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A note on the $\gamma$-coefficients of the tree Eulerian polynomial

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Abstract

We consider the generating polynomial of the number of rooted trees on the set $\{1,2,\ldots,n\}$ counted by the number of descending edges (a parent with a greater label than a child). This polynomial is an extension of the descent generating polynomial of the set of permutations of a totally ordered $n$-set, known as the Eulerian polynomial. We show how this extension shares some of the properties of the classical one. A classical product formula shows that this polynomial factors completely over the integers. From this product formula it can be concluded that this polynomial has positive coefficients in the $\gamma$-basis and we show that a formula for these coefficients can also be derived. We discuss various combinatorial interpretations of these coefficients in terms of leaf-labeled binary trees and in terms of the Stirling permutations introduced by Gessel and Stanley. These interpretations are derived from previous results of Liu, Dotsenko-Khoroshkin, Bershtein-Dotsenko-Khoroshkin, González D’León-Wachs and González D’León related to the free multibracketed Lie algebra and the poset of weighted partitions.

Keywords: Gamma positivity; Eulerian polynomial; Rooted trees

1 introduction

A labeled rooted tree $T$ on the set $[n] := \{1,2,\ldots,n\}$ is a tree whose nodes or vertices are the elements of $[n]$ and such that one of its nodes has been distinguished and called the root. For nodes $x$ and $y$ in $T$ we say that $x$ is the child of $y$ or $y$ is the parent of $x$ if $y$ is the first node following $x$ in the unique path from $x$ to the root of $T$ and we

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say that \( y = p(x) \). Nodes that have children are said to be internal otherwise we call a
node without children a leaf. If \( y \) is the parent of \( x \), we say that the edge \( \{x, y\} \) of \( T \) is
descending (and we call \( x \) a descent of \( T \)) if the label of \( y \) is greater than the label of \( x \).
We denote \( \text{des}(T) \) the number of descents in \( T \). Figure 1 shows all the rooted trees on \([3]\) grouped by the number of descents. We draw the trees with the convention that parents
come higher than their children and the root is the highest node. We denote \( \mathcal{T}_n \) the set
of rooted trees on \([n]\) and \( \mathcal{T}_{n,i} \) the set of trees in \( \mathcal{T}_n \) with exactly \( i \) descents.

For a given \( n \geq 1 \) define

\[
T_n(t) := \sum_{T \in \mathcal{T}_n} t^{\text{des}(T)} = \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i, \tag{1.1}
\]

the descent generating polynomial of \( \mathcal{T}_n \). We call \( T_n(t) \) the tree Eulerian polynomial in
analog with the classical polynomial \( A_n(t) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{des}(\sigma)} \), that is the descent generating
polynomial of the set \( \mathcal{S}_n \) of permutations of \([n]\). We can identify the permutations in \( \mathcal{S}_n \) with the set of rooted trees on \([n]\) that have \( n - 1 \) internal nodes, each of them
having a unique child (and so containing a unique leaf). It is not hard to see that our
definition of a descent on this set of trees coincides with the classical definition of descent
in a permutation so the polynomial \( T_n(t) \) is an extension of the polynomial \( A_n(t) \). The
polynomials \( A_n(t) \) have been extensively studied in the literature and are known with the
name of Eulerian polynomials since Euler was one of the first in studying them (see [20]).
The Eulerian polynomial \( A_n(t) = \sum_{k=0}^{n-1} A_{n,i} t^i \) have degree \( n - 1 \) and its coefficients satisfy
the relation

\[
A_{n,i} = A_{n,n-1-i}. \tag{1.2}
\]

For example, the Eulerian polynomial for \( n = 3 \) is \( A_3(t) = 1 + 4t + t^2 \). A polynomial that
satisfies Equation 1.2 is called symmetric or palindromic.

