problem for the Laplacian. Verchota [40] studied the Dirichlet problem with data having one derivative in $L^p(\partial\Omega)$ in 1984 by appealing to the method of layer potentials, thus extending the results of Jerison and Kenig [17]. Dahlberg, Kenig and Verchota [6] obtained results in 1986 analogous to those of Dahlberg [7] on the Dirichlet problem for the so-called biharmonic operator $\Delta^2$.

Regularity results on the pure Neumann problem for the Laplacian on Lipschitz domains were first studied in 1981 by Jerison and Kenig [17] in the case where the Neumann data is in $L^2(\partial\Omega)$. In the same paper, the authors also studied regularity for the Dirichlet problem with data having one derivative in $L^2(\partial\Omega)$. In 1987 Dahlberg and Kenig [9] considered optimal conditions for the solvability of the pure Neumann problem on a Lipschitz domain when the data lies in $L^p(\partial\Omega)$, including endpoint results involving Hardy spaces on $\partial\Omega$.

In lecture notes published in 1994, Kenig [19] states the $L^p$ mixed problem as an open problem. Indeed, certain difficulties arise when studying the mixed problem, even when $\Omega$ is smoother than Lipschitz. For example, let $\Omega \subset \mathbf{R}^2$ be a bounded smooth domain in the upper half plane whose boundary contains the segment $[-1,1] \times \{0\}$, choose $D, N \subseteq \partial\Omega$ which satisfy $[-1,0) \times \{0\} \subseteq D$ and $(0,1] \times \{0\} \subseteq N$, and set $u(x_1,x_2) := \mathrm{Re}(\sqrt{z}) = r^{\frac{1}{2}} \cos\frac{\theta}{2}$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\tan\theta = \frac{x_2}{x_1}$. Since $u$ is the real part of a function holomorphic on $\mathbf{C} \setminus \{0\}$, we have $-\Delta u = 0$ on $\Omega$. Further, $u$ satisfies the boundary conditions $\frac{\partial u}{\partial \nu} = 0$ on $N \cap ([-1,1] \times \{0\})$ and $u = 0$ on $D \cap ([-1,1] \times \{0\})$. However, $\frac{\partial u}{\partial \nu}$ and $\nabla u$ are not in $L^p(\partial\Omega)$ for any $p \geq 2$. Indeed, on $D \cap ([-1,1] \times \{0\})$ we have

$$\frac{\partial u}{\partial \nu} = -\frac{1}{2}|x_1|^{-\frac{1}{2}} - |\nabla u|,$$

hence, for any $0 < \varepsilon < 1$ the integral

$$\int_{D\cap([-\varepsilon,\varepsilon]\times\{0\})} \left|\frac{\partial u}{\partial \nu}\right|^p \, d\sigma \geq \frac{1}{4}\int_0^{\varepsilon} \frac{1}{x^{p/2}}\, dx$$

diverges. We can also construct for any $p > 2$ domains where the mixed problem has no solution if we insist on $|\nabla u|$ being in $L^p(\partial\Omega)$ even non-tangentially (cf. Kenig [18] for relevant examples). In 1997, Savaré [34] showed that solutions of the homogeneous mixed problem are, however, in the Besov space $B^{3/2}_{2,\infty}(\Omega)$. In this same paper, Savaré studies the effects of perturbation of the set $D$ on solutions of the mixed problem.

Since 1994, much work has been done regarding the $L^p$ mixed problem. In 1994, Brown [2] gave mild conditions on $\partial\Omega$ and the Dirichlet data for which $\Omega$ admits a solution $u$ to the mixed problem satisfying $\nabla u \in L^2(\partial\Omega)$ non-tangentially. Under the same restrictions on $\partial\Omega$, Sykes and Brown [37] prove existence and uniqueness of solutions for $1 < p < 2$, assuming the Dirichlet data and Neumann data are elements of $L^{1,p}(D)$ and $L^p(N)$, respectively. The results of Brown [2] and Sykes and Brown [37] are valid when $d \geq 3$. Lanzani, Capogna, and Brown [21] extended these results to hold when $d = 2$, but only in Lipschitz hypergraphs with Lipschitz constant less than 1. In 2012, Ott and Brown [30] give sufficient conditions on $D$, $N$ and $\Lambda$ so that for a general bounded Lipschitz domain $\Omega$ and an exponent $p_0 > 1$ depending on $\Omega$, existence and uniqueness of solutions to the $L^p$ mixed problem is guaranteed when $p \in (1, p_0)$. In particular, Ott and Brown [30] require that $\Lambda$ locally be the graph of a Lipschitz function $\varphi : \mathbf{R}^{d-2} \to \mathbf{R}$. In 2013, Taylor, Ott and Brown [38] improve the previous result by replacing the condition on $\Lambda$ with an Ahlfors regularity condition which is less restrictive.

### 1.2 Formulation and History of the Approximate Mixed Problem

Formally, (MP) can be written with a single boundary condition

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ \chi_N \frac{\partial u}{\partial \nu} + \chi_D u = \chi_N G & \text{on } \partial\Omega, \end{cases}$$

where $\chi_D$ and $\chi_N$ are the characteristic functions of $D$ and $N$, respectively, and $G$ satisfies $G|_N = g$. Now, suppose for some small $\varepsilon > 0$ we set $\chi_N^\varepsilon$ to be a continuous approximation of $\chi_N$ which is 0 on $D$, non-zero on $N$, and 1 when $\operatorname{dist}(x, D) > \varepsilon$, and set $a = 1 - \chi_N^\varepsilon$. The questions we look to answer are as follows: Do solutions of the Robin problem converge to solutions of the mixed problem in some function space as $\varepsilon \to 0$? If so, at what rate? Also, is there any advantage to using this method of approximation?

Before answering these questions, we first formulate the problem more rigorously. If we once again impose the Dirichlet boundary condition $u = 0$ on $D$, we can divide both sides of the boundary condition by $\chi_N^\varepsilon$ on $N$, leaving us with the so-called *approximate mixed problem*

$$\begin{cases} -\Delta u_\varepsilon = f & \text{on } \Omega \\ u_\varepsilon = 0 & \text{on } D \\ \frac{\partial u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon = g & \text{on } N, \end{cases} \tag{AMP}$$

where $a_\varepsilon := \frac{1-\chi_N^\varepsilon}{\chi_N^\varepsilon}$. Note that $u_\varepsilon$ now depends on the parameter $\varepsilon$, and satisfies a Robin boundary condition on $N$.

In the case $d \geq 3$, Medková [27] considers the *Robin problem*

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + bu = f & \text{on } \partial\Omega \end{cases}$$

when $\partial\Omega$ is piecewise smooth, obtaining a solution by layer potentials under certain conditions on $b$ and $f$. Lanzani and Shen [23] obtain existence and uniqueness of solutions to the $L^p$ *Robin problem*

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + bu = f \in L^p(\partial\Omega) & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p(\partial\Omega) \end{cases}$$

for $1 < p \leq 2$ and $b$ a given non-negative function on $\partial\Omega$ satisfying $b \in L^{d-1}(\partial\Omega) \cap L^q(\partial\Omega)$ for some $q > 2$. Here $(\nabla u)^*$ is the non-tangential maximal function of $\nabla u$, defined for $x \in \partial\Omega$ by

$$(\nabla u)^*(x) := \sup_{y \in \Gamma(x)} |\nabla u(y)|,$$

where

$$\Gamma(x) := \{z \in \Omega \,:\, |x - z| < 2\,\mathrm{dist}(z, \partial\Omega)\}$$

is the *interior non-tangential approach region*. Lanzani and Mendez [22] give corresponding results for the inhomogeneous equation $-\Delta u = g$, and Agranovich [1] extends these results to general strongly elliptic systems of equations. In most of the above publications, the Robin function $b$ is assumed to be at least in $L^{d-1}(\partial\Omega)$. One of the novelties of our results is that $a_\varepsilon$ is not in general an element of $L^{d-1}(N)$. In certain cases we have $a_\varepsilon \in L^p(\partial\Omega)$ for at least one $p > 1$, but in others we do not even have $a_\varepsilon \in L^1(\partial\Omega)$. In the latter cases, certain layer-potential results which hold for the former no longer apply.

Later we shall see that when $a_\varepsilon \in L^p(\partial\Omega)$ for some $p > 1$, (AMP) is a compact perturbation of (MP). As far as can be easily determined, many similar perturbation problems appear in the literature, none of which are precisely (AMP). In 1996, Costabel [5] studies a similar problem on smooth domains in $\mathbf{R}^2$:

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega \\ \frac{\partial u_\varepsilon}{\partial nu} = 0 & \text{on } N \\ \varepsilon \frac{\partial u_\varepsilon}{\partial \nu} + u_\varepsilon = g & \text{on } D. \end{cases}$$

The author is able to determine an asymptotic expansion of $u_\varepsilon$ in terms of $\varepsilon$, as well as estimates near $\Lambda = \{c_1, c_2\}$ which describe how $u_\varepsilon$ approximates the singularities in the limiting mixed problem. Given operators $A, B : X \to X$ on a Banach space $X$, Friedman [12] considers equations of the form

$$\varepsilon A u_\varepsilon + B u_\varepsilon = f_\varepsilon,$$

and discusses the rate at which solutions $u_\varepsilon$ converge to a solution $u_0 \in X$ of the equation $Bu_0 = f_0$ as $f_\varepsilon \to f_0$. Paltsev [31] studies equations with mixed Dirichlet and Robin data of the form

$$\begin{cases} Lu(x) + \mu^2 q(x) = f(x) & \text{on } \Omega, \\ u = g & \text{on } D, \\ \frac{\partial u}{\partial \nu} + bu = h & \text{on } N, \end{cases}$$

where $L$ is an elliptic operator in divergence form, $\mu \in \mathbf{C}$ is a parameter which may have large modulus, $q_0$ is a given constant, and $q(x) \geq q_0 > 0$ is a given function. However, the effect of $\mu$ on solutions $u$ is only discussed in the sense that large $\mu$ results in a rapid rate of convergence for iterative methods of approximating $u$.

### 1.3 Main Results

Our first result is a Hardy inequality on $H^1_D(\Omega)$ which holds when $D$ satisfies the corkscrew condition (2.3) defined in Section 2.1 (cf. Hardy [15] and Grisvard [14]).

**Lemma 1.1** (Hardy Inequality)**.** *Suppose $D$ satisfies the corkscrew condition* (2.3)*. If $u \in H^1_D(\Omega)$, then there is a constant $C$ depending only on $d$ and $D$ such that*

$$\int_N \frac{(\operatorname{Tr} u)^2}{\delta}\, d\sigma \leq C \int_\Omega |\nabla u|^2\, dx.$$

Though not difficult to prove, Lemma 1.1 proves to be the key estimate for studying (AMP). In particular, it implies the following existence and uniqueness theorem based on the famous lemma of Lax and Milgram [24].

**Theorem 1.2** (Existence and Uniqueness)**.** *Suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition* (2.3)*. Let $f \in H^{-1}_D(\Omega)$ and $g \in \widetilde{H}^{-1/2}(N)$, and let $a_\varepsilon$ be a standard family of functions. Take $a_0$ to be identically 0 on $\partial\Omega$. There is an $\varepsilon_0 > 0$ and a constant $C$ not depending on $\varepsilon$ such that for each $0 \leq \varepsilon < \varepsilon_0$, a unique weak solution $u_\varepsilon \in H^1_D(\Omega)$ of* (AMP) *(or* (MP) *in the case $\varepsilon = 0$) exists which satisfies*

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C \left( \|f\|_{H^{-1}_D(\Omega)} + \|g\|_{H^{-1/2}(N)} \right). \tag{1.1}$$

Under some extra regularity assumptions on $f$, $g$, and $\Lambda$, we obtain our next main result at the end of Chapter 3, which is an upper bound on the rate at which solutions of (AMP) converge to those of (MP).

**Theorem 1.3.** *Suppose $D$ satisfies the corkscrew condition* (2.3)*, $\Lambda$ satisfies the Ahlfors regularity condition* (2.4)*, $a_\varepsilon$ is a standard family of functions, $f \in L^q(\Omega)$ for some $q > \frac{d}{2}$, and $g \in L^p(N)$ for some $p > d-1$. If $u_\varepsilon \in H^1_D(\Omega)$ is a weak solution of* (AMP)*, $u_0 \in H^1_D(\Omega)$ is a weak solution of* (MP)*, and $0 < \varepsilon < \varepsilon_0$, then there is a constant $C$ not depending on $\varepsilon$ such that*

$$\|u_\varepsilon - u_0\|_{H^1(\Omega)} \leq C\varepsilon^{1+\alpha}.$$

Note that Theorem 1.3 essentially answers the first two questions asked immediately before stating AMP rigorously: *Do solutions of the Robin problem converge to solutions of the mixed problem in some function space as $\varepsilon \to 0$? If so, at what rate?* Its proof relies on the fact that weak solutions of (AMP) are globally Hölder continuous *uniformly in $\varepsilon$*. Interior estimates were proven independently by Ennio De Giorgi [10] and John Nash [29] in 1957, and again by Jürgen Moser [28] in 1960. Also in 1960, Stampacchia [35] applied the method of De Giorgi [10] and Nash [29] to prove that weak solutions to (MP) are in fact Hölder continuous up to the boundary. A.F.M. ter Elst and Rehberg [39] give a thorough treatment of Hölder continuity of solutions to (MP) under more general conditions on $\Omega$ and $D$ than what we use.

Our final main results concern integral representation of $u_\varepsilon \in H^1_D(\Omega)$. Theorem 1.4 says that if we can solve a certain system of boundary integral equations, then we have a representation formula for weak solutions of (MP) and (AMP), and Theorem 1.5 confirms that the aforementioned system is in fact uniquely solvable.

**Theorem 1.4.** *Let $f \in H^{-1}_D(\Omega^-)$, $g \in H^{-1/2}(N)$, and $a_\varepsilon$ a standard family of functions. Set $a_\varepsilon$ to be identically 0 when $\varepsilon = 0$. Choose $\Gamma_D \in \widetilde{H}^{1/2}(N)$ and $\Gamma_N \in H^{-1/2}(\partial\Omega)$ with $g := \Gamma_N|_N$. Define $\boldsymbol{h}_\varepsilon = \begin{bmatrix} h_{D,\varepsilon} \\ h_{N,\varepsilon} \end{bmatrix} \in \boldsymbol{H}^*_\varepsilon$ by*

$$h_{D,\varepsilon} := \left(-\operatorname{Tr}\mathcal{G}f + \mathcal{A}(a_\varepsilon\Gamma_D - \Gamma_N) - \frac{1}{2}(\Gamma_D - \mathcal{C}\Gamma_D)\right)\bigg|_{D_\varepsilon}, \quad \textit{and}$$
$$h_{N,\varepsilon} := \left(-\frac{\partial^-}{\partial\nu}\mathcal{G}f + \frac{1}{2}\big(g + a_\varepsilon\Gamma_D + \mathcal{B}(a_\varepsilon\Gamma_D - \Gamma_N)\big) - \mathcal{D}\Gamma_D\right)\bigg|_N.$$

*For fixed $\varepsilon$, $0 \leq \varepsilon < \varepsilon_0$, if $\boldsymbol{\psi}_\varepsilon := \begin{bmatrix} \psi_{D,\varepsilon} \\ \psi_{N,\varepsilon} \end{bmatrix} \in \boldsymbol{H}_\varepsilon$ solves the system of integral equations*

$$\boldsymbol{A}_\varepsilon\,\boldsymbol{\psi}_\varepsilon = \boldsymbol{h}_\varepsilon, \tag{1.2}$$

*then the weak solution $u_\varepsilon \in H^1_D(\Omega^-)$ of* (AMP) *has integral representation*

$$u_\varepsilon = \mathcal{G}f + \operatorname{SL}(\psi_{D,\varepsilon} - a_\varepsilon\psi_{N,\varepsilon} + \Gamma_N - a_\varepsilon\Gamma_D) - \operatorname{DL}(\psi_{N,\varepsilon} + \Gamma_D) \quad \textit{on } \Omega^-.$$

*Conversely, if $u_\varepsilon \in H^1_D(\Omega^-)$ solves* (AMP)*, then $\boldsymbol{\psi}_\varepsilon := \begin{bmatrix} \psi_{D,\varepsilon} \\ \psi_{N,\varepsilon} \end{bmatrix} \in \boldsymbol{H}_\varepsilon$ given by*

$$\psi_{D,\varepsilon} := \frac{\partial^- u_\varepsilon}{\partial\nu} + a_\varepsilon \operatorname{Tr}^- u_\varepsilon - \Gamma_N \quad \textit{and} \quad \psi_{N,\varepsilon} := \operatorname{Tr}^- u_\varepsilon - \Gamma_D$$

*solves the system* (1.2)*.*

**Theorem 1.5.** *Let $0 \leq \varepsilon < \varepsilon_0$. If $a_\varepsilon \in L^p(\partial\Omega)$ for some $p > 1$, the map $\boldsymbol{A}_\varepsilon : \boldsymbol{H}_\varepsilon \to \boldsymbol{H}_\varepsilon^*$ has a bounded inverse $\boldsymbol{A}_\varepsilon^{-1} : \boldsymbol{H}_\varepsilon^* \to \boldsymbol{H}_\varepsilon$, and the system $\boldsymbol{A}_\varepsilon \boldsymbol{\psi}_\varepsilon = \boldsymbol{h}_\varepsilon$ has a unique solution for each $\boldsymbol{h}_\varepsilon \in \boldsymbol{H}_\varepsilon$.*

**Chapter 2 Notation and Weak Formulation of the Approximate Mixed Problem**

In this chapter, we develop the notation and preliminary results required to state precisely the weak formulations of (MP) and (AMP). Section 2.1 deals with the geometric requirements on $\Omega$, $D$, and $N$. In Section 2.2 we define Sobolev spaces on $\mathbf{R}^d$, $\Omega$, and subsets of $\partial\Omega$). Of particular interest are the spaces $H^1_D(\Omega)$, $\widetilde{H}^{1/2}(\partial\Omega)$, and their duals. Section 2.2 concludes with our main estimate, a Hardy inequality on $N$. We wrap up the chapter with Section 2.3, in which we give the weak formulations of (MP) and (AMP), and prove existence and uniqueness of weak solutions to both.

## 2.1 Lipschitz Domains

For $x = (x', x_d) \in \mathbf{R}^{d-1} \times \mathbf{R}$, $M > 0$, and $r > 0$, define coordinate cylinders

$$Z_r(x) := \{(y', y_d) \in \mathbf{R}^{d-1} \times \mathbf{R} \,:\, |y' - x'| < r,\ |y_d - x_d| < (1 + 2M)r\}.$$

We say $\Omega \subset \mathbf{R}^d$ is a Lipschitz domain with constant $M$ if $\partial\Omega$ is compact and for each $x \in \partial\Omega$ there is an $r_x > 0$ and a Lipschitz function $\psi_x : \mathbf{R}^{d-1} \to \mathbf{R}$ with constant $M$ such that (after a possible rigid change of coordinates)

$$\Omega_{r_x}(x) := \Omega \cap Z_{r_x}(x) = \{(y', y^d) \in \mathbf{R}^{d-1} \times \mathbf{R} \,:\, y^d > \psi_x(y')\} \cap Z_{r_x}(x), \quad \text{and} \quad (2.1)$$

$$\Psi_{r_x}(x) := \partial\Omega \cap Z_{r_x}(x) = \{(y', y^d) \in \mathbf{R}^{d-1} \times \mathbf{R} \,:\, y^d = \psi_x(y')\} \cap Z_{r_x}(x). \quad (2.2)$$

Since $\partial\Omega$ is compact, we can always choose a collection of finitely many cylinders $\{Z_{r_0}(x_j)\}_{j=1}^J$ and a corresponding collection $\{\psi_{x_j}\}_{j=1}^J$ of Lipschitz functions satisfying $\partial\Omega = \bigcup_{j=1}^J \Psi_{r_0}(x_j)$, where $r_0$ is chosen small enough so that conditions (2.1) and (2.2) are still met by $\Omega_{4r_0}$ and $\Psi_{4r_0}$, respectively. Note that the measure of a coordinate cylinder is $2(1 + 2M)\omega_{d-1} r^d$, where $\omega_{d-1}$ is the measure of the unit ball in $\mathbf{R}^{d-1}$. When an estimate depends on the specific choice of cylinders and Lipschitz functions in addition to the Lipschitz constant $M$ and the constants in (2.3) and (2.4) below, we say that our estimate depends on the *global character of* $\Omega$.

*Remark.* Since $\partial\Omega$ is compact, $\mathbf{R}^d \setminus \partial\Omega$ may be unbounded, though it can have at most one unbounded component. In Chapter 3 we work almost exclusively with bounded Lipschitz domains, but in Chapter 4 we discuss both bounded and unbounded domains. When a distinction must be made, we let $\Omega^-$ denote an arbitrary *bounded* Lipschitz domain, and $\Omega^+ := \mathbf{R}^d \setminus \overline{\Omega^-}$ the complementary unbounded domain.

*Remark.* By Rademacher's theorem [32] a Lipschitz function $\psi : \mathbf{R}^{d-1} \to \mathbf{R}$ with constant $M$ is differentiable a.e. with $|\nabla\psi| \leq M$. Hence, $\partial\Omega$ has a well-defined surface measure $\sigma$ and for $\sigma$-a.e. $x \in \partial\Omega$ there is a well-defined tangent plane to $\Omega$ with unit normal vectors $\pm\nu$. We shall adopt the convention of choosing $\nu$ to point *out of* $\Omega^-$ and *into* $\Omega^+$.

Let $D \subseteq \partial\Omega$ be nonempty and relatively open in $\partial\Omega$, $N := \partial\Omega \setminus \bar{D}$, and $\Lambda = \bar{D} \cap \bar{N}$ the shared boundary of $D$ and $N$. Set $\delta(x) := \operatorname{dist}(x, \Lambda)$. We say that the set $D$ satisfies the corkscrew condition if there are constants $C > 1$ and $R_0 > 0$ such that

$$\forall\, \ell \in \Lambda,\ 0 < r < R_0,\ \exists\, x \in D \text{ such that } |x - \ell| < r \text{ and } \delta(x) > \frac{r}{C}. \tag{2.3}$$

We say $\Lambda$ is Ahlfors $(d-2)$-regular if there are constants $C \geq 1$ and $R_0 > 0$ such that

$$\frac{1}{C} r^{d-2} \leq \mathcal{H}^{d-2}(B_r(\ell) \cap \Lambda) \leq C r^{d-2} \quad \forall\, \ell \in \Lambda \text{ and } 0 < r < R_0, \tag{2.4}$$

where $\mathcal{H}^{d-2}$ denotes $(d-2)$-dimensional Hausdorff measure and $B_r(\ell)$ is a ball of radius $r$ centered at $\ell$. We will always indicate when conditions (2.3) and (2.4) are necessary.

## 2.2 Sobolev Spaces

### 2.2.1 Sobolev Spaces on $\Omega$

Let $1 \leq p \leq \infty$. The Sobolev spaces $W^{1,p}(\mathbf{R}^d)$ are those $u \in L^p(\mathbf{R}^d)$ with finite norm

$$\begin{aligned} \|u\|_{W^{1,p}(\mathbf{R}^d)} &:= \left( \|u\|^p_{L^p(\mathbf{R}^d)} + \|\nabla u\|^p_{L^p(\mathbf{R}^d)} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \text{ and} \\ \|u\|_{W^{1,\infty}(\mathbf{R}^d)} &:= \|u\|_{L^\infty(\mathbf{R}^d)} + \|\nabla u\|_{L^\infty(\mathbf{R}^d)}, \end{aligned}$$

where $\nabla u$ denotes the weak gradient of $u$. The dual space of $W^{1,p}(\mathbf{R}^d)$ is denoted $W^{-1,p}(\mathbf{R}^d)$, and has norm

$$\|u\|_{W^{-1,p}(\mathbf{R}^d)} := \sup \left\{ |\langle u, v \rangle_p| \,:\, v \in W^{1,p}(\Omega),\ \|v\|_{W^{1,p}(\mathbf{R}^d)} = 1 \right\},$$

where $\langle \cdot, \cdot \rangle_p : W^{-1,p}(\mathbf{R}^d) \times W^{1,p}(\mathbf{R}^d) \to \mathbf{R}$ denotes the dual pairing. Note that $C_c^\infty(\mathbf{R}^d)$ is dense in $W^{\pm 1,p}(\mathbf{R}^d)$ for $1 \leq p < \infty$.

We define spaces $W^{1,p}(\Omega)$ to be the restriction spaces

$$W^{1,p}(\Omega) := \{u = U|_\Omega \,:\, U \in W^{1,p}(\Omega)\},$$

whose norms are given by

$$\|u\|_{W^{1,p}(\Omega)} := \inf\{\|U\|_{W^{1,p}(\mathbf{R}^d)} \,:\, U \in W^{1,p}(\mathbf{R}^d),\ U|_\Omega = u\}.$$

The dual space of $W^{1,p}(\Omega)$ is denoted $W_0^{-1,p}(\Omega)$, and consists of those elements $F \in W^{-1,p}(\mathbf{R}^d)$ with $\operatorname{supp} F \subseteq \bar{\Omega}$. The dual pairing is denoted $\langle \cdot, \cdot \rangle_{p,\Omega} : W_0^{-1,p}(\Omega) \times W^{1,p}(\Omega) \to \mathbf{R}$, and is given by $\langle F, u \rangle_{p,\Omega} := \langle F, U \rangle_p$, where $U \in W^{1,p}(\mathbf{R}^d)$ is any element of $W^{1,p}(\mathbf{R}^d)$ such that $U|_\Omega = u$. It is important to note that, as the next example indicates, $F \in W_0^{-1,p}(\Omega)$ may be a non-zero distribution on $\mathbf{R}^d$ which is supported on $\partial\Omega$, a set with $d$-dimensional measure 0. This will become more apparent in Section 2.3.2, as well as Chapter 4.

*Example.* Consider the distribution $F \in C_c^\infty(\mathbf{R}^d)^*$ given by $\langle F, u\rangle_{2,\Omega} := \int_{\partial\Omega} \operatorname{Tr} u \, d\sigma$, where the space $H^{1/2}(\partial\Omega)$ and the trace operator $\operatorname{Tr} : W^{1,2}(\Omega) \to H^{1/2}(\partial\Omega)$ are discussed in more detail in Section 2.2.2. On the one hand, if $U \in C_c^\infty(\mathbf{R}^d)$ is supported away from $\partial\Omega$, then $\langle F, u\rangle_{2,\Omega} = \langle F, U\rangle_2 = 0$, implying $\operatorname{supp} F \subseteq \partial\Omega$. On the other hand, for all $u \in W^{1,2}(\Omega)$ we have

$$\begin{aligned} \left| \int_{\partial\Omega} \operatorname{Tr} u \, d\sigma \right| &\le \left( \int_{\partial\Omega} 1 \, d\sigma \right)^{\frac{1}{2}} \left( \int_{\partial\Omega} |\operatorname{Tr} u|^2 \, d\sigma \right)^{\frac{1}{2}} \\ &\le C\sigma(\partial\Omega)^{\frac{1}{2}} \|u\|_{W^{1,2}(\Omega)}, \end{aligned}$$

implying $\|F\|_{W^{-1,2}(\mathbf{R}^d)} \le C\sigma(\partial\Omega)^{\frac{1}{2}}$.

Now, consider the space $\hat{W}^{1,p}(\Omega) := \{u \in L^p(\Omega) \,:\, \nabla u \in L^p(\Omega)\}$ with norm

$$\begin{aligned} \|u\|_{\hat{W}^{1,p}(\Omega)} &:= \left( \|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \text{ and} \\ \|u\|_{\hat{W}^{1,\infty}(\Omega)} &:= \|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)}. \end{aligned}$$

Since $\Omega$ is a Lipschitz domain, there is a bounded extension operator $E : \hat{W}^{1,p}(\Omega) \to W^{1,p}(\mathbf{R}^d)$ satisfying $(Eu)|_\Omega = u$ for all $u \in \hat{W}^{1,p}(\Omega)$ (cf. Calderón [3] or the monograph of Stein [36, p. 181]). The existence of such an operator guarantees $W^{1,p}(\Omega) = \hat{W}^{1,p}(\Omega)$ and that these spaces have equivalent norms, since

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega)} &\le \|Eu\|_{W^{1,p}(\mathbf{R}^d)} \\ &\le C\|u\|_{\hat{W}^{1,p}(\Omega)} \\ &\le C\|u\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Since these spaces are equal, we discard the notation $\hat{W}^{1,p}(\Omega)$ and write $W^{1,p}(\Omega)$ when writing about either space.

Given a $\sigma$-measurable subset $F \subseteq \partial\Omega$ and $1 \le p < \infty$, let $W_F^{1,p}(\Omega)$ be the closure in $W^{1,p}(\Omega)$ of the set of functions in $C^\infty(\bar{\Omega})$ which vanish on a neighborhood of $\bar{F}$, and let $W_F^{-1,p}(\Omega)$ denote the dual of $W_F^{1,p}(\Omega)$. In the special cases $F = \emptyset$ and $F = \partial\Omega$, we write $W_\emptyset^{1,p}(\Omega) =: W^{1,p}(\Omega)$, $W_\emptyset^{-1,p}(\Omega) =: W_0^{-1,p}(\Omega)$, $W_{\partial\Omega}^{1,p}(\Omega) =: W_0^{1,p}(\Omega)$, and $W_{\partial\Omega}^{-1,p}(\Omega) =: W^{-1,p}(\Omega)$.

The spaces $W^{1,p}(\Omega)$ are Banach spaces, and the space $W^{1,2}(\Omega)$ is in fact a Hilbert space when endowed with the inner product

$$(u, v)_{H^1(\Omega)} := \int_\Omega (uv + \nabla u \cdot \nabla v) \, dx.$$

For this reason, we denote $H^{\pm 1}(\mathbf{R}^d) := W^{\pm 1,2}(\mathbf{R}^d)$, $H^{\pm 1}(\Omega) := W^{\pm 1,2}(\Omega)$, and $H_F^{\pm 1}(\Omega) := W_F^{\pm 1,2}(\Omega)$. When considering the case $p = 2$, we drop $p$ from our dual pairings and write $\langle \cdot, \cdot \rangle : H^{-1}(\mathbf{R}^d) \times H^1(\mathbf{R}^d) \to \mathbf{R}$ and $\langle \cdot, \cdot \rangle_\Omega : H_D^{-1}(\Omega) \times H_D^1(\Omega) \to \mathbf{R}$.

Throughout the sequel we will make use of estimates on functions $u \in H_D^1(\Omega)$ which involve only the seminorm $\|\nabla u\|_{L^2(\Omega)}$. The following simple lemma states that $\|\nabla(\cdot)\|_{L^2(\Omega)}$ is, in fact, a norm on $H_D^1(\Omega)$ equivalent to $\|\cdot\|_{H^1(\Omega)}$ when $D$ satisfies a weak condition.

**Lemma 2.1** (Poincaré inequality on $W_D^{1,p}(\Omega)$)**.** *Let $\emptyset \neq D \subseteq \partial\Omega$ be relatively open and intersect every component of $\partial\Omega$. There is a constant $C$ depending only on $d$, $D$, $\Omega$, and $p$, such that*

$$\int_\Omega u^p \, dx \leq C \int_\Omega |\nabla u|^p \, dx \quad \forall\, u \in H_D^1(\Omega). \tag{2.5}$$

*Proof.* Suppose estimate (2.5) does not hold. Then we may construct a sequence $\{u_n\}_{n=1}^\infty \subset W_D^{1,p}(\Omega)$ such that $\|u_n\|_{W^{1,p}(\Omega)} = 1$ and $\|u_n\|_{L^p(\Omega)} \geq n\|\nabla u\|_{L^p(\Omega)}$ for every $n \in \mathbb{N}$. By passing to a subsequence, we can assume $\{u_n\}$ converges in $L^p(\Omega)$ to some $u \in L^p(\Omega)$ satisfying $\|u\|_{W^{1,p}(\Omega)} = 1$. However, since $\|\nabla u_n\|_{L^p(\Omega)} \leq \frac{1}{n}$, we have $\nabla u = 0$ a.e. in $\Omega$, i.e., $u$ is a constant function. The only constant function in $W_D^{1,p}(\Omega)$ is 0, which contradicts the fact that $\|u\|_{W^{1,p}(\Omega)} = 1$. $\square$

### 2.2.2 Sobolev Spaces on $\partial\Omega$

Let $H^{1/2}(\partial\Omega)$ be those elements of $L^2(\partial\Omega)$ with finite seminorm

$$|\varphi|_{H^{1/2}(\partial\Omega)} := \left( \int_{\partial\Omega} \int_{\partial\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^d} \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{2}}. \tag{2.6}$$

A norm on $H^{1/2}(\partial\Omega)$ is given by

$$\|\varphi\|_{H^{1/2}(\partial\Omega)} := \left( \|\varphi\|^2_{L^2(\partial\Omega)} + |\varphi|^2_{H^{1/2}(\partial\Omega)} \right)^{\frac{1}{2}}.$$

Now, consider the trace map $\mathrm{Tr} : C(\bar{\Omega}) \to C(\partial\Omega)$ given by $\mathrm{Tr}\, u = u|_{\partial\Omega}$. It is well-known (cf. the monograph of McLean [26, pp. 100-102]) that Tr has a continuous extension $\mathrm{Tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ with continuous right inverse $P : H^{1/2}(\partial\Omega) \to H^1(\Omega)$. It follows that there is a constant $C$ depending on $\partial\Omega$ which satisfies

$$\frac{1}{C}\|\varphi\|_{H^{1/2}(\partial\Omega)} \leq \|P\varphi\|_{H^1(\Omega)} \leq C\|\varphi\|_{H^{1/2}(\partial\Omega)} \quad \forall\, \varphi \in H^{1/2}(\partial\Omega),$$

i.e., $\|\cdot\|_{H^{1/2}(\partial\Omega)}$ and $\|P(\cdot)\|_{H^1(\Omega)}$ are equivalent norms on $H^{1/2}(\partial\Omega)$. Moreover, the space $H_0^1(\partial\Omega)$ can be characterized as the closed supspace $\{u \in H^1(\Omega) : \mathrm{Tr}\, u = 0\}$, and if $D \subseteq \partial\Omega$ satisfies the corkscrew condition, $H_D^1(\Omega)$ is the space of those $u \in H^1(\Omega)$ with zero trace on $D$.

