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Optical Tomography on Graphs

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PAPER

Optical tomography on graphs

OPTICAL TOMOGRAPHY ON GRAPHS

FRANCIS J. CHUNG, ANNA C. GILBERT, JEREMY G. HOSKINS, AND JOHN C. SCHOTLAND

ABSTRACT. We present an algorithm for solving inverse problems on graphs analogous to those arising in diffuse optical tomography for continuous media. In particular, we formulate and analyze a discrete version of the inverse Born series, proving estimates characterizing the domain of convergence, approximation errors, and stability of our approach. We also present a modification which allows additional information on the structure of the potential to be incorporated, facilitating recovery for a broader class of problems.

1. INTRODUCTION

Inverse problems arise in numerous settings within discrete mathematics, including graph tomography [42, 28, 29, 25, 19] and resistor networks [20, 21, 22, 23, 30, 11, 10]. In such problems, one is typically interested in reconstructing a function defined on edges of a fixed graph or, in some cases, the edges themselves. In this paper, we focus on recovering vertex properties of a graph from boundary measurements. The problem we consider is the discrete analog of optical tomography. Optical tomography is a biomedical imaging modality that uses scattered light as a probe of structural variations in the optical properties of tissue [4]. The inverse problem of optical tomography consists of recovering the potential of a Schrödinger operator from boundary measurements.

Let $G = (V, E)$ be a finite locally connected loop-free graph with vertex boundary δV. We consider the time-independent diffusion equation [36]

$$(Lu)(x) + \alpha_0[1 + \eta(x)]u(x) = f(x), \quad x \in V, \tag{1}$$

$$t\, u(x) + \partial u(x) = g(x), \quad x \in \delta V, \tag{2}$$

which, in the continuous setting, describes the transport of the energy density of an optical field in an absorbing medium. Here we assume that the absorption of the medium is nearly constant with background absorption α_0 and inhomogeneities represented by the *vertex potential* η. In place of the Laplace-Beltrami operator, we introduce the combinatorial Laplacian L defined by

$$(Lu)(x) = \sum_{y \sim x} [u(x) - u(y)], \tag{3}$$

where $y \sim x$ if the vertices x and y are adjacent. We make use of the graph analog of Robin boundary conditions, where the normal derivative is defined by

$$\partial u(x) = \sum_{\substack{y \in V \\ y \sim x}} [u(x) - u(y)], \tag{4}$$

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and t is an arbitrary nonnegative parameter, which interpolates between Dirichlet and Neumann boundary conditions. If the vertex potential η is non-negative, then there exists a unique solution to the diffusion equation (1) satisfying the boundary condition (2), [see 24 and the references therein].

In [24] we presented an algorithm for solving the forward problem of determining u, given η. Our approach was a perturbative one, making use of known Green's functions for the time-independent diffusion equation (or Schrödinger equation) [3, 8, 9, 7, 13, 14, 15, 16, 41, 43], with η identically zero. The corresponding inverse problem, which we refer to as graph optical tomography, is to recover the potential η from measurements of u on the boundary of the graph. More precisely, let $G = (V, E)$ be a connected subgraph of a finite graph $\Gamma = (\mathcal{V}, \mathcal{E})$ and let δV denote those vertices in $\mathcal{V}$ adjacent to a vertex in V. In addition, let S, R denote fixed subsets of δV. We will refer to elements of S and R as sources and receivers, respectively. For a fixed potential η, source $s \in S$ and receiver $r \in R$, let $u(r, s; \eta)$ be the solution to (1) with vertex potential η and boundary condition (2), where

$$g(x) = \begin{cases} 1 & x = s, \\ 0 & x \neq s. \end{cases} \tag{5}$$

We define the Robin-to-Dirichlet map Λ_η by

$$\Lambda_\eta(s, r) = u(r, s; \eta). \tag{6}$$

The inverse problem is to recover η from the Robin-to-Dirichlet map Λ_η.

Eqs. (1) and (2) also arise when considering the Schrödinger equation on graphs and related inverse problems [39, 31, 2, 11, 12]. For circular planar graphs, or lattice graphs in two or more dimensions, [39, 31, 2, 11] outline an algorithm that can be used to recover the vertex potential. In particular, the first three employ special combinations of boundary sources which force the solution in the interior to be zero except on a small, controllable set of vertices. Using this approach, the potential at each vertex can be calculated. Then, starting at the boundary, the entire potential can be recovered. The resulting algorithm relies on the lattice structure of the graphs and is unstable for potentials with large support.

In this paper we present a reconstruction method for graph optical tomography that is based on inversion of the Born series solution to the forward problem [36, 6, 34, 37, 35, 5, 32]. Using this approach, we show that it is possible to recover vertex potentials for a general class of graphs under certain smallness conditions on the boundary measurements. Our results are complementary to those in [12], where a discrete analog of complex geometrical optics solutions are used to show that if the linearized problem is solvable, then the Robin-to-Dirichlet map is invertible almost everywhere. We also note that our algorithm applies to complex η, a case that arises in optical tomography. In addition, we obtain sufficient conditions under which the inverse Born series converges to the vertex potential. We also obtain a corresponding stability estimate, which is independent of the support of the potential. In numerical studies of the inverse Born series for large potentials or large graphs, where exact recovery is not guaranteed, we nevertheless find that good qualitative recovery of large scale features of the potential is possible. Moreover, our approach can

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be easily modified to incorporate additional information on the structure of the potential, improving both the speed and accuracy of the algorithm. As an application of this idea, we show how to determine the potential η using data for multiple values of α_0, assuming η is independent of α_0. This allows us to apply our method to graphs whose structure makes exact potential recovery otherwise impossible.

The remainder of this paper is organized as follows. In Section 2 we briefly review key results on the solvability of the forward problem and introduce the Born series. We obtain necessary conditions for the convergence of the inverse Born series depending on the measurement data and the graph. We also describe related stability and error estimates. In Section 3 we discuss the numerical implementation of the inverse series and present the results of numerical simulations. Finally, in Section 4 we extend our results to the case where measurements can be taken at multiple values of α_0.

2. Inverse Born series

2.1. **Forward Born series.** In this section we formulate the inverse Born series. We begin by reviewing some important properties of the Born series, based in part on [24, 36].

We recall that the *background Green's function* [24] for (1) is the matrix G_0 whose i,jth entry is the solution to (1), with $\eta \equiv 0$, at the ith vertex for a unit source at the jth vertex. Under suitable restrictions this matrix can be used to construct the *Robin-to-Dirichlet map* Λ_η giving the solution of (1) on $R \subset \delta V$ to unit sources located in $S \subset \delta V$. To write a compact expression for Λ_η in terms of G_0, let D_η denote the matrix with entries given by

$$(D_\eta)_{i,j} = \begin{cases} \eta_i & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Additionally, for any two sets $U, W \subset V \cup \delta V$, let $G_0^{U;W}$ denote the submatrix of G_0 formed by taking the rows indexed by U and the columns indexed by W. For η sufficiently small we may write the Robin-to-Dirichlet map as a Neumann series

$$\Lambda_\eta(s,r) = G_0(r,s) - \sum_{j=1}^{\infty} K_j(\eta, \cdots, \eta)(r,s), \quad r \in R, \ s \in S, \tag{7}$$

where $K_j : \ell^p(V^n) \to \ell^p(R \times S)$ is defined by

$$K_j(\eta_1, \cdots, \eta_j)(r,s) = (-\alpha_0)^j G_0^{r;V} D_{\eta_1} G_0^{V;V} D_{\eta_2} \cdots G_0^{V;V} D_{\eta_j} G_0^{V;s}. \tag{8}$$

We refer to the series (7) as the *forward Born series.*

In order to establish the convergence and stability of (7), we seek appropriate bounds on the operators $K_j : \ell^p(V \times \cdots \times V) \to \ell^p(\delta V \times \delta V)$. Note that if $|V|$ and $|\delta V|$ are finite then all norms are equivalent. However, since we are interested in the rate of convergence of the inverse series it will prove useful to establish bounds for arbitrary ℓ_p norms.