It is a simple observation that a symmetric polynomial \( f(t) = \sum_{i=0}^{d} f_i t^i \) of degree \( d \)
with \( f_i \in \mathbb{Z} \) can be written in the form

\[
\sum_{i=0}^{d} f_i t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1 + t)^{d - 2i}, \tag{1.3}
\]

where the coefficients \( \gamma_i \in \mathbb{Z} \), i.e., the set \( \{t^i (1 + t)^{d - 2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor} \), where \( \lfloor \cdot \rfloor \) is the integer floor function, is a basis (known as the \( \gamma \)-basis) for the space of symmetric polynomials of
degree \( d \) with integer coefficients. If \( \gamma_i \geq 0 \) for all \( i \) then we say that the polynomial
\( f(t) \) is \( \gamma \)-nonnegative and if \( \gamma_i > 0 \) for all \( i \leq \lfloor \frac{d}{2} \rfloor \) then we say that the polynomial \( f(t) \)
is \( \gamma \)-positive. It is a result of Foata and Schützenberger ([8]) that \( A_n(t) \) is \( \gamma \)-positive. And it follows from the work of Foata and Strehl in [9] that its coefficients \( \gamma_i \) have a nice
combinatorial interpretation (see also [18]). Indeed, let \( \mathcal{S}_n \) be the set of permutations in \( \mathcal{S}_n \) that have no two adjacent descents and no descent in the last position. Then

\[
\gamma_j = \{ \sigma \in \mathcal{S}_n \mid \text{des}(\sigma) = j \}.
\]
For example, $A_3(t) = (1 + t)^2 + 2t$ with $\gamma_0 = 1$ and $\gamma_1 = 2$. The permutations in $\hat{S}_3$ are, 123 with no descents and; 213 and 312 with one descent. Gal [10] and Brändén [2, 3] have introduced the use of the $\gamma$-basis in different contexts. Gal conjectured that the $\gamma$-coefficients of the $h$-polynomial of a flag simple polytope are all nonnegative. In particular, $A_n(t)$ is the $h$-vector of the permutahedron that is a flag simple polytope so Gal’s conjecture is confirmed in this case. Postnikov, Reiner and Williams [17] have confirmed Gal’s conjecture for the family of chordal nestohedra that is a large family of flag simple polytopes. For more information about $\gamma$-nonnegativity see [4].

We will show that the properties discussed above for the Eulerian polynomial $A_n(t)$ are also shared by the polynomial $T_n(t)$ in a similar fashion. The degree of $T_n(t)$ is also $n - 1$ and it is easy to see from the definition of a descent that

$$|T_{n,i}| = |T_{n,n-1-i}|,$$

so $T_n(t)$ is also symmetric. Indeed there is a natural bijection $T_{n,i} \simeq T_{n,n-1-i}$ where the image of a labeled rooted tree $T \in T_{n,i}$, is the tree in $T_{n,n-1-i}$ with the same shape of $T$ but where each label $i$ has been replaced by $n + 1 - i$. For the example in Figure 1, $T_3(t) = 2 + 5t + 2t^2$.

Figure 1: All labeled rooted trees on $[3]$

The following nice product formula for $T_n(t)$ can be obtained from the results in [7] (see also [12] and [6]).

**Theorem 1.** For $n \geq 1$,

$$\sum_{i=0}^{n-1} |T_{n,i}|t^i = \prod_{i=1}^{n-1} ((n - i) + it). \quad (1.5)$$

In particular, setting $t = 1$ in (1.5) reduces to the classical formula $|T_n| = n^{n-1}$. Equation (1.5) implies that all the roots of this polynomial are real and negative. It is known and not difficult to show that a real-rooted symmetric polynomial with nonnegative real coefficients is $\gamma$-nonnegative (see [4, 10]). For example $T_3(t) = 2(1 + t)^2 + t$, so $\gamma_0 = 2$ and $\gamma_1 = 1$. Although $T_n(t)$ is not in general an $h$-vector of a convex polytope (for example...
$(h_0 \neq 1$ for $n \geq 3$), it is of interest to find combinatorial formulas and interpretations of nonnegative $\gamma$-coefficients of general symmetric polynomials.

Let $\gamma_j(T_n(t))$ denote the $j$-th $\gamma$-coefficient of the symmetric polynomial $T_n(t)$. Equation (1.5) can be used to find a formula for the coefficients $\gamma_j$ of $T_n(t)$. The values of $\gamma_j(T_n(t))$ for $n = 1, \ldots, 7$ appear in Table 1.

