9-1-2017

On τ_σ-Quasinormal Subgroups of Finite Groups

James C. Beidleman
University of Kentucky, clark@ms.uky.edu

Alexander N. Skiba
Francisk Skorina Gomel State University, Belarus

J. Group Theory **20** (2017), 955–969
DOI 10.1515/jgth-2017-0016

On τ_σ-quasinormal subgroups of finite groups

James C. Beidleman and Alexander N. Skiba

Communicated by Evgenii I. Khukhro

Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$.

Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \backslash \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \emptyset$ for a Hall σ_i-subgroup H of $G\}$. We say that a subgroup A of G is *τ_σ-permutable* or *τ_σ-quasinormal in G with respect to* $\mathcal{H}$ if $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$, and *$\tau_\sigma$-permutable* or *$\tau_\sigma$-quasinormal* in G if A is *τ_σ-permutable* in G with respect to some complete Hall σ-set of G.

We study G assuming that τ_σ-quasinormality is a transitive relation in G.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. The group G is called *π-supersoluble* provided every chief factor of G is either cyclic or a π'-group.

If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. The symbol H_G denotes the largest normal subgroup of G contained in $H \leq G$.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

The symbol $\sigma(n)$ denotes the set $\{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$. The group G is said to be

(i) *σ-primary* [16] if G is a σ_i-group for some $i \in I$,

(ii) *σ-decomposable* (Shemetkov [14]) or *σ-nilpotent* (Guo and Skiba [6]) if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$,

(iii) *σ-soluble* [16] if every chief factor of G is σ-primary.

A set $\mathcal{H}$ of subgroups of G is a *complete Hall σ-set* of G [15, 17] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$.

Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \setminus \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \emptyset$ for a Hall σ_i-subgroup $H \in \mathcal{H}\}$.

Definition 1.1. We say that a subgroup A of G is

(i) τ_σ*-permutable* or τ_σ*-quasinormal in* G *with respect to* $\mathcal{H}$ if $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$,

(ii) τ_σ*-permutable* or τ_σ*-quasinormal* in G if A is τ_σ*-permutable* in G with respect to some complete Hall σ-set $\mathcal{H}$ of G.

Recall that a subgroup A of G is said to be

(i) σ*-permutable* or σ*-quasinormal* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$ (cf. [16]),

(ii) σ*-semipermutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ with $\sigma(A) \cap \sigma(H) = \emptyset$ (cf. [7]).

In the classical case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, σ-permutable, σ-semipermutable and τ_σ-quasinormal subgroups are also called respectively S*-permutable* [1, 5], S*-semipermutable* [5] and τ*-quasinormal* [11, 12].

It is clear that every σ-quasinormal subgroup is also σ-semipermutable and every σ-semipermutable subgroup is τ_σ-quasinormal.

Example 1.2. Let $p > q > r$ be primes, C_r a group of order r and $H = Q \rtimes C_r$, where Q is a simple $\mathbb{F}_q C_r$-module which is faithful for C_r. Let $G = P \rtimes H$, where P is a simple $\mathbb{F}_p H$-module which is faithful for H. Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{p\}$, $\sigma_2 = \{q\}$, and $\sigma_3 = \{p, q\}'$. Then every Sylow r-subgroup of G is τ_σ-quasinormal but not σ-semipermutable. A Hall $\{q, r\}$-subgroup of G is σ-semipermutable in G but not σ-quasinormal.

We say that G is a $T\sigma T$*-group* if τ_σ-quasinormality is a transitive relation in G, that is, if K is a τ_σ-quasinormal subgroup of H and H is a τ_σ-quasinormal subgroup of G, then K is a τ_σ-quasinormal subgroup of G. Our purpose here is to establish the structure of σ-soluble $T\sigma T$-groups.

Theorem A. *Let* $D = G^{\mathfrak{N}_\sigma}$ *and* $\pi = \pi(D)$. *Suppose that* G *possesses a complete Hall* σ*-set* $\mathcal{H}$ *such that all members of* $\mathcal{H}$ *are* π*-supersoluble. Then* G *is a* σ*-soluble* $T\sigma T$*-group if and only if either* G *is* σ*-nilpotent or* $G = D \rtimes M$*, where:*

(i) D *and* M *are Hall subgroups of* G *and the smallest prime divisor of* $|G|$ *divides* $|M|$.

(ii) D *is abelian and every element of* M *induces a power automorphism in* D.

(iii) *Every subgroup of G is τ_σ-quasinormal in G.*

(iv) *If A and B are respectively a Hall σ_i-subgroup and a Hall σ_j-subgroup of G, where $i \neq j$ and $\sigma_i, \sigma_j \in \sigma(G) \setminus \sigma(D)$, then the order of $[A, B]$ divides a prime. Moreover, if $|[A, B]| = r \neq 1$, then $r \in \pi(D)$ and either the Sylow r-subgroup R of G is cyclic or $[A, R] = 1 = [B, R]$; if, also, A is a p-group and B is a q-group for some primes p and q, then $r > p$ and $r > q$.*

In this theorem $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual of G*, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

One of the main objectives of this paper is to give a correct proof of [12, Theorem 1.1]. The proof given in [12] for this theorem has several gaps. However, Theorem A and its proof allows us to eliminate all of those gaps. Corollary 1.3, our next result, is a statement of [12, Theorem 1.1].

