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Zhongwei Shen
University of Kentucky, zshen2@uky.edu

Jinping Zhuge
University of Kentucky, jinping.zhuge@uky.edu

Zhongwei Shen, Jinping Zhuge

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Convergence Rates in Periodic Homogenization of Systems of Elasticity

Zhongwei Shen* Jinping Zhuge†

Abstract

This paper is concerned with homogenization of systems of linear elasticity with rapidly oscillating periodic coefficients. We establish sharp convergence rates in L^2 for the mixed boundary value problems with bounded measurable coefficients.

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1 Introduction and main results

This paper is concerned with convergence rates in periodic homogenization of systems of linear elasticity with mixed boundary conditions. More precisely, we consider the operator

$$\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right\}, \qquad \varepsilon > 0. \tag{1.1}$$

(The summation convention is used throughout this paper). We will assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j, \alpha, \beta \le d$ is real, bounded measurable, and satisfies the elasticity condition,

$$\begin{aligned} a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \\ \kappa_1|\xi + \xi^T|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \le \kappa_2|\xi|^2, \end{aligned} \tag{1.2}$$

for a.e. $y \in \mathbb{R}^d$ and matrix $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}$, where $\kappa_1, \kappa_2 > 0$. We also assume that A satisfies the 1-periodic condition:

$$A(y + z) = A(y) \qquad \text{for a.e. } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.3}$$

We shall be interested in the mixed boundary value problems (or mixed problems) for the elliptic system $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in a bounded Lipschitz domain Ω. Let D be a closed subset of $\partial\Omega$ and $N = \partial\Omega\backslash D$. Denote by $H_D^1(\Omega;\mathbb{R}^d)$ the closure in $H^1(\Omega;\mathbb{R}^d)$ of the set $C_0^\infty(\mathbb{R}^d\backslash D;\mathbb{R}^d)$ and $H_D^{-1}(\Omega;\mathbb{R}^d)$ the dual of $H_D^1(\Omega;\mathbb{R}^d)$. Assume that $F \in H_D^{-1}(\Omega;\mathbb{R}^d)$, $f \in H^1(\Omega;\mathbb{R}^d)$ and $g \in H^{-1/2}(\partial\Omega;\mathbb{R}^d)$ (the dual of $H^{1/2}(\partial\Omega;\mathbb{R}^d)$). We call $u \in H^1(\Omega;\mathbb{R}^d)$ a weak solution of the mixed boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } D, \\ n\cdot A(x/\varepsilon)\nabla u_\varepsilon = g & \text{on } N, \end{cases} \tag{1.4}$$

*Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506, USA.
†Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506, USA.

if $u_\varepsilon - f \in H^1_D(\Omega;\mathbb{R}^d)$ and

$$\int_\Omega A^\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi = \langle F, \varphi\rangle_{H^{-1}_D(\Omega)\times H^1_D(\Omega)} + \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \tag{1.5}$$

holds for any $\varphi \in H^1_D(\Omega;\mathbb{R}^d)$. Here and throughout this paper, we define $h^\varepsilon(x) = h(x/\varepsilon)$ for any function h and use n to denote the outward unit normal to $\partial\Omega$.

The existence and uniqueness of the weak solution to the mixed problem (1.4) follow readily from the Lax-Milgram theorem, with the help of Korn's inequalities. It can also be shown that under the elasticity condition (1.2) and the periodicity condition (1.3), the weak solutions u_ε converge to some function u_0 weakly in $H^1(\Omega;\mathbb{R}^d)$ and thus strongly in $L^2(\Omega;\mathbb{R}^d)$, as $\varepsilon \to 0$. Furthermore, the function u_0 is the weak solution to the mixed problem:

$$\begin{cases} \mathcal{L}_0 u_0 = F & \text{in } \Omega, \\ u_0 = f & \text{on } D, \\ n \cdot \widehat{A}\nabla u_0 = g & \text{on } N, \end{cases} \tag{1.6}$$

where

$$\mathcal{L}_0 = -\text{div}(\widehat{A}\nabla) = -\frac{\partial}{\partial x_i}\left\{\widehat{a}_{ij}^{\alpha\beta}\frac{\partial}{\partial x_j}\right\} \tag{1.7}$$

is a system of linear elasticity with constant matrix $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$, known as the homogenized (or effective) matrix of A.

The primary purpose of this paper is to establish the optimal rate of convergence of u_ε to u_0 in $L^2(\Omega;\mathbb{R}^d)$. More precisely, we are interested in the estimate,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}, \tag{1.8}$$

for the mixed problem (1.4) with nonsmooth coefficients, where C depends at most on d, κ_1, κ_2, Ω, and D. The problem of convergence rates is central in quantitative homogenization and has been studied extensively in various settings. We refer the reader to [1, 7, 10] for references on earlier work in this area. More recent work on the problem of convergence rates in periodic homogenization may be found in [17, 4, 5, 13, 11, 8, 9, 12, 15, 16, 14, 6] and their references. In particular, the estimate (1.8) was proved by Griso in [4, 5] for scalar elliptic equations with either Dirichlet or Neumann boundary conditions, using the method of periodic unfolding [2, 3]. In [15, 16] the results were extended by Suslina to a broader class of elliptic systems in C^2 domains, which includes the systems of elasticity considered in this paper, with either Dirichlet or Neumann boundary conditions. We mention that for systems of elasticity, the results were further extended by the first author in [14], where the estimate $\|u_\varepsilon - u_0\|_{L^p(\Omega)} \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}$, with $p = \frac{2d}{d-1}$, was proved in Lipschitz domains for solutions with either Dirichlet or Neumann boundary conditions. As far as we know, there are no results on the estimate (1.8) for the mixed problems, even for scalar elliptic equations.

The following is our main result.

Theorem 1.1. *Let Ω be a bounded $C^{1,1}$ domain and D a closed subset of $\partial\Omega$ with a nonempty interior. Let u_ε, u_0 be the weak solutions of mixed boundary value problems (1.4) and (1.6), respectively. Assume that $u_0 \in H^2(\Omega;\mathbb{R}^d)$. Then the estimate (1.8) holds with constant C depending at most on d, κ_1, κ_2, D, and Ω.*

Let $\chi = (\chi_j^{\alpha\beta})$ denote the correctors for the operator $\mathcal{L}_\varepsilon$. Let S_ε be a smoothing operator at ε-scale and $\widetilde{u}_0$ an extension of u_0 from $H^2(\Omega;\mathbb{R}^d)$ to $H^2(\mathbb{R}^d;\mathbb{R}^d)$. The key step in the proof of

Theorem 1.1 is the following estimate,

$$
\begin{aligned}
&\left|\int_\Omega A^\varepsilon \nabla\Big(u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\Big)\cdot\nabla\psi\right| \\
&\qquad\le C\Big\{\varepsilon\,\|\nabla\psi\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla\psi\|_{L^2(\Omega_{2\varepsilon})}\Big\}\|u_0\|_{H^2(\Omega)},
\end{aligned}
\tag{1.9}
$$

where $\psi\in H^1_D(\Omega;\mathbb{R}^d)$ and $\Omega_{2\varepsilon}=\{x\in\Omega:\text{dist}(x,\partial\Omega)<2\varepsilon\}$ (see Lemma 3.5). We point out that some analogous estimates were proved in [5] by the method of periodic unfolding, which is not used in this paper. Our approach to (1.9), which involves a standard smoothing operator at the scale ε, is much more direct and flexible and allows us to handle different boundary conditions in a uniform fashion. We also mention that the use of smoothing operators as well as the duality argument in our proof of Theorem 1.1 is motivated by the work [5, 15, 16]. However, in comparison with [15, 16], our proof does not rely on the sharp convergence estimates for the whole space $\mathbb{R}^d$ and thus avoids the estimates of terms that are used to correct the boundary discrepancies. As a result, this significantly simplifies the argument.