**Theorem 2.** For $n \geq 1$,

$$
\gamma_j(T_n(t)) = \begin{cases}
\sum_{J \subset \left[\frac{n-1}{2}\right]} \prod_{i \in J} (n-2i)^2 \prod_{s \in \left[\frac{n-1}{2}\right] \setminus J} s(n-s) & \text{if } n \text{ is odd} \\
\frac{n}{2} \sum_{J \subset \left[\frac{n-3}{2}\right]} \prod_{i \in J} (n-2i)^2 \prod_{s \in \left[\frac{n-3}{2}\right] \setminus J} s(n-s) & \text{if } n \text{ is even}
\end{cases}
$$

Equation 1.5 can be written as

$$
T_n(t) = \begin{cases}
\prod_{i=1}^{\frac{n-1}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is odd,} \\
\frac{n}{2}(1+t) \prod_{i=1}^{\frac{n-3}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is even,}
\end{cases}
$$

implying Formula 1.6.

**Proof.** If we multiply

$$(n-i+it)(i+(n-i)t) = (n-i)i + [(n-i)^2 + i^2]t + (n-i)it^2$$

$$= (n-i)i(1+t^2) + [(n-i)^2 + i^2]t$$

$$= (n-i)i(1+2t + i^2) + [(n-i)^2 - 2i(n-i) + i^2]t$$

$$= (n-i)i(1+t)^2 + (n-2i)^2t.$$

Equation 1.5 can be written as

$$T_n(t) = \begin{cases}
\prod_{i=1}^{\frac{n-1}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is odd,} \\
\frac{n}{2}(1+t) \prod_{i=1}^{\frac{n-3}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is even,}
\end{cases}$$

implying Formula 1.6. 

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Table 1: Values of $\gamma_j(T_n(t))$ for $n = 1, \ldots, 7$. 

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The purpose of this note is to present four different combinatorial interpretations for the coefficients $\gamma_j$ that are consequences of results in the work of Liu [15], Dotsenko-Khoroshkin [5], Bershtein-Dotsenko-Khoroshkin [1], González D’León-Wachs [14] and González D’León [13]. We present now one of these combinatorial interpretations, whose proof will be given in Section 2.

A planar leaf-labeled binary tree with label set $[n]$ is a rooted tree (a priori without labels) in which the set of children of every internal node is a totally ordered set with exactly two elements (the left and right children) and where each leaf has been assigned a unique element from the set $[n]$. By a subtree in a rooted tree $T$ we mean the rooted tree induced by the descendents of any node $x$ of $T$, including and rooted at $x$. We say that a planar leaf-labeled binary tree with label set $[n]$ is normalized if in each subtree, the leftmost leaf is the one with the smallest label. We denote the set of normalized binary trees with label set $[n]$ by $\text{Nor}_n$. All normalized trees with leaf labels in $[3]$ are illustrated in Figure 2.

A right descent in a normalized tree is an internal node that is the right child of its parent. For $T \in \text{Nor}_n$ we define $\text{rdes}(T) := |\{\text{right descents of } T\}|$. A double right descent is a right descent whose parent is also a right descent. We denote by $\text{NDRD}_n$ the set of trees in $\text{Nor}_n$ with no double right descents.

**Theorem 3.** For $n \geq 1$ and $j \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}$,

$$\gamma_j(T_n(t)) = |\{T \in \text{NDRD}_n \mid \text{rdes}(T) = j\}|.$$

As it is illustrated in Figure 2, there are two trees in $\text{NDRD}_3$ (for $n = 3$ it happens to be equal to $\text{Nor}_3$) with $\text{rdes}(T) = 0$ and one with $\text{rdes}(T) = 1$, corresponding to $\gamma_0 = 2$ and $\gamma_1 = 1$ respectively.

![Figure 2: Set of trees in $\text{NDRD}_3 = \text{Nor}_3$](image)

In Section 2 we provide the proof of Theorem 3 and an additional version of Theorem 3 also in terms of normalized trees but with a statistic different than rdes. In Section 3 we provide two additional versions of Theorem 3 in terms of the Stirling permutations introduced by Gessel and Stanley in [11]. In Section 4 we discuss a generalization of the $\gamma$-positivity of $T_n(t)$ to the positivity of certain symmetric function in the basis of elementary symmetric functions.
2 Combinatorial interpretations in terms of binary trees

Now we consider normalized trees $T$ where every internal node $x$ of $T$ has been assigned an element $\text{color}(x) \in \{0, 1\}$. We call an element of this set of trees a bicolored normalized tree on $[n]$. A bicolored comb is a bicolored normalized tree $T$ satisfying the following coloring restriction:

(C) If $x$ is a right descent of $T$ then $\text{color}(x) = 0$ and $\text{color}(p(x)) = 1$.