Similar to what we did in Section 2.2.1, given a $\sigma$-measurable set $F \subseteq \partial\Omega$, we define spaces

$$H^{1/2}(F) := \{\Phi|_F \,:\, \Phi \in H^{1/2}(\partial\Omega)\}$$

and

$$\widetilde{H}^{1/2}(F) := \{u \in H^{1/2}(\partial\Omega) \,:\, \operatorname{supp} u \subseteq \bar{F}\}.$$

The space $\widetilde{H}^{1/2}(F)$ inherits the norm from $H^{1/2}(\partial\Omega)$, while the space $H^{1/2}(F)$ is given the norm

$$\|u\|_{H^{1/2}(F)} := \inf\{\|U\|_{H^{1/2}(\partial\Omega)} : U|_F = u\}.$$

Note that if $D \subseteq \partial\Omega$ is open, $\widetilde{H}^{1/2}(N) = \{\operatorname{Tr} u : u \in H^1_D(\Omega)\}$.

Let $H^{-1/2}(\partial\Omega)$ denote the dual of $H^{1/2}(\partial\Omega)$ with norm

$$\|\psi\|_{H^{-1/2}(\partial\Omega)} := \sup\{|\langle\psi,\varphi\rangle_{\partial\Omega}| : \varphi \in H^{1/2}(\partial\Omega),\ \|\varphi\|_{H^{1/2}(\partial\Omega)} = 1\},$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega} : H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \to \mathbf{R}$ is the dual pairing. Given $F \subseteq \partial\Omega$ relatively open, we define spaces $H^{-1/2}(F)$ and $\widetilde{H}^{-1/2}(F)$ and their norms in the same fashion as $H^{1/2}(F)$ and $\widetilde{H}^{1/2}(F)$ are given above. Observe that if $\emptyset \neq D \subsetneq \bar{D} \subsetneq \partial\Omega$ and $D$ satisfies the corkscrew condition (2.3), the dual spaces $H^{\pm 1/2}(D)^*$ and $H^{\pm 1/2}(N)^*$ are isometrically isomorphic to $\widetilde{H}^{\mp 1/2}(D)$ and $\widetilde{H}^{\mp 1/2}(N)$, respectively (cf. the monograph of McLean [26, pp. 92, 99]).

### 2.2.3 Hardy Inequality

The main estimate we will use in the sequel is the following Hardy inequality on $N$. See the monograph of Grisvard [14, p. 33] for an analogous result.

**Lemma 2.2** (Hardy Inequality)**.** *Suppose $D$ satisfies the corkscrew condition* (2.3)*. If $u \in H^1_D(\Omega)$, then there is a constant $C$ depending only on $d$ and $D$ such that*

$$\int_N \frac{(\operatorname{Tr} u)^2}{\delta}\, d\sigma \le C \int_\Omega |\nabla u|^2\, dx. \tag{2.7}$$

*Proof.* First recall that $\operatorname{Tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is continuous, so if $u \in H^1_D(\Omega)$ we have the estimate

$$\begin{aligned}\int_N |\operatorname{Tr} u(y)|^2 \int_D |x-y|^{-d}\, d\sigma(x)\, d\sigma(y) &\le \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\operatorname{Tr} u(x) - \operatorname{Tr} u(y)|^2}{|x-y|^d}\, d\sigma(x)\, d\sigma(y) \\ &\le \|\operatorname{Tr} u\|^2_{H^{1/2}(\partial\Omega)} \\ &\le C \int_\Omega |\nabla u|^2\, dx.\end{aligned}$$

Fix $y \in N$ and note that $\delta(y) > 0$. Further, because $\Lambda$ is closed, there is a point $\ell_y \in \Lambda$ such that $|y - \ell_y| = \delta(y)$. Then, since $D$ satisfies the corkscrew condition there is a constant $C > 1$ independent of $y$ and a point $\widetilde{y} \in D$ such that $|\widetilde{y} - \ell| < \delta(y)$ and $B_{\delta(y)/C}(\widetilde{y}) \cap \bar{N} = \emptyset$, i.e., $\Psi_{\delta(y)/C}(\widetilde{y}) = D \cap B_{\delta(y)/C}(\widetilde{y}) \subseteq D \cap B_{3\delta(y)}(y)$. Hence,

$$\begin{aligned}\int_D |x-y|^{-d}\, d\sigma(x) &\ge \int_{D\cap B_{3\delta(y)}(y)} |x-y|^{-d}\, d\sigma(x) \\ &\ge 3^{-d}\delta(y)^{-d}\sigma(D \cap B_{3\delta(y)}(y)) \\ &\ge 3^{-d}\delta(y)^{-d}\sigma(\Psi_{\delta(y)/C}(\widetilde{y})) \\ &\ge C^{1-d}3^{-d}\omega_{d-1}\delta(y)^{-1},\end{aligned}$$

which gives the desired result. $\square$

## 2.3 The Approximate Mixed Problem

In Chapter 1 we took a very informal approach to formulating (AMP). In Section 2.3.2 we give a rigorous definition to the normal derivative $\frac{\partial u}{\partial \nu}$ of a function $u \in H^1(\Omega)$, and in Section 2.3.1 we give precise general conditions on $a_\varepsilon$ which will ensure that (AMP) has a consistent weak formulation, that a unique weak solution $u_\varepsilon \in H^1_D(\Omega)$ of (AMP) exists, and that for such a weak solution estimate (1.1) is valid.

### 2.3.1 Standard Families of Functions

Let $\Lambda_\varepsilon := \{x \in N \,:\, 0 < \delta(x) < \varepsilon\}$ and $D_\varepsilon := \bar{D} \cup \Lambda_\varepsilon$. Fix $\mu \in (0, 1]$ and $\varepsilon_0 > 0$, and for each $0 < \varepsilon < \varepsilon_0$, consider an example function $a_\varepsilon : \partial\Omega \to \mathbf{R}$ given by

$$a_\varepsilon(x) := \begin{cases} \frac{\varepsilon^\mu - \delta(x)^\mu}{\delta(x)^\mu}, & 0 < \delta(x) \le \varepsilon, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

Observe that for $0 < \varepsilon < \varepsilon_0$, $a_\varepsilon$ satisfies the following general conditions:

$$\operatorname{supp} a_\varepsilon \subseteq \overline{\Lambda_\varepsilon}, \tag{2.9}$$

$$0 \le a_\varepsilon \le \left(\frac{\varepsilon}{\delta}\right)^\mu \quad \sigma\text{-a.e. on } \Lambda_\varepsilon, \quad \text{and} \tag{2.10}$$

$$|a_{\varepsilon+h}(x) - a_\varepsilon(x)| \le \frac{C_\mu \varepsilon^{\mu-1}}{\delta(x)^\mu} |h| \quad \text{for all } x \in N,\ |h| < \varepsilon,\ 0 < \varepsilon + h < \varepsilon_0. \tag{2.11}$$

Further, for each $x \in N$, the function $\widetilde{a}_x : (0, \varepsilon_0) \to \left[0, \frac{\varepsilon_0^\mu}{\delta(x)^\mu}\right)$ given by $\widetilde{a}_x(\varepsilon) := a_\varepsilon(x)$ satisfies

$$\widetilde{a}_x \in C^\infty\big((0, \delta(x)) \cup (\delta(x), \varepsilon_0)\big) \cap C^{0,1}(0, \varepsilon_0) \text{ with } \left|\frac{d^k \widetilde{a}}{d\varepsilon^k}\right| \le \frac{C_\mu \varepsilon^{\mu-k}}{\delta^\mu},\ k = 0, 1, \ldots. \tag{2.12}$$

In general if a family of functions $\{a_\varepsilon\}_{\varepsilon>0}$ satisfies conditions (2.9) and (2.10), we say $a_\varepsilon$ is a *standard family of functions*. If in addition $\{a_\varepsilon\}_{\varepsilon>0}$ satisfies condition (2.11), we say $a_\varepsilon$ is a *continuous standard family*, and if $a_\varepsilon$ satisfies condition (2.12) we say $a_\varepsilon$ is a *smooth standard family*.

*Remark.* When $a_\varepsilon$ is a standard family, the Hardy inequality implies for each $0 < \varepsilon < \varepsilon_0$ that the operator $\varphi \mapsto a_\varepsilon \varphi$ is bounded as a map from $\widetilde{H}^{1/2}(N)$ to $\widetilde{H}^{-1/2}(N)$.

### 2.3.2 The Laplacian and Normal Derivative

For $u \in C^2(\Omega)$ the Laplacian of $u$, written $-\Delta u$, is given by

$$-\Delta u := \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_i^2}.$$

Now let $u \in H^1(\Omega)$. Clearly the above definition no longer makes sense as an element of $L^2(\Omega)$, but if we define $-\Delta u \in H^{-1}(\partial\Omega)$ as the distribution given by

$$\langle -\Delta u, v\rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad v \in H_0^1(\Omega),$$

then we have a definition which is consistent with the usual integration by parts formula when $u \in C^2(\Omega)$ and $v \in C_c^1(\Omega)$. If we have the added assumptions $u \in C^2(\bar{\Omega})$ and $v \in C^1(\bar{\Omega})$, integration by parts yields Green's identity

$$\int_{\Omega^\pm} (-\Delta u) v \, dx = \int_{\Omega^\pm} \nabla u \cdot \nabla v \, dx \pm \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma,$$

where $\frac{\partial u}{\partial \nu} := \nabla u|_{\partial\Omega} \cdot \nu$ is the *normal derivative* of $u$. Note the sign change due to our choice of $\nu$. Once again we cannot use this formula directly for general $u \in H^1(\Omega)$, since $\nabla u \in L^2(\Omega)$ does not have a well-defined trace on the set $\partial\Omega$ of $d$-dimensional measure 0. Thus we need to be careful in defining $\frac{\partial u}{\partial \nu}$. The following lemma allows us to extend the notion of a normal derivative to elements of $H^1(\Omega)$.

**Lemma 2.3** (Green's Identity for $H^1(\Omega)$)**.** *Suppose $u \in H^1(\Omega)$ and $F \in H_0^{-1}(\Omega)$ satisfy $-\Delta u = F$ on $\Omega$, i.e., $\langle -\Delta u, v\rangle_\Omega = \langle F, v\rangle_\Omega$ for all $v \in H_0^1(\Omega)$. If $\Omega$ is bounded, then there is an element $g^- \in H^{-1/2}(\partial\Omega)$ uniquely determined by $u$ and $F$ such that*

$$\|g^-\|_{H^{-1/2}(\partial\Omega)} \le C \left( \|u\|_{H^1(\Omega)} + \|F\|_{H_0^{-1}(\Omega)} \right), \tag{2.13}$$

*and for all $v \in H^1(\Omega)$ satisfies*

$$\langle F, v\rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, dx - \left\langle g^-, \operatorname{Tr} v \right\rangle_{\partial\Omega}. \tag{2.14}$$

*If $\Omega$ is unbounded, then there is an element $g^+ \in H^{-1/2}(\partial\Omega)$ for which the estimate* (2.13) *holds and which satisfies for every $v \in H^1(\Omega)$ the identity*

$$\langle F, v\rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, dx + \left\langle g^+, \operatorname{Tr} v \right\rangle_{\partial\Omega}. \tag{2.15}$$

*In either case, we refer to $g^-$ or $g^+$ as the* normal derivative of $u$ with respect to $F$.

*Remark.* We have the two separate equations (2.14) and (2.15) to ensure that $g^- = g^+$ when $u \in C_c^2(\mathbf{R}^d)$, which is consistent with our convention of having $\nu$ point *out of* a bounded domain $\Omega^-$ and *into* its unbounded complementary domain $\Omega^+$.

*Proof of Lemma 2.3.* Let $P : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ be a continuous right inverse of the trace map $\operatorname{Tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ (See Section 2.2.2), and define $g \in H^{-1/2}(\partial\Omega)$ by

$$\langle g, \varphi\rangle_{\partial\Omega} = \int_\Omega \nabla u \cdot \nabla(P\varphi) \, dx - \langle F, P\varphi\rangle_\Omega, \quad \forall\, \varphi \in H^{1/2}(\partial\Omega).$$

Clearly $|\langle g,\varphi\rangle_{\partial\Omega}| \le \|P\|(\|\nabla u\|_{L^2(\Omega)} + \|F\|_{H_0^{-1}(\Omega)})$ whenever $\|\varphi\|_{H^{1/2}(\partial\Omega)} = 1$, which proves the estimate (2.13). Next, for $v \in H^1(\Omega)$ set $v_0 := v - P(\operatorname{Tr} v) \in H_0^1(\Omega)$. By how we defined $g$, and the fact that $-\Delta u = F$ on $\Omega$, we have

$$\begin{aligned}\langle F, v\rangle_\Omega &= \langle -\Delta u, v_0\rangle_\Omega + \langle F, P(\operatorname{Tr} v)\rangle_\Omega \\ &= \int_\Omega \nabla u\cdot\nabla v\,dx - \int_\Omega \nabla u\cdot\nabla[P(\operatorname{Tr} v)]\,dx + \langle F, P(\operatorname{Tr} v)\rangle_\Omega \\ &= \int_\Omega \nabla u\cdot\nabla v\,dx - \langle g, \operatorname{Tr} v\rangle_{\partial\Omega}\,.\end{aligned}$$

Finally, suppose $\widetilde{g} \in H^{-1/2}(\partial\Omega)$ also satisfies (2.14) for all $v \in H^1(\Omega)$. Then for all $\varphi \in H^{1/2}(\partial\Omega)$, $\langle g-\widetilde{g},\varphi\rangle_{\partial\Omega} = \langle g-\widetilde{g}, \operatorname{Tr}(P\varphi)\rangle_{\partial\Omega} = 0$, i.e., $g = \widetilde{g}$. □

*Remark.* As we mentioned in Section 2.2.1, even if $F, \widetilde{F} \in H_0^{-1}(\Omega)$ satisfy $F = \widetilde{F}$ on $\Omega$, $F - \widetilde{F}$ may still be a non-zero distribution on $\mathbf{R}^d$ supported in $\partial\Omega$. Hence the distribution $g \in H^{-1/2}(\partial\Omega)$ indeed depends upon both $u$ and the choice of $F$. Thus, when refering to the normal derivative of $u \in H^1(\Omega)$, we are referring specifically to the normal derivative of $u$ with respect to the distribution

$$F = \begin{cases} -\Delta u & \text{on } \Omega, \\ 0 & \text{on } \mathbf{R}^d \setminus \Omega. \end{cases}$$

### 2.3.3 Weak Formulation, Existence and Uniqueness

Recall from Chapter 1 the mixed problem (MP)

$$\begin{cases} -\Delta u_0 = f & \text{on } \Omega \\ u_0 = 0 & \text{on } D \\ \frac{\partial u_0}{\partial \nu} = g & \text{on } N, \end{cases}$$

and for a standard family $a_\varepsilon$, the approximate mixed problem (AMP)

$$\begin{cases} -\Delta u_\varepsilon = f & \text{on } \Omega \\ u_\varepsilon = 0 & \text{on } D \\ \frac{\partial u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon = g & \text{on } N. \end{cases}$$

For $f \in H_D^{-1}(\Omega)$ and $g \in \widetilde{H}^{-1/2}(N)$, we call $u_0 \in H_D^1(\Omega)$ a weak solution of (MP) if

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\,dx = \langle f,\varphi\rangle_\Omega + \langle g, \operatorname{Tr}\varphi\rangle_{\partial\Omega} \quad \forall \varphi \in H_D^1(\Omega),$$

and we call $u_\varepsilon \in H_D^1(\Omega)$ a weak solution of (AMP) if

$$\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi\,dx + \int_N a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr}\varphi\,d\sigma = \langle f,\varphi\rangle_\Omega + \langle g,\operatorname{Tr}\varphi\rangle_{\partial\Omega} \quad \forall\varphi \in H_D^1(\Omega).$$

Here, $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denote dual pairings on $H_D^{-1}(\Omega)$ and $\widetilde{H}^{-1/2}(N)$, respectively.

**Theorem 2.4** (Existence and Uniqueness)**.** *Suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition* (2.3)*. Let $f \in H_D^{-1}(\Omega)$ and $g \in \widetilde{H}^{-1/2}(N)$, and let $a_\varepsilon$ be a standard family of functions. Take $a_0$ to be identically 0 on $\partial\Omega$. There is an $\varepsilon_0 > 0$ and a constant $C$ not depending on $\varepsilon$ such that for each $0 \le \varepsilon < \varepsilon_0$, a unique weak solution $u_\varepsilon \in H_D^1(\Omega)$ of* (AMP) *(or* (MP) *in the case $\varepsilon = 0$) exists which satisfies*

$$\|u_\varepsilon\|_{H^1(\Omega)} \le C\left(\|f\|_{H_D^{-1}(\Omega)} + \|g\|_{H^{-1/2}(N)}\right). \tag{2.16}$$

*Proof.* Fix $\theta > 1$ and $0 \le \varepsilon < \varepsilon_0 \le \frac{1}{\theta C}$, where $C$ is the constant in the Hardy inequality (2.7). Let $F \in H_D^{-1}(\Omega)$ be the distribution given by $\langle F, v\rangle_\Omega := \langle f, v\rangle_\Omega + \langle g, \operatorname{Tr} v\rangle_{\partial\Omega}$, and define the bilinear form $B_\varepsilon : H_D^1(\Omega) \times H_D^1(\Omega) \to \mathbf{R}$ by

$$B_\varepsilon[u,v] := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_N a_\varepsilon \operatorname{Tr} u \operatorname{Tr} v \, d\sigma.$$

We will show that $B_\varepsilon$ is bounded and coercive on $H_D^1(\Omega)$.

We first prove coercivity of $B_\varepsilon$. Since $a_\varepsilon$ is a standard family, condition (2.10) on $a_\varepsilon$ and Lemma 2.2 imply that

$$\begin{aligned} B_\varepsilon[u,u] &> \int_\Omega |\nabla u|^2 - \varepsilon_0 \int_N \frac{\operatorname{Tr} u^2}{\delta} \, d\sigma \\ &\ge \left(1 - \frac{1}{\theta}\right) \|\nabla u\|_{L^2(\Omega)}^2. \end{aligned}$$

To deduce boundedness of $B_\varepsilon$, we appeal again to the Hardy inequality (2.7) to compute for all $u, \varphi \in H_D^1(\Omega)$

$$\begin{aligned} B_\varepsilon[u,v] &\le \int_\Omega |\nabla u||\nabla v| + \varepsilon_0 \int_N \frac{u\varphi}{\delta} \, d\sigma \\ &\le \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \frac{1}{\theta C}\left(\int_N \frac{u^2}{\delta}\right)^{\frac{1}{2}} \left(\int_N \frac{v^2}{\delta}\right)^{\frac{1}{2}} \\ &\le \left(1 + \frac{1}{\theta}\right) \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}. \end{aligned}$$

Hence, by the Lax-Milgram theorem there is a unique $u \in H_D^1(\Omega)$ satisfying $B_\varepsilon[u,v] = \langle F, v\rangle_\Omega$ for all $v \in H_D^1(\Omega)$.

Finally, since $u_\varepsilon \in H_D^1(\Omega)$ satisfies $B_\varepsilon[u_\varepsilon, u_\varepsilon] = \langle F, u_\varepsilon\rangle_\Omega$, it follows from coercivity that

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 &\le \frac{\theta}{\theta - 1} B_\varepsilon[u_\varepsilon, u_\varepsilon] \\ &= \frac{\theta}{\theta - 1} \langle F, u_\varepsilon\rangle_\Omega \\ &\le \frac{\theta}{\theta - 1} \left(\|f\|_{H_D^{-1}(\Omega)} + \|g\|_{H^{-1/2}(N)}\right) \|u_\varepsilon\|_{H^1(\Omega)}. \end{aligned}$$

Dividing by $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$, we arrive at estimate (2.16). □

The goal of this chapter is to prove Theorem 1.1. To do so, we first prove a Sobolev inequality which holds uniformly on a family of star-shaped convex domains, as well as a trace theorem, both of which will be used throughout the chapter. Next, we introduce the notation of the so-called "$\mathcal{B}$-spaces" found in Ladyzhenskaya and Ural'tseva [20, pp. 81-95], and we show that elements of $\mathcal{B}(\Omega_r(x), \gamma, Q)$ are in fact Hölder continuous. Then we prove that weak solutions of (AMP) satisfy a Caccioppoli inequality, after which we adapt the famous result of De Giorgi [10], Nash [29], and Moser [28], to deduce that weak solutions $u_\varepsilon \in H^1_D(\Omega)$ of (AMP) have finite $L^\infty(\Omega)$ norm independent of $\varepsilon$, and are in fact in $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1]$. Finally, we conclude by deriving an upper bound for the rate at which $u_\varepsilon$ converges to $u_0$ in $H^1(\Omega)$ as $\varepsilon \to 0$, and discuss an asymptotic expansion of $u_\varepsilon$ in $\varepsilon$. Unless noted otherwise, $\Omega \subset \mathbf{R}^d$ will always be taken to be a bounded Lipschitz domain with constant $M$.

## 3.1 Preliminary Inequalities

### 3.1.1 Sobolev Inequality on Star-shaped Convex Domains

First, let us state what it means for a bounded open set $\Upsilon \subseteq \mathbf{R}^d$ to be star-shaped Lipschitz convex and star-shaped with respect to a ball. We say $\Upsilon$ is *star-shaped Lipschitz convex with constant $M$, scale $r$, and star-center $x$* if there is a Lipschitz function $\varphi : \mathbf{S}^{d-1} \to [1, 1 + M]$ with Lipschitz constant $M$ such that

$$\Upsilon = \left\{ y \in \mathbf{R}^d \,:\, |x - y| < r\varphi(\hat{y}) \right\},$$

and we say $\Upsilon$ is *star-shaped convex with respect to a ball $B_r(x)$* if we have

$$(1 - t)y + tz \in \Upsilon \quad \forall\, y \in B_r(x),\; z \in \Upsilon,\; t \in [0, 1].$$

*Remark.* When defining a Lipschitz domain, we chose to consider intersections with cylinders rather than balls precisely because for each $x \in \partial\Omega$, the set $\Omega_r(x)$ is star-shaped convex with respect to the ball $B_{r/4}\big(x', \psi_x(x') + (2M + 1/2)r\big) \subseteq \Omega_r(x)$. This fact will be crucial in Sections 3.2 and 3.3, where Lemma 3.4 and Corollary 3.5 are used extensively. See Appendix A for more information concerning star-shaped domains.

The main results of this subsection are Lemma 3.4 and Corollary 3.5. To obtain them, we first prove a series of three intermediate lemmas. Recall that $\omega_d$ denotes the measure of the unit ball in $\mathbf{R}^d$.

**Lemma 3.1** (Poincaré inequality on star-shaped convex domains)**.** *Let $1 \le p < \infty$ and suppose $u \in W^{1,p}(\Upsilon)$, where $\Upsilon \subseteq \mathbf{R}^d$ is star-shaped convex with constant $M$ and scale $r$. Let $\omega_d$ denote the volume of the unit ball in $\mathbf{R}^d$, and $\bar{u}_G := |G|^{-1} \int_G u(x)\, dx$*

*the average of $u$ on $G$. If $F, G \subseteq \Upsilon$ are measurable with $|G| > 0$, then*

$$\|u - \bar{u}_G\|_{L^p(F)} \le \frac{2^{d+1}(1+M)^d \left|\Upsilon\right|^{\frac{d+p-2}{dp}} \left|F\right|^{\frac{1}{dp}}}{\omega_d^{1/d} \left|G\right|^{\frac{d-1}{dp}}} \|\nabla u\|_{L^p(\Upsilon)}. \tag{3.1}$$

*Proof.* Without loss of generality we assume $\Upsilon$ is star-shaped with respect to every point in a ball $B_{Cr}$ centered at the origin, and that $u \in W^{1,p}(\Upsilon) \cap C^1(\Upsilon)$. By the fundamental theorem of calculus, if $y \neq x$, $x \in \Upsilon$, and $y \in B_{Cr}$, we have

$$u(y) - u(x) = \int_0^{|y-x|} \frac{\partial}{\partial s} \left[u(x + s\omega)\right] ds, \quad \omega := \frac{y-x}{|y-x|} \in \mathbf{S}^{d-1}. \tag{3.2}$$

Taking the average of (3.2) over $y \in B_{Cr}$ gives

$$\bar{u}_{B_{Cr}} - u(x) = \frac{1}{|B_{Cr}|} \int_{B_{Cr}} \int_0^{|y-x|} \omega \cdot \nabla u(x + s\omega)\, ds\, dy. \tag{3.3}$$

Now, define $V : \mathbf{R}^d \to \mathbf{R}$ to be $|\nabla u|$ on $\Upsilon$ and 0 elsewhere. From (3.3) we deduce

$$\begin{aligned}
|\bar{u}_{B_{Cr}} - u(x)| &\le \frac{1}{|B_{Cr}|} \int_{B_{Cr}} \int_0^{|y-x|} V(x + s\omega)\, ds\, dy \\
&\le C \frac{1}{\omega_d r^d} \int_{B_{2C(1+M)r}(x)} \int_0^{\infty} V(x + s\omega)\, ds\, dy \\
&= C \frac{1}{\omega_d r^d} \int_0^{2C(1+M)r} \int_{\mathbf{S}^{d-1}} \int_0^{\infty} V(x + s\omega)\, ds\, d\omega\ t^{d-1}\, dt \\
&= C \frac{2^d (1+M)^d}{d\omega_d} \int_0^{\infty} \int_{\mathbf{S}^{d-1}} \frac{V(x + s\omega)}{s^{d-1}}\, d\omega\ s^{d-1}\, ds \\
&= C \frac{2^d (1+M)^d}{d\omega_d} \int_{\Upsilon} \frac{|\nabla u(y)|}{|x-y|^{d-1}}\, dy.
\end{aligned}$$

Set $t = \left(\frac{|F|}{\omega_d}\right)^{1/d}$ so that $|B_t(0)| = |F|$. It is well-known (cf. the monograph of Gilbarg and Trudinger [13, p. 159]) that

$$\begin{aligned}
\int_F |x-y|^{1-d}\, dx &\le \int_{B_t(0)} |x|^{1-d}\, dx \\
&= t d\omega_d \\
&= d\omega_d^{1-1/d} \left|F\right|^{1/d}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\|u-\bar{u}_{B_{Cr}}\|_{L^p(F)}^p &\le C\left[\frac{2^d(1+M)^d}{d\omega_d}\right]^p \int_F \left|\int_\Upsilon \frac{|\nabla u(y)|}{|x-y|^{d-1}}\,dy\right|^p\,dx \\
&\le C\left[\frac{2^d(1+M)^d}{d\omega_d}\right]^p \times \\
&\quad\times \int_F \left[\left(\int_\Upsilon \frac{1}{|x-y|^{d-1}}\,dy\right)^{\frac{p-1}{p}} \left(\int_\Upsilon \frac{|\nabla u(y)|^p}{|x-y|^{d-1}}\,dy\right)^{\frac{1}{p}}\right]^p\,dx \\
&\le C\left[\frac{2^d(1+M)^d}{d\omega_d}\right]^p \times \\
&\quad\times \left(d\omega_d^{1-1/d}\,|\Upsilon|^{\frac{1}{d}}\right)^{p-1} \int_\Upsilon \left(\int_F \frac{1}{|x-y|^{d-1}}\,dx\right)|\nabla u(y)|^p\,dy \\
&\le C\left[2^d(1+M)^d\,|\Upsilon|^{\frac{p-1}{dp}}\,|F|^{\frac{1}{dp}}\,\omega_d^{-1/d}\right]^p \|\nabla u\|_{L^p(\Upsilon)}^p.
\end{aligned}
$$

Next, we invoke the previous estimate two times to deduce

$$
\begin{aligned}
\|u-\bar{u}_G\|_{L^p(F)} &\le \|u-\bar{u}_{B_{Cr}}\|_{L^p(F)} + \|\bar{u}_G-\bar{u}_{B_{Cr}}\|_{L^p(F)} \\
&\le 2^dC(1+M)^d\,|\Upsilon|^{\frac{p-1}{dp}}\,|F|^{\frac{1}{dp}}\,\omega_d^{-1/d}\|\nabla u\|_{L^p(\Upsilon)} + \frac{|F|^{\frac{1}{p}}}{|G|^{\frac{1}{p}}}\|u-\bar{u}_{B_{Cr}}\|_{L^p(G)} \\
&\le 2^dC(1+M)^d\,|\Upsilon|^{\frac{p-1}{dp}}\,|F|^{\frac{1}{dp}}\,\omega_d^{-1/d}\left[1+\left(\frac{|F|}{|G|}\right)^{\frac{d-1}{dp}}\right]\|\nabla u\|_{L^p(\Upsilon)}.
\end{aligned}
$$

Finally, since $|F| \le |\Upsilon|$ and $1 \le \frac{|\Upsilon|}{|G|}$, we obtain

$$
\frac{|\Upsilon|^{\frac{p-1}{dp}}\,|F|^{\frac{1}{dp}}}{\omega_d^{1/d}}\left[1+\left(\frac{|F|}{|G|}\right)^{\frac{d-1}{dp}}\right] \le 2\frac{|\Upsilon|^{\frac{d+p-2}{dp}}\,|F|^{\frac{1}{dp}}}{\omega_d^{1/d}\,|G|^{\frac{d-1}{dp}}},
$$

from which (3.1) now follows. □

**Lemma 3.2.** *Let $1 \le p < \infty$ and suppose $u \in W^{1,p}_T(\Upsilon)$, where $\Upsilon \subseteq \mathbf{R}^d$ is star-shaped convex with constant $M$ and scale $r$, and $T \subseteq \partial\Upsilon$ satisfies $\sigma(T) \ge cr^{d-1}$ for some $c$ independent of $r$. There is a constant $C$ depending only on $c, d, M$, and $p$ such that*

$$
\|u\|_{L^p(\Upsilon)} \le Cr\|\nabla u\|_{L^p(\Upsilon)}.
$$

*Proof.* Without loss of generality, we assume that $u \in W^{1,p}_T(\Upsilon) \cap C^1(\bar{\Upsilon})$, and that $\Upsilon$ has star-center 0. By the fundamental theorem of calculus, for $y \in T$, $\hat{y} := \frac{y}{|y|} \in \mathbf{S}^{d-1}$ and $s \in [0, r\varphi(\hat{y}))$ we know

$$
\begin{aligned}
u(s\hat{y}) &= -\int_s^{r\varphi(\hat{y})} \frac{d}{dt}\left[u(t\hat{y})\right]\,dt \\
&= -\int_s^{r\varphi(\hat{y})} \nabla u(t\hat{y}) \cdot \hat{y}\,dt.
\end{aligned}
$$

Let $\bar{T} := \{sy \,:\, s \in [0,1), y \in T\}$ and $\hat{T} := \{\hat{y} \,:\, y \in T\}$. Integrating over $\bar{T}$ we obtain

$$\begin{aligned}
\int_{\bar{T}} u(x)\,dx &= -\int_{\hat{T}} \left[\int_0^{r\varphi(\hat{y})} \left(\int_s^{\varphi(\hat{y})} \nabla u(t\hat{y}) \cdot \hat{y}\,dt\right) s^{d-1}\,ds\right] d\sigma(\hat{y}) \\
&= -\int_{\hat{T}} \left[\int_0^{r\varphi(\hat{y})} \left(\int_0^{t} s^{d-1}\,ds\right) \nabla u(t\hat{y}) \cdot \hat{y}\,dt\right] d\sigma(\hat{y}) \\
&= -\frac{1}{d}\int_{\hat{T}} \int_0^{r\varphi(\hat{y})} \nabla u(t\hat{y}) \cdot (t\hat{y}) t^{d-1}\,dt\,d\sigma(\hat{y}) \\
&= -\frac{1}{d}\int_{\bar{T}} \nabla u(x) \cdot x\,dx.
\end{aligned}$$

Now let $\Phi(y) := \frac{|y|}{r\varphi(\hat{y})}$. Observe that the level sets $\Phi^{-1}(s)$ of $\Phi$ are precisely $s\partial\Upsilon := \{sy \,:\, y \in \Upsilon\}$. We compute

$$\begin{aligned}
|\nabla\Phi(x)| &= \left|\frac{\hat{x}}{r}\left[\frac{1}{\varphi(\hat{x})} - \frac{\hat{x}\cdot\nabla\varphi(\hat{x})}{\varphi(\hat{x})^2}\right] - \frac{\nabla\varphi(\hat{x})}{r\varphi(\hat{x})^2}\right| \\
&\leq \frac{\varphi(\hat{x}) + 2|\nabla\varphi(\hat{x})|}{r\varphi(\hat{x})^2} \\
&\leq \frac{2(1+M)}{r}.
\end{aligned}$$

To estimate $|\bar{T}|$, we appeal to the coarea formula to obtain

$$\begin{aligned}
|\bar{T}| &= \int_0^1 \int_{\Phi^{-1}(s)} \frac{1}{|\nabla\Phi(y)|}\,d\sigma(y)\,ds \\
&= \int_0^1 \int_T \frac{1}{|\nabla\Phi(y)|} s^{d-1}\,d\sigma(y)\,ds \\
&\geq \frac{r}{2d(1+M)}\sigma(T) \\
&\geq \frac{cr^d}{2d(1+M)}.
\end{aligned}$$

