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REGULARITY AND UNIQUENESS OF SOME GEOMETRIC HEAT FLOWS AND IT'S APPLICATIONS

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REGULARITY AND UNIQUENESS OF SOME GEOMETRIC HEAT FLOWS AND IT'S APPLICATIONS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Tao Huang
Lexington, Kentucky

Director: Dr. Changyou Wang, Professor of Mathematics
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ABSTRACT OF DISSERTATION

REGULARITY AND UNIQUENESS OF SOME GEOMETRIC HEAT FLOWS AND IT'S APPLICATIONS

This manuscript demonstrates the regularity and uniqueness of some geometric heat flows with critical nonlinearity. First, under the assumption of smallness of renormalized energy, several issues of the regularity and uniqueness of heat flow of harmonic maps into a unit sphere or a compact Riemannian homogeneous manifold without boundary are established. For a class of heat flow of harmonic maps to any compact Riemannian manifold without boundary, satisfying the Serrin's condition, the regularity and uniqueness is also established. As an application, the hydrodynamic flow of nematic liquid crystals in Serrin's class is proved to be regular and unique. The natural extension of all the results to the heat flow of biharmonic maps is also presented in this manuscript.

KEYWORDS: Heat flow of harmonic maps, nematic liquid crystal flows, heat flow of biharmonic maps, regularity, uniqueness

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Date: 09. July 2013

REGULARITY AND UNIQUENESS OF SOME GEOMETRIC HEAT FLOWS AND IT'S APPLICATIONS

By
Tao Huang

Director of Dissertation: Changyou Wang

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Date: 09. July 2013

This document is dedicated to my wife, Xiaoli, my son, Sean, and my parents. Without their love and support I would have not gotten this far!

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Chapter 1 Introduction

It is well known that for geometric nonlinear evolution equations with critical nonlinearity the uniqueness and regularity of weak solutions is often a very challenging question. One of the most important examples is harmonic maps and its heat flows. It is a very important object in the study of topology and geometry and provide a rich of phenomena in both fields (see e.g. [63] and the references therein). In this thesis, the issues of uniqueness for heat flow of harmonic maps in higher dimensions and its applications will be investigated.

1.1 Overview of previous studies

Heat flow of harmonic maps

Let (M, g) be a n-dimensional compact Riemannian manifold possibly with $\partial M \neq \emptyset$ or complete Riemannian manifold with $\partial M = \emptyset$, and let $(N, h) \subset \mathbb{R}^k$ be a compact Riemannian manifold without boundary. For $m \geq 1$, $p \geq 1$, the Sobolev space $W^{m,p}(M, N)$ is defined by

$$W^{m,p}(M, N) = \left\{ v \in W^{m,p}(M, \mathbb{R}^{L+1}) \ : \ v(x) \in N \text{ for a.e. } x \in M \right\}.$$

For any $u : M \to N$, the Dirichlet energy is given by

$$E_1(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dx.$$

Definition 1.1.1. *A map $u \in W^{1,2}(M, N)$ is called a weakly harmonic map if it is a critical point of the Dirichlet energy functional $E_1(u)$ and the Euler-Lagrange equation of the weakly harmonic maps is*

$$-\Delta_g u = A(u)(\nabla u, \nabla u), \tag{1.1}$$

where Δ_g is the Laplacian operator on (M, g) and $A(u)(\nabla u, \nabla u)$ is the second fundamental form of N at u, which satisfies the following estimate

$$|A(u)(\nabla u, \nabla u)| \leq C|\nabla u|^2. \tag{1.2}$$

Harmonic maps are nonlinear extension of harmonic functions and are important objects in the study of topology and calculus of variations. They provide a rich phenomena in both differential geometry and analysis and play a very important role in the study of geometric analysis. It is easy to see from (1.2) that the nonlinearity of (1.1) is critical, which makes the regularity and uniqueness of weak solutions in higher dimensions a very challenge question. The regularity of weakly harmonic maps has been studied in dimension two by Hélein's famous works [33, 34, 35]. The partial regularity of weak solutions in higher dimensions has been established by Evans [23], Bethuel [2], Chang-Wang-Yang [9] and Riviere-Struwe [73].

The uniqueness of weakly harmonic maps is not true in general for higher dimensions as pointed out in Struwe [85]. In fact, in Struwe [85] (for $n = 3$) and Moser [70] (for $n \geq 4$), the uniqueness of weak solutions to (1.1) has been proved under the following smallness assumption

$$\sup_{x\in M, r>0} \frac{1}{r} \int_{B_r(x)\cap M} |\nabla u|^2 \, dx < \epsilon_0,$$

for some $\epsilon_0 > 0$, where $B_r(x) = \{y \in M : \ d_g(y, x) \leq r\}$ for $x \in M$ and d_g denotes the distance function on M induced by g.

The study of heat flow of harmonic maps began with the ground breaking work of Eells-Sampson [20]. For $0 < T \leq +\infty$, the heat flow of harmonic maps $u : M \times [0, T) \to N$ is defined by:

$$\begin{cases} \partial_t u - \Delta_g u = A(u)(\nabla u, \nabla u) & \text{in } M \times (0, T) \\ u = u_0 & \text{on } \partial_p(M \times [0, T]) \end{cases} \tag{1.3}$$

where $\partial_p(M \times [0, T]) = (M \times \{0\}) \cup (\partial M \times (0, T))$ denotes the parabolic boundary of $M \times [0, T]$, and $u_0 : M \to N$ is a given map. Denote

$$H^1\Big(M \times [0, T], N\Big) = \ \Big\{v \in W^{1,2}(M \times [0, T], \mathbb{R}^k) \ \Big| \ v(x, t) \in N, \\ \text{a.e. } (x, t) \in M \times [0, T]\Big\}.$$

Definition 1.1.2. *For $u_0 \in W^{1,2}(M, N)$ and $0 < T \leq +\infty$, $u \in H^1(M \times [0, T], N)$ is a weak solution of (1.3) if u satisfies $(1.3)_1$ in the sense of distribution and $(1.3)_2$ in the sense of trace.*

Here we denote $(\cdot)_i$ for the i-th equation of the system $(\cdot)$.

For the Cauchy problem of (1.3) ($\partial M = \emptyset$), Eells-Sampson [20] have proved that any homotopy class of maps from M to N contains a smooth harmonic mas whenever N is nonpositively curved. More precisely, they proved the following theorem:

Theorem 1.1.3. *If M is a compact Riemannian manifold without boundary and the sectional curvature K^N of N is nonpositive. Then for any $u_0 \in C^\infty(M, N)$, the Cauchy problem (1.3) has a unique smooth solution $u \in C^\infty(M \times [0, +\infty), N)$. As $t \to +\infty$, $u(x, t)$ converges to a harmonic map $u_\infty \in C^\infty(M, N)$ in $C^2(M, N)$.*

Later on, the same conclusion was proved for the case of compact Riemannian manifolds with boundary by Hamilton [31]. In [36], Hildebrandt-Kaul-Widman proved the similar conclusion under the assumption $u_0(M)$ belongs to a convex geodesic ball of N.

For general Reimannian manifold N, the existence of a unique, global weak solution to (1.3) with finitely many singularities has been obtained by Struwe [82] and Chang [5] for $n = 2$, i.e.,

Theorem 1.1.4. *For any $u_0 \in W^{1,2}(M,N)$, there exists a unique weak solution $u \in H^1(M \times [0,T], N)$ satisfying the following energy inequality*

$$\sup_{0<t<T} \int_M |\nabla u|^2(t)\, dx + 2\int_0^T |u_t|^2\, dxdt \leq \int_M |\nabla u_0|^2\, dx,$$

and $u \in C^\infty(M \times (0,\infty) \setminus S, N)$, where $S := \{(x_i, T_i)\}_{i=1}^{\overline{K}}$ is the collection of finite singularity points. Any (x_i, T_i) is characterized by

$$\lim_{t\uparrow T_i} \int_{B_r(x_i)\cap M} |\nabla u|^2(t)\, dx \geq \epsilon_0$$

for any $r > 0$ and some $\epsilon_0 > 0$.

The examples of finite and infinite time blow up have been established by Coron-Ghidaglia [16] and Chen-Ding [7] for $n \geq 3$ and Chang-Ding-Ye [8] for $n = 2$, so that finite time blow-ups of weak solutions to (1.3) do exist.

For higher dimensions ($n \geq 3$), Chen-Struwe [13] and Chen-Lin [11] have established the existence of global weak solutions $u \in H^1(M \times [0,T], N)$ to (1.3) for any $u_0 \in W^{1,2}(M,N)$. Also u is smooth away from a closed set $V \subset M \times (0,\infty)$ with $\mathcal{P}^m(V) < +\infty$, where $\mathcal{P}^m$ denotes m-dimensional Hausdorff measure with respect to the parabolic metric $\delta((x,t),(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$.

