9-1-2017

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Repository Citation

Beidleman, James C. and Skiba, Alexander N., "On τ_σ -Quasinormal Subgroups of Finite Groups" (2017).
Mathematics Faculty Publications. 29.
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Digital Object Identifier (DOI)

<https://doi.org/10.1515/jgth-2017-0016>

Notes/Citation Information

Published in *Journal of Group Theory*, v. 20, issue 5, p. 955-969.

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On τ_σ -quasinormal subgroups of finite groups

James C. Beidleman and Alexander N. Skiba

Communicated by Evgenii I. Khukhro

Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ -set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i -subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i -subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$.

Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \setminus \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \emptyset \text{ for a Hall } \sigma_i\text{-subgroup } H \text{ of } G\}$. We say that a subgroup A of G is τ_σ -permutable or τ_σ -quasinormal in G with respect to $\mathcal{H}$ if $AH^x = H^x A$ for all $x \in G$ and all $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$, and τ_σ -permutable or τ_σ -quasinormal in G if A is τ_σ -permutable in G with respect to some complete Hall σ -set of G .

We study G assuming that τ_σ -quasinormality is a transitive relation in G .

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. The group G is called π -supersoluble provided every chief factor of G is either cyclic or a π' -group.

If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n ; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G . The symbol H_G denotes the largest normal subgroup of G contained in $H \leq G$.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

The symbol $\sigma(n)$ denotes the set $\{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$. The group G is said to be

- (i) σ -primary [16] if G is a σ_i -group for some $i \in I$,
- (ii) σ -decomposable (Shemetkov [14]) or σ -nilpotent (Guo and Skiba [6]) if $G = G_1 \times \cdots \times G_n$ for some σ -primary groups $G_1, \dots, G_n$,
- (iii) σ -soluble [16] if every chief factor of G is σ -primary.

A set $\mathcal{H}$ of subgroups of G is a *complete Hall σ -set* of G [15, 17] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i -subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i -subgroup of G for every $\sigma_i \in \sigma(G)$.

Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \setminus \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \emptyset \text{ for a Hall } \sigma_i\text{-subgroup } H \in \mathcal{H}\}$.

Definition 1.1. We say that a subgroup A of G is

- (i) τ_{σ} -permutable or τ_{σ} -quasinormal in G with respect to $\mathcal{H}$ if $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$,
- (ii) τ_{σ} -permutable or τ_{σ} -quasinormal in G if A is τ_{σ} -permutable in G with respect to some complete Hall σ -set $\mathcal{H}$ of G .

Recall that a subgroup A of G is said to be

- (i) σ -permutable or σ -quasinormal in G if G possesses a complete Hall σ -set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$ (cf. [16]),
- (ii) σ -semipermutable in G if G possesses a complete Hall σ -set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ with $\sigma(A) \cap \sigma(H) = \emptyset$ (cf. [7]).

In the classical case when $\sigma = \{\{2\}, \{3\}, \dots\}$, σ -permutable, σ -semipermutable and τ_{σ} -quasinormal subgroups are also called respectively S -permutable [1, 5], S -semipermutable [5] and τ -quasinormal [11, 12].

It is clear that every σ -quasinormal subgroup is also σ -semipermutable and every σ -semipermutable subgroup is τ_{σ} -quasinormal.

Example 1.2. Let $p > q > r$ be primes, C_r a group of order r and $H = Q \rtimes C_r$, where Q is a simple $\mathbb{F}_q C_r$ -module which is faithful for C_r . Let $G = P \rtimes H$, where P is a simple $\mathbb{F}_p H$ -module which is faithful for H . Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{p\}$, $\sigma_2 = \{q\}$, and $\sigma_3 = \{p, q\}'$. Then every Sylow r -subgroup of G is τ_{σ} -quasinormal but not σ -semipermutable. A Hall $\{q, r\}$ -subgroup of G is σ -semipermutable in G but not σ -quasinormal.

We say that G is a $T\sigma T$ -group if τ_{σ} -quasinormality is a transitive relation in G , that is, if K is a τ_{σ} -quasinormal subgroup of H and H is a τ_{σ} -quasinormal subgroup of G , then K is a τ_{σ} -quasinormal subgroup of G . Our purpose here is to establish the structure of σ -soluble $T\sigma T$ -groups.

Theorem A. Let $D = G^{\mathfrak{R}\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete Hall σ -set $\mathcal{H}$ such that all members of $\mathcal{H}$ are π -supersoluble. Then G is a σ -soluble $T\sigma T$ -group if and only if either G is σ -nilpotent or $G = D \rtimes M$, where:

- (i) D and M are Hall subgroups of G and the smallest prime divisor of $|G|$ divides $|M|$.
- (ii) D is abelian and every element of M induces a power automorphism in D .