Recall that G is said to be a *TQT-group* [12] if τ-quasinormality is a transitive relation in G.

Corollary 1.3. *Let P be a Sylow p-subgroup of G, Q a Sylow q-subgroup of G such that $p \neq q$. The following statements are equivalent:*

(1) *G is a soluble TQT-group.*

(2) *G is a supersoluble group which has an abelian normal Hall subgroup of odd order D such that G/D is nilpotent, every subgroup of D is normal in G, and every subgroup of G is τ-quasinormal in G. Moreover, if $p \notin \pi(D)$ and $q \notin \pi(D)$, then the order of $[P, Q]$ divides r, where $r \in \pi(D)$, and if $|[P, Q]| = r$, then $r > p$, $r > q$ and either the Sylow r-subgroup R of G is cyclic or $[P, R] = 1 = [Q, R]$.*

The following observation covers many steps in the proof of Theorem A.

Theorem B. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ such that all members of $\mathcal{H}$ are π-supersoluble. If all maximal subgroups of every Sylow p-subgroup of G are τ_σ-quasinormal in G for all $p \in \pi$, then D is a nilpotent Hall subgroup of G, the smallest prime divisor of $|G|$ divides $|G : D|$ and every chief factor of G below D is cyclic.*

Corollary 1.4 (Srinivasan [18]). *If every maximal subgroup of every Sylow subgroup of G is S-permutable in G, then G is supersoluble.*

2 Preliminaries

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

Lemma 2.1 (see [16, Lemma 2.5]). *The class $\mathfrak{N}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/E \cap \Phi(G)$ is σ-nilpotent, then E is σ-nilpotent.*

In view of [2, Proposition 2.2.8], we get from Lemma 2.1 the following:

Lemma 2.2. *If N is a normal subgroup of G, then*

$$(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N.$$

Lemma 2.3 (Knyagina and Monakhov [10]). *Let H, K and N be pairwise permutable subgroups of G and H is a Hall subgroup of G. Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Suppose that G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$. For any subgroup H of G we write $H \cap \mathcal{H}$ to denote the set $\{H \cap H_1, \ldots, H \cap H_t\}$. If $H \cap \mathcal{H}$ is a complete Hall σ-set of H, then we say that $\mathcal{H}$ *reduces into H*.

Recall that G is said to be

(i) a D_π-group if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E,

(ii) a *σ-full group of Sylow type* [16] if every subgroup E of G is a D_{σ_i}-group for every $\sigma_i \in \sigma(E)$,

(iii) *σ-full* [15, 17] provided G possesses a complete Hall σ-set.

In view of [15, Theorems A and B], the following fact is true.

Lemma 2.4. *If G is σ-soluble, then G is a σ-full group of Sylow type.*

Lemma 2.5 (see [16, Lemma 3.1]). *Let H be a σ_i-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_i}(G) \leq N_G(H)$.*

Lemma 2.6. *Suppose that G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that the subgroups H and K of G are τ_σ-quasinormal in G with respect to $\mathcal{H}$. Let R be a normal subgroup of G and $H \leq L \leq G$. Then:*

(1) *$\mathcal{H}_0 = \{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R. Moreover, if $\sigma(H) = \sigma(HR/R)$, then HR/R is τ_σ-quasinormal in G/N with respect to $\mathcal{H}_0$.*

(2) *If $HK = KH$ and $\sigma(H \cap K) = \sigma(H) = \sigma(K)$, then $H \cap K$ is τ_σ-quasinormal in G with respect to $\mathcal{H}$.*

(3) *If for some i we have $H \leq O_{\sigma_i}(G)$, then H is σ-quasinormal in G.*

(4) *If $\mathscr{H}$ reduces into L, then H is τ_σ-quasinormal in L with respect to $L \cap \mathscr{H}$.*

(5) *If G is a σ-full group of Sylow type, then H is τ_σ-quasinormal in L.*

Proof. Without loss of generality we can assume that H_i is a σ_i-group for all $i = 1, \dots, t$.

(1) It is clear that $\mathscr{H}_0 = \{H_1R/R, \dots, H_tR/R\}$ is a complete Hall σ-set of G/R. Now let $\sigma_i \in \tau_{\mathscr{H}_0}(HR/R)$, that is, $\sigma_i \notin \sigma(HR/R)$ and

$$\sigma(HR/R) \cap \sigma((H_iR/R)^G) \neq \emptyset.$$

Then $\sigma_i \notin \sigma(H)$ since $\sigma(H) = \sigma(HR/R)$. On the other hand, we have

$$(H_iR/R)^G = H_i^G R/R \simeq H_i^G/(H_i^G \cap R),$$

so $\sigma(H) \cap \sigma(H_i^G) \neq \emptyset$. Hence $\sigma_i \in \tau_{\mathscr{H}}(H)$ and so

$$\begin{aligned}(HR/R)(H_iR/R) &= HH_iR/R \\ &= H_iHR/R \\ &= (H_iR/R)(HR/R).\end{aligned}$$

Thus HR/R is τ_σ-quasinormal in G/R with respect to $\mathscr{H}_0$.