One of the important tools in their construction is the following energy monotonicity inequality which was first discovered by Struwe [83] for smooth solutions to (1.3), i.e.,

$$\Phi_{(\bar{x},\bar{t})}(\rho) \leq \Phi_{(\bar{x},\bar{t})}(r),\ \forall \bar{x} \in \mathbb{R}^n, \bar{t} > 0,\ 0 < \rho \leq r \leq \sqrt{\bar{t}} \tag{1.4}$$

where

$$\Phi_{(\bar{x},\bar{t})}(\rho) = \rho^2 \int_{\mathbb{R}^n \times \{\bar{t}-\rho^2\}} |\nabla u|^2(x,t) G(x-\bar{x}, t-\bar{t})\, dx$$

and

$$G(y,s) = \frac{1}{(4\pi|s|)^{\frac{n}{2}}} \exp\Big(-\frac{|y|^2}{4|s|}\Big),\ y \in \mathbb{R}^n, s < 0$$

is the fundamental solution to the backward heat equation on $\mathbb{R}^n$.

Concerning the uniqueness for weak solutions to (1.3), Freire [26] first proved that in dimension $n = 2$, uniqueness holds for the weak solutions whose Dirichlet energy is monotone decreasing with respect to t (see L.Wang [96] and L. Z. Lin [64] for a new simple proof). For $n \geq 3$, there are non-uniqueness for weak solutions to (1.3) constructed by Coron [15] and Bethuel-Coron-Ghidaglia-Soyeur [3]. In fact, Coron [15] proved that for $n = 3$ and $u_0 = \phi(\frac{x}{|x|}) : \mathbb{R}^3 \to S^2$, where $\phi : S^2 \to S^2$ is a harmonic map satisfying

$$\int_{S^2} x|\nabla\phi(x)|^2\, d\mathcal{S}(x) \neq 0,$$

there are infinitely many weak solutions to the Cauchy problem of (1.3) with initial data u_0. Since u_0 (as a stationary weak solution to (1.3)) does not satisfy the monotonicity formula (1.4), it is different from those constructed by Chen-Struwe.

Partially motivated by [15], Struwe [81] has raised the following question:

Struwe's Question: *For $M = \mathbb{R}^n$, exhibit a class of functions within which (1.3) posses a unique solution. A potential candinate is the class of functions satisfying the strong monotonicity formula* (1.4).

To the best of my knowledge, this question is largely open. In this thesis, some uniqueness results for the heat flow of harmonic maps (1.3) will be presented, that may shed light on the validity of Struwe's conjecture as above.

Hydrodynamic flow of nematic liquid crystals

As a very important application of heat flow of harmonic maps, the hydrodynamic flow of nematic liquid crystals in $\mathbb{R}^n \times [0, T]$, for any $n \geq 3$ and some $0 < T < +\infty$, is given by

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_n) & \text{in } \mathbb{R}^n \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0, T) \\ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d & \text{in } \mathbb{R}^n \times (0, T) \\ (u, d) = (u_0, d_0) & \text{on } \mathbb{R}^n \times \{0\} \end{cases} \tag{1.5}$$

where $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ is the velocity field of underlying incompressible fluid, $d : \mathbb{R}^n \times [0, T] \to S^2$ is the director field of nematic liquid crystal molecules, $P : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is the pressure function, $\nabla \cdot$ denotes the divergence operator on $\mathbb{R}^n$, $\nabla d \otimes \nabla d = \left(\frac{\partial d}{\partial x_i} \cdot \frac{\partial d}{\partial x_j}\right)_{1 \leq i,j \leq n}$ is the stress tensor induced by the director field d, $\mathbb{I}_n$ is the identity matrix of order n, $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the initial velocity field with $\nabla \cdot u_0 = 0$, and $d_0 : \mathbb{R}^n \to S^2$ is the initial director field.

Definition 1.1.5. $(u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)$ *is a weak solution to (1.5) if* (u, d) *satisfies* $(1.5)_1$*-*$(1.5)_3$ *in the sense of distributions and* $(1.5)_4$ *in the sense of trace.*

The system (1.5) was studied by Lin [59] and Lin-Liu [61] around the 1990's, which is a simplified version of the Ericksen-Leslie system modeling the hydrodynamics of liquid crystal materials proposed by Ericksen [21] and Leslie [57] in the 1960's. It is a macroscopic continuum description of the time evolution of the material under the influence of both the flow field and the macroscopic description of the microscopic orientation configurations of rod-like liquid crystals. (see [21], [57], [59], and [61] for more details). Mathematically, the system (1.5) is a strong coupling of the Navier-Stokes equations and the (transported) heat flow of harmonic maps into S^2.

For $n = 2$, Lin-Lin-Wang [60] have proved the existence of global Leray-Hopf type weak solutions to (1.5) with initial and boundary conditions, which are smooth away from finitely many possible singular times (see Hong [39] and Xu-Zhang [97] for related works). Here Leray-Hopf type refers to a suitable version of the following energy inequality:

$$\int_\Omega \left(|u|^2+|\nabla d|^2\right)(T)\,dx + 2\int_0^T\int_\Omega \left(\mu|\nabla u|^2+|\Delta d+|\nabla d|^2 d|^2\right)\,dxdt$$
$$\le \int_\Omega \left(|u_0|^2+|\nabla d_0|^2\right)\,dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Lin-Wang [62] have proved the uniqueness for such weak solutions. It remains a very challenging open problem to prove the global existence of Leray-Hopf type weak solutions and partial regularity of suitable weak solutions to (1.5) in higher dimensions. A blow-up criterion was obtained for the local strong solution to (1.5) for $n = 3$ in [46], i.e., if $0 < T_* < +\infty$ is the maximum time interval of the strong solution to (1.5), then

$$\int_0^{T_*} \left(\|\nabla \times u\|_{L^\infty} + \|\nabla d\|^2_{L^\infty}\right)\,dt = +\infty.$$

Recently, the local well-posedness of (1.5) was obtained for initial data (u_0, d_0) with $(u_0, \nabla d_0) \in L^3_{\text{uloc}}(\mathbb{R}^3)$, the space of uniformly locally L^3-integrable functions, of small norm for $n = 3$ in [38]. While the global well-posedness of (1.5) was obtained by [95] for $(u_0, d_0) \in \text{BMO}\times\ \text{BMO}^{-1}$ of small norm for $n \ge 3$. Here for any open set $U \subset \mathbb{R}^{n+1}$, BMO$(U)$ denotes the space of functions of bounded mean oscillations: $f \in \text{BMO}(U)$ if

$$[f]_{\text{BMO}(U)} := \sup\left\{ ⨍_{P_r(z)} |f - f_{P_r(z)}| :\ P_r(z) \subset U \right\} < +\infty,$$

where $⨍_{P_r(z)} = \frac{1}{|P_r(z)|}\int_{P_r(z)}$ and $f_{P_r(z)} = ⨍_{P_r(z)} f$ denotes the average of f over $P_r(z)$. And $f \in \text{BMO}^{-1}(U)$ if

$$[f]_{\text{BMO}^{-1}(U)} := \inf\left\{ \sum_{i=1}^n [f_i]_{\text{BMO}(U)} \,|\, f_i \in \text{BMO}(U) \text{ and } f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \right\} < +\infty,$$

The existence of global Leray-Hopf weak solutions to the Navier-Stokes equations has long been established by Leray [56] and Hopf [42]. However the uniqueness (regularity) of Leray-Hopf solutions in dimension three remains largely open. In [78], Serrin proved the so called 'weak-strong' uniqueness, i.e., the uniqueness holds for Leray-Hopf solutions u, v with the same initial data, if $u \in L^p_t L^q_x(\mathbb{R}^n \times [0,T])$, where the $L^q_t L^p_x$-space is defined by

$$L^q_t L^p_x\Big(M\times[0,T],\mathbb{R}^k\Big) := \Big\{ f : M\times[0,T] \to \mathbb{R}^k \;\Big|\; f \in L^q([0,T], L^p(M)) \Big\},$$

and $p \ge 2$ and $q \ge n$ satisfy

$$\frac{2}{p} + \frac{n}{q} = 1. \tag{1.6}$$

The smoothness of such solutions was established by Ladyzhenskaya in [51] for $p > 2$ and $q > n$. In the fundamental work [22], Escauriaza-Seregin-Šverák have proved the smoothness of Serrin's solutions for the endpoint case $(p, q) = (+\infty, n)$ when $n = 3$ (see also [17] for $n \geq 4$). Wang [90] proved smoothness of weak solutions u to the heat flow of harmonic maps such that $\nabla u \in L_t^p L_x^q(\mathbb{R}^n \times [0, T])$ with $\frac{2}{p} + \frac{n}{q} = 1$ for $n \geq 4$ (or $q \geq 4$ for $2 \leq n < 4$, see [46] for the case $2 < q < 4$ when $2 \leq n < 4$). In [46], the uniqueness of Serrin's solutions to the heat flow of harmonic maps is also established when $p > 2$ and $q > n$. Motivated by these results, the regularity and uniqueness of Serrin's (p, q)-solutions to the system (1.5) of nematic liquid crystal flows will be investigated in this dissertation.