We denote by $\text{Comb}_n$ the set of bicolored combs and by $\text{Comb}_{n,i}$ the set of bicolored combs where $i$ internal nodes have been colored 1 (and $n - 1 - i$ colored 0). Figure 3 illustrates the bicolored combs on [3] grouped by the number of internal nodes that have been colored 1.

![Figure 3: Set of bicolored combs on [3]](image)

Denote by $\tilde{T} \in \text{Nor}_n$ the underlying uncolored normalized tree associated to a tree $T \in \text{Comb}_n$. Note that the coloring condition (C) implies that $\tilde{T} \in \text{NDRD}_n$. Indeed, in a double right descent the coloring condition (C) cannot be satisfied since the parent of a double right descent is also a right descent. Also note that the monochromatic combs in $\text{Comb}_{n,0}$ and $\text{Comb}_{n,n-1}$ are just the traditional left combs that are described in [21] and that index a basis for the space $\text{Lie}(n)$, the multilinear component of the free Lie algebra over $\mathbb{C}$ on $n$ generators (see [21] for details). Liu [15] and Dotsenko-Khoroshkin [5] independently proved a conjecture of Feigin regarding the dimension of the multilinear component $\text{Lie}_2(n)$ of the free Lie algebra with two compatible brackets, a generalization of $\text{Lie}(n)$.

**Theorem 4** ([5, 15]). For $n \geq 1$, $\dim \text{Lie}_2(n) = |T_n|$.

In particular, the space $\text{Lie}_2(n)$ has the decomposition

$$\text{Lie}_2(n) = \bigoplus_{i=0}^{n-1} \text{Lie}_2(n, i),$$

where the subspace $\text{Lie}_2(n, i)$ is the component generated by certain “bracketed permutations” with exactly $i$ brackets of one of the types. Liu finds the following formula for the dimension of $\text{Lie}_2(n, i)$.

**Theorem 5** ([15, Proposition 11.3]). For $n \geq 1$ and $i \in \{0, 1, \cdots, n-1\}$,

$$\dim \text{Lie}_2(n, i) = |T_{n,i}|.$$
In [1] Bershtein, Dotsenko and Khoroshkin found a basis for $\mathcal{L}ie_2(n,i)$ indexed by the elements of $\text{Comb}_{n,i}$ giving the following alternative description for the dimension of $\mathcal{L}ie_2(n,i)$.

**Theorem 6** ([1, Lemma 5.2]). For $n \geq 1$ and $i \in \{0, 1, \cdots, n-1\}$,
\[
\dim \mathcal{L}ie_2(n,i) = |\text{Comb}_{n,i}|.
\]

**Corollary 7.** For every $n \geq 1$ and $i \in \{0, \cdots, n-1\}$,
\[
|\text{Comb}_{n,i}| = |\mathcal{T}_{n,i}|.
\]

**Problem 8.** Find an explicit bijection $\text{Comb}_{n,i} \rightarrow \mathcal{T}_{n,i}$, for every $n \geq 1$ and $i \in \{0, \cdots, n-1\}$.

**Theorem 9** (Theorem 3). For $n \geq 1$ and $j \in \{0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor \}$,
\[
\gamma_j(T_n(t)) = |\{T \in \text{NDRD}_n \mid \text{rdes}(T) = j\}|.
\]

**Proof.** First note that by the comments above if $T \in \text{Comb}_n$ then its underlying uncolored tree $\tilde{T} \in \text{NDRD}_n$.

