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A Note on the γ-Coefficients of the Tree Eulerian Polynomial

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Abstract

We consider the generating polynomial of the number of rooted trees on the set $\{1, 2, \ldots, n\}$ counted by the number of descending edges (a parent with a greater label than a child). This polynomial is an extension of the descent generating polynomial of the set of permutations of a totally ordered n-set, known as the Eulerian polynomial. We show how this extension shares some of the properties of the classical one. A classical product formula shows that this polynomial factors completely over the integers. From this product formula it can be concluded that this polynomial has positive coefficients in the γ-basis and we show that a formula for these coefficients can also be derived. We discuss various combinatorial interpretations of these coefficients in terms of leaf-labeled binary trees and in terms of the Stirling permutations introduced by Gessel and Stanley. These interpretations are derived from previous results of Liu, Dotsenko-Khoroshkin, Bershtein-Dotsenko-Khoroshkin, González D'León-Wachs and González D'León related to the free multibracketed Lie algebra and the poset of weighted partitions.

Keywords: Gamma positivity; Eulerian polynomial; Rooted trees

1 introduction

A *labeled rooted tree* T on the set $[n] := \{1, 2, \cdots, n\}$ is a tree whose nodes or vertices are the elements of $[n]$ and such that one of its nodes has been distinguished and called the *root*. For nodes x and y in T we say that *x is the child of y* or *y is the parent of x* if y is the first node following x in the unique path from x to the root of T and we

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say that $y = \mathrm{p}(x)$. Nodes that have children are said to be *internal* otherwise we call a node without children a *leaf.* If y is the parent of x, we say that the edge $\{x, y\}$ of T is *descending* (and we call x a *descent* of T) if the label of y is greater than the label of x. We denote $\mathrm{des}(T)$ the number of descents in T. Figure 1 shows all the rooted trees on [3] grouped by the number of descents. We draw the trees with the convention that parents come higher than their children and the root is the highest node. We denote $\mathcal{T}_n$ the set of rooted trees on $[n]$ and $\mathcal{T}_{n,i}$ the set of trees in $\mathcal{T}_n$ with exactly i descents.

For a given $n \geqslant 1$ define

$$T_n(t) := \sum_{T \in \mathcal{T}_n} t^{\mathrm{des}(T)} = \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i, \tag{1.1}$$

the descent generating polynomial of $\mathcal{T}_n$. We call $T_n(t)$ the *tree Eulerian polynomial* in analogy with the classical polynomial $A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}$, that is the descent generating polynomial of the set $\mathfrak{S}_n$ of permutations of $[n]$. We can identify the permutations in $\mathfrak{S}_n$ with the set of rooted trees on $[n]$ that have $n-1$ internal nodes, each of them having a unique child (and so containing a unique leaf). It is not hard to see that our definition of a descent on this set of trees coincides with the classical definition of descent in a permutation so the polynomial $T_n(t)$ is an extension of the polynomial $A_n(t)$. The polynomials $A_n(t)$ have been extensively studied in the literature and are known with the name of *Eulerian polynomials* since Euler was one of the first in studying them (see [20]). The Eulerian polynomial $A_n(t) = \sum_{k=0}^{n-1} A_{n,i} t^i$ have degree $n-1$ and its coefficients satisfy the relation

$$A_{n,i} = A_{n,n-1-i}. \tag{1.2}$$

For example, the Eulerian polynomial for $n = 3$ is $A_3(t) = 1 + 4t + t^2$. A polynomial that satisfies Equation 1.2 is called *symmetric* or *palindromic.*

It is a simple observation that a symmetric polynomial $f(t) = \sum_{k=0}^{d} f_i t^i$ of degree d with $f_i \in \mathbb{Z}$ can be written in the form

$$\sum_{i=0}^{d} f_i t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i}, \tag{1.3}$$