Heat flow of biharmonic maps

For $n \geq 4$ and $L \geq k \geq 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $N \subset \mathbb{R}^{L+1}$ be a k-dimensional compact Riemannian manifold without boundary. For $m \geq 1$, $p \geq 1$, the Sobolev space $W^{m,p}(\Omega, N)$ is defined by

$$W^{m,p}(\Omega, N) = \left\{ v \in W^{m,p}(\Omega, \mathbb{R}^{L+1}) \ : \ v(x) \in N \text{ for a.e. } x \in \Omega \right\}.$$

On $W^{2,2}(\Omega, N)$, there are two second order energy functionals:

$$E_2(u) = \int_\Omega |\Delta u|^2 \quad \text{and} \quad F_2(u) = \int_\Omega |(\Delta u)^T|^2,$$

where $(\Delta u)^T$ is the tangential component of Δu to $T_u N$ at u, which is also called the tension field of u (see [19]). A map $u \in W^{2,2}(\Omega, N)$ is called an extrinsic (or intrinsic) biharmonic map, if u is a critical point of $E_2(\cdot)$ (or $F_2(\cdot)$ respectively). It is well known that biharmonic maps are higher-order extensions of harmonic maps, which are critical points of the Dirichlet energy $E_1(u) = \int_\Omega |\nabla u|^2$ over $W^{1,2}(\Omega, N)$. Recall that the Euler-Lagrange equation of (extrinsic) biharmonic maps is (see [89] Lemma 2.1):

$$\Delta^2 u = \mathcal{N}_{\text{bh}}[u] := [\Delta(A(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla(P(u)) \rangle - \langle \Delta(P(u)), \Delta u \rangle] \perp T_u N, \tag{1.7}$$

where $P(y) : \mathbb{R}^{L+1} \to T_y N$ is the orthogonal projection for $y \in N$, and $A(y)(\cdot, \cdot) = \nabla P(y)(\cdot, \cdot)$ is the second fundamental form of N at $y \in N$. Throughout this thesis, denote $\mathcal{N}_{\text{bh}}[u]$ to be the nonlinearity in the right hand side of the biharmonic map equation (1.7). If $N = \mathbb{S}^L := \{y \in \mathbb{R}^{L+1} : \ |y| = 1\}$, then direct calculations yield

$$\mathcal{N}_{\text{bh}}[u] = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla \Delta u \rangle) u.$$

Motivated by the regularity theory of harmonic maps by Schoen-Uhlenbeck [86], Hélein [34], Evans [23], Bethuel [2], Lin [58], Rivière [72], and many others, the study of biharmonic maps has attracted considerable interest and prompted a large number of interesting works by analysts during the last several years. The regularity of biharmonic maps to $N = \mathbb{S}^L$ – the unit sphere in $\mathbb{R}^{L+1}$ – was first studied by Chang-Wang-Yang [10]. Wang [89, 91, 92] extended the main theorems of [10] to any compact Riemannian manifold N without boundary. He proves smoothness of biharmonic maps when the dimension $n = 4$, and the partial regularity of stationary biharmonic maps when $n \geq 5$. Here we mention in passing the interesting works on biharmonic maps by Angelsberg [1], Strzelecki [80], Hong-Wang [40], Lamm-Rivière [54], Struwe [84], Ku [49], Gastel-Scheven [28], Scheven [75, 76], Lamm-Wang [55], Moser [68, 69], Gastel-Zorn [29], Hong-Yin [41], and Gong-Lamm-Wang [30].

The initial and boundary value problem for the heat flow of biharmonic maps is described as follows: For $0 < T \leq +\infty$, and $u_0 \in W^{2,2}(\Omega, N)$, a map $u \in W_2^{1,2}(\Omega \times [0,T], N)$, i.e. $\partial_t u, \nabla^2 u \in L^2(\Omega \times [0,T])$, is called a weak solution of the heat flow of biharmonic maps, if u satisfies in the sense of distributions

$$\begin{cases} \partial_t u + \Delta^2 u = \mathcal{N}_{\text{bh}}[u] & \text{in } \Omega \times (0,T) \\ u = u_0 & \text{on } \partial_p(\Omega \times [0,T]) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u_0}{\partial \nu} & \text{on } \partial\Omega \times [0,T), \end{cases} \tag{1.8}$$

where ν denotes the outward unit normal of $\partial\Omega$. Denote the parabolic boundary of $\Omega \times [0,T]$ by $\partial_p(\Omega \times [0,T]) = (\Omega \times \{0\}) \cup (\partial\Omega \times (0,T))$. So when $N = \mathbb{S}^L$, $(1.8)_1$ can be written as

$$\partial_t u + \Delta^2 u = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\langle \nabla u, \nabla \Delta u \rangle)u. \tag{1.9}$$

The formulation of heat flow of biharmonic maps (1.8) remains unchanged, if Ω is replaced by a n-dimensional compact Riemannian manifold M with boundary ∂M. On the other hand, if Ω is replaced by a n-dimensional compact Riemannian manifold without boundary or a complete, non-compact Riemannian manifold without boundary M, then the Cauchy problem of heat flow of biharmonic maps is considered. More precisely, if $\partial M = \emptyset$, then (1.8) becomes

$$\begin{cases} \partial_t u + \Delta^2 u = \mathcal{N}_{\text{bh}}[u] & \text{in } M \times (0,T) \\ u = u_0 & \text{on } M \times \{0\}. \end{cases} \tag{1.10}$$

The Cauchy problem (1.10) was first studied by Lamm [52], [53] for $u_0 \in C^\infty(M, N)$ in dimension $n = 4$, where the existence of a unique, global smooth solution is established under the condition that $\|u_0\|_{W^{2,2}(M)}$ is sufficiently small. For any $u_0 \in W^{2,2}(M, N)$, the existence of a unique, global weak solution of (1.10), that is smooth away from finitely many times, has been independently proved by Gastel [27] and Wang [93]. With suitable modifications of their proofs, the existence theorem in [27] and [93] can be extended to (1.8) for any compact 4-dimensional Rimannian manifold M with boundary ∂M, if, in addition, the trace of u_0 on ∂M for $u_0 \in W^{2,2}(M, N)$ satisfies $u_0|_{\partial M} \in W^{\frac{7}{2},2}(\partial M, N)$ (see [43]). Namely, there is a unique, global weak solution $u \in W_2^{1,2}(M \times [0,\infty), N)$ of (1.8) such that (i) $E_2(u(t))$ is monotone decreasing for $t \geq 0$; and (ii) there exist $T_0 = 0 < T_1 < \ldots < T_k < T_{k+1} = +\infty$ such that

$$u \in \bigcap_{i=0}^{k} C^\infty(M \times (T_i, T_{i+1}), N) \ \text{ and } \ \nabla u \in \bigcap_{i=0}^{k} C^\alpha(\overline{M} \times (T_i, T_{i+1}), N), \ \forall\, \alpha \in (0,1).$$

For dimensions $n \geq 4$, Wang [94] established the well-posedness of (1.10) on $\mathbb{R}^n$ for any $u_0 : \mathbb{R}^n \to N$ that has sufficiently small BMO norm. Moser [71] showed the existence of global weak solutions $u \in W_2^{1,2}(\Omega \times [0,\infty), N)$ to (1.8) on any bounded smooth domain $\Omega \subset \mathbb{R}^n$ for $n \leq 8$ and $u_0 \in W^{2,2}(\Omega, N)$.

1.2 Main results and structure of the thesis

Heat flow of harmonic maps

The main result of the uniqueness of weak solutions to (1.3) is the following theorem .

Theorem 1.2.1. *For $n \geq 2$ and $1 < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$ and $R_0 = R_0(M, g, \epsilon_0) > 0$ such that if*
(i) (M, g) is a n-dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;
(ii) $(N, h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and
(iii) $u_1, u_2 \in H^1(M \times [0,T], N)$ are two weak solutions of (1.3), with $u_1 = u_2 = u_0$ on $\partial_p(M \times [0,T])$ for some $u_0 \in W^{1,2}(M, N)$, that satisfy

$$\max_{i=1,2} \left[\|\nabla u_i\|_{M_{R_0}^{p,p}(M\times(0,T))} + \|\partial_t u_i\|_{M_{R_0}^{p,2p}(M\times(0,T))} \right] \leq \epsilon_0, \tag{1.11}$$

then $u_1 \equiv u_2$ on $M \times [0,T]$.