Proposition 1. Let $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ and define the constants ν_p and μ_p by

$$\nu_p = \alpha_0 \|G_0^{R;V}\|_{\ell^q(V) \times \ell^p(R)} \|G_0^{V;S}\|_{\ell^q(V) \times \ell^p(S)}, \quad \text{and} \quad \mu_p = \alpha_0 \, C_{G_0^{V;V}, q}, \tag{9}$$

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where

$$C_{G_0^{V;V},q} = \max_{v \in V} \|G_0^{V;v}\|_{\ell^q(V)}. \tag{10}$$

The forward Born series (7) converges if

$$\mu_p \|\eta\|_p < 1. \tag{11}$$

Moreover, the N-term truncation error has the following bound,

$$\left\| \Lambda_\eta - \left(G_0 - \sum_{j=N}^{\infty} K_j(\eta, \cdots, \eta) \right) \right\|_{\ell^p(R \times S)} \leq \nu_p \|\eta\|_p^{N+1} \mu_p^N \frac{1}{1 - \mu_p \|\eta\|_p}. \tag{12}$$

Remark 2. The bounds we obtain are similar to those found in the continuous setting [36], though here we present a novel proof of ℓ^2-boundedness and extend our results to include $p \in [1, 2)$; a case not previously considered.

Before proving the proposition, we first establish the following useful identities.

Lemma 3. Let M be an $n \times n$ matrix, and D_a, D_b be $n \times n$ diagonal matrices with diagonal entries given by vectors a and b, respectively. Let $M(k)$ denote the kth row of M, and

$$C_{M,q} = \max_k \|M(k)\|_q, \tag{13}$$

for $1 \leq q \leq \infty$. Then for any vectors u^T and v, and $p, q \in [1, \infty]$, such that $1/p + 1/q = 1$,

$$|u^T D_a M D_b v| \leq C_{M,q} \|u\|_q \|a\|_p \|b\|_p \|v\|_\infty. \tag{14}$$

Proof. We begin by observing that if e_k is the kth canonical basis vector, $a = \sum_k D_a e_k$ and $I = \sum_k e_k e_k^T$, where I is the $n \times n$ identity matrix. Hence

$$|u^T D_a M D_b v| \leq \left(\sum_k |u^T D_a e_k| \right) \max_k |M(k) D_b v|,$$

$$\leq \|u\|_q \|a\|_p \max_k \sum_j |M(k) D_b e_j| \max_j |e_j^T v|, \tag{15}$$

$$\leq \|u\|_q \|a\|_p \|b\|_p \max_k \|M(k)\|_q \max_j |e_j^T v|.$$

$$\square$$

We can iterate the result of the Lemma to obtain the following corollary.

Corollary 4. Let $M_1, \cdots, M_{j-1}$ be $n \times n$ matrices and $D_{a_1}, \cdots, D_{a_j}$ be $n \times n$ diagonal matrices with diagonal elements given by the vectors $a_1, \cdots, a_j$. If $M_i(k)$ and $C_{M_i,q}$ are defined as in the previous Lemma, then for all u and v,

$$\left| u^T D_{a_1} M_1 D_{a_2} \cdots M_{j-1} D_{a_j} v \right| \leq \|a_1\|_p \cdots \|a_j\|_p \, C_{M_1,q} \cdots C_{M_{j-1},q} \|u\|_q \|v\|_\infty, \tag{16}$$

where once again $p, q \in [1, \infty]$ and $1/p + 1/q = 1$.

We now return to the proof of Proposition 1.

Proof. Since D_{η_i} is a diagonal matrix, $D_{\eta_i} = \sum_{k \in V} \eta_i(k) e_k e_k^T$, where $\eta_i(k)$ is the kth component of the vector η_i and e_k is the canonical basis vector corresponding to the vertex k. From the definition of K_j, we see that

$$\|K_j(\eta_1, \cdots, \eta_j)\|_p \leq \alpha_0^j \left(\sum_{r \in R, \, s \in S} \left[G_0^{r;V} D_{\eta_1} G_0^{V;V} D_{\eta_2} \cdots G_0^{V;V} D_{\eta_j} G_0^{V;s} \right]^p \right)^{1/p}. \tag{17}$$

The previous Corollary implies that

$$\left| G_0^{r;V} D_{\eta_1} \cdots D_{\eta_j} G_0^{V;s} \right| \leq \|\eta_1\|_p \cdots \|\eta_j\|_p C_{G_0^{V;V},q}^{j-1} \|G_0^{r,V}\|_q \|G_0^{V,s}\|_\infty. \tag{18}$$

Thus

$$\|K_j\|_p \leq \alpha_0^j \|G_0^{R,V}\|_{\ell^p(R) \times \ell^q(V)} \|G_0^{V,S}\|_{\ell^q(V) \times \ell^p(S)} C_{G_0^{V;V},q}^{j-1}, \tag{19}$$

$$\leq \nu_p \, \mu_p^{j-1},$$

where $\nu_p = \alpha_0 \|G_0^{R,V}\|_{\ell^q(V) \times \ell^p(R)} \|G_0^{V,S}\|_{\ell^q(V) \times \ell^p(S)}$ and $\mu_p = \alpha_0 C_{G_0^{V;V},q}$, from which the result follows immediately.

$\square$

2.2. **Inverse Born series.** Proceeding as in [36], let $\phi \in \ell^2(R \times S)$ denote the *scattering data*,

$$\phi(r, s) = G_0(r, s) - \Lambda_\eta(r, s), \tag{20}$$

corresponding to the difference between the measurements in the background medium and those in the medium with the potential present. Note that if the forward Born series converges, we have

$$\phi(r, s) = \sum_{j=1}^\infty K_j(\eta, \ldots, \eta). \tag{21}$$

Next, we introduce the ansatz

$$\eta = \mathcal{K}_1(\phi) + \mathcal{K}_2(\phi, \phi) + \mathcal{K}_3(\phi, \phi, \phi) + \cdots, \tag{22}$$

where each $\mathcal{K}_n$ is a multilinear operator. Though ϕ can be thought of as an operator from $\ell^2(R)$ to $\ell^2(S)$, in (22) we treat it as a vector of length $|R| \cdot |S|$. Similarly, though it is often convenient to think of η as a (diagonal) matrix, in (22) it should be thought of as a vector of length $|V|$. Treating η and ϕ as matrices results in a different inverse problem related to matrix completion [33]. With a slight abuse of notation, we also use K_1 to denote the $|R||S| \times |V|$ matrix mapping η (viewed as a vector) to $K_1\eta$, once again thought of as a vector.

To derive the inverse Born series, we substitute the ansatz (22) into the forward series (21) and equate tensor powers of ϕ. We thus obtain the following recursive expressions for

the operators $\mathcal{K}_j$ [36]:

$$
\begin{aligned}
\mathcal{K}_1 &= K_1^+, \\
\mathcal{K}_2 &= -\mathcal{K}_1 K_2 \mathcal{K}_1 \otimes \mathcal{K}_1, \\
\mathcal{K}_3 &= -\left(\mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3\right) \mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1, \\
\mathcal{K}_j &= -\left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m}\right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1,
\end{aligned}
\tag{23}
$$

where K_1^+ denotes the (regularized) pseudoinverse of K_1.

The following result provides sufficient conditions for the convergence of the inverse Born series for graphs where $|V| = |R \times S|$, corresponding to the case of a formally determined inverse problem.