where the coefficients $\gamma_i \in \mathbb{Z}$, i.e., the set $\{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor}$, where $\lfloor \cdot \rfloor$ is the integer floor function, is a basis (known as the γ-*basis*) for the space of symmetric polynomials of degree d with integer coefficients. If $\gamma_i \geqslant 0$ for all i then we say that the polynomial $f(t)$ is γ-*nonnegative* and if $\gamma_i > 0$ for all $i \leqslant \lfloor \frac{d}{2} \rfloor$ then we say that the polynomial $f(t)$ is γ-*positive.* It is a result of Foata and Schützenberger ([8]) that $A_n(t)$ is γ-positive. And it follows from the work of Foata and Strehl in [9] that its coefficients γ_i have a nice combinatorial interpretation (see also [18]). Indeed, let $\widehat{\mathfrak{S}}_n$ be the set of permutations in $\mathfrak{S}_n$ that have no two adjacent descents and no descent in the last position. Then

$$\gamma_j = \{\sigma \in \widehat{\mathfrak{S}}_n \mid \mathrm{des}(\sigma) = j\}.$$

For example, $A_3(t) = (1+t)^2 + 2t$ with $\gamma_0 = 1$ and $\gamma_1 = 2$. The permutations in $\widehat{\mathfrak{S}}_3$ are, 123 with no descents and; 213 and 312 with one descent. Gal [10] and Brändén [2, 3] have introduced the use of the γ-basis in different contexts. Gal conjectured that the γ-coefficients of the h-polynomial of a flag simple polytope are all nonnegative. In particular, $A_n(t)$ is the h-vector of the permutahedron that is a flag simple polytope so Gal's conjecture is confirmed in this case. Postnikov, Reiner and Williams [17] have confirmed Gal's conjecture for the family of chordal nestohedra that is a large family of flag simple polytopes. For more information about γ-nonnegativity see [4].

We will show that the properties discussed above for the Eulerian polynomial $A_n(t)$ are also shared by the polynomial $T_n(t)$ in a similar fashion. The degree of $T_n(t)$ is also $n-1$ and it is easy to see from the definition of a descent that

$$|\mathcal{T}_{n,i}| = |\mathcal{T}_{n,n-1-i}|, \tag{1.4}$$

so $T_n(t)$ is also symmetric. Indeed there is a natural bijection $\mathcal{T}_{n,i} \simeq \mathcal{T}_{n,n-1-i}$ where the image of a labeled rooted tree $T \in \mathcal{T}_{n,i}$, is the tree in $\mathcal{T}_{n,n-1-i}$ with the same shape of T but where each label i has been replaced by $n+1-i$. For the example in Figure 1, $T_3(t) = 2 + 5t + 2t^2$.

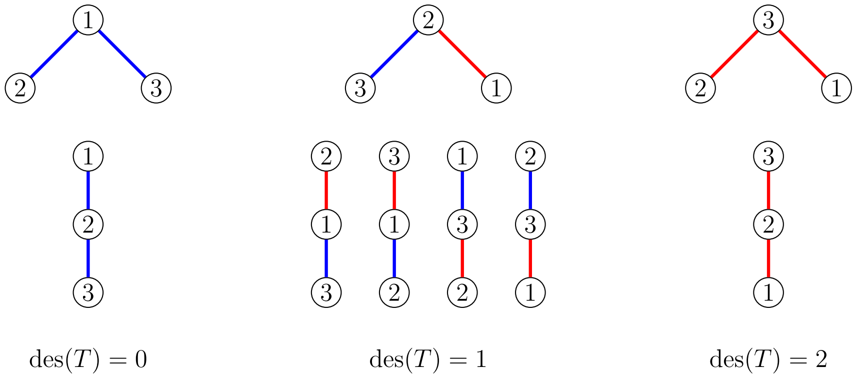

Figure 1: All labeled rooted trees on [3]

The following nice product formula for $T_n(t)$ can be obtained from the results in [7] (see also [12] and [6]).

Theorem 1. *For $n \geqslant 1$,*

$$\sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{1.5}$$

In particular, setting $t = 1$ in (1.5) reduces to the classical formula $|\mathcal{T}_n| = n^{n-1}$. Equation (1.5) implies that all the roots of this polynomial are real and negative. It is known and not difficult to show that a real-rooted symmetric polynomial with nonnegative real coefficients is γ-nonnegative (see [4, 10]). For example $T_3(t) = 2(1+t)^2 + t$, so $\gamma_0 = 2$ and $\gamma_1 = 1$. Although $T_n(t)$ is not in general an h-vector of a convex polytope (for example

$h_0 \neq 1$ for $n \geqslant 3$), it is of interest to find combinatorial formulas and interpretations of nonnegative γ-coefficients of general symmetric polynomials.