Here $M_R^{p,\lambda}$ denote the parabolic Morrey space . For any $1 \leq p < +\infty$, $0 \leq \lambda \leq n+2$, $0 < R \leq +\infty$, and any open set $U = U_1 \times U_2 \subset M \times \mathbb{R}$,

$$M_R^{p,\lambda}(U) = \left\{ f \in L^p_{\text{loc}}(U) : \Big\| f \Big\|_{M_R^{p,\lambda}(U)} < +\infty \right\},$$

where

$$\Big\| f \Big\|_{M_R^{p,\lambda}(U)} = \Big(\sup_{(x,t)\in U} \sup_{0<r<\min\{R, d_g(x,\partial U_1), \sqrt{t}\}} r^{\lambda-n-2} \int_{P_r(x,t)\cap U} |f|^p \Big)^{\frac{1}{p}}.$$

Here

$$P_r(x,t) = B_r(x) \times (t - r^2, t], \text{ with } B_r(x) = \{y \in M : \ d_g(y,x) \leq r\}$$

for $(x,t) \in U$.

Recall that N is a Riemannian homogeneous manifold if there exists a finite dimensional Lie group $\mathcal{G}$ ($\dim \mathcal{G} = s < +\infty$) that acts transitively on N by isometries.

There are two main ideas in the proof of Theorem 1.2.1:
(i) an ϵ_0-regularity theorem for the heat flow of harmonic maps that satisfy the smallness condition (2.22), which is new and improves the regularity theorem previously

obtained by Chen-Li-Lin [12], Feldman [25], and Chen-Wang [14]. In particular, for $i = 1, 2$ $u_i \in C^\infty(M \times (0, T])$ and satisfies the gradient estimate:

$$\max_{i=1,2} |\nabla u_i|(x,t) \le C\epsilon_0 \Big(\frac{1}{R_0} + \frac{1}{d_g(x, \partial M)} + \frac{1}{\sqrt{t}} \Big), \ \forall (x,t) \in M \times (0, T], \tag{1.12}$$

(ii) applications of (1.12), the Hardy inequality, and a generalized Gronwall inequality type argument.

Now a few remarks are in order.

Remark 1.2.2. *i) Note that by the Hölder inequality, the Morrey norm* $\mathcal{E}(p) := (\|\nabla u\|_{M^{p,p}(\cdot)} + \|\partial_t u\|_{M^{p,2p}(\cdot)})$ *is monotone increasing for* $1 < p \le 2$*. The bound of* $\mathcal{E}(2)$ *for solutions* u *to (1.3) holds if* u *satisfies*
(a) a local energy inequality (assume $M = \mathbb{R}^n$ *for simplicity):*

$$\int_{P_r(x,t)} |\partial_t u|^2 \le \frac{C}{(R-r)^2} \int_{P_R(x,t)} |\nabla u|^2, \quad \forall (x,t) \in \mathbb{R}^{n+1}_+, \ 0 < r \le \frac{R}{2}, \ R \le \sqrt{t}, \tag{1.13}$$

(b) a local energy monotonicity inequality:

$$r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \le C R^{-n} \int_{P_R(x,t)} |\nabla u|^2, \quad \forall (x,t) \in \mathbb{R}^{n+1}_+, \quad 0 < r \le \frac{R}{2}, \ R \le \sqrt{t}. \tag{1.14}$$

Both properties hold if u *is either a smooth solution (see [82] and [13]) or a stationary solution of (1.3) (see [12], [25], and [14]). Therefore, under (1.13) and (1.14), the condition (1.11) is satisfied, provided that there exists* $R_0 > 0$ *such that there holds*

$$\sup \Big\{ R_1^{-n} \int_{P_{R_1}(x,t)} |\nabla u|^2 \ \Big| \ x \in \mathbb{R}^n, \ R_1 = \min\{R_0, \sqrt{t}\} \Big\} \le \epsilon_0^2. \tag{1.15}$$

Hence Theorem 1.2.1 implies that uniqueness does hold for the class of solutions that satisfy, in addition to (1.13) and (1.14), the smallness condition (1.15), which partially answers the Struwe's question.
ii) For any compact or complete noncompact (M, g) *without boundary, there exists* $\epsilon_0 > 0$ *such that if the initial data* $u_0 : M \to N$ *satisfies for some* $R_0 > 0$*,*

$$\sup \Big\{ r^{2-n} \int_{B_r(x)} |\nabla u_0|^2 \ \Big| x \in M, \ r \le R_0 \Big\} \le \epsilon_0^2,$$

then as a consequence of the local well-posedness theorem by Wang [95], there exists $0 < T_0 (\approx R_0^2)$ *and a solution* $u \in C^\infty(M \times (0, T_0), N)$ *of (1.3) that satisfies condition (1.11).*

Motivated by the proof of Theorem 1.2.1, the following convexity property of the Dirichlet energy $E_1(u)$ holds.

Theorem 1.2.3. *For* $n \ge 2$, $1 < p \le 2$, *and* $1 \le T \le \infty$, *there exist* $\epsilon_0 = \epsilon_0(p, n) > 0$, $R_0 = R_0(M, g, \epsilon_0) > 0$, *and* $0 < T_0 = T_0(\epsilon_0) < T$ *such that if*

(i) (M,g) is a n-dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;
(ii) $(N,h)\subset\mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and
(iii) $u\in H^1(M\times[0,T],N)$ is a weak solution of (1.3), with $u=u_0$ on $\partial_p(M\times[0,T])$ for some $u_0\in W^{1,2}(M,N)$, that satisfies

$$\|\nabla u\|_{M^{p,p}_{R_0}(M\times(0,T))}+\|\partial_t u\|_{M^{p,2p}_{R_0}(M\times(0,T))}\le\epsilon_0, \tag{1.16}$$

then
(i) the Dirichlet energy $E(u(t)):=\frac{1}{2}\int_M|\nabla u|^2$ is monotone decreasing for $t\ge T_0$; and
(ii) for any $t_2\ge t_1\ge T_0$,

$$\int_M|\nabla(u(t_1)-u(t_2))|^2\le C\Big[\int_M|\nabla u(t_1)|^2-\int_M|\nabla u(t_2)|^2\Big].$$

The convexity property was first observed by Schoen [77] for the Dirichlet energy of harmonic maps into manifolds N with nonpositive sectional curvatures. In Chapter 2, it will be shown that the convexity property is also true for harmonic maps with small renormalized energy, which yields a new proof of the uniqueness theorem by Struwe [85] and Moser [70].

A direct consequence of Theorem 1.2.3 is the following uniqueness of limit at $t=\infty$ for (1.3).

Corollary 1.2.4. *For $n\ge 2$ and $1<p\le 2$, there exist $\epsilon_0=\epsilon_0(p,n)>0$, and $R_0=R_0(M,g,\epsilon_0)>0$ such that if*
(i) (M,g) is a n-dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;
(ii) $(N,h)\subset\mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and
(iii) $u\in H^1(M\times[0,\infty),N)$ is a weak solution of (1.3), with $u=u_0$ on $\partial_p(M\times[0,\infty])$ for some $u_0\in W^{1,2}(M,N)$, that satisfies the condition (1.16).
Then there exists a harmonic map $u_\infty\in C^\infty(M,N)\cap W^{1,2}(M,N)$, with $u_\infty=u_0$ on ∂M, such that

$$\lim_{t\uparrow\infty}\|u(t)-u_\infty\|_{W^{1,2}(M)}=0,$$

and, for any compact subset $K\subset\subset M$ and $m\ge 1$,

$$\lim_{t\uparrow\infty}\|u(t)-u_\infty\|_{C^m(K)}=0.$$

The uniqueness of limit at $t=\infty$ has been proved by Hartman [32] for the smooth solutions to (1.3) when N has nonpositive sectional curvatures. L. Simon in his celebrated work [79] has shown the unique limit at $t=\infty$ for smooth solutions to (1.3) into a target manifold (N,h) that is real analytic. Note that the solution u in our theorem is allowed to be singular near the parabolic boundary $\partial_p(M\times(0,\infty))$, as the initial-boundary data u_0 may be in $W^{1,2}(M,N)$. Also, the proof of Theorem 1.2.3 depends only on the smallness condition (1.16) and the small energy regularity theorem. There are two very interesting articles by L.Wang [96] and L.Z. Lin [64], in which Theorems 1.2.1, 1.2.3, and Corollary 1.2.4 were proven for Struwe's almost regular solution u to (1.3) in dimension $n = 2$ when the Dirichlet energy of u_0 is sufficiently small. Since Struwe's solution u to (1.3) satisfies the energy inequality, the condition in [64] yields the global smallness:

$$\sup_{t\geq 0} E(u(t)) + \int_{M\times[0,t]} |\partial_t u|^2 \leq E(u_0) \leq \epsilon_0^2, \ \forall t > 0,$$

which is stronger than (1.16) in dimension $n = 2$. There is also an interesting paper by Topping [88] that addressed the rigidity at $t = \infty$ of the heat flow of harmonic maps from S^2 to S^2.

A class of weak solutions that satisfy the smallness condition (1.16) are the so-called Serrin (l, q)-solutions.