(2) Let H_i be a member of $\mathscr{H}$ such that $\sigma_i \in \tau_{\mathscr{H}}(H \cap K)$, that is, we have $\sigma_i \in \sigma(G) \setminus \sigma(H \cap K)$ and $\sigma(H \cap K) \cap \sigma(H_i^G) \neq \emptyset$. Then $\sigma_i \in \sigma(G) \setminus \sigma(H)$ (since $\sigma(H \cap K) = \sigma(H)$) and $\sigma(H) \cap \sigma(H_i^G) \neq \emptyset$, so $\sigma_i \in \tau_{\mathscr{H}}(H)$. Similarly we get that $\sigma_i \in \tau_{\mathscr{H}}(K)$. Hence $HH_i^x = H_i^xH$ and $KH_i^x = H_i^xK$ for all $x \in G$. It is clear also that $H \cap H_i^x = 1$. Therefore for every $x \in G$ we have

$$\begin{aligned}HH_i^x \cap KH_i^x &= H_i^x(H \cap KH_i^x) \\ &= H_i^x(H \cap K)(H \cap H_i^x) \\ &= H_i^x(H \cap K) \\ &= (H \cap K)H_i^x\end{aligned}$$

by Lemma 2.3. Hence $H \cap K$ is τ_σ-quasinormal in G with respect to $\mathscr{H}$.

(3) Let $j \neq i$. Suppose that for some $x \in G$ we have $HH_j^x \neq H_j^xH$. Then $\sigma_j \notin \tau_{\mathscr{H}}(H)$. Hence $\sigma(H) \cap \sigma(H_j^G) = \emptyset$ since H is τ_σ-quasinormal in G with respect to $\mathscr{H}$ by hypothesis. But then H_j^G is a σ_i'-group, so $H \leq O_{\sigma_i}(G) \leq C_G(H_j^G)$, which implies that $HH_j^x = H_j^xH$. This contradiction shows that we have (3).

(4) Let $L_i = H_i \cap L$ for all $i = 1, \dots, t$ and $\mathscr{L} = \{L_1, \dots, L_t\}$. By hypothesis, $\mathscr{L}$ is a complete Hall σ-set of L. Let $\sigma_i \in \tau_{\mathscr{L}}(H)$, that is, $\sigma_i \in \sigma(L) \setminus \sigma(H)$ and $\sigma(H) \cap \sigma((L_i)^L) \neq \emptyset$. Then $\sigma_i \in \sigma(G) \setminus \sigma(H)$ and $\sigma(H) \cap \sigma((H_i)^G) \neq \emptyset$

since $(L_i)^L \leq L_i^G \leq (H_i \cap L)^G \leq H_i^G$. Hence $\sigma_i \in \tau_{\mathcal{H}}(H)$, so $HH_i^a = H_i^a H$ for all $a \in L$, so $L \cap HH_i^a = H(L \cap H_i^a) = H(L \cap H_i)^a = HL_i^a = L_i^a H$. This shows that H is τ_σ-quasinormal in L with respect to $L \cap \mathcal{H}$.

(5) Since G is a σ-full group of Sylow type, H is τ_σ-quasinormal in G with respect to each complete Hall σ-set of G. Moreover, this condition implies also that some complete Hall σ-set of G reduces into L, so we (5) by part (4).

The lemma is proved. □

Lemma 2.7 (Kegel [9]). *Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^x A$ for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

The following lemma is a corollary of [4, Chapter IV, (6.7)] (see also [3, Lemma 2.12]).

Lemma 2.8. *Let $N \leq E$ be normal subgroups of G such that $N \leq \Phi(E)$ and every chief factor of G between E and N is cyclic. Then every chief factor of G below E is cyclic.*

Lemma 2.9. *If $\mathcal{H} = \{H_1, \dots, H_t\}$ is a complete Hall σ-set of G and every member of $\mathcal{H}$ is $\pi(G^{\mathfrak{N}_\sigma})$-supersoluble, then*

$$\mathcal{H}_0 = \{H_1N/N, \dots, H_tN/N\}$$

is a complete Hall σ-set of G/N and every member of $\mathcal{H}_0$ is $\pi((G/N)^{\mathfrak{N}_\sigma})$-supersoluble.

Proof. It is clear that $\mathcal{H}_0$ is a complete Hall σ-set of G/N. Now let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Then $(G/N)^{\mathfrak{N}_\sigma} = DN/N$ by Lemma 2.2, so

$$\pi_0 = \pi((G/N)^{\mathfrak{N}_\sigma}) = \pi(DN/N) = \pi(D/D \cap N) \subseteq \pi(D) = \pi.$$

Hence every member H_i of $\mathcal{H}$ is π_0-supersoluble, so H_iN/N is π_0-supersoluble. The lemma is proved. □

Lemma 2.10. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that D is a nilpotent Hall subgroup of G and G possesses a complete Hall σ-set $\mathcal{H}$ such that all members of $\mathcal{H}$ are π-supersoluble. If every σ_i-subgroup of G is τ_σ-quasinormal in G for all $\sigma_i \in \sigma(D)$, then every subgroup of D is normal in G.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \dots, t$.

