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Adolfo Ballester-Bolinches
*Universitat de València, Spain*

James C. Beidleman
*University of Kentucky, clark@ms.uky.edu*

Arnold D. Feldman
*Franklin and Marshall College*

Matthew F. Ragland
*Auburn University at Montgomery*

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Finite groups in which pronormality and $\mathcal{F}$-pronormality coincide

Adolfo Ballester-Bolinches, James C. Beidleman, Arnold D. Feldman and Matthew F. Ragland

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Abstract. For a formation $\mathcal{F}$, a subgroup $U$ of a finite group $G$ is said to be $\mathcal{F}$-pronormal in $G$ if for each $g \in G$, there exists $x \in \langle U, U^g \rangle^\mathcal{F}$ such that $U^x = U^g$. If $\mathcal{F}$ contains $\mathcal{N}$, the formation of nilpotent groups, then every $\mathcal{F}$-pronormal subgroup is pronormal and, in fact, $\mathcal{N}$-pronormality is just classical pronormality. The main aim of this paper is to study classes of finite soluble groups in which pronormality and $\mathcal{F}$-pronormality coincide.

1 Introduction and statements of results

Note that all groups considered in this paper are finite and we use the notation of [1, 5, 9].

The third author [6] and Müller [8] independently generalised pronormality of a subgroup of a finite soluble group to $\mathcal{F}$-pronormality, where $\mathcal{F}$ is a subgroup-closed saturated formation containing $\mathcal{N}$, the class of all nilpotent groups. Both used the concept of an $\mathcal{F}$-base, a generalisation of a Hall system of a soluble group. But $\mathcal{F}$-bases cannot be defined for non-soluble groups, so we use the following definition due to Müller:

Definition 1.1. Let $G$ be a group and let $U$ be a subgroup of $G$. Then $U$ is said to be $\mathcal{F}$-pronormal in $G$ if, for each $g \in G$, there exists $x \in \langle U, U^g \rangle^\mathcal{F}$ such that $U^x = U^g$.

See [3, remark after Definition 2] for a brief proof that $\mathcal{N}$-pronormality is simply pronormality, so that this is a legitimate generalisation.

It is easy to see that when $\mathcal{F}$ contains $\mathcal{N}$, every $\mathcal{F}$-pronormal subgroup is pronormal. This leads immediately to the question of how to characterise the groups $G$ such that a subgroup is $\mathcal{F}$-pronormal in $G$ if and only if it is pronormal in $G$. In the soluble case, Müller showed that for $\mathcal{F}$ saturated and containing $\mathcal{N}$, the class

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of groups whose maximal subgroups are all $\mathcal{F}$-pronormal is the same as the class of groups whose $\mathcal{F}$-normalisers are nilpotent if and only if the latter class is a saturated formation [8, Satz 7.3].

In [2, 3] we dealt with generalisations of pronormality and subnormality. For recent work by others on generalisations of subnormality, see [7, 10, 11]. This paper is a natural continuation of [4], which dealt principally with generalisations of pronormality. Our aim here is to investigate conditions for soluble and supersoluble groups to belong to the class $\mathcal{P}_{\mathcal{F}}$ of all groups in which pronormality and $\mathcal{F}$-pronormality coincide, where $\mathcal{F}$ is a formation containing $\mathcal{N}$ satisfying a certain closure property. A precise description of the largest subgroup-closed classes of soluble and supersoluble groups contained in $\mathcal{P}_{\mathcal{F}}$ appears as a consequence of our study.

We begin by introducing some definitions and notation that are needed to state our main results.

**Definition 1.2.** The class $\mathcal{E}_{\mathcal{F}}$ consists of all soluble groups $G$ such that if $H \leq L \leq G$, then $H$ is $\mathcal{F}$-pronormal in $L$ if and only if $H$ is pronormal in $L$.