As a bi-product, we also obtain an $O(\varepsilon^{1/2})$ estimate in $H^1(\Omega)$ as well as an interior $O(\varepsilon)$ estimate in H^1.

Theorem 1.2. *Under the same conditions as in Theorem 1.1, we have*

$$
\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)}\le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}, \tag{1.10}
$$

where C depends at most on d, κ_1, κ_2, D, and Ω.

Theorem 1.3. *Under the same condition as Theorem 1.1, we have*

$$
\|\delta\nabla(u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0))\|_{L^2(\Omega)}\le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}, \tag{1.11}
$$

where $\delta(x)=\text{dist}(x,\partial\Omega)$ and C depends at most on d, κ_1, κ_2, D, and Ω.

We should point out that unlike the Neumann and Dirichlet problems, solutions to the mixed problems in general are not necessarily in $H^2(\Omega)$, even if the domains and data are smooth. However, any function in $H^2(\Omega)$ is a solution of the mixed problem with the Dirichlet and Neumann data given by the function. We mention that our argument also yields the estimates in Theorems 1.1, 1.2 and 1.3 for the Neumann problem, where $D=\emptyset$, and thus provides a unified approach to the Neumann, Dirichlet, and mixed problems. We further point out that the approach works equally well for the strongly elliptic systems $-\text{div}(A(x/\varepsilon)\nabla u_\varepsilon)=F$, where $A(y)=(a_{ij}^{\alpha\beta}(y))$ with $1\le i,j\le d$ and $1\le\alpha,\beta\le m$ is real, bounded measurable, 1-periodic, and satisfies the ellipticity condition $a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta\ge\mu|\xi|^2$ for a.e. $y\in\mathbb{R}^d$ and $\xi=(\xi_i^\alpha)\in\mathbb{R}^{m\times d}$.

2 Preliminaries

In this section we give a brief review of the solvability and the homogenization theory for the mixed problem (1.4). We begin with a Korn inequality [10, Theorem 2.7].

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and D a closed subset of $\partial\Omega$ with a nonempty interior. Then for any vector field $u\in H^1_D(\Omega;\mathbb{R}^d)$,*

$$
\|u\|_{H^1(\Omega)}\le C\,\|\nabla u+(\nabla u)^T\|_{L^2(\Omega)}, \tag{2.1}
$$

where C depends only on d, D, and Ω.

Theorem 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and D a closed subset of $\partial\Omega$ with a nonempty interior. For $F \in H_D^{-1}(\Omega;\mathbb{R}^d)$, $f \in H^1(\Omega;\mathbb{R}^d)$ and $g \in H^{-1/2}(\partial\Omega;\mathbb{R}^d)$, there exists a unique weak solution $u_\varepsilon \in H^1(\Omega;\mathbb{R}^d)$ to the mixed problem (1.4). Moreover, the solution u_ε satisfies*

$$\|u_\varepsilon\|_{H^1(\Omega)} \le C\Big\{\|F\|_{H_D^{-1}(\Omega)} + \|f\|_{H^1(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}\Big\}, \tag{2.2}$$

where C depends only on d, κ_1, κ_2, Ω, and D.

Proof. By considering the bilinear form

$$\int_\Omega A^\varepsilon \nabla\psi \cdot \nabla\varphi$$

and the bounded linear functional

$$\langle F, \varphi\rangle_{H_D^{-1}(\Omega)\times H_D^1(\Omega)} + \langle g, \varphi\rangle_{H^{-1/2}(\partial\Omega)\times H^{/2}(\partial\Omega)} - \int_\Omega A^\varepsilon \nabla f \cdot \nabla\varphi$$

on $H_D^1(\Omega;\mathbb{R}^d)$, Theorem 2.2 follows readily from the Lax-Milgram theorem, using the elasticity condition (1.2) and the Korn inequality in Lemma 2.1. □

Assume that A satisfies (1.2) and (1.3). Let $\chi = (\chi_j^\beta) = (\chi_j^{\alpha\beta})$ denote the correctors for $\mathcal{L}_\varepsilon$, where $1 \le j \le d$ and $1 \le \alpha, \beta \le d$. This means that $\chi_j^\beta \in H^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ is the 1-periodic function such that $\int_Q \chi_j^\beta = 0$ and

$$\mathcal{L}_1(\chi_j^\beta + P_j^\beta) = 0 \quad \text{in } \mathbb{R}^d, \tag{2.3}$$

where $Q = [-1/2, 1/2]^d$, $P_j^\beta(y) = y_j e^\beta$, and $e^\beta = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^d$ with 1 in the βth position. For the existence of correctors χ, see e.g. [7, 10]. The homogenized operator $\mathcal{L}_0$ is given by (1.7), where the homogenized matrix $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$ is defined by

$$\widehat{A} = \fint_Q A(I + \nabla\chi) \quad \text{or precisely} \quad \widehat{a}_{ij}^{\alpha\beta} = \fint_Q \left\{ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k}(\chi_j^{\gamma\beta}) \right\}. \tag{2.4}$$

It is known that $\widehat{A}$ satisfies the elasticity condition (1.2) (with possible different κ_1, κ_2) [7].

Theorem 2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and D a closed subset of $\partial\Omega$ with a nonempty interior. For $\varepsilon > 0$, let u_ε, u_0 be the weak solutions of the mixed boundary value problems (1.4) and (1.6), respectively, where $F \in H_D^{-1}(\Omega;\mathbb{R}^d)$, $f \in H^1(\Omega;\mathbb{R}^d)$, and $g \in H^{-1/2}(\partial\Omega;\mathbb{R}^d)$. Then*

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \textit{weakly in } H^1(\Omega;\mathbb{R}^d),\\ A^\varepsilon\nabla u_\varepsilon &\rightharpoonup \widehat{A}\nabla u_0 && \textit{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}), \end{aligned} \tag{2.5}$$

as $\varepsilon \to 0$.