For a tree $T \in \text{Comb}_n$ call $\text{free}(T)$ the number of internal nodes that are not right descents and whose right child is a leaf. Then in $\text{NDRD}_n$ we have that
\[
\text{free}(T) + 2 \text{rdes}(T) = n - 1.
\]

Over the set of bicolored combs with $m$ free nodes there is a free action of $(\mathbb{Z}_2)^m$ by toggling the colors of the free nodes. Then there are $2^m$ bicolored combs with the same underlying tree $\tilde{T} \in \text{NDRD}_n$. By Corollary 7 we can write $T_n(t)$ as
\[
T_n(t) = \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i
= \sum_{i=0}^{n-1} |\text{Comb}_{n,i}| t^i
= \sum_{T \in \text{Comb}_n} t^i |\{x \text{ internal in } T \mid \text{color}(x)=1\}|
= \sum_{\mathcal{T} \in \text{NDRD}_n} \sum_{T \in \text{Comb}_n, \tilde{T}=\mathcal{T}} t^i |\{x \text{ internal in } T \mid \text{color}(x)=1\}|
= \sum_{\mathcal{T} \in \text{NDRD}_n} t^{\text{rdes}(|\mathcal{T}|)} (1 + t)^{\text{free}(|\mathcal{T}|)}
= \sum_{\mathcal{T} \in \text{NDRD}_n} t^{\text{rdes}(|\mathcal{T}|)} (1 + t)^{n-1 - 2 \text{rdes}(|\mathcal{T}|)}. \quad \Box
\]
2.1 A second description in terms of normalized trees

In [14] the author and Wachs studied the relation between $\mathcal{L}ie_2(n, i)$ and the cohomology of the maximal intervals of a poset of weighted partitions. Using poset topology techniques they found an alternative description for the dimension of $\mathcal{L}ie_2(n, i)$.

Define the valency $v(x)$ of a node (internal or leaf) $x$ of $T \in \text{Nor}_n$ to be the minimal label in the subtree of $T$ rooted at $x$. For an internal node $x$ of $T$ let $L(x)$ and $R(x)$ denote the left and right children of $x$ respectively. A Lyndon node is an internal node $x$ of $T$ such that

$$v(R(L(x))) > v(R(x)).$$

(2.1)

A Lyndon tree is a normalized tree in which all its internal nodes are Lyndon. We denote $\text{nlyn}(T)$ the number of non-Lyndon nodes in $T$. A double non-Lyndon node is a non-Lyndon node that is the left child of its parent and its parent is also a non-Lyndon node. We denote the set of trees in $\text{Nor}_n$ with no double non-Lyndon nodes by $\text{NDNL}_n$. A bicolored Lyndon tree is a bicolored normalized tree satisfying the coloring condition:

(L) For every non-Lyndon node $x$ of $T$ then $\text{color}(x) = 0$ and $\text{color}(L(x)) = 1$.

The set of bicolored Lyndon trees is denoted $\text{Lyn}_n$ and the set of the ones with exactly $i$ nodes with color 1 is denoted $\text{Lyn}_{n,i}$.

Figure 4: Set of bicolored Lyndon trees on [3]

Theorem 10 ([14, Section 5]). For every $n \geq 1$ and $i \in \{0, \ldots, n - 1\}$,

$$\dim \mathcal{L}ie_2(n, i) = |\text{Lyn}_{n,i}|.$$

Hence,

$$|\text{Lyn}_{n,i}| = |T_{n,i}|.$$

The proof of the following theorem follows the same arguments of the proof of Theorem 3.

Theorem 11. For $n \geq 1$ and $j \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$,

$$\gamma_j(T_n(t)) = |\{T \in \text{NDNL}_n \mid \text{nlyn}(T) = j\}|.$$
3 Combinatorial interpretation in terms of Stirling permutations

Consider now the set of multipermutations of the multiset \( \{1, 1, 2, 2, \ldots, n, n\} \) such that all numbers between the two occurrences of any number \( m \) are larger than \( m \). To this family belongs for example the permutation 12234431 but not 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. This family (denoted \( \mathcal{Q}_n \)) of permutations was introduced by Gessel and Stanley in [11] and the permutations in \( \mathcal{Q}_n \) are known as **Stirling Permutations**.

