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Removable singularities in C*-algebras of real rank zero

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Abstract
Let $\mathfrak{A}$ be a C*-algebra with identity and real rank zero. Suppose a complex-valued function is holomorphic and bounded on the intersection of the open unit ball of $\mathfrak{A}$ and the identity component of the set of invertible elements of $\mathfrak{A}$. We give a short transparent proof that the function has a holomorphic extension to the entire open unit ball of $\mathfrak{A}$. The author previously deduced this from a more general fact about Banach algebras.

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1. Preliminary definitions and theorems.

Recall [1] that a C*-algebra is a closed complex subalgebra $\mathfrak{A}$ of the Banach algebra $B(H)$ of all bounded linear operators on a Hilbert space with the operator norm such that $\mathfrak{A}$ contains the adjoints of each of its elements. All our C*-algebras contain the identity operator $I$.

To give a basic example, let $S$ be a compact Hausdorff space and let $C(S)$ be the algebra of all continuous complex-valued functions on $S$ with the sup norm. Then there exist a Hilbert space $H$, a C*-algebra $\mathfrak{A}$ in $B(H)$ and an isomorphism $\rho : C(S) \rightarrow \mathfrak{A}$ that preserves norms and adjoints. To see this, let $H$ be the Hilbert space having the same dimension as the cardinality of $S$ and let $\{e_s : s \in S\}$ be an orthonormal basis for $H$. Then we may take $\rho(f)$ to be the multiplication operator defined by $\rho(f)(e_s) = f(s)e_s$ for all $s \in S$ and $f \in C(S)$.

More generally, one can define a Banach algebra that is an abstraction of a C*-algebra and show that an isomorphism like the above exists. Specifically, a B*-algebra is a complex Banach algebra $A$ with an involution * such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. Then a norm and adjoint preserving isomorphism $\rho$ of $A$ onto a C*-algebra exists by the Gelfand-Naimark theorem [1, p. 209].

* Dedicated to Richard Aron with gratitude
We now turn to some basic facts about complex-valued holomorphic functions defined on a domain \( D \) in a complex Banach space \( X \). We say that a function \( f : D \to \mathbb{C} \) is holomorphic if for each \( x \in D \) there exists a continuous complex-linear functional \( \ell \in X^* \) such that

\[
\lim_{y \to 0} \frac{f(x + y) - f(x) - \ell(y)}{\|y\|} = 0.
\]

Clearly, if \( f \) is holomorphic in \( D \) then the function \( \phi(\lambda) = f(x + \lambda y) \) is holomorphic (in the usual sense) in a neighborhood of the origin for each \( x \in D \) and \( y \in X \). It is well known [7, Theorem 3.17.1] that this property also implies holomorphy when \( f \) is locally bounded in \( D \). One can extend many classical results about holomorphic functions by applying the above property. For example, this is true for the following elementary form of the identity theorem [7, Theorem 3.16.4].

**Proposition 1.** Let \( D \) be a domain in a complex Banach space \( X \) and let \( f : D \to \mathbb{C} \) be holomorphic in \( D \). If \( f \) vanishes on a ball in \( D \) then \( f \) vanishes everywhere in \( D \).

By definition, a ball is a set of the form

\[
B_r(x_0) = \{ x \in X : \|x - x_0\| < r \},
\]

where \( x_0 \in X \) and \( r > 0 \).

We will need the following elementary version of Taylor’s theorem, which can be proved as in [7, Theorem 3.17.1], and a simple converse, which can be obtained from the Weierstrass M-test and [7, Theorem 3.18.1].

**Proposition 2.** Let \( X \) be a complex Banach space and let \( x_0 \in X \) and \( r > 0 \). If \( f : B_r(x_0) \to \mathbb{C} \) is a bounded holomorphic function, then for each \( n \) there is a continuous complex-homogeneous polynomial \( P_n : X \to \mathbb{C} \) of degree \( n \) such that

\[
f(x) = \sum_{n=0}^{\infty} P_n(x - x_0) \quad \text{for } x \in B_r(x_0).
\]

(1)

Conversely, if for each \( n \) there is a continuous complex-homogeneous polynomial \( P_n : X \to \mathbb{C} \) of degree \( n \) and if

\[
\|P_n\| \leq \frac{M}{r^n}, \quad n = 0, 1, \ldots
\]

(2)

for some positive constants \( r \) and \( M \), then the function \( f \) given by (1) is holomorphic in \( B_r(x_0) \).

For example, if (1) holds then

\[
P_n(y) = \frac{1}{n!} \frac{d^n}{dt^n} f(x_0 + ty) \bigg|_{t=0}, \quad n = 0, 1, \ldots
\]

(3)
for all $y \in X$. If $f$ is holomorphic on $B_r(x_0)$ and $M$ is a bound for $f$, then (2) is a consequence of the classical Cauchy estimates. As usual,

$$||P_n|| = \sup\{|P_n(x)| : ||x|| \le 1, x \in X\}.$$ .