- (iii) Every subgroup of G is τ_σ -quasinormal in G .
- (iv) If A and B are respectively a Hall σ_i -subgroup and a Hall σ_j -subgroup of G , where $i \neq j$ and $\sigma_i, \sigma_j \in \sigma(G) \setminus \sigma(D)$, then the order of $[A, B]$ divides a prime. Moreover, if $|[A, B]| = r \neq 1$, then $r \in \pi(D)$ and either the Sylow r -subgroup R of G is cyclic or $[A, R] = 1 = [B, R]$; if, also, A is a p -group and B is a q -group for some primes p and q , then $r > p$ and $r > q$.

In this theorem $G^{\mathfrak{N}_\sigma}$ denotes the σ -nilpotent residual of G , that is, the intersection of all normal subgroups N of G with σ -nilpotent quotient G/N .

One of the main objectives of this paper is to give a correct proof of [12, Theorem 1.1]. The proof given in [12] for this theorem has several gaps. However, Theorem A and its proof allows us to eliminate all of those gaps. Corollary 1.3, our next result, is a statement of [12, Theorem 1.1].

Recall that G is said to be a TQT -group [12] if τ -quasinormality is a transitive relation in G .

Corollary 1.3. *Let P be a Sylow p -subgroup of G , Q a Sylow q -subgroup of G such that $p \neq q$. The following statements are equivalent:*

- (1) G is a soluble TQT -group.
- (2) G is a supersoluble group which has an abelian normal Hall subgroup of odd order D such that G/D is nilpotent, every subgroup of D is normal in G , and every subgroup of G is τ -quasinormal in G . Moreover, if $p \notin \pi(D)$ and $q \notin \pi(D)$, then the order of $[P, Q]$ divides r , where $r \in \pi(D)$, and if $|[P, Q]| = r$, then $r > p$, $r > q$ and either the Sylow r -subgroup R of G is cyclic or $[P, R] = 1 = [Q, R]$.

The following observation covers many steps in the proof of Theorem A.

Theorem B. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete Hall σ -set $\mathcal{H}$ such that all members of $\mathcal{H}$ are π -supersoluble. If all maximal subgroups of every Sylow p -subgroup of G are τ_σ -quasinormal in G for all $p \in \pi$, then D is a nilpotent Hall subgroup of G , the smallest prime divisor of $|G|$ divides $|G : D|$ and every chief factor of G below D is cyclic.*

Corollary 1.4 (Srinivasan [18]). *If every maximal subgroup of every Sylow subgroup of G is S -permutable in G , then G is supersoluble.*

2 Preliminaries

We use $\mathfrak{N}_\sigma$ to denote the class of all σ -nilpotent groups.

Lemma 2.1 (see [16, Lemma 2.5]). *The class $\mathfrak{N}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ -nilpotent, then E is σ -nilpotent.*

In view of [2, Proposition 2.2.8], we get from Lemma 2.1 the following:

Lemma 2.2. *If N is a normal subgroup of G , then*

$$(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N.$$

Lemma 2.3 (Knyagina and Monakhov [10]). *Let H , K and N be pairwise permutable subgroups of G and H is a Hall subgroup of G . Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Suppose that G has a complete Hall σ -set $\mathcal{H} = \{H_1, \dots, H_t\}$. For any subgroup H of G we write $H \cap \mathcal{H}$ to denote the set $\{H \cap H_1, \dots, H \cap H_t\}$. If $H \cap \mathcal{H}$ is a complete Hall σ -set of H , then we say that $\mathcal{H}$ *reduces into* H .

Recall that G is said to be

- (i) a D_π -group if G possesses a Hall π -subgroup E and every π -subgroup of G is contained in some conjugate of E ,
- (ii) a σ -full group of Sylow type [16] if every subgroup E of G is a D_{σ_i} -group for every $\sigma_i \in \sigma(E)$,
- (iii) σ -full [15, 17] provided G possesses a complete Hall σ -set.

In view of [15, Theorems A and B], the following fact is true.

Lemma 2.4. *If G is σ -soluble, then G is a σ -full group of Sylow type.*

Lemma 2.5 (see [16, Lemma 3.1]). *Let H be a σ_i -subgroup of a σ -full group G . Then H is σ -permutable in G if and only if $O^{\sigma_i}(G) \leq N_G(H)$.*

Lemma 2.6. *Suppose that G has a complete Hall σ -set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that the subgroups H and K of G are τ_σ -quasinormal in G with respect to $\mathcal{H}$. Let R be a normal subgroup of G and $H \leq L \leq G$. Then:*

- (1) $\mathcal{H}_0 = \{H_1 R/R, \dots, H_t R/R\}$ is a complete Hall σ -set of G/R . Moreover, if $\sigma(H) = \sigma(HR/R)$, then HR/R is τ_σ -quasinormal in G/N with respect to $\mathcal{H}_0$.
- (2) If $HK = KH$ and $\sigma(H \cap K) = \sigma(H) = \sigma(K)$, then $H \cap K$ is τ_σ -quasinormal in G with respect to $\mathcal{H}$.
- (3) If for some i we have $H \leq O_{\sigma_i}(G)$, then H is σ -quasinormal in G .