For a permutation \( \theta = \theta_1 \theta_2 \cdots \theta_{2n} \) in \( \mathcal{Q}_n \) we say that the position \( i \) contains a first occurrence of a letter if \( \theta_j \neq \theta_i \) for all \( j < i \), otherwise we say that it contains a second occurrence. An ascending adjacent pair in \( \theta \) is a pair \((a, b)\) such that \( a < b \) and in \( \theta \) the second occurrence of \( a \) is the immediate predecessor of the first occurrence of \( b \). An ascending adjacent sequence (of length 2) is a sequence \( a < b < c \) such that \((a, b)\) and \((b, c)\) are both ascending adjacent pairs. For example, in \( \theta = 13344155688776 \) the ascending adjacent pairs are \((1, 5)\), \((5, 6)\) and \((3, 4)\) but the only ascending adjacent sequence is \(1 < 5 < 6\). We denote \( \text{NAAS}_n \) the set of all Stirling permutations in \( \mathcal{Q}_n \) that do not contain ascending adjacent sequences. Similarly, a terminally nested pair in \( \theta \) is a pair \((a, b)\) such that \( a < b \) and in \( \theta \) the second occurrence of \( a \) is the immediate successor of the second occurrence of \( b \). A terminally nested sequence (of length 2) is a sequence \( a < b < c \) such that \((a, b)\) and \((b, c)\) are both terminally nested pairs. For example, in \( \theta = 13443566188776 \) the terminally nested pairs are \((1, 5)\), \((5, 6)\) and \((3, 4)\) but the only terminally nested sequence is \(1 < 5 < 6\). We denote \( \text{NTNS}_n \) the set of all Stirling permutations in \( \mathcal{Q}_n \) that do not contain terminally nested sequences. For \( \sigma \in \mathcal{Q}_n \), we denote \( \text{aapair}(\sigma) \) the number of ascending adjacent pairs in \( \sigma \) and \( \text{tnpair}(\sigma) \) the number of terminally nested pairs in \( \sigma \).

The following result in [13] relates the statistics above in \( \mathcal{Q}_{n-1} \) with the ones previously discussed for \( \text{Nor}_n \).

**Proposition 12** ([13, Proposition 4.8]). There is a bijection \( \phi : \text{Nor}_n \to \mathcal{Q}_{n-1} \) such that for every \( T \in \text{Nor}_n \),

1. \( \text{rdes}(T) = \text{tnpair}(\phi(T)) \)
2. \( \text{nlyn}(T) = \text{aapair}(\phi(T)) \)
3. \( \phi(NDRD_n) = NAAS_{n-1} \)

4. \( \phi(NDNL_n) = NTNS_{n-1} \).

**Corollary 13.** For \( n \geq 1 \) and \( j \in \{0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor \} \),

\[
\gamma_j(T_n(t)) = \left| \{ T \in NTNS_{n-1} \mid \tnpair(T) = j \} \right|
\]

\[
= \left| \{ T \in NAAS_{n-1} \mid \aapair(T) = j \} \right|
\]

**Example 14.** The Stirling permutations in \( Q_2 \) are 1122, 1221 and 2211. In this particular case \( Q_2 = NAAS_2 = NTNS_2 \) and the statistics in Table 2 imply that \( \gamma_0 = 2 \) and \( \gamma_1 = 1 \) are the \( \gamma \)-coefficients of the polynomial \( T_3(t) \).

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Table 2: \( \tnpair \) and \( \aapair \) statistics in \( Q_2 \).

## 4 A comment about \( \gamma \)-positivity and \( e \)-positivity

Let \( x = x_1, x_2, \ldots \) be an infinite set of variables and \( \Lambda = \Lambda_0 \) the ring of symmetric functions with rational coefficients on the variables \( x \), that is, the ring of power series on \( x \) of bounded degree that are invariant under permutation of the variables.

Define \( e_0 := 1 \), for \( n \geq 1 \)

\[
e_n := \sum_{1 \leq i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},
\]

and for an integer partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \) (i.e., a weakly decreasing finite sequence of positive integers) \( e_\lambda := \prod_i e_{\lambda_i} \). We call \( e_\lambda \) the *elementary symmetric function* corresponding to the partition \( \lambda \).

It is known that for \( n \geq 0 \), the set \( \{ e_\lambda \mid \lambda \vdash n \} \) is a basis for the \( n \)-th homogeneous graded component of \( \Lambda \), where the grading is with respect to degree. See [16] and [19] for more information about symmetric functions.