**Definition 1.3.** Given distinct prime numbers $p$ and $q$, let $\mathcal{M}_{(p,q)}$ be the class of groups $G$ such that $G = PQ$, where $P$ is an elementary abelian $p$-group, $Q$ is cyclic of order $q$, $P$ is normal in $G$, $C_Q(P) = 1$ and $P$, regarded as a $Q$-module, is homogeneous. Furthermore, let $\mathcal{X}_{(p,q)}$ be the subclass of $\mathcal{M}_{(p,q)}$ in which $Q$ acts irreducibly on $P$.

It is clear that a group $G \in \mathcal{M}_{(p,q)}$ always has an epimorphic image in $\mathcal{X}_{(p,q)}$. Let $\mathcal{M}$ be the union of the classes $\mathcal{M}_{(p,q)}$ for all pairs $(p, q)$, and let $\mathcal{X}$ be the union of the classes $\mathcal{X}_{(p,q)}$. Write $\mathcal{M}_{\mathcal{F}} = \mathcal{M} \cap \mathcal{F}$ and $\mathcal{X}_{\mathcal{F}} = \mathcal{X} \cap \mathcal{F}$.

For each prime $p$, let $\pi_{\mathcal{F}}(p)$ be the set of primes $q$ such that $\mathcal{X}_{(p,q)}$ is contained in $\mathcal{F}$. Then set $g_{\mathcal{F}}(p)$ to be the class of soluble groups whose orders are not divisible by any prime in $\pi_{\mathcal{F}}(p)$, i.e., the soluble $\pi_{\mathcal{F}}(p)'$-groups.

**Definition 1.4.** We say a formation $\mathcal{F}$ possesses property $\delta$ if $\mathcal{F}$ is closed under taking normal $\mathcal{M}$-subgroups of soluble groups.

**Definition 1.5.** If $\mathcal{F}$ is a formation, a soluble group $G$ is said to be a $\mathcal{B}_{\mathcal{F}}$-group if no section of $G$ is isomorphic to a member of $\mathcal{X}_{\mathcal{F}}$.

**Theorem 1.6.** Let $\mathcal{F}$ be a formation that contains the class $\mathcal{N}$. Then $\mathcal{B}_{\mathcal{F}}$ is a saturated formation which is locally defined by the formation function $f$ given by $f(p) = g_{\mathcal{F}}(p)$ for all primes $p$. Furthermore, $\mathcal{E}_{\mathcal{F}}$ is contained in $\mathcal{B}_{\mathcal{F}}$. If $\mathcal{F}$ has property $\delta$, then $\mathcal{E}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}}$. 
Finite groups in which pronormality and \( \Xi \)-pronormality coincide

Property (\( \delta \)) is necessary in Theorem 1.6, as the following example constructed in [2] shows.

**Example 1.7.** Let \( h \) be a function that takes each prime to a class of groups such that \( h(5) \) is the class of groups that are trivial or cyclic of order 4 and \( h(r) \) is trivial for all primes other than 5. As in [5, Chapter IV, Proposition (1.3)], let \( \mathcal{S} \) be the formation of soluble groups \( G \) such that for each \( p \)-chief factor \( S \) of \( G \), \( \text{Aut}_G(S) \in h(p) \). Then \( \mathcal{X}_{(5,2)} \) is not contained in \( \mathcal{S} \), and, in fact, neither is any other member of \( \mathcal{X} \). Thus \( \mathcal{X}_{\mathcal{S}} \) is empty, and \( \mathcal{P}_{\mathcal{S}} \) is the class of all soluble groups. Now consider the semidirect product \( G \) of a cyclic group of order 5 and its full automorphism group, which is cyclic of order 4, and let \( T \) be a Sylow 2-subgroup of \( G \). Then \( T \) is maximal and pronormal in \( G \), and \( G \in \mathcal{S} \). So if \( G \in \mathcal{E}_{\mathcal{S}} \), \( T \) is \( \mathcal{S} \)-pronormal in \( G \), and therefore normal in \( G \) by Lemma 2.1 below. But \( T \) is not normal in \( G \), so \( G \) is in \( \mathcal{P}_{\mathcal{S}} \) and not \( \mathcal{E}_{\mathcal{S}} \). Note that \( G \) has a subgroup of index 2 that is in \( \mathcal{X}_{(5,2)} \) but not in \( \mathcal{S} \), violating property (\( \delta \)).