Proof. The proof is the same as in the case of the Dirichlet problem [7]. By Theorem 2.2 the solutions u_ε are uniformly bounded in $H^1(\Omega;\mathbb{R}^d)$. Let $\{u_{\varepsilon'}\}$ be a subsequence such that

$$\begin{aligned} u_{\varepsilon'} &\rightharpoonup w \text{ weakly in } H^1(\Omega;\mathbb{R}^d),\\ A^{\varepsilon'}\nabla u_{\varepsilon'} &\rightharpoonup G \text{ weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{aligned}$$

Since $u_\varepsilon - f \in H^1_D(\Omega;\mathbb{R}^d)$, we have $w - f \in H^1_D(\Omega;\mathbb{R}^d)$. Next we will show that $G = \widehat{A}\nabla w$. To this end we consider the identity

$$\int_\Omega A^{\varepsilon'}(x)\nabla u_{\varepsilon'}\cdot\nabla_x\Big(P_j^\beta(x)+\varepsilon'\chi_j^\beta(x/\varepsilon')\Big)\phi = \int_\Omega \nabla u_{\varepsilon'}\cdot A^{\varepsilon'}(x)\nabla_x\Big(P_j^\beta(x)+\varepsilon'\chi_j^\beta(x/\varepsilon')\Big)\phi, \quad (2.6)$$

where $\phi \in C_0^\infty(\Omega)$ and we have used the symmetry condition $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$. By the Div-Curl Lemma (see e.g. [7, p.4]), the LHS of (2.6) converges to

$$\int_\Omega G\cdot\Big(\nabla P_j^\beta\Big)\phi = \int_\Omega G_j^\beta\phi, \quad (2.7)$$

as $\varepsilon \to 0$, where $G = (G_i^\alpha)$. Similarly, by the Div-Curl Lemma, the RHS of (2.6) converges to

$$\int_\Omega \nabla w\cdot\left(\fint_Q A\big(\nabla_y P_j^\beta(y)+\nabla_y\chi_j^\beta\big)\right)\phi = \int_\Omega \frac{\partial w^\alpha}{\partial x_i}\cdot\widehat{a}_{ij}^{\alpha\beta}\phi, \quad (2.8)$$

as $\varepsilon \to 0$. Since $\phi \in C_0^\infty(\Omega)$ is arbitrary, we obtain

$$G_j^\beta = \frac{\partial w^\alpha}{\partial x_i}\widehat{a}_{ij}^{\alpha\beta} = \widehat{a}_{ji}^{\beta\alpha}\frac{\partial w^\alpha}{\partial x_i};$$

i.e. $G = \widehat{A}\nabla w$ in Ω.

Finally, note that for any $\varphi \in H^1_D(\Omega;\mathbb{R}^d)$,

$$\begin{aligned}\int_\Omega \widehat{A}\nabla w\cdot\nabla\varphi &= \int_\Omega G\cdot\nabla\varphi = \lim_{\varepsilon'\to 0}\int_\Omega A^{\varepsilon'}\nabla u_{\varepsilon'}\cdot\nabla\varphi \\ &= \langle F,\varphi\rangle_{H_D^{-1}(\Omega)\times H_D^1(\Omega)} + \langle g,\varphi\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.\end{aligned}$$

This shows that w is a solution of the mixed problem (1.6) for the homogenized system. By the uniqueness of (1.6) it follows that the whole sequence u_ε converges weakly to u_0 in $H^1(\Omega;\mathbb{R}^d)$. The argument above also shows that the whole sequence $A^\varepsilon\nabla u_\varepsilon$ converges weakly to $\widehat{A}\nabla u_0$ in $L^2(\Omega;\mathbb{R}^{d\times d})$. □

3 Convergence rates in $H^1(\Omega)$

In this section we give the proof of the estimate (1.9) and Theorem 1.2. Let S_ε be the operator on $L^2(\mathbb{R}^d)$ given by

$$S_\varepsilon u(x) = u * \phi_\varepsilon(x) = \int_{\mathbb{R}^d} u(x-y)\phi_\varepsilon(y)dy, \quad (3.1)$$

where $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$, $\phi \in C_0^\infty(B(0,1/2))$, $\phi \geq 0$, and $\int\phi = 1$. We will call S_ε the smoothing operator at ε-scale. Note that

$$\|S_\varepsilon u\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)}, \quad (3.2)$$

and $D^\alpha S_\varepsilon u = S_\varepsilon D^\alpha u$ for $u \in H^s(\mathbb{R}^d)$ and $|\alpha| \leq s$.

Lemma 3.1. *Let $u \in H^1(\mathbb{R}^d)$. Then*

$$\|S_\varepsilon u - u\|_{L^2(\mathbb{R}^d)} \leq C\,\varepsilon\,\|\nabla u\|_{L^2(\mathbb{R}^d)}, \quad (3.3)$$

for any $\varepsilon > 0$.

Proof. This is well known. See e.g. [17] or [14] for a proof. $\square$

Lemma 3.2. *Let $f \in L^2_{loc}(\mathbb{R}^d)$ be a 1-periodic function. Then for any $u \in L^2(\mathbb{R}^d)$,*

$$\|f^\varepsilon S_\varepsilon u\|_{L^2(\mathbb{R}^d)} \le C\, \|f\|_{L^2(Q)} \|u\|_{L^2(\mathbb{R}^d)}, \tag{3.4}$$

where $f^\varepsilon(x) = f(x/\varepsilon)$ and $Q = [-1/2, 1/2]^d$.

Proof. See e.g. [17] or [14] for a proof. $\square$

Let $\widetilde{\Omega}_\varepsilon = \big\{ x \in \mathbb{R}^d : \text{dist}(x, \partial\Omega) < \varepsilon \big\}$.

Lemma 3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then for any $u \in H^1(\mathbb{R}^d)$,*

$$\int_{\widetilde{\Omega}_\varepsilon} |u|^2 \le C\, \varepsilon\, \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)}, \tag{3.5}$$

where the constant C depends only on d and Ω.

Proof. This is known. See e.g. [12]. We provide a proof for the reader's convenience. Note that the desired estimate is invariant under Lipschitz homeomorphism. By covering $\partial\Omega$ with coordinate patches, it suffices to prove a local estimate for the upper half-space with $0 < \varepsilon < 1$.