Let $\gamma_j(T_n(t))$ denote the j-th γ-coefficient of the symmetric polynomial $T_n(t)$. Equation (1.5) can be used to find a formula for the coefficients γ_j of $T_n(t)$. The values of $\gamma_j(T_n(t))$ for $n = 1, \ldots, 7$ appear in Table 1.

Theorem 2. *For $n \geqslant 1$,*

$$\gamma_j(T_n(t)) = \begin{cases} \displaystyle\sum_{\substack{J \subset [\frac{n-1}{2}] \\ |J|=j}} \prod_{i \in J} (n-2i)^2 \prod_{s \in [\frac{n-1}{2}] \setminus J} s(n-s) & \text{if } n \text{ is odd} \\ \displaystyle\frac{n}{2} \sum_{\substack{J \subset [\frac{n-2}{2}] \\ |J|=j}} \prod_{i \in J} (n-2i)^2 \prod_{s \in [\frac{n-2}{2}] \setminus J} s(n-s) & \text{if } n \text{ is even} \end{cases} \tag{1.6}$$

Proof. If we multiply

$$\begin{aligned} (n-i+it)(i+(n-i)t) &= (n-i)i + [(n-i)^2 + i^2]t + (n-i)it^2 \\ &= (n-i)i(1+t^2) + [(n-i)^2 + i^2]t \\ &= (n-i)i(1+2t+t^2) + [(n-i)^2 - 2i(n-i) + i^2]t \\ &= (n-i)i(1+t)^2 + (n-2i)^2 t. \end{aligned}$$

Equation 1.5 can be written as

$$T_n(t) = \begin{cases} \displaystyle\prod_{i=1}^{\frac{n-1}{2}} [(n-i)i(1+t)^2 + (n-2i)^2 t] & \text{if } n \text{ is odd,} \\ \displaystyle\frac{n}{2}(1+t) \prod_{i=1}^{\frac{n-2}{2}} [(n-i)i(1+t)^2 + (n-2i)^2 t] & \text{if } n \text{ is even,} \end{cases}$$

implying Formula 1.6. ∎

n/j	**0**	**1**	**2**	**3**
1	1			
2	1			
3	2	1		
4	6	8		
5	24	58	9	
6	120	444	192	
7	720	3708	3004	225

Table 1: Values of $\gamma_j(T_n(t))$ for $n = 1, \ldots, 7$.

The purpose of this note is to present four different combinatorial interpretations for the coefficients γ_j that are consequences of results in the work of Liu [15], Dotsenko-Khoroshkin [5], Bershtein-Dotsenko-Khoroshkin [1], González D'León-Wachs [14] and González D'León [13]. We present now one of these combinatorial interpretations, whose proof will be given in Section 2.

A *planar leaf-labeled binary tree* with label set $[n]$ is a rooted tree (a priori without labels) in which the set of children of every internal node is a totally ordered set with exactly two elements (the left and right children) and where each leaf has been assigned a unique element from the set $[n]$. By a subtree in a rooted tree T we mean the rooted tree induced by the descendents of any node x of T, including and rooted at x. We say that a planar leaf-labeled binary tree with label set $[n]$ is *normalized* if in each subtree, the leftmost leaf is the one with the smallest label. We denote the set of normalized binary trees with label set $[n]$ by Nor_n. All normalized trees with leaf labels in [3] are illustrated in Figure 2.

A *right descent* in a normalized tree is an internal node that is the right child of its parent. For $T \in \mathsf{Nor}_n$ we define $\text{rdes}(T) := |\{\text{right descents of T}\}|$. A *double right descent* is a right descent whose parent is also a right descent. We denote by NDRD_n the set of trees in Nor_n with no double right descents.

Theorem 3. *For $n \geqslant 1$ and $j \in \{0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor\}$,*

$$\gamma_j(T_n(t)) = |\{T \in \mathsf{NDRD}_n \mid \text{rdes}(T) = j\}|.$$

As it is illustrated in Figure 2, there are two trees in NDRD_3 (for $n = 3$ it happens to be equal to Nor_3) with $\text{rdes}(T) = 0$ and one with $\text{rdes}(T) = 1$, corresponding to $\gamma_0 = 2$ and $\gamma_1 = 1$ respectively.