Definition 1.2.5. *A weak solution $u \in H^1_{loc}(M \times [0,T], N)$ of (1.3) is called a Serrin (l, q)-solution if, in addition, $\nabla u \in L^q_t L^l_x(M \times [0,T])$ for some $l \geq n$ and $q \geq 2$ satisfying*

$$\frac{n}{l} + \frac{2}{q} = 1. \tag{1.17}$$

In Chapter 2, it will be verified that if u is a Serrin (l, q)-solution to (1.3) with $l > n$, and $u|_{\partial_p(M\times[0,T])} = u_0$ for a given $u_0 : M \to N$ with $\nabla u_0 \in L^r(M)$ for some $n < r < \infty$, then u satisfies (1.16) for some $p_0 > 1$. (For such initial and boundary data u_0, the local existence of Serrin's (l, q)-solutions of (1.3) can be shown by the standard fixed point theory. In fact, it can be verified by the argument in Fabes-Jones-Riviere [24] §4). The following is the uniqueness result for Serrin's (l, q)-solutions of the heat flow of harmonic maps into a general Riemannian manifold.

Theorem 1.2.6. *For $n \geq 2$, $0 < T \leq +\infty$, let (M, g) be either a compact or complete Riemannian manifold without boundary or a compact Riemannian manifold with boundary, and N be a compact Riemannian manifold without boundary. Let $u_1, u_2 \in H^1(M \times [0,T], N)$ be two weak solutions of (1.3), with $u_1 = u_2 = u_0$ on $\partial_p(M \times [0,T])$ for some $u_0 \in W^{1,2}(M,N)$, such that $\nabla u_1, \nabla u_2 \in L^q_t L^l_x(M \times [0,T])$ for some (l, q) satisfying (1.17) with $l > n, q > 2$. Then $u_1, u_2 \in C^\infty(M \times (0,T))$, and $u_1 \equiv u_2$ on $M \times [0,T]$.*

Theorem 1.2.6 remains open for the end point case $l = n, q = +\infty$. Lin-Wang [62] have proved that uniqueness holds for two weak solutions u_1, u_2 to (1.3) with the same initial data, provided that $\nabla u_1, \nabla u_2 \in C([0,T], L^n(M))$. Wang [90] has proved that for any $n \geq 4$, any weak solution $u \in H^1(M \times [0,T], N)$ with $\nabla u \in L^\infty_t L^n_x(M \times [0,T])$ belongs to $C^\infty(M \times (0,T])$. However, since $\|\nabla u(t)\|_{L^n(M)}$ may lack continuity at $t = 0$, the issue of uniqueness for the end point case remains unsolved.

Hydrodynamic flow of nematic liquid crystals

Motivated by the study of heat flow of harmonic maps, the following regularity and uniqueness of the weak solutions to the hydrodynamic flow of nematic liquid crystals holds for the Serrin's solutions, which is defined as follows:

Definition 1.2.7. *A weak solution* $(u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)$ *of (1.5) is called a Serrin (p, q)-solution, if* $(u, \nabla d) \in L_t^p L_x^q(\mathbb{R}^n \times [0, T])$ *for some* (p, q) *satisfying (1.6).*

The main result on the regularity and uniqueness of Serrin's (p, q)-solutions to (1.5) is the following.

Theorem 1.2.8. *For* $n \geq 2$, $0 < T < +\infty$, *and* $i = 1, 2$, *let* $(u_i, d_i) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n \times S^2$ *be two weak solutions to (1.5) with the same initial data* $(u_0, d_0) : \mathbb{R}^n \to \mathbb{R}^n \times S^2$. *Suppose, in addition, there exists* $p > 2$ *and* $q > n$ *satisfying (1.6) such that* $(u_1, \nabla d_1), (u_2, \nabla d_2) \in L_t^p L_x^q(\mathbb{R}^n \times [0, T])$. *Then* $(u_i, \nabla d_i) \in C^\infty(\mathbb{R}^n \times (0, T])$ *and* $(u_1, d_1) \equiv (u_2, d_2)$ *on* $\mathbb{R}^n \times [0, T]$.

For $n = 2$, Lin-Wang [62] have proved the uniqueness of (1.5) for $p = q = 4$. More precisely, if, for $i = 1, 2$,

$$\begin{cases} u_i \in L^\infty([0, T], L^2(\mathbb{R}^2, \mathbb{R}^2)) \cap L^2([0, T], H^1(\mathbb{R}^2, \mathbb{R}^2)); \\ \nabla d_i \in L^\infty([0, T], L^2(\mathbb{R}^2)) \cap L^2([0, T], H^1(\mathbb{R}^2)) \end{cases}$$

are weak solutions to (1.5) under the same initial condition, then $(u_1, d_1) \equiv (u_2, d_2)$ on $\mathbb{R}^2 \times [0, T]$. For $n \geq 3$, Lin-Wang [62] proved the uniqueness, provided that $u_i \in C([0, T], L^n(\mathbb{R}^n))$ and $\nabla d_i \in C([0, T], L^n(\mathbb{R}^n))$ for $i = 1, 2$.

Heat flow of biharmonic maps

Because of the critical nonlinearity in the equation $(1.8)_1$, the question of regularity and uniqueness for weak solutions of (1.8) is very challenging for dimensions $n \geq 4$. There has been very few works in this direction. This is one of my motivations to study these issues for the equation (1.8) in this thesis. Another motivation comes from the study of the heat flow of harmonic maps. Similar to heat flow of harmonic maps, several interesting results concerning regularity, uniqueness, convexity, and unique limit at time infinity of the equation (1.8), under a smallness condition of renormalized total energy, will be established.

The first result concerns the regularity of (1.9).

Theorem 1.2.9. *For* $\frac{3}{2} < p \leq 2$ *and* $0 < T < +\infty$, *there exists* $\epsilon_p > 0$ *such that if* $u \in W_2^{1,2}(\Omega \times [0, T], \mathbb{S}^L)$ *is a weak solution of (1.9) and satisfies for* $z_0 = (x_0, t_0) \in \Omega \times (0, T]$ *and* $0 < R_0 \leq \frac{1}{2} \min\{d(x_0, \partial\Omega), \sqrt{t_0}\}$,

$$\|\nabla^2 u\|_{M_{R_0}^{p,2p}(P_{R_0}(z_0))} + \|\partial_t u\|_{M_{R_0}^{p,4p}(P_{R_0}(z_0))} \leq \epsilon_p, \tag{1.18}$$

then $u \in C^\infty\left(P_{\frac{R_0}{16}}(z_0), \mathbb{S}^L\right)$, *and*

$$\left|\nabla^m u(z_0)\right| \leq \frac{C\epsilon_p}{R_0^m}, \ \forall\, m \geq 1.$$

To avoid conflicts, we will let $M_R^{p,\lambda}$ denotes the Morrey space for $0 \le \lambda \le n+4$, $0 < R \le \infty$, and $U = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}$:

$$M_R^{p,\lambda}(U) = \left\{ f \in L^p_{\text{loc}}(U) : \left\| f \right\|_{M_R^{p,\lambda}(U)} < +\infty \right\},$$

where

$$\left\| f \right\|_{M_R^{p,\lambda}(U)} = \Big(\sup_{(x,t)\in U} \sup_{0<r<\min\{R, d(x,\partial U_1), \sqrt{t}\}} r^{\lambda-n-4} \int_{P_r(x,t)} |f|^p \Big)^{\frac{1}{p}},$$

and

$$B_r(x) = \{y \in \mathbb{R}^n : |y-x| \le r\},\ P_r(x,t) = B_r(x) \times [t-r^4, t],\ d(x, \partial U_1) = \inf_{y\in\partial U_1} |x-y|.$$

Denote B_r (or P_r) for $B_r(0)$ (or $P_r(0)$ respectively), and $M^{p,\lambda}(U) = M_\infty^{p,\lambda}(U)$ for $R = \infty$.

Remark 1.2.10. *It is an open question whether Theorem 1.2.9 holds for any compact Riemannian manifold N without boundary (with $p = 2$).*

Utilizing Theorem 1.2.9, one can obtain the following uniqueness theorem.

Theorem 1.2.11. *For $n \ge 4$ and $\frac{3}{2} < p \le 2$, there exist $\epsilon_0 = \epsilon_0(p,n) > 0$ and $R_0 = R_0(\Omega, \epsilon_0) > 0$ such that if $u_1, u_2 \in W_2^{1,2}(\Omega \times [0,T], \mathbb{S}^L)$ are weak solutions of (1.8), with the same initial and boundary value $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$, that satisfy*

$$\max_{i=1,2} \left[\|\nabla^2 u_i\|_{M_{R_0}^{p,2p}(\Omega\times(0,T))} + \|\partial_t u_i\|_{M_{R_0}^{p,4p}(\Omega\times(0,T))} \right] \le \epsilon_0, \tag{1.19}$$

then $u_1 \equiv u_2$ on $\Omega \times [0,T]$.