Theorem 5. Let $|V| = |R \times S|$ and $p \in [1, \infty]$. Suppose that the operator K_1 is invertible. Then the inverse Born series converges to the true potential η if $\|\phi\|_p < r_p$. Here the radius of convergence r_p is defined by

$$
r_p = \frac{C_p}{\mu_p}\left[1 - 2\frac{\nu_p}{C_p}\left(\sqrt{1 + \frac{C_p}{\nu_p}} - 1\right)\right],
\tag{24}
$$

where

$$
C_p = \min_{\|\eta\|_p=1} \|K_1(\eta)\|_p
\tag{25}
$$

and ν_p, μ_p are defined in (9).

Remark 6. The convergence of the inverse Born series in the continuous setting was analyzed in [35]. It was found that certain smallness conditions on both $\|\mathcal{K}_1\|_p$ and $\|\mathcal{K}_1\phi\|_p$ are sufficient to guarantee convergence. Note that such a condition on $\|\mathcal{K}_1\|_p$ is not present in Theorem 5, Proposition 10 or Theorem 11. As explained below, this is due to the use of different techniques than in [35].

The proof of Theorem 5 depends on the following multi-dimensional version of Rouché's theorem.

Theorem 7. [Theorem 2.5, 1] Let D be a domain in $\mathbb{C}^n$ with a piecewise smooth boundary ∂D. Suppose that $f, g : \mathbb{C}^n \to \mathbb{C}^n$ are holomorphic on $\bar{D}$. If for each point $z \in \partial D$ there is at least one index j, $j = 1, \ldots, n$, such that $|g_j(z)| < |f_j(z)|$, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros in D, counting multiplicity.

Proof of Theorem 5. Put $n = |V| = |R \times S|$. Let $F : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be the function defined by

$$
F(\eta, \phi) = \phi - \sum_{j=1}^{\infty} K_j(\eta, \ldots, \eta).
\tag{26}
$$

Note that F has n components $F_1, \ldots, F_n$, each of which is well-defined and holomorphic for all ϕ if $\|\eta\|_p < 1/\mu_p$, since they are defined by a convergent Taylor series in ϕ and η.

Let

$$C_p = \min_{\|\eta\|_p = 1} \|K_1(\eta)\|_p, \tag{27}$$

which is non-zero for all p since K_1 is invertible. Then

$$\|F(\eta, 0)\|_p \geq C_p \|\eta\|_p - \sum_{j=2}^{\infty} \|K_j(\eta, \ldots, \eta)\|_p,$$

$$\geq C_p \|\eta\|_p - \nu_p \sum_{j=2}^{\infty} \mu_p^{j-1} \|\eta\|_p^j, \tag{28}$$

$$\geq C_p \|\eta\|_p - \nu_p \mu_p \|\eta\|_p^2 \frac{1}{1 - \mu_p \|\eta\|_p},$$

where the second inequality follows from the bounds on the forward operators obtained in the proof of Proposition 1. For $0 < \|\eta\|_p < 1/\mu_p$, $\|F(\eta, 0)\|_p$ is non-vanishing if

$$\|\eta\|_p < \frac{1}{\mu_p} \frac{C_p}{C_p + \nu_p}. \tag{29}$$

Suppose $\lambda \geq 1$. We then define

$$R_\lambda = \frac{1}{\mu_p} \frac{C_p}{C_p + \nu_p \lambda}, \tag{30}$$

and let $\Omega_{1,\lambda} = \{\eta \in \mathbb{C}^n \,|\, \|\eta\|_p < R_\lambda\}$.

Next, we observe that $F(\eta, \phi) - F(\eta, 0) = \phi$ and hence if

$$\|\phi\|_p < \|F(\eta, 0)\|_p, \tag{31}$$

then

$$\|F(\eta, \phi) - F(\eta, 0)\|_p < \|F(\eta, 0)\|_p. \tag{32}$$

Note that

$$\|F(\eta, 0)\|_p \geq C_p \|\eta\|_p - \nu_p \mu_p \frac{\|\eta\|_p^2}{1 - \mu_p \|\eta\|_p}, \tag{33}$$

and thus (31) holds if

$$\|\phi\|_p < C_p \|\eta\|_p - \nu_p \mu_p \frac{\|\eta\|_p^2}{1 - \mu_p \|\eta\|_p}. \tag{34}$$

If $\eta \in \partial \Omega_{1,\lambda}$, (31) holds if

$$\|\phi\|_p < R_\lambda C_p \left(1 - \frac{1}{\lambda}\right) \equiv r_{p,\lambda}. \tag{35}$$

Defining $\Omega_{2,\lambda} = \{\phi \in \mathbb{C}^n \,|\, \|\phi\|_p < r_{p,\lambda}\}$, we note the following: for all $(\eta, \phi) \in \Omega_{1,\lambda} \times \Omega_{2,\lambda}$, $F(\eta, 0) \neq 0$; and, for all $(\eta, \phi) \in \partial \Omega_{1,\lambda} \times \Omega_{2,\lambda}$, $\|F(\eta, \phi) - F(\eta, 0)\|_p < \|F(\eta, 0)\|_p$. By Theorem 7, $F(\eta, 0)$ and $F(\eta, \phi)$ have the same number of zeroes counting multiplicity on $\Omega_{1,\lambda} \times \Omega_{2,\lambda}$, namely precisely one. Thus, for all $\phi \in \Omega_{2,\lambda}$ there exists a unique $\eta = \psi(\phi)$ such that $F(\psi(\phi), \phi) = 0$. Since the unique zero must have multiplicity one,

$$\det\left(\{\partial_{\eta_j} F_i(\psi(\phi), \phi)\}_{i,j=1}^n\right) \neq 0. \tag{36}$$

Consequently, by the analytic implicit function theorem [Theorem 3.1.3, 40], ψ is analytic in a neighborhood of each $\phi \in \Omega_{2,\lambda}$, which is sufficient to prove that ψ is analytic on all of $\Omega_{2,\lambda}$. Hence ψ has a Taylor series converging absolutely for all $\phi \in \Omega_{2,\lambda}$. By construction, the terms in this series must be the same as those of the inverse Born series, since they are both power series for the same function. It follows that the inverse Born series must also converge for all $\phi \in \Omega_{2,\lambda}$. Optimizing over $\lambda \geq 1$, the inverse Born series converges for all $\phi \in \mathbb{C}^n$, such that

$$\|\phi\|_p < \frac{C_p}{\mu_p} \left[1 - 2\frac{\nu_p}{C_p} \left(\sqrt{1 + \frac{C_p}{\nu_p}} - 1 \right) \right], \tag{37}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Remark 8. We note that Theorem 6 is closely related to the problem of determining the domain of biholomorphy of a function of several complex variables, where the radii of analyticity of the function and its inverse are referred to as Bloch radii or Bloch constants [27, 26, 18]. In the context of nonlinear optimization a related result was obtained in [17], which also made use of Rouché's theorem.

Remark 9. The bound constructed in Theorem 5 is only a lower bound for the radius of convergence. In practice, the series converges well outside this range, as the example in the next section confirms. Additionally, if in the proof of Theorem 5 we instead define $F(\eta, \phi)$ by

$$F(\eta, \phi) = \mathcal{K}_1 \phi - \sum_{j=1}^{\infty} \mathcal{K}_1 K_j(\eta, \ldots, \eta), \tag{38}$$

then it can easily be shown that the inverse series converges if

$$\|\mathcal{K}_1 \phi\|_p < \tilde{r}_p := \frac{1}{\mu_p} \left[1 - 2\frac{\nu_p}{C_p} \left(\sqrt{1 + \frac{C_p}{\nu_p}} - 1 \right) \right]. \tag{39}$$

Though the right-hand side is slightly more complicated, it is often easily computed and gives a better bound.