We show that the hypothesis holds on G/N for every minimal normal subgroup N of G. First note that

$$(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/D \cap N$$

is a nilpotent Hall subgroup of G/N by Lemma 2.2, and G/N possesses a complete Hall σ-set $\mathcal{H}_0$ such that all members of $\mathcal{H}_0$ are $\pi((G/N)^{\mathfrak{N}_\sigma})$-supersoluble by Lemma 2.9.

Now let V/N be a non-identity σ_i-subgroup of G/N for some

$$\sigma_i \in \sigma((G/N)^{\mathfrak{N}_\sigma}) = \sigma(DN/N) = \sigma(D/D \cap N) \subseteq \sigma(D).$$

And let U be a minimal supplement to N in V. Then $U \cap N \leq \Phi(U)$, so U is a σ_i-subgroup of G since $V/N = UN/N \simeq U/U \cap N$. Therefore U is τ_σ-quasinormal in G by hypothesis and $\sigma(U) = \sigma(UN/N) = \{\sigma_i\}$, which implies that $V/N = UN/N$ is τ_σ-quasinormal in G/N by Lemma 2.6 (1). Hence the hypothesis holds on G/N.

Let H be a subgroup of the Sylow p-subgroup P of D for some prime $p \in \pi$. We show that H is normal in G. For some i we have $P \leq O_{\sigma_i}(D) = H_i \cap D$. On the other hand, we have $D = O_{\sigma_i}(D) \times O^{\sigma_i}(D)$ since D is nilpotent. Assume that $O^{\sigma_i}(D) \neq 1$ and let N be a minimal normal subgroup of G contained in $O^{\sigma_i}(D)$. Then $HN/N \leq DN/N = (G/N)^{\mathfrak{N}_\sigma}$, so the choice of G implies that HN/N is normal in G/N. Hence $H = H(N \cap O_{\sigma_i}(D)) = HN \cap O_{\sigma_i}(D)$ is normal in G.

Now assume that $O^{\sigma_i}(D) = 1$, so D is a σ_i-group. Since G/D is σ-nilpotent by Lemma 2.1, H_i/D is normal in G/D and hence H_i is normal in G. Therefore all subgroups of H_i are σ-quasinormal in G by Lemma 2.6 (3) and hypothesis. Since D is a normal Hall subgroup of H_i, it has a complement S in H_i by the Schur–Zassenhaus theorem. Lemma 2.5 implies that $D \leq O^{\sigma_i}(G) \leq N_G(S)$. Hence $H_i = D \times S$. Therefore

$$G = H_i O^{\sigma_i}(G) = SO^{\sigma_i}(G) \leq N_G(H),$$

so H is normal in G.

Therefore every subgroup of D is normal in G since D is nilpotent by hypothesis. The lemma is proved. □

3 Proof of Theorems A and B

Proof of Theorem B. Suppose that this theorem is false and let G be a counterexample of minimal order. Then $D \neq 1$. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$. Let R be a minimal normal subgroup of G.

Claim 1. *The conclusion of the theorem holds for G/R.*

Let V/R be a maximal subgroup of a Sylow p-subgroup P/R of G/R, where $p \in \pi((G/R)^{\mathfrak{N}_\sigma})$. Then for some Sylow p-subgroup G_p of G we have $P/R = G_pR/R$ and $V = R(V \cap G_p)$. It is clear that $V \cap G_p$ is a maximal subgroup of G_p. Therefore $V \cap G_p$ is τ_σ-quasinormal in G by hypothesis since

$$\pi((G/R)^{\mathfrak{N}_\sigma}) = \pi(DR/R) = \pi(D/D \cap R) \subseteq \pi$$

by Lemma 2.2, so $V/R = R(V \cap G_p)/R$ is τ_σ-quasinormal in G/R by Lemma 2.6 (1). Consequently, the hypothesis holds for G/R by Lemma 2.9. Hence we have Claim 1 by the choice of G.

Claim 2. *The group D is soluble, so G is σ-soluble. Hence G is a σ-full group of Sylow type.*

Assume that this is false. By Claim 1 and Lemma 2.2,

$$(G/R)^{\mathfrak{N}_\sigma} = DR/R \simeq D/D \cap R$$

is nilpotent. Hence $R \leq D$. Moreover, if G has a minimal normal subgroup $N \neq R$, then $N \leq D$ and $D \simeq D/R \cap N = D/1$ is nilpotent. Therefore $C_G(R) = 1$. Then 2 divides $|R|$ by the Feit–Thompson theorem, and a Sylow 2-subgroup Q of R is not cyclic by [8, Chapter IV, Section 2.8]. Hence $|Q| > 2$.