For the class \( \mathcal{U} \) of supersoluble groups, the solution of the problem is much nicer.

**Theorem 1.8.** Let \( \Xi \) be a formation that contains \( \mathcal{N} \) and has property (\( \delta \)). Then \( \mathcal{E}_{\Xi} \cap \mathcal{U} = \mathcal{P}_{\Xi} \cap \mathcal{U} \), that is, if \( G \) is a supersoluble group, then the following statements are pairwise equivalent:

(i) \( G \in \mathcal{E}_{\Xi} \).

(ii) For all primes \( p \) dividing \( |G| \), if \( S \) is a \( p \)-chief factor of \( G \), then we have \( \text{Aut}_G(S) \in g_{\Xi}(p) \).

(iii) If \( U \) is pronormal in \( G \), then \( U \) is \( \Xi \)-pronormal in \( G \).

(iv) \( G \in \mathcal{P}_{\Xi} \).

2 Preliminaries

We assume here that \( \Xi \) contains \( \mathcal{N} \). Note that in this case, all \( \Xi \)-pronormal subgroups are pronormal. The following lemmas will be used in the proofs of our main results.

**Lemma 2.1** ([4, Lemma 3]). Let \( U \) be a subgroup of a group \( G \) and let \( \Xi \) be a formation.

(i) If \( U \leq H \) and \( U \) is \( \Xi \)-pronormal in \( G \), then \( U \) is \( \Xi \)-pronormal in \( H \).

(ii) If \( N \) is a normal subgroup of \( G \) and \( U \) is \( \Xi \)-pronormal in \( G \), then \( UN/N \) is \( \Xi \)-pronormal in \( G/N \).
(iii) If $N$ is a normal subgroup of $G$ and $U/N$ is $\mathcal{F}$-pronormal in $G/N$, then $U$ is $\mathcal{F}$-pronormal in $G$.

(iv) If $U$ is maximal, $\mathcal{F}$-pronormal, and $G \in \mathcal{F}$, then $U$ is normal in $G$.

Lemma 2.2 ([3, Proposition 1]). Let $U$ be a subgroup of a group $G$ and let $N$ be a normal subgroup of $G$ such that $U \leq N \leq G$. Then if $\mathcal{F}$ is a formation, the following conditions are equivalent:

(i) $U$ is $\mathcal{F}$-pronormal in $G$.

(ii) $U$ is $\mathcal{F}$-pronormal in $N$ and $G = N_G(U)N$.

Lemma 2.3. The class $\mathcal{E}_\mathcal{F}$ is closed under the taking of sections.

Proof. It is clear from the definition that $\mathcal{E}_\mathcal{F}$ is subgroup-closed, so suppose $N$ is normal in $G \in \mathcal{E}_\mathcal{F}$ and $H/N$ is pronormal in $L/N \leq G/N$. Then $H$ is pronormal in $L \leq G$, so by hypothesis, $H$ is $\mathcal{F}$-pronormal in $L$, and by Lemma 2.1, $H/N$ is $\mathcal{F}$-pronormal in $L/N$. Hence $G/N \in \mathcal{E}_\mathcal{F}$. Thus if $A/B$ is a section of $G \in \mathcal{E}_\mathcal{F}$, then $A \leq G$ implies $A \in \mathcal{E}_\mathcal{F}$ as remarked above, so $A/B \in \mathcal{E}_\mathcal{F}$ as just proved. \qed