Figure 1 shows a plot of the bound on the radius of convergence,

$$r_p = \frac{C_p}{\mu_p} \left[1 - 2\frac{\nu_p}{C_p} \left(\sqrt{1 + \frac{C_p}{\nu_p}} - 1 \right) \right]$$

for various values of C_p/ν_p. For large graphs we expect the determinant of K_1 to be small, corresponding to a small value of C_p. In this regime we observe that the first term in the asymptotic expansion of (37) is

$$r_p = \frac{C_p^2}{4\nu_p\mu_p} + \mathrm{O}\left(C_p^3\right). \tag{40}$$

We now consider the stability of the limit of the inverse scattering series under perturbations in the scattering data. The following stability estimate follows immediately from Theorem 5.
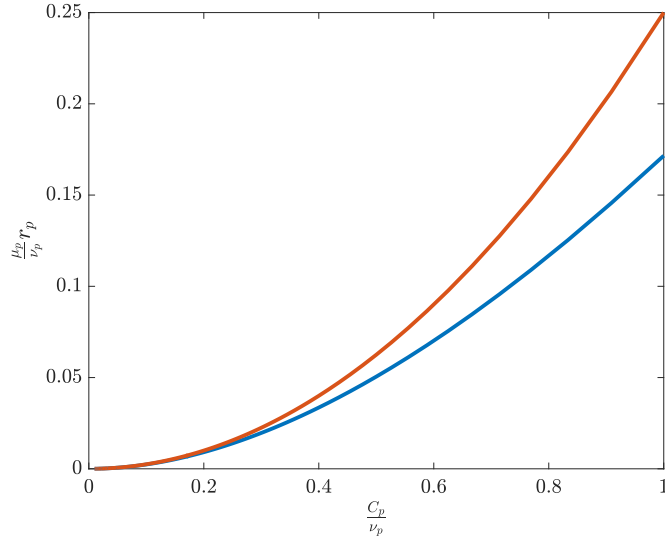
FIGURE 1. The bound on the radius of convergence of the inverse Born series as a function of C_p/ν_p. The radius r_p (multiplied by μ_p/ν_p) is shown in blue. The red curve is the asymptotic estimate given in (40).

Proposition 10. Let E be a compact subset of $\Omega_p = \{\phi \in \mathbb{C}^n \,|\, \|\phi\|_p < r_p\}$, where r_p is defined in (24) and $p \in [1, \infty]$. Let ϕ_1 and ϕ_2 be scattering data belonging to E and ψ_1 and ψ_2 denote the corresponding limits of the inverse Born series. Then the following stability estimate holds:

$$\|\psi_1 - \psi_2\|_p \leq M \|\phi_1 - \phi_2\|_p \ ,$$

where $M = M(E, p)$ is a constant which is otherwise independent of ϕ_1 and ϕ_2.

Proof. In the proof of Theorem 5 it was shown that ψ is analytic on Ω_p. In particular, it follows that there exists an $M < \infty$ such that

$$\|D\psi\|_p \leq M, \tag{41}$$

for all $\phi \in \Omega$. Here $D\psi$ is the differential of ψ and $\| \cdot \|_p$ is its induced matrix p-norm. By the mean value theorem,

$$\|\psi_1 - \psi_2\|_p \leq M \|\phi_1 - \phi_2\|_p, \tag{42}$$

for all $\phi_1, \phi_2 \in \Omega$. $\qquad\square$

Theorem 5 guarantees convergence of the inverse Born series, but does not provide an estimate of the approximation error. Such an estimate is provided in the next theorem.

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Theorem 11. Suppose that the hypotheses of Theorem 5 hold and $\|\phi\|_p < \tau r_p$, where $\tau < 1$. If η is the true vertex potential corresponding to the scattering data ϕ, then

$$\left\| \eta - \sum_{m=1}^{N} \mathcal{K}_m(\phi, \ldots, \phi) \right\|_\infty < M \left(\frac{1}{1-\tau} \right)^n \left(\frac{\|\phi\|_p}{\tau r_p} \right)^N \frac{1}{1 - \frac{\|\phi\|_p}{\tau r_p}}.$$

Proof. The proof follows a similar argument to one used to show uniform convergence of analytic functions on polydiscs, see [Lemma 1.5.8 and Corollary 1.5.9, 40] for example. By Theorem 5, since $\|\phi\|_p < r_p$, the inverse Born series converges. Moreover, the value to which it converges is precisely the unique potential η corresponding to the scattering data ϕ.

Let ψ_j be the jth component of the sum of the inverse Born series, which is of the form

$$\psi_j = \sum_{|\alpha|=0}^{\infty} c_\alpha^{(j)} \phi^\alpha, \tag{43}$$

for suitable $c_\alpha^{(j)}$, consistent with (22). Here we have used the following notational convention: if $\alpha = (\alpha_1, \ldots, \alpha_n)$ then $\phi^\alpha \equiv \phi_1^{\alpha_1} \ldots \phi_n^{\alpha_n}$. Additionally, for a given multi-index α we define $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Note that each α in the sum has exactly n elements, though any number of them may be zero.

Let

$$\psi_j^{(N)} = \sum_{|\alpha|=0}^{N} c_\alpha^{(j)} \phi^\alpha,$$

and Δ_ϕ be the polydisc

$$\Delta_\phi = \left\{ z \in \mathbb{C}^n \,|\, |z_s| < |\phi_s| \frac{r_p}{\|\phi\|_p}, \ s = 1, \ldots, n \right\}.$$

We note that $\phi \in \Delta_\phi \subseteq \{ \phi \,|\, \|\phi\|_p < r_p \}$. It follows by Cauchy's estimate [Theorem 1.3.3, 40] that

$$|c_\alpha^{(j)}| \leq M \left(\frac{\|\phi\|_p}{r_p} \right)^{|\alpha|} \frac{1}{|\phi|^\alpha}, \tag{44}$$

where $M = \max_{\|\phi\|_p < r_p} \|\psi\|_p$. To proceed, we employ the following combinatorial identity, [Example 1.5.7, 40],

$$\sum_{|\alpha|=0}^{\infty} t^{|\alpha|} = \frac{1}{(1-t)^n}, \tag{45}$$

for all $t \in (-1, 1)$. In light of the above, we see that

$$M \sum_{|\alpha|=0}^{\infty} \left(\frac{\|\phi\|_p}{r_p} \right)^{|\alpha|} = M \left(\sum_{m=1}^{\infty} \left(\frac{\|\phi\|_p}{r_p} \right)^m \right)^n = M \left(\frac{1}{1 - \frac{\|\phi\|_p}{r_p}} \right)^n.$$

The function $1/(1-t)^n$ is bounded by

$$M \left(\frac{1}{1-\tau} \right)^n$$

for all $|t| < \tau < 1$. Thus the one-dimensional Cauchy estimate implies that the kth coefficient of its Taylor series, b_k, is bounded by

$$|b_k| \leq M \left(\frac{1}{1-\tau} \right)^n \frac{1}{\tau^k},$$

and so

$$\sum_{|\alpha|>N} \left(\frac{\|\phi\|_p}{r_p} \right)^{|\alpha|} \leq M \sum_{k>N}^{\infty} \left(\frac{1}{1-\tau} \right)^n \left(\frac{\|\phi\|_p}{\tau r_p} \right)^k,$$
$$= M \left(\frac{1}{1-\tau} \right)^n \left(\frac{\|\phi\|_p}{\tau r_p} \right)^N \frac{1}{1 - \frac{\|\phi\|_p}{\tau r_p}}. \tag{46}$$

Hence, independent of j,

$$\left| \psi_j - \psi_j^{(N)} \right| = \left| \sum_{|\alpha|>N} c_\alpha^{(j)} \phi^\alpha \right|,$$
$$\leq \sum_{|\alpha|>N} |c_\alpha^{(j)}| |\phi|^\alpha,$$
$$\leq M \sum_{|\alpha|>N} \left(\frac{\|\phi\|_p}{r_p} \right)^{|\alpha|}, \tag{47}$$
$$\leq M \left(\frac{1}{1-\tau} \right)^n \left(\frac{\|\phi\|_p}{\tau r_p} \right)^N \frac{1}{1 - \frac{\|\phi\|_p}{\tau r_p}},$$

from which the result follows immediately. □

Remark 12. Note that in the previous theorem we can minimize our bound over $\tau \in (\|\phi\|_p/r_p, 1)$. Letting $\gamma = \|\phi\|_p/r_p$ the minimum occurs at

$$\tau_* = \frac{\gamma}{2} \left[\left(1 + \frac{N-\gamma}{\gamma(n+N)} \right) + \sqrt{\left(1 - \frac{N-\gamma}{\gamma(N+n)} \right)^2 + 4 \frac{1-\gamma}{\gamma(N+n)}} \right].$$

Finally, we conclude our discussion of the convergence of the inverse Born series by proving an asymptotic estimate for the truncation error. Specifically, we show that for a fixed number of terms N the error in the N-term inverse Born series goes to zero as η goes to zero. We note that our estimate does not apply to the case of fixed ϕ and $N \to \infty$ since $C_{N,a} x^N \to \infty$ for any fixed positive x.