Let P be a Sylow 2-subgroup of G such that $Q = P \cap R$. Then for some maximal subgroup V of P we have $Q \nleq V$ by the Tate theorem [8, Chapter IV, Section 4.7], which implies that $P = QV$ and so $V \cap R < P \cap R = Q$. Moreover, $V \cap R \neq 1$ since otherwise we have $V \cap R = P \cap V \cap R = Q \cap V = 1$ and so $|Q| = 2$. Since $R = R_1 \times \cdots \times R_n$, where $R_1 \simeq \cdots \simeq R_n$ are non-abelian simple groups, it follows that $Q = (P \cap R_1) \times \cdots \times (P \cap R_n)$ and so for some i we have $V \cap R_i < P \cap R_i$. Note also that $V \cap R_i \neq 1$ since otherwise from the isomorphism

$$V(P \cap R_i)/V \simeq (P \cap R_i)/(V \cap (P \cap R_i)) = P \cap R_i/1$$

we get that the order of a Sylow 2-subgroup of $P \cap R_i$ divides 2 and so $P \cap R_i$ is 2-nilpotent by [8, Chapter IV, Section 2.8], which implies that R is 2-nilpotent.

Assume that $2 \in \sigma_k$. First we show that R is σ-primary. Suppose that this is false. We can assume without loss of generality that V is τ_σ-quasinormal in G with respect to $\mathcal{H}$. Then for some $j \neq k$ and for $H = H_j$ we have $H \cap R_i \neq 1$ since R is not σ-primary. Note also that $\sigma_k \in \sigma(H^G)$ since otherwise we have $R \cap H^G = 1$, which implies that $1 < H^G \leq C_G(R) = 1$. Therefore $\sigma_k \in \tau_{\mathcal{H}}(V)$, so $VH^x = H^xV$ for all $x \in G$. By [4, Chapter A, Section 14.1 (a)], we have $L = VH^x \cap R_i$ is a subnormal subgroup of VH^x, where V is a Hall σ_k-subgroup of VH^x and H^x is a Hall σ_j-subgroup of VH^x. Therefore, $L = (L \cap V)(L \cap H^x)$

by [4, Chapter I, Section 3.2]. Hence

$$\begin{aligned} L &= (L \cap V)(L \cap H^x) \\ &= (VH^x \cap R_i \cap V)(VH^x \cap R_i \cap H^x) \\ &= (R_i \cap V)(R_i \cap H^x) \\ &= (V \cap R_i)(H \cap R_i)^x \\ &= (H \cap R_i)^x(V \cap R_i) \end{aligned}$$

for all $x \in R_i$, where $(H \cap R_i)(V \cap R_i) \neq R_i$ since $V \cap R_i < P \cap R_i$. Therefore, R_i is not simple by Lemma 2.7 since $H \cap R_i \neq 1$ and $V \cap R_i \neq 1$. This contradiction shows that R is σ-primary.

Now assume that $R \leq D$ and R is not abelian. Then

$$\sigma(V \cap R) = \sigma(V) = \sigma(R) = \{\sigma_k\}$$

since R is σ-primary, $2 \in \sigma_k$ and $V \cap R \neq 1$. Therefore $V \cap R$ is τ_σ-quasinormal in G by Lemma 2.6 (2). But $V \cap R \leq R \leq O_{\sigma_k}(G)$ and so $V \cap R$ is σ-quasinormal in G by Lemma 2.6 (3). Hence $R \leq N_G(V \cap R)$ by Lemma 2.5 since $R \leq D \leq O^{\sigma_i}(G)$. Therefore $V \cap R \leq O_2(R) = 1$, a contradiction. Thus R is abelian. Hence D is soluble by Claim 1. Therefore G is σ-soluble and so G is a σ-full group of Sylow type by Lemma 2.4.

Claim 3. *The group D is nilpotent.*

Assume that this is false. Note that $RD/R = (G/R)^{\mathfrak{N}}$ is nilpotent by Claim 1. Therefore $R \leq D$, R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Lemma 2.1. Claim 2 implies that R is a p-group for some prime p. Therefore $R = C_G(R)$ by [4, Chapter A, Section 15.2], and $G = R \rtimes M$ for some maximal subgroup M of G. If $|R| = p$, then $G/C_G(R) = G/R$ is a cyclic group. Hence G is supersoluble and therefore D is nilpotent, which contradicts our assumption on G. Therefore $|R| > p$.

For some i we have $R \leq H_i \cap D$. Then $H_i = R \rtimes (H_i \cap M)$ and H_i is π-supersoluble by hypothesis. It follows that some maximal subgroup V of R is normal in H_i. Let P be a Sylow p-subgroup of $H_i \cap M$. Then $RP \in \mathrm{Syl}_p(G)$ and VP is a maximal subgroup of RP, so VP is τ_σ-quasinormal in G by hypothesis. Then $V = V(R \cap P) = R \cap VP$ is σ-quasinormal in G by Lemma 2.6 (2)–(3). Therefore $O^{\sigma_i}(G) \leq N_G(V)$ by Lemma 2.5. Hence $G = H_i O^{\sigma_i}(G) \leq N_G(V)$. The minimality of R implies that $V = 1$, so $|R| = p$, a contradiction. Hence we have Claim 3.

Claim 4. *If E is a subgroup of G, then $E^{\mathfrak{N}} \leq D$.*

Note that since $E/E \cap D \simeq ED/D \in \mathfrak{N}$ and $\mathfrak{N}$ is a hereditary class by Lemma 2.1, $E/E \cap D \in \mathfrak{N}$. Hence $E^{\mathfrak{N}} \leq E \cap D$.