Let $\theta \in C^\infty(\mathbb{R})$ such that $0 \le \theta \le 1$, $\theta(t) = 1$ for $t \le 1$, and $\theta(t) = 0$ for $t \ge 2$. For any (x', t) with $x' \in \mathbb{R}^{d-1}$ and $-\varepsilon < t < \varepsilon < 1$, we have

$$\begin{aligned} u^2(x', t) &= -\int_t^2 \frac{\partial}{\partial s}\big[\theta(s) u^2(x', s)\big]\, ds \\ &= -\int_t^2 \frac{\partial}{\partial s}\big[\theta(s)\big] u^2(x', s)\, ds - 2\int_t^2 \theta(s) u(x', s) \frac{\partial}{\partial s} u(x', s)\, ds. \end{aligned}$$

It follows that

$$u^2(x', t) \le C \int_{-2}^2 u^2(x', s)\, ds + 2\int_{-2}^2 |u(x', s)||\nabla u(x', s)|\, ds. \tag{3.6}$$

Let Δ be a surface ball in $\mathbb{R}^{d-1}$. Then

$$\begin{aligned} &\int_{-\varepsilon}^{\varepsilon} \int_\Delta u^2(x', t)\, dx' dt \\ &\quad \le C\varepsilon \int_{-2}^2 \int_\Delta u^2(x', s)\, dx' ds + 4\varepsilon \int_{-2}^2 \int_\Delta |u(x', s)||\nabla u(x', s)|\, dx' ds \\ &\quad \le C\, \varepsilon\, \|u\|_{L^2(\Delta \times [-2,2])} \|u\|_{H^1(\Delta \times [-2,2])}. \end{aligned}$$

This completes the proof. $\square$

Lemma 3.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $f \in L^2_{loc}(\mathbb{R}^d)$ a 1-periodic function. Then for any $u \in H^1(\mathbb{R}^d)$,*

$$\int_{\widetilde{\Omega}_\varepsilon} |f^\varepsilon|^2 |S_\varepsilon u|^2 \le C\, \varepsilon\, \|f\|^2_{L^2(Q)} \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)}, \tag{3.7}$$

where C depends only on d and Ω.

Proof. This is known and similar estimates may be found in [17, 12]. Note that

$$S_\varepsilon u(x) = \int_{B(0,1/2)} u(x-\varepsilon y)\phi(y)\,dy. \tag{3.8}$$

By Minkowski's integral inequality and Fubini's theorem,

$$\begin{aligned}
\int_{\widetilde{\Omega}_\varepsilon} |f^\varepsilon(x)|^2 |S_\varepsilon u(x)|^2\,dx &\le C\int_{\widetilde{\Omega}_\varepsilon}\int_{B(0,1/2))} |f^\varepsilon(x)|^2|u(x-\varepsilon y)|^2\,dydx\\
&\le C\int_{B(0,1/2)}\int_{\widetilde{\Omega}_\varepsilon-\varepsilon y} |f^\varepsilon(x+\varepsilon y)|^2|u(x)|^2\,dxdy\\
&\le C\int_{B(0,1/2))}\int_{\widetilde{\Omega}_{2\varepsilon}} |f^\varepsilon(x+\varepsilon y)|^2|u(x)|^2\,dxdy\\
&\le C\int_{\widetilde{\Omega}_{2\varepsilon}} |u(x)|^2\,dx \sup_{x\in\mathbb{R}^d}\int_{B(0,1/2)} |f^\varepsilon(x+\varepsilon y)|^2\,dy\\
&\le C\,\varepsilon\,\|f\|^2_{L^2(Q)}\|u\|_{H^1(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where we have used Lemma 3.3 for the last inequality. □

Let u_0 be the solution of (1.6). Suppose that $u_0 \in H^2(\Omega;\mathbb{R}^d)$. Since Ω is Lipschitz, there exists a bounded extension operator $E : H^2(\Omega;\mathbb{R}^d) \to H^2(\mathbb{R}^d;\mathbb{R}^d)$ so that $\widetilde{u}_0 = Eu_0$ is an extension of u_0 and $\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)}$. Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0, \tag{3.9}$$

where $u_\varepsilon \in H^1(\Omega;\mathbb{R}^d)$ is the solution of (1.4). Then w_ε satisfies

$$\begin{cases}
\mathcal{L}_\varepsilon w_\varepsilon = F_\varepsilon = \mathcal{L}_0 u_0 - \mathcal{L}_\varepsilon u_0 - \mathcal{L}_\varepsilon(\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0) & \text{in } \Omega,\\
w_\varepsilon = h_\varepsilon = -\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0 & \text{on } D,\\
n\cdot A^\varepsilon\nabla w_\varepsilon = g_\varepsilon = n\cdot\widehat{A}\nabla u_0 - n\cdot A^\varepsilon\nabla u_0 - n\cdot A^\varepsilon\nabla(\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0) & \text{on } N.
\end{cases} \tag{3.10}$$

Recall that $\Omega_{2\varepsilon} = \{x\in\Omega :\ \text{dist}(x,\partial\Omega) < 2\varepsilon\}$. The following lemma plays a key role in this paper.

Lemma 3.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and D a closed subset of $\partial\Omega$. For any $\psi\in H^1_D(\Omega;\mathbb{R}^d)$, we have*

$$\left|\int_\Omega A^\varepsilon\nabla w_\varepsilon\cdot\nabla\psi\right| \le C\,\|u_0\|_{H^2(\Omega)}\Big\{\varepsilon\,\|\nabla\psi\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla\psi\|_{L^2(\Omega_{2\varepsilon})}\Big\},$$

where w_ε is given by (3.9) and C depends only on d, κ_1, κ_2, D, and Ω.

Proof. By a density argument we may assume $\psi\in C_0^\infty(\mathbb{R}^d\setminus D;\mathbb{R}^d)$. Using

$$\int_\Omega A^\varepsilon\nabla u_\varepsilon\cdot\nabla\psi = \int_\Omega \widehat{A}\nabla u_0\cdot\nabla\psi,$$

we obtain

$$\int_\Omega A^\varepsilon\nabla w_\varepsilon\cdot\nabla\psi = \int_\Omega\Big[\widehat{A}\nabla u_0 - A^\varepsilon\nabla u_0 - \varepsilon A^\varepsilon\nabla(\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0)\Big]\cdot\nabla\psi. \tag{3.11}$$

A direct calculation shows that

$$
\begin{aligned}
&\widehat{A}\nabla u_0 - A^\varepsilon \nabla u_0 - \varepsilon A^\varepsilon \nabla(\chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0) \\
&\quad = B^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + \Big[(\widehat{A}\nabla u_0 - \widehat{A} S_\varepsilon \nabla \widetilde{u}_0) - (A^\varepsilon \nabla u_0 - A^\varepsilon S_\varepsilon \nabla \widetilde{u}_0) - \varepsilon A^\varepsilon \chi^\varepsilon S_\varepsilon \nabla^2 \widetilde{u}_0 \Big] \\
&\quad = B^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + T_\varepsilon,
\end{aligned}
$$

where $B(y) = \widehat{A} - A(y) - A(y)\nabla\chi(y)$. As a result, we have

$$
\begin{aligned}
\int_\Omega A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi &= \int_\Omega B^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 \cdot \nabla \psi + \int_\Omega T_\varepsilon \cdot \nabla \psi \\
&= J_1 + J_2.
\end{aligned} \tag{3.12}
$$

For J_2, it follows from Lemmas 3.1 and 3.2 that

$$
\|T_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}. \tag{3.13}
$$

Thus,

$$
|J_2| \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)} \|\nabla\psi\|_{L^2(\Omega)}. \tag{3.14}
$$