Hence, we may estimate the average of $u$ on $\bar{T}$ by

$$\begin{aligned}
|\bar{u}_{\bar{T}}| &= \frac{1}{|\bar{T}|}\left|\int_{\bar{T}} u(x)\,dx\right| \\
&\leq \frac{2(1+M)}{cr^d}\int_{\bar{T}} |\nabla u(x)||x|\,dx \\
&\leq \frac{2(1+M)^2}{cr^{d-1}}\int_{\bar{T}} |\nabla u(x)|\,dx.
\end{aligned}$$

Finally, by Lemma 3.1 with $F = \Upsilon$ and $G = \bar{T}$, as well as the previous estimate and Hölder's inequality, we deduce

$$\begin{aligned}
\|u\|_{L^p(\Upsilon)} &\le \|u - \bar{u}_{\bar{T}}\|_{L^p(\Upsilon)} + \|\bar{u}_{\bar{T}}\|_{L^p(\Upsilon)} \\
&\le C\frac{2^{d+1}(1+M)^d\,|\Upsilon|^{\frac{d+p-2}{dp}}\,|F|^{\frac{1}{dp}}}{\omega_d^{1/d}|\bar{T}|^{\frac{d-1}{dp}}}\|\nabla u\|_{L^p(\Upsilon)} + |\bar{u}_{\bar{T}}|\,|\Upsilon|^{\frac{1}{p}} \\
&\le 2^{d+1}C\left(2c^{-1}d\omega_d\right)^{\frac{d-1}{dp}}(1+M)^{\frac{dp+d+p-1}{p}}r\|\nabla u\|_{L^p(\Upsilon)} \\
&\quad + 2c^{-1}(1+M)^{\frac{2p+d}{p}}\omega_d^{1/p}r^{\frac{d}{p}-d+1}\|\nabla u\|_{L^1(\Upsilon)} \\
&\le \left[2^{d+1}C\left(2c^{-1}d\omega_d\right)^{\frac{d-1}{dp}}(1+M)^{\frac{dp+d+p-1}{p}} + 2c^{-1}(1+M)^{2+d}\omega_d\right]r\|\nabla u\|_{L^p(\Upsilon)}. \square
\end{aligned}$$

**Lemma 3.3** (Extension Lemma). *Suppose $\Upsilon \subseteq \mathbf{R}^d$ is star-shaped convex with constant $M$ and scale $r$. For $1 \le p < \infty$ there is an extension operator $E : W^{1,p}(\Upsilon) \to W^{1,p}(\mathbf{R}^d)$ such that*

$$\|\nabla(Eu)\|_{L^p(\mathbf{R}^d)} \le C\left(\frac{1}{r}\|u\|_{L^p(\Upsilon)} + \|\nabla u\|_{L^p(\Upsilon)}\right),$$

*where $C$ depends only on $d, M$ and $p$.*

*Proof.* Assume $\Upsilon$ has star-center 0. For $x \neq 0$ set $\hat{x} := \frac{x}{|x|} \in \mathbf{S}^{d-1}$ and let

$$x^* := \frac{\hat{x}}{|x|}r^2\varphi(\hat{x})^2$$

denote the *reflection of $x$ over $\partial\Upsilon$*. Observe that $(x^*)^* = x$ for all $x \neq 0$, $x^* = x$ iff $x \in \partial\Upsilon$, and $x \in \mathbf{R}^d \setminus \bar{\Upsilon}$ iff $x^* \in \Upsilon \setminus \{0\}$. If $u : \bar{\Upsilon} \to \mathbf{R}$, we define $u^* : \mathbf{R}^d \setminus \Upsilon \to \mathbf{R}$ by $u^*(x) := u(x^*)$. Fix $\eta \in C_c^\infty(\mathbf{R}^d)$ so that $\eta \equiv 1$ on $B_{(1+M)r}$, $\eta \le 1$, $\operatorname{supp}\eta \subseteq B_{(2+2M)r}$, and $|\nabla\eta| \le \frac{C}{r}$. Define an extension operator $E$ on $W^{1,p}(\Upsilon) \cap C^1(\bar{\Upsilon})$ by

$$Eu(x) := \begin{cases} u(x), & x \in \bar{\Upsilon} \\ \eta(x)u^*(x), & x \in \mathbf{R}^d \setminus \Upsilon. \end{cases}$$

Observe that $Eu \in C_c(\mathbf{R}^d)$. Further, when $x \in \mathbf{R}^d \setminus \bar{\Upsilon}$ we have

$$\nabla(Eu)(x) = \nabla\eta(x)u^*(x) + \eta(x)[J(x^*)](x)\nabla u(x^*),$$

where $[J(x^*)](x)$ is the $d \times d$ Jacobian matrix of $x^*$ defined in Lemma A.4. Now, by the estimates of Lemma A.4 and a change of variables, we write

$$\begin{aligned}
\int_{\mathbf{R}^d\setminus\Upsilon}|\nabla\eta(x)||u^*(x)|^p\,dx &\le \frac{C^p}{r^p}\int_{B_{(2+2M)r}\setminus\Upsilon}|u^*(x)|^p\,dx \\
&\le \frac{C^p}{r^p}\int_{\Upsilon\setminus\{0\}}|u(x)|^p|\det[J(x^*)]|\,dx \\
&\le \frac{C^p}{r^p}d!\,5^d(3M+1)^{2d}\int_{\Upsilon\setminus\{0\}}|u(x)|^p\,dx,
\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}^d\setminus\Upsilon} |\eta(x)[J(x^*)]\nabla u(x^*)|^p\,dx &\le 5^p(3M+1)^{2p}\int_{B_{(2+2M)r}\setminus\Upsilon} |\nabla u(x^*)|^p\,dx \\ &\le 5^p(3M+1)^{2p}\int_{\Upsilon\setminus\{0\}} |\nabla u(x)|^p|\det[J(x^*)]|\,dx \\ &\le d!\,5^{p+d}(3M+1)^{2(p+d)}\int_{\Upsilon\setminus\{0\}} |\nabla u(x)|^p\,dx.\end{aligned}$$

Hence, we have $\|\nabla(Eu)\|_{L^p(\mathbf{R}^d)} \le C\left(\frac{1}{r}\|u\|_{L^p(\Upsilon)} + \|\nabla u\|_{L^p(\Upsilon)}\right)$. This fact, in conjunction with our earlier observation that $Eu \in C_c(\mathbf{R}^d)$, also confirms $Eu \in W^{1,p}(\mathbf{R}^d)$. □

We are now equipped to prove our main results for the subsection.

**Lemma 3.4.** *Let $1 \le p < d$ and suppose $u \in W^{1,p}_T(\Upsilon)$, where $\Upsilon \subseteq \mathbf{R}^d$ is star-shaped convex with constant $M$ and scale $r$, and $T \subseteq \partial\Upsilon$ satisfies $\sigma(T) \ge cr^{d-1}$ for some $c$ independent of $r$. There is a constant $C$ depending only on $c, d, M$, and $p$ such that*

$$\|u\|_{L^{\frac{dp}{d-p}}(\Upsilon)} \le C\|\nabla u\|_{L^p(\Upsilon)}.$$

*Proof.* Clearly $\|u\|_{L^{\frac{dp}{d-p}}(\Upsilon)} \le \|Eu\|_{L^{\frac{dp}{d-p}}(\mathbf{R}^d)}$ holds. By the usual Sobolev inequality, followed by Lemmas 3.3 and 3.2, we conclude

$$\begin{aligned}\|Eu\|_{L^{\frac{dp}{d-p}}(\mathbf{R}^d)} &\le C\|\nabla(Eu)\|_{L^p(\mathbf{R}^d)} \\ &\le C\left(\frac{1}{r}\|u\|_{L^p(\Upsilon)} + \|\nabla u\|_{L^p(\Upsilon)}\right) \\ &\le C\|\nabla u\|_{L^p(\Upsilon)}.\end{aligned}$$

□

**Corollary 3.5.** *Let $x \in \partial\Omega$ and $0 < r < r_0$. For $u \in W^{1,1}(\Omega)$, let $\Omega_{k,r}(x) := \{y \in \Omega_r(x) : u(y) > k\}$. If $\ell > k$, then*

$$(\ell - k)\,|\Omega_{\ell,r}(x)|^{1-1/d} \le C\frac{|\Omega_r(x)|}{|\Omega_r(x)\setminus\Omega_{k,r}(x)|}\int_{\Omega_{k,r}(x)\setminus\Omega_{\ell,r}(x)} |\nabla u(y)|\,dy, \tag{3.4}$$

*where $C$ depends only on $d$ and $M$. Moreover, if $D \subseteq \partial\Omega$ satisfies the corkscrew condition (2.3), $u \in W^{1,1}_D(\Omega)$, $\Psi_{r/2}(x) \cap D \ne \emptyset$, and $k \ge 0$, then the estimate is uniform, that is,*

$$(\ell - k)\,|\Omega_{\ell,r}(x)|^{1-1/d} \le C\int_{\Omega_{k,r}(x)\setminus\Omega_{\ell,r}(x)} |\nabla u(y)|\,dy, \tag{3.5}$$

*where $C$ depends only on $d$, and $M$.*

*Proof.* As we remarked at the beginning of the section, $\Omega_r$ is star-shaped with respect to every point in a ball $B_{Cr}(y)$ contained within $\Omega_r$. To obtain 3.4, set $p = 1$, $F = \Omega_{\ell,r}$, $G = \Omega_r \setminus \Omega_{k,r}$, and $\Upsilon = \Omega_r$, and apply Lemma 3.1 to the function

$$v(x) := \begin{cases} 0, & u(x) < k \\ u(x) - k, & k \le u(x) \le \ell \\ \ell - k, & \ell < u(x) \end{cases}$$

and divide by $|\Omega_{\ell,r}|$. Observe that $\bar{v}_G = 0$ and $\nabla v = \chi_{\Omega_{k,r} \setminus \Omega_{\ell,r}} \nabla u$.

To obtain estimate (3.5), we first note that the corkscrew condition on $D$ and our assumption that $\Psi_{r/2}(x) \cap D \neq 0$ guarantees $\sigma(D \cap \Psi_r(x)) \ge cr^{d-1}$ for some $c$ depending only on $d$ and $M$. Further, the assumption that $k \ge 0$ ensures that the function $v$ defined above satisfies $v \in W^{1,1}_{D \cap \Psi_r(x)}(\Omega_r(x))$. The desired estimate now follows by applying Lemma 3.4 with $p = 1$ and $\Upsilon = \Omega_r(x)$ to the function $v$. □

### 3.1.2 Trace Theorem

To prove that solutions of (AMP) are Hölder continuous, we will require an inequality which states that the $L^{p^*}$-norm of $u_\varepsilon$ on $\Psi_r(x)$ is controlled by the $L^p$-norm of $\nabla u_\varepsilon$ on $\Omega_r(x)$, where $p^*$ depends only on $d$ and $p$. The following lemma fills this requirement.

**Lemma 3.6** (Trace Theorem)**.** *Let $0 < r < r_0$. Suppose $v \in W^{1,\tau}(\Omega)$ for $1 \le \tau < d$, and $\eta \in C_c^\infty(Z_r(x))$. Then $\mathrm{Tr}(\eta v) \in L^{\frac{(d-1)\tau}{d-\tau}}(\Psi_r(x))$, and there is a constant $C$ depending only on $d, M$, and $\tau$ such that*

$$\| \mathrm{Tr}(v\eta)\|_{L^{\frac{(d-1)\tau}{d-\tau}}(\Psi_r(x))} \le C \|\nabla(v\eta)\|_{L^\tau(\Omega_r(x))}.$$

*Proof.* We begin by proving the case $1 < \tau < d$. Without loss of generality we assume $x = 0$ and write $\Psi_r$ and $\Omega_r$. We may choose a constant unit vector $w \in \mathbf{R}^d$ such that $w \cdot \nu \ge \delta > 0$ $\sigma$-a.e. on $\Psi_r$, where $\nu$ is the outward-pointing unit normal to $\Omega$. Set $Q = \frac{(d-1)\tau}{d-\tau}$ and $\mathrm{sgn}(x)$ to be 1 when $x > 0$, $-1$ when $x < 0$, and 0 when $x = 0$. By the divergence theorem and Hölder's inequality,

$$\begin{aligned} \delta \| \mathrm{Tr}(v\eta)\|^Q_{L^Q(\Psi_r)} &\le \int_{\Psi_r} w \cdot \nu |\mathrm{Tr}(v\eta)|^Q \, d\sigma \\ &= \int_{\Omega_r} Q |v\eta|^{Q-1} \mathrm{sgn}(v\eta) w \cdot \nabla(v\eta) \, dx \\ &\le Q \| |v\eta|^{Q-1} \|_{L^{\tau'}(\Omega_r)} \|\nabla(v\eta)\|_{L^\tau(\Omega_r)}, \end{aligned}$$

where $\tau' = \frac{\tau}{\tau - 1}$. Now note that $(Q-1)\tau' = \frac{d\tau}{d-\tau}$, which is precisely the exponent which appears in the Sobolev inequality of Lemma 3.4. Hence,

$$\delta \| \mathrm{Tr}(v\eta)\|^Q_{L^Q(\Psi_r)} \le CQ \|\nabla(v\eta)\|^{Q-1}_{L^\tau(\Omega_r)} \|\nabla(v\eta)\|_{L^\tau(\Omega_r)}.$$

Dividing by $\delta$ and raising both sides to the $1/Q$ power gives the desired result.

In the case $\tau = 1$, we have $\tau' = \infty$ and $Q = 1$, which greatly simplifies the above calculations as we no longer require the use of Hölder's inequality to obtain the desired estimate. □

As a corollary to Lemma 3.6, we now state a fractional order Sobolev inequality which holds on $H^{1/2}(\partial\Omega)$.

**Corollary 3.7** (Fractional-order Sobolev inequality)**.** *Suppose $\varphi \in H^{1/2}(\partial\Omega)$. Then $\varphi \in L^{\frac{2(d-1)}{d-2}}(\partial\Omega)$, and there is a constant $C$ depending only on the global character of $\Omega$ such that*

$$\|\varphi\|_{L^{\frac{2(d-1)}{d-2}}(\partial\Omega)} \leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}.$$

*Proof.* Recall that $P : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ denotes a bounded right inverse of the trace map Tr. Let $\{Z_{r_0}(x_j)\}_1^J$ be as in the definition of a Lipschitz domain. Let $\eta_j \in C_c^\infty(Z_{2r_0}(x_j))$ satisfy $\eta_j \equiv 1$ on $Z_{r_0}(x_j)$, $0 \leq \eta_j \leq 1$, and $|\nabla \eta_j| \leq \frac{A}{r_0}$ for some constant $A > 0$. By Lemma 3.6 we have

$$\begin{aligned}
\|\varphi\|_{L^{\frac{2(d-1)}{d-2}}(\partial\Omega)} &\leq \sum_{j=1}^{J} \|\operatorname{Tr}(P\varphi)\|_{L^{\frac{2(d-1)}{d-2}}(\Psi_{r_0}(x_j))} \\
&\leq \sum_{j=1}^{J} \|\operatorname{Tr}(\eta_j P\varphi)\|_{L^{\frac{2(d-1)}{d-2}}(\Psi_{2r_0}(x_j))} \\
&\leq C \sum_{j=1}^{J} \|\nabla(\eta_j P\varphi)\|_{L^2(\Omega)} \\
&\leq CJ\left(\frac{A}{r_0}\|\nabla(P\varphi)\|_{L^2(\Omega)} + C\|P\varphi\|_{L^2(\Omega)}\right) \\
&\leq C\|P\varphi\|_{H^1(\Omega)} \\
&\leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}.
\end{aligned}$$

□

### 3.2 $\mathcal{B}$-spaces and Hölder Continuity

Suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition (2.3). Given $\gamma > 0$, $0 < r < r_0$, and $Q > \frac{d}{2}$, we say $u \in \mathcal{B}(\Omega_r(x), \gamma, Q)$ if $u \in H_D^1(\Omega) \cap L^\infty(\Omega)$, and for all $\Omega_s(y) \subseteq \Omega_r(x)$, $\varsigma \in (0, 1)$, and $k$ as below we have

$$\int_{\Omega_{k,s-\varsigma s}(y)} |\nabla u|^2 \leq \gamma \left[\frac{1}{\varsigma^2 s^{2\left(1-\frac{d}{2Q}\right)}} \sup_{\Omega_{k,s}(y)} (u-k)^2 + 1\right] |\Omega_{k,s}(y)|^{1-\frac{1}{Q}}. \tag{3.6}$$

Here, $k \in \mathbf{R}$ if $\partial\Omega_s(y) \cap D_\varepsilon = \emptyset$ and $k \geq 0$ if $\partial\Omega_s(y) \cap D_\varepsilon \neq \emptyset$.

In this section, we will closely follow the exposition given in the monograph of Ladyzhenskaya and Ural'tseva [20, pp. 81-95], though much of the notation is changed. The goal of this section is to show that elements $u \in \mathcal{B}(\Omega_r(x), \gamma, Q)$ are Hölder continuous. We begin by proving Lemma 3.8, a technical result which describes how $\sup_{\Omega_r(x)}(u-k)$ reacts to changes in $r$, assuming certain size conditions on $|\Omega_{k,r}(x)|$ and $\sup_{\Omega_{k,r}(x)}(u-k)$. We then use the aforementioned lemma to prove a result describing the oscillation of $u$ on $\Omega_r(x)$, from which Hölder continuity follows.

**Lemma 3.8.** *Let $x \in \partial\Omega$, $0 < r < r_0$, and suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition (2.3). If $u \in \mathcal{B}(\Omega_r(x), \gamma, Q)$ and $k$ is as in the definition of this space, then there is a number $\theta_1 > 0$ depending only on $d$, $\gamma$, $M$, $p$, and $q$, such that if $|\Omega_{k,r}| \leq \theta_1 r^d$ and $H := \sup_{\Omega_{k,r}}(u-k) \geq r^{1-\frac{d}{2Q}}$, then $\left|\Omega_{k+H/2,r/2}\right| = 0$, i.e.,*

$$\sup_{\Omega_{r/2}(x)} (u-k) \leq \frac{H}{2}.$$

*Proof.* Fix $\Omega_r(x)$ and $k$, and let $u \in \mathcal{B}(\Omega_r(x), \gamma, Q)$. As usual, we suppress the point $x$ in our notation and write $\Omega_r$ and $\Omega_{k,r}$. For $i = 0, 1, 2, \ldots$, set

$$r_i = \frac{r}{2} + \frac{r}{2^{i+1}} \quad \text{and} \quad k_i = k + \frac{H}{2} - \frac{H}{2^{i+1}}.$$

Set $\varsigma_i = \frac{r_i - r_{i+1}}{r_i} \in (0,1)$. Note that $r_i - \varsigma_i r_i = r_{i+1}$ and $k_{i+1} - k_i = H/2^{i+2}$. Thus, substituting $r_i$ for $s$ and $\varsigma_i$ for $\varsigma$ in formula (3.6), we obtain

$$\begin{aligned}\int_{\Omega_{k_i,r_{i+1}}} |\nabla u|^2 &\leq \gamma \left( \frac{r_i^{\frac{d}{Q}}}{(r_i - r_{i+1})^2} \sup_{\Omega_{k_i,r_i}} (u-k_i)^2 + 1 \right) |\Omega_{k_i,r_i}|^{1-\frac{1}{Q}} \\ &\leq \gamma \left( r^{\frac{d}{Q}-2} 2^{2i+4} H^2 + 1 \right) |\Omega_{k_i,r_i}|^{1-\frac{1}{Q}}. \qquad (3.7)\end{aligned}$$

By Corollary 3.5 we have for some constant $C_1$

$$(k_{i+1} - k_i) \left|\Omega_{k_{i+1},r_{i+1}}\right|^{1-\frac{1}{d}} \leq C_1 \frac{\left|\Omega_{r_{i+1}}\right|}{\left|\Omega_{r_{i+1}} \setminus \Omega_{k_i,r_{i+1}}\right|} \int_{\Omega_{k_i,r_{i+1}} \setminus \Omega_{k_{i+1},r_{i+1}}} |\nabla u| \, dy. \qquad (3.8)$$

Now, choose $\theta_1 \leq \frac{(M+1)\omega_{d-1}}{2^{d+1}d}$. A geometric argument shows $|\Omega_\rho| \geq \frac{(M+1)\omega_{d-1}}{d}\rho^d$ for any $0 < \rho \leq r$, so if $|\Omega_{k,r}| \leq \theta_1 r^d$, then

$$\begin{aligned}\left|\Omega_{k_i,r_{i+1}}\right| &\leq |\Omega_{k,r}| \\ &\leq \frac{1}{2} \frac{(M+1)\omega_{d-1}}{d} \left(\frac{r}{2}\right)^d \\ &\leq \frac{1}{2} \left|\Omega_{r/2}\right| \\ &\leq \frac{1}{2} \left|\Omega_{r_{i+1}}\right|.\end{aligned}$$

Hence, we deduce from (3.8) the estimate

$$\begin{aligned}\frac{H}{2^{i+2}} \left|\Omega_{k_{i+1},r_{i+1}}\right|^{1-\frac{1}{d}} &\leq 2C_1 \int_{\Omega_{k_i,r_{i+1}}} |\nabla u| \, dy \\ &\leq 2C_1 \left( \int_{\Omega_{k_i,r_{i+1}}} |\nabla u|^2 \, dy \right)^{\frac{1}{2}} |\Omega_{k_i,r_i}|^{\frac{1}{2}} \\ &\leq 2C_1 \gamma^{1/2} \left( r^{\frac{d}{Q}-2} 2^{2i+4} H^2 + 1 \right)^{\frac{1}{2}} |\Omega_{k_i,r_i}|^{1-\frac{1}{2Q}}.\end{aligned}$$

Rearranging constants then gives us

$$\left|\Omega_{k_{i+1},r_{i+1}}\right|^{1-\frac{1}{d}} \leq 2^{i+3}C_1\gamma^{1/2}\left(r^{\frac{d}{Q}-2}2^{2i+4}+H^{-2}\right)^{\frac{1}{2}}\left|\Omega_{k_i,r_i}\right|^{1-\frac{1}{2Q}}.$$

Now, if $H \geq r^{1-\frac{d}{2Q}}$, i.e., $H^{-2} \leq r^{\frac{d}{Q}-2}$, we may write

$$\left|\Omega_{k_{i+1},r_{i+1}}\right|^{1-\frac{1}{d}} \leq 2^{2i+6}C_1\gamma^{1/2}r^{\frac{d}{2Q}-1}\left|\Omega_{k_i,r_i}\right|^{1-\frac{1}{2Q}}. \tag{3.9}$$

Dividing both sides of (3.9) by $r^{d-1}$ and raising both sides to the $\frac{d}{d-1}$ power, we obtain

$$\frac{\left|\Omega_{k_{i+1},r_{i+1}}\right|}{r^d} \leq \left(2^6C_1\gamma^{1/2}\right)^{\frac{d}{d-1}}\left(2^{\frac{2d}{d-1}}\right)^i\left(\frac{\left|\Omega_{k_i,r_i}\right|}{r^d}\right)^{1+\frac{2Q-d}{2Q(d-1)}}. \tag{3.10}$$

Since $2Q-d>0$, by choosing

$$\theta_1 = \min\left\{\frac{(M+1)\omega_{d-1}}{2^{d+1}d}, 2^{\frac{12dQ}{d-2Q}+\frac{8dQ^2(1-d)}{(2Q-d)^2}}C_1^{\frac{2dQ}{d-2Q}}\gamma^{\frac{dQ}{d-2Q}}\right\},$$

we may conclude from Proposition 3.9 below that $\left|\Omega_{k_{i+1},r_{i+1}}\right| \to 0$ as $i\to\infty$, i.e., $\left|\Omega_{k+H/2,r/2}\right|=0$ as desired. □

**Proposition 3.9** (Lemma 4.7 of Ladyzhenskaya and Ural'tseva [20, p. 66])**.** *Let $C,\alpha>0$ and $B>1$ be fixed constants, and suppose a non-negative sequence $\{x_i\}_{i=0}^{\infty}$ satisfies the recursion relation*

$$x_{i+1} \leq B^iCx_i^{1+\alpha} \quad \textit{for all } i=0,1,\ldots.$$

*If $x_0 \leq B^{-\frac{1}{\alpha^2}}C^{-\frac{1}{\alpha}}$, then $x_i \leq B^{-\frac{1+i\alpha}{\alpha^2}}C^{-\frac{1}{\alpha}}$, and consequently $x_i\to 0$ as $i\to\infty$.*

**Lemma 3.10** (Oscillation Lemma)**.** *Suppose $D\subseteq\partial\Omega$ satisfies the corkscrew condition. There exists a natural number $s$ so that for any $x\in\partial\Omega$, $0<r<r_0$, and $u\in\mathcal{B}(\Omega_{4r}(x),\gamma,Q)$, at least one of the following holds:*

$$\operatorname*{osc}_{\Omega_r(x)} u \leq 2^sr^{1-\frac{d}{2Q}} \tag{3.11}$$

$$\operatorname*{osc}_{\Omega_r(x)} u \leq \left(1-2^{1-s}\right)\operatorname*{osc}_{\Omega_{4r}(x)} u. \tag{3.12}$$

*Here $\operatorname{osc}_E u := \sup_E u - \inf_E u$ is the* oscillation *of $u$ on the set $E$.*

*Proof.* First, note $\operatorname{osc}_E u = \operatorname{osc}_E(-u)$, so it suffices to prove the lemma for $u$ or $-u$. Also, as usual we suppress the point $x$ in our notation and write $\Omega_r$, $\Omega_{4r}$, etc.... Define

$$\begin{aligned}
M_r &:= \sup_{\Omega_r} u,\\
m_r &:= \inf_{\Omega_r} u,\\
\bar{M}_r &:= \frac{M_r+m_r}{2},\\
\mathcal{O}_r &:= M_r-m_r=\operatorname*{osc}_{\Omega_r} u,\\
\mathcal{O} &:= \mathcal{O}_{4r}, \quad \text{and}\\
A_t &:= \left\{y\in\Omega_{2r} : M_{4r}-\frac{\mathcal{O}}{2^t} < u(y) \leq M_{4r}-\frac{\mathcal{O}}{2^{t-1}}\right\}.
\end{aligned}$$

Set $k_t := M_{4r} - \frac{\mathcal{O}}{2^t}$ and observe that $A_t = \Omega_{k_t,2r} \setminus \Omega_{k_{t+1},2r}$ and $k_{t+1} - k_t = \frac{\mathcal{O}}{2^{t+1}}$. Finally, let $\theta_1$ be the constant from Lemma 3.8, $C_1$ the max of the two constants of Corollary 3.5, $C_0 = 2^{d+6-\frac{2d+1}{Q}} \cdot 5C_1^2 \gamma \omega_{d-1}^{2-1/Q}(1+M)^{2-\frac{1}{Q}}$, and

$$s = \max\left\{4, \frac{C_0}{\theta_1^{2-2/d}} + 3\right\}.$$

Now, let $u \in \mathcal{B}(\Omega_{4r}(x), \gamma, Q)$ and suppose (3.11) does not hold, i.e., $\mathcal{O}_r > 2^s r^{1-\frac{d}{2Q}}$. This implies $r^{1-\frac{d}{2Q}} < \mathcal{O}/2^s \leq \mathcal{O}/2^t$ for $t \leq s$. Combined with the facts $|\Omega_{4r}| \leq |Z_1|\,(4r)^d = 2^{2d+1}\omega_{d-1}(1+M)r^d$, $d/Q - 1 < 1$, and (3.6), we deduce

$$\begin{aligned}
\int_{\Omega_{k_t,2r}} |\nabla u|^2\,dy &\leq \gamma\left[4(4r)^{\frac{d}{Q}-2}\sup_{\Omega_{k_t,4r}}(u-k_t)^2 + 1\right]|\Omega_{k_t,4r}|^{1-\frac{1}{Q}} \\
&\leq \gamma\left[4^{\frac{d}{Q}-1}\left(\frac{\mathcal{O}}{2^t}\right)^2 + r^{2\left(1-\frac{d}{2Q}\right)}\right] r^{2\left(\frac{d}{2Q}-1\right)}|\Omega_{4r}|^{1-\frac{1}{Q}} \\
&\leq \gamma\left(\frac{\mathcal{O}}{2^t}\right)^2\left[4^{\frac{d}{Q}-1}+1\right]4^{d\left(1-\frac{1}{Q}\right)}|Z_1|^{1-\frac{1}{Q}}\,r^{d-2} \\
&\leq 2^{2d+1-\frac{2d+1}{Q}} \cdot 5\gamma\omega_{d-1}^{1-1/Q}(1+M)^{1-\frac{1}{Q}}\left(\frac{\mathcal{O}}{2^t}\right)^2 r^{d-2}. \qquad (3.13)
\end{aligned}$$

We now consider the following two cases: $\Omega_{2r} \cap D_\varepsilon = \emptyset$ and $\Omega_{2r} \cap D_\varepsilon \neq \emptyset$. In both cases, we claim

$$\left(\frac{\mathcal{O}}{2^{t+1}}\right)^2 \left|\Omega_{k_{t+1},2r}\right|^{2-\frac{2}{d}} \leq C_1^2\,|A_t| \int_{\Omega_{k_t,2r}} |\nabla u|^2\,dy. \qquad (3.14)$$

On the one hand, when $\Omega_{2r} \cap D_\varepsilon = \emptyset$ the sign of $k$ and $\ell$ are irrelevant and, after possibly replacing $u$ with $-u$, we may assume $\left|\Omega_{\bar{M}_{4r},2r}\right| \leq \frac{1}{2}|\Omega_{2r}|$. Referring to estimate (3.4) from Corollary 3.5, we obtain

$$\frac{\mathcal{O}}{2^{t+1}}\left|\Omega_{\ell,2r}\right|^{1-\frac{1}{d}} \leq C_1 \int_{A_t} |\nabla u|\,dy \qquad (3.15)$$

by observing that $\left|\Omega_{k_{t+1},2r}\right| \leq \left|\Omega_{\bar{M}_{4r},2r}\right| \leq \frac{1}{2}|\Omega_{2r}|$ and arguing as in the proof of Lemma 3.8. On the other hand, when $\Omega_{2r} \cap D_\varepsilon \neq \emptyset$ we assume $\bar{M}_{4r} \geq 0$, replacing $u$ with $-u$ if necessary, to ensure $k_t \geq 0$ for all $t$. Under this assumption, the condition $\left|\Omega_{\bar{M}_{4r},2r}\right| \leq \frac{1}{2}|\Omega_{2r}|$ is not necessarily true. However, we may instead use estimate (3.5) from Corollary 3.5 to obtain estimate (3.15). In each case, since $A_t \subseteq \Omega_{k_t,2r}$ we may apply the Cauchy-Schwarz inequality to (3.15) to obtain (3.14).

Now, combining estimates (3.13) and (3.14) and dividing by $\left(\frac{\mathcal{O}}{2^{t+1}}\right)^2$, we obtain for each $t \leq s-3$

$$\begin{aligned}
\left|\Omega_{k_{s-2},2r}\right|^{2-\frac{2}{d}} &\leq \left|\Omega_{k_{t+1},2r}\right|^{2-\frac{2}{d}} \\
&\leq 2^{2d+3-\frac{2d+1}{Q}} \cdot 5C_1^2\gamma\omega_{d-1}^{1-1/Q}(1+M)^{1-\frac{1}{Q}}\,|A_t|\,r^{d-2}. \qquad (3.16)
\end{aligned}$$

Summing (3.16) as $t$ goes from 1 to $s-3$ yields

$$\begin{aligned}(s-3)\left|\Omega_{k_{s-2},2r}\right|^{2-\frac{2}{d}} &\leq 2^{2d+3-\frac{2d+1}{Q}}\cdot 5C_1^2\gamma\omega_{d-1}^{1-1/Q}(1+M)^{1-\frac{1}{Q}}\left|\Omega_{k_1,2r}\setminus\Omega_{k_{s-2},2r}\right|r^{d-2}\\ &\leq 2^{3d+4-\frac{2d+1}{Q}}\cdot 5C_1^2\gamma\omega_{d-1}^{2-1/Q}(1+M)^{2-\frac{1}{Q}}r^{2d-2}\\ &= C_0(2r)^{2d-2}.\end{aligned}$$

From the above estimate and our choice of $s$, we immediately obtain

$$\left|\Omega_{k_{s-2},2r}\right| \leq \theta_1(2r)^d.$$

As in Lemma 3.8, set

$$H := \sup_{\Omega_{k_{s-2},2r}} (u-k_{s-2}).$$

We now consider what happens when $H < (2r)^{1-\frac{d}{2Q}}$ and when $H \geq (2r)^{1-\frac{d}{2Q}}$. In the first case, our initial assumption that (3.11) does not hold gives us

$$\begin{aligned}M_{2r} &< M_{4r} - \frac{\mathcal{O}}{2^{s-2}} + 2^{1-\frac{d}{2Q}}r^{1-\frac{d}{2Q}}\\ &< M_{4r} - \frac{\mathcal{O}}{2^{s-2}} + 2^{1-\frac{d}{2Q}}\frac{\mathcal{O}_r}{2^s}\\ &\leq M_{4r} - \frac{\mathcal{O}}{2^{s-1}},\end{aligned}$$

from which (3.12) follows immediately. In the second case, we refer to Lemma 3.8 in order to obtain

$$\begin{aligned}M_r &\leq M_{4r} - \frac{\mathcal{O}}{2^{s-2}} + \frac{1}{2}\left(M_{2r} - M_{4r} + \frac{\mathcal{O}}{2^{s-2}}\right)\\ &\leq M_{4r} - \frac{\mathcal{O}}{2^{s-1}},\end{aligned}$$

which is precisely (3.12). □

**Theorem 3.11** (Hölder continuity of functions in $\mathcal{B}(\Omega_r(x),\gamma,Q)$)**.** *Let $x\in\partial\Omega$ and $0<r<r_0$. If $u\in H^1_D(\Omega)$ is in $\mathcal{B}(\Omega_r(x),\gamma,Q)$, then $\operatorname{osc}_{\Omega_\rho(x)} u \leq C\left(\frac{\rho}{r}\right)^\alpha$ for each $0<\rho\leq r$, where $\alpha = \min\left\{1-\frac{d}{2Q}, -\log_4(1-2^{1-s})\right\}\in(0,1)$ and $s$ is as in Lemma 3.10.*

*Proof.* By Lemma 3.10, either (3.11) or (3.12) holds (with $r$ replaced by $\rho$ in each). In the first case, by choosing $C = 2^s r^{1-d/(2Q)}$ and $\alpha = 1-\frac{d}{2Q}$ we deduce $\operatorname{osc}_{\Omega_r(x)} u \leq C\left(\frac{\rho}{r}\right)^\alpha$, and the theorem follows easily. Thus, we assume (3.12) holds.