Claim 5. *The group D is a Hall subgroup of G.*

Suppose that this is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p \in \mathrm{Syl}_p(G)$. We can assume without loss of generality that $G_p \leq H_1$ and that $R \leq D$.

Claim (a). *The group $D = P$ is a minimal normal subgroup of G.*

Since D is nilpotent by Claim 3, it follows that R is a q-group for some prime q. Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G/R by Claim 1. Suppose that $PR/R \neq 1$. Then $PR/R \in \mathrm{Syl}_p(G/R)$. If $q \neq p$, then $P \in \mathrm{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence we have $q = p$ and so $R \leq P$, therefore $P/R \in \mathrm{Syl}_p(G/R)$ and we again get that $P \in \mathrm{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of G contained in D. Since D is nilpotent by Claim 3, a p'-complement E of D is characteristic in D and so it is normal in G. Hence $E = 1$, which implies that $R = D = P$.

Claim (b). *We have $D \nleq \Phi(G)$. Hence for some maximal subgroup M of G we have $G = D \rtimes M$.*

This follows from Claim 2 and Lemma 2.1 since G is not σ-nilpotent

Claim (c). *If G has a minimal normal subgroup $L \neq D$, then $G_p = D \times (L \cap G_p)$. Hence $O_{p'}(G) = 1$.*

Indeed, $DL/L \simeq D$ is a Hall subgroup of G/L by Claims 2 and (a). Hence $G_pL/L = DL/L$, so $G_p = D \times (L \cap G_p)$. Thus $O_{p'}(G) = 1$ since $D < G_p$ by Claim (a).

Claim (d). *The group $V = C_G(D) \cap M$ is a normal subgroup of G and we have $C_G(D) = D \times V \leq H_1$.*

In view of Claims (a) and (b), $C_G(D) = D \times V$, where $V = C_G(D) \cap M$ is a normal subgroup of G. By Claim (a), $V \cap D = 1$ and hence $V \simeq DV/D$ is σ-nilpotent by Lemma 2.1. Let W be a σ_1-complement of V. Then W is characteristic in V and so it is normal in G. Therefore we have Claim (d) by Claim (c).

Claim (e). *We have $G_p \neq H_1$.*

Assume that $G_p = H_1$. Let Z be a subgroup of order p in $Z(G_p) \cap D$. Then, since $D \leq O^{\sigma_1}(G) = O^p(G)$, Z is normal in G by Lemma 2.5. Hence we have

$D = Z < G_p$ and so $D < C_G(D)$. Then $V = C_G(D) \cap M \neq 1$ is a normal subgroup of G and $V \leq H_1 = G_p$ by Claim (d). Let L be a minimal normal subgroup of G contained in V. Then $G_p = D \times L$ is a normal elementary abelian subgroup of G by Claim (c). Hence every maximal subgroup of G_p is normal in G by Lemmas 2.6 (3) and 2.5. It follows that every subgroup of G_p is normal in G. Hence $|D| = |L| = p$. Let $D = \langle a \rangle$, $L = \langle b \rangle$ and $N = \langle ab \rangle$. Then $N \not\leq D$, so in view of the G-isomorphisms

$$DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D$$

we get that $G/C_G(D) = G/C_G(N)$ is a p-group since G/D is σ-nilpotent by Lemma 2.1. But then Claim (d) implies that G is a p-group. This contradiction shows that we have Claim (e).

Final contradiction for Claim 5. In view of [15, Theorem A], G has a σ_1-complement E such that $W = EG_p = G_pE$. Let $V = W^{\mathfrak{N}_\sigma}$. In view of Claims 2 and 4 and Lemma 2.6 (5), the hypothesis holds for W. Moreover, Claim (e) implies that $W \neq G$. Hence the conclusion of the theorem holds on W by the choice of G, which implies that V is a Hall subgroup of W. Moreover, Claim 4 implies that $V \leq D$, so for a Sylow p-subgroup V_p of V we have $|V_p| \leq |P| < |G_p|$. Hence V is a p'-group and so $V \leq C_G(D) \leq H_1 \cap W$ by Claim (d). Hence $V = 1$. Therefore $W = EG_p = E \times G_p$ is σ-nilpotent and so $E \leq C_G(D) \leq H_1$. Hence $E = 1$ and so $D = 1$, a contradiction. Thus, D is a Hall subgroup of G.

Claim 6. *If p is a prime such that $(p-1, |G|) = 1$, then p does not divide $|D|$. In particular, the smallest prime divisor of $|G|$ divides $|G : D|$.*

Assume that this is false and let P be the Sylow p-subgroup of D. Then, arguing similarly as in the proof of Claim 3, one can show that some maximal subgroup E of P is normal in G. Hence $C_G(D/E) = G$ since $(p-1, |G|) = 1$ by hypothesis. Since D is a Hall subgroup of G by Claim 5, it has a complement M in G. Hence $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is σ-nilpotent. Therefore G/E is σ-nilpotent. But then $D \leq E$, a contradiction. Hence p does not divide $|D|$. In particular, the smallest prime divisor of $|G|$ divides $|G : D|$.