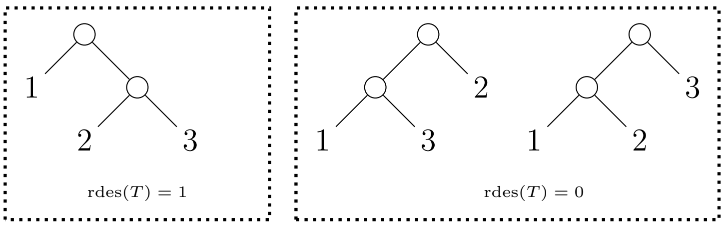

Figure 2: Set of trees in $\mathsf{NDRD}_3 = \mathsf{Nor}_3$

In Section 2 we provide the proof of Theorem 3 and an additional version of Theorem 3 also in terms of normalized trees but with a statistic different than rdes. In Section 3 we provide two additional versions of Theorem 3 in terms of the Stirling permutations introduced by Gessel and Stanley in [11]. In Section 4 we discuss a generalization of the γ-positivity of $T_n(t)$ to the positivity of certain symmetric function in the basis of elementary symmetric functions.

2 Combinatorial interpretations in terms of binary trees

Now we consider normalized trees T where every internal node x of T has been assigned an element $\mathbf{color}(x) \in \{0, 1\}$. We call an element of this set of trees a *bicolored normalized tree* on $[n]$. A *bicolored comb* is a bicolored normalized tree T satisfying the following coloring restriction:

(C) If x is a right descent of T then $\mathbf{color}(x) = 0$ and $\mathbf{color}(\mathrm{p}(x)) = 1$.

We denote by Comb_n the set of bicolored combs and by $\mathsf{Comb}_{n,i}$ the set of bicolored combs where i internal nodes have been colored 1 (and $n - 1 - i$ colored 0). Figure 3 illustrates the bicolored combs on [3] grouped by the number of internal nodes that have been colored 1.

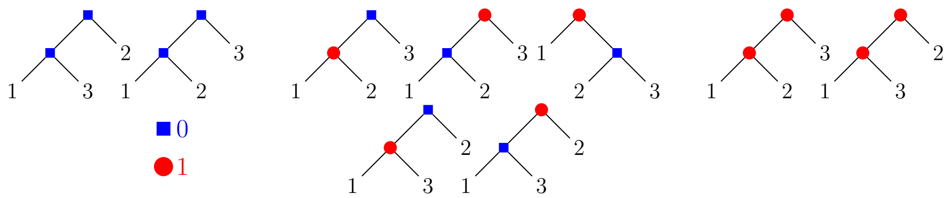

Figure 3: Set of bicolored combs on [3]

Denote by $\widetilde{T} \in \mathsf{Nor}_n$ the underlying uncolored normalized tree associated to a tree $T \in \mathsf{Comb}_n$. Note that the coloring condition (C) implies that $\widetilde{T} \in \mathsf{NDRD}_n$. Indeed, in a double right descent the coloring condition (C) cannot be satisfied since the parent of a double right descent is also a right descent. Also note that the monochromatic combs in $\mathsf{Comb}_{n,0}$ and $\mathsf{Comb}_{n,n-1}$ are just the traditional left combs that are described in [21] and that index a basis for the space $\mathcal{L}ie(n)$, the multilinear component of the free Lie algebra over $\mathbb{C}$ on n generators (see [21] for details). Liu [15] and Dotsenko-Khoroshkin [5] independently proved a conjecture of Feigin regarding the dimension of the multilinear component $\mathcal{L}ie_2(n)$ of the free Lie algebra with two compatible brackets, a generalization of $\mathcal{L}ie(n)$.

Theorem 4 ([5, 15])**.** *For $n \geqslant 1$, $\dim \mathcal{L}ie_2(n) = |\mathcal{T}_n|$.*

In particular, the space $\mathcal{L}ie_2(n)$ has the decomposition

$$\mathcal{L}ie_2(n) = \bigoplus_{i=0}^{n-1} \mathcal{L}ie_2(n, i),$$

where the subspace $\mathcal{L}ie_2(n, i)$ is the component generated by certain "bracketed permutations" with exactly i brackets of one of the types. Liu finds the following formula for the dimension of $\mathcal{L}ie_2(n, i)$.