- (4) If $\mathcal{H}$ reduces into L , then H is τ_σ -quasinormal in L with respect to $L \cap \mathcal{H}$.
 (5) If G is a σ -full group of Sylow type, then H is τ_σ -quasinormal in L .

Proof. Without loss of generality we can assume that H_i is a σ_i -group for all $i = 1, \dots, t$.

(1) It is clear that $\mathcal{H}_0 = \{H_1 R/R, \dots, H_t R/R\}$ is a complete Hall σ -set of G/R . Now let $\sigma_i \in \tau_{\mathcal{H}_0}(HR/R)$, that is, $\sigma_i \notin \sigma(HR/R)$ and

$$\sigma(HR/R) \cap \sigma((H_i R/R)^G) \neq \emptyset.$$

Then $\sigma_i \notin \sigma(H)$ since $\sigma(H) = \sigma(HR/R)$. On the other hand, we have

$$(H_i R/R)^G = H_i^G R/R \simeq H_i^G / (H_i^G \cap R),$$

so $\sigma(H) \cap \sigma(H_i^G) \neq \emptyset$. Hence $\sigma_i \in \tau_{\mathcal{H}}(H)$ and so

$$\begin{aligned} (HR/R)(H_i R/R) &= HH_i R/R \\ &= H_i HR/R \\ &= (H_i R/R)(HR/R). \end{aligned}$$

Thus HR/R is τ_σ -quasinormal in G/R with respect to $\mathcal{H}_0$.

(2) Let H_i be a member of $\mathcal{H}$ such that $\sigma_i \in \tau_{\mathcal{H}}(H \cap K)$, that is, we have $\sigma_i \in \sigma(G) \setminus \sigma(H \cap K)$ and $\sigma(H \cap K) \cap \sigma(H_i^G) \neq \emptyset$. Then $\sigma_i \in \sigma(G) \setminus \sigma(H)$ (since $\sigma(H \cap K) = \sigma(H)$) and $\sigma(H) \cap \sigma(H_i^G) \neq \emptyset$, so $\sigma_i \in \tau_{\mathcal{H}}(H)$. Similarly we get that $\sigma_i \in \tau_{\mathcal{H}}(K)$. Hence $HH_i^x = H_i^x H$ and $KH_i^x = H_i^x K$ for all $x \in G$. It is clear also that $H \cap H_i^x = 1$. Therefore for every $x \in G$ we have

$$\begin{aligned} HH_i^x \cap KH_i^x &= H_i^x (H \cap KH_i^x) \\ &= H_i^x (H \cap K)(H \cap H_i^x) \\ &= H_i^x (H \cap K) \\ &= (H \cap K)H_i^x \end{aligned}$$

by Lemma 2.3. Hence $H \cap K$ is τ_σ -quasinormal in G with respect to $\mathcal{H}$.

(3) Let $j \neq i$. Suppose that for some $x \in G$ we have $HH_j^x \neq H_j^x H$. Then $\sigma_j \notin \tau_{\mathcal{H}}(H)$. Hence $\sigma(H) \cap \sigma(H_j^G) = \emptyset$ since H is τ_σ -quasinormal in G with respect to $\mathcal{H}$ by hypothesis. But then H_j^G is a σ'_i -group, so $H \leq O_{\sigma_i}(G) \leq C_G(H_j^G)$, which implies that $HH_j^x = H_j^x H$. This contradiction shows that we have (3).

(4) Let $L_i = H_i \cap L$ for all $i = 1, \dots, t$ and $\mathcal{L} = \{L_1, \dots, L_t\}$. By hypothesis, $\mathcal{L}$ is a complete Hall σ -set of L . Let $\sigma_i \in \tau_{\mathcal{L}}(H)$, that is, $\sigma_i \in \sigma(L) \setminus \sigma(H)$ and $\sigma(H) \cap \sigma((L_i)^L) \neq \emptyset$. Then $\sigma_i \in \sigma(G) \setminus \sigma(H)$ and $\sigma(H) \cap \sigma((H_i)^G) \neq \emptyset$

since $(L_i)^L \leq L_i^G \leq (H_i \cap L)^G \leq H_i^G$. Hence $\sigma_i \in \tau_{\mathcal{H}}(H)$, so $HH_i^a = H_i^a H$ for all $a \in L$, so $L \cap HH_i^a = H(L \cap H_i^a) = H(L \cap H_i)^a = HL_i^a = L_i^a H$. This shows that H is τ_{σ} -quasinormal in L with respect to $L \cap \mathcal{H}$.

(5) Since G is a σ -full group of Sylow type, H is τ_{σ} -quasinormal in G with respect to each complete Hall σ -set of G . Moreover, this condition implies also that some complete Hall σ -set of G reduces into L , so we (5) by part (4).

The lemma is proved. $\square$

Lemma 2.7 (Kegel [9]). *Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^x A$ for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

The following lemma is a corollary of [4, Chapter IV, (6.7)] (see also [3, Lemma 2.12]).