To handle J_1, we write

$$
\begin{aligned}
J_1 &= \int_\Omega B^\varepsilon (1-\theta_\varepsilon) S_\varepsilon \nabla \widetilde{u}_0 \cdot \nabla\psi + \int_\Omega B^\varepsilon \theta_\varepsilon S_\varepsilon \nabla \widetilde{u}_0 \cdot \nabla \psi \\
&= J_{11} + J_{12},
\end{aligned} \tag{3.15}
$$

where $\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ is a smooth function such that $\theta_\varepsilon(x) = 1$ if $x \in \widetilde{\Omega}_\varepsilon$, $\theta_\varepsilon(x) = 0$ if $x \notin \widetilde{\Omega}_{2\varepsilon}$, and $|\nabla\theta_\varepsilon| \le C\varepsilon^{-1}$. Since $B(y)$ is 1-periodic and locally square integrable, by Lemma 3.4, we obtain

$$
\begin{aligned}
|J_{12}| &\le \int_{\Omega_{2\varepsilon}} |B^\varepsilon S_\varepsilon \nabla \tilde{u}_0 \cdot \theta_\varepsilon \nabla \psi| \\
&\le C\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} \|\nabla\psi\|_{L^2(\Omega_{2\varepsilon})}.
\end{aligned} \tag{3.16}
$$

It remains to estimate J_{11}. To this end we let $B = (b_{ij}^{\alpha\beta}(y))$. Note that $b_{ij}^{\alpha\beta}$ is 1-periodic and $b_{ij}^{\alpha\beta} \in L^2_{\text{loc}}(\mathbb{R}^d)$. Also, by (2.3) and (2.4),

$$
\frac{\partial}{\partial y_i} b_{ij}^{\alpha\beta} = 0 \quad \text{and} \quad \int_Q b_{ij}^{\alpha\beta} = 0.
$$

It follows that there exist 1-periodic functions $\phi_{kij}^{\alpha\beta} \in H^1_{\text{loc}}(\mathbb{R}^d)$, where $1 \le \alpha, \beta, i, j, k \le d$, such that

$$
b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} \phi_{kij}^{\alpha\beta} \qquad \text{and} \qquad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}. \tag{3.17}
$$

(see [7] or [8]). Using integration by parts, this allows us to write J_{11} as

$$
\begin{aligned}
J_{11} &= \int_\Omega \frac{\partial}{\partial x_k} \left(\varepsilon \phi_{kij}^{\alpha\beta\varepsilon}\right) (1-\theta_\varepsilon) S_\varepsilon \left(\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) \cdot \frac{\partial \psi^\alpha}{\partial x_i} \\
&= -\varepsilon \int_\Omega \phi_{kij}^{\alpha\beta\varepsilon} \frac{\partial}{\partial x_k} (1-\theta_\varepsilon) S_\varepsilon \left(\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) \cdot \frac{\partial \psi^\alpha}{\partial x_i} - \varepsilon \int_\Omega \phi_{kij}^{\alpha\beta\varepsilon} (1-\theta_\varepsilon) S_\varepsilon \left(\frac{\partial^2 \widetilde{u}_0}{\partial x_k \partial x_j}\right) \cdot \frac{\partial \psi^\alpha}{\partial x_i} \\
&\quad - \varepsilon \int_\Omega \phi_{kij}^{\alpha\beta\varepsilon} (1-\theta_\varepsilon) S_\varepsilon \left(\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) \cdot \frac{\partial^2 \psi^\alpha}{\partial x_i \partial x_k},
\end{aligned}
$$

where $\phi_{kij}^{\alpha\beta\varepsilon}(x) = \phi_{kij}^{\alpha\beta}(x/\varepsilon)$. Note that the last term vanishes in view of the second equation in (3.17). Therefore, by Lemmas 3.2 and 3.4, we obtain

$$\begin{aligned}|J_{11}| &\le C\int_{\Omega_{2\varepsilon}} |\Phi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0||\nabla\psi| + C\,\varepsilon\int_\Omega |\Phi^\varepsilon S_\varepsilon \nabla^2\widetilde{u}_0||\nabla\psi|\\ &\le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega_{2\varepsilon})} + C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)},\end{aligned}$$

where $\Phi = (\phi_{kij}^{\alpha\beta})$. Thus, in view of (3.16), we have proved that

$$|J_1| \le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega_{2\varepsilon})} + C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}. \tag{3.18}$$

The lemma now follows by combining (3.12), (3.14), and (3.18). □

We are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let w_ε be defined by (3.9). Set $r_\varepsilon = \varepsilon\theta_\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)$ and $\psi_\varepsilon = w_\varepsilon + r_\varepsilon$, where $\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ is the same as in the proof of Lemma 3.4. Then

$$\psi_\varepsilon = u_\varepsilon - u_0 - \varepsilon(1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) \in H_D^1(\Omega;\mathbb{R}^d).$$

It follows from Lemma 3.5 that

$$\left|\int_\Omega A^\varepsilon\nabla w_\varepsilon\cdot\nabla\psi_\varepsilon\right| \le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\psi_\varepsilon\|_{L^2(\Omega)}. \tag{3.19}$$

This, together with the observation $w_\varepsilon = \psi_\varepsilon - r_\varepsilon$ and

$$\|r_\varepsilon\|_{H^1(\Omega)} \le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}, \tag{3.20}$$

gives

$$\left|\int_\Omega A^\varepsilon\nabla\psi_\varepsilon\cdot\nabla\psi_\varepsilon\right| \le C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\psi_\varepsilon\|_{L^2(\Omega)}. \tag{3.21}$$

By the Korn inequality (2.1), the elasticity condition (1.2), and (3.21), we obtain

$$\|\psi_\varepsilon\|_{H^1(\Omega)} \le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}. \tag{3.22}$$

Finally, by (3.20) and (3.22),

$$\|w_\varepsilon\|_{H^1(\Omega)} \le \|\psi_\varepsilon\|_{H^1(\Omega)} + \|r_\varepsilon\|_{H^1(\Omega)} \le C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}. \tag{3.23}$$

This completes the proof. □

Remark 3.6. If $D = \partial\Omega$, Theorem 1.2 gives the $O(\varepsilon^{1/2})$ error estimate in H^1 for the Dirichlet problem. In the case of the Neumann problem where $D = \emptyset$, Lemma 3.5 as well as the estimate (3.21) continues to hold. We now use the second Korn inequality,

$$\|u\|_{H^1(\Omega)} \le C\Big\{\|\nabla u + (\nabla u)^T\|_{L^2(\Omega)} + \sum_{j=1}^m \Big|\int_\Omega u\cdot\phi_j\Big|\Big\}, \tag{3.24}$$

for any $u \in H^1(\Omega;\mathbb{R}^d)$, where $m = d(d+1)/2$, $\{\phi_j : j = 1,\dots,m\}$ is an orthonormal basis of $\mathcal{R}$, and $\mathcal{R} = \{u = Cx + D : C^T = -C \in \mathbb{R}^{d\times d} \text{ and } D \in \mathbb{R}^d\}$ denotes the space of rigid displacements. This, together with (1.2) and (3.21), gives

$$\|\psi_\varepsilon\|_{H^1(\Omega)} \le C\Big\{\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)} + \sum_{j=1}^m \Big|\int_\Omega \psi_\varepsilon\cdot\phi_j\Big|\Big\}.$$

Thus, if we require that $u_\varepsilon, u_0 \perp \mathcal{R}$ in $L^2(\Omega;\mathbb{R}^d)$, the estimate (3.23) still holds.