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Theorem 13. Let $\|\eta\|_p \mu_p < a < 1$. Then there exists a constant $C_{N,a}$, depending on N such that

$$\left\| \eta - \sum_{j=1}^{N} \mathcal{K}_j(\phi, \cdots, \phi) \right\|_p \le C_{N,a} \|\eta\|_p^{N+1}. \tag{48}$$

Proof. We begin by considering the truncated inverse Born series,

$$\eta_N(\phi) = \sum_{j=1}^{N} \mathcal{K}_j(\phi, \cdots, \phi). \tag{49}$$

If $\mu_p \|\eta\|_p < 1$, ϕ is equal to its forward Born series, and hence

$$\eta_N - \eta = \sum_{j=1}^{N} \sum_{i_1, \cdots, i_j=1}^{\infty} \mathcal{K}_j[K_{i_1}(\eta, \ldots, \eta), \cdots, K_{i_N}(\eta, \ldots, \eta)] - \eta. \tag{50}$$

Using (23) we find that

$$\eta_N - \eta = \sum_{j=1}^{n} \sum_{i_1 + \cdots + i_j > N}^{\infty} \mathcal{K}_j[K_{i_1}(\eta, \ldots, \eta), \cdots, K_{i_N}(\eta, \ldots, \eta)], \tag{51}$$

which follows from the construction of the inverse Born series. Therefore

$$\begin{aligned}
\|\eta_N - \eta\|_p &\le \sum_{j=1}^{N} \|\mathcal{K}_j\|_p \, \nu_p^j \sum_{k>N}^{\infty} \mu_p^{k-j} \|\eta\|_p^k, \\
&\le \sum_{j=1}^{N} \|\mathcal{K}_j\|_p \left(\frac{\nu_p}{\mu_p} \right)^j \sum_{k>N}^{\infty} \mu_p^k \|\eta\|_p^k, \\
&\le \sum_{j=1}^{N} \|\mathcal{K}_j\|_p \left(\frac{\nu_p}{\mu_p} \right)^j \|\eta\|_p^{N+1} \mu_p^{N+1} \sum_{k=0}^{\infty} (\mu_p \|\eta\|_p)^k, \\
&= \sum_{j=1}^{N} \|\mathcal{K}_j\|_p \, \nu_p^j \mu_p^{N+1-j} \|\eta\|_p^{N+1} \frac{1}{1 - \mu_p \|\eta\|_p}.
\end{aligned} \tag{52}$$

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In order to proceed, we require a bound on $\|\mathcal{K}_j\|_p$. As in [36], we begin by observing that if $p \in [1, \infty]$, $j > 2$,

$$
\begin{aligned}
\|\mathcal{K}_j\|_p &\leq \|\mathcal{K}_1\|_p^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p \sum_{i_1 + \cdots + i_m = j} \|K_{i_1}\|_p \cdots \|K_{i_m}\|_p, \\
&\leq \|\mathcal{K}_1\|_p^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p \sum_{i_1 + \cdots + i_m = j} \left(\frac{\nu_p}{\mu_p}\right)^m \mu_p^j, \\
&= \|\mathcal{K}_1\|_p^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p \binom{j-1}{m-1} \left(\frac{\nu_p}{\mu_p}\right)^m \mu_p^j, \\
&\leq \nu_p \mu_p^{j-1} \|\mathcal{K}_1\|_p^j \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p\right) \sum_{m=0}^{j-2} \binom{j-1}{m} \left(\frac{\nu_p}{\mu_p}\right)^m,
\end{aligned}
\tag{53}
$$

where we have shifted the index m in the last expression. It follows immediately from the binomial theorem that

$$
\begin{aligned}
\|\mathcal{K}_j\|_p &\leq \|\mathcal{K}_1\|_p^j \nu_p \left[(\mu_p + \nu_p)^{j-1} - \nu_p^{j-1}\right] \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p\right), \\
&\leq [\|\mathcal{K}_1\|_p (\mu_p + \nu_p)]^j \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p\right), \\
&\leq \|\mathcal{K}_1\|_p (\mu_p + \nu_p) \|\mathcal{K}_{j-1}\|_p + \frac{\nu_p}{\mu_p + \nu_p} [\|\mathcal{K}_1\|_p (\mu_p + \nu_p)]^j \|\mathcal{K}_{j-1}\|_p, \\
&\leq \|\mathcal{K}_1\|_p (\mu_p + \nu_p) \left[1 + (\mu_p + \nu_p)^{j-1} \|\mathcal{K}_1\|_p^{j-1}\right] \|\mathcal{K}_{j-1}\|_p.
\end{aligned}
\tag{54}
$$

Further note that if $j = 2$, then

$$
\|\mathcal{K}_2\|_p \leq \|\mathcal{K}_1\|_p^3 \nu_p^2 \leq \|\mathcal{K}_1\|_p^3 (\mu_p + \nu_p)^2.
\tag{55}
$$

For ease of notation, let $r = (\mu_p + \nu_p) \|\mathcal{K}_1\|_p$ and note that

$$
\begin{aligned}
\|\mathcal{K}_j\|_p &\leq \|\mathcal{K}_1\|_p (\mu_p + \nu_p) \left[1 + (\mu_p + \nu_p)^{j-1} \|\mathcal{K}_1\|_p^{j-1}\right] \|\mathcal{K}_{j-1}\|_p, \\
&\leq r \left[1 + r^{j-1}\right] \|\mathcal{K}_{j-1}\|_p, \\
&\leq \|\mathcal{K}_1\|_p r^j 2^j \left[\max\{1, r\}\right]^{\frac{j(j-1)}{2}}.
\end{aligned}
\tag{56}
$$

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If we define $C = \max\{1, r\}$, then it follows from (52) and (56)

$$\|\eta_N - \eta\|_p \leq \frac{\|\eta\|_p^{N+1}\mu_p^{N+1}\|\mathcal{K}_1\|_p}{1 - \mu_p\|\eta\|_p} \sum_{j=1}^{N} \left(\frac{2\nu_p r}{\mu_p}\right)^j C^{\frac{j^2}{2}},$$

$$\leq \frac{\|\eta\|_p^{N+1}\mu_p^{N+1}\|\mathcal{K}_1\|_p}{1 - \mu_p\|\eta\|_p} \frac{1 - \left(2\nu_p\mu_p^{-1}r\right)^{N+1}}{1 - 2\nu_p\mu_p^{-1}r} C^{\frac{N^2}{2}}, \tag{57}$$

$$\leq \tilde{C}(N)\frac{\|\eta\|_p^{N+1}}{1 - \mu_p\|\eta\|_p}.$$

Thus, for $\|\eta\|_p < \mu_p^{-1}a < \mu_p^{-1}$,

$$\|\eta_N - \eta\|_p \leq C'(N)\|\eta\|_p^{N+1} \tag{58}$$

for some constant $C'(N)$. □

3. Implementation

3.1. **Regularizing** $\mathcal{K}_1$. In the previous section we found that the norm of $\mathcal{K}_1$ plays an essential role in controlling the convergence of the inverse Born series. In practice, for large graphs $\|\mathcal{K}_1\|_p$ is too large to guarantee convergence of the inverse series. Moreover, even if the series converges, a modest amount of noise can lead to large changes in the recovered potential. Regularization improves the stability and radius of convergence of the inverse Born series by employing a regularized pseudoinverse, K_1^+ in place of the true inverse K_1^{-1} in the definition of $\mathcal{K}_1$. In our numerical studies we compute $\mathcal{K}_1$ using a Tikhonov-regularized singular value decomposition of K_1 [38]. In the following we denote the regularization parameter by ϵ, noting that when $\epsilon = 0$ no regularization has been performed.