Lemma 2.8. *Let $N \leq E$ be normal subgroups of G such that $N \leq \Phi(E)$ and every chief factor of G between E and N is cyclic. Then every chief factor of G below E is cyclic.*

Lemma 2.9. *If $\mathcal{H} = \{H_1, \dots, H_t\}$ is a complete Hall σ -set of G and every member of $\mathcal{H}$ is $\pi(G^{\mathfrak{N}_{\sigma}})$ -supersoluble, then*

$$\mathcal{H}_0 = \{H_1 N / N, \dots, H_t N / N\}$$

is a complete Hall σ -set of G/N and every member of $\mathcal{H}_0$ is $\pi((G/N)^{\mathfrak{N}_{\sigma}})$ -supersoluble.

Proof. It is clear that $\mathcal{H}_0$ is a complete Hall σ -set of G/N . Now let $D = G^{\mathfrak{N}_{\sigma}}$ and $\pi = \pi(D)$. Then $(G/N)^{\mathfrak{N}_{\sigma}} = DN/N$ by Lemma 2.2, so

$$\pi_0 = \pi((G/N)^{\mathfrak{N}_{\sigma}}) = \pi(DN/N) = \pi(D/D \cap N) \subseteq \pi(D) = \pi.$$

Hence every member H_i of $\mathcal{H}$ is π_0 -supersoluble, so $H_i N / N$ is π_0 -supersoluble. The lemma is proved. $\square$

Lemma 2.10. *Let $D = G^{\mathfrak{N}_{\sigma}}$ and $\pi = \pi(D)$. Suppose that D is a nilpotent Hall subgroup of G and G possesses a complete Hall σ -set $\mathcal{H}$ such that all members of $\mathcal{H}$ are π -supersoluble. If every σ_i -subgroup of G is τ_{σ} -quasinormal in G for all $\sigma_i \in \sigma(D)$, then every subgroup of D is normal in G .*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume without loss of generality that H_i is a σ_i -group for all $i = 1, \dots, t$.

We show that the hypothesis holds on G/N for every minimal normal subgroup N of G . First note that

$$(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/D \cap N$$

is a nilpotent Hall subgroup of G/N by Lemma 2.2, and G/N possesses a complete Hall σ -set $\mathcal{H}_0$ such that all members of $\mathcal{H}_0$ are $\pi((G/N)^{\mathfrak{N}_\sigma})$ -supersoluble by Lemma 2.9.

Now let V/N be a non-identity σ_i -subgroup of G/N for some

$$\sigma_i \in \sigma((G/N)^{\mathfrak{N}_\sigma}) = \sigma(DN/N) = \sigma(D/D \cap N) \subseteq \sigma(D).$$

And let U be a minimal supplement to N in V . Then $U \cap N \leq \Phi(U)$, so U is a σ_i -subgroup of G since $V/N = UN/N \simeq U/U \cap N$. Therefore U is τ_σ -quasinormal in G by hypothesis and $\sigma(U) = \sigma(UN/N) = \{\sigma_i\}$, which implies that $V/N = UN/N$ is τ_σ -quasinormal in G/N by Lemma 2.6(1). Hence the hypothesis holds on G/N .

Let H be a subgroup of the Sylow p -subgroup P of D for some prime $p \in \pi$. We show that H is normal in G . For some i we have $P \leq O_{\sigma_i}(D) = H_i \cap D$. On the other hand, we have $D = O_{\sigma_i}(D) \times O^{\sigma_i}(D)$ since D is nilpotent. Assume that $O^{\sigma_i}(D) \neq 1$ and let N be a minimal normal subgroup of G contained in $O^{\sigma_i}(D)$. Then $HN/N \leq DN/N = (G/N)^{\mathfrak{N}_\sigma}$, so the choice of G implies that HN/N is normal in G/N . Hence $H = H(N \cap O_{\sigma_i}(D)) = HN \cap O_{\sigma_i}(D)$ is normal in G .

Now assume that $O^{\sigma_i}(D) = 1$, so D is a σ_i -group. Since G/D is σ -nilpotent by Lemma 2.1, H_i/D is normal in G/D and hence H_i is normal in G . Therefore all subgroups of H_i are σ -quasinormal in G by Lemma 2.6(3) and hypothesis. Since D is a normal Hall subgroup of H_i , it has a complement S in H_i by the Schur–Zassenhaus theorem. Lemma 2.5 implies that $D \leq O^{\sigma_i}(G) \leq N_G(S)$. Hence $H_i = D \times S$. Therefore

$$G = H_i O^{\sigma_i}(G) = S O^{\sigma_i}(G) \leq N_G(H),$$

so H is normal in G .

Therefore every subgroup of D is normal in G since D is nilpotent by hypothesis. The lemma is proved. $\square$

3 Proof of Theorems A and B

Proof of Theorem B. Suppose that this theorem is false and let G be a counterexample of minimal order. Then $D \neq 1$. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume without loss of generality that H_i is a σ_i -group for all $i = 1, \dots, t$. Let R be a minimal normal subgroup of G .