Observe that $s$ does not depend on $r$, so we may iterate this inequality to obtain

$$\operatorname*{osc}_{\Omega_{r/4^k}} u \leq \left(1-2^{1-s}\right)^k \operatorname*{osc}_{\Omega_r} u, \quad k=1,2,\ldots.$$

Choose $\alpha = -\log_4(1-2^{1-s})$ so that the above becomes

$$\begin{aligned}\operatorname*{osc}_{\Omega_{r/4^k}} u &\le 4^{-k\alpha} \operatorname*{osc}_{\Omega_r} u \\ &\le 2\cdot 4^{\alpha}\|u\|_{L^\infty(\Omega)} 4^{-\alpha(k+1)}.\end{aligned}$$

Now, let $0 < \rho \le r$ and let $k$ be such that $4^{-k-1} \le \frac{\rho}{r} < 4^{-k}$. It follows that

$$\begin{aligned}\operatorname*{osc}_{\Omega_\rho} u &\le \operatorname*{osc}_{\Omega_{r/4^k}} u \\ &\le 2\cdot 4^{\alpha}\|u\|_{L^\infty(\Omega)} 4^{-\alpha(k+1)} \\ &\le C\left(\frac{\rho}{r}\right)^{\alpha}.\end{aligned}$$

Note that $C = 4^{\alpha+\frac{1}{2}}\|u\|_{L^\infty(\Omega)}$ is independent of $\rho$ and $r$. □

### 3.3 Hölder Continuity of Solutions to (AMP)

From the previous section, we know that the problem of proving Hölder continuity of solutions $u_\varepsilon \in H^1_D(\Omega)$ to (AMP) can be reduced to showing that $u_\varepsilon \in \mathcal{B}(\Omega_{r_0}(x_j), \gamma, Q)$ for each $x_j$ from the definition of a Lipschitz domain (See Section 2.1). To do this, we will first prove a Caccioppoli inequality on $u_\varepsilon$ which holds when $f$ and $g$ are in certain $L^p$ spaces. From this inequality we deduce that $u_\varepsilon \in L^\infty(\Omega)$ uniformly in $\varepsilon$, and shortly thereafter, that $u_\varepsilon \in C^{0,\alpha}$, where $\alpha$ and $\|u_\varepsilon\|_{C^{0,\alpha}}$ are independent of $\varepsilon$.

**Lemma 3.12** (Caccioppoli Inequality)**.** *Suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition* (2.3)*, $a_\varepsilon$ is a standard family of functions, $f \in L^q(\Omega)$ for some $q > \frac{d}{2}$, and $g \in L^p(N)$ for some $p > d-1$. Let $u_\varepsilon \in H^1_D(\Omega)$ be a weak solution of* (AMP) *and $y \in \partial\Omega$. If $\Psi_r(y) \cap D_\varepsilon \ne \emptyset$ let $k \ge 0$; otherwise, let $k \in \mathbf{R}$ be arbitrary. The positive part of $u_\varepsilon - k$, denoted $v = (u_\varepsilon - k)^+$, satisfies*

$$\int_{\Omega_{k,r}(y)} \eta^2 |\nabla v|^2\,dx \le C\left(\int_{\Omega_{k,r}(y)} v^2|\nabla\eta|^2\,dx + \|f\|_q^2\,|\Omega_{k,r}|^{1-\frac{2}{q}+\frac{2}{d}} + \|g\|_p^2\,|\Omega_{k,r}|^{\frac{2}{\tau}-1}\right)$$

*whenever $\eta \in C^1_c(Z_r(y))$ satisfies $\eta \ge 0$ on $Z_r(y)$. Here, $\Omega_{k,r}(y) := \{x \in \Omega_r(y) : u_\varepsilon > k\}$, and $\tau = \frac{dp}{dp-d+1}$. Note that $C$ depends only on $d$, $M$, $p$ and $q$.*

*Proof.* Without loss of generality, we assume $y = 0$ and drop the dependence on $y$ from our notation, writing $\Omega_{k,r}$ rather than $\Omega_{k,r}(y)$. First, observe that $v \in H^1(\Omega)$ satisfies $\nabla v = \nabla u_\varepsilon$ when $u_\varepsilon > k$ and $\nabla v = 0$ when $u_\varepsilon < k$, but may not necessarily be an element of $H^1_D(\Omega)$ if $k < 0$. By how we chose $k$ and $\eta$, however, $\varphi := \eta^2 v \in H^1_D(\Omega)$ is a valid test function to use in the weak formulation of (AMP). Using Hölder's inequality and Cauchy's inequality with $\varepsilon = \frac{1}{2}$, we obtain the estimate

$$
\begin{aligned}
\int_{\Omega} \nabla u_\varepsilon \cdot \nabla \varphi \, dx &= \int_{\Omega_{k,r}} \eta^2 \nabla v \cdot \nabla v \, dx + 2 \int_{\Omega_{k,r}} v\eta (\nabla v \cdot \nabla \eta) \, dx \\
&\geq \int_{\Omega_{k,r}} \eta^2 |\nabla v|^2 \, dx - 2 \int_{\Omega_{k,r}} v |\eta| |\nabla v| |\nabla \eta| \, dx \\
&\geq \int_{\Omega_{k,r}} \eta^2 |\nabla v|^2 \, dx \\
&\quad - 2 \left( \int_{\Omega_{k,r}} v^2 |\nabla \eta|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{k,r}} |\eta|^2 |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \\
&\geq \frac{1}{2} \int_{\Omega_{k,r}} \eta^2 |\nabla v|^2 \, dx - \frac{1}{2} \int_{\Omega_{k,r}} v^2 |\nabla \eta|^2 \, dx.
\end{aligned}
$$

By the above calculation, and the fact that $u_\varepsilon$ is a weak solution of (AMP), we deduce

$$
\begin{aligned}
\int_{\Omega_{k,r}} \eta^2 |\nabla v|^2 \, dx &\leq \int_{\Omega_{k,r}} v^2 |\nabla \eta|^2 \, dx + 2 \int_{\Omega_{k,r}} \nabla u_\varepsilon \cdot \nabla (\eta^2 v) \, dx \\
&\leq 2 \left( \int_{\Omega_{k,r}} v^2 |\nabla \eta|^2 \, dx + \int_{\Omega_{k,r}} |f| v \eta^2 \, dx + \int_N |g| \operatorname{Tr}(v\eta^2) \, d\sigma \right. \\
&\quad \left. - \int_{\Psi_r \cap N} a_\varepsilon \operatorname{Tr}(u_\varepsilon) \operatorname{Tr}(v\eta^2) \, d\sigma \right).
\end{aligned}
$$

Now, if $\Psi_r \cap \Lambda_\varepsilon = \emptyset$ then $a_\varepsilon = 0$ on $N \cap \Psi_r$. If, on the other hand, $\Psi_r \cap \Lambda_\varepsilon \neq \emptyset$, the assumptions $a_\varepsilon \geq 0$ and $k \geq 0$ guarantee $a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr} v\eta^2 \geq 0$. In each case we obtain $-\int_{N \cap \Psi_r} a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr} v\eta^2 \, d\sigma \leq 0$, giving us

$$
\int_{\Omega_{k,r}} \eta^2 |\nabla v|^2 \, dx \leq 2 \left( \int_{\Omega_{k,r}} v^2 |\nabla \eta|^2 \, dx + \int_{\Omega_{k,r}} |f| v \eta^2 \, dx + \int_N |g| \operatorname{Tr}(v\eta^2) \, d\sigma \right).
$$

Now, if $d \geq 3$ let $t = 2$, and if $d = 2$ let $t \in [1, 2) \cap \left( \frac{2q}{3q-2}, 2 \right)$ be arbitrary. These conditions guarantee that the inequality $1 - \frac{1}{q} - \frac{1}{t} + \frac{1}{d} > 0$ holds for all $d \geq 2$. Using Hölder's inequality, the Sobolev inequality of Lemma 3.4 with $t$ as above, and Cauchy's inequality, we estimate

$$
\begin{aligned}
\int_{\Omega_{k,r}} |f| v\eta^2\,dx &\le \left(\int_{\Omega_{k,r}} |f|^q\,dx\right)^{\frac{1}{q}} \left(\int_{\Omega_{k,r}} |v\eta|^{\frac{dt}{d-t}}\,dx\right)^{\frac{1}{t}-\frac{1}{d}} |\Omega_{k,r}|^{1-\frac{1}{q}-\frac{1}{t}+\frac{1}{d}} \\
&\le C\|f\|_{L^q(\Omega)} \left(\int_{\Omega_{k,r}} |\nabla(v\eta)|^t\,dx\right)^{\frac{1}{t}} |\Omega_{k,r}|^{1-\frac{1}{q}-\frac{1}{t}+\frac{1}{d}} \\
&\le C\|f\|_{L^q(\Omega)} \left(\int_{\Omega_{k,r}} |\nabla(v\eta)|^2\,dx\right)^{\frac{1}{2}} |\Omega_{k,r}|^{\frac{1}{2}-\frac{1}{q}+\frac{1}{d}} \\
&\le \frac{1}{8}\int_{\Omega_{k,r}} |\nabla(v\eta)|^2\,dx + 2C^2\|f\|^2_{L^q(\Omega)} |\Omega_{k,r}|^{1-\frac{2}{q}+\frac{2}{d}} \\
&\le \frac{1}{4}\int_{\Omega_{k,r}} \left(\eta^2|\nabla v|^2 + v^2|\nabla\eta|^2\right)\,dx + 2C^2\|f\|^2_{L^q(\Omega)} |\Omega_{k,r}|^{1-\frac{2}{q}+\frac{2}{d}}.
\end{aligned}
$$

Thus,

$$
\int_{\Omega_{k,r}} \eta^2|\nabla v|^2\,dx \le C\left(\int_{\Omega_{k,r}} v^2|\nabla\eta|^2\,dx + \|f\|^2_{L^q(\Omega)} |\Omega_{k,r}|^{1-\frac{2}{q}+\frac{2}{d}} + \int_N |g|\operatorname{Tr}(v\eta^2)\,d\sigma\right).
$$

Next, observe that $p = \frac{\tau(d-1)}{d(\tau-1)}$ for $\tau = \frac{dp}{dp-d+1} \in \left[1, \frac{d}{d-1}\right) \subseteq [1,2)$, and $\frac{p}{p-1}$ is precisely $\frac{(d-1)\tau}{d-\tau}$. By two more applications of Hölder's inequality, as well as the trace inequality in Lemma 3.6 with $\tau$ as above and Cauchy's inequality, we deduce

$$
\begin{aligned}
\int_N |g|\operatorname{Tr}(v\eta^2)\,d\sigma &\le \left(\int_N |g|^p\,d\sigma\right)^{\frac{1}{p}} \left(\int_N |\operatorname{Tr}(v\eta)|^{\frac{p}{p-1}}\,d\sigma\right)^{1-\frac{1}{p}} \\
&\le C\|g\|_{L^p(N)} \left(\int_{\Omega_{k,r}} |\nabla(v\eta)|^\tau\,dx\right)^{\frac{1}{\tau}} \\
&\le C\|g\|_{L^p(N)} \left(\int_{\Omega_{k,r}} |\nabla(v\eta)|^2\,dx\right)^{\frac{1}{2}} |\Omega_{k,r}|^{\frac{1}{\tau}-\frac{1}{2}} \\
&\le \lambda\int_{\Omega_{k,r}} |\nabla(v\eta)|^2\,dx + \frac{C^2}{4\lambda}\|g\|^2_{L^p(N)} |\Omega_{k,r}|^{\frac{2}{\tau}-1} \\
&\le 2\lambda\int_{\Omega_{k,r}} \left(\eta^2|\nabla v|^2 + v^2|\nabla\eta|^2\right)\,dx + \frac{C^2}{4\lambda}\|g\|^2_{L^p(N)} |\Omega_{k,r}|^{\frac{2}{\tau}-1}.
\end{aligned}
$$

Hence, when $\lambda$ is chosen small enough we obtain the desired estimate. □

**Theorem 3.13.** *Suppose $D$ satisfies the corkscrew condition (2.3), $a_\varepsilon$ is a standard family of functions, $f \in L^q(\Omega)$ for some $q > \frac{d}{2}$, $g \in L^p(N)$ for some $p > d-1$, and $u_\varepsilon \in H^1_D(\Omega)$ is a weak solution of (AMP). For each $\Omega_{r_0}(x_j)$ as in the definition of $\Omega$ as a Lipschitz domain, there is a constant $C$ not depending on $\varepsilon$ such that*

$$
\|u_\varepsilon\|_{L^\infty(\Omega_{r_0}(x_j))} \le C\left(\|f\|_{L^q(\Omega)} + \|g\|_{L^p(N)}\right). \tag{3.17}
$$

*Proof.* Let $\rho = 2r_0$. As in Chapter 2, we may assume without loss of generality that $x_j = 0$ and write $Z_\rho$, $\Omega_\rho$ and $\Omega_{k,\rho}$ without dependence on $x_j$. Let $v := (u_\varepsilon - k)^+$ for $k \geq 0$. To find an upper bound for $u_\varepsilon$, we will show that there is a $k$ such that $v = 0$ a.e. in $\Omega_{\rho/2}$. To find a lower bound for $u_\varepsilon$, we observe that $-u_\varepsilon$ solves (AMP) with $f$ and $g$ replaced by $-f$ and $-g$, respectively, so we may apply the argument that follows to $(-u_\varepsilon - k)^+$ for $k \geq 0$.

Choose $\eta \in C_c^1(Z_\rho)$ so that $0 \leq \eta$ on $Z_\rho$. By Hölder's inequality, followed by Lemmas 3.4 and 3.12, we obtain

$$\begin{aligned}\int_{\Omega_{k,\rho}} |v\eta|^2\,dx &\leq \left(\int_{\Omega_{k,\rho}} |v\eta|^{\frac{2d}{d-2}}\,dx\right)^{1-\frac{2}{d}} |\Omega_{k,\rho}|^{\frac{2}{d}} \\ &\leq C\int_{\Omega_{k,\rho}} |\nabla(v\eta)|^2\,dx\,|\Omega_{k,\rho}|^{\frac{2}{d}} \\ &\leq C\left(\int_{\Omega_{k,\rho}} v^2|\nabla\eta|^2\,dx\,|\Omega_{k,\rho}|^{\frac{2}{d}} + \|f\|_q^2\,|\Omega_{k,\rho}|^{1-\frac{2}{q}+\frac{4}{d}} + \|g\|_p^2\,|\Omega_{k,\rho}|^{\frac{2}{\tau}-1+\frac{2}{d}}\right).\end{aligned}$$

Since $q > \frac{d}{2}$ it follows that $\frac{4}{d} - \frac{2}{q} > 0$, and since $\tau \in (1, \frac{d}{d-1})$ we have $\frac{2}{\tau} - 1 + \frac{2}{d} > 1$. Set $\theta := \min\left\{\frac{2}{d}, \frac{2}{\tau} + \frac{2}{d} - 2, \frac{4}{d} - \frac{2}{q}\right\} > 0$ and $k_0 := \max\left\{1, \|u_\varepsilon^+\|_{L^2(\Omega)}^2\right\}$. If $k \geq k_0$, then Tchebyshev's inequality implies

$$|\Omega_{k,R}| \leq \frac{1}{k}\int_{\Omega_{k,R}} u_\varepsilon^+\,dx \leq \frac{1}{k}\int_\Omega (u_\varepsilon^+)^2\,dx \leq 1,$$

so for all $k \geq k_0$ we have

$$\int_{\Omega_{k,\rho}} |v\eta|^2\,dx \leq C\left[\int_{\Omega_{k,\rho}} v^2|\nabla\eta|^2\,dx + \left(\|f\|_q^2 + \|g\|_p^2\right)|\Omega_{k,\rho}|\right]|\Omega_{k,\rho}|^\theta. \tag{3.18}$$

Now, for $0 < r < R \leq \rho$ choose $\eta \in C_c^\infty(Z_\rho)$ so that $\eta \equiv 1$ on $Z_r$, $\eta \equiv 0$ outside of $Z_R$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq \frac{C}{R-r}$ on $Z_\rho$. If $k \geq k_0$, we deduce from (3.18) that

$$\int_{\Omega_{k,r}} (u_\varepsilon - k)^2\,dx \leq C\left[\frac{1}{(R-r)^2}\int_{\Omega_{k,R}} (u_\varepsilon - k)^2\,dx + \left(\|f\|_q^2 + \|g\|_p^2\right)|\Omega_{k,R}|\right]|\Omega_{k,R}|^\theta. \tag{3.19}$$

Let $h > k \geq k_0$. Since $\Omega_{h,R} \subseteq \Omega_{k,R}$ and $\frac{u_\varepsilon - k}{h-k} \geq 1$ on $\Omega_{h,R}$, we have

$$\begin{aligned}|\Omega_{h,R}| &= \int_{\Omega_{h,R}} 1\,dx \\ &\leq \int_{\Omega_{h,R}} \frac{u_\varepsilon - k}{h-k}\,dx \\ &\leq \frac{1}{(h-k)^2}\int_{\Omega_{k,R}} (u_\varepsilon - k)^2\,dx.\end{aligned}$$

Therefore, if $h > k \geq k_0$ and $\frac{\rho}{2} \leq r < R \leq \rho$, estimate (3.19) gives us

$$\int_{\Omega_{h,r}} (u_\varepsilon - h)^2\, dx \leq C \left( \frac{1}{(R-r)^2} + \frac{\|f\|_q^2 + \|g\|_p^2}{(h-k)^2} \right) \frac{1}{(h-k)^{2\theta}} \left( \int_{\Omega_{k,R}} (u_\varepsilon - k)^2\, dx \right)^{1+\theta}. \tag{3.20}$$

Set $I(h,r) = \int_{\Omega_{h,r}} (u_\varepsilon - h)^2$, and let $k > 0$ be fixed. For $j = 0, 1, \ldots$, define

$$k_j := k_0 + k - \frac{k}{2^j},$$
$$r_j := \frac{\rho}{2} + \frac{\rho}{2^{j+1}}.$$

Observe that $k_j - k_{j-1} = \frac{k}{2^j}$ and $r_{j-1} - r_j = \frac{\rho}{2^{j+1}}$. We show by induction on $j$ that

$$I(k_j, r_j) \leq \frac{I(k_0, r_0)}{\gamma^j}, \qquad j = 0, 1, \ldots, \tag{3.21}$$

where $\gamma := 2^{\frac{2(1+\theta)}{\theta}} > 1$.

In the base case $j = 0$, (3.21) clearly holds, so assume the induction hypothesis $I(k_{j-1}, r_{j-1}) \leq I(k_0, r_0)/\gamma^{j-1}$ holds for some $j > 0$. By (3.20), we may write

$$\begin{aligned} I(k_j, r_j) &\leq C \left( \frac{1}{(r_{j-1} - r_j)^2} + \frac{\|f\|_q^2 + \|g\|_p^2}{(k_j - k_{j-1})^2} \right) \frac{1}{(k_j - k_{j-1})^{2\theta}} I(k_{j-1}, r_{j-1})^{1+\theta} \\ &= C \left( \frac{2^{2j+2}}{\rho^2} + \frac{2^{2j}(\|f\|_q^2 + \|g\|_p^2)}{k^2} \right) \frac{2^{2j\theta}}{k^{2\theta}} I(k_{j-1}, r_{j-1})^{1+\theta} \\ &\leq C \cdot \frac{k^2 + \rho^2(\|f\|_q^2 + \|g\|_p^2)}{\rho^2 k^{2(1+\theta)}} \cdot 2^{2j(1+\theta)} \cdot I(k_{j-1}, r_{j-1})^{1+\theta}. \end{aligned} \tag{3.22}$$

Next, by our induction hypothesis we have

$$\begin{aligned} I(k_{j-1}, r_{j-1})^{1+\theta} &\leq \left( \frac{I(k_0, r_0)}{\gamma^{j-1}} \right)^{1+\theta} \\ &\leq \frac{I(k_0, r_0)^\theta}{\gamma^{j\theta - (1+\theta)}} \cdot \frac{I(k_0, r_0)}{\gamma^j}. \end{aligned}$$

Placing this into (3.22), we obtain

$$I(k_j, r_j) \leq C\gamma^{1+\theta} \cdot \frac{k^2 + \rho^2(\|f\|_q^2 + \|g\|_p^2)}{\rho^2 k^2} \cdot \frac{I(k_0, r_0)^\theta}{k^{2\theta}} \cdot \frac{2^{2j(1+\theta)}}{\gamma^{j\theta}} \cdot \frac{I(k_0, r_0)}{\gamma^j}.$$

By how we chose $\gamma$, it follows that $\frac{2^{2j(1+\theta)}}{\gamma^{j\theta}} = 1$. Further, as $k \to \infty$ we have

$$\frac{k^2 + \rho^2(\|f\|_q^2 + \|g\|_p^2)}{\rho^2 k^2} \to \frac{1}{\rho^2} \quad \text{and} \quad \frac{I(k_0, r_0)}{k^2} \to 0.$$

Choosing $k = \max\left\{\rho(\|f\|_q^2 + \|g\|_p^2)^{1/2}, \frac{\sqrt{2^{1/\theta} I(k_0,r_0)}}{\rho^{1/\theta}}\right\}$ guarantees that for all $j$,

$$\gamma^{1+\theta} \cdot \frac{k^2 + \rho^2(\|f\|_q^2 + \|g\|_p^2)}{\rho^2 k^{2(1+\theta)}} \cdot I(k_0, r_0)^\theta \leq 1.$$

This completes the induction proof.

Hence, as $j \to \infty$ we find $I(k_j, r_j) \to 0$, i.e., $I(k_0 + k, \rho/2) = 0$. Consequently,

$$\left(\sup_{\Omega_{\rho/2}} u_\varepsilon\right)^2 \leq C\big(\|f\|_q^2 + \|g\|_p^2 + I(k_0, r_0)\big),$$

and since $I(k_0, r_0) \leq \|u_\varepsilon\|_{L^2(\Omega)}^2 \leq C(\|f\|_q + \|g\|_p)$, we also have estimate (3.17). $\square$

**Theorem 3.14.** *Suppose $D$ satisfies the corkscrew condition* (2.3)*, $a_\varepsilon$ is a standard family of functions, $f \in L^q(\Omega)$ for some $q > \frac{d}{2}$, and $g \in L^p(N)$ for some $p > d-1$. If $u_\varepsilon \in H_D^1(\Omega)$ is a weak solution of* (AMP)*, then $u_\varepsilon \in \mathcal{B}(\Omega_{r_0}(x_j), \gamma, Q)$ for each $\Omega_{r_0}(x_j)$ as in the definition of $\Omega$ as a Lipschitz domain. Here $Q$ depends only on $d$, $p$, and $q$, and $\gamma$ depends only on $d$, $\|f\|_q$, $\|g\|_p$, $M$, $p$, $q$, and $r_0$.*

*Proof.* Fix $\Omega_s(y) \subseteq \Omega_r(x)$ and $\varsigma \in (0,1)$. Throughout the proof $\Omega_s$ and $\Omega_{k,s}$ will always be centered at $y$. Let $\eta \in C^\infty(Z_s(y))$ be a cutoff function satisfying $\eta \equiv 1$ on $Z_{s-\varsigma s}(y)$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq \frac{C_0}{\varsigma s}$ on $Z_s(y)$. Set $v := (u_\varepsilon - k)^+ \in H^1(\Omega)$ denote the positive part of $u_\varepsilon - k$. By Lemma 3.12 and how we chose $\eta$, we know

$$\begin{aligned}
\int_{\Omega_{k,s-\varsigma s}} |\nabla v|^2\,dx &\leq \int_{\Omega_{k,s}} \eta^2|\nabla v|^2\,dx \\
&\leq C\left(\int_{\Omega_{k,s}(y)} v^2|\nabla\eta|^2\,dx + \|f\|_q^2\,|\Omega_{k,s}|^{1-\frac{2}{q}+\frac{2}{d}} + \|g\|_p^2\,|\Omega_{k,s}|^{\frac{2}{\tau}-1}\right) \\
&\leq C\left(\frac{1}{s^2\varsigma^2}\,|\Omega_{k,s}|\sup_{\Omega_{k,s}} v^2 + \|f\|_q^2\,|\Omega_{k,s}|^{1-\frac{2}{q}+\frac{2}{d}} + \|g\|_p^2\,|\Omega_{k,s}|^{\frac{2}{\tau}-1}\right),
\end{aligned}$$

where $\tau \in \left(1, \frac{d}{d-1}\right)$ solves $p = \frac{\tau(d-1)}{d(\tau-1)}$. Choose $Q := \min\left\{q, \frac{\tau}{2(\tau-1)}\right\} = \min\left\{q, \frac{dp}{2(d-1)}\right\}$. We make a number of observations:

- Both $1 - \frac{1}{q}$ and $\frac{2}{\tau} - 1$ are bounded below by $1 - \frac{1}{Q}$,
- $q > \frac{d}{2}$ by assumption, and $p > d-1$ implies $\frac{\tau}{2(\tau-1)} > \frac{d}{2}$, so $Q > \frac{d}{2}$,
- $|\Omega_{k,s}| \leq |Z_s| = 2(1+M)\omega_{d-1}s^d$, where $\omega_{d-1}$ is the Lebesgue measure of the unit ball in $\mathbf{R}^{d-1}$, and
- $|\Omega_{k,s}| \leq |Z_{r_0}| = 2(1+M)\omega_{d-1}r_0^d$.

Based on these observations, we conclude

$$\begin{aligned}\int_{\Omega_{k,s-\varsigma s}} |\nabla v|^2\,dx \le C\Bigg[&\frac{|Z_1|^{\frac{1}{Q}}}{\varsigma^2 s^{2(1-\frac{d}{2Q})}}\sup_{\Omega_{k,s}} v^2 + \|f\|_q^2\,|Z_{r_0}|^{\frac{2}{d}-\frac{2}{q}+\frac{1}{Q}}\\ &+ \|g\|_p^2\,|Z_{r_0}|^{\frac{2}{\tau}+\frac{1}{Q}-2}\Bigg]\,|\Omega_{k,s}|^{1-\frac{1}{Q}}\\ &\le \gamma\left(\frac{1}{\varsigma^2 s^{2(1-\frac{d}{2Q})}}\sup_{\Omega_{k,s}}(u-k)^2+1\right)|\Omega_{k,s}|^{1-\frac{1}{Q}}.\end{aligned}$$

□

### 3.4 Approximating Solutions to the Mixed Problem

**Proposition 3.15** (Lemma 2.2 of Taylor, Ott and Brown [38, p. 2900])**.** *Suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition* (2.3) *and $\Lambda$ satisfies the Ahlfors regularity condition* (2.4)*. If $x \in \Lambda$ and $-1 < s < \infty$, then there is a constant $C$ depending only on $M$ and $d$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$\frac{1}{C}\varepsilon^{d-1+s} \le \int_{\Psi_\varepsilon(x)} \delta(y)^s\,d\sigma(y) \le C\varepsilon^{d-1+s}.$$

**Corollary 3.16.** *Suppose $D \subseteq \partial\Omega$ satisfies the corkscrew condition* (2.3) *and $\Lambda$ satisfies the Ahlfors regularity condition* (2.4)*. There is a constant $C$ depending only on $d$, $M$, and $\mathcal{H}^{d-2}(\Lambda)$ so that*

$$\int_{\Lambda_\varepsilon} \delta(y)^s\,d\sigma(y) \le C\varepsilon^{s+1}.$$

*Proof.* Let $\mathcal{N}_\varepsilon$ be the smallest natural number such that there exists a collection of $\mathcal{N}_\varepsilon$ points $\{x_n\}_{n=1}^{\mathcal{N}_\varepsilon} \subset \Lambda$ so that $\Lambda \subseteq \bigcup_{n=1}^{\mathcal{N}_\varepsilon} \Psi_\varepsilon(x_n)$. It follows that $\Lambda_\varepsilon \subseteq \bigcup_{n=1}^{\mathcal{N}_\varepsilon} \Psi_{2\varepsilon}(x_n)$. Since $\Lambda$ is $(d-2)$-Ahlfors regular, it follows that $\mathcal{N}_\varepsilon = O(\varepsilon^{2-d})$ as $\varepsilon \to 0$. Hence, by Proposition 3.15 we conclude

$$\begin{aligned}\int_{\Lambda_\varepsilon} \delta^s\,d\sigma &\le \sum_{n=1}^{\mathcal{N}_\varepsilon}\int_{\Psi_\varepsilon(x_n)} \delta^s\,d\sigma\\ &\le C\varepsilon^{2-d}\varepsilon^{d-1+s}\\ &= C\varepsilon^{s+1}.\end{aligned}$$

□

**Corollary 3.17.** *Suppose $D$ satisfies the corkscrew condition* (2.3)*, $\Lambda$ satisfies the Ahlfors regularity condition* (2.4)*, $a_\varepsilon$ is a standard family of functions, $f \in L^q(\Omega)$ for some $q > \frac{d}{2}$, and $g \in L^p(N)$ for some $p > d-1$. If $u_\varepsilon \in H^1_D(\Omega)$ is a weak solution of* (AMP) *and $0 < \varepsilon < \varepsilon_0$, then there is a constant $C$ not depending on $\varepsilon$ such that*

$$\int_{\Lambda_\varepsilon} \frac{(\operatorname{Tr} u_\varepsilon)^2}{\delta}\,d\sigma \le C\varepsilon^{2\alpha},$$

*where $\alpha \in (0,1)$ is the minimum exponent obtained from Theorem 3.11.*

*Proof.* Since $\operatorname{Tr} u_\varepsilon$ vanishes on $D$ by assumption, and is Hölder continuous with exponent $\alpha$ on $\partial\Omega$ by Theorem 3.11, we know $|\operatorname{Tr} u_\varepsilon| \le C\delta^\alpha$, where $C$ does not depend on $\varepsilon$. Finally, since $-1 < 2\alpha - 1$ we may conclude from Corollary 3.17

$$\begin{aligned}\int_{\Lambda_\varepsilon} \frac{(\operatorname{Tr} u_\varepsilon)^2}{\delta}\, d\sigma &\le C \int_{\Lambda_\varepsilon} \delta^{2\alpha-1}\, d\sigma \\ &\le C\varepsilon^{2\alpha}.\end{aligned}$$

□

**Theorem 3.18.** *Under the same hypotheses as Corollary 3.17, if $u_\varepsilon \in H^1_D(\Omega)$ is a weak solution of* (AMP) *and $u_0 \in H^1_D(\Omega)$ is a weak solution of* (MP)*, then there is a constant $C$ not depending on $\varepsilon$ so that*

$$\|u_\varepsilon - u_0\|_{H^1(\Omega)} \le C\varepsilon^{1+\alpha}.$$

*Proof.* Recall that since $u_0 \in H^1_D(\Omega)$ is a weak solution of (MP) and $u_\varepsilon \in H^1_D(\Omega)$ is a weak solution of (AMP), for all $\varphi \in H^1_D(\Omega)$ we have

$$\int_\Omega \nabla u_0 \cdot \nabla\varphi\, dx = \int_\Omega f\varphi\, dx + \int_N g \operatorname{Tr}\varphi\, d\sigma \tag{3.23}$$

and

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \int_N a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr}\varphi\, d\sigma = \int_\Omega f\varphi\, dx + \int_N g \operatorname{Tr}\varphi\, d\sigma. \tag{3.24}$$

Let $\varphi := u_0 - u_\varepsilon \in H^1_D(\Omega)$. Subtracting (3.24) from (3.23) yields

$$\int_\Omega \nabla(u_0 - u_\varepsilon)\cdot\nabla(u_0 - u_\varepsilon)\, dx = \int_{\Lambda_\varepsilon} a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr}(u_0 - u_\varepsilon)\, d\sigma.$$

Then, by Lemmas 2.7 and 3.17, we write

$$\begin{aligned}\int_\Omega |\nabla(u_\varepsilon - u_0)|^2\, dx &\le \varepsilon \int_{\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon||\operatorname{Tr}(u_0 - u_\varepsilon)|}{\delta}\, d\sigma \\ &\le \left(\int_{\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon|^2}{\delta}\, d\sigma\right)^{\frac12} \left(\int_N \frac{|\operatorname{Tr}(u_0 - u_\varepsilon)|^2}{\delta}\, d\sigma\right)^{\frac12} \\ &\le C\varepsilon^{1+\alpha}\left(\int_\Omega |\nabla(u_0 - u_\varepsilon)|^2\, dx\right)^{\frac12}.\end{aligned}$$

Dividing both sides by $\left(\int_\Omega |\nabla(u_\varepsilon - u_0)|^2\, dx\right)^{1/2}$ gives the desired result. □

**Lemma 3.19.** *Suppose in addition to the hypotheses of Corollary 3.17 that $a_\varepsilon$ is a continuous family of functions (See* (2.11)*). If $u_\varepsilon, u_{\varepsilon+h} \in H^1_D(\Omega)$ are weak solutions of* (AMP) *for $0 < \varepsilon < \varepsilon_0$, $0 < \varepsilon + h < \varepsilon_0$, and $|h| < \varepsilon$, then there is a constant $C$ not depending on $\varepsilon$ or $h$ so that*

$$\|u_{\varepsilon+h} - u_\varepsilon\|_{H^1(\Omega)} \le C|h|\big(\|f\|_{L^q(\Omega)} + \|g\|_{L^p(N)}\big).$$

*Proof.* Let $\varphi := u_{\varepsilon+h} - u_\varepsilon \in H^1_D(\Omega)$. Then $u_{\varepsilon+h}, u_\varepsilon \in H^1_D(\Omega)$ satisfy

$$\int_\Omega \nabla u_{\varepsilon+h} \cdot \nabla\varphi\, dx + \int_N a_{\varepsilon+h} \operatorname{Tr} u_{\varepsilon+h} \operatorname{Tr}\varphi\, d\sigma = \int_\Omega f\varphi\, dx + \int_N g \operatorname{Tr}\varphi\, d\sigma \tag{3.25}$$

and

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \int_N a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr}\varphi\, d\sigma = \int_\Omega f\varphi\, dx + \int_N g \operatorname{Tr}\varphi\, d\sigma, \tag{3.26}$$

respectively. Subtracting equation (3.26) from (3.25) yields

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2\, dx &= -\int_N (a_{\varepsilon+h} \operatorname{Tr} u_{\varepsilon+h} - a_\varepsilon \operatorname{Tr} u_\varepsilon) \operatorname{Tr}\varphi\, d\sigma \\ &= -\int_N (a_{\varepsilon+h} - a_\varepsilon) \operatorname{Tr} u_{\varepsilon+h} \operatorname{Tr}\varphi\, d\sigma - \int_N a_\varepsilon |\operatorname{Tr}\varphi|^2\, d\sigma.\end{aligned}$$

Now, recall from Theorem 2.4 that the bilinear form $B_\varepsilon[u,\varphi] = \int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_N a_\varepsilon \operatorname{Tr} u \operatorname{Tr}\varphi\, d\sigma$ is coercive on $H^1_D(\Omega)$ whenever $0 < \varepsilon < \varepsilon_0$. In conjunction with condition (2.11) for a continuous family $a_\varepsilon$ and the Hardy inequality, this implies that for some $C > 0$ we have

$$\begin{aligned}C\|\nabla\varphi\|^2_{L^2(\Omega)} &\le \|\nabla\varphi\|^2_{L^2(\Omega)} + \int_N a_\varepsilon |\operatorname{Tr}\varphi|^2\, d\sigma \\ &= -\int_N (a_{\varepsilon+h} - a_\varepsilon) \operatorname{Tr} u_{\varepsilon+h} \operatorname{Tr}\varphi\, d\sigma \\ &\le C_\mu |h| \int_N \frac{|\operatorname{Tr} u_{\varepsilon+h}||\operatorname{Tr}\varphi|}{\delta}\, d\sigma \\ &\le C_\mu |h| \left(\int_N \frac{|\operatorname{Tr} u_{\varepsilon+h}|^2}{\delta}\, d\sigma\right)^{\frac12} \left(\int_N \frac{\operatorname{Tr}\varphi^2}{\delta}\, d\sigma\right)^{\frac12} \\ &\le C_\mu |h| \|\nabla u_{\varepsilon+h}\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} \\ &\le C_\mu |h| \left(\|f\|_{H^{-1}_D(\Omega)} + \|g\|_{H^{-1/2}(N)}\right) \|\nabla\varphi\|_{L^2(\Omega)}.\end{aligned}$$

Dividing the above inequality by $C\|\nabla\varphi\|_{L^2(\Omega)}$ completes the proof. □

### 3.5 Asymptotic Expansion of $u_\varepsilon$ in $\varepsilon$

Suppose $a_\varepsilon \ge 0$ is a *smooth* standard family of functions. We now consider weak solutions for the following boundary value problem:

$$\int_\Omega \nabla(u'_\varepsilon) \cdot \nabla\varphi\, dx + \int_N a_\varepsilon \operatorname{Tr}(u'_\varepsilon) \operatorname{Tr}\varphi\, d\sigma = -\left\langle a'_\varepsilon \operatorname{Tr} u_\varepsilon, \operatorname{Tr}\varphi\right\rangle_N. \tag{3.27}$$

Here, $a'_\varepsilon : \partial\Omega \to \mathbf{R}$ is the function given by $a'_\varepsilon(x) := \frac{\partial \widetilde{a}_x}{\partial\varepsilon}(\varepsilon)$, and $\widetilde{a}_x(\varepsilon)$ was defined in (2.12). In the case of the example function (2.8), we have

$$a'_\varepsilon(x) := \begin{cases} \frac{\mu\varepsilon^{\mu-1}}{\delta(x)^\mu}, & 0 < \delta(x) \le \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \tag{3.28}$$

which satisfies the bound $0 \le a'_\varepsilon(x) \le \mu/\delta(x)$ for all $x \in \partial\Omega \setminus \Lambda$.