Claim 7. *Every chief factor of G below D is cyclic.*

Suppose that this is false. Assume that $\Phi(D) \neq 1$ and let $R \leq \Phi(D)$. Claim 1 implies that every chief factor of G/R below $(G/R)^{\mathfrak{N}_\sigma} = D/R$ is cyclic, so every chief factor of G below D is cyclic by Lemma 2.8. Hence $\Phi(D) = 1$, so every Sylow subgroup of D is elementary. Moreover, there is a prime $p \in \pi(D)$ such that the Sylow p-subgroup P of D contains a minimal normal subgroup N of G

such that $|N| > p$. Let V be a maximal subgroup of P such that $P = NV$. Then $N \cap V \neq 1$. Since D is a Hall subgroup of G, $P \in \mathrm{Syl}_p(G)$. Therefore V is τ_σ-quasinormal in G, so $N \cap V$ is σ-quasinormal in G by Lemma 2.6 (2)–(3). Now, arguing similarly as in the proof of Claim 3, one can show that $N \cap V$ is normal in G. The minimality of N implies that $N \cap V = 1$, so $|N| = p$. This contradiction completes the proof of Claim 7.

Claims 3, 5, 6 and 7 show that the conclusion of the theorem holds for G, which contradicts the choice of G.

The theorem is proved. □

Proof of Theorem A. It is enough to show that if G is a σ-soluble $T\sigma T$-group and G is not σ-nilpotent, then conditions (i)–(iv) hold on G. Suppose that this is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \dots, t$.

Claim 1. *Every subgroup H of G is τ_σ-quasinormal in G.*

It is enough to consider the case when H is a maximal subgroup of G. But since G is σ-soluble by hypothesis, $|G : H|$ is a σ_i-number for some i and so for a Hall σ_i-subgroup H_i of G we have $HH_i = G$. Hence $G = HH_i^x = H_i^x H$ for all $x \in G$. Thus we have Claim 1.

Claim 2. *We have $G = D \rtimes M$, where D is nilpotent and condition* (i) *holds for* (D, M).

In view of the Schur–Zassenhaus theorem this directly follows from Theorem B and Claim 1.

Claim 3. *Every subgroup of D is normal in G. Hence D is abelian and every element of M induces a power automorphism in D.*

In view of Lemma 2.10 this follows from Claims 1 and 2.

Claim 4. *Condition* (iv) *holds on G.*

First note that since G is σ-soluble, it is a σ-full group of Sylow type by Lemma 2.4 and so every two Hall σ_k-subgroups of G are conjugate for all $\sigma_k \in \sigma(G)$.

Since G/D is σ-nilpotent, it follows that DA and DB are normal in G. Hence $A^G = A(A^G \cap D)$ and $B^G = B(B^G \cap D)$, so $A^G \cap B = 1$ and $B^G \cap A = 1$. Let $r \in \pi(D)$.

Claim (a). *If $r \in \pi(A^G)$ and C_r is a group of order r, then $C_r BA = AC_r B$ is a subgroup of G. If $r \in \pi(B^G)$, then $C_r AB = BC_r A$ is a subgroup of G.*

First note that $C_r \leq D$ since $r \in \pi(D)$ and D is a Hall subgroup of G by Claim 2. Therefore C_r is normal in G by Claim 3. Claim 1 implies that $C_r B$ is τ_σ-quasinormal in G. On the other hand, we have $\sigma(C_r B) \cap \sigma(A) = \emptyset$ and also $\sigma(C_r B) \cap \sigma(A^G) \neq \emptyset$ since $r \in \pi(A^G)$, so $(C_r B)A = A(C_r B)$ is a subgroup of G. Similarly one can get the second assertion of Claim (a).

Claim (b). *If* $C_r \leq A^G$ *and* $C_r \nleq B^G$ *or* $C_r \leq B^G$ *and* $C_r \nleq A^G$*, then* $[A, B] = 1$.

Assume, for example, that $C_r \leq A^G$ and $C_r \nleq B^G$. Then

$$\begin{aligned} A^G \cap C_r BA &= C_r(A^G \cap BA) \\ &= C_r(A^G \cap B)A = C_r A, \end{aligned}$$

so $B \leq N_G(C_r A)$. On the other hand,

$$\begin{aligned} B^G \cap C_r BA &= B^G \cap AC_r B \\ &= (B^G \cap AC_r)B \\ &= (B^G \cap A)(B^G \cap C_r)B = B \end{aligned}$$

by Lemma 2.3, so $C_r A \leq N_G(B)$. Hence $[A, B] \leq [C_r A, B] \leq C_r A \cap B = 1$.

Claim (c). *If* $C_r \leq A^G \cap B^G$ *and* $[A, B] \neq 1$*, then* $[A, B] = C_r$.

Claim (a) implies that $C_r BA$ and $C_r AB$ are subgroups of G, so

$$C_r AB = C_r BA.$$

Hence

$$A^G \cap C_r AB = C_r(A^G \cap AB) = C_r A(A^G \cap B) = C_r A$$

and $B^G \cap C_r AB = C_r B$, so $C_r B \leq N_G(C_r A)$ and $C_r A \leq N_G(C_r B)$. Hence we have $[A, B] \leq [C_r A, C_r B] \leq C_r A \cap C_r B = C_r$, so $[A, B] = C_r$ since $[A, B] \neq 1$.