Claim 1. *The conclusion of the theorem holds for G/R .*

Let V/R be a maximal subgroup of a Sylow p -subgroup P/R of G/R , where $p \in \pi((G/R)^{\mathfrak{N}_\sigma})$. Then for some Sylow p -subgroup G_p of G we have $P/R = G_p R/R$ and $V = R(V \cap G_p)$. It is clear that $V \cap G_p$ is a maximal subgroup of G_p . Therefore $V \cap G_p$ is τ_σ -quasinormal in G by hypothesis since

$$\pi((G/R)^{\mathfrak{N}_\sigma}) = \pi(DR/R) = \pi(D/D \cap R) \subseteq \pi$$

by Lemma 2.2, so $V/R = R(V \cap G_p)/R$ is τ_σ -quasinormal in G/R by Lemma 2.6(1). Consequently, the hypothesis holds for G/R by Lemma 2.9. Hence we have Claim 1 by the choice of G .

Claim 2. *The group D is soluble, so G is σ -soluble. Hence G is a σ -full group of Sylow type.*

Assume that this is false. By Claim 1 and Lemma 2.2,

$$(G/R)^{\mathfrak{N}_\sigma} = DR/R \simeq D/D \cap R$$

is nilpotent. Hence $R \leq D$. Moreover, if G has a minimal normal subgroup $N \neq R$, then $N \leq D$ and $D \simeq D/R \cap N = D/1$ is nilpotent. Therefore $C_G(R) = 1$. Then 2 divides $|R|$ by the Feit–Thompson theorem, and a Sylow 2-subgroup Q of R is not cyclic by [8, Chapter IV, Section 2.8]. Hence $|Q| > 2$.

Let P be a Sylow 2-subgroup of G such that $Q = P \cap R$. Then for some maximal subgroup V of P we have $Q \not\leq V$ by the Tate theorem [8, Chapter IV, Section 4.7], which implies that $P = QV$ and so $V \cap R < P \cap R = Q$. Moreover, $V \cap R \neq 1$ since otherwise we have $V \cap R = P \cap V \cap R = Q \cap V = 1$ and so $|Q| = 2$. Since $R = R_1 \times \cdots \times R_n$, where $R_1 \simeq \cdots \simeq R_n$ are non-abelian simple groups, it follows that $Q = (P \cap R_1) \times \cdots \times (P \cap R_n)$ and so for some i we have $V \cap R_i < P \cap R_i$. Note also that $V \cap R_i \neq 1$ since otherwise from the isomorphism

$$V(P \cap R_i)/V \simeq (P \cap R_i)/(V \cap (P \cap R_i)) = P \cap R_i/1$$

we get that the order of a Sylow 2-subgroup of $P \cap R_i$ divides 2 and so $P \cap R_i$ is 2-nilpotent by [8, Chapter IV, Section 2.8], which implies that R is 2-nilpotent.

Assume that $2 \in \sigma_k$. First we show that R is σ -primary. Suppose that this is false. We can assume without loss of generality that V is τ_σ -quasinormal in G with respect to $\mathcal{H}$. Then for some $j \neq k$ and for $H = H_j$ we have $H \cap R_i \neq 1$ since R is not σ -primary. Note also that $\sigma_k \in \sigma(H^G)$ since otherwise we have $R \cap H^G = 1$, which implies that $1 < H^G \leq C_G(R) = 1$. Therefore $\sigma_k \in \tau_{\mathcal{H}}(V)$, so $VH^x = H^xV$ for all $x \in G$. By [4, Chapter A, Section 14.1(a)], we have $L = VH^x \cap R_i$ is a subnormal subgroup of VH^x , where V is a Hall σ_k -subgroup of VH^x and H^x is a Hall σ_j -subgroup of VH^x . Therefore, $L = (L \cap V)(L \cap H^x)$

by [4, Chapter I, Section 3.2]. Hence

$$\begin{aligned}
 L &= (L \cap V)(L \cap H^x) \\
 &= (VH^x \cap R_i \cap V)(VH^x \cap R_i \cap H^x) \\
 &= (R_i \cap V)(R_i \cap H^x) \\
 &= (V \cap R_i)(H \cap R_i)^x \\
 &= (H \cap R_i)^x(V \cap R_i)
 \end{aligned}$$

for all $x \in R_i$, where $(H \cap R_i)(V \cap R_i) \neq R_i$ since $V \cap R_i < P \cap R_i$. Therefore, R_i is not simple by Lemma 2.7 since $H \cap R_i \neq 1$ and $V \cap R_i \neq 1$. This contradiction shows that R is σ -primary.

Now assume that $R \leq D$ and R is not abelian. Then

$$\sigma(V \cap R) = \sigma(V) = \sigma(R) = \{\sigma_k\}$$

since R is σ -primary, $2 \in \sigma_k$ and $V \cap R \neq 1$. Therefore $V \cap R$ is τ_σ -quasinormal in G by Lemma 2.6(2). But $V \cap R \leq R \leq O_{\sigma_k}(G)$ and so $V \cap R$ is σ -quasinormal in G by Lemma 2.6(3). Hence $R \leq N_G(V \cap R)$ by Lemma 2.5 since $R \leq D \leq O^{\sigma_i}(G)$. Therefore $V \cap R \leq O_2(R) = 1$, a contradiction. Thus R is abelian. Hence D is soluble by Claim 1. Therefore G is σ -soluble and so G is a σ -full group of Sylow type by Lemma 2.4.