2. Real rank zero.

**Definition 1.** (See [2] ) Let $\mathfrak{A}$ be a $C^*$-algebra and let $S$ be the set of self-adjoint elements of $\mathfrak{A}$. Then $\mathfrak{A}$ has real rank zero if the elements of $S$ with finite spectra are dense in $S$.

As shown by Brown and Pedersen [2], many interesting $C^*$-algebras have real rank zero. For example, the $C^*$-algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space $H$ has real rank zero. More generally, any von Neumann algebra has real rank zero. The space $C(S)$ of all continuous functions on a compact Hausdorff space $S$ has real rank zero if and only if $S$ is totally disconnected. (It is a von Neumann algebra only if $S$ is extremely disconnected.) Also, any AF-algebra has real rank zero. If $\mathcal{BC}(H)$ is the $C^*$-algebra of all compact operators on $H$, then $\mathbb{C}I + \mathcal{BC}(H)$ has real rank zero as does the Calkin algebra $\mathcal{B}(H)/\mathcal{BC}(H)$. Note that the set of invertible elements of the Calkin algebra has a different component for each value of the Fredholm index and thus is not connected. See [3] for further details and references.

Let $\mathfrak{A}$ be a $C^*$-algebra with identity, let

$$\mathfrak{A}_0 = \{ A \in \mathfrak{A} : ||A|| < 1 \}$$

be the open unit ball of $\mathfrak{A}$ and let $\mathfrak{A}_\text{inv}$ be the identity component of the set of invertible elements of $\mathfrak{A}$. Our main result is the following:

**Theorem 1.** Suppose $\mathfrak{A}$ has real rank zero and let $f$ be a complex-valued function that is holomorphic and bounded on the intersection of the domains $\mathfrak{A}_0$ and $\mathfrak{A}_\text{inv}$. Then $f$ has a holomorphic extension to $\mathfrak{A}_0$.

The author does not know even in the commutative case whether the removable singularity property of Theorem 1 characterizes $C^*$-algebras of real rank zero. However, it is shown in [4] that $C(S)$ does not have this property when $S$ contains the homeomorphic image of an interval.

The proof given below of the previous theorem depends on two important facts about the identity component $\mathcal{U}$ of the set of unitary operators in $\mathfrak{A}$. The first is a maximum principle that is a special case of [6, Theorem 8] and [5, Theorem 9] and the second is a density theorem due to Huaxin Lin [8].

**Proposition 3.** Let $f : \mathfrak{A}_0 \to \mathbb{C}$ be a holomorphic function having a continuous extension to the closed unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$. If $|f(U)| \le 1$ for all $U \in \mathcal{U}$ then $|f(A)| \le 1$ for all $A \in \mathfrak{A}_1$. 

3
Proposition 4. If $\mathfrak{A}$ has real rank zero then the set of unitaries in $U$ with finite spectrum is dense in $U$.

Proof of Theorem 1. Given any $\epsilon$ with $0 < \epsilon < 1/2$, let $r = 1 - \epsilon$. The set $D = B_r(\epsilon I) \cap S^c_{\Inv}$ is open since $S^c_{\Inv}$ is open and one can deduce that $D$ is connected from the fact that $B_r(\epsilon I)$ contains a neighborhood of 0. By Proposition 1, it suffices to show that there exists a function $f_\epsilon$ that is holomorphic in the ball $B_r(\epsilon I)$ and satisfies $f_\epsilon(A) = f(A)$ for all $A \in D$. Since the function $f$ is holomorphic in a ball with center at $x_0 = \epsilon I$, it follows from Proposition 2 that

$$f(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I)$$

for all $A$ in this ball. Thus by the converse part of Proposition 2, it suffices to show that

$$\|P_n\| \leq \frac{M}{r^n}, \quad n = 0, 1, \ldots,$$

where $M$ satisfies $|f| \leq M$ on $A_0 \cap S^c_{\Inv}$, since then the function

$$f_\epsilon(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I)$$

is holomorphic on $B_r(\epsilon I)$ and agrees with $f$ on $D$ by Proposition 1.

Let $B \in \mathfrak{A}$ with $\|B\| \leq 1$ and suppose the spectrum $\sigma(B)$ is finite. Define $\phi(\lambda) = f(\epsilon I + \lambda B)$. If $|\lambda| < r$ then $\epsilon I + \lambda B \in A_0$, $\epsilon I + \lambda B \in S^c_{\Inv}$ and $|\phi(\lambda)| \leq M$ for all but finitely many $\lambda$. By the classical Riemann removable singularity theorem, the function $\phi$ has a holomorphic extension to the disc $|\lambda| < r$ with $|\phi| \leq M$. Hence $|\phi^{(n)}(0)| \leq n!M/r^n$ by the Cauchy estimates so

$$|P_n(B)| \leq \frac{M}{r^n}$$

by (3).

By Proposition 4, inequality (6) holds whenever $B$ is in the identity component of the set of unitary elements of $\mathfrak{A}$ and hence for all $B \in \mathfrak{A}$ with $\|B\| \leq 1$ by Proposition 3. This establishes (5) and completes the proof.

The proof of Theorem 1 given in [4] does not require Proposition 4 but the argument is less straightforward. See [4] for further results, examples and references.

References