Note that if we make the specialization \( x_i \mapsto 0 \) in \( \Lambda \) for all \( i \geq 3 \) then \( e_1 \mapsto x_1 + x_2 \), \( e_2 \mapsto x_1 x_2 \) and \( e_i \mapsto 0 \) for all \( i \geq 3 \). Thus for a partition \( \lambda \) of \( n \) (i.e., \( \sum_i \lambda_i = n \)) the symmetric function \( e_\lambda \mapsto 0 \) unless \( \lambda = (2^j, 1^{n-2j}) \) for some \( j \in \mathbb{N} \). In that case,

\[
e_{(2^j, 1^{n-2j})} \mapsto (x_1 x_2)^j (x_1 + x_2)^{n-2j}.
\]

If we further replace \( x_1 \mapsto 1 \) and \( x_2 \mapsto t \) we obtain

\[
e_{(2^j, 1^{n-2j})} \mapsto t^j (1 + t)^{n-2j}.
\]
In other words, the elementary basis in two variables is equivalent to the $\gamma$ basis. A consequence of this observation is that another possible approach to conclude the $\gamma$-nonnegativity of a palindromic polynomial $f(t)$ is to find an $e$-nonnegative symmetric function $F(x_1, x_2, \ldots)$ such that $f(t) = F(1, t, 0, 0, \ldots)$.

4.1 Colored combs and comb type of a normalized tree

A colored comb is a normalized binary tree $T$ together with a function color that assigns positive integers in $\mathbb{P}$ to the internal nodes of $T$ and that satisfies the following coloring restriction: for each internal node $x$ whose right child $R(x)$ is not a leaf,

$$\text{color}(x) > \text{color}(R(x)).$$ (4.1)

Note that the set of colored combs that only use the colors 1 and 2 are the same as the bicolored combs defined in Section 2. We denote $\mathbf{MComb}_n$ the set of colored combs with $n$ leaves. Figure 6 shows an example of a colored comb.

![Figure 6: Example of a colored comb of comb type (2, 2, 1, 1, 1, 1)](image)

We can associate a type to each $\Upsilon \in \text{Nor}_n$ in the following way: Let $\pi(\Upsilon)$ be the finest (set) partition of the set of internal nodes of $\Upsilon$ satisfying

- for every pair of internal nodes $x$ and $y$ such that $y$ is a right child of $x$, $x$ and $y$ belong to the same block of $\pi(\Upsilon)$.

We define the comb type $\lambda(\Upsilon)$ of $\Upsilon$ to be the integer partition whose parts are the sizes of the blocks of $\pi(\Upsilon)$.

Note that the coloring condition (4.1) is closely related to the comb type of a normalized tree. The coloring condition implies that in a colored comb $\Upsilon$ there are no repeated colors in each block $B$ of the partition $\pi(\Upsilon)$ associated to $\Upsilon$. So after choosing $|B|$ different colors for the internal nodes of $\Upsilon$ in $B$, there is a unique way to assign the colors such that $\Upsilon$ is a colored comb (the colors must decrease towards the right in each block of $\pi(\Upsilon)$). In Figure 6 this relation is illustrated.

For a colored comb $C$ denote $\mu(C)$ the sequence of nonnegative integers such that

$$\mu(C)(j) := |\{x \text{ a internal node in } C \mid \text{color}(x) = j\}|.$$
Let $x^\mu := x_1^{\mu(1)} x_2^{\mu(2)} \cdots$ and

$$F_{\text{MComb}_n}(x) := \sum_{C \in \text{MComb}_n} x^{\mu(C)}.$$ 

The following theorem is a consequence of the definition of a colored comb, the definition of the symmetric functions $e_i(x)$ and the observations above (see [13]).

**Theorem 15** ([13]). For $n \geq 1$

$$F_{\text{MComb}_n}(x) = \sum_{T \in \text{Nor}_n} e_{\lambda(T)}(x).$$

Note that $F_{\text{MComb}_n}(1, t, 0, 0, \ldots) = \sum_{C \in \text{Comb}_n} t^{\text{red} C} = T_n(t)$ and so Theorem 15 is a generalization of Theorem 3.

**Remark 16.** Versions of Theorem 15 can also be given in terms of a completely different type on the set $\text{Nor}_n$ corresponding to a family of multicolored Lyndon trees and also in terms of colored Stirling permutations, see [13].

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**References**