4 Convergence rates in $L^2(\Omega)$

In this section we give the proof of Theorem 1.1. We begin by considering the Neumann boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon \rho_\varepsilon = G & \text{in } \Omega, \\ n \cdot A^\varepsilon \nabla \rho_\varepsilon = h & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $G \in L^2(\Omega; \mathbb{R}^d)$, $h \in L^2(\partial\Omega; \mathbb{R}^d)$, and

$$\int_\Omega G \cdot \phi + \int_{\partial\Omega} h \cdot \phi = 0 \quad \text{for any } \phi \in \mathcal{R}. \tag{4.2}$$

Recall that a function $\rho_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ is called a weak solution of (4.1) if

$$\int_\Omega A^\varepsilon \nabla \rho_\varepsilon \cdot \nabla \psi = \int_\Omega G \cdot \psi + \int_{\partial\Omega} h \cdot \psi \tag{4.3}$$

for any $\psi \in H^1(\Omega; \mathbb{R}^d)$. Under the elasticity condition (1.2), it is well known that the Neumann problem (4.1) has a unique solution $\rho_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ such that $\rho_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$.

The homogenized problem for (4.1) is given by

$$\begin{cases} \mathcal{L}_0 \rho_0 = G & \text{in } \Omega, \\ n \cdot \widehat{A} \nabla \rho_0 = h & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

If Ω is $C^{1,1}$, $G \in L^2(\Omega; \mathbb{R}^d)$ and $h \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$, it is known that the unique weak solution of (4.4) in $H^1(\Omega; \mathbb{R}^d)$ with the property $\rho_0 \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$ satisfies

$$\|\rho_0\|_{H^2(\Omega)} \leq C \Big\{ \|G\|_{L^2(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)} \Big\}. \tag{4.5}$$

For the proof of Theorem 1.1 we will need to construct a function $h \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ satisfying (4.2) and

$$h = 0 \quad \text{on } N = \partial\Omega \setminus D, \tag{4.6}$$

for each $G \in L^2(\Omega; \mathbb{R}^d)$. This is done in the following lemma.

Lemma 4.1. *Let Ω be a bounded Lipschitz domain and D a closed subset of $\partial\Omega$ with a nonempty interior. Let $G \in L^2(\Omega; \mathbb{R}^d)$. Then there is $h \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ such that h satisfies (4.2), (4.6), and*

$$\|h\|_{H^{1/2}(\partial\Omega)} \leq C \, \|G\|_{L^2(\Omega)}, \tag{4.7}$$

where C depends only on Ω and D.

Proof. By our assumption on D there exist $x_0 \in D$ and $r_0 > 0$ such that $B(x_0, r_0) \cap \partial\Omega \subset D$. We fix a nonnegative function $h_0 \in C_0^\infty(B(x_0, r_0))$ satisfying $h_0 \geq 1$ in $B(x_0, r_0/2)$. Let

$$h = (\alpha_1 \phi_1 + \alpha_2 \phi_2 + \cdots + \alpha_m \phi_m) h_0, \tag{4.8}$$

where $m = d(d+1)/2$ and $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is an orthonormal basis of $\mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. Clearly, $h = 0$ on $N = \partial\Omega \setminus D$. We claim that it is possible to choose $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$ such that h satisfies (4.2) and (4.7). To find $(\alpha_1, \ldots, \alpha_m)$, we solve the $m \times m$ system of linear equations

$$\alpha_i \int_{\partial\Omega} \phi_i \cdot \phi_j h_0 = \int_\Omega G \cdot \phi_j, \qquad j = 1, 2, \ldots, m, \tag{4.9}$$

which is uniquely solvable, provided

$$\det\left(\int_{\partial\Omega} \phi_i \cdot \phi_j h_0\right) \neq 0. \tag{4.10}$$

It is easy to see that in this case, the function h in (4.8) with $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ given by (4.9) satisfies condition (4.2) and estimate (4.7).

Finally, suppose that (4.10) is not true. Then there exists some $(\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^m \setminus \{0\}$ such that

$$\beta_i \int_{\partial\Omega} \phi_i \cdot \phi_j h_0 = 0, \qquad j = 1, 2, \dots, m.$$

Let $u = \beta_1\phi_1 + \cdots \beta_m\phi_m$. Then $\int_{\partial\Omega} |u|^2 h_0 = 0$. Since $h_0 \geq 1$ on $B(x_0, r_0/2)$, it follows that $u = 0$ on $B(x_0, r_0/2) \cap \partial\Omega$. Using $\nabla u + (\nabla u)^T = 0$ in $\mathbb{R}^d$ and the Korn inequality (2.1), we obtain $u = 0$ in Ω. This implies that $\beta_1 = \beta_2 = \cdots \beta_m = 0$ and gives us a contradiction. □

Suppose that Ω is $C^{1,1}$. By Lemma 4.1 and (4.5), for each $G \in L^2(\Omega; \mathbb{R}^d)$, we can construct h so that the weak solution ρ_0 of (4.4) with the property $\rho_0 \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$ satisfies

$$\|\rho_0\|_{H^2(\Omega)} \leq C\,\|G\|_{L^2(\Omega)}. \tag{4.11}$$

Let $\widetilde{\rho}_0 = E\rho_0$ be an extension of ρ_0 in $H^2(\mathbb{R}^d; \mathbb{R}^d)$ and set $\eta_\varepsilon = \rho_\varepsilon - \rho_0 - \varepsilon\chi^\varepsilon S_\varepsilon \nabla\widetilde{\rho}_0$. By Remark 3.6 we see that

$$\|\eta_\varepsilon\|_{H^1(\Omega)} \leq C\,\varepsilon^{1/2}\|\rho_0\|_{H^2(\Omega)} \leq C\,\varepsilon^{1/2}\|G\|_{L^2(\Omega)}. \tag{4.12}$$

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let ψ_ε, w_ε, and r_ε be the same functions as in the proof of Theorem 1.2. Note that $\psi_\varepsilon = w_\varepsilon + r_\varepsilon = u_\varepsilon - u_0 - \varepsilon(1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0$. Clearly, by Lemma 3.2,

$$\|\varepsilon(1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0\|_{L^2(\Omega)} \leq C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}. \tag{4.13}$$

Thus, to prove Theorem 1.1, it suffices to show $\|\psi_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}$. This will be done by a duality argument, using Lemma 3.5.