Theorem 5 ([15, Proposition 11.3])**.** *For $n \geqslant 1$ and $i \in \{0, 1, \cdots, n - 1\}$,*

$$\dim \mathcal{L}ie_2(n, i) = |\mathcal{T}_{n,i}|.$$

In [1] Bershtein, Dotsenko and Khoroshkin found a basis for $\mathcal{L}ie_2(n,i)$ indexed by the elements of $\mathsf{Comb}_{n,i}$ giving the following alternative description for the dimension of $\mathcal{L}ie_2(n,i)$.

Theorem 6 ([1, Lemma 5.2])**.** *For $n \geqslant 1$ and $i \in \{0, 1, \cdots, n-1\}$,*

$$\dim \mathcal{L}ie_2(n,i) = |\mathsf{Comb}_{n,i}|.$$

Corollary 7. *For every $n \geqslant 1$ and $i \in \{0, \cdots, n-1\}$,*

$$|\mathsf{Comb}_{n,i}| = |\mathcal{T}_{n,i}|.$$

Problem 8. Find an explicit bijection $\mathsf{Comb}_{n,i} \to \mathcal{T}_{n,i}$, for every $n \geqslant 1$ and $i \in \{0, \cdots, n-1\}$.

Theorem 9 (Theorem 3)**.** *For $n \geqslant 1$ and $j \in \{0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor\}$,*

$$\gamma_j(T_n(t)) = |\{T \in \mathsf{NDRD}_n \mid \operatorname{rdes}(T) = j\}|.$$

Proof. First note that by the comments above if $T \in \mathsf{Comb}_n$ then its underlying uncolored tree $\widetilde{T} \in \mathsf{NDRD}_n$.

For a tree $T \in \mathsf{Comb}_n$ call $\operatorname{free}(T)$ the number of internal nodes that are not right descents and whose right child is a leaf. Then in NDRD_n we have that

$$\operatorname{free}(T) + 2\operatorname{rdes}(T) = n - 1.$$

Over the set of bicolored combs with m free nodes there is a free action of $(\mathbb{Z}_2)^m$ by toggling the colors of the free nodes. Then there are 2^m bicolored combs with the same underlying tree $\widetilde{T} \in \mathsf{NDRD}_n$. By Corollary 7 we can write $T_n(t)$ as

$$\begin{aligned}
T_n(t) &= \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i \\
&= \sum_{i=0}^{n-1} |\mathsf{Comb}_{n,i}| t^i \\
&= \sum_{T \in \mathsf{Comb}_n} t^{|\{x \text{ internal in } T \,|\, \mathbf{color}(x)=1\}|} \\
&= \sum_{\mathfrak{T} \in \mathsf{NDRD}_n} \sum_{\substack{T \in \mathsf{Comb}_n \\ \widetilde{T} = \mathfrak{T}}} t^{|\{x \text{ internal in } T \,|\, \mathbf{color}(x)=1\}|} \\
&= \sum_{\mathfrak{T} \in \mathsf{NDRD}_n} t^{\operatorname{rdes}(\mathfrak{T})} (1+t)^{\operatorname{free}(\mathfrak{T})} \\
&= \sum_{\mathfrak{T} \in \mathsf{NDRD}_n} t^{\operatorname{rdes}(\mathfrak{T})} (1+t)^{n-1-2\operatorname{rdes}(\mathfrak{T})}.
\end{aligned}$$

□

2.1 A second description in terms of normalized trees

In [14] the author and Wachs studied the relation between $\mathcal{L}ie_2(n,i)$ and the cohomology of the maximal intervals of a poset of weighted partitions. Using poset topology techniques they found an alternative description for the dimension of $\mathcal{L}ie_2(n,i)$.