There are two main ingredients in the proof of Theorem 1.2.11:
(i) The interior regularity of u_i $(i = 1,2)$: $u_i \in C^\infty(\Omega \times (0,T), \mathbb{S}^L)$ and

$$\max_{i=1,2} |\nabla^m u_i|(x,t) \lesssim \epsilon_0 \left(\frac{1}{R_0^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}} \right)$$

for any $(x,t) \in \Omega \times (0,T)$ and $m \ge 1$.
(ii) The energy method, with suitable applications of the Poincaré inequality and the second order Hardy inequality in Lemma 4.2.2 below.

Remark 1.2.12. *(i) A novel feature of Theorem 1.2.11 is that solutions may have singularities at the parabolic boundary $\partial_p(\Omega \times [0,T])$ so that the standard argument to prove uniqueness for classical solutions is not applicable.*
(ii) For $\Omega = \mathbb{R}^n$, if the initial data $u_0 : \mathbb{R}^n \to N$ satisfies for some $R_0 > 0$,

$$\sup \left\{ r^{4-n} \int_{B_r(x)} |\nabla^2 u_0|^2 :\ x \in \mathbb{R}^n, r \le R_0 \right\} \le \epsilon_0^2,$$

then by the local well-posedness theorem of Wang [94] there exists $0 < T_0 (\approx R_0^4)$ and a solution $u \in C^\infty(\mathbb{R}^n \times (0,T_0), N)$ of (1.10) that satisfies the condition (1.18).

Prompted by the ideas in the proof of Theorem 1.2.11, the convexity property of the E_2-energy along the heat flow of biharmonic maps to $\mathbb{S}^L$ can be obtained.

Theorem 1.2.13. *For $n \geq 4$, $\frac{3}{2} < p \leq 2$, and $1 \leq T \leq \infty$, there exist $\epsilon_0 = \epsilon_0(p,n) > 0$, $R_0 = R_0(\Omega, \epsilon_0) > 0$, and $0 < T_0 = T_0(\epsilon_0) < T$ such that if $u \in W_2^{1,2}(\Omega \times [0,T], \mathbb{S}^L)$ is a weak solution of (1.8), with the initial and boundary value $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$, satisfying*

$$\|\nabla^2 u\|_{M_{R_0}^{p,2p}(\Omega\times(0,T))} + \|\partial_t u\|_{M_{R_0}^{p,4p}(\Omega\times(0,T))} \leq \epsilon_0,$$

then
(i) $E_2(u(t))$ is monotone decreasing for $t \geq T_0$; and
(ii) for any $t_2 \geq t_1 \geq T_0$,

$$\int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2 \leq C\Big[\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2\Big]$$

for some $C = C(n, \epsilon_0) > 0$.

Schoen [77] proved the convexity of Dirichlet energy for harmonic maps into N with nonpositive sectional curvature. The convexity for harmonic maps into any compact manifold N with small renormalized energy was proved in [46]. In this dissertation, the convexity for biharmonic maps with small renormalized E_2-energy will be proved which seems to be the first convexity result for the biharmonic maps.

A direct consequence of the convexity property of E_2-energy is the unique limit at $t = \infty$ of (1.8).

Corollary 1.2.14. *For $n \geq 4$ and $\frac{3}{2} < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p,n) > 0$, and $R_0 = R_0(\Omega, \epsilon_0) > 0$ such that if $u \in W_2^{1,2}(\Omega \times [0,\infty), \mathbb{S}^L)$ is a weak solution of (1.8), with the initial and boundary value $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$, satisfying the condition (1.18), then there exists a biharmonic map $u_\infty \in C^\infty \cap W^{2,2}(\Omega, \mathbb{S}^L)$, with $(u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu})$ on $\partial\Omega$, such that*

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{W^{2,2}(\Omega)} = 0,$$

and, for any compact subset $K \subset\subset \Omega$ and $m \geq 1$,

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{C^m(K)} = 0.$$

Remark 1.2.15. *(i) If Theorem 1.2.9 has been proved for any compact Riemannian manifold N without boundary, then Theorem 1.2.11, Theorem 1.2.13, and Corollary 1.2.14 would be true for any compact Riemannian manifold N without boundary.*
(ii) With slight modifications of the proofs, Theorem 1.2.9, Theorem 1.2.11, Theorem 1.2.13, and Corollary 1.2.14 remain true, if Ω is replaced by a compact Riemannian manifold M with boundary ∂M.
(iii) If Ω is replaced by a compact or complete, non-compact Riemannian manifold M

with $\partial M = \emptyset$ then Theorem 1.2.9, Theorem 1.2.11, Theorem 1.2.13, and Corollary 1.2.14 remain true for the Cauchy problem (1.10). In fact, the proof is slightly simpler than the one here, since the Hardy inequalities are not necessary.
(iv) In general, it is a difficult question to ask whether the unique limit at $t = \infty$ holds for geometric evolution equations. Simon in his celebrated work [79] showed the unique limit at $t = \infty$ for smooth solutions to the heat flow of harmonic maps into a real analytic manifold (N, h). Corollary 1.2.14 seems to be first result on the unique limit at time infinity for the heat flow of biharmonic maps.

A natural class of weak solutions of (1.8) that satisfy the smallness condition (1.18) is:

Definition 1.2.16. *A weak solutions $u \in W_2^{1,2}(\Omega \times [0,T], N)$ of (1.8) is called a Serrin's (p,q)-solution, if $\nabla^2 u \in L_t^q L_x^p(\Omega \times [0,T])$ for some $p \geq \frac{n}{2}$ and $q \leq \infty$ satisfying*

$$\frac{n}{p} + \frac{4}{q} = 2. \tag{1.20}$$

In chapter 4, it will be proved that if u is a weak solution of (1.8) such that $\nabla^2 u \in L_t^q L_x^p(\Omega \times [0,T])$ for some $p > \frac{n}{2}$ and $q > 3$ satisfying (1.20) and $u_0 \in W^{2,r}(\Omega, N)$ for some $r > \frac{n}{2}$, then u satisfies (1.18) for some $p_0 > \frac{3}{2}$. Thus, for $N = \mathbb{S}^L$, the regularity and uniqueness for such solutions of (1.8) follow from Theorem 1.2.9 and Theorem 1.2.11. However, for a compact Riemannian manifold N without boundary, the regularity and uniqueness for such a class of weak solutions of (1.8) require different arguments. More precisely,

Theorem 1.2.17. *For $n \geq 4$ and $0 < T \leq \infty$, let $u_1, u_2 \in W_2^{1,2}(\Omega \times [0,T], N)$ be weak solutions of (1.8), with the same initial and boundary value $u_0 \in W^{2,2}(\Omega, N)$. If, in addition, $\nabla^2 u_1, \nabla^2 u_2 \in L_t^q L_x^p(\Omega \times [0,T])$ for some $p > \frac{n}{2}$ and $q < \infty$ satisfying (1.20), then $u_1, u_2 \in C^\infty(\Omega \times (0,T), N)$, and $u_1 \equiv u_2$ in $\Omega \times [0,T]$.*

Remark 1.2.18. *(i) It is a very interesting question to ask whether Theorem 1.2.17 holds for the end-point case $p = \frac{n}{2}$ and $q = \infty$.*
(ii) If $u_0 \in W^{2,r}(\Omega, N)$ for some $r > \frac{n}{2}$, then the local existence of solutions u of (1.8) such that $\nabla^2 u \in L_t^q L_x^p(\Omega \times [0,T])$ for some $p > \frac{n}{2}$ and $q < \infty$ satisfying (1.20) can be shown by the fixed point argument similar to [24] §4.

The remainder of the thesis is written as follows:

- Chapter 2 is devoted to prove the regularity and uniqueness of heat flow of harmonic maps.
- Chapter 3 is devoted to prove the regularity and uniqueness of Serrin's (p,q)-solutions to nematic liquid crystal flows.
- Chapter 4 is devoted to prove the regularity and uniqueness of heat flow of biharmonic maps.

Chapter 2 Regularity and uniqueness of heat flow of harmonic maps

In this chapter, the regularity and uniqueness of heat flow of harmonic maps (1.3) will be considered by some new and elementary argument based on the Riesz potential estimates between Morrey spaces, the Hardy inequality and the general Gronwall argument. As applications, the uniqueness of weakly harmonic maps, uniqueness of limit at $t = +\infty$ of heat flow of harmonic maps and uniqueness of Serrin's-(p, q) solutions will also be considered.

2.1 ϵ-regularity Theorem

This section will be devoted to establish an ϵ-regularity theorem for the heat flow of harmonic maps (1.3), which plays a crucial role in the proof of our main theorems. This regularity theorem seems to be new, whose proof is rather elementary and mainly motivated by [9]. It improves the regularity theorem previously obtained by Chen-Li-Lin [12], Feldman [25], Chen-Wang [14] (see also Moser [67, 66] for more general results).

