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# Removable Singularities in C*-Algebras of Real Rank Zero

Lawrence A. Harris
*University of Kentucky*, larry@uky.edu

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Lawrence A. Harris

*Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027*

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**Abstract**

Let $\mathfrak{A}$ be a C*-algebra with identity and real rank zero. Suppose a complex-valued function is holomorphic and bounded on the intersection of the open unit ball of $\mathfrak{A}$ and the identity component of the set of invertible elements of $\mathfrak{A}$. We give a short transparent proof that the function has a holomorphic extension to the entire open unit ball of $\mathfrak{A}$. The author previously deduced this from a more general fact about Banach algebras.

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### 1. Preliminary definitions and theorems.

Recall [1] that a C*-algebra is a closed complex subalgebra $\mathfrak{A}$ of the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space with the operator norm such that $\mathfrak{A}$ contains the adjoints of each of its elements. All our C*-algebras contain the identity operator $I$.

To give a basic example, let $S$ be a compact Hausdorff space and let $C(S)$ be the algebra of all continuous complex-valued functions on $S$ with the sup norm. Then there exist a Hilbert space $H$, a C*-algebra $\mathfrak{A}$ in $\mathcal{B}(H)$ and an isomorphism $\rho : C(S) \to \mathfrak{A}$ that preserves norms and adjoints. To see this, let $H$ be the Hilbert space having the same dimension as the cardinality of $S$ and let $\{e_s : s \in S\}$ be an orthonormal basis for $H$. Then we may take $\rho(f)$ to be the multiplication operator defined by $\rho(f)(e_s) = f(s)e_s$ for all $s \in S$ and $f \in C(S)$.

More generally, one can define a Banach algebra that is an abstraction of a C*-algebra and show that an isomorphism like the above exists. Specifically, a B*-algebra is a complex Banach algebra $A$ with an involution * such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. Then a norm and adjoint preserving isomorphism $\rho$ of $A$ onto a C*-algebra exists by the Gelfand-Naimark theorem [1, p. 209].

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☆Dedicated to Richard Aron with gratitude

We now turn to some basic facts about complex-valued holomorphic functions defined on a domain $D$ in a complex Banach space $X$. We say that a function $f : D \to \mathbb{C}$ is holomorphic if for each $x \in D$ there exists a continuous complex-linear functional $\ell \in X^*$ such that

$$\lim_{y \to 0} \frac{f(x+y) - f(x) - \ell(y)}{\|y\|} = 0.$$

Clearly, if $f$ is holomorphic in $D$ then the function $\phi(\lambda) = f(x + \lambda y)$ is holomorphic (in the usual sense) in a neighborhood of the origin for each $x \in D$ and $y \in X$. It is well known [7, Theorem 3.17.1] that this property also implies holomorphy when $f$ is locally bounded in $D$. One can extend many classical results about holomorphic functions by applying the above property. For example, this is true for the following elementary form of the identity theorem [7, Theorem 3.16.4].

**Proposition 1.** *Let $D$ be a domain in a complex Banach space $X$ and let $f : D \to \mathbb{C}$ be holomorphic in $D$. If $f$ vanishes on a ball in $D$ then $f$ vanishes everywhere in $D$.*

By definition, a ball is a set of the form

$$B_r(x_0) = \{x \in X : \ \|x - x_0\| < r\},$$

where $x_0 \in X$ and $r > 0$.

We will need the following elementary version of Taylor's theorem, which can be proved as in [7, Theorem 3.17.1], and a simple converse, which can be obtained from the Weierstrass M-test and [7, Theorem 3.18.1].

**Proposition 2.** *Let $X$ be a complex Banach space and let $x_0 \in X$ and $r > 0$. If $f : B_r(x_0) \to \mathbb{C}$ is a bounded holomorphic function, then for each $n$ there is a continuous complex-homogeneous polynomial $P_n : X \to \mathbb{C}$ of degree $n$ such that*

$$f(x) = \sum_{n=0}^{\infty} P_n(x - x_0) \quad \text{for } x \in B_r(x_0). \tag{1}$$

*Conversely, if for each $n$ there is a continuous complex-homogeneous polynomial $P_n : X \to \mathbb{C}$ of degree $n$ and if*

$$\|P_n\| \le \frac{M}{r^n}, \quad n = 0, 1, \ldots \tag{2}$$

*for some positive constants $r$ and $M$, then the function $f$ given by (1) is holomorphic in $B_r(x_0)$.*

For example, if (1) holds then

$$P_n(y) = \frac{1}{n!} \, \frac{d^n}{dt^n} f(x_0 + ty)\bigg|_{t=0}, \quad n = 0, 1, \ldots \tag{3}$$

for all $y \in X$. If $f$ is holomorphic on $B_r(x_0)$ and $M$ is a bound for $f$, then (2) is a consequence of the classical Cauchy estimates. As usual,

$$\|P_n\| = \sup\{|P_n(x)| : \|x\| \le 1, x \in X\}.$$

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## 2. Real rank zero.