Define the *valency* $v(x)$ of a node (internal or leaf) x of $T \in \mathsf{Nor}_n$ to be the minimal label in the subtree of T rooted at x. For an internal node x of T let $L(x)$ and $R(x)$ denote the left and right children of x respectively. A *Lyndon node* is an internal node x of T such that

$$v(R(L(x))) > v(R(x)). \tag{2.1}$$

A *Lyndon tree* is a normalized tree in which all its internal nodes are Lyndon. We denote $\mathrm{nlyn}(T)$ the number of non-Lyndon nodes in T. A *double non-Lyndon node* is a non-Lyndon node that is the left child of its parent and its parent is also a non-Lyndon node. We denote the set of trees in Nor_n with no double non-Lyndon nodes by NDNL_n. A *bicolored Lyndon tree* is a bicolored normalized tree satisfying the coloring condition:

(L) For every non-Lyndon node x of T then $\mathbf{color}(x) = 0$ and $\mathbf{color}(L(x)) = 1$.

The set of bicolored Lyndon trees is denoted Lyn_n and the set of the ones with exactly i nodes with color 1 is denoted $\mathsf{Lyn}_{n,i}$.

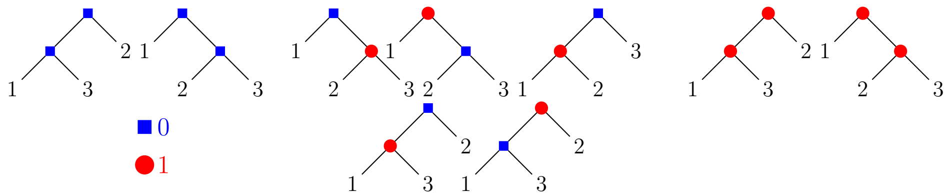

Figure 4: Set of bicolored Lyndon trees on [3]

Theorem 10 ([14, Section 5]). *For every $n \geqslant 1$ and $i \in \{0, \cdots, n-1\}$,*

$$\dim \mathcal{L}ie_2(n,i) = |\mathsf{Lyn}_{n,i}|.$$

Hence,

$$|\mathsf{Lyn}_{n,i}| = |\mathcal{T}_{n,i}|.$$

The proof of the following theorem follows the same arguments of the proof of Theorem 3.

Theorem 11. *For $n \geqslant 1$ and $j \in \{0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor\}$,*

$$\gamma_j(T_n(t)) = |\{T \in \mathsf{NDNL}_n \mid \mathrm{nlyn}(T) = j\}|.$$

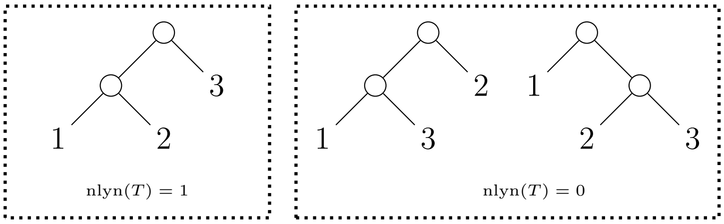

Figure 5: Set of trees in $\mathsf{NDNL}_3 = \mathsf{Nor}_3$

3 Combinatorial interpretation in terms of Stirling permutations

Consider now the set of multipermutations of the multiset $\{1, 1, 2, 2, \cdots, n, n\}$ such that all numbers between the two occurrences of any number m are larger than m. To this family belongs for example the permutation 12234431 but not 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. This family (denoted $\mathcal{Q}_n$) of permutations was introduced by Gessel and Stanley in [11] and the permutations in $\mathcal{Q}_n$ are known as *Stirling Permutations*.