Fix $G \in L^2(\Omega; \mathbb{R}^d)$ and let $h \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ be the function given in Lemma 4.1. Let ρ_ε, ρ_0 be the weak solutions of (4.1) and (4.4), respectively, such that $\rho_\varepsilon, \rho_0 \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. Since $\psi_\varepsilon \in H^1_D(\Omega; \mathbb{R}^d)$ and $n \cdot A^\varepsilon \nabla\rho_\varepsilon = h = 0$ on N, by (4.3),

$$\int_\Omega \psi_\varepsilon \cdot G = \int_\Omega A^\varepsilon \nabla\psi_\varepsilon \cdot \nabla\rho_\varepsilon. \tag{4.14}$$

Write

$$\int_\Omega A^\varepsilon \nabla\psi_\varepsilon \cdot \nabla\rho_\varepsilon = \int_\Omega A^\varepsilon \nabla w_\varepsilon \cdot \nabla\rho_\varepsilon + \int_\Omega A^\varepsilon \nabla r_\varepsilon \cdot \nabla\rho_\varepsilon = J_3 + J_4. \tag{4.15}$$

We estimate J_4 first. Note that,

$$\begin{aligned} J_4 &= \int_\Omega A^\varepsilon \nabla r_\varepsilon \cdot \nabla\eta_\varepsilon + \int_\Omega A^\varepsilon \nabla r_\varepsilon \cdot \nabla\rho_0 + \int_\Omega A^\varepsilon \nabla r_\varepsilon \cdot \nabla(\varepsilon\chi^\varepsilon S_\varepsilon \nabla\widetilde{\rho}_0) \\ &= J_{41} + J_{42} + J_{43}. \end{aligned}$$

In view of (3.20) and (4.12), we obtain

$$|J_{41}| \leq C\|\nabla r_\varepsilon\|_{L^2\Omega)}\|\nabla\eta_\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}. \tag{4.16}$$

For J_{42}, note that r_ε is supported in $\widetilde{\Omega}_{2\varepsilon}$. Hence,

$$\begin{aligned} |J_{42}| &\le C\,\|\nabla r_\varepsilon\|_{L^2(\Omega)}\|\nabla\rho_0\|_{L^2(\Omega_{2\varepsilon})} \\ &\le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}, \end{aligned}$$

where we have used Lemma 3.3 for the last inequality. Similarly,

$$\begin{aligned} |J_{43}| &\le C\,\|\nabla r_\varepsilon\|_{L^2(\Omega)}\|\nabla(\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0)\|_{L^2(\Omega_{2\varepsilon})} \\ &\le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}, \end{aligned} \tag{4.17}$$

where we have used Lemma 3.4. As a result, we have proved that

$$|J_4| \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}. \tag{4.18}$$

It remains to estimate J_3. Again, we write

$$\begin{aligned} J_3 &= \int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla\eta_\varepsilon + \int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla\rho_0 + \int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla(\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0) \\ &= J_{31} + J_{32} + J_{33}. \end{aligned}$$

Note that J_{31} can be easily handled by the H^1 estimates of w_ε and η_ε. Since the estimate of J_{32} is similar to that of J_{33}, we will only give the estimate for J_{33}. To this end, we write

$$\begin{aligned} &\int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla(\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0) \\ &\quad= \int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla(\theta_{2\varepsilon}\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0) + \int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla\big((1-\theta_{2\varepsilon})\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0\big), \end{aligned} \tag{4.19}$$

where $\theta_{2\varepsilon} \in C_0^\infty(\mathbb{R}^d)$ is a smooth function such that $\theta_{2\varepsilon}(x) = 1$ if $\text{dist}(x, \partial\Omega) \le 2\varepsilon$, $\theta_{2\varepsilon}(x) = 0$ if $\text{dist}(x, \partial\Omega) \ge 4\varepsilon$, and $|\nabla\theta_{2\varepsilon}| \le C\varepsilon^{-1}$. It follows by Theorem 1.2 and Lemma 3.4 that

$$\begin{aligned} \left|\int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla(\theta_{2\varepsilon}\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0)\right| &\le C\,\varepsilon\,\|w_\varepsilon\|_{H^1(\Omega)}\|\theta_{2\varepsilon}\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0\|_{H^1(\Omega)} \\ &\le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}. \end{aligned} \tag{4.20}$$

For the second term in the RHS of (4.19), note that $(1-\theta_\varepsilon)\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0 \in H^1_D(\Omega; \mathbb{R}^d)$. This allows us to apply Lemma 3.5 and obtain

$$\begin{aligned} &\left|\int_\Omega A^\varepsilon\nabla w_\varepsilon\nabla\Big((1-\theta_{2\varepsilon})\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0\Big)\right| \\ &\quad\le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\nabla\big((1-\theta_{2\varepsilon})\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0\big)\|_{L^2(\Omega)} \\ &\qquad + C\,\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\big((1-\theta_{2\varepsilon})\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0\big)\|_{L^2(\Omega_{2\varepsilon})}. \end{aligned} \tag{4.21}$$

Note that the second term vanishes, as $1-\theta_{2\varepsilon}$ is supported in $\mathbb{R}^d \setminus \Omega_{2\varepsilon}$. Also,

$$\|\nabla\big((1-\theta_{2\varepsilon})\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{\rho}_0\big)\|_{L^2(\Omega)} \le C\,\|\rho_0\|_{H^2(\Omega)}. \tag{4.22}$$

This, together with (4.20) and (4.21), leads to

$$|J_{33}| \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}. \tag{4.23}$$

Combining this with the estimates of J_{31}, J_{32}, we obtain

$$|J_3| \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)}. \tag{4.24}$$

Hence, in view of (4.14), (4.15), (4.18) and (4.24), we have proved

$$\left|\int_\Omega \psi_\varepsilon \cdot G\right| \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|\rho_0\|_{H^2(\Omega)} \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}\|G\|_{L^2(\Omega)}, \tag{4.25}$$

where C depends only on d, κ_1, κ_2, D, and Ω. Therefore, by duality,

$$\|\psi_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)}, \tag{4.26}$$

which completes the proof of Theorem 1.1. □

Remark 4.2. If $D = \partial\Omega$, Theorem 1.1 gives the sharp $O(\varepsilon)$ estimate in L^2 for the Dirichlet problem. In the case of the Neumann problem, our proof also gives the estimate (1.8), if we further require that $u_\varepsilon, u_0 \perp \mathcal{R}$ in $L^2(\Omega;\mathbb{R}^d)$. To see this, we consider the Neumann problem (4.1) with $G \in L^2(\Omega;\mathbb{R}^d)$, $G \perp \mathcal{R}$, and $h = 0$ on $\partial\Omega$. The same argument as in the proof of Theorem 1.1 gives the estimate (4.25). By duality this implies that

$$\|\psi_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon\,\|u_0\|_{H^2(\Omega)} + C\sum_{j=1}^m \Big|\int_\Omega \psi_\varepsilon \cdot \phi_j\Big|,$$

where $m = d(d+1)/2$ and $\{\phi_j : j = 1,\dots,m\}$ forms an orthonormal basis for $\mathcal{R}$ in $L^2(\Omega;\mathbb{R}^d)$. Using $u_\varepsilon, u_0 \perp \mathcal{R}$ in $L^2(\Omega;\mathbb{R}^d)$, it follows that $\|\psi_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon\|u_0\|_{L^2(\Omega)}$, from which the estimate (1.8) follows.