**Definition 1.** *(See [2].) Let $\mathfrak{A}$ be a C*-algebra and let $\mathcal{S}$ be the set of self-adjoint elements of $\mathfrak{A}$. Then $\mathfrak{A}$ has real rank zero if the elements of $\mathcal{S}$ with finite spectra are dense in $\mathcal{S}$.*

As shown by Brown and Pedersen [2], many interesting C*-algebras have real rank zero. For example, the C*-algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space $H$ has real rank zero. More generally, any von Neumann algebra has real rank zero. The space $C(S)$ of all continuous functions on a compact Hausdorff space $S$ has real rank zero if and only if $S$ is totally disconnected. (It is a von Neumann algebra only if $S$ is extremely disconnected.) Also, any AF-algebra has real rank zero. If $\mathcal{BC}(H)$ is the C*-algebra of all compact operators on $H$, then $\mathbb{C}I + \mathcal{BC}(H)$ has real rank zero as does the Calkin algebra $\mathcal{B}(H)/\mathcal{BC}(H)$. Note that the set of invertible elements of the Calkin algebra has a different component for each value of the Fredholm index and thus is not connected. See [3] for further details and references.

Let $\mathfrak{A}$ be a C*-algebra with identity, let

$$\mathfrak{A}_0 = \{A \in \mathfrak{A} : \|A\| < 1\}$$

be the open unit ball of $\mathfrak{A}$ and let $\mathfrak{A}^e_{\mathrm{inv}}$ be the identity component of the set of invertible elements of $\mathfrak{A}$. Our main result is the following:

**Theorem 1.** *Suppose $\mathfrak{A}$ has real rank zero and let $f$ be a complex-valued function that is holomorphic and bounded on the intersection of the domains $\mathfrak{A}_0$ and $\mathfrak{A}^e_{\mathrm{inv}}$. Then $f$ has a holomorphic extension to $\mathfrak{A}_0$.*

The author does not know even in the commutative case whether the removable singularity property of Theorem 1 characterizes C*-algebras of real rank zero. However, it is shown in [4] that $C(S)$ does not have this property when $S$ contains the homeomorphic image of an interval.

The proof given below of the previous theorem depends on two important facts about the identity component $\mathcal{U}$ of the set of unitary operators in $\mathfrak{A}$. The first is a maximum principle that is a special case of [6, Theorem 8] and [5, Theorem 9] and the second is a density theorem due to Huaxin Lin [8].

**Proposition 3.** *Let $f : \mathfrak{A}_0 \to \mathbb{C}$ be a holomorphic function having a continuous extension to the closed unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$. If $|f(U)| \le 1$ for all $U \in \mathcal{U}$ then $|f(A)| \le 1$ for all $A \in \mathfrak{A}_1$.*

**Proposition 4.** *If $\mathfrak{A}$ has real rank zero then the set of unitaries in $\mathcal{U}$ with finite spectrum is dense in $\mathcal{U}$.*

**Proof of Theorem 1.** Given any $\epsilon$ with $0 < \epsilon < 1/2$, let $r = 1-\epsilon$. The set $D = B_r(\epsilon I) \cap \mathfrak{A}^e_{\text{inv}}$ is open since $\mathfrak{A}^e_{\text{inv}}$ is open and one can deduce that $D$ is connected from the fact that $B_r(\epsilon I)$ contains a neighborhood of 0. By Proposition 1, it suffices to show that there exists a function $f_\epsilon$ that is holomorphic in the ball $B_r(\epsilon I)$ and satisfies $f_\epsilon(A) = f(A)$ for all $A \in D$. Since the function $f$ is holomorphic in a ball with center at $x_0 = \epsilon I$, it follows from Proposition 2 that

$$f(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I) \tag{4}$$

for all $A$ in this ball. Thus by the converse part of Proposition 2, it suffices to show that

$$\|P_n\| \le \frac{M}{r^n}, \quad n = 0, 1, \ldots, \tag{5}$$

where $M$ satisfies $|f| \le M$ on $\mathfrak{A}_0 \cap \mathfrak{A}^e_{\text{inv}}$, since then the function

$$f_\epsilon(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I)$$

is holomorphic on $B_r(\epsilon I)$ and agrees with $f$ on $D$ by Proposition 1.

Let $B \in \mathfrak{A}$ with $\|B\| \le 1$ and suppose the spectrum $\sigma(B)$ is finite. Define $\phi(\lambda) = f(\epsilon I + \lambda B)$. If $|\lambda| < r$ then $\epsilon I + \lambda B \in \mathfrak{A}_0$, $\epsilon I + \lambda B \in \mathfrak{A}^e_{\text{inv}}$ and $|\phi(\lambda)| \le M$ for all but finitely many $\lambda$. By the classical Riemann removable singularity theorem, the function $\phi$ has a holomorphic extension to the disc $|\lambda| < r$ with $|\phi| \le M$. Hence $|\phi^{(n)}(0)| \le n!M/r^n$ by the Cauchy estimates so

$$|P_n(B)| \le \frac{M}{r^n} \tag{6}$$

by (3).

By Proposition 4, inequality (6) holds whenever $B$ is in the identity component of the set of unitary elements of $\mathfrak{A}$ and hence for all $B \in \mathfrak{A}$ with $\|B\| \le 1$ by Proposition 3. This establishes (5) and completes the proof.

The proof of Theorem 1 given in [4] does not require Proposition 4 but the argument is less straightforward. See [4] for further results, examples and references.

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