For a permutation $\theta = \theta_1\theta_2 \dots \theta_{2n}$ in $\mathcal{Q}_n$ we say that the position i contains a *first occurrence* of a letter if $\theta_j \neq \theta_i$ for all $j < i$, otherwise we say that it contains a *second occurrence*. An *ascending adjacent pair* in θ is a pair (a, b) such that $a < b$ and in θ the second occurrence of a is the immediate predecessor of the first occurrence of b. An *ascending adjacent sequence (of length* 2*)* is a sequence $a < b < c$ such that (a, b) and (b, c) are both ascending adjacent pairs. For example, in $\theta = 13344155688776$ the ascending adjacent pairs are $(1, 5)$, $(5, 6)$ and $(3, 4)$ but the only ascending adjacent sequence is $1 < 5 < 6$. We denote NAAS_n the set of all Stirling permutations in $\mathcal{Q}_n$ that do not contain ascending adjacent sequences. Similarly, a *terminally nested pair* in θ is a pair (a, b) such that $a < b$ and in θ the second occurrence of a is the immediate successor of the second occurrence of b. A *terminally nested sequence (of length* 2*)* is a sequence $a < b < c$ such that (a, b) and (b, c) are both terminally nested pairs. For example, in $\theta = 13443566518877$ the terminally nested pairs are $(1, 5)$, $(5, 6)$ and $(3, 4)$ but the only terminally nested sequence is $1 < 5 < 6$. We denote NTNS_n the set of all Stirling permutations in $\mathcal{Q}_n$ that do not contain terminally nested sequences. For $\sigma \in \mathcal{Q}_n$, we denote $\mathrm{aapair}(\sigma)$ the number of ascending adjacent pairs in σ and $\mathrm{tnpair}(\sigma)$ the number of terminally nested pairs in σ.

The following result in [13] relates the statistics above in $\mathcal{Q}_{n-1}$ with the ones previously discussed for Nor_n.

Proposition 12 ([13, Proposition 4.8])**.** *There is a bijection $\phi : \mathsf{Nor}_n \to \mathcal{Q}_{n-1}$ such that for every $T \in \mathsf{Nor}_n$,*

1. $\mathrm{rdes}(T) = \mathrm{tnpair}(\phi(T))$
2. $\mathrm{nlyn}(T) = \mathrm{aapair}(\phi(T))$

3. $\phi(\mathsf{NDRD}_n) = \mathsf{NAAS}_{n-1}$

4. $\phi(\mathsf{NDNL}_n) = \mathsf{NTNS}_{n-1}$.

Corollary 13. *For $n \geqslant 1$ and $j \in \{0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor\}$,*

$$\begin{aligned}\gamma_j(T_n(t)) =& |\{T \in \mathsf{NTNS}_{n-1} \mid \text{tnpair}(T) = j\}| \\ =& |\{T \in \mathsf{NAAS}_{n-1} \mid \text{aapair}(T) = j\}|.\end{aligned}$$

Example 14. The Stirling permutations in $\mathcal{Q}_2$ are 1122, 1221 and 2211. In this particular case $\mathcal{Q}_2 = \mathsf{NAAS}_2 = \mathsf{NTNS}_2$ and the statistics in Table 2 imply that $\gamma_0 = 2$ and $\gamma_1 = 1$ are the γ-coefficients of the polynomial $T_3(t)$.

$\boldsymbol{\sigma}$	**tnpair**	**aapair**
1122	0	1
1221	1	0
2211	0	0

Table 2: tnpair and aapair statistics in $\mathcal{Q}_2$.

4 A comment about γ-positivity and e-positivity

Let $\mathbf{x} = x_1, x_2, \ldots$ be an infinite set of variables and $\Lambda = \Lambda_{\mathbb{Q}}$ the ring of symmetric functions with rational coefficients on the variables $\mathbf{x}$, that is, the ring of power series on $\mathbf{x}$ of bounded degree that are invariant under permutation of the variables.

Define $e_0 := 1$, for $n \geqslant 1$

$$e_n := \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and for an integer partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ (i.e., a weakly decreasing finite sequence of positive integers) $e_\lambda := \prod_i e_{\lambda_i}$. We call e_λ the *elementary symmetric function* corresponding to the partition λ.

It is known that for $n \geqslant 0$, the set $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for the n-th homogeneous graded component of Λ, where the grading is with respect to degree. See [16] and [19] for more information about symmetric functions.

Note that if we make the specialization $x_i \mapsto 0$ in Λ for all $i \geqslant 3$ then $e_1 \mapsto x_1 + x_2$, $e_2 \mapsto x_1 x_2$ and $e_i \mapsto 0$ for all $i \geqslant 3$. Thus for a partition λ of n (i.e., $\sum_i \lambda_i = n$) the symmetric function $e_\lambda \mapsto 0$ unless $\lambda = (2^j, 1^{n-2j})$ for some $j \in \mathbb{N}$. In that case,

$$e_{(2^j, 1^{n-2j})} \mapsto (x_1 x_2)^j (x_1 + x_2)^{n-2j}.$$

If we further replace $x_1 \mapsto 1$ and $x_2 \mapsto t$ we obtain

$$e_{(2^j, 1^{n-2j})} \mapsto t^j (1 + t)^{n-2j}.$$

In other words, the elementary basis in two variables is equivalent to the γ basis. A consequence of this observation is that another possible approach to conclude the γ-nonnegativity of a palindromic polynomial $f(t)$ is to find an e-nonnegative symmetric function $F(x_1, x_2, \dots)$ such that $f(t) = F(1, t, 0, 0, \dots)$.