3.2. **Numerical examples.** In the following we present numerical reconstructions for a 12×12 lattice with boundary vertices connected to the outermost layer of vertices, as illustrated in Figures 2-6. Note that each outgoing edge connects to a boundary vertex. The scattering data is obtained by solving the forward problem by applying a direct solver to to the linear system (1). As is often the case in biomedical applications, we consider a homogeneous medium with a small number of large inclusions.

Figures 2–6 show typical results of the inverse Born series reconstruction. Note that due to the rapid increase in the number of terms at each order in the inverse Born series, it is seldom practical to proceed beyond the first few terms of the series. As such, in each of our experiments it is not possible to say whether the series converges, since we are well beyond the radius of convergence guaranteed by Theorem 5. Instead, we consider the behavior of the first five terms. If the sum grows exponentially in the order of truncation then we say that the series diverges.

In Figure 3 the potential η is scaled by a factor of 10 compared to Figure 2. The inverse Born series diverges for the larger potential, and regularization is necessary to ensure convergence. Though this regularization improves the rate of convergence of the inverse

Born series, it no longer converges to the true potential. We note, however, that there is still good qualitative recovery of the potential. Moreover, the method of regularization we have used here, Tikhonov regularization, has a smoothing effect on the recovered potential in the continuous setting. The same effect is evident in the transition from Figure 3 to Figure 4 where the regularization parameter, ϵ, has been increased from 10^{-7} to 10^{-5}. Figure 5 shows the effect of changing the boundary condition parameter t. In particular, decreasing t appears to shrink the radius of convergence, necessitating a larger regularization parameter. Finally, in Figure 6 we see the effect of partial boundary data. Note that a larger regularization parameter is required since the forward operator K_1 is more ill-conditioned.

FIGURE 2. a) True potential b) first term of the inverse Born series, c) first two terms of the inverse Born series, d) first five terms of the inverse Born series. Here $\alpha_0 = 0.1$, $t = 1$, $\epsilon = 0$, and every boundary vertex is both a source and a receiver. $\mu_2 = 0.0874$, $\nu_2 = 0.4702$, and $\tilde{r}_2 = 3.7 \times 10^{-7}$.

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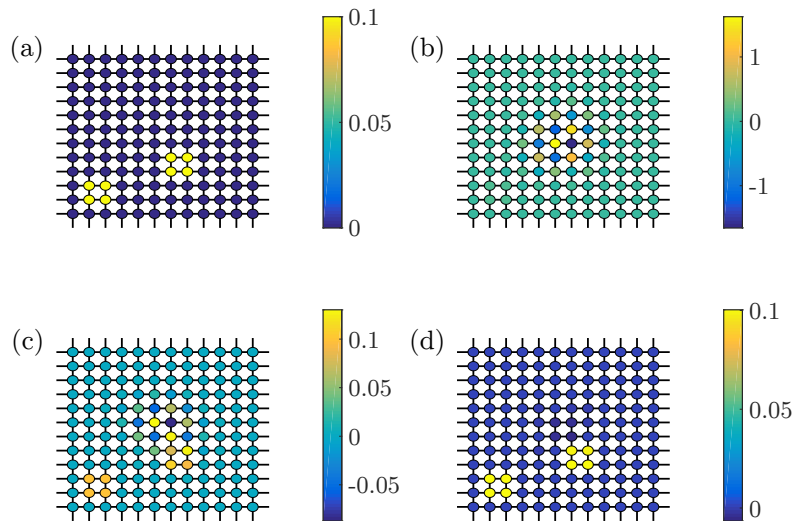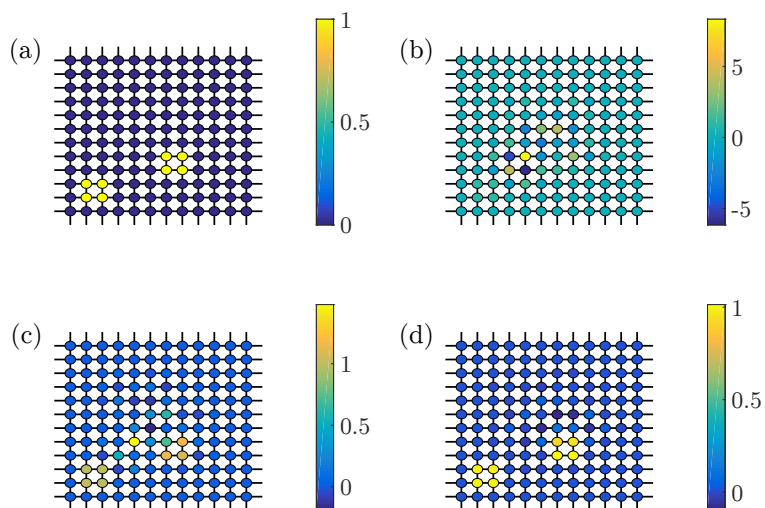

FIGURE 3. a) True potential b) first term of the inverse Born series, c) first two terms of the inverse Born series, d) first five terms of the inverse Born series. Here $\alpha_0 = 0.1$, $t = 1$, $\epsilon = 10^{-7}$, and every boundary vertex is both a source and a receiver. $\mu_2 = 0.0874$, $\nu_2 = 0.4702$, and $\tilde{r}_2 = 1.215 \times 10^{-6}$. Note that without the regularization the inverse Born series diverges.

FIGURE 4. a) True potential b) first term of the inverse Born series, c) first two terms of the inverse Born series, d) first five terms of the inverse Born series. Here $\alpha_0 = 0.1$, $t = 1$, $\epsilon = 10^{-5}$, and every boundary vertex is both a source and a receiver. $\mu_2 = 0.0874$, $\nu_2 = 0.4702$, and $\tilde{r}_2 = 1.224 \times 10^{-4}$.

FIGURE 5. a) True potential b) first term of the inverse Born series, c) first two terms of the inverse Born series, d) first five terms of the inverse Born series. Here $\alpha_0 = 0.1$, $t = 0$, $\epsilon = 10^{-5}$, and every boundary vertex is both a source and a receiver. $\mu_2 = 0.1738$, $\nu_2 = 10.7463$, and $\tilde{r}_2 = 2.677 \times 10^{-6}$.
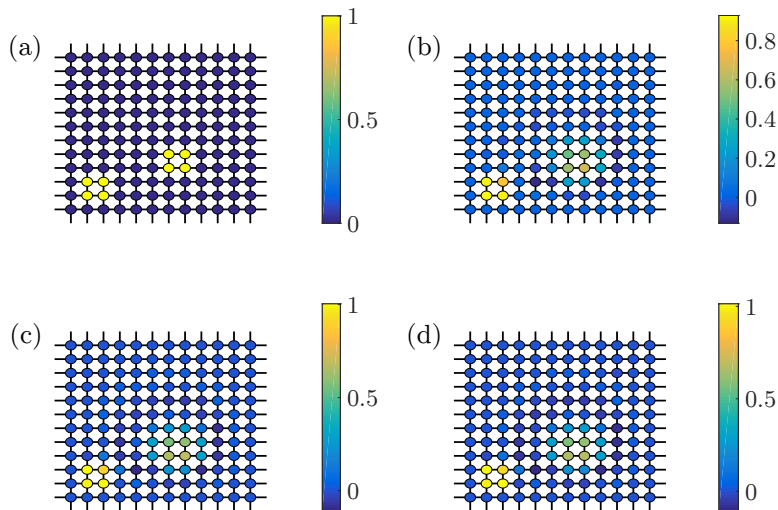