3 Proofs of the main results

Proof of Theorem 1.6. Let $\mathcal{Y} = LF(f)$ be the saturated formation locally defined by the formation function $f$ given by $f(p) = g_{\mathcal{F}}(p)$ for all primes $p$. Next, we see that $\mathcal{Y} = \mathcal{E}_\mathcal{F}$. Assume, arguing by contradiction, that $\mathcal{E}_\mathcal{F}$ is not contained in $\mathcal{Y}$ and let $G \in \mathcal{E}_\mathcal{F} \setminus \mathcal{Y}$ of minimal order. By Lemma 2.3, $G/N \in \mathcal{Y}$ for every non-trivial normal subgroup $N$ of $G$. Therefore, $G$ is a primitive group. Applying [5, Chapter A, Theorem 15.2], $N = \text{Soc}(G)$ is a minimal normal subgroup of $G$ which is complemented by a maximal subgroup $M$ in $G$, $C_G(N) = N$ and $G/N \in \mathcal{Y}$. Let $p$ be the prime dividing the order of $N$. Then $G/N \notin f(p)$ and so there exists a prime $q \neq p$ dividing the order of $M$ such that $\mathcal{X}_{(p,q)}$ is contained in $\mathcal{F}$. Let $A$ be a subgroup of order $q$ of $M$. According to [5, Chapter A, Proposition 12.5], $N = C_N(A) \times [A, N]$. Since $C_G(N) = N$, it follows that $[A, N] \neq 1$. Let $N_0$ be a minimal normal subgroup of $AN$ contained in $[A, N]$. Then $C_{N_0}(A) = 1$ and so $B = AN_0 \in \mathcal{X}_{(p,q)}$. In particular, $B \in \mathcal{F}$. Since $A$ is maximal in $B$ and $B \in \mathcal{E}_\mathcal{F}$, it follows that $A$ is normal in $B$ by Lemma 2.1, contrary to the choice of $N_0$. Consequently, $\mathcal{E}_\mathcal{F}$ is contained in $\mathcal{Y}$.

Applying [5, Chapter IV, Proposition 3.14], $\mathcal{Y}$ is subgroup-closed. Assume that $\mathcal{Y}$ is not contained in $\mathcal{P}_\mathcal{F}$ and let $G \in \mathcal{Y} \setminus \mathcal{P}_\mathcal{F}$ be a group of minimal order. Then every proper section of $G$ belongs to $\mathcal{P}_\mathcal{F}$. Since $G \notin \mathcal{P}_\mathcal{F}$, we conclude that $G$ has a section which belongs to some $\mathcal{X}_{(p,q)} \cap \mathcal{F}$. The minimal choice of $G$ implies that $G \in \mathcal{X}_{(p,q)} \cap \mathcal{F}$. Since $G \in \mathcal{Y}$, it follows that $G/\text{Soc}(G) \in f(p) = g_{\mathcal{F}}(p)$, against supposition. Therefore $\mathcal{Y}$ is contained in $\mathcal{P}_\mathcal{F}$. 

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Suppose that $\mathcal{P}_G$ is not contained in $\mathcal{P}$, and let $G \in \mathcal{P}_G \setminus \mathcal{P}$ be of minimal order. Then $G$ is an $\mathcal{Y}$-critical group, that is, every proper subgroup of $G$ belongs to $\mathcal{Y}$. Since $\mathcal{Y}$ is saturated, it follows that $G$ is primitive. Let $N = \text{Soc}(G)$ and $p$ the prime dividing $|N|$. Let $M$ be a core-free maximal subgroup of $G$ complementing $N$ in $G$. Then $NM_1 \in \mathcal{Y}$ for every maximal subgroup $M_1$ of $M$. Since $O_{p'p}(NM_1) = O_p(NM_1)$, we have that the Hall $p'$-subgroups of $NM_1$ belong to $g\mathcal{P}(p)$. Hence, if $M$ is not of prime order, it follows that $G/N \in f(p)$. Applying [5, Chapter IV, Remark 3.5 e)], $G \in \mathcal{Y}$, against the choice of $G$. Hence $G \in \mathcal{X}(p,q)$ for some $q \neq p$, and $q \in \pi\mathcal{P}(p)$, i.e. $G \in \mathcal{F}$. This contradiction yields $\mathcal{P}_G \subseteq \mathcal{Y}$. Therefore we have $\mathcal{E}_G \subseteq \mathcal{P}_G = \mathcal{Y}$.