4.1 Colored combs and comb type of a normalized tree

A *colored comb* is a normalized binary tree T together with a function **color** that assigns positive integers in $\mathbb{P}$ to the internal nodes of T and that satisfies the following coloring restriction: for each internal node x whose right child $R(x)$ is not a leaf,

$$\mathbf{color}(x) > \mathbf{color}(R(x)). \tag{4.1}$$

Note that the set of colored combs that only use the colors 1 and 2 are the same as the bicolored combs defined in Section 2. We denote MComb_n the set of colored combs with n leaves. Figure 6 shows an example of a colored comb.

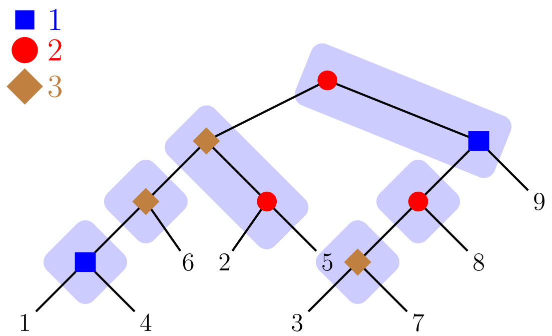

Figure 6: Example of a colored comb of comb type $(2, 2, 1, 1, 1, 1)$

We can associate a type to each $\Upsilon \in \mathsf{Nor}_n$ in the following way: Let $\pi(\Upsilon)$ be the finest (set) partition of the set of internal nodes of Υ satisfying

- for every pair of internal nodes x and y such that y is a right child of x, x and y belong to the same block of $\pi(\Upsilon)$.

We define the *comb type* $\lambda(\Upsilon)$ of Υ to be the integer partition whose parts are the sizes of the blocks of $\pi(\Upsilon)$.

Note that the coloring condition (4.1) is closely related to the comb type of a normalized tree. The coloring condition implies that in a colored comb Υ there are no repeated colors in each block B of the partition $\pi(\Upsilon)$ associated to Υ. So after choosing $|B|$ different colors for the internal nodes of Υ in B, there is a unique way to assign the colors such that Υ is a colored comb (the colors must decrease towards the right in each block of $\pi(\Upsilon)$). In Figure 6 this relation is illustrated.

For a colored comb C denote $\mu(C)$ the sequence of nonnegative integers such that

$$\mu(C)(j) := |\{x \text{ a internal node in } C \mid \mathbf{color}(x) = j\}|.$$

Let $\mathbf{x}^\mu := x_1^{\mu(1)} x_2^{\mu(2)} \cdots$ and

$$F_{\mathsf{MComb}_n}(\mathbf{x}) := \sum_{C \in \mathsf{MComb}_n} \mathbf{x}^{\mu(C)}.$$

The following theorem is a consequence of the definition of a colored comb, the definition of the symmetric functions $e_i(\mathbf{x})$ and the observations above (see [13]).

Theorem 15 ([13])**.** *For $n \geqslant 1$*

$$F_{\mathsf{MComb}_n}(\mathbf{x}) = \sum_{T \in \mathsf{Nor}_n} e_{\lambda(T)}(\mathbf{x}).$$

Note that $F_{\mathsf{MComb}_n}(1, t, 0, 0, \dots) = \sum_{C \in \mathsf{Comb}_n} t^{\mathrm{red}\, C} = T_n(t)$ and so Theorem 15 is a generalization of Theorem 3.

Remark 16. Versions of Theorem 15 can also be given in terms of a completely different type on the set Nor_n corresponding to a family of multicolored Lyndon trees and also in terms of colored Stirling permutations, see [13].

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