Theorem 2.1.1. *Assume that N is either a unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary. For $1 < p \leq 2$ and $0 < T < +\infty$, there exists $\epsilon_p > 0$ such that if $u \in H^1(\Omega \times [0, T], N)$ is a weak solution of (1.3)$_1$ and satisfies that, for $z_0 = (x_0, t_0) \in \Omega \times (0, T]$ and $0 < R_0 \leq \frac{1}{2}\min\{d(x_0, \partial\Omega), \sqrt{t_0}\}$,*

$$\|\nabla u\|_{M^{p,p}_{R_0}(P_{R_0}(z_0))} + \|\partial_t u\|_{M^{p,2p}_{R_0}(P_{R_0}(z_0))} \leq \epsilon_p. \tag{2.1}$$

Then $u \in C^\infty(P_{\frac{R_0}{4}}(z_0), N)$, and

$$|\nabla^m u|(z_0) \leq \frac{C\epsilon_p}{R_0^m},\ \forall m \geq 1. \tag{2.2}$$

Remark 2.1.2. *It remains an open question whether Theorem 2.1.1 holds for any compact Riemannian manifold N without boundary, under the condition (2.1) for $p = 2$ (see Moser [65] and Moser [66] for related works.).*

The proof of Theorem 2.1.1 is based on the following lemma.

Lemma 2.1.3. *For any $1 < p \leq 2$, there exists $\epsilon_p > 0$ such that if $N = S^{k-1}$ or a compact Riemannian homogeneous manifold without boundary, and $u \in H^1(P_4, N)$ is a weak solution of (1.3) satisfying*

$$\sup_{(x,t)\in P_2, 0<r\leq 2} r^{p-(n+2)} \int_{P_r(x,t)} (|\nabla u|^p + r^p|\partial_t u|^p) \leq \epsilon^p. \tag{2.3}$$

Then $u \in C^\infty(P_{\frac{1}{2}}, S^{k-1})$ and satisfies

$$\|\nabla^m u\|_{C^0(P_{\frac{1}{2}})} \leq C(n, p, \epsilon, m),\ \forall m \geq 1. \tag{2.4}$$

Proof. The crucial step to establish (2.4) is the following decay estimate:
Claim: There exists $q > \max\{\frac{p}{p-1}, n+2\}$ such that for any $\theta \in (0, \frac{1}{2})$, $z_0 = (x_0, t_0) \in P_1$, and $0 < r \leq 1$, it holds

$$\frac{1}{(\theta r)^{n+2}} \int_{P_{\theta r}(z_0)} |u - u_{z_0,\theta r}| \leq C\Big(\theta^{-(n+2)}\epsilon_p + \theta\Big)\Big(\frac{1}{r^{n+2}} \int_{P_r(z_0)} |u - u_{z_0,r}|^q\Big)^{\frac{1}{q}}, \tag{2.5}$$

where $f_{z_0,r} = \frac{1}{|P_r(z_0)|}\int_{P_r(z_0)} f$ is the average of f over $P_r(z_0)$.

For $z_0 = (x_0, t_0) \in P_1$ and $0 < r \leq 1$, since $v(y, s) = u(z_0 + (ry, r^2 s)) : P_2 \to N$ satisfies (1.3), and the condition (2.3) yields that v satisfies

$$\sup_{(x,t)\in P_1, 0<r\leq 1} r^{p-(n+2)} \int_{P_r(x,t)} |\nabla v|^p + r^p |\partial_t v|^p \leq \epsilon^p. \tag{2.6}$$

Thus it suffices to show (2.5) for $z_0 = (0, 0)$ and $r = 2$.

The proof of the Claim can be divided into two cases:
Case 1: $N = S^{k-1}$ is the unit sphere.
Step 1. Rewriting of (1.3). Since $|u| = 1$, we have $u^i u^i_\alpha = 0$. Also, it follows (1.3) that

$$(u^i u^j_\alpha - u^j u^i_\alpha)_\alpha = u^i \Delta u^j - u^j \Delta u^i = u^i \partial_t u^j - u^j \partial_t u^i.$$

Hence we have

$$\begin{aligned}
\partial_t u^i - \Delta u^i &= A^i(u)(\nabla u, \nabla u) = |\nabla u|^2 u^i \\
&= u^j_\alpha u^j_\alpha u^i - u^j_\alpha u^j u^i_\alpha = u^j_\alpha (u^i u^j_\alpha - u^j u^i_\alpha) \\
&= \Big[(u^j - c^j)(u^i u^j_\alpha - u^j u^i_\alpha)\Big]_\alpha - (u^j - c^j)(u^i \partial_t u^j - u^j \partial_t u^i),
\end{aligned} \tag{2.7}$$

where $c^j \in \mathbb{R}$ is an arbitrary constant. For the convenience, set

$$W^{ij} = u^i \partial_t u^j - u^j \partial_t u^i, \ V^{ij}_\alpha = u^i u^j_\alpha - u^j u^i_\alpha, \ 1 \leq i, j \leq k, 1 \leq \alpha \leq n.$$

Step 2. Construction of auxiliary functions. Let $\eta \in C^\infty_0(P_2)$ such that

$$0 \leq \eta \leq 1, \ \eta = 1 \text{ on } P_1, \text{ and } |\nabla \eta| \leq C.$$

Define $v, w : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^k$ by

$$\partial_t v^i - \Delta v^i = \Big[\eta^2 (u^j - c^j) V^{ij}_\alpha\Big]_\alpha; \ v^i\Big|_{t=0} = 0, \tag{2.8}$$

and

$$\partial_t w^i - \Delta w^i = -\eta^2 (u^j - c^j) W^{ij}; \ w^i\Big|_{t=0} = 0. \tag{2.9}$$

Set $h = u - (v + w) : P_1 \to \mathbb{R}^k$. Then

$$\partial_t h - \Delta h = 0 \ \text{ in } \ P_1. \tag{2.10}$$

Step 3. Estimation of v, w, and u. By the Duhamel formula, we have that

$$\begin{aligned} v^i(x,t) &= \int_0^t \int_{\mathbb{R}^n} H(x-y,t-s)\Big[\eta^2(u^j-c^j)V_\alpha^{ij}\Big]_\alpha(y,s) \\ &= \int_0^t \int_{\mathbb{R}^n} \nabla_x H(x-y,t-s)(\eta^2(u^j-c^j)V_\alpha^{ij})(y,s), \end{aligned}$$

where H denotes the heat kernel on $\mathbb{R}^n$. Then, as in [45], we have

$$|\nabla_x H|(x-y,t-s) \lesssim \delta((x,t),(y,s))^{-(n+1)}, \quad (x,t),\ (y,s) \in \mathbb{R}^{n+1},$$

where $\delta((x,t),(y,s))$ is the parabolic distance on $\mathbb{R}^{n+1}$. Hence

$$|v^i|(x,t) \lesssim I_1(\eta^2|u^j-c^j||V_\alpha^{ij}|)(x,t),$$

where

$$I_1(f)(x,t) := \int_{\mathbb{R}^{n+1}} \frac{f(y,s)}{\delta((x,t),(y,s))^{n+1}}, \ \forall f \in L^1_{\text{loc}}(\mathbb{R}^{n+1}),$$

is the parabolic Riesz potential of order 1. By the Riesz potential estimate (see [45]), we have

$$\begin{aligned} &\|v\|_{L^1(P_2)} \lesssim \|v\|_{L^p(P_2)} \le \|v\|_{L^p(\mathbb{R}^{n+1})} \\ \lesssim\ & \sum_{i,j} \|\eta^2|u^j-c^j||V_\alpha^{ij}|\|_{L^{\frac{(n+2)p}{n+2+p}}(\mathbb{R}^{n+1})} \lesssim \sum_{i,j} \|V_\alpha^{ij}\|_{L^p(P_2)}\|u^j-c^j\|_{L^{n+2}(P_2)}. \end{aligned} \tag{2.11}$$

For w, since

$$w^i(x,t) = \sum_j \int_0^t \int_{\mathbb{R}^n} H(x-y,t-s)(\eta^2(u^j-c^j)W^{ij})(y,s),$$

applying the Young inequality we obtain

$$\begin{aligned} \|w\|_{L^1(P_2)} &\le \|w\|_{L^{\widetilde{q_1}}(P_2)} \le \|w\|_{L^{\widetilde{q_1}}(\mathbb{R}^n\times[0,1])} \\ &\lesssim \sum_{i,j} \Big\|\eta^2(u^j-c^j)W^{ij}\Big\|_{L^{\widetilde{q_1}}(\mathbb{R}^n\times[0,1])} \lesssim \sum_{i,j} \Big\||u^j-c^j||W^{ij}|\Big\|_{L^{\widetilde{q_1}}(P_2)} \\ &\lesssim \sum_{i,j} \Big\|u^j-c^j\Big\|_{L^{q_1}(P_2)}\Big\|W^{ij}\Big\|_{L^p(P_2)}, \end{aligned} \tag{2.12}$$

where $1 < \widetilde{q_1} < p$ and $q_1 > \frac{p}{p-1}$ satisfy $\dfrac{1}{\widetilde{q_1}} = \dfrac{1}{p} + \dfrac{1}{q_1}$.