Claim 3. *The group D is nilpotent.*

Assume that this is false. Note that $RD/R = (G/R)^{\mathfrak{N}}$ is nilpotent by Claim 1. Therefore $R \leq D$, R is the unique minimal normal subgroup of G and $R \not\leq \Phi(G)$ by Lemma 2.1. Claim 2 implies that R is a p -group for some prime p . Therefore $R = C_G(R)$ by [4, Chapter A, Section 15.2], and $G = R \rtimes M$ for some maximal subgroup M of G . If $|R| = p$, then $G/C_G(R) = G/R$ is a cyclic group. Hence G is supersoluble and therefore D is nilpotent, which contradicts our assumption on G . Therefore $|R| > p$.

For some i we have $R \leq H_i \cap D$. Then $H_i = R \rtimes (H_i \cap M)$ and H_i is π -supersoluble by hypothesis. It follows that some maximal subgroup V of R is normal in H_i . Let P be a Sylow p -subgroup of $H_i \cap M$. Then $RP \in \text{Syl}_p(G)$ and VP is a maximal subgroup of RP , so VP is τ_σ -quasinormal in G by hypothesis. Then $V = V(R \cap P) = R \cap VP$ is σ -quasinormal in G by Lemma 2.6(2)–(3). Therefore $O^{\sigma_i}(G) \leq N_G(V)$ by Lemma 2.5. Hence $G = H_i O^{\sigma_i}(G) \leq N_G(V)$. The minimality of R implies that $V = 1$, so $|R| = p$, a contradiction. Hence we have Claim 3.

Claim 4. *If E is a subgroup of G , then $E^{\mathfrak{N}} \leq D$.*

Note that since $E/E \cap D \simeq ED/D \in \mathfrak{N}$ and $\mathfrak{N}$ is a hereditary class by Lemma 2.1, $E/E \cap D \in \mathfrak{N}$. Hence $E^{\mathfrak{N}} \leq E \cap D$.

Claim 5. *The group D is a Hall subgroup of G .*

Suppose that this is false and let P be a Sylow p -subgroup of D such that $1 < P < G_p \in \text{Syl}_p(G)$. We can assume without loss of generality that $G_p \leq H_1$ and that $R \leq D$.

Claim (a). *The group $D = P$ is a minimal normal subgroup of G .*

Since D is nilpotent by Claim 3, it follows that R is a q -group for some prime q . Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G/R by Claim 1. Suppose that $PR/R \neq 1$. Then $PR/R \in \text{Syl}_p(G/R)$. If $q \neq p$, then $P \in \text{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence we have $q = p$ and so $R \leq P$, therefore $P/R \in \text{Syl}_p(G/R)$ and we again get that $P \in \text{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of G contained in D . Since D is nilpotent by Claim 3, a p' -complement E of D is characteristic in D and so it is normal in G . Hence $E = 1$, which implies that $R = D = P$.

Claim (b). *We have $D \not\leq \Phi(G)$. Hence for some maximal subgroup M of G we have $G = D \rtimes M$.*

This follows from Claim 2 and Lemma 2.1 since G is not σ -nilpotent

Claim (c). *If G has a minimal normal subgroup $L \neq D$, then $G_p = D \times (L \cap G_p)$. Hence $O_{p'}(G) = 1$.*

Indeed, $DL/L \simeq D$ is a Hall subgroup of G/L by Claims 2 and (a). Hence $G_p L/L = DL/L$, so $G_p = D \times (L \cap G_p)$. Thus $O_{p'}(G) = 1$ since $D < G_p$ by Claim (a).

Claim (d). *The group $V = C_G(D) \cap M$ is a normal subgroup of G and we have $C_G(D) = D \times V \leq H_1$.*

In view of Claims (a) and (b), $C_G(D) = D \times V$, where $V = C_G(D) \cap M$ is a normal subgroup of G . By Claim (a), $V \cap D = 1$ and hence $V \simeq DV/D$ is σ -nilpotent by Lemma 2.1. Let W be a σ_1 -complement of V . Then W is characteristic in V and so it is normal in G . Therefore we have Claim (d) by Claim (c).

Claim (e). *We have $G_p \neq H_1$.*

Assume that $G_p = H_1$. Let Z be a subgroup of order p in $Z(G_p) \cap D$. Then, since $D \leq O^{\sigma_1}(G) = O^p(G)$, Z is normal in G by Lemma 2.5. Hence we have

$D = Z < G_p$ and so $D < C_G(D)$. Then $V = C_G(D) \cap M \neq 1$ is a normal subgroup of G and $V \leq H_1 = G_p$ by Claim (d). Let L be a minimal normal subgroup of G contained in V . Then $G_p = D \times L$ is a normal elementary abelian subgroup of G by Claim (c). Hence every maximal subgroup of G_p is normal in G by Lemmas 2.6 (3) and 2.5. It follows that every subgroup of G_p is normal in G . Hence $|D| = |L| = p$. Let $D = \langle a \rangle$, $L = \langle b \rangle$ and $N = \langle ab \rangle$. Then $N \not\leq D$, so in view of the G -isomorphisms

$$DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D$$

we get that $G/C_G(D) = G/C_G(N)$ is a p -group since G/D is σ -nilpotent by Lemma 2.1. But then Claim (d) implies that G is a p -group. This contradiction shows that we have Claim (e).