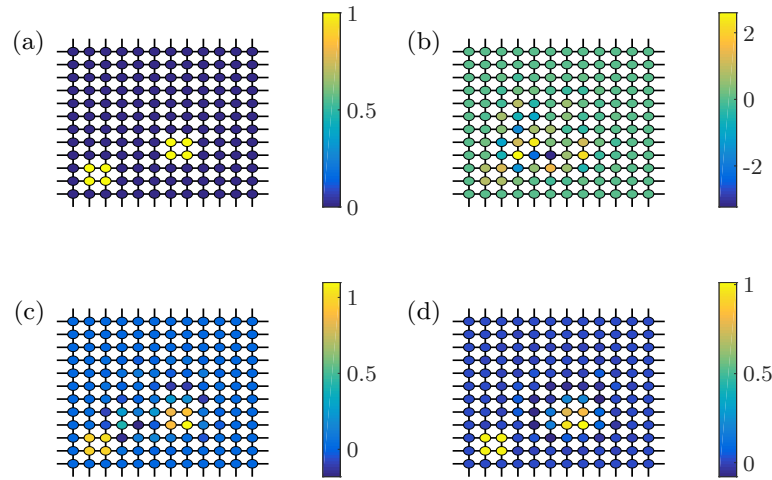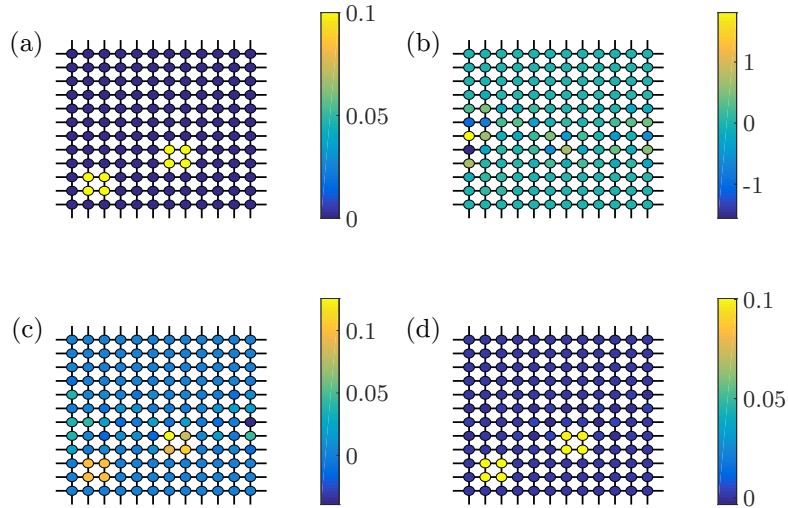FIGURE 6. a) True potential b) first term of the inverse Born series, c) first two terms of the inverse Born series, d) first five terms of the inverse Born series. Here $\alpha_0 = 0.1$, $t = 1$, $\epsilon = 10^{-9}$, and every boundary vertex on the top and bottom edges of the lattice is both a source and a receiver. $\mu_2 = 0.0874$, $\nu_2 = 0.2351$, and $\tilde{r}_2 = 2.656 \times 10^{-7}$.

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4. Incorporating potential structure

The inverse Born series algorithm can be extended to take into account additional constraints on the vertex potential η, such as restrictions on its support or requirements that it is constant on some subset of the domain, allowing the recovery of vertex potentials which would otherwise be unrecoverable using the inverse Born series described above.

Theorem 14. Let F be a linear mapping from $\mathbb{R}^k \to \mathbb{R}^{|V|}$, where $k \leq |V|$ and suppose that η is in the image of F. Let η' be its pre-image,

$$\eta' = F^{-1}(\eta). \tag{59}$$

Then

$$\eta' = \mathcal{K}'_1(\phi) + \mathcal{K}'_2(\phi, \phi) + \cdots + \mathcal{K}'_n(\phi, \ldots, \phi) + \cdots, \tag{60}$$

and

$$\begin{aligned}
\mathcal{K}'_1 &= (K_1 \circ F)^+, \\
\mathcal{K}'_2 &= -\mathcal{K}'_1 \circ K_2 \circ ((F \circ \mathcal{K}_1) \otimes (F \circ \mathcal{K}_1)), \\
\mathcal{K}'_n &= -\sum_{j=1}^{n-1} \mathcal{K}'_j \circ \left(\sum_{i_1 + \cdots + i_j = n} K_{i_1} \otimes K_{i_2} \otimes \cdots \otimes K_{i_j} \right) \circ ((F \circ \mathcal{K}_1) \otimes \cdots \otimes (F \circ \mathcal{K}_1)),
\end{aligned} \tag{61}$$

where $(K_1 \circ F)^+$ denotes the (regularized) pseudoinverse of $(K_1 \circ F)$.

Proof. We begin by rewriting the discrete time-independent diffusion equation as

$$\begin{aligned}
Lu + \alpha_0 [I + D_{F(\eta')}] u - A_{V, \delta V}^T v &= \mathbf{0}, \\
-A_{V, \delta V} u + Dv &= \mathbf{g},
\end{aligned} \tag{62}$$

where $D_{F(\eta')}$ is the diagonal matrix whose diagonal elements are given by the vector $F(\eta')$. If $\Lambda'_{\eta'} : \ell^2(\mathbb{R}^k) \to \ell^2(R \times S)$ denotes the Robin-to-Dirichlet map for the modified system (62) and η is in the image of F, then

$$\Lambda'_{\eta'} = \Lambda_\eta. \tag{63}$$

Thus the forward Born series of (62) is given by

$$\Lambda'_{\eta'}(s, r) = G_0(r, s) - \sum_{n=1}^{\infty} K_n \left(F(\eta'), \ldots, F(\eta') \right). \tag{64}$$

Following the construction of the inverse Born series, we let ϕ represent the measured data, and consider the ansatz

$$\eta' = \mathcal{K}'_1(\phi) + \mathcal{K}'_2(\phi, \phi) + \cdots + \mathcal{K}'_n(\phi, \ldots, \phi) + \ldots \tag{65}$$

We see immediately that

$$\mathcal{K}_1' \circ K_1 \circ F = I,$$
$$\mathcal{K}_1' \circ K_2 \circ (F \otimes F) + \mathcal{K}_2' \circ ((K_1 \circ F) \otimes (K_1 \circ F)) = 0,$$
$$\cdots \tag{66}$$
$$\sum_{j=1}^{n} \mathcal{K}_j' \circ \left(\sum_{i_1 + \ldots i_j = n} K_{i_1} \otimes K_{i_2} \otimes \cdots \otimes K_{i_j} \right) \circ (F \otimes \cdots \otimes F) = 0.$$

If $(K_1 \circ F)^+$ denotes the (regularized) pseudoinverse of $(K_1 \circ F)$, then we obtain

$$\mathcal{K}_1' = (K_1 \circ F)^+,$$
$$\mathcal{K}_2' = -\mathcal{K}_1' \circ K_2 \circ ((F \circ \mathcal{K}_1) \otimes (F \circ \mathcal{K}_1)),$$
$$\mathcal{K}_n' = -\sum_{j=1}^{n-1} \mathcal{K}_j' \circ \left(\sum_{i_1 + \cdots + i_j = n} K_{i_1} \otimes K_{i_2} \otimes \cdots \otimes K_{i_j} \right) \circ ((F \circ \mathcal{K}_1) \otimes \cdots \otimes (F \circ \mathcal{K}_1)). \tag{67}$$

$\square$

We observe that bounds on the radius of convergence, truncation error, and stability of the modified inverse Born series can be easily obtained using arguments similar to those made in Section 2. Theorem 14 can easily be applied to incorporate measurements from multiple values of α_0, provided the vertex potential η is independent of the value of α_0. In optical tomography, this corresponds to varying the optical wave wavelength so that the absorption coefficients of the background medium and the inhomogeneities to be imaged have the same wavelength dependence.