Assume now that $\mathcal{F}$ has property (δ). We prove that $\mathcal{E}_G = \mathcal{P}_G$. Suppose, arguing by contradiction, that $G$ is a soluble group of minimal order with respect to being a member of $\mathcal{P}_G$ but not $\mathcal{E}_G$. Then there exists $U \leq L \leq G$ such that $U$ is pronormal in $L$ but not $\mathcal{F}$-pronormal in $L$. If $L < G$, by minimality of $G$, $U$ is $\mathcal{F}$-pronormal in $L$. Hence $L = G$, and $U$ is pronormal in $G$ but not $\mathcal{F}$-pronormal in $G$. Let $C = \text{Core}_G(U)$. Then if $C > 1$, by minimality of $G$, $G/C \in \mathcal{P}_G$ implies $G/C \in \mathcal{E}_G$, so that, since $U/C$ is pronormal in $G/C$, $U/C$ is $\mathcal{F}$-pronormal in $G/C$, implying $U$ $\mathcal{F}$-pronormal in $G$ by Lemma 2.1. Hence $\text{Core}_G(U) = 1$.

Now consider the normal closure $V = U^G$ of $U$ in $G$. If $V < G$, then by minimality of $G$, since $U$ is pronormal in $V$, $U$ is $\mathcal{F}$-pronormal in $V$. Also, the pronormality of $U$ in $G$ implies $G = N_G(U)V$, so by Lemma 2.2, $U$ is $\mathcal{F}$-pronormal in $G$. Hence $V = G$, i.e., $U$ is contained in no proper normal subgroup of $G$.

Since $U$ is not $\mathcal{F}$-pronormal in $G$, there exists $g \in G$ such that $U$ and $U^g$ are not conjugate via an element of $J\mathcal{F}$, where $J = \langle U, U^g \rangle$. But $U$ is pronormal in $G$, so there exists $x \in J$ such that $U^g = U^x$. Thus if $J < G$, by minimality of $G$, $U$ pronormal in $J$ implies $U$ $\mathcal{F}$-pronormal in $J$, so there does exist $y \in J\mathcal{F}$ conjugating $U$ to $U^y = U^g$. Hence $J = G$, $J\mathcal{F} = G\mathcal{F}$, and we are assuming that $U$ and $U^g$ are not conjugate via any element of $G\mathcal{F}$.

Assume now that $G$ is not in $\mathcal{F}$, so that $G\mathcal{F} > 1$, and let $N$ be any minimal normal subgroup of $G$ contained in $G\mathcal{F}$. Note that $UN/N$ is pronormal in $G/N$, so by minimality of $G$, $UN/N$ is $\mathcal{F}$-pronormal in $G/N$, so that $UN$ is $\mathcal{F}$-pronormal in $G$ by Lemma 2.1. Hence $UN$ and $(UN)^g$ are conjugate by an element of $\langle UN, (UN)^g \rangle\mathcal{F} = G\mathcal{F}$. Thus there exists $z \in G\mathcal{F}$ such that $(UN)^z = (UN)^g$. Therefore, $gz^{-1} \in N_G(UN)$. But because $U$ is pronormal in $G$, $N_G(UN)$ is equal to $N_G(U)N$, so $gz^{-1} = hw$, where $h \in N_G(U)$ and $w \in N$. Therefore, $gz^{-1}w^{-1} \in N_G(U)$, so $U^g = U^{wz}$. But $N \leq G\mathcal{F}$, so $wz \in G\mathcal{F}$, a contradiction.