5 Interior H^1 estimates

In this section we study the interior H^1 convergence and give the proof of Theorem 1.3.

Lemma 5.1. *Let w_ε be defined by (3.9). Let $\zeta \in W^{1,\infty}(\Omega)$ be a nonnegative function in Ω such that $\zeta = 0$ on $\partial\Omega$. Then,*

$$\|\zeta\nabla w_\varepsilon\|_{L^2(\Omega)} \le C\|u_0\|_{H^2(\Omega)}\Big\{\varepsilon\,\|\zeta\|_{W^{1,\infty}(\Omega)} + \varepsilon^{1/2}\|\zeta\|_{L^\infty(\Omega_{2\varepsilon})} + \varepsilon^{3/4}\|\zeta\nabla\zeta\|^{1/2}_{L^\infty(\Omega_{2\varepsilon})}\Big\},$$

where C depends only on d, κ_1, κ_2, D, and Ω.

Proof. Since $\zeta w_\varepsilon \in H^1_0(\Omega;\mathbb{R}^d)$, it follows from the elasticity condition and the first Korn inequality that

$$\begin{aligned}
\|\zeta\nabla w_\varepsilon\|^2_{L^2(\Omega)} &\le 2\|\nabla(\zeta w_\varepsilon)\|^2_{L^2(\Omega)} + 2\|w_\varepsilon\nabla\zeta\|^2_{L^2(\Omega)}\\
&\le C\int_\Omega A^\varepsilon\nabla(\zeta w_\varepsilon)\cdot\nabla(\zeta w_\varepsilon) + 2\|w_\varepsilon\|^2_{L^2(\Omega)}\|\nabla\zeta\|^2_{L^\infty(\Omega)}\\
&\le C\int_\Omega A^\varepsilon\nabla w_\varepsilon\cdot\nabla(\zeta^2 w_\varepsilon) + C\|w_\varepsilon\|^2_{L^2(\Omega)}\|\nabla\zeta\|^2_{L^\infty(\Omega)},
\end{aligned} \tag{5.1}$$

where we also used the identity

$$A^\varepsilon\nabla(\zeta w_\varepsilon)\cdot\nabla(\zeta w_\varepsilon) = A^\varepsilon\nabla w_\varepsilon\cdot\nabla(\zeta^2 w_\varepsilon) + A^\varepsilon(w_\varepsilon\nabla\zeta)\cdot(w_\varepsilon\nabla\zeta).$$

Note that by Lemma 3.5,

$$\int_\Omega A^\varepsilon \nabla w_\varepsilon \cdot \nabla(\zeta^2 w_\varepsilon) \\ \le C\,\varepsilon\, \|u_0\|_{H^2(\Omega)} \|\nabla(\zeta^2 w_\varepsilon)\|_{L^2(\Omega)} + C\,\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} \|\nabla(\zeta^2 w_\varepsilon)\|_{L^2(\Omega_{2\varepsilon})}. \tag{5.2}$$

This, together with (5.1), gives

$$\begin{aligned} \|\zeta \nabla w_\varepsilon\|^2_{L^2(\Omega)} \le\ & C\,\varepsilon\, \|u_0\|_{H^2(\Omega)} \|\zeta \nabla w_\varepsilon\|_{L^2(\Omega)} \|\zeta\|_{L^\infty(\Omega)} \\ & + C\,\varepsilon\, \|u_0\|_{H^2(\Omega)} \|w_\varepsilon\|_{L^2(\Omega)} \|\zeta \nabla \zeta\|_{L^\infty(\Omega)} \\ & + C\,\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} \|\zeta \nabla w_\varepsilon\|_{L^2(\Omega_{2\varepsilon})} \|\zeta\|_{L^\infty(\Omega_{2\varepsilon})} \\ & + C\,\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} \|w_\varepsilon\|_{L^2(\Omega)} \|\zeta \nabla \zeta\|_{L^\infty(\Omega_{2\varepsilon})} \\ & + C\, \|w_\varepsilon\|^2_{L^2(\Omega)} \|\nabla \zeta\|^2_{L^\infty(\Omega)}. \end{aligned} \tag{5.3}$$

By the Cauchy inequality with an $\varepsilon > 0$ we obtain

$$\begin{aligned} \|\zeta \nabla w_\varepsilon\|^2_{L^2(\Omega)} \le\ & C\,\varepsilon^2 \|u_0\|^2_{H^2(\Omega)} \|\zeta\|^2_{L^\infty(\Omega)} + C\,\varepsilon\, \|u_0\|_{H^2(\Omega)} \|w_\varepsilon\|_{L^2(\Omega)} \|\zeta \nabla \zeta\|_{L^\infty(\Omega)} \\ & + C\,\varepsilon\, \|u_0\|^2_{H^2(\Omega)} \|\zeta\|^2_{L^\infty(\Omega_{2\varepsilon})} \\ & + C\varepsilon^{1/2} \|u_0\|_{H^2(\Omega)} \|w_\varepsilon\|_{L^2(\Omega)} \|\zeta \nabla \zeta\|_{L^\infty(\Omega_{2\varepsilon})} \\ & + C\, \|w_\varepsilon\|^2_{L^2(\Omega)} \|\nabla \zeta\|^2_{L^\infty(\Omega)}. \end{aligned} \tag{5.4}$$

It then follows by the estimate $\|w_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}$ that

$$\|\zeta \nabla w_\varepsilon\|^2_{L^2(\Omega)} \le C\, \|u_0\|^2_{H^2(\Omega)} \Big\{ \varepsilon^2 \|\zeta\|^2_{W^{1,\infty}(\Omega)} + \varepsilon\, \|\zeta\|^2_{L^\infty(\Omega_{2\varepsilon})} + \varepsilon^{3/2} \|\zeta \nabla \zeta\|_{L^\infty(\Omega_{2\varepsilon})} \Big\}.$$

This completes the proof. □

Proof of Theorem 1.3. Let $\zeta(x) = \delta(x) = \text{dist}(x, \partial\Omega)$. Note that $\zeta = 0$ on $\partial\Omega$ and $\|\zeta\|_{W^{1,\infty}(\Omega)} \le C$, where C depends only on Ω. Theorem 1.3 now follows readily from Lemma 5.1. □

As a corollary, we obtain the following interior estimate.

Corollary 5.2. *Let Ω' be an open subset of Ω such that dist$(\Omega', \partial\Omega) > 0$. Under the same conditions as in Theorem 1.1, we have*

$$\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0\|_{H^1(\Omega')} \le C\,\varepsilon\, \|u_0\|_{H^2(\Omega)}, \tag{5.5}$$

where C depends only on d, κ_1, κ_2, D, Ω' and Ω.

Remark 5.3. The estimates in Lemma 5.1 and Theorem 1.3 as well as in Corollary 5.2 continue to hold for the Neumann boundary value problems, if we further require $u_\varepsilon, u_0 \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. The proof is exactly the same.

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