Final contradiction for Claim 5. In view of [15, Theorem A], G has a σ_1 -complement E such that $W = EG_p = G_pE$. Let $V = W^{\mathfrak{N}_\sigma}$. In view of Claims 2 and 4 and Lemma 2.6 (5), the hypothesis holds for W . Moreover, Claim (e) implies that $W \neq G$. Hence the conclusion of the theorem holds on W by the choice of G , which implies that V is a Hall subgroup of W . Moreover, Claim 4 implies that $V \leq D$, so for a Sylow p -subgroup V_p of V we have $|V_p| \leq |P| < |G_p|$. Hence V is a p' -group and so $V \leq C_G(D) \leq H_1 \cap W$ by Claim (d). Hence $V = 1$. Therefore $W = EG_p = E \times G_p$ is σ -nilpotent and so $E \leq C_G(D) \leq H_1$. Hence $E = 1$ and so $D = 1$, a contradiction. Thus, D is a Hall subgroup of G .

Claim 6. *If p is a prime such that $(p-1, |G|) = 1$, then p does not divide $|D|$. In particular, the smallest prime divisor of $|G|$ divides $|G : D|$.*

Assume that this is false and let P be the Sylow p -subgroup of D . Then, arguing similarly as in the proof of Claim 3, one can show that some maximal subgroup E of P is normal in G . Hence $C_G(D/E) = G$ since $(p-1, |G|) = 1$ by hypothesis. Since D is a Hall subgroup of G by Claim 5, it has a complement M in G . Hence $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is σ -nilpotent. Therefore G/E is σ -nilpotent. But then $D \leq E$, a contradiction. Hence p does not divide $|D|$. In particular, the smallest prime divisor of $|G|$ divides $|G : D|$.

Claim 7. *Every chief factor of G below D is cyclic.*

Suppose that this is false. Assume that $\Phi(D) \neq 1$ and let $R \leq \Phi(D)$. Claim 1 implies that every chief factor of G/R below $(G/R)^{\mathfrak{N}_\sigma} = D/R$ is cyclic, so every chief factor of G below D is cyclic by Lemma 2.8. Hence $\Phi(D) = 1$, so every Sylow subgroup of D is elementary. Moreover, there is a prime $p \in \pi(D)$ such that the Sylow p -subgroup P of D contains a minimal normal subgroup N of G

such that $|N| > p$. Let V be a maximal subgroup of P such that $P = NV$. Then $N \cap V \neq 1$. Since D is a Hall subgroup of G , $P \in \text{Syl}_p(G)$. Therefore V is τ_σ -quasinormal in G , so $N \cap V$ is σ -quasinormal in G by Lemma 2.6(2)–(3). Now, arguing similarly as in the proof of Claim 3, one can show that $N \cap V$ is normal in G . The minimality of N implies that $N \cap V = 1$, so $|N| = p$. This contradiction completes the proof of Claim 7.

Claims 3, 5, 6 and 7 show that the conclusion of the theorem holds for G , which contradicts the choice of G .

The theorem is proved. $\square$

Proof of Theorem A. It is enough to show that if G is a σ -soluble $T\sigma T$ -group and G is not σ -nilpotent, then conditions (i)–(iv) hold on G . Suppose that this is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume without loss of generality that H_i is a σ_i -group for all $i = 1, \dots, t$.

Claim 1. *Every subgroup H of G is τ_σ -quasinormal in G .*

It is enough to consider the case when H is a maximal subgroup of G . But since G is σ -soluble by hypothesis, $|G : H|$ is a σ_i -number for some i and so for a Hall σ_i -subgroup H_i of G we have $HH_i = G$. Hence $G = HH_i^x = H_i^x H$ for all $x \in G$. Thus we have Claim 1.

Claim 2. *We have $G = D \rtimes M$, where D is nilpotent and condition (i) holds for (D, M) .*

In view of the Schur–Zassenhaus theorem this directly follows from Theorem B and Claim 1.

Claim 3. *Every subgroup of D is normal in G . Hence D is abelian and every element of M induces a power automorphism in D .*

In view of Lemma 2.10 this follows from Claims 1 and 2.

Claim 4. *Condition (iv) holds on G .*

First note that since G is σ -soluble, it is a σ -full group of Sylow type by Lemma 2.4 and so every two Hall σ_k -subgroups of G are conjugate for all $\sigma_k \in \sigma(G)$.