Thus we may assume $G \in \mathcal{F}$. Let $N$ be any minimal normal subgroup of $G$, so that $G/N \in \mathcal{F}$, and by minimality of $G$, any maximal subgroup $L/N$ of $G/N$, being pronormal in $G/N$, is $\mathcal{F}$-pronormal in $G/N$. Then by Lemma 2.1 (iv), $L/N$ is normal in $G/N$. Thus $G/N$ is nilpotent. If $N$ is not unique, then $G$ itself is...
nilpotent, so that any pronormal subgroup of \(G\), being subnormal as well, is normal in \(G\) and therefore \(\mathcal{B}\)-pronominal in \(G\). Hence \(N\) is the unique minimal normal subgroup of \(G\) and \(UN/N\), being pronormal and subnormal in \(G/N\), is normal in \(G/N\). Thus \(UN\) is normal in \(G\). But \(U^G = G\), so \(UN = G\), \(U\) is a core-free maximal subgroup of \(G\) and \(C_G(N) = N\). Then \(U\) belongs to \(\mathfrak{P}(p)\). Let \(q\) be a prime dividing \(|U|\) and let \(A\) be a subgroup of \(U\) of order \(q\) in \(\mathbb{Z}(U)\). Then \(N\), regarded as \(A\)-module, is homogeneous by [5, Chapter B, Corollary 9.4]. Therefore \(C_A(N) = 1\) and \(B = NA\) is a normal subgroup of \(G\) in \(\mathcal{M}(p,q)\). Since \(\mathcal{B}\) has property (\(\delta\)), it follows that \(B \in \mathcal{B}\), contrary to \(G \not\in \mathcal{P}_{\mathcal{B}}\).

The proof of the theorem is complete.

\(\square\)

**Proof of Theorem 1.8.** Conditions (i), (ii) and (iv) are equivalent by Theorem 1.6, and clearly (i) implies (iii), so we need only prove that (iii) implies (iv). Hence suppose \(G\) is a supersoluble group of minimal order such that every pronormal subgroup of \(G\) is \(\mathcal{B}\)-pronominal in \(G\), but \(G\) is not a member of \(\mathcal{P}_{\mathcal{B}}\). Applying Lemma 2.1, the condition on \(G\) is inherited in every epimorphic image of \(G\). Therefore, \(G/N \not\in \mathcal{P}_{\mathcal{B}}\) for all non-trivial normal subgroups of \(G\). Applying Theorem 1.6, \(\mathcal{P}_{\mathcal{B}}\) is a saturated formation. Hence \(G\) is a supersoluble primitive group. Applying [5, Chapter A, Theorem 15.2], \(N = \text{Soc}(G)\) is a minimal normal subgroup of \(G\) of prime order, \(p\) say, which is complemented in \(G\) by a core-free maximal subgroup \(M\) of \(G\). Moreover, \(C_G(N) = N\) and so \(M\) is a cyclic group of order dividing \(p - 1\). Since \(G/N \not\in g_{\mathcal{B}}(p)\), there exists a prime \(q\) dividing \(|M|\) such that \(X_{(p,q)} \cap G\) is not empty. Note that if \(A\) is a subgroup of order \(q\) of \(M\), we have that \(NA \in X_{(p,q)}\). Therefore \(NA\) is normal in \(G\) and so \(A\) is pronominal in \(G\) since \(A\) is a Sylow \(q\)-subgroup of \(NA\). By hypothesis, \(A\) is \(\mathcal{B}\)-pronominal in \(G\) and so is \(\mathcal{B}\)-pronominal in \(NA\).

By Lemma 2.1, \(A\) is normal in \(AN\), which is not possible. Consequently, we have \(\mathcal{P}_{\mathcal{B}} \cap U = \mathcal{P}_{\mathcal{B}} \cap \mathcal{U}\). 

\(\square\)

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**Bibliography**


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**Author information**

Adolfo Ballester-Bolinches, Departament d’Àlgebra, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain.
E-mail: adolfo.ballester@uv.es

James C. Beidleman, University of Kentucky, Lexington, KY 40506-0027, USA.
E-mail: james.beidleman@uky.edu

Arnold D. Feldman, Department of Mathematics, Franklin and Marshall College, Lancaster, PA 17604-3003, USA.
E-mail: afeldman@fandm.edu

Matthew F. Ragland, Department of Mathematics and Computer Science, Auburn University at Montgomery, P.O. Box 244023, Montgomery, AL 36124-4023, USA.
E-mail: mragland@aum.edu