By the Hardy inequality and the fact that $u_\varepsilon \in H^1_D(\Omega)$ is known, $a'_\varepsilon \operatorname{Tr} u_\varepsilon \in H^{-1/2}(N)$ with norm uniformly bounded in $\varepsilon$ by

$$\begin{aligned}\|a'_\varepsilon \operatorname{Tr} u_\varepsilon\|_{H^{-1/2}(\partial\Omega)} &\le C\mu\|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\le C\mu\big(\|f\|_{H_D^{-1}(\Omega)} + \|g\|_{H^{-1/2}(N)}\big).\end{aligned}$$

Hence, each $u'_\varepsilon \in H^1_D(\Omega)$ is a well-defined weak solution of (AMP) with $f = 0$ and $g = a'_\varepsilon \operatorname{Tr} u_\varepsilon$, that is,

$$\begin{cases} -\Delta u'_\varepsilon = 0 & \text{on } \Omega \\ u'_\varepsilon = 0 & \text{on } D \\ \frac{\partial u'_\varepsilon}{\partial \nu} + a_\varepsilon u'_\varepsilon = -a'_\varepsilon \operatorname{Tr} u_\varepsilon & \text{on } N. \end{cases} \tag{AMP$'$}$$

The problem (AMP$'$) and its weak formulation (3.27) are the result of formally differentiating (AMP) with respect to $\varepsilon$. We now wish to know in what sense $u'_\varepsilon$ can be understood as a derivative of $u_\varepsilon$, and whether there is a $u'_0 \in H^1_D(\Omega)$ such that $\|u'_\varepsilon - u'_0\|_{H^1(\Omega)} \to 0$ as $\varepsilon \to 0$. The following two theorems address these questions.

**Theorem 3.20.** *Under the same hypotheses as Corollary 3.17, there is a constant $C$ not depending on $\varepsilon$ so that*

$$\|u'_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^\alpha.$$

*Proof.* Since $u'_\varepsilon \in H^1_D(\Omega)$ is a weak solution of (AMP$'$), for some $C_0 > 0$ we have by the Hardy inequality, Theorem 3.14, and Corollary 3.16

$$\begin{aligned} C_0\|\nabla u'_\varepsilon\|^2_{L^2(\Omega)} &\le \int_\Omega |\nabla u'_\varepsilon|^2\,dx + \int_N a_\varepsilon|\operatorname{Tr} u'_\varepsilon|^2\,d\sigma \\ &= -\int_N a'_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr} u'_\varepsilon\,d\sigma \\ &\le \varepsilon^{\mu-1}\int_{\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon||\operatorname{Tr} u'_\varepsilon|}{\delta^\mu}\,d\sigma \\ &\le C\varepsilon^{\mu-1+\alpha}\int_{\Lambda_\varepsilon}\frac{|\operatorname{Tr} u'_\varepsilon|}{\delta^\mu}\,d\sigma \\ &\le C\varepsilon^{\mu-1+\alpha}\left(\int_{\Lambda_\varepsilon}\delta^{1-2\mu}\,d\sigma\right)^{\frac12}\left(\int_{\Lambda_\varepsilon}\frac{|\operatorname{Tr} u'_\varepsilon|}{\delta^\mu}\,d\sigma\right)^{\frac12} \\ &\le C\varepsilon^\alpha\|\nabla u'_\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

□

**Theorem 3.21.** *Suppose that in addition to the hypotheses of lemma 3.19, $a_\varepsilon$ is a smooth family of functions (See* (2.12)*). If $u'_\varepsilon \in H^1_D(\Omega)$ is a weak solution of* (AMP$'$)*, then there is a constant $C$ depending on $\varepsilon$ such that*

$$\|u_{\varepsilon+h} - u_\varepsilon - hu'_\varepsilon\|_{H^1(\Omega)} \le C|h|^{1+\alpha}.$$

*Proof.* Without loss of generality, assume $h > 0$. Let $\varphi := u_{\varepsilon+h} - u_\varepsilon - hu'_\varepsilon \in H^1_D(\Omega)$. Then $u_{\varepsilon+h}$, $u_\varepsilon$, and $hu'_\varepsilon$ satisfy, respectively,

$$\int_\Omega \nabla u_{\varepsilon+h} \cdot \nabla\varphi \, dx + \int_N a_{\varepsilon+h} \operatorname{Tr} u_{\varepsilon+h} \operatorname{Tr}\varphi \, d\sigma = \int_\Omega f\varphi \, dx + \int_N g \operatorname{Tr}\varphi \, d\sigma, \tag{3.29}$$

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi \, dx + \int_N a_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr}\varphi \, d\sigma = \int_\Omega f\varphi \, dx + \int_N g \operatorname{Tr}\varphi \, d\sigma, \tag{3.30}$$

and

$$\int_\Omega \nabla(hu'_\varepsilon) \cdot \nabla\varphi \, dx + \int_N a_\varepsilon \operatorname{Tr}(hu'_\varepsilon) \operatorname{Tr}\varphi \, d\sigma = -\int_N ha'_\varepsilon \operatorname{Tr} u_\varepsilon \operatorname{Tr}\varphi \, d\sigma. \tag{3.31}$$

Subtracting (3.30) and (3.31) from (3.29) and adding $\int_N a_{\varepsilon+h} |\operatorname{Tr}\varphi|^2 \, d\sigma$ yields

$$\begin{aligned}
C\|\nabla\varphi\|^2_{L^2(\Omega)} &\le \|\nabla\varphi\|^2_{L^2(\Omega)} + \int_N a_{\varepsilon+h} |\operatorname{Tr}\varphi|^2 \, d\sigma \\
&= -\int_N (a_{\varepsilon+h} - a_\varepsilon - ha'_\varepsilon) \operatorname{Tr} u_\varepsilon \operatorname{Tr}\varphi \, d\sigma \\
&\quad - h \int_N (a_{\varepsilon+h} - a_\varepsilon) \operatorname{Tr}(u'_\varepsilon) \operatorname{Tr}\varphi \, d\sigma \\
&\le \int_{\Lambda_\varepsilon} |a_{\varepsilon+h} - a_\varepsilon - ha'_\varepsilon| |\operatorname{Tr} u_\varepsilon| |\operatorname{Tr}\varphi| \, d\sigma \\
&\quad + \int_{\Lambda_{\varepsilon+h} \setminus \Lambda_\varepsilon} |a_{\varepsilon+h} - a_\varepsilon - ha'_\varepsilon| |\operatorname{Tr} u_\varepsilon| |\operatorname{Tr}\varphi| \, d\sigma \\
&\quad + |h| \int_N |a_{\varepsilon+h} - a_\varepsilon| |\operatorname{Tr}(u'_\varepsilon)| |\operatorname{Tr}\varphi| \, d\sigma \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Now, for $x \in \Lambda_\varepsilon$ we appeal to condition (2.12) for a smooth family $a_\varepsilon$ to write $|a_{\varepsilon+h}(x) - a_\varepsilon(x) - ha'_\varepsilon(x)| \le \frac{C_\mu \varepsilon^{\mu-2}}{\delta(x)^\mu} h^2$, and when $x \in \Lambda_{\varepsilon+h} \setminus \Lambda_\varepsilon$ we write $a_\varepsilon(x) = a'_\varepsilon(x) = 0$ and $|a_{\varepsilon+h}(x) - a_\varepsilon(x) - ha'_\varepsilon(x)| = a_{\varepsilon+h}(x) \le \frac{C_\mu \varepsilon^{\mu-1}}{\delta(x)^\mu}$. Thus, by Corollary 3.17 and the fact that $0 < h < \varepsilon$ we have

$$\begin{aligned}
I_1 &\le C_\mu \varepsilon^{-1} h^2 \int_{\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon| |\operatorname{Tr}\varphi|}{\delta} \, d\sigma \\
&\le C_\mu \varepsilon^{-1} h^2 \left( \int_{\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon|^2}{\delta} \, d\sigma \right)^{\frac{1}{2}} \left( \int_{\Lambda_\varepsilon} \frac{|\operatorname{Tr}\varphi|^2}{\delta} \, d\sigma \right)^{\frac{1}{2}} \\
&\le C_\mu \varepsilon^{\alpha - \frac{1}{2}} h^{\frac{3}{2}} \|\nabla\varphi\|_{L^2(\Omega)} \\
&\le C_\mu h^{1+\alpha} \|\nabla\varphi\|_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\le C_\mu h \int_{\Lambda_{\varepsilon+h}\setminus\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon||\operatorname{Tr}\varphi|}{\delta}\, d\sigma \\
&\le C_\mu h \left( \int_{\Lambda_{\varepsilon+h}\setminus\Lambda_\varepsilon} \frac{|\operatorname{Tr} u_\varepsilon|^2}{\delta}\, d\sigma \right)^{\frac{1}{2}} \left( \int_{\Lambda_{\varepsilon+h}\setminus\Lambda_\varepsilon} \frac{|\operatorname{Tr}\varphi|^2}{\delta}\, d\sigma \right)^{\frac{1}{2}} \\
&\le C_\mu h \left( (\varepsilon+h)^{2\alpha} - \varepsilon^{2\alpha} \right)^{\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega)} \\
&\le \alpha^{\frac{1}{2}} C_\mu h^{\frac{3}{2}} \varepsilon^{\alpha-\frac{1}{2}} \|\nabla\varphi\|_{L^2(\Omega)} \\
&\le \alpha^{\frac{1}{2}} C_\mu h^{1+\alpha} \|\nabla\varphi\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, we estimate $I_3 \le C_\mu h^2 \left( \|f\|_{H_D^{-1}(\Omega)} + \|g\|_{H^{-1/2}(N)} \right) \|\nabla\varphi\|_{L^2(\Omega)}$ as in the proof of Theorem 3.19. Hence,

$$\|\nabla\varphi\|_{L^2(\Omega)} \le C \left( \|f\|_{H_D^{-1}(\Omega)} + \|g\|_{H^{-1/2}(N)} + 1 \right) h^{1+\alpha}. \qquad \square$$

*Remark.* In Theorems 3.20 and 3.21 above, it suffices to let $a_\varepsilon$ be merely a "once-differentiable family," i.e., a continuous family for which condition (2.12) holds for $k = 0$ and 1, but not for $k \ge 2$.

# Chapter 4 Solving the Approximate Mixed Problem by Layer Potentials

The goal of this chapter is to establish a method for finding weak solutions of (AMP) by the method of layer potentials. Section 4 introduces the single- and double-layer potential operators as bounded operators on Sobolev spaces. In Section 4.1.4, we find a system of boundary integral equations which are equivalent to (AMP), and show that the system is uniquely solvable on an appropriate function space, provided the exponent $\mu$ from Section 2.3.1 is in the interval $(0, 1)$.

## 4.1 Layer Potentials

This section lays the groundwork for the chapter's main results. We begin with some notation, then deduce a representation formula for solutions $u \in H^1(\Omega)$ of $-\Delta u = f$. We conclude by obtaining estimates on the layer potentials.

### 4.1.1 Preliminary Notation

Recall that $\Omega^-$ denotes a bounded Lipschitz domain and $\Omega^+ := \mathbf{R}^d \setminus \bar{\Omega}$ its complementary unbounded domain. Also recall our use of $\omega_d$ to denote the measure of the unit ball in $\mathbf{R}^d$. Let $\operatorname{Tr}^\pm : H^1(\Omega^\pm) \to H^{1/2}(\partial\Omega)$ and $\frac{\partial^\pm}{\partial\nu} : H^1(\Omega^\pm) \to H^{-1/2}(\partial\Omega)$ be the one-sided trace and normal derivatives, respectively. Note that $\frac{\partial^\pm}{\partial\nu}$ may not necessarily be a continuous operator due to the dependence of $\frac{\partial^\pm u}{\partial\nu}$ on $-\Delta u$ (See Section 2.3.2). If $u \in L^2(\mathbf{R}^d)$ and $u|_{\Omega^\pm} \in H^1(\Omega^\pm)$, we denote the jumps in the trace and normal derivative of $u$ by

$$[u]_{\partial\Omega} := \operatorname{Tr}^+ u - \operatorname{Tr}^- u \quad \text{and} \quad \left[\frac{\partial u}{\partial\nu}\right]_{\partial\Omega} := \frac{\partial^+ u}{\partial\nu} - \frac{\partial^- u}{\partial\nu},$$

respectively. If such a $u$ satisfies $[u]_{\partial\Omega} = 0$ or $\left[\frac{\partial u}{\partial\nu}\right] = 0$, we usually drop the sign on the operator and write $\operatorname{Tr} u$ or $\frac{\partial u}{\partial\nu}$. Observe that $u \in H^1(\mathbf{R}^d)$ if and only if $[u]_{\partial\Omega} = 0$. Further, since we have adopted the convention of choosing $\nu$ to always point *out of* $\Omega^-$ and *into* $\Omega^+$, we have $\left[\frac{\partial v}{\partial\nu}\right]_{\partial\Omega} = 0$ for all $v \in C_c^\infty(\mathbf{R}^d)$. Recall, however, that there is a sign change in Green's identity (Lemma 2.3).

The two-sided trace $\operatorname{Tr} : H^1(\mathbf{R}^d) \to H^{1/2}(\partial\Omega)$ and normal derivative $\frac{\partial}{\partial\nu} : H^1(\mathbf{R}^d) \to H^{-1/2}(\partial\Omega)$ give rise to well-defined adjoint operators $\operatorname{Tr}^* : H^{-1/2}(\partial\Omega) \to H^{-1}(\mathbf{R}^d)$ and $\left(\frac{\partial}{\partial\nu}\right)^* : H^{1/2}(\partial\Omega) \to H^{-1}(\mathbf{R}^d)$, respectively, given by

$$\langle \operatorname{Tr}^* \psi, v\rangle_{\mathbf{R}^d} = \langle \psi, \operatorname{Tr} v\rangle_{\partial\Omega} \quad \forall\ \psi \in H^{-1/2}(\partial\Omega),\ v \in H^1(\mathbf{R}^d), \text{ and}$$
$$\left\langle \frac{\partial^*\varphi}{\partial\nu}, v\right\rangle_{\mathbf{R}^d} = \left\langle \frac{\partial v}{\partial\nu}, \varphi\right\rangle_{\partial\Omega} \quad \forall\ \varphi \in H^{1/2}(\partial\Omega),\ v \in H^1(\mathbf{R}^d).$$

Observe that $\operatorname{supp} \operatorname{Tr}^* \psi \subseteq \partial\Omega$ and $\operatorname{supp} \frac{\partial^*\varphi}{\partial\nu} \subseteq \partial\Omega$ for all $\psi \in H^{-1/2}(\partial\Omega)$ and $\varphi \in H^{1/2}(\partial\Omega)$. Indeed, if we choose $v \in C_c^\infty(\mathbf{R}^d)$ supported away from $\partial\Omega$, the above definitions give $\langle \operatorname{Tr}^* \psi, v\rangle_{\mathbf{R}^d} = \left\langle \frac{\partial^*\varphi}{\partial\nu}, v\right\rangle_{\mathbf{R}^d} = 0$. Thus, when it is convenient to do so we may consider $\operatorname{Tr}^* \psi$ and $\frac{\partial^*\varphi}{\partial\nu}$ as elements of $H_0^{-1}(\Omega^\pm)$ rather than $H^{-1}(\mathbf{R}^d)$.

### 4.1.2 Representation Formula

Let $\mathcal{G} : C_c^\infty(\mathbf{R}^d) \to C^\infty(\mathbf{R}^d)$ be an operator which satisfies $-\Delta(\mathcal{G}u) = u = \mathcal{G}(-\Delta u)$ for every $u \in C_c^\infty(\mathbf{R}^d)$. It can be shown (cf. the monographs of Evans [11] and Gilbarg & Trudinger [13]) that $\mathcal{G}u(x) = \int_{\mathbf{R}^d} G(x-y)u(y)\,dy$ is such an operator, where the convolution kernel $G : \mathbf{R}^d \setminus \{0\}$ is given by

$$G(z) := \begin{cases} -\frac{1}{2\pi} \ln\left(\frac{|z|}{R}\right), & d = 2,\ R > 0 \\ \frac{1}{d(d-2)\omega_d}|z|^{2-d}, & d \geq 3. \end{cases}$$

The function $G$ (and often the operator $\mathcal{G}$) is called the *fundamental solution of* $-\Delta$. Note that $\mathcal{G}$ has a natural extension $\mathcal{G} : (C^\infty(\mathbf{R}^d))^* \to (C_0^\infty(\mathbf{R}^d))^*$ given by

$$\langle \mathcal{G}\psi, \varphi\rangle_{\mathbf{R}^d} := \langle \psi, \mathcal{G}\varphi\rangle_{\mathbf{R}^d}, \quad \psi \in (C^\infty(\mathbf{R}^d))^*,\ \varphi \in C_0^\infty(\mathbf{R}^d).$$

Now, define the *single-* and *double-layer potential* operators $\mathrm{SL} : H^{-1/2}(\partial\Omega) \to C^\infty(\mathbf{R}^d)^*$ and $\mathrm{DL} : H^{1/2}(\partial\Omega) \to C^\infty(\mathbf{R}^d)^*$ by

$$\mathrm{SL}\,\psi := \mathcal{G}\,\mathrm{Tr}^*\,\psi,\ \psi \in H^{-1/2}(\partial\Omega), \quad \text{and} \quad \mathrm{DL}\,\varphi := \mathcal{G}\frac{\partial^*\varphi}{\partial\nu},\ \varphi \in H^{1/2}(\partial\Omega),$$

respectively. The following theorem gives a representation formula for solutions of Poisson's equation.

**Theorem 4.1** (Representation Formula). *Let $u = u^+ + u^- \in L^2(\mathbf{R}^d)$ and $f = f^+ + f^- \in H^{-1}(\mathbf{R}^d)$, where $u^\pm \in H^1(\Omega^\pm)$ and $f^\pm \in H_0^{-1}(\Omega^\pm)$, and suppose $-\Delta u^\pm = f^\pm$ on $\Omega^\pm$. If $\operatorname{supp} u$ and $\operatorname{supp} f$ are compact, then*

$$u = \mathcal{G}f - \mathrm{SL}\left[\frac{\partial u}{\partial\nu}\right]_{\partial\Omega} + \mathrm{DL}[u]_{\partial\Omega} \quad \text{on } \mathbf{R}^d. \tag{4.1}$$

*In particular, if $u^+ = 0$ and $f^+ = 0$, formula (4.1) reduces to*

$$u = \mathcal{G}f + \mathrm{SL}\frac{\partial^- u}{\partial\nu} - \mathrm{DL}\,\mathrm{Tr}^-\,u \quad \text{on } \mathbf{R}^d. \tag{4.2}$$

*Proof.* By Green's identity, for each $v \in H^1(\mathbf{R}^d)$ we have on the one hand

$$\begin{aligned}\int_{\mathbf{R}^d} \nabla u\cdot\nabla v\,dx &= \int_{\Omega^+}\nabla u\cdot\nabla v\,dx + \int_{\Omega^-}\nabla u\cdot\nabla v\,dx \\ &= \left\langle \frac{\partial^- u}{\partial\nu}, \mathrm{Tr}\,v\right\rangle_{\partial\Omega} - \left\langle \frac{\partial^+ u}{\partial\nu}, \mathrm{Tr}\,v\right\rangle_{\partial\Omega} + \langle f^+, v\rangle_{\Omega^+} + \langle f^-, v\rangle_{\Omega^-} \\ &= \langle f, v\rangle_{\mathbf{R}^d} - \left\langle \left[\frac{\partial u}{\partial\nu}\right]_{\partial\Omega}, \mathrm{Tr}\,v\right\rangle_{\partial\Omega},\end{aligned}$$

and on the other hand

$$\begin{aligned}\int_{\mathbf{R}^d} \nabla u \cdot \nabla v \, dx &= \int_{\Omega^+} \nabla u \cdot \nabla v \, dx + \int_{\Omega^-} \nabla u \cdot \nabla v \, dx \\ &= \left\langle \frac{\partial v}{\partial \nu}, \operatorname{Tr}^- u \right\rangle_{\partial\Omega} - \left\langle \frac{\partial v}{\partial \nu}, \operatorname{Tr}^+ u \right\rangle_{\partial\Omega} + \langle -\Delta v, u \rangle_{\Omega^+} + \langle -\Delta v, u \rangle_{\Omega^+} \\ &= \langle -\Delta v, u \rangle_{\mathbf{R}^d} - \left\langle \frac{\partial v}{\partial \nu}, [u]_{\partial\Omega} \right\rangle_{\partial\Omega}.\end{aligned}$$

Then, since $-\Delta$ is formally self-adjoint on $\mathbf{R}^d$, we deduce

$$\begin{aligned}\langle -\Delta u, v \rangle_{\mathbf{R}^d} &= \langle f, v \rangle_{\mathbf{R}^d} - \left\langle \left[\frac{\partial u}{\partial \nu}\right]_{\partial\Omega}, \operatorname{Tr} v \right\rangle_{\partial\Omega} + \left\langle \frac{\partial v}{\partial \nu}, [u]_{\partial\Omega} \right\rangle_{\partial\Omega} \\ &= \langle f, v \rangle_{\mathbf{R}^d} - \left\langle \operatorname{Tr}^* \left[\frac{\partial u}{\partial \nu}\right]_{\partial\Omega}, v \right\rangle_{\mathbf{R}^d} + \left\langle \left(\frac{\partial}{\partial \nu}\right)^* [u]_{\partial\Omega} \ , \ v \right\rangle_{\mathbf{R}^d}.\end{aligned}$$

Thus, we have shown that $-\Delta u \in H^{-1}(\mathbf{R}^d)$ is given by

$$-\Delta u = f - \operatorname{Tr}^* \left[\frac{\partial u}{\partial \nu}\right]_{\partial\Omega} + \left(\frac{\partial}{\partial \nu}\right)^* [u]_{\partial\Omega} \quad \text{on } \mathbf{R}^d. \tag{4.3}$$

Finally, since $\operatorname{supp} u$ and $\operatorname{supp} f$ are compact we can apply $\mathcal{G}$ to both sides of (4.3) to obtain formula (4.1). Equation (4.2) is an obvious consequence. □

### 4.1.3 Mapping Properties and Jump Relations

Since we saw in Section 2.3.2 that $-\Delta u \in H_0^{-1}(\Omega)$ whenever $u \in H^1(\Omega)$, we can say that in a certain sense that the Laplacian $-\Delta$ "takes away two derivatives" from the function on which it operates. The next lemma, in contrast, essentially says that $\mathcal{G}$ "gives two extra derivatives" to the objects on which it operates, and will help us to prove Lemmas (4.3) and (4.4).

**Lemma 4.2.** *Let $\eta_1, \eta_2 \in C_c^\infty(\mathbf{R}^d)$ be fixed. There is a constant $C$ depending on $d$, $\eta_1$, and $\eta_2$ (and the choice of $G$ when $d = 2$) such that for all $u \in H^{-1}(\mathbf{R}^d)$,*

$$\|\eta_1 \mathcal{G} \eta_2 u\|_{H^1(\mathbf{R}^d)} \leq C \|u\|_{H^{-1}(\mathbf{R}^d)}.$$

*Proof.* As noted in Chapter 2, $C_c^\infty(\mathbf{R}^d)$ is dense in $H^{-1}(\mathbf{R}^d)$, so it suffices to prove the estimate when $u \in C_c^\infty(\mathbf{R}^d)$. Further, by the Riesz representation theorem, $\|u\|_{H^{-1}(\mathbf{R}^d)} = \|u\|_{H^1(\mathbf{R}^d)}$ in this case.

Now, suppose $\eta_1, \eta_2$ are supported in balls $B_{R_1}(0), B_{R_2}(0)$, respectively, and let $x \in B_{R_1}(0)$. When $d \geq 3$,

$$\int_{B_{R_2}(0)} |x - y|^{2-d} \, dy \leq \frac{1}{2} d\omega_d R_2^2,$$

and when $d = 2$,

$$\int_{B_{R_2}(0)} \left| \ln \frac{|x-y|}{R} \right| \, dy \le \begin{cases} \frac{1}{4}(R_1+R_2)^2 \left[ 2\ln \frac{R}{R_1+R_2} + 1 \right], & R_1 + R_2 \le R, \\ \frac{1}{4}(R_1+R_2)^2 \left[ 2\ln \frac{R_1+R_2}{R} - 1 \right] + \frac{1}{4}R^2, & R_1 + R_2 > R. \end{cases}$$

Hence,

$$\begin{aligned} |\eta_1 \mathcal{G} \eta_2 u(x)|^2 &= \left| \eta_1(x) \int_{\mathbf{R}^d} G(x-y)\eta_2(y)u(y)\, dy \right|^2 \\ &\le \|\eta_2\|_\infty^2 |\eta_1(x)|^2 \left( \int_{B_{R_1+R_2}} |G(x-y)|\, dy \right) \left( \int_{B_{R_1+R_2}} |G(x-y)||u(y)|^2\, dy \right), \end{aligned}$$

and by Fubini's theorem,

$$\begin{aligned} \|\eta_1 \mathcal{G} \eta_2 u\|_{L^2(\mathbf{R}^d)} &\le C \|\eta_1\|_\infty \|\eta_2\|_\infty \left( \int_{B_{R_1}} \int_{B_{R_1+R_2}} |G(x-y)||u(y)|^2\, dy \right)^{\frac{1}{2}} \\ &\le C \|\eta_1\|_\infty \|\eta_2\|_\infty \|u\|_{L^2(\mathbf{R}^d)}. \end{aligned}$$

Finally,

$$\begin{aligned} \|\nabla[\eta_1 \mathcal{G} \eta_2 u](x)\|_{L^2(\mathbf{R}^d)} &\le \|\nabla(\eta_1)\mathcal{G}\eta_2 u(x)\|_{L^2(\mathbf{R}^d)} + \|\eta_1 \mathcal{G}(\nabla\eta_2) u(x)\|_{L^2(\mathbf{R}^d)} \\ &\quad + \|\eta_1 \mathcal{G} \eta_2 (\nabla u)(x)\|_{L^2(\mathbf{R}^d)} \\ &\le C \left( \|\nabla \eta_1\|_\infty \|\eta_2\|_\infty + \|\eta_1\|_\infty \|\nabla \eta_2\|_\infty \right) \|u\|_{L^2(\mathbf{R}^d)} \\ &\quad + C \|\eta_1\|_\infty \|\eta_2\|_\infty \|\nabla u\|_{L^2(\mathbf{R}^d)}. \end{aligned}$$

Hence, we have

$$\|\eta_1 \mathcal{G} \eta_2 u\|_{H^1(\mathbf{R}^d)} \le C \|u\|_{H^1(\mathbf{R}^d)}$$

as desired. □

Lemmas 4.3 and 4.4 are originally due to Costabel [4], though we use much of the notation found in the monograph of McLean [26].

**Lemma 4.3** (Mapping Properties and Jump Relations for SL)**.** *Fix* $\eta_1 \in C_c^\infty(\mathbf{R}^d)$*. The single-layer potential gives rise to continuous operators*

$$\eta_1 \,\mathrm{SL} : H^{-1/2}(\partial\Omega) \to H^1(\mathbf{R}^d), \tag{4.4}$$

$$\mathrm{Tr}\,\mathrm{SL} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \text{ and} \tag{4.5}$$

$$\frac{\partial^\pm}{\partial \nu} \mathrm{SL} : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \tag{4.6}$$

*and satisfies for all* $\psi \in H^{-1/2}(\partial\Omega)$ *the jump relations*

$$[\mathrm{SL}\,\psi]_{\partial\Omega} = 0 \quad \text{and} \quad \left[ \frac{\partial}{\partial \nu} \mathrm{SL}\,\psi \right]_{\partial\Omega} = -\psi. \tag{4.7}$$

*Proof.* Let $\psi \in H^{-1/2}(\partial\Omega)$ and let $\eta_2 \in C_c^\infty(\mathbf{R}^d)$ be a cutoff function which is 1 on a neighborhood of $\overline{\Omega^-}$. By Lemma 4.2 and the fact that $\operatorname{Tr} : H^1(\mathbf{R}^d) \to H^{1/2}(\partial\Omega)$ and $\operatorname{Tr}^* : H^{-1/2}(\partial\Omega) \to H^{-1}(\mathbf{R}^d)$ are bounded, we have

$$\begin{aligned}\|\eta_1 \operatorname{SL}\psi\|_{H^1(\mathbf{R}^d)} &= \|\eta_1 \mathcal{G} \operatorname{Tr}^* \psi\|_{H^1(\mathbf{R}^d)}\\ &\le C\|\operatorname{Tr}^*\psi\|_{H^{-1}(\mathbf{R}^d)}\\ &\le C\|\psi\|_{H^{-1/2}(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}\|\operatorname{Tr}\operatorname{SL}\psi\|_{H^{1/2}(\partial\Omega)} &= \|\operatorname{Tr}\eta_2 \operatorname{SL}\psi\|_{H^{1/2}(\partial\Omega)}\\ &\le C\|\eta_2 \operatorname{SL}\psi\|_{H^1(\mathbf{R}^d)}\\ &\le C\|\psi\|_{H^{-1/2}(\partial\Omega)},\end{aligned}$$

and thus properties (4.4) and (4.5) are established. Property (4.4) also gives the first jump relation of (4.7), since $\eta_2 \operatorname{SL}\psi \in H^1(\mathbf{R}^d)$ implies $[\operatorname{SL}\psi]_{\partial\Omega} = [\eta_2 \operatorname{SL}\psi]_{\partial\Omega} = 0$.