Since G/D is σ -nilpotent, it follows that DA and DB are normal in G . Hence $A^G = A(A^G \cap D)$ and $B^G = B(B^G \cap D)$, so $A^G \cap B = 1$ and $B^G \cap A = 1$. Let $r \in \pi(D)$.

Claim (a). *If $r \in \pi(A^G)$ and C_r is a group of order r , then $C_r BA = AC_r B$ is a subgroup of G . If $r \in \pi(B^G)$, then $C_r AB = BC_r A$ is a subgroup of G .*

First note that $C_r \leq D$ since $r \in \pi(D)$ and D is a Hall subgroup of G by Claim 2. Therefore C_r is normal in G by Claim 3. Claim 1 implies that $C_r B$ is τ_σ -quasinormal in G . On the other hand, we have $\sigma(C_r B) \cap \sigma(A) = \emptyset$ and also $\sigma(C_r B) \cap \sigma(A^G) \neq \emptyset$ since $r \in \pi(A^G)$, so $(C_r B)A = A(C_r B)$ is a subgroup of G . Similarly one can get the second assertion of Claim (a).

Claim (b). *If $C_r \leq A^G$ and $C_r \not\leq B^G$ or $C_r \leq B^G$ and $C_r \not\leq A^G$, then $[A, B] = 1$.*

Assume, for example, that $C_r \leq A^G$ and $C_r \not\leq B^G$. Then

$$\begin{aligned} A^G \cap C_r B A &= C_r(A^G \cap B A) \\ &= C_r(A^G \cap B)A = C_r A, \end{aligned}$$

so $B \leq N_G(C_r A)$. On the other hand,

$$\begin{aligned} B^G \cap C_r B A &= B^G \cap A C_r B \\ &= (B^G \cap A C_r)B \\ &= (B^G \cap A)(B^G \cap C_r)B = B \end{aligned}$$

by Lemma 2.3, so $C_r A \leq N_G(B)$. Hence $[A, B] \leq [C_r A, B] \leq C_r A \cap B = 1$.

Claim (c). *If $C_r \leq A^G \cap B^G$ and $[A, B] \neq 1$, then $[A, B] = C_r$.*

Claim (a) implies that $C_r B A$ and $C_r A B$ are subgroups of G , so

$$C_r A B = C_r B A.$$

Hence

$$A^G \cap C_r A B = C_r(A^G \cap A B) = C_r A(A^G \cap B) = C_r A$$

and $B^G \cap C_r A B = C_r B$, so $C_r B \leq N_G(C_r A)$ and $C_r A \leq N_G(C_r B)$. Hence we have $[A, B] \leq [C_r A, C_r B] \leq C_r A \cap C_r B = C_r$, so $[A, B] = C_r$ since $[A, B] \neq 1$.

Claim (d). *If $C_t \leq [A, B]$, where C_t is a group of prime order t , then we have $C_t \leq A^G \cap B^G$, $t \in \pi(D)$ and $[A, B] = C_t$. Moreover, if the Sylow t -subgroup P of D is not cyclic, then $[A, P] = 1 = [B, P]$.*

Note that $C_t \leq A^G \cap B^G$ since $[A, B] \leq [A^G, B^G] \leq A^G \cap B^G$. But we have $A^G \cap B^G = A(A^G \cap D) \cap B(B^G \cap D)$. Hence $t \in \pi(D)$, so from Claim (c) we get $[A, B] = C_t$. Suppose that P is not cyclic. Then, by [8, Chapter I, Section 2.20], P possesses a subgroup $C_1 \neq C_t$ of order t since the smallest prime divisor of $|G|$ divides $|M|$ by Claim 2. Claim (c) implies that $C_1 \not\leq A^G$ or $C_1 \not\leq B^G$

since $[A, B] = C_t$. Hence we have $C_1 \not\leq A^G$ and $C_1 \not\leq B^G$ by Claim (b). Then $[C_1, A] = [C_1, B] = 1$. But every element $a \in A$ induces a power automorphism α in P , so in the case when α is non-trivial, it is fixed-point-free by [13, Section 13.4.3 (ii)]. Therefore every element of A acts trivially on P , so $[A, P] = 1$. Similarly we get $[B, P] = 1$.

Finally, suppose that $|[A, B]| = r$ is a prime and, also, A is a p -group and B is a q -group for some primes p and q . Then $[A, B] \leq D$ and $E = [A, B]AB$ is a subgroup of G by Claims (a) and (d). Let V be a Hall $\{p, q\}$ -subgroup of E . Then $V \simeq DV/D$ is nilpotent, so E is supersoluble. Assume that $p > r$. Then A or B is normal in E and so either $[A, B] \leq A$ or $[A, B] \leq B$, contrary to the fact that $1 < [A, B] \leq D$. Hence $r > p$. Similarly we get that $r > q$. Therefore we have (iv).

Thus conditions (i)–(iv) hold on G , contrary to our assumption on G . The theorem is proved. $\square$

Acknowledgments. The authors are grateful to the helpful suggestions of the referee.

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Received December 22, 2016.

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