In particular, let $\Gamma = (E, V)$ be a graph and suppose we have measurements for $\alpha_0 = (\alpha_i)_1^m$. Let $\Gamma' = \{\Gamma_1, \ldots, \Gamma_m\}$ be the graph with vertices $V' = \{V_1, \ldots, V_m\}$ and edges $E' = \{E_1, \ldots, E_m\}$, consisting of m copies of Γ. Here the subscript denotes the copy of E, V, or Γ to which we are referring. Let $\pi : V' \to V$ denote the projection map taking a vertex in V_i or δV_i to the corresponding vertex in V or δV, respectively. Finally, for a given vertex potential, η, on Γ let η' denote the corresponding potential on Γ'. Thus, for each vertex $v \in V'$,

$$\eta'(v) = \eta(\pi(v)). \tag{68}$$

Next we construct the following modified time-independent diffusion equation

$$L_i u_i + \alpha_i [I + D_{\eta'}] u_i - (A_i)_{V, \delta V}^T v_i = \mathbf{0},$$
$$-(A_i)_{V, \delta V} u_i + D v_i = \mathbf{g_i}, \tag{69}$$

where u_i and v_i, $i = 1, \ldots, m$, are supported on V_i and δV_i, respectively, and L_i is the Laplacian corresponding to the ith subgraph. As before $D_{\eta'}$ denotes the diagonal matrix with entries given by η'.

Note that Γ' consists of m disconnected components, and hence the solution in one component is independent of the solution in another. If $W, U \subset V_i \times \delta V_i$ let $G_i^{W;U}$ denote

the submatrix of G_i consisting of the rows indexed by W and the columns indexed by U. It follows that the background Green's function for (69) is given by

$$
G_0 = \left(
\begin{array}{cccc:cccc}
G_1^{V_1;V_1} & & & & G_1^{V_1;\delta V_1} & & & \\
 & G_2^{V_2;V_2} & & & & G_2^{V_2;\delta V_2} & & \\
 & & \ddots & & & & \ddots & \\
 & & & G_m^{V_m;V_m} & & & & G_m^{V_m;\delta V_m} \\ \hdashline
G_1^{\delta V_1;V_1} & & & & G_1^{\delta V_1;\delta V_1} & & & \\
 & G_2^{\delta V_2;V_2} & & & & G_2^{\delta V_2;\delta V_2} & & \\
 & & \ddots & & & & \ddots & \\
 & & & G_m^{\delta V_m;V_m} & & & & G_m^{\delta V_m;\delta V_m}
\end{array}
\right) \tag{70}
$$

Thus, if $\mathbf{u} = (u_1, \ldots, u_m, v_1, \ldots, v_m)^T$ solves (69) when $\eta' \equiv 0,$ and

$$
\mathbf{g} = (0, \ldots, 0, g_1, \ldots, g_m)^T,
$$

then

$$
\mathbf{u} = G_0 \mathbf{g}. \tag{71}
$$

Using this we can define the operators $K_1, \ldots, K_n$ for (69), where we replace G_0 by

$$
G_0' = \left(
\begin{array}{cccc:cccc}
\frac{\alpha_1}{\alpha_0}G_1^{V_1;V_1} & & & & \frac{\alpha_1}{\alpha_0}G_1^{V_1;\delta V_1} & & & \\
 & \frac{\alpha_2}{\alpha_0}G_2^{V_2;V_2} & & & & \frac{\alpha_2}{\alpha_0}G_2^{V_2;\delta V_2} & & \\
 & & \ddots & & & & \ddots & \\
 & & & \frac{\alpha_m}{\alpha_0}G_m^{V_m;V_m} & & & & \frac{\alpha_m}{\alpha_0}G_m^{V_m;\delta V_m} \\ \hdashline
\frac{\alpha_1}{\alpha_0}G_1^{\delta V_1;V_1} & & & & \frac{\alpha_1}{\alpha_0}G_1^{\delta V_1;\delta V_1} & & & \\
 & \frac{\alpha_2}{\alpha_0}G_2^{\delta V_2;V_2} & & & & \frac{\alpha_2}{\alpha_0}G_2^{\delta V_2;\delta V_2} & & \\
 & & \ddots & & & & \ddots & \\
 & & & \frac{\alpha_m}{\alpha_0}G_m^{\delta V_m;V_m} & & & & \frac{\alpha_m}{\alpha_0}G_m^{\delta V_m;\delta V_m}
\end{array}
\right), \tag{72}
$$

to account for the different α value in each component.

We now enforce the condition that η is identical on each copy of Γ, and hence is independent of α. The map $F : \ell^p(V_1) \to \ell^p(V_1 \times \cdots \times V_m)$ in (59) is defined by

$$
F\{\eta\}(v) = \eta(\pi(v)). \tag{73}
$$

Using this we form the modified inverse Born series operators in (67) and thus construct the modified inverse Born series. Provided that $(\mathcal{K}_1 \circ F)$ is invertible and the measured data ϕ is sufficiently small, by Theorem 5 the inverse Born series converges to the true (unique) value of η. Since η is the α-independent absorption of the vertices in Γ, we have constructed a reconstruction algorithm using data from multiple α_0. To illustrate this algorithm we consider a path of length 10, noting that it cannot be imaged using the standard inverse Born series, that is with one value of α_0. Observe that, more generally, any graph containing a path of length greater than six in its interior, connected to the remainder of the graph only at its endpoints, the corresponding K_1 is not invertible. In fact, it can be shown that for such graphs that the absorption η cannot be uniquely determined from the data ϕ.

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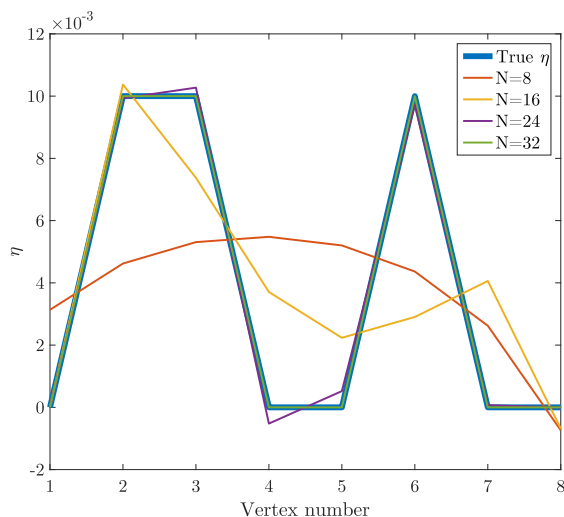OPTICAL TOMOGRAPHY ON GRAPHS   21

FIGURE 7. Reconstructions using the multi-frequency inverse Born series for $\alpha_i = 0.1 \left(1 + 4\frac{i}{N-1}\right)$, $i = 0, \ldots, N-1$, $t = 0.01$, and $\epsilon = 10^{-10}$.

To illustrate the effect of the number of α_i on recovery, we choose one boundary vertex to act both as source and receiver and take $\alpha_i = 0.1 \left(1 + 4\frac{i}{N-1}\right)$, $i = 0, \ldots, N-1$, for $N = 8, 16, 24$, and 32; see Figure 7. Here η is chosen to be a function supported on the interior vertices $2, 3$ and 6, with a height of 0.01. In each case the sum of the first 6 terms of the inverse Born series is taken with the Tikhonov regularization parameter $\epsilon = 10^{-10}$). The effect of regularization on the recovery of the potential is similar to that obtained in the results presented in Section 3.

5. ACKNOWLEDGEMENTS

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