Now, since $\operatorname{supp}(\operatorname{Tr}^*\psi) \subseteq \partial\Omega$ we have $-\Delta(\operatorname{SL}\psi) = 0$ on $\Omega^-$, implying by (2.13) from Lemma 2.3 that

$$\begin{aligned}\left\|\frac{\partial^-}{\partial\nu}\operatorname{SL}\psi\right\|_{H^{-1/2}(\partial\Omega)} &= \left\|\frac{\partial^-}{\partial\nu}[\eta_2 \operatorname{SL}\psi]\right\|_{H^{-1/2}(\partial\Omega)}\\ &\le C\|\eta_2 \operatorname{SL}\psi\|_{H^1(\Omega^-)}\\ &\le C\|\psi\|_{H^{-1/2}(\partial\Omega)}.\end{aligned}$$

This is property (4.6) for $\frac{\partial^-}{\partial\nu}\operatorname{SL}\psi$. To obtain the analogous result for $\frac{\partial^+}{\partial\nu}\operatorname{SL}\psi$, we first observe that $-\Delta(\eta_2 \operatorname{SL}\psi) = -(\Delta\eta_2)\operatorname{SL}\psi - 2\nabla\eta_2 \cdot \nabla(\operatorname{SL}\psi)$ on $\Omega^+$. Let $\eta_3 \in C_c^\infty(\mathbf{R}^d)$ be 1 on a neighborhood of $\operatorname{supp}\eta_2$ and satisfy $\|\eta_3\|_\infty \le 1$ and $\|\nabla\eta_3\|_\infty \le 1$. Then Lemmas 2.3 and 4.2 give

$$\begin{aligned}\left\|\frac{\partial^+}{\partial\nu}\operatorname{SL}\psi\right\|_{H^{-1/2}(\partial\Omega)} &= \left\|\frac{\partial^+}{\partial\nu}[\eta_2 \operatorname{SL}\psi]\right\|_{H^{-1/2}(\partial\Omega)}\\ &\le C(\|\eta_2 \operatorname{SL}\psi\|_{H^1(\Omega^+)} + \|(\Delta\eta_2)\operatorname{SL}\psi\|_{H^1(\Omega^+)}\\ &\quad + \|\nabla\eta_2 \cdot \nabla \operatorname{SL}\psi\|_{L^2(\Omega^+)})\\ &\le C\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|\nabla\eta_2 \cdot \nabla(\eta_3\mathcal{G}\eta_3 \operatorname{Tr}^*\psi)\|_{L^2(\Omega^+)}\right)\\ &\le C\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|\eta_3\mathcal{G}\eta_3 \operatorname{Tr}^*\psi\|_{H^1(\mathbf{R}^d)}\right)\\ &\le C\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + C\|\operatorname{Tr}^*\psi\|_{H^{-1}(\mathbf{R}^d)}\right)\\ &\le C\|\psi\|_{H^{-1/2}(\partial\Omega)}.\end{aligned}$$

Finally, we prove the jump relation for $\left[\frac{\partial}{\partial\nu}\operatorname{SL}\psi\right]_{\partial\Omega}$ from (4.7). Let $u = \eta_2 \operatorname{SL}\psi$ and suppose $\Omega'$ is the neighborhood of $\overline{\Omega^-}$ on which $\eta_2 = 1$. On the one hand, $-\Delta u = \operatorname{Tr}^*\psi$ on $\Omega'$ by the definition of $\mathcal{G}$. On the other hand, $-\Delta u = -\operatorname{Tr}^*\left[\frac{\partial u}{\partial\nu}\right]_{\partial\Omega} + \left(\frac{\partial}{\partial\nu}\right)^*[u]_{\partial\Omega}$ by formula (4.3). Since we have already shown that $[u]_{\partial\Omega} = 0$, and we know $\left[\frac{\partial u}{\partial\nu}\right]_{\partial\Omega} = \left[\frac{\partial}{\partial\nu}\operatorname{SL}\psi\right]_{\partial\Omega}$, we deduce

$$\operatorname{Tr}^*\left(\psi + \left[\frac{\partial}{\partial\nu}\operatorname{SL}\psi\right]_{\partial\Omega}\right) = 0 \quad \text{on } \mathbf{R}^d.$$

Hence, $\langle \psi + \left[\frac{\partial}{\partial \nu} \operatorname{SL} \psi\right]_{\partial\Omega}, v\rangle_{\mathbf{R}^d} = 0$ for all $v \in H^1(\mathbf{R}^d)$, i.e., $\left[\frac{\partial}{\partial \nu} \operatorname{SL} \psi\right]_{\partial\Omega} = -\psi$. $\square$

**Lemma 4.4** (Mapping Properties and Jump Relations for DL)**.** *Fix $\eta_1 \in C_c^\infty(\mathbf{R}^d)$. The double-layer potential gives rise to continuous operators*

$$\eta_1 \operatorname{DL} : H^{1/2}(\partial\Omega) \to H^1(\Omega^\pm), \tag{4.8}$$

$$\operatorname{Tr}^\pm \operatorname{DL} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \text{ and} \tag{4.9}$$

$$\frac{\partial}{\partial \nu} \operatorname{DL} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \tag{4.10}$$

*and satisfies for all $\varphi \in H^{1/2}(\partial\Omega)$ the jump relations*

$$[\operatorname{DL} \varphi]_{\partial\Omega} = \varphi \quad \text{and} \quad \left[\frac{\partial}{\partial \nu} \operatorname{DL} \varphi\right]_{\partial\Omega} = 0. \tag{4.11}$$

*Proof.* By Theorem 2.4, for each $\varphi \in H^{1/2}(\partial\Omega)$ the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega^-, \\ u = \varphi & \text{on } \partial\Omega^- \end{cases}$$

has a unique solution $u_\varphi \in H^1(\Omega^-)$ with $\|u_\varphi\|_{H^1(\Omega^-)} \leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}$. Set $\widetilde{u}_\varphi \in L^2(\mathbf{R}^d)$ to be $u_\varphi$ on $\Omega^-$ and 0 on $\overline{\Omega^+}$, so that $[\widetilde{u}_\varphi]_{\partial\Omega} = -\varphi$ and $\left[\frac{\partial \widetilde{u}_\varphi}{\partial \nu}\right]_{\partial\Omega} = -\frac{\partial^- u_\varphi}{\partial \nu}$. By the representation formula (4.2) we have

$$\operatorname{DL} \varphi = \operatorname{SL} \frac{\partial^- u_\varphi}{\partial \nu} - \widetilde{u}_\varphi \quad \text{on } \mathbf{R}^d.$$

We also have $-\Delta \widetilde{u}_\varphi = 0$ on $\Omega^-$, which gives $\left\|\frac{\partial^- u_\varphi}{\partial \nu}\right\|_{H^{-1/2}(\partial\Omega)} \leq C\|u_\varphi\|_{H^1(\Omega^-)} \leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}$ by Lemma 2.3. Hence, the mapping property (4.4) gives

$$\begin{aligned} \|\eta_1 \operatorname{DL} \varphi\|_{H^1(\Omega^-)} &\leq \left\|\eta_1 \operatorname{SL} \frac{\partial^- u_\varphi}{\partial \nu}\right\|_{H^1(\mathbf{R}^d)} + \|\widetilde{u}_\varphi\|_{H^1(\Omega^-)} \\ &\leq C\left\|\frac{\partial^- u_\varphi}{\partial \nu}\right\|_{H^{-1/2}(\partial\Omega)} + \|u_\varphi\|_{H^1(\Omega^-)} \\ &\leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

and

$$\begin{aligned} \|\eta_1 \operatorname{DL} \varphi\|_{H^1(\Omega^+)} &\leq \left\|\eta_1 \operatorname{SL} \frac{\partial^- u_\varphi}{\partial \nu}\right\|_{H^1(\mathbf{R}^d)} + \|\widetilde{u}_\varphi\|_{H^1(\Omega^+)} \\ &\leq C\left\|\frac{\partial^- u_\varphi}{\partial \nu}\right\|_{H^{-1/2}(\partial\Omega)} \\ &\leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

which confirms property (4.8). Property (4.9) then follows immediately from the boundedness of the trace maps $\mathrm{Tr}^\pm : H^1(\Omega^\pm) \to H^{1/2}(\partial\Omega)$. To prove (4.10), we again appeal to Lemma 2.3, as well as property (4.6), to write

$$\begin{aligned}
\left\| \frac{\partial^\pm}{\partial\nu} \operatorname{DL} \varphi \right\|_{H^{-1/2}(\partial\Omega)} &\leq \left\| \frac{\partial^\pm}{\partial\nu} \operatorname{SL} \frac{\partial^- u_\varphi}{\partial\nu} \right\|_{H^{-1/2}(\partial\Omega)} + \left\| \frac{\partial^\pm \widetilde{u}_\varphi}{\partial\nu} \right\|_{H^{-1/2}(\partial\Omega)} \\
&\leq C \left\| \frac{\partial^- u_\varphi}{\partial\nu} \right\|_{H^{-1/2}(\partial\Omega)} \\
&\leq C \|\varphi\|_{H^{1/2}(\partial\Omega)}.
\end{aligned}$$

Finally, we prove the jump relations (4.11). By the mapping properties (4.7) for the single-layer potential, $\left[\operatorname{SL} \frac{\partial^- u_\varphi}{\partial\nu}\right]_{\partial\Omega} = 0$ and $\left[\frac{\partial}{\partial\nu} \operatorname{SL} \frac{\partial^- u_\varphi}{\partial\nu}\right]_{\partial\Omega} = -\frac{\partial^- u_\varphi}{\partial\nu}$. Hence,

$$\begin{aligned}
[\operatorname{DL} \varphi]_{\partial\Omega} &= \left[\operatorname{SL} \frac{\partial^- u_\varphi}{\partial\nu}\right]_{\partial\Omega} - [\widetilde{u}_\varphi]_{\partial\Omega} \\
&= \varphi
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{\partial}{\partial\nu} \operatorname{DL} \varphi\right]_{\partial\Omega} &= \left[\frac{\partial}{\partial\nu} \operatorname{SL} \frac{\partial^- u_\varphi}{\partial\nu}\right]_{\partial\Omega} - \left[\frac{\partial \widetilde{u_\varphi}}{\partial\nu}\right] \\
&= -\frac{\partial^- u_\varphi}{\partial\nu} + \frac{\partial^- u_\varphi}{\partial\nu} \\
&= 0.
\end{aligned}$$

□

**Corollary 4.5.** *The operators*

$$\begin{aligned}
\mathcal{A} &:= \operatorname{Tr} \operatorname{SL} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \\
\mathcal{B} &:= \frac{\partial^+}{\partial\nu} \operatorname{SL} + \frac{\partial^-}{\partial\nu} \operatorname{SL} : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \\
\mathcal{C} &:= \mathrm{Tr}^+ \operatorname{DL} + \mathrm{Tr}^- \operatorname{DL} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \quad \textit{and} \\
\mathcal{D} &:= -\frac{\partial}{\partial\nu} \operatorname{DL} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)
\end{aligned}$$

*are all bounded. Further, $\mathcal{B}$ and $\mathcal{C}$ satisfy the jump relations*

$$\frac{\partial^\pm}{\partial\nu} \operatorname{SL} \psi = \frac{1}{2}(\mp\psi + \mathcal{B}\psi), \quad \textit{and} \tag{4.12}$$

$$\mathrm{Tr}^\pm \operatorname{DL} \varphi = \frac{1}{2}(\pm\varphi + \mathcal{C}\varphi) \tag{4.13}$$

*for all $\psi \in H^{-1/2}(\partial\Omega)$ and $\varphi \in H^{1/2}(\partial\Omega)$.*

### 4.1.4 The Associated System of Boundary Integral Equations

The main result of this subsection is Theorem 4.6, in which we derive a system of boundary integral equations associated with (AMP), aptly named the *associated system*, whose solution (or solutions) determine the weak solution of (AMP). We begin by defining bounded operators

$$\mathcal{A}^{\varepsilon}_{DD} : \widetilde{H}^{-1/2}(D) \to H^{1/2}(D_\varepsilon) \qquad \mathcal{B}_{ND} : \widetilde{H}^{-1/2}(D) \to H^{-1/2}(N)$$
$$\mathcal{C}^{\varepsilon}_{DN} : \widetilde{H}^{1/2}(N) \to H^{1/2}(D_\varepsilon) \qquad \mathcal{D}_{NN} : \widetilde{H}^{1/2}(N) \to H^{-1/2}(N)$$

by

$$\mathcal{A}^{\varepsilon}_{DD}\psi := (\mathcal{A}\psi)|_{D_\varepsilon} \quad \text{and} \quad \mathcal{B}_{ND}\psi := (\mathcal{B}\psi)|_N \quad \text{for } \psi \in \widetilde{H}^{-1/2}(D),$$

and

$$\mathcal{C}^{\varepsilon}_{DN}\varphi := (\mathcal{C}\varphi)|_{D_\varepsilon} \quad \text{and} \quad \mathcal{D}_{NN}\varphi := (\mathcal{D}\varphi)|_N \quad \text{for } \varphi \in \widetilde{H}^{1/2}(N).$$

For $0 < \varepsilon < \varepsilon_0$ let $a_\varepsilon$ be a standard family of functions as in (2.9) and (2.10), and for $\varepsilon = 0$ let $a_0$ be identically 0 on $\partial\Omega$. Let $\boldsymbol{H}_\varepsilon = \widetilde{H}^{-1/2}(D_\varepsilon) \times \widetilde{H}^{1/2}(N)$ and $\boldsymbol{H}^*_\varepsilon = H^{1/2}(D_\varepsilon) \times H^{-1/2}(N)$, and for $\boldsymbol{\varphi} := \left[\begin{smallmatrix}\varphi_D\\ \varphi_N\end{smallmatrix}\right] \in \boldsymbol{H}_\varepsilon$ and $\boldsymbol{\psi} := \left[\begin{smallmatrix}\psi_D\\ \psi_N\end{smallmatrix}\right] \in \boldsymbol{H}^*_\varepsilon$ let

$$\langle \boldsymbol{\psi}, \boldsymbol{\varphi}\rangle_{D_\varepsilon \times N} := \langle \psi_D, \varphi_D\rangle_{D_\varepsilon} + \langle \psi_N, \varphi_N\rangle_N .$$

Define operators $\boldsymbol{A}^{\varepsilon}_0, \boldsymbol{A}_\varepsilon, \boldsymbol{K}_\varepsilon : \boldsymbol{H}_\varepsilon \to \boldsymbol{H}^*_\varepsilon$ by

$$\boldsymbol{A}^{\varepsilon}_0 = \begin{bmatrix} \mathcal{A}^{\varepsilon}_{DD} & -\frac{1}{2}\mathcal{C}^{\varepsilon}_{DN} \\ \frac{1}{2}\mathcal{B}_{ND} & \mathcal{D}_{NN} \end{bmatrix}, \quad \boldsymbol{K}_\varepsilon = \begin{bmatrix} 0 & -\mathcal{A}a_\varepsilon \\ 0 & -\frac{1}{2}\mathcal{B}a_\varepsilon \end{bmatrix}, \quad \text{and} \quad \boldsymbol{A}_\varepsilon = \boldsymbol{A}^{\varepsilon}_0 + \boldsymbol{K}_\varepsilon,$$

where the operators $\mathcal{A}a_\varepsilon$ and $\mathcal{B}a_\varepsilon$ are understood to be the composition of the multiplication operator $\psi \mapsto a_\varepsilon\psi$ with $\mathcal{A}$ and $\mathcal{B}$, respectively.

**Theorem 4.6.** *Let $f \in H^{-1}_D(\Omega^-)$, $g \in H^{-1/2}(N)$, and $a_\varepsilon$ a standard family of functions. Set $a_\varepsilon$ to be identically 0 when $\varepsilon = 0$. Choose $\Gamma_D \in \widetilde{H}^{1/2}(N)$ and $\Gamma_N \in H^{-1/2}(\partial\Omega)$ with $g := \Gamma_N$ on $N$. Define $\boldsymbol{h}_\varepsilon = \left[\begin{smallmatrix}h_{D,\varepsilon}\\ h_{N,\varepsilon}\end{smallmatrix}\right] \in \boldsymbol{H}^*_\varepsilon$ by*

$$h_{D,\varepsilon} := \left(-\operatorname{Tr}\mathcal{G}f + \mathcal{A}(a_\varepsilon\Gamma_D - \Gamma_N) - \frac{1}{2}(\Gamma_D - \mathcal{C}\Gamma_D)\right)\Big|_{D_\varepsilon}, \quad \textit{and}$$
$$h_{N,\varepsilon} := \left(-\frac{\partial^-}{\partial\nu}\mathcal{G}f + \frac{1}{2}\big(g + a_\varepsilon\Gamma_D + \mathcal{B}(a_\varepsilon\Gamma_D - \Gamma_N)\big) - \mathcal{D}\Gamma_D\right)\Big|_N .$$

*For fixed $\varepsilon$, $0 \le \varepsilon < \varepsilon_0$, if $\boldsymbol{\psi}_\varepsilon := \left[\begin{smallmatrix}\psi_{D,\varepsilon}\\ \psi_{N,\varepsilon}\end{smallmatrix}\right] \in \boldsymbol{H}_\varepsilon$ solves the system of integral equations*

$$\boldsymbol{A}_\varepsilon\,\boldsymbol{\psi}_\varepsilon = \boldsymbol{h}_\varepsilon, \tag{4.14}$$

*then the weak solution $u_\varepsilon \in H^1_D(\Omega^-)$ of* (AMP) *has integral representation*

$$u_\varepsilon = \mathcal{G}f + \mathrm{SL}\,(\psi_{D,\varepsilon} - a_\varepsilon\psi_{N,\varepsilon} + \Gamma_N - a_\varepsilon\Gamma_D) - \mathrm{DL}(\psi_{N,\varepsilon} + \Gamma_D) \quad \textit{on } \Omega^-. \tag{4.15}$$

*Conversely, if* $u_\varepsilon \in H^1_D(\Omega^-)$ *is a weak solution of* (AMP)*, then* $\boldsymbol{\psi}_\varepsilon := \begin{bmatrix} \psi_{D,\varepsilon} \\ \psi_{N,\varepsilon} \end{bmatrix} \in \boldsymbol{H}_\varepsilon$ *given by*

$$\psi_{D,\varepsilon} := \frac{\partial^- u_\varepsilon}{\partial \nu} + a_\varepsilon \operatorname{Tr}^- u_\varepsilon - \Gamma_N \quad \textit{and} \quad \psi_{N,\varepsilon} := \operatorname{Tr}^- u_\varepsilon - \Gamma_D \tag{4.16}$$

*solves the system 4.14.*

*Proof.* Suppose $\boldsymbol{\psi}_\varepsilon \in \boldsymbol{H}_\varepsilon$ solves the system (4.14), and let $u_\varepsilon$ be given by (4.15) on $\Omega^-$. It follows easily from how we defined $\mathcal{G}$ that $-\Delta u_\varepsilon = f$ on $\Omega^-$, and from the mapping properties (4.4) and (4.8) that $u_\varepsilon \in H^1(\Omega^-)$.

By the definition of $\mathcal{A}$ and the jump relation (4.13) from Corollary 4.5 we have

$$\begin{aligned}
\operatorname{Tr}^- u_\varepsilon &= \operatorname{Tr} \mathcal{G} f + \operatorname{Tr} \operatorname{SL}(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon} + \Gamma_N + a_\varepsilon \Gamma_D) - \operatorname{Tr}^- \operatorname{DL}(\psi_{N,\varepsilon} + \Gamma_D) \\
&= \operatorname{Tr} \mathcal{G} f + \mathcal{A}\left(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon} + \Gamma_N - a_\varepsilon \Gamma_D\right) + \frac{1}{2}\left(\psi_{N,\varepsilon} + \Gamma_D - \mathcal{C}\psi_{N,\varepsilon} - \mathcal{C}\Gamma_D\right) \\
&= -\left(-\operatorname{Tr} \mathcal{G} f + \mathcal{A}(a_\varepsilon \Gamma_D - \Gamma_N) - \frac{1}{2}(\Gamma_D - \mathcal{C}\Gamma_D)\right) \\
&\quad + \left(\mathcal{A}\psi_{D,\varepsilon} - \left(\mathcal{A} a_\varepsilon + \frac{1}{2}\mathcal{C}\right)\psi_{N,\varepsilon}\right) + \frac{1}{2}\psi_{N,\varepsilon}.
\end{aligned}$$

Restricting to $D_\varepsilon$ and recalling that $\boldsymbol{\psi}_\varepsilon$ solves (4.14), we obtain

$$\begin{aligned}
\operatorname{Tr}^- u_\varepsilon &= -h_{D,\varepsilon} + \mathcal{A}^\varepsilon_{DD}\psi_{D,\varepsilon} - \left(\mathcal{A} a_\varepsilon + \frac{1}{2}\mathcal{C}^\varepsilon_{DN}\right)\psi_{N,\varepsilon} + \frac{1}{2}\psi_{N,\varepsilon} \\
&= \frac{1}{2}\psi_{N,\varepsilon}.
\end{aligned} \tag{4.17}$$

In particular, this implies $\operatorname{Tr}^- u_\varepsilon = 0$ on $D$, i.e., $u_\varepsilon \in H^1_D(\Omega^-)$.

It remains to show that $u_\varepsilon$ satisfies the boundary condition $\frac{\partial^- u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon = g$ on $N$. By the definition of $\mathcal{D}$ and the identity (4.12) from Corollary 4.5 we have

$$\begin{aligned}
\frac{\partial^- u_\varepsilon}{\partial \nu} &= \frac{\partial^-}{\partial \nu}\mathcal{G} f + \frac{\partial^-}{\partial \nu}\operatorname{SL}(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon} + \Gamma_N - a_\varepsilon \Gamma_D) - \frac{\partial}{\partial \nu}\operatorname{DL}(\psi_{N,\varepsilon} + \Gamma_D) \\
&= \frac{\partial^-}{\partial \nu}\mathcal{G} f + \frac{1}{2}(I + \mathcal{B})\big(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon} + \Gamma_N - a_\varepsilon \Gamma_D\big) + \mathcal{D}(\psi_{N,\varepsilon} + \Gamma_D) \\
&= -\left(-\frac{\partial^-}{\partial \nu}\mathcal{G} f + \frac{1}{2}\big(\Gamma_N + a_\varepsilon \Gamma_D + \mathcal{B}(a_\varepsilon \Gamma_D - \Gamma_N)\big) - \mathcal{D}\Gamma_D\right) \\
&\quad + \frac{1}{2}\mathcal{B}\psi_{D,\varepsilon} + \left(\mathcal{D} - \frac{1}{2}\mathcal{B} a_\varepsilon\right)\psi_{N,\varepsilon} + \Gamma_N + \frac{1}{2}\psi_{D,\varepsilon} - \frac{1}{2}a_\varepsilon \psi_{N,\varepsilon}.
\end{aligned}$$

Restricting to $N$ and recalling that $\boldsymbol{\psi}_\varepsilon$ solves (4.14), we compute

$$\frac{\partial^- u_\varepsilon}{\partial \nu} = -h_{N_\varepsilon} + \frac{1}{2}\mathcal{B}_{ND}\psi_{D,\varepsilon} + \left(\mathcal{D}_{NN} - \frac{1}{2}\mathcal{B} a_\varepsilon\right)\psi_{N,\varepsilon} + g - \frac{1}{2}a_\varepsilon \psi_{N,\varepsilon}.$$

Finally, $\frac{1}{2}a_\varepsilon\psi_{N,\varepsilon} = 0$ on $N \setminus D_\varepsilon$ by the definition of $a_\varepsilon$, and $\frac{1}{2}a_\varepsilon\psi_{N,\varepsilon} = a_\varepsilon \operatorname{Tr}^- u_\varepsilon$ on $N \cap D_\varepsilon$ by (4.17). Hence,

$$\frac{\partial^- u_\varepsilon}{\partial\nu} + a_\varepsilon \operatorname{Tr}^- u_\varepsilon = g \quad \text{on } N$$

as desired.

Conversely, suppose $u_\varepsilon \in H^1_D(\Omega^-)$ is a weak solution of (AMP) on $\Omega^-$ and let $\boldsymbol{\psi}_\varepsilon$ be given by (4.16). Set $u = 0$ and $f = 0$ on $\Omega^+$. Since $u_\varepsilon$ satisfies the boundary conditions of (AMP), it follows that $\boldsymbol{\psi}_\varepsilon \in \boldsymbol{H}_\varepsilon$, and by (4.2) we have the representation formula (4.15). We now work through the previous calculations in reverse, with $\boldsymbol{\psi}_\varepsilon$ given by (4.16), to conclude that $\boldsymbol{\psi}_\varepsilon$ indeed solves the system (4.14). □

*Remark.* In Theorem 4.6, if we choose $\Gamma_D$ to be identically 0 on $\partial\Omega$, the only dependence $\boldsymbol{h}_\varepsilon$ has on $\varepsilon$ is the space $\boldsymbol{H}_\varepsilon$ in which $\boldsymbol{h}_\varepsilon$ lies.

### 4.2 Unique Solvability of the Associated System when $d \geq 3$

Concisely, Theorem 4.6 states that if we can solve the system (4.14) then we can produce a weak solution $u_\varepsilon \in H^1_D(\Omega^-)$ of (AMP) and, conversely, that we can construct a solution of (4.14) given a weak solution of (AMP). Thus, Theorem 2.4 guarantees that at least one solution of (4.14) exists, but says nothing about the uniqueness of such a solution. We address this issue presently.

To show that for fixed $0 \leq \varepsilon < \varepsilon_0$ we can uniquely solve (4.14), we begin by showing that $\boldsymbol{A}_0^\varepsilon$ is coercive on $\boldsymbol{H}_\varepsilon$ and positive and bounded below on a subspace $\boldsymbol{V} \subseteq \boldsymbol{H}_\varepsilon$, i.e., $\langle \boldsymbol{A}_0^\varepsilon \boldsymbol{\psi}, \boldsymbol{\psi}\rangle_{D_\varepsilon\times N} \geq c\|\boldsymbol{\psi}\|_{\boldsymbol{H}_\varepsilon} - C\|\psi_N\|_{L^2(\partial\Omega)}$ for all $\boldsymbol{\psi} \in \boldsymbol{H}_\varepsilon$ and $\langle \boldsymbol{A}_0^\varepsilon \boldsymbol{\psi}, \boldsymbol{\psi}\rangle_{D_\varepsilon\times N} \geq C\|\boldsymbol{\psi}\|^2_{\boldsymbol{V}}$ for all $\boldsymbol{\psi} \in \boldsymbol{V}$. To accomplish this, we begin by showing that the operator $\mathcal{A}^\varepsilon_{DD}$ is positive and bounded below on $\widetilde{H}^{-1/2}(D)$, and the operator $\mathcal{D}_{NN}$ is positive and bounded below on $\widetilde{H}^{1/2}(N) \cap (\ker \mathcal{D}_{NN})^\perp$. We conclude by showing that $\mathcal{C}_{DN}$ and $\mathcal{B}_{ND}$ act as adjoints of each other in a certain sense.

Next, we show that $\boldsymbol{K}_\varepsilon : \boldsymbol{H}_\varepsilon \to \boldsymbol{H}^*_\varepsilon$ is compact, and thus the operator $\boldsymbol{A}_\varepsilon : \boldsymbol{H}_\varepsilon \to \boldsymbol{H}^*_\varepsilon$ is Fredholm and has index 0. We finish the section and the chapter by proving that $\boldsymbol{A}_\varepsilon : \boldsymbol{H}_\varepsilon \to \boldsymbol{H}^*_\varepsilon$ is injective, and thus the system $\boldsymbol{A}_\varepsilon \boldsymbol{\psi}_\varepsilon = \boldsymbol{h}$ has a unique solution $\boldsymbol{\psi}_\varepsilon \in \boldsymbol{H}_\varepsilon$ for each $\boldsymbol{h} \in \boldsymbol{H}^*_\varepsilon$ and $0 \leq \varepsilon < \varepsilon_0$ by the Fredholm alternative.

#### 4.2.1 Positive Definiteness of the Operator $\mathcal{A}^\varepsilon_{DD}$

To show that $\mathcal{A}^\varepsilon_{DD}$ is positive and bounded below, we prove that it is Fredholm of index 0, then show that $\ker \mathcal{A}^\varepsilon_{DD} = \{0\}$, at which point we deduce the desired result from the Fredholm alternative.

**Lemma 4.7.** *For $d \geq 2$, $\mathcal{A} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is Fredholm of index 0.*

*Proof.* Let $\eta \in C_c^\infty(\mathbf{R}^d)$ be identically 1 on a neighborhood of $\overline{\Omega^-}$, $\psi_1, \psi_2 \in H^{-1/2}(\partial\Omega)$, and set $u = \eta \operatorname{SL}\psi_1$ and $v = \eta \operatorname{SL}\psi_2$. By the jump relations (4.7) for the single-layer potential, as well as Green's identity (Lemma (2.3)), we have

$$\begin{aligned}\langle \psi_2, \mathcal{A}\psi_1\rangle_{\partial\Omega} &= \left\langle \frac{\partial^- v}{\partial\nu} - \frac{\partial^+ v}{\partial\nu}, \operatorname{Tr} u \right\rangle_{\partial\Omega} \\ &= \int_{\mathbf{R}^d} \nabla u \cdot \nabla v \, dx - \langle -\Delta v, u\rangle_{\Omega^+} .\end{aligned}$$

Let $K : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ be given by $\langle \psi_2, K\psi_1\rangle_{\partial\Omega} := - \langle -\Delta v, u\rangle_{\Omega^+} - \int_{\mathbf{R}^d} uv \, dx$ and write $\mathcal{A}_0 := \mathcal{A} - K$ so that

$$\langle \psi_1, \mathcal{A}_0\psi_1\rangle_{\partial\Omega} = \|u\|^2_{H^1(\mathbf{R}^d)}.$$

To complete the proof, we will show that $A_0$ is positive definite and $K$ is compact. Since $-\Delta(\operatorname{SL}\psi_1) = 0$ on $\mathbf{R}^d \setminus \partial\Omega$, by the jump relation $\left[\frac{\partial u}{\partial\nu}\right]_{\partial\Omega} = -\psi_1$ we deduce in similar fashion to how we proved (4.6)

$$\begin{aligned}\|\psi_1\|^2_{H^{-1/2}(\partial\Omega)} &= \left\| \frac{\partial^- u}{\partial\nu} - \frac{\partial^+ u}{\partial\nu} \right\|^2_{H^{-1/2}(\partial\Omega)} \\ &\leq C\|u\|^2_{H^1(\mathbf{R}^d)},\end{aligned}$$

from which positive definiteness of $\mathcal{A}_0$ immediately follows, since

$$\langle \psi_1, \mathcal{A}_0\psi_1, \rangle_{\partial\Omega} \geq C\|\psi_1\|^2_{H^{-1/2}(\partial\Omega)}.$$

When $\psi_1 \in C(\partial\Omega)$, the operator $\langle \psi_2, K_1\psi_1\rangle_{\partial\Omega} := \langle -\Delta v, u\rangle_{\Omega^+}$ can be written as an integral operator

$$K_1\psi_1(z) = \int_{\partial\Omega} \mathcal{K}(z, y)\psi_1(y) \, d\sigma(y),$$

where

$$\mathcal{K}(z, y) = \int_{\Omega^+} -\Delta_x(\eta(x)G(x, z))\eta(x)G(x, y) \, dx.$$

The integral kernel $\mathcal{K}$ is $C^\infty$ in a neighborhood of $\partial\Omega \times \partial\Omega$, since $G \in C^\infty(\mathbf{R}^d \setminus \{0\})$ and $-\Delta v$ is compactly supported in $\Omega^+$. From this we may conclude that $K_1$ is densely defined as a Hilbert-Schmidt integral operator, and consequently is compact.