Claim (d). *If* $C_t \leq [A, B]$*, where* C_t *is a group of prime order* t*, then we have* $C_t \leq A^G \cap B^G$*,* $t \in \pi(D)$ *and* $[A, B] = C_t$*. Moreover, if the Sylow* t*-subgroup* P *of* D *is not cyclic, then* $[A, P] = 1 = [B, P]$.

Note that $C_t \leq A^G \cap B^G$ since $[A, B] \leq [A^G, B^G] \leq A^G \cap B^G$. But we have $A^G \cap B^G = A(A^G \cap D) \cap B(B^G \cap D)$. Hence $t \in \pi(D)$, so from Claim (c) we get $[A, B] = C_t$. Suppose that P is not cyclic. Then, by [8, Chapter I, Section 2.20], P possesses a subgroup $C_1 \neq C_t$ of order t since the smallest prime divisor of $|G|$ divides $|M|$ by Claim 2. Claim (c) implies that $C_1 \nleq A^G$ or $C_1 \nleq B^G$

since $[A, B] = C_t$. Hence we have $C_1 \nleq A^G$ and $C_1 \nleq B^G$ by Claim (b). Then $[C_1, A] = [C_1, B] = 1$. But every element $a \in A$ induces a power automorphism α in P, so in the case when α is non-trivial, it is fixed-point-free by [13, Section 13.4.3 (ii)]. Therefore every element of A acts trivially on P, so $[A, P] = 1$. Similarly we get $[B, P] = 1$.

Finally, suppose that $|[A, B]| = r$ is a prime and, also, A is a p-group and B is a q-group for some primes p and q. Then $[A, B] \leq D$ and $E = [A, B]AB$ is a subgroup of G by Claims (a) and (d). Let V be a Hall $\{p, q\}$-subgroup of E. Then $V \simeq DV/D$ is nilpotent, so E is supersoluble. Assume that $p > r$. Then A or B is normal in E and so either $[A, B] \leq A$ or $[A, B] \leq B$, contrary to the fact that $1 < [A, B] \leq D$. Hence $r > p$. Similarly we get that $r > q$. Therefore we have (iv).

Thus conditions (i)–(iv) hold on G, contrary to our assumption on G. The theorem is proved. □

Acknowledgments. The authors are grateful to the helpful suggestions of the referee.

Bibliography

[1] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of Finite Groups*, Walter de Gruyter, Berlin, 2010.

[2] A. Ballester-Bolinches and L. M. Ezquerro, *Classes of Finite Groups*, Springer, New York, 2006.

[3] X. Chen, W. Guo and A. N. Skiba, Some conditions under which a finite group belongs to a Baer-local formation, *Comm. Algebra* **42** (2014), 4188–4203.

[4] K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin, 1992.

[5] W. Guo, *Structure Theory for Canonical Classes of Finite Groups*, Springer, Berlin, 2015.

[6] W. Guo and A. N. Skiba, Finite groups with permutable complete Wielandt sets of subgroups, *J. Group Theory* **18** (2015), 191–200.

[7] W. Guo and A. N. Skiba, On σ-semipermutable subgroups of finite groups, preprint (2016), `http://arxiv.org/abs/1609.08815v1`.

[8] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.

[9] O. H. Kegel, Produkte nilpotenter Gruppen, *Arch. Math.* **12** (1961), 90–93.

[10] V. N. Knyagina and V. S. Monakhov, On π'-properties of finite groups having a Hall π-subgroup, *Siberian Math. J.* **522** (2011), 398–309.

[11] V. O. Lukyanenko and A. N. Skiba, On weakly τ-quasinormal subgroups of finite groups, *Acta Math. Hungar.* **125** (2009), no. 3, 237–248.

[12] V. O. Lukyanenko and A. N. Skiba, Finite groups in which τ-quasinormality is a transitive relation, *Rend. Semin. Mat. Univ. Padova* **124** (2010), 1–15.

[13] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer, New York, 1982.

[14] L. A. Shemetkov, *Formations of Finite Groups*, Nauka, Moscow, 1978.

[15] A. N. Skiba, A generalization of a Hall theorem, *J. Algebra Appl.* **15** (2015), no. 4, 21–36.

[16] A. N. Skiba, On σ-subnormal and σ-permutable subgroups of finite groups, *J. Algebra* **436** (2015), 1–16.

[17] A. N. Skiba, On some results in the theory of finite partially soluble groups, *Commun. Math. Stat.* **4** (2016), no. 3, 281–309.

[18] S. Srinivasan, Two sufficient conditions for supersolubility of finite groups, *Israel J. Math.* **3** (1980), no. 35, 210–214.

Received December 22, 2016.

Author information

James C. Beidleman, Department of Mathematics, University of Kentucky,
Lexington, KY 40506-0027, USA.
E-mail: `james.beidleman@uky.edu`

Alexander N. Skiba, Department of Mathematics and Technologies of Programming,
Francisk Skorina Gomel State University, Gomel 246019, Belarus.
E-mail: `alexander.skiba49@gmail.com`