It is beyond the scope of the main text to prove that the operator $K_2 : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ given by $\langle \psi_2, K_2\psi_1\rangle_{\partial\Omega} := \int_{\mathbf{R}^d} uv \, dx$ is compact. The relevant result is Lemma B.3 which can be found in Appendix B. □

**Lemma 4.8.** *For $d \geq 3$, $\mathcal{A} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is positive definite, i.e.,*

$$\langle \psi, \mathcal{A}\psi\rangle_{\partial\Omega} \geq C\|\psi\|^2_{H^{-1/2}(\partial\Omega)} \quad \forall \ \psi \in H^{-1/2}(\partial\Omega).$$

*Proof.* We begin in similar fashion to how we proved Lemma 4.7. Let $\psi_1, \psi_2 \in C(\partial\Omega)$, set $u = \mathrm{SL}\,\psi_1$ and $v = \mathrm{SL}\,\psi_2$, and let $\rho > 0$ be large enough that $\overline{\Omega^-} \subset B_\rho(0)$. Let $\nu_r(x) = \frac{x}{r}$ denote the outward unit normal to $B_r(0)$ at the point $x \in \partial B_r(0)$. By Green's identity, we have

$$\langle \psi_2, \mathcal{A}\psi_1 \rangle_{\partial\Omega} = \int_{B_\rho(0)} \nabla u \cdot \nabla v \, dx + \int_{B_\rho(0)} u \frac{\partial v}{\partial \nu_\rho} \, d\sigma.$$

For $x \notin \partial\Omega$ we have

$$\begin{aligned} u(x) &= \int_{\partial\Omega} G(x-y)\psi_1(y)\, d\sigma(y) \\ &= \frac{1}{d(2-d)\omega_d} \int_{\partial\Omega} \frac{\psi_1(y)}{|x-y|^{d-2}} \, d\sigma(y), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial v}{\partial \nu_r}(x) &= \int_{\partial\Omega} \nu_r(x) \cdot \nabla_x G(x,y)\psi_2(y)\, d\sigma(y) \\ &= \frac{1}{r d\omega_d} \int_{\partial\Omega} \frac{x\cdot(x-y)}{|x-y|^d} \psi_2(y)\, d\sigma(y). \end{aligned}$$

Here, $\nabla_x G(x,y)$ denotes the gradient of $G(x,y)$ with respect to $x$. Observe that for $r = |x| > 2\rho$ and $|y| < \rho < \frac{r}{2}$,

$$|u(x)| \le \frac{2^{d-2}\sigma(\partial\Omega)^{\frac{1}{2}} \|\psi_1\|_{L^2(\partial\Omega)}}{d(d-2)\omega_d} r^{2-d}$$

and

$$\left| \frac{\partial v}{\partial \nu_r}(x) \right| \le \frac{2^{d-1}\sigma(\partial\Omega)^{\frac{1}{2}} \|\psi_2\|_{L^2(\partial\Omega)}}{d\omega_d} r^{1-d}.$$

Hence,

$$\begin{aligned} \left| \int_{B_r(0)} u \frac{\partial v}{\partial \nu_r} \, d\sigma \right| &\le \frac{\sigma(\partial\Omega) \|\psi_1\|_{L^2(\partial\Omega)} \|\psi_2\|_{L^2(\partial\Omega)}}{d^2(d-2)\omega_d^2} r^{3-2d} \sigma(\partial B_r(0)) \\ &= \frac{\sigma(\partial\Omega) \|\psi_1\|_{L^2(\partial\Omega)} \|\psi_2\|_{L^2(\partial\Omega)}}{d(d-2)\omega_d} r^{2-d}. \end{aligned}$$

Letting $r \to \infty$ we conclude $\langle \psi_2, \mathcal{A}\psi_1 \rangle_{\partial\Omega} = \int_{\mathbf{R}^d} \nabla u \cdot \nabla v \, dx$.

The fact that $\langle \psi_1, \mathcal{A}\psi_1 \rangle_{\partial\Omega} \ge 0$ follows immediately from the above calculation. Moreover, if $\langle \psi_1, \mathcal{A}\psi_1 \rangle_{\partial\Omega} = 0$ then $|\nabla u| = 0$ on $\mathbf{R}^d \setminus \partial\Omega$. As a consequence of the jump relations (4.7), $-\psi_1 = \left[\frac{\partial u}{\partial \nu}\right]_{\partial\Omega} = 0$, so $\mathcal{A}$ is strictly positive, i.e., $\langle \psi_1, \mathcal{A}\psi_1 \rangle_{\partial\Omega} > 0$ when $\psi_1 \neq 0$. From this observation we gain $\ker \mathcal{A} = \{0\}$, and since $\mathcal{A}$ is Fredholm of index 0 by Lemma 4.7, $\mathcal{A}$ is bijective and therefore has a bounded inverse $\mathcal{A}^{-1} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by the inverse mapping theorem.

Next, for each $\psi_1 \in H^{-1/2}(\partial\Omega)$ we have

$$\begin{aligned}\|\psi_1\|_{H^{-1/2}(\partial\Omega)} &= \|\mathcal{A}^{-1}\mathcal{A}\psi_1\|_{H^{-1/2}(\partial\Omega)} \\ &\le C_1\|\mathcal{A}\psi_1\|_{H^{1/2}(\partial\Omega)},\end{aligned}$$

i.e., $\|\mathcal{A}\psi_1\|_{H^{1/2}(\partial\Omega)} \ge C_1\|\psi_1\|_{H^{-1/2}(\partial\Omega)}$. For $\psi_1 \neq 0$ this gives

$$\begin{aligned}C_1 &\le \frac{\|\mathcal{A}\psi_1\|_{H^{1/2}(\partial\Omega)}}{\|\psi_1\|_{H^{-1/2}(\partial\Omega)}} \\ &= \sup_{\substack{\psi_2\in H^{-1/2}(\partial\Omega)\\ \psi_2\neq 0}} \frac{|\left\langle \mathcal{A}\psi_1, \psi_2\right\rangle_{\partial\Omega}|}{\|\psi_1\|_{H^{-1/2}(\partial\Omega)}\|\psi_2\|_{H^{-1/2}(\partial\Omega)}} \\ &\le \sup_{\substack{\psi_2\in H^{-1/2}(\partial\Omega)\\ \psi_2\neq 0}} \frac{\sqrt{\left\langle \mathcal{A}\psi_1, \psi_1\right\rangle_{\partial\Omega}}\sqrt{\left\langle \mathcal{A}\psi_2, \psi_2\right\rangle_{\partial\Omega}}}{\|\psi_1\|_{H^{-1/2}(\partial\Omega)}\|\psi_2\|_{H^{-1/2}(\partial\Omega)}} \\ &\le C_2\frac{\sqrt{\left\langle \mathcal{A}\psi_1, \psi_1\right\rangle_{\partial\Omega}}}{\|\psi_1\|_{H^{-1/2}(\partial\Omega)}},\end{aligned}$$

i.e., $\left\langle \mathcal{A}\psi_1, \psi_1\right\rangle_{\partial\Omega} \ge C\|\psi_1\|^2_{H^{-1/2}(\partial\Omega)}$ as desired. □

*Remark.* The inequality

$$|\left\langle \mathcal{A}\psi_1, \psi_2\right\rangle_{\partial\Omega}| \le \sqrt{\left\langle \mathcal{A}\psi_1, \psi_1\right\rangle_{\partial\Omega}}\sqrt{\left\langle \mathcal{A}\psi_2, \psi_2\right\rangle_{\partial\Omega}}$$

relies on the fact that $\mathcal{A}$ is self-adjoint and strictly positive

**Corollary 4.9.** *For $d \ge 3$, $\mathcal{A}^{\varepsilon}_{DD} : \widetilde{H}^{-1/2}(D) \to H^{1/2}(D_\varepsilon)$ is positive definite.*

*Proof.* This is clear from the fact that

$$\left\langle \psi, \mathcal{A}^{\varepsilon}_{DD}\psi\right\rangle_{D_\varepsilon} = \left\langle \psi, \mathcal{A}\psi\right\rangle_{\partial\Omega}.$$ □

### 4.2.2 Positive Definiteness of the Operator $\mathcal{D}_{NN}$

This subsection proceeds much as Section 4.2.1 did. First we prove that $\mathcal{D}_{NN}$ is Fredholm of index 0, then show that $\ker \mathcal{D}^{\varepsilon}_{NN}$ is the set of functions constant on each component of $\partial\Omega$, and that $\mathcal{D}_{NN}$ is positive definite on $\widetilde{H}^{1/2}(N) \cap (\ker \mathcal{D}_{NN})^{\perp}$.

**Lemma 4.10.** *When $d \ge 2$, $\mathcal{D} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is coercive, i.e.,*

$$\left\langle \mathcal{D}\varphi, \varphi\right\rangle_{\partial\Omega} \ge c\|\varphi\|^2_{H^{1/2}(\partial\Omega)} - C\|\varphi\|^2_{L^2(\partial\Omega)} \quad \forall\, \varphi \in H^{1/2}(\partial\Omega),$$

*hence $\mathcal{D}$ is Fredholm with index 0.*

*Proof.* The proof follows similarly to that of Lemma 4.7. Let $\eta \in C_c^\infty(\mathbf{R}^d)$ be identically 1 on a neighborhood of $\overline{\Omega^-}$, $\varphi_1, \varphi_2 \in H^{1/2}(\partial\Omega)$, and set $u = \eta\,\mathrm{DL}\,\varphi_1$ and $v = \eta\,\mathrm{DL}\,\varphi_2$. By the jump relations (4.11) for the double-layer potential, as well as Green's identity, we have

$$\begin{aligned}\langle \mathcal{D}\varphi_1, \varphi_2\rangle_{\partial\Omega} &= \left\langle \frac{\partial u}{\partial \nu}, \mathrm{Tr}^- v - \mathrm{Tr}^+ v \right\rangle_{\partial\Omega} \\ &= \int_{\Omega^-} \nabla u \cdot \nabla v\, dx + \int_{\Omega^+} \nabla u \cdot \nabla v\, dx - \langle -\Delta u, v\rangle_{\Omega^+}.\end{aligned}$$

The jump relations (4.11) also imply

$$\begin{aligned}\|\varphi_1\|^2_{H^{1/2}(\partial\Omega)} &= \|\,\mathrm{Tr}^+ u - \mathrm{Tr}^- u\|_{H^{1/2}(\partial\Omega)} \\ &\leq c\left(\|u\|^2_{H^1(\Omega^-)} + \|u\|^2_{H^1(\Omega^+)}\right).\end{aligned}$$

Now, let $K : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be given by $\langle K\varphi_1, \varphi_2\rangle_{\partial\Omega} := -\langle -\Delta u, v\rangle_{\Omega^+} - \int_{\mathbf{R}^d} uv\, dx$ so that

$$\begin{aligned}\langle \mathcal{D}\varphi_1, \varphi_1\rangle_{\partial\Omega} &= \|u\|^2_{H^1(\Omega^-)} + \|u\|^2_{H^1(\Omega^+)} + \langle K\varphi_1, \varphi_1\rangle_{\partial\Omega} \\ &\geq c\|\varphi_1\|^2_{H^{1/2}(\partial\Omega)} + \langle K\varphi_1, \varphi_1\rangle_{\partial\Omega}.\end{aligned}$$

We showed in the proof of Lemma 4.7 that the operator $K_1 : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by $\langle K_1\varphi_1, \varphi_2\rangle_{\partial\Omega} = -\langle(-\Delta u, v\rangle_{\Omega^+}$ is an integral operator whose kernel is $C^\infty$ on a neighborhood of $\partial\Omega \times \partial\Omega$. Here, we use this fact to assert that $K_1$ is also bounded as an operator from $L^2(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$, i.e.,

$$|\langle K_1\varphi_1, \varphi_1\rangle_{\partial\Omega}| \leq C\|\varphi_1\|^2_{L^2(\partial\Omega)}.$$

Finally, by Lemma B.4 we conclude that the operator $K_2 : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by $\langle K_2\varphi_1, \varphi_2\rangle_{\partial\Omega} := -\int_{\mathbf{R}^d} uv\, dx$ satisfies

$$\begin{aligned}|\langle K_2\varphi_1, \varphi_1\rangle_{\partial\Omega}| &\leq C\|\varphi_1\|^2_{H^{-1/2}(\partial\Omega)} \\ &\leq C\|\varphi_1\|^2_{L^2(\partial\Omega)},\end{aligned}$$

and therefore,

$$\langle \mathcal{D}\varphi_1, \varphi_1\rangle_{\partial\Omega} \geq c\|u\|^2_{H^1(\Omega^-)} - C\|\varphi_1\|^2_{L^2(\partial\Omega)}. \qquad \square$$

**Lemma 4.11.** *Suppose $\partial\Omega$ has $K$ components, denoted $\partial\Omega_1, \ldots, \partial\Omega_K$. Let $\chi_n$ be the characteristic function of $\partial\Omega_n$, and let*

$$V := \left\{\varphi \in H^{1/2}(\partial\Omega) : \int_{\partial\Omega_k} \varphi\, d\sigma = 0,\ 1 \leq k \leq K\right\}.$$

*When $d \geq 3$, $\mathcal{D} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is positive definite on $V$, i.e.,*

$$\langle \mathcal{D}\varphi, \varphi\rangle_{\partial\Omega} \geq \|\varphi\|^2_{H^{1/2}(\partial\Omega)} \quad \forall\, \varphi \in V.$$

*Proof.* The proof begins almost identically to that of Lemma 4.8. Let $\varphi_1, \varphi_2 \in H^{1/2}(\partial\Omega)$, set $u = \mathrm{DL}\,\varphi_1$ and $v = \mathrm{DL}\,\varphi_2$, and let $\rho > 0$ be large enough that $\overline{\Omega^-} \subset B_\rho(0)$. Let $\nu_r(x) = \frac{x}{r}$ denote the outward unit normal to $B_r(0)$ at the point $x \in \partial B_r(0)$. By Green's identity, we have

$$\langle \mathcal{D}\varphi_1, \varphi_2\rangle_{\partial\Omega} = \int_{B_\rho(0)\setminus\partial\Omega} \nabla u \cdot \nabla v \, dx + \int_{B_\rho(0)} v \frac{\partial u}{\partial \nu_\rho} \, d\sigma.$$

For $x \notin \partial\Omega$ we have

$$\begin{aligned} v(x) &= \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x-y)\varphi_2(y)\, d\sigma(y) \\ &= \frac{1}{d\omega_d} \int_{\partial\Omega} \frac{\nu(y)\cdot(x-y)}{|x-y|^d} \varphi_2(y)\, d\sigma(y), \end{aligned}$$

and

$$\frac{\partial u}{\partial \nu_r}(x) = \frac{1}{r\omega_d} \int_{\partial\Omega} \left( \frac{x\cdot\nu(y)}{d|x-y|^d} - \frac{x\cdot(x-y)\nu(y)\cdot(x-y)}{|x-y|^{d+2}} \right) \varphi_1(y)\, d\sigma(y).$$

Hence, $|v(x)| = O(|x|^{1-d})$ and $\left|\frac{\partial u}{\partial \nu_r}(x)\right| = O(|x|^{-d})$ as $|x| \to \infty$, i.e., $\left|\int_{B_\rho(0)} v \frac{\partial u}{\partial \nu_\rho} \, d\sigma\right| = O(\rho^{-d})$ as $\rho \to \infty$. By letting $\rho \to \infty$ we obtain

$$\langle \mathcal{D}\varphi_1, \varphi_2\rangle_{\partial\Omega} = \int_{\mathbf{R}^d\setminus\partial\Omega} \nabla u \cdot \nabla v \, dx.$$

The fact that $\langle \mathcal{D}\varphi_1, \varphi_1\rangle_{\partial\Omega} \geq 0$ follows immediately. Moreover, if $\langle \mathcal{D}\varphi_1, \varphi_1\rangle_{\partial\Omega} = 0$ then $u = \mathrm{DL}\,\varphi_1$ is constant on each component of $\mathbf{R}^d \setminus \partial\Omega$. Consequently, $\mathcal{D}\varphi_1 = -\frac{\partial}{\partial\nu}\mathrm{DL}\,\varphi_1 = 0$, and for some real numbers $a_1, \ldots, a_K$ we have

$$\varphi_1 = [u]_{\partial\Omega} = \sum_{k=1}^{K} a_k \chi_k.$$

Observe that the only such $\varphi_1$ which is an element of $V$ is $\varphi_1 = 0$.

Up to this point, we have shown that $V = H^{1/2}(\partial\Omega) \cap (\ker\mathcal{D})^\perp$, where the orthogonal complement is taken with respect to the $L^2(\partial\Omega)$ inner product, and that $\varphi \in V \setminus \{0\}$ implies $\langle \mathcal{D}\varphi, \varphi\rangle_{\partial\Omega} > 0$. Hence, by a similar argument to that which concluded the proof of Lemma 4.8, the bounded, self-adjoint, positive-definite, Fredholm operator $\mathcal{D}|_V : V \to H^{-1/2}(\partial\Omega)$ of index 0 is positive and bounded below on $V$, i.e.,

$$\langle \mathcal{D}\varphi, \varphi\rangle_{\partial\Omega} \geq \|\varphi\|^2_{H^{1/2}(\partial\Omega)} \quad \forall\, \varphi \in V. \qquad \square$$

**Corollary 4.12.** *Given $K$, $\partial\Omega_1, \ldots, \partial\Omega_K$, and $\chi_1, \ldots, \chi_K$ as in Lemma 4.11, let*

$$V := \left\{ \varphi \in \widetilde{H}^{1/2}(N) \, : \, \int_{\partial\Omega_n} \varphi \, d\sigma = 0, \ 1 \leq k \leq K, \ \partial\Omega_k \cap D = \emptyset \right\}.$$

*When $d \geq 3$, $\mathcal{D}_{NN} : \widetilde{H}^{1/2}(N) \to H^{-1/2}(N)$ is positive definite on $V$, i.e.,*

$$\langle \mathcal{D}_{NN}\varphi, \varphi\rangle_N \geq \|\varphi\|^2_{H^{1/2}(\partial\Omega)} \quad \forall\, \varphi \in V.$$

*Proof.* The inclusion $(\widetilde{H}^{1/2}(N) \cap \ker \mathcal{D}) \subseteq \ker \mathcal{D}_{NN}$ is obvious, so suppose $\varphi \in \ker \mathcal{D}_{NN}$. On the one hand, $\mathcal{D}_{NN}\varphi = 0$ on $N$, and on the other, $\varphi = 0$ on $\bar{D}$. Therefore, we have $\langle \mathcal{D}\varphi, \varphi \rangle_{\partial\Omega} = \langle \mathcal{D}_{NN}\varphi, \varphi \rangle_N = 0$, i.e., $\ker \mathcal{D}_{NN} = (\widetilde{H}^{1/2}(N) \cap \ker \mathcal{D})$. Now, $\varphi \in \ker \mathcal{D}_{NN}$ implies $\varphi$ is identically 0 on any component of $\partial\Omega$ which has nonempty intersection with $D$. On the remaining components, $\varphi|_{\partial\Omega_k}$ is identically equal to a constant $a_k$. Finally, for all $\varphi \in V$,

$$\begin{aligned}\langle \mathcal{D}_{NN}\varphi, \varphi \rangle_N &= \langle \mathcal{D}\varphi, \varphi \rangle_{\partial\Omega} \\ &\geq C\|\varphi\|^2_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

□

### 4.2.3 Positive Definiteness of the Operator $A_0^\varepsilon$

**Lemma 4.13.** *For $d \geq 2$, the operator $\mathcal{C} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is the adjoint of $\mathcal{B} : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, i.e., for $\varphi \in H^{1/2}(\partial\Omega)$ and $\psi \in H^{-1/2}(\partial\Omega)$ we have*

$$\langle \psi, \mathcal{C}\varphi \rangle_{\partial\Omega} = \langle \mathcal{B}\psi, \varphi \rangle_{\partial\Omega}.$$

*Proof.* Let $u \in H^1(\mathbf{R}^d)$, $\varphi := \operatorname{Tr} u \in H^{1/2}(\partial\Omega)$, and $\psi \in H^{-1/2}(\partial\Omega)$. First observe that $\nabla G(z) = \frac{-z}{d\omega_d|z|^d}$ is locally integrable on $\mathbf{R}^d$, since for any $U \subset \mathbf{R}^d$ contained in a ball $B_r(0)$ we have $\int_U |G(z)|\,dz \leq r$. Hence, the functions $x \mapsto \int_{\Omega^\pm} \nabla_x G(x-y) \cdot \nabla u(x)\,dx$ are continuous on $\mathbf{R}^d$. Now when $y \in \Omega^\pm$ we have by Green's identity

$$\begin{aligned}\int_{\Omega^+} \nabla_x G(x-y) \cdot \nabla u(x)\,dx &= \mp \int_{\partial\Omega} \frac{\partial G}{\partial \nu_z}(z-y) \operatorname{Tr} u(z)\,d\sigma(z) \\ &= \mp \operatorname{DL}\psi(y).\end{aligned}$$

By another application of Green's identity we obtain

$$\begin{aligned}\left\langle \frac{\partial^\pm}{\partial\nu} \operatorname{SL}\psi, \varphi \right\rangle_{\partial\Omega} &= \mp \int_{\Omega^\pm} \nabla \operatorname{SL}\psi \cdot \nabla u\,dx \\ &= \mp \int_{\Omega^\pm} \langle \psi, \nabla_x G(x-\cdot) \cdot \nabla u(x) \rangle_{\partial\Omega}\,dx \\ &= \mp \left\langle \psi, \int_{\Omega^\pm} \nabla_x G(x-\cdot) \cdot \nabla u(x)\,dx \right\rangle_{\partial\Omega} \\ &= \left\langle \psi, \operatorname{Tr}^\pm \operatorname{DL}\varphi \right\rangle_{\partial\Omega}.\end{aligned}$$

Taking the sum of both cases gives the desired result. □

**Corollary 4.14.** *For $d \geq 2$, $\varphi \in \widetilde{H}^{1/2}(N)$ and $\psi \in \widetilde{H}^{-1/2}(D)$, the maps $\mathcal{C}^\varepsilon_{DN} : \widetilde{H}^{1/2}(N) \to H^{1/2}(D_\varepsilon)$ and $\mathcal{B}_{ND} : \widetilde{H}^{-1/2}(D) \to H^{-1/2}(N)$ satisfy*

$$\langle \psi, \mathcal{C}^\varepsilon_{DN}\varphi \rangle_{D_\varepsilon} = \langle \mathcal{B}_{ND}\psi, \varphi \rangle_N.$$

*Proof.* Since $\psi$ is supported in $\bar{D}$ and $\varphi = 0$ on $\bar{D}$, we compute

$$\begin{aligned}\langle \psi, \mathcal{C}^\varepsilon_{DN}\varphi \rangle_{D_\varepsilon} &= \langle \psi, \mathcal{C}^\varepsilon_{DN}\varphi \rangle_{\partial\Omega} \\ &= \langle \mathcal{B}_{ND}\psi, \varphi \rangle_{\partial\Omega} \\ &= \langle \mathcal{B}_{ND}\psi, \varphi \rangle_N.\end{aligned}$$

□

**Corollary 4.15.** *Let* $\boldsymbol{V} = (\{0\} \times \ker \mathcal{D}_{NN})^{\perp} = \widetilde{H}^{-1/2}(D) \times (\ker \mathcal{D}_{NN})^{\perp} \subseteq \boldsymbol{H}_{\varepsilon}$. *For* $d \geq 3$, $\boldsymbol{A}_0^{\varepsilon} : \boldsymbol{H}_{\varepsilon} \to \boldsymbol{H}_{\varepsilon}^*$ *is coercive on* $\boldsymbol{H}_{\varepsilon}$ *and positive and bounded below on* $\boldsymbol{V}$, *i.e.,*

$$\begin{aligned} &\langle \boldsymbol{A}_0^{\varepsilon} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{D_{\varepsilon} \times N} \geq c\|\boldsymbol{\psi}\|_{\boldsymbol{H}_{\varepsilon}} - C\|\psi_N\|_{L^2(\partial\Omega)} \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_{\varepsilon}, \text{ and} \\ &\langle \boldsymbol{A}_0^{\varepsilon} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{D_{\varepsilon} \times N} \geq C\|\boldsymbol{\psi}\|_{\boldsymbol{H}_{\varepsilon}} \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{V}. \end{aligned}$$

*Consequently,* $\boldsymbol{A}_0^{\varepsilon}$ *is Fredholm of index 0.*

*Proof.* First, by Corollaries 4.9 and 4.14, as well as Lemma 4.10, we write

$$\begin{aligned} \langle \boldsymbol{A}_0^{\varepsilon} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{D_{\varepsilon} \times N} &= \langle \psi_D, \mathcal{A}_{DD}^{\varepsilon} \psi_D \rangle_{D_{\varepsilon}} - \frac{1}{2} \langle \psi_D, \mathcal{C}_{DN}^{\varepsilon} \psi_N \rangle_{D_{\varepsilon}} \\ &\quad + \frac{1}{2} \langle \mathcal{B}_{ND} \psi_D, \psi_N \rangle_N + \langle \mathcal{D}_{NN} \psi_N, \psi_N \rangle_N \\ &\geq C\|\psi_D\|_{H^{-1/2}(\partial\Omega)}^2 + c\|\psi_N\|_{H^{1/2}(\partial\Omega)}^2 - C\|\psi_N\|_{L^2(\partial\Omega)}^2, \end{aligned}$$

which proves coercivity. Next, $\mathcal{A}_{DD}^{\varepsilon}$ and $\mathcal{D}_{NN}$ are self-adjoint, so $\langle \boldsymbol{A}_0^{\varepsilon} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{D_{\varepsilon} \times N} \geq 0$. Moreover, if $\langle \boldsymbol{A}_0^{\varepsilon} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{D_{\varepsilon} \times N} = 0$, then $\langle \psi_D, \mathcal{A}_{DD}^{\varepsilon} \psi_D \rangle_{D_{\varepsilon}} = \langle \mathcal{D}_{NN} \psi_N, \psi_N \rangle_N = 0$, implying $\boldsymbol{\varphi} \in \boldsymbol{V}$ by Corollaries 4.9 and 4.12. □

*Remark.* In particular, Corollary 4.15 says that if $D$ intersects every component of $\partial\Omega$, then $\boldsymbol{A}_0^{\varepsilon}$ is positive and bounded below on $\boldsymbol{H}_{\varepsilon}$, i.e.,

$$\langle \boldsymbol{A}_0^{\varepsilon} \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{D_{\varepsilon} \times N} \geq C\|\boldsymbol{\psi}\|_{\boldsymbol{H}_{\varepsilon}} \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_{\varepsilon}.$$

### 4.2.4 Compactness of the Operator $\boldsymbol{K}_{\varepsilon}$

**Lemma 4.16.** *The embedding* $\widetilde{H}^{1/2}(N) \hookrightarrow L^2\left(N, \frac{1}{\delta^{\mu}}\, d\sigma\right)$ *is compact for* $\mu \in [0, 1)$, *but is not compact for* $\mu = 1$.

*Proof.* Let $\mu \in [0, 1)$ be fixed, and let $\{F_n\}_{n=1}^{\infty} \subset \widetilde{H}^{1/2}(N)$ be a bounded sequence. Since $H^{1/2}(\partial\Omega)$ is compactly embedded in $L^2(\partial\Omega)$, by passing to a subsequence we can assume $\|F_n - F\|_{L^2(\partial\Omega)} \to 0$ as $n \to \infty$ for some $F \in L^2(\partial\Omega)$. By Hölder's inequality, we compute

$$\begin{aligned} \int_N |F_n - F|^2 \frac{1}{\delta^{\mu}}\, d\sigma &\leq \left( \int_N |F_n - F|^2 \frac{1}{\delta}\, d\sigma \right)^{\mu} \left( \int_N |F_n - F|^2\, d\sigma \right)^{1-\mu} \\ &\leq C\|F_n - F\|_{H^{1/2}(\partial\Omega)}^{2\mu} \|F_n - F\|_{L^2(\partial\Omega)}^{2(1-\mu)}. \end{aligned}$$

Since $\|F_n - F\|_{H^{1/2}(\partial\Omega)}$ is bounded and $\|F_n - F\|_{L^2(\partial\Omega)} \to 0$ as $n \to \infty$, it follows that $F_n \to F$ in $L^2\left(N, \frac{1}{\delta^{\mu}}\, d\sigma\right)$.

When $\mu = 1$, we can always find a bounded sequence $\{F_n\}_{n=1}^{\infty} \subset \widetilde{H}^{1/2}(N)$ which has no convergent subsequence in $L^2\left(N, \frac{1}{\delta}\, d\sigma\right)$. As a particular counterexample, let $\Omega \subseteq \mathbf{R}^d$ be a bounded Lipschitz domain with $\Lambda, N \subset \partial\Omega$ such that $\Lambda \supseteq [0,1]^{d-2} \times \{0\}^2$ and $N \supseteq \bigcup_{n=0}^{\infty} \bar{Q}_n$, where $Q_n := (0, 2^{-n})^{d-2} \times (2^{-n}, 2^{1-n}) \times \{0\}$. Now, fix $F \in C^{\infty}(\bar{\Omega})$

so that $\operatorname{supp}(F|_{\partial\Omega}) \subset\subset Q_0$, $\|F\|_{L^2(\partial\Omega)} = 1$, and $\|F\|^2_{H^{1/2}(\partial\Omega)} = C$, and set $F_n(x) := 2^{\frac{(d-2)n}{2}} F(2^n x)$. We compute

$$\begin{aligned}\int_{\partial\Omega} |F_n(x)|^2\, d\sigma(x) &= 2^{(d-2)n} \int_{Q_n} |F(2^n x)|^2\, d\sigma(x)\\ &= 2^{-n} \int_{Q_0} |F(u)|^2\, du\\ &= 2^{-n},\end{aligned}$$

and

$$\begin{aligned}|F_n|^2_{H^{1/2}(2^{-n}\partial\Omega)} &= \int_{2^{-n}\partial\Omega}\int_{2^{-n}\partial\Omega} \frac{|F_n(x)-F_n(y)|^2}{|x-y|^d}\, d\sigma(x)\, d\sigma(y)\\ &= 2^{(d-2)n}\int_{2^{-n}\partial\Omega}\int_{2^{-n}\partial\Omega} \frac{|F(2^n x)-F(2^n y)|^2}{|x-y|^d}\, d\sigma(x)\, d\sigma(y)\\ &= 2^{-dn}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|F(u)-F(v)|^2}{|2^{-n}(u-v)|^d}\, d\sigma(u)\, d\sigma(v)\\ &= \int_{\partial\Omega}\int_{\partial\Omega} \frac{|F(u)-F(v)|^2}{|u-v|^d}\, d\sigma(u)\, d\sigma(v)\\ &= C-1.\end{aligned}$$

Hence, $\|F_n\|^2_{H^{1/2}(\partial\Omega)} = C - 1 + 2^{-n} \le C$ for all $n = 0, 1, \ldots$. However,

$$\begin{aligned}\int_N \frac{|F_n(x)|^2}{\delta(x)}\, d\sigma(x) &= 2^{(d-2)n}\int_{Q_n} \frac{|F(2^n x)|^2}{\delta(x)}\, d\sigma(x)\\ &\ge 2^{(d-1)n-1}\int_{Q_n} |F(2^n x)|^2\, d\sigma(x)\\ &= \frac{1}{2}\int_{Q_0} |F(u)|^2\, d\sigma(u)\\ &= \frac{1}{2}.\end{aligned}$$

Moreover, observe that $\operatorname{supp}(F_m|_{\partial\Omega}) \cap \operatorname{supp}(F_n|_{\partial\Omega}) = \emptyset$ when $m \ne n$, i.e.,

$$\begin{aligned}\|F_m - F_n\|^2_{L^2(N,\delta^{-1}\,d\sigma)} &= \|F_m\|^2_{L^2(N,\delta^{-1}\,d\sigma)} + \|F_n\|^2_{L^2(N,\delta^{-1}\,d\sigma)}\\ &\ge 1\end{aligned}$$

We have thus shown that $\{F_n\}_1^\infty$ has no Cauchy subsequence in $L^2\left(N, \frac{1}{\delta}\, d\sigma\right)$, and hence the injection $\widetilde{H}^{1/2}(N) \hookrightarrow L^2\left(N, \frac{1}{\delta}\, d\sigma\right)$ is not compact. □

**Corollary 4.17.** *If $\mu \in [0,1)$, the operator $\boldsymbol{K}_\varepsilon : H \to H^*$ is compact for each $0 < \varepsilon < \varepsilon_0$.*

*Proof.* First, we observe that $a_\varepsilon : L^2(N, \delta^{-\mu}\, d\sigma) \to L^2(N, \delta^{\mu}\, d\sigma)$. Indeed, if $\psi \in L^2(N, \delta^{-\mu}\, d\sigma)$, then

$$\begin{aligned} \|a_\varepsilon \psi\|^2_{L^2(N,\delta^\mu\, d\sigma)} &= \int_N |a_\varepsilon \psi|^2 \delta^\mu \, d\sigma \\ &\le \varepsilon^{2\mu} \int_N \frac{|\psi|^2}{\delta^\mu}\, d\sigma \\ &\le \varepsilon^{2\mu} \|\psi\|^2_{L^2(N,\delta^{-\mu}\, d\sigma)}. \end{aligned}$$

Now, by Lemma 4.16 we have that the injection $i : \widetilde{H}^{1/2}(N) \hookrightarrow L^2(N, \delta^{-\mu}\, d\sigma)$ is compact. Further, it is well-known that the adjoint of a compact operator is compact, hence the adjoint $i^* : L^2(N, \delta^{\mu}\, d\sigma) \hookrightarrow H^{-1/2}(N)$ is also compact. Hence, the operator $a_\varepsilon = i^* \circ a_\varepsilon \circ i : \widetilde{H}^{1/2}(N) \to H^{-1/2}(N)$ is compact. The compactness of $\boldsymbol{K}_\varepsilon$ follows immediately. □

**Corollary 4.18.** *If $\mu \in [0, 1)$, the operator $\boldsymbol{A}_\varepsilon : H \to H^*$ is Fredholm of index 0.*

*Proof.* By Corollaries 4.15 and 4.17, $\boldsymbol{A}_\varepsilon$ is the sum of a coercive operator and a compact operator, implying $\boldsymbol{A}_\varepsilon$ is Fredholm of index 0. □

**Theorem 4.19.** *Let $0 \le \varepsilon < \varepsilon_0$ and $\mu \in [0, 1)$. The map $\boldsymbol{A}_\varepsilon : \boldsymbol{H}_\varepsilon \to \boldsymbol{H}^*_\varepsilon$ has a bounded inverse $\boldsymbol{A}_\varepsilon^{-1} : \boldsymbol{H}^*_\varepsilon \to \boldsymbol{H}_\varepsilon$, and the system $\boldsymbol{A}_\varepsilon\, \boldsymbol{\psi}_\varepsilon = \boldsymbol{h}_\varepsilon$ has a unique solution for each $\boldsymbol{h}_\varepsilon \in \boldsymbol{H}_\varepsilon$.*

*Proof.* Since Corollary 4.18 says $\boldsymbol{A}_\varepsilon$ is Fredholm of index 0, by the Fredholm alternative it suffices to show that $\boldsymbol{A}_\varepsilon$ is injective, i.e., $\ker \boldsymbol{A}_\varepsilon = \{\boldsymbol{0}\}$.

Suppose $\boldsymbol{\psi} \in \boldsymbol{H}_\varepsilon$ solves the homogeneous system $\boldsymbol{A}_\varepsilon\, \boldsymbol{\psi}_\varepsilon = \boldsymbol{0}$. By Theorem 4.6, for any $f \in H^{-1}_D(\Omega)$, $\Gamma_N \in H^{-1/2}(N)$ and $\Gamma_D \in \widetilde{H}^{1/2}(D)$ satisfying

$$\begin{aligned} &\Big( -\operatorname{Tr} \mathcal{G} f + \mathcal{A}(a_\varepsilon \Gamma_D - \Gamma_N) - \frac{1}{2}(\Gamma_D - \mathcal{C}\Gamma_D) \Big) = 0 \text{ on } D_\varepsilon, \quad \text{and} \\ &\Big( -\frac{\partial^-}{\partial \nu} \mathcal{G} f + \frac{1}{2}\big(g + a_\varepsilon \Gamma_D + \mathcal{B}(a_\varepsilon \Gamma_D - \Gamma_N)\big) - \mathcal{D}\Gamma_D \Big) = 0 \text{ on } N, \end{aligned}$$

the function $u_\varepsilon \in H^1_D(\Omega^-)$ given by

$$u_\varepsilon = \mathcal{G} f + \mathrm{SL}\,(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon} + \Gamma_N - a_\varepsilon \Gamma_D) - \mathrm{DL}(\psi_{N,\varepsilon} + \Gamma_D) \quad \text{on } \Omega^- \tag{4.18}$$

is a weak solution of (AMP) with data $f$ and $g = \Gamma_N|_N$ given above. In particular, $u_\varepsilon = \mathrm{SL}(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon}) - \mathrm{DL}\, \psi_{N,\varepsilon}$ solves the interior homogeneous problem

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{on } \Omega^- \\ u_\varepsilon = 0 & \text{on } D \\ \frac{\partial u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon = 0 & \text{on } N. \end{cases} \tag{4.19}$$

Theorem 2.4 then implies that $u_\varepsilon = 0$ on $\Omega^-$.

Now, set $u_\varepsilon := \mathrm{SL}(\psi_{D,\varepsilon} - a_\varepsilon \psi_{N,\varepsilon}) - \mathrm{DL}\,\psi_{N,\varepsilon}$ on $\mathbf{R}^d$. By definition, $-\Delta u_\varepsilon = 0$ on $\Omega^+$. Further, the jump relations (4.7) and (4.11) imply $[u_\varepsilon]_{\partial\Omega} = -\psi_{N,\varepsilon} = 0$ on $D$ and

$$\begin{aligned}\left[\frac{\partial u_\varepsilon}{\partial \nu}\right]_{\partial\Omega} &= a_\varepsilon \psi_{N,\varepsilon} - \psi_{D,\varepsilon} \\ &= -[a_\varepsilon u_\varepsilon]_{\partial\Omega} - \psi_{D,\varepsilon},\end{aligned}$$

i.e., $\left[\frac{\partial u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon\right]_{\partial\Omega} = 0$ on $N$. Hence, $u_\varepsilon$ solves the exterior homogeneous problem

$$\begin{cases} \Delta u_\varepsilon = 0 & \text{on } \Omega^+ \\ u_\varepsilon = 0 & \text{on } D \\ \frac{\partial u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon = 0 & \text{on } N. \end{cases}$$

Finally, as we showed in the proofs of Lemmas 4.8 and 4.11, $u(x) = O(|x|^{2-d})$ as $|x| \to \infty$. Given this estimate near $\infty$, we deduce that $u_\varepsilon = 0$ on $\Omega^+$, from which the conclusion $\psi_{D,\varepsilon} = \psi_{N,\varepsilon} = 0$ on $\partial\Omega$ follows immediately. □

## Chapter 5 Future Problems

First, in Theorem 3.18 we obtained an upper bound on $\|u_\varepsilon - u_0\|_{H^1_D(\Omega)}$, but did not produce a similar estimate for $\|\boldsymbol{\psi}_\varepsilon - \boldsymbol{\psi}_0\|_{\boldsymbol{H}_\varepsilon}$, if such an upper bound exists. Also, in Chapter 4 we did not consider solvability of $\boldsymbol{A}_\varepsilon\, \boldsymbol{\psi}_\varepsilon = \boldsymbol{h}_\varepsilon$ when $d = 2$.

Next, we considered only the equation $-\Delta u = f$ for the sake of simplicity. A future project will be to show that much (if not all) of Chapters 2, 3, and 4 remain valid when $-\Delta$ is replaced by an arbitrary elliptic operator $L$ in divergence form with, say, bounded and measurable coefficients. Certainly the results still hold if $L = -\operatorname{div}(A\nabla(\cdot))$, with $A$ a positive-definite $d \times d$ constant matrix, but difficulties arise in Chapter 4 when dealing with fundamental solutions. These difficulties will be a source of additional results.

Also, we only considered homogeneous boundary data, again in the interest of simplicity. Consider a solution $u_\varepsilon \in H^1(\Omega)$ of the fully inhomogeneous problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{on } \Omega \\ u_\varepsilon = g_D & \text{on } D \\ \frac{\partial u_\varepsilon}{\partial \nu} + a_\varepsilon u_\varepsilon = g_N & \text{on } N. \end{cases}$$

Let $w \in H^1(\Omega)$ be an arbitrary function which satisfies $\operatorname{Tr} w = g_D$ on $D$, and let $v_\varepsilon = u_\varepsilon - w \in H^1_D(\Omega)$. We then have $v_\varepsilon$ satisfying the approximate mixed problem

$$\begin{cases} -\Delta v_\varepsilon = \widetilde{f} & \text{on } \Omega \\ v_\varepsilon = 0 & \text{on } D \\ \frac{\partial v_\varepsilon}{\partial \nu} + a_\varepsilon v_\varepsilon = \widetilde{g} & \text{on } N, \end{cases}$$

where $\widetilde{f} = f + \Delta w$ and $\widetilde{g} = g_N - \frac{\partial w}{\partial \nu} - a_\varepsilon w$. The problem which arises is the fact that $a_\varepsilon w$ does not make sense in general as an element of $H^{-1/2}(N)$. This is detrimental because it means we cannot use our approximation scheme for general Dirichlet data $g_D \in H^{1/2}(D)$. However, if $\operatorname{Tr} w|_D \in H^{1/2}(D)$ can be made into an element of $\widetilde{H}^{1/2}(D)$ by extending by zero, then $a_\varepsilon w$ does make sense as an element of $H^{-1/2}(N)$. The questions to address are these: Is the "extension property" described above necessary to make $a_\varepsilon w$ make sense? Can we alter our approximation scheme enough to take a wider range of Dirichlet data while still maintaining the rate of convergence from Theorem 3.18?

Finally, though we mentioned $L^p$ problems extensively in Chapter 1, we did not find any such estimates for (AMP). Thus, a possible source of future work would be to formulate (AMP) as an $L^p$ approximate mixed problem and attempt to find non-tangential estimates on the gradient of solutions. As a related thought, can we show that solutions of (AMP) are smoother than those of (MP)? If so, does this translate to numerical methods being more effective or converging more quickly to solutions of (AMP)?

# Chapter A Facts Regarding Star-Shaped Convex Domains

## A.1 Equivalence of the Definitions of Star-Shaped Domains

In Section 3.1.1 we proved a number of inequalities which hold either on star-shaped Lipschitz convex domains, or domains which are star-shaped convex with respect to a ball. Recall that $\Upsilon \subseteq \mathbf{R}^d$ is *star-shaped Lipschitz convex with constant* $M$ and *scale* $r$ if there is a point $x \in \mathbf{R}^d$ and a Lipschitz function $\varphi : \mathbf{S}^{d-1} \to [1, 1+M]$ with Lipschitz constant $M$ such that

$$\Upsilon = \left\{ y \in \mathbf{R}^d \, : \, |x - y| < r\varphi(\hat{y}) \right\}, \tag{A.1}$$

and $\Upsilon$ is *star-shaped convex with respect to a ball* $B_r(x)$ if we have

$$(1-t)y + tz \in \Upsilon \quad \forall\, y \in B_r(x),\; z \in \Upsilon,\; t \in [0,1]. \tag{A.2}$$

If $\Omega \subset \mathbf{R}^d$ is a Lipschitz domain with constant $M$, the sets $\Omega_r(x)$ given by (2.1) are in fact star-shaped convex with respect to the ball $B_{r/4}(x', \psi_x(x') + (2M + \frac{1}{2})r)$. See Figure A.1 below.

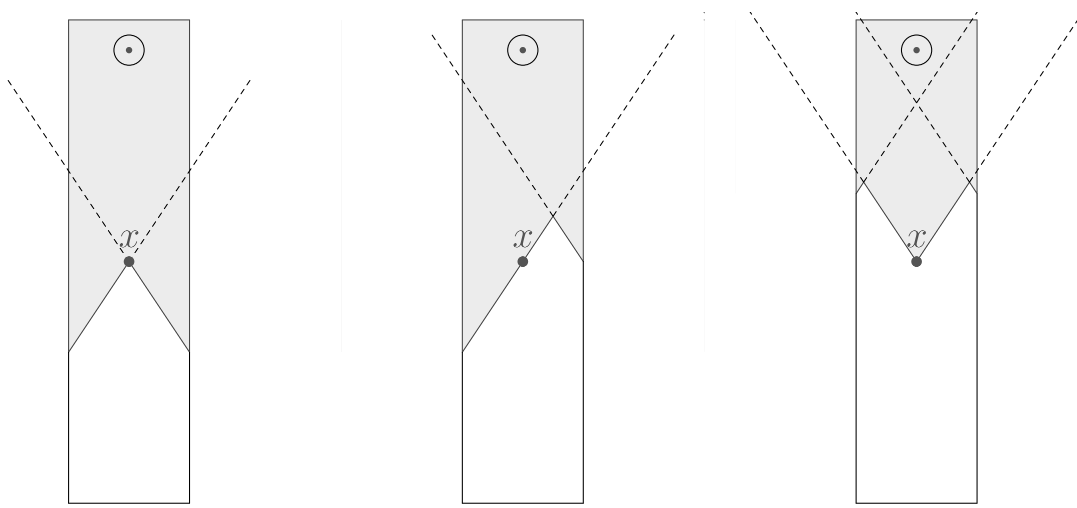


Figure A.1: The sets $\Omega_r(x)$ are star-shaped convex with respect to a ball.

As the next two lemmas show, the definitions given above are in fact equivalent. In general, for a domain $\Upsilon$ the values of $r$ in definitions (A.1) and (A.2) will not be the same.

**Lemma A.1.** *Suppose $\Upsilon$ is star-shaped convex with respect to every point in a ball $B_r(x)$. Then there is a number $M$ depending only on $r$ and $R := \sup_{y \in \Upsilon} |x - y| > r$ such that $\Upsilon$ is star-shaped Lipschitz convex with constant $M$ and scale $r$.*

*Proof.* Without loss of generality we may assume $x = 0$. For $y \in \mathbf{R}^d \setminus \{0\}$, set $y = (y', y_d) \in \mathbf{R}^{d-1} \times \mathbf{R}$ and $\hat{y} := \frac{y}{|y|} = (\hat{y}', \hat{y}_d) \in \mathbf{S}^{d-1}$. By our initial assumption that $\Upsilon$ is star-shaped convex with respect to the origin, for each $\hat{y} \in \mathbf{S}^{d-1}$ there is a unique number $\varphi(\hat{y}) \in \left[1, \frac{R}{r}\right]$ such that $r\,\hat{y}\,\varphi(\hat{y}) \in \partial\Upsilon$. Now, let $e_d = (0, 0, \ldots, 0, 1) \in \mathbf{R}^d$ and consider Figure A.2 below.

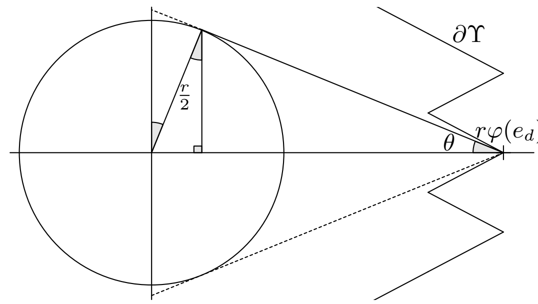


Figure A.2: Bounds for the constant $M$

Since $\varphi(e_d) \geq 1$ we have $\frac{r/2}{r\varphi(e_d)} \leq \frac{1}{2}$. Now let

$$\Gamma := \left\{(1-t)r\varphi(e_d)e_d + tz \;:\; z \in B_{r/2}(0),\, t \in [0,1]\right\},$$

and let $\Gamma_1$ be the solid cone with boundary given by

$$\partial\Gamma_1 := \left\{z \in \mathbf{R}^d \;:\; \frac{r|z'|}{R-r} + \frac{z_d}{r\varphi(e_d)} = 1,\; z_d < \varphi(e_d)\right\}.$$

Observe that $\Gamma \subseteq \bar{\Upsilon}$ by our initial assumption on $\Upsilon$, and

$$\begin{aligned}\Gamma &= B_{r/2}(0) \cup \left(\Gamma_1 \cap \left\{|z'| < \frac{r}{2}\cos\theta\right\}\right)\\ &= B_{r/2}(0) \cup \left(\Gamma_1 \cap \left\{z_d < \frac{1}{2}\sin\theta\right\}\right).\end{aligned}$$

When $\hat{z}_d \geq \frac{1}{2} \geq \sin\theta$ and $|\hat{z}'| \leq \frac{\sqrt{3}}{2} \leq \cos\theta$, we have

$$\begin{aligned}\varphi(\hat{z}) - \varphi(e_d) &\geq 1 - \varphi(e_d)\\ &\geq \frac{r|\hat{z}'|\varphi(e_d) + (R-r)\hat{z}_d}{(R-r)\varphi(e_d)} - \varphi(e_d)\\ &= \frac{(R-r)\varphi(e_d)(1 - \hat{z}_d - r|\hat{z}'|\varphi(e_d)^2}{r|\hat{z}'|\varphi(e_d) + (R-r)\hat{z}_d}\\ &\geq \frac{-2r|\hat{z}'|\varphi(e_d)^2}{R-r}\\ &\geq \frac{-2R^2}{r(R-r)}|\hat{z} - e_d|.\end{aligned}$$

The above calculation gives us a lower bound for $\varphi(\hat{z}) - \varphi(e_d)$ in a neighborhood of $\varphi(e_d)$, and a similar calculation with cones lying outside of $\Upsilon$ gives a corresponding upper bound. Thus we have $|\varphi(\hat{z}) - \varphi(e_d)| \leq C|\hat{z} - e_d|$ for $\hat{z}$ near $e_d$. For a general point $y \in \partial\Upsilon$, by rotating $\Upsilon$ about the origin we can assume $\hat{y} = e_d$ and perform the previous calculations without any loss of generality. □

**Lemma A.2.** *Suppose $\Upsilon$ is star-shaped Lipschitz convex with constant $M$ and scale $r$ as in* (A.1). *If $x$ is the star-center, then there is a number $s$ depending on $M$ and $r$ so that $\Upsilon$ is star-shaped convex with respect to the ball $B_s(x)$.*

*Proof.* If $\Upsilon$ has scale $r$, then $r\Upsilon = \{ry : y \in \Upsilon\}$ has scale 1, so without loss of generality we may assume $r = 1$. Using the same notation as in the proof of Lemma A.1, let us restrict our attention to the portion of $\mathbf{S}^{d-1}$ where $|\hat{z}'| \leq \frac{1}{2}$ and $\hat{z}_d \geq \frac{\sqrt{3}}{2}$, i.e., $|\hat{z}'| = sin\theta$ and $\hat{z}_d = \cos\theta$ for some $\theta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$. In this way,

$$\begin{aligned}
|\hat{z}'| &= \sin\theta \\
&\leq \sqrt{\sin^2\theta + (1-\cos\theta)^2} \\
&= |\hat{z} - e_d| \\
&= \sqrt{\sin^2\theta + (1-\cos\theta)^2} \\
&\leq \sqrt{2}\sin\theta \\
&= \sqrt{2}|\hat{z}'|.
\end{aligned}$$

It follows that the set

$$\Gamma_1 := \left\{ z \in \mathbf{R}^d \,:\, |\hat{z}'| \leq \frac{1}{2},\ |\hat{z}'| \leq \frac{1}{2M}(\varphi(e_d) - |z|) \right\} \cup B_1(0)$$

is a subset of $\Upsilon$. See Figure A.3

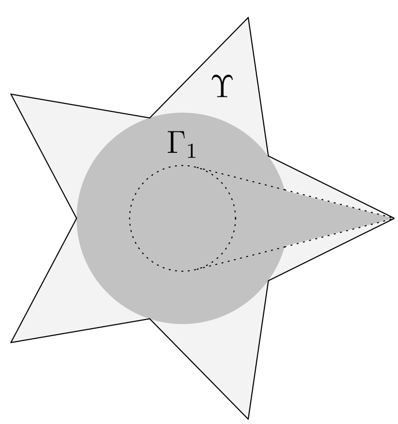


Figure A.3: The shaded set $\Gamma_1$ inside the star-shaped Lipschitz convex domain $\Upsilon$, with dotted lines to emphasize star-shaped convexity.

Note that the ball tangent to the boundary of the solid cone

$$\Gamma_2 = \left\{ z \,:\, |\hat{z}'| \leq \frac{1}{2M}(\varphi(e_d) - |z|) \right\}$$

is $B_{\frac{\sqrt{2}}{2}}(0)$ and is contained in $\Upsilon$. By rotating $\Upsilon$ about the origin so that $\hat{y} = e_d$, we can see that this ball will be contained in $\Gamma_2$ when $e_d$ is replaced by $\hat{y}$ above. Thus, $\Upsilon$ is in fact star-shaped convex with respect to the ball $B_{\frac{\sqrt{2}}{2}}(0)$. □

## A.2 Estimates on the Reflection of $x$ over $\partial\Upsilon$

**Lemma A.3.** *Suppose $\varphi : \mathbf{S}^{d-1} \to [1, M+1]$ is Lipschitz with constant $M$. There is a function $\Phi : \mathbf{R}^d \to \mathbf{R}$ which is Lipschitz with constant $3M+1$ such that $\Phi(\hat{x}) = \varphi(\hat{x})$ for all $\hat{x} \in \mathbf{S}^{d-1}$.*

*Proof.* For $x \in \mathbf{R}^d$, let $\hat{x} = \frac{x}{|x|}$ and set

$$\Phi(x) := \begin{cases} |x|\varphi(\hat{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

For $x \in \mathbf{R}^d$ and $y = 0$ we clearly have $|\Phi(x) - \Phi(y)| \leq M|x - y|$, so assume $x, y \neq 0$. We compute

$$\begin{aligned} |\Phi(x) - \Phi(y)| &= ||x|\varphi(\hat{x}) - |y|\varphi(\hat{y})| \\ &= ||x|\varphi(\hat{x}) - |x|\varphi(\hat{y}) + |x|\varphi(\hat{y}) - |y|\varphi(\hat{y})| \\ &\leq |x||\varphi(\hat{x})\varphi(\hat{y})| + ||x| - |y|||\varphi(\hat{y})| \\ &\leq M|x||\hat{x} - \hat{y}| + (M+1)|x - y| \\ &= M\left|\frac{x|y| - y|y| + y|y| - y|x|}{|y|}\right| + (M+1)|x - y| \\ &\leq M|x - y| + M||x| - |y|| + (M+1)|x - y| \\ &= (3M+1)|x - y|. \end{aligned}$$

□

**Lemma A.4.** *Let $\varphi, \Phi$ be as in Lemma* (A.3)*, and set $\hat{x} := \frac{x}{|x|}$ for $x \neq 0$. We write $\nabla\varphi(\hat{x}) := \nabla\Phi(x)|_{\mathbf{S}^{d-1}}$ and $\varphi_{x_i}(\hat{x}) := \Phi_{x_i}(x)|_{\mathbf{S}^{d-1}}$ for $1 \leq i \leq d$, all of which are defined $\sigma$-a.e. on $\mathbf{S}^{d-1}$. Suppose $\Upsilon \subseteq \mathbf{R}^d$ is star-shaped convex with constant $M$ and scale $r$, with $\varphi$ as its defining Lipschitz function. Let*

$$x^* := \frac{\hat{x}}{|x|} r^2 \varphi(\hat{x})^2$$

*denote the* reflection of $x$ over $\partial\Upsilon$*. The $d \times d$ Jacobian matrix $[J(x^*)]$ of $x^*$ has entries*

$$[J(x^*)](x)_{ij} = r^2 \left( \frac{-2x_i x_j}{|x|^4} \left[\varphi(\hat{x})\nabla\varphi(\hat{x}) \cdot \hat{x} + \varphi(\hat{x})^2\right] + \frac{2x_j}{|x|^3}\varphi(\hat{x})\varphi_{x_i}(\hat{x}) + \frac{\delta_{ij}}{|x|^2}\varphi(\hat{x})^2 \right). \tag{A.3}$$

*Moreover, for $x \notin \Upsilon$ the entries of $[J(x^*)]$ satisfy the estimate*

$$|[J(x^*)](x)_{ij}| \leq 5(3M+1)^2, \tag{A.4}$$

*and the determinant consequently satisfies the crude estimate*

$$|\det[J(x^*)](x)| \leq d!5^d(3M+1)^{2d}. \tag{A.5}$$

*Proof.* The formula (A.3) comes from direct computation. Further, $x \notin \Upsilon$ implies $|x| \geq r$, giving us the estimate (A.4) by appealing to Lemma A.3. The estimate (A.5) follows by induction on $d$. □

*Remark.* The estimate (A.4) can be improved to

$$|[J(x^*)](x)_{ij}| \leq (15M+7)(M+1),$$

improving the estimate (A.5) as well. However, the estimate in Lemma A.4 is sufficient for our purposes.

## Chapter B Sobolev Spaces $H^s(\partial\Omega)$ and $H^t(\Omega)$ for $-1 < s < 1$ and $-\frac{3}{2} < t < \frac{3}{2}$

Let $0 < s < 1$, $-\frac{3}{2} < t < \frac{3}{2}$, and $\tau = t - \lfloor t \rfloor$. Define seminorms $|\varphi|_{H^s(\partial\Omega)}$ and $|u|_{H^\tau(\Omega)}$ by

$$|\varphi|_{H^s(\partial\Omega)} = \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{d-1+2s}}\,d\sigma(x)\,d\sigma(y)\right)^{\frac{1}{2}}, \quad \text{and}$$
$$|u|_{H^\tau(\Omega)} = \left(\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\tau}}\,d\sigma(x)\,d\sigma(y)\right)^{\frac{1}{2}},$$

respectively. For $0 < s, t < 1$ we define $H^s(\partial\Omega)$ and $H^t(\Omega)$ as the spaces of all $\varphi \in L^2(\partial\Omega)$ with finite norms

$$\|\varphi\|_{H^s(\partial\Omega)} := \left(\|\varphi\|^2_{L^2(\partial\Omega)} + |\varphi|^2_{H^s(\partial\Omega)}\right)^{\frac{1}{2}}$$
$$\|u\|_{H^t(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{H^t(\Omega)}\right)^{\frac{1}{2}}.$$

When $1 < t < \frac{3}{2}$, the space $H^t(\Omega)$ is the space of all $u \in H^1(\Omega)$ with finite norm

$$\|u\|_{H^t(\Omega)} := \left(\|u\|^2_{H^1(\Omega)} + |\nabla u|^2_{H^\tau(\Omega)}\right)^{\frac{1}{2}}.$$

When $s = \frac{1}{2}$ and $t = 1$, we have the usual Sobolev spaces $H^{1/2}(\partial\Omega)$ and $H^1(\Omega)$. Let $H^0(\partial\Omega) = L^2(\partial\Omega)$ and $H^0(\Omega) = L^2(\Omega)$, and let $H^{-s}(\partial\Omega)$ and $H^{-t}(\Omega)$ denote the duals of $H^s(\partial\Omega)$ and $H^t(\Omega)$, respectively, for $0 \le s < 1$ and $0 \le t < \frac{3}{2}$.

**Proposition B.1** (Rellich [33])**.** *If $-1 < s_1 < s_2 < 1$ and $-\frac{3}{2} < t_1 < t_2 < \frac{3}{2}$, then the inclusions $H^{s_2}(\partial\Omega) \hookrightarrow H^{s_1}(\partial\Omega)$ and $H^{t_2}(\Omega) \hookrightarrow H^{t_1}(\Omega)$ are compact.*

Next, by a result originally due to Costabel [4] we can improve the mapping properties of Lemma 4.3 by extending them to a range of Sobolev spaces.

**Proposition B.2** (Theorem 6.12 (i) and Exercise 6.4 of Mclean [26, pp. 205+206])**.** *Let $\eta \in C_c^\infty(\mathbf{R}^d)$. For $0 < s < 1$, the single- and double layer potentials* SL *and* DL *give rise to bounded operators*

$$\eta\,\mathrm{SL} : H^{-s}(\partial\Omega) \to H^{s+\frac{1}{2}}(\mathbf{R}^d) \quad \textit{and} \quad \eta\,\mathrm{DL} : H^{-s}(\partial\Omega) \to H^{\frac{1}{2}-s}(\Omega^\pm).$$

We are now prepared to prove that the operator $K^2 : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ from the proof of Lemma 4.7 is in fact compact.

**Lemma B.3.** *Fix a function $\eta \in C_c^\infty(\mathbf{R}^d)$ which is 1 on a neighborhood of $\overline{\Omega^-}$, let $\psi_1, \psi_2 \in H^{-1/2}(\partial\Omega)$, and set $u := \eta\,\mathrm{SL}\,\psi_1$, $v := \eta\,\mathrm{SL}\,\psi_2$. The operator $K$ : $H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ given by $\langle\psi_2, K\psi_1\rangle_{\partial\Omega} := \int_{\mathbf{R}^d} uv\,dx$ is compact.*

*Proof.* Observe that for $0 < s < 1$, $H^s(\partial\Omega) \subseteq H^{-s}(\partial\Omega)$, i.e., $\|\psi_2\|_{H^{-s}(\partial\Omega)} \leq C\|\psi_2\|_{H^s(\partial\Omega)}$. Fix $0 < s < 1$ and suppose $\|\psi_2\|_{H^s(\partial\Omega)} = 1$. By the Cauchy-Schwarz inequality and Proposition B.2,

$$\begin{aligned}
|\langle \psi_2, K\psi_1\rangle_{\partial\Omega}| &\leq \|u\|_{L^2(\mathbf{R}^d)}\|v\|_{L^2(\mathbf{R}^d)} \\
&\leq \|u\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}\|v\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \\
&\leq C\|\psi_1\|_{H^{-s}(\partial\Omega)}\|\psi_2\|_{H^{-s}(\partial\Omega)} \\
&\leq C\|\psi_1\|_{H^{-s}(\partial\Omega)}.
\end{aligned}$$

Thus, we have shown that $\|K\psi_1\|_{H^s(\partial\Omega)} \leq C\|\psi_1\|_{H^{-s}(\partial\Omega)}$, i.e., $K$ is bounded as a map from $H^{-s}(\partial\Omega)$ to $H^s(\partial\Omega)$. Finally, by Proposition B.1 we conclude that $K : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is compact, since for $\frac{1}{2} < s < 1$,

$$K = H^{-1/2}(\partial\Omega) \hookrightarrow H^{-s}(\partial\Omega) \xrightarrow{K} H^s(\partial\Omega) \hookrightarrow H^{1/2}(\partial\Omega). \qquad \square$$

**Lemma B.4.** *Fix a function $\eta \in C_c^\infty(\mathbf{R}^d)$ which is 1 on a neighborhood of $\overline{\Omega^-}$, let $\varphi_1, \varphi_2 \in H^{1/2}(\partial\Omega)$, and set $u := \eta\,\mathrm{DL}\,\varphi_1$, $v := \eta\,\mathrm{DL}\,\varphi_2$. The operator $K : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by $\langle K\varphi_1, \varphi_2\rangle_{\partial\Omega} := \int_{\mathbf{R}^d} uv\,dx$ is bounded as an operator from $H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$.*

*Proof.* By the Cauchy-Schwarz inequality and Proposition B.2, we have

$$\begin{aligned}
\langle K\varphi_1, \varphi_2\rangle_{\partial\Omega} &\leq \|u\|_{L^2(\Omega^-)}\|v\|_{L^2(\Omega^-)} + \|u\|_{L^2(\Omega^+)}\|v\|_{L^2(\Omega^+)} \\
&\leq C\|\varphi_1\|_{H^{-1/2}(\partial\Omega)}\|\varphi_2\|_{H^{-1/2}(\partial\Omega)}.
\end{aligned}$$

Hence, $\|K\varphi_1\|_{H^{1/2}(\partial\Omega)} \leq C\|\varphi_1\|_{H^{-1/2}(\partial\Omega)}$ as desired. $\square$

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**Vitae**

## EDUCATION

- Ph.D., Mathematics, University of Kentucky, expect to graduate May 2017.
  - Thesis Topic: Approximation of Solutions to the Mixed Dirichlet Neumann Boundary-Value Problem on Lipschitz Domains
  - Thesis Advisor: Dr. Russell Brown
  - Written Exams passed January 2014
  - Oral Exam passed December 2014
- M.A., Mathematics, University of Kentucky, December 2013.
- B.A., Mathematics, Millersville University, Millersville, Pennsylvania, May 2011.
  - Minor: Philosophy
  - Cum Laude
  - Senior Thesis Topic: Fixed-Point Theorems in Topology and Geometry
  - Senior Thesis Advisor: Dr. Ronald Umble

## RESEARCH INTERESTS

Mixed Problem, Robin Problem, Layer Potentials, Approximation, Sobolev Spaces

## TEACHING EXPERIENCE

- 2011-Present *Teaching Assistant, University of Kentucky Department of Mathematics*
  - Primary Instructor of MA 109, MA 111, MA 112, and MA 114.
  - Recitation Instructor for MA 113, MA 114, MA 123, and MA 213.
  - Substitute Instructor for MA 214 and MA 471.
- Summers of 2012, 2013, 2014, 2016, *Primary Instructor, Freshman Summer Program (FSP) through the Center for Academic Resources and Enrichment Services (CARES), University of Kentucky*

## WORK EXPERIENCE

- Summer 2012, *Assisted in writing web homework, University of Kentucky*

  Proofread questions and tested answers to Calculus I and II problems for mathclass.org, a web-based homework system used at the University of Kentucky.

## AWARDS AND HONORS

- Recipient of a *Max Steckler Fellowship*, 2011, awarded by the Graduate School of the University of Kentucky for the 2011-2012 academic year.
- Received 10 points in the *William Lowell Putnam Mathematical Competition*, December 2010.
- Awarded Course Honors (roughly equivalent to an A+) in MATH 393 (Number Theory) and MATH 566 (Complex Variables) at Millersville University.

## SERVICE

- Co-chair, Math Department Graduate Student Council (GSC), *University of Kentucky* August 2013-May 2014. Responsibilities include representing the graduate student body at faculty meetings, hosting departmental teas, greeting and guiding visiting potential graduate students, and raising funds.
- Spring 2010-Spring 2011, *Tutor, Millersville University Athletics Department*

  Tutored student athletes in a group setting. Particularly, athletes with a C or lower in their current math class were required to spend two or more hours a week in my sessions. Courses covered include Algebra, Trigonometry, Calculus I and II, and Survey of Mathematical Ideas.
- Spring 2010-Spring 2011, *Tutor, Millersville Office of Learning Services*

  Gave individual supplemental instruction to students with documented learning disabilities. Courses covered include Contemporary Math, Calculus, and Introductory Statistics.
- Spring 2008-Spring 2009, *Tutor, Harrisburg Area Community College*

  Tutored students in group and individual settings in the "Math Lab." Courses covered include Beginning, Intermediate, and College Algebra, Trigonometry, the Calculus sequence, and Linear Algebra.

## CONFERENCES, PROGRAMS, AND PRESENTATIONS

- Presented *Proof of a Hardy Inequality on the Boundary of Lipschitz Domains* to the UK Math Department Analysis/PDE Seminar (October 15, 2013)
- Attended the *Fourth Ohio River Analysis Meeting* at the University of Kentucky (March 2014)
- Attended the *2014 NSF-CBMS Conference on Inverse Scattering and Transmission Eigenvalues* at the University of Texas at Arlington (May 27-31 2014)
- Presented *Solving the Mixed Problem on Lipschitz Domains by the Method of Layer Potentials* to the UK Math Department Analysis/PDE Seminar (December 2, 2014)
- Attended the *Fifth Ohio River Analysis Meeting* at the University of Cincinnati (March 2015)

- Attended the *Sixth Symposium on Analysis and PDEs* at Purdue University (June 1-4, 2015)
- Participated in the *IMA Mathematical Modeling in Industry Workshop* at the University of Minnesota (August 5-14, 2015)
- Attended the *Nineteenth Annual Rivière-Fabes Symposium on Analysis and PDE* at the University of Minnesota (April 15-17, 2016)
- Attended the *Seventy-Seventh Midwest PDE Seminar* at the University of Cincinnati (May 7-8, 2016)
- Attended *MAA MathFest 2016* in Columbus, OH (August 3-6, 2016)
- Presented *Generalized Functions* at the University of Kentucky (*Graduate Student Colloquium*, September 14, 2016)
- Presented *Approximation of Solutions to the Mixed Problem on Lipschitz Domains* at the University of Kansas (*Prairie Analysis Seminar*, September 16-17, 2016)
- Presented *Approximation of Solutions to the Mixed Problem on Lipschitz Domains* at Florida Gulf Coast University (*SEARCDE*, November 5-6, 2016)
- Attended *2017 Joint Mathematics Meetings* in Atlanta, GA (January 4-7, 2017)
- Presented *Approximation of Solutions to the Mixed Problem on Lipschitz Domains* at the University of Cincinnati (*Ohio River Analysis Meeting 7*, March 25-26, 2017)