For h, by the standard theory on the heat equation we have that for any $0 < \theta < 1$,

$$\frac{1}{\theta^{n+2}} \int_{P_\theta} |h-h_\theta| \lesssim \theta \int_{P_1} |h-h_1| \lesssim \theta\Big[\|v\|_{L^1(P_2)} + \|w\|_{L^1(P_2)} + \|u-u_2\|_{L^1(P_2)}\Big], \tag{2.13}$$

where $f_r = \frac{1}{|P_r|}\int_{P_r} f$ is the average of a function f over P_r.

Now we let $c^j = u^j_2$, the average of u^j over P_2 and set $q = \max\{q_1, n+2\}$. Combining the estimates (2.11), (2.12), and (2.13) and applying Hölder's inequality together yields

$$\begin{aligned}
&\frac{1}{\theta^{n+2}}\int_{P_\theta}|u-u_\theta| \le \frac{1}{\theta^{n+2}}\int_{P_\theta}(|v|+|w|) + \frac{1}{\theta^{n+2}}\int_{P_\theta}|h-h_\theta| \\
&\lesssim \theta^{-(n+2)}\Big[\|v\|_{L^1(P_2)}+\|w\|_{L^1(P_2)}\Big] + \theta\Big[\|v\|_{L^1(P_2)}+\|w\|_{L^1(P_2)}+\|u-u_2\|_{L^1(P_2)}\Big] \\
&\lesssim \Big[\theta+\theta^{-(n+2)}(\|V^{ij}_\alpha\|_{L^p(P_2)}+\|W^{ij}\|_{L^p(P_2)})\Big]\Big(\frac{1}{2^{n+2}}\int_{P_2}|u-u_2|^q\Big)^{\frac{1}{q}} \\
&\le C\Big[\theta+\theta^{-(n+2)}\epsilon_p\Big]\Big(\frac{1}{2^{n+2}}\int_{P_2}|u-u_2|^q\Big)^{\frac{1}{q}}, \qquad (2.14)
\end{aligned}$$

where we have used in the last step the condition (2.3) so that

$$\|V^{ij}_\alpha\|_{L^p(P_2)} + \|W^{ij}\|_{L^p(P_2)} \le C\epsilon_p.$$

This yields (2.5). It follows from (2.3) and the Poincaré inequality that $u \in \mathrm{BMO}(P_2)$, and

$$\Big[u\Big]_{\mathrm{BMO}(P_2)} := \sup_{P_r(z)\subset P_2}\Big\{\frac{1}{r^{n+2}}\int_{P_r(z)}|u-u_{z,r}|\Big\} \le C\epsilon_p. \qquad (2.15)$$

By the celebrated John-Nirenberg's inequality [47], (2.15) implies that for any $q > 1$, it holds

$$\sup_{P_r(z)\subset P_2}\Big\{\Big(\frac{1}{\theta^{n+2}}\int_{P_r(z)}|u-u_{z,r}|^q\Big)^{\frac{1}{q}}\Big\} \le C(q)\Big[u\Big]_{\mathrm{BMO}(P_2)}. \qquad (2.16)$$

By (2.16), we see that (2.5) implies that

$$\frac{1}{(\theta r)^{n+2}}\int_{P_{\theta r}(z_0)}|u-u_{z_0,\theta r}| \le C\left(\theta^{-(n+2)}\epsilon_p+\theta\right)\Big[u\Big]_{\mathrm{BMO}(P_2)} \qquad (2.17)$$

holds for any $\theta \in (0, \frac{1}{2}), z_0 \in P_1, 0 < r \le 1$. Taking supremum of (2.17) over all $z_0 \in P_\theta$ and $0 < r \le 1$, we obtain

$$\Big[u\Big]_{\mathrm{BMO}(P_\theta)} \le C\left(\theta^{-(n+2)}\epsilon_p+\theta\right)\Big[u\Big]_{\mathrm{BMO}(P_2)}. \qquad (2.18)$$

If we choose $\theta = \theta_0 \in (0, \frac{1}{2})$ and $\epsilon_p > 0$ so small that

$$C\left(\theta_0^{-(n+2)}\epsilon_p+\theta_0\right) \le \frac{1}{2},$$

then (2.18) implies

$$\Big[u\Big]_{\mathrm{BMO}(P_{\theta_0})} \le \frac{1}{2}\Big[u\Big]_{\mathrm{BMO}(P_2)}. \qquad (2.19)$$

It is standard that by iterations and the Campanato theory [4], (2.19) implies that there exists $\alpha \in (0, 1)$ such that $u \in C^\alpha(P_{\frac{3}{4}})$ and

$$\Big[u\Big]_{C^\alpha(P_{\frac{3}{4}})} \le C(p, \epsilon_p).$$

The higher regularity and the estimate (2.4) then follow from the parabolic hole filling type argument and the bootstrap argument (see also [45]).

Case 2: N is a compact Riemannian homogeneous manifold without boundary. We will indicate that (1.3) can be written into the same form as (2.7). In fact, according to Hélein [35], there exist s smooth tangential vector fields $Y_1, \cdots, Y_s$ and s smooth tangential killing vector fields $X_1, \cdots, X_s$ on N such that for any $y \in N$ and $V \in T_yN$, it holds

$$V = \sum_{i=1}^{s} \langle V, X_i(y)\rangle Y_i(y).$$

Thus, as in [14] Lemma 4.2, (1.3) is equivalent to

$$\begin{aligned} \partial_t u - \Delta u &= -\sum_{i=1}^{s} \langle \nabla u, X_i(u)\rangle \nabla(Y_i(u)) \\ &= -\sum_{i=1}^{s} \operatorname{div}(\langle \nabla u, X_i(u)\rangle (Y_i(u) - c^i)) - \sum_{i=1}^{s} \langle \partial_t u, X_i(u)\rangle (Y_i(u) - c^i), \quad (2.20) \end{aligned}$$

where $c^i \in \mathbb{R}^k$ is an arbitrary constant. Here we have used the killing property of X_i that yields $\langle \nabla u, \nabla(X_i(u))\rangle = 0$ in the derivation of (2.20). It is clear that the rest of proof follows exactly as in Case 1. This completes the proof. □

Proof of Theorem 2.1.1. It is easy to see that (2.1) implies

$$r^{p-(n+2)} \int_{P_r(z)} (|\nabla u|^p + r^p |\partial_t u|^p) \le \epsilon_p^p, \ \forall z = (x,t) \in P_{\frac{R_0}{2}}(z_0) \ \text{ and } \ 0 < r \le \frac{R_0}{2}. \quad (2.21)$$

Hence Lemma 2.1.3 implies that $u \in C^\infty(P_{\frac{R_0}{4}}(z_0))$, and (2.2) holds. □

2.2 Uniqueness of heat flow of harmonic maps

Now it is ready to prove the main theorem on the uniqueness of weak solutions to (1.3).

Theorem 2.2.1. *For $n \ge 2$ and $1 < p \le 2$, there exist $\epsilon_0 = \epsilon_0(p,n) > 0$ and $R_0 = R_0(M, g, \epsilon_0) > 0$ such that if*
(i) (M, g) is a n-dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;
(ii) $(N, h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and
(iii) $u_1, u_2 \in H^1(M \times [0,T], N)$ are two weak solutions of (1.3), with $u_1 = u_2 = u_0$ on $\partial_p(M \times [0,T])$ for some $u_0 \in W^{1,2}(M, N)$, that satisfy

$$\max_{i=1,2} \left[\|\nabla u_i\|_{M^{p,p}_{R_0}(M\times(0,T))} + \|\partial_t u_i\|_{M^{p,2p}_{R_0}(M\times(0,T))} \right] \le \epsilon_0, \quad (2.22)$$

then $u_1 \equiv u_2$ on $M \times [0,T]$.

Remark 2.2.2. *i) Due to technical difficulties, it is unknown whether the ϵ-regularity Theorem 2.1.1 (with $p = 2$) holds for a general Riemannian manifold N. Hence it is an open question that Theorem 2.2.1, Theorem 2.4.1, and Corollary 2.4.2 in the following remain to hold for a general manifold N.*
(ii) The proof of Theorem 2.2.1 is based on Theorem 2.1.1, and application of both the Hardy inequality and a generalized Gronwall inequality.

Proof of Theorem 2.2.1. For simplicity, we will focus on the case that (M, g) is a compact Riemannian manifold with boundary and remark on the other two cases at the end of the proof.

Assume $(M, g) = (\Omega, g_0)$, with $\Omega \subset \mathbb{R}^n$ and g_0 the standard metric. By Theorem 2.1.1, we have that $u_i \in C^\infty(\Omega \times (0, T])$ for $i = 1, 2$, and

$$\max\Big\{|\nabla u_1|(x,t),\ |\nabla u_2|(x,t)\Big\} \le C\epsilon_0\Big(\frac{1}{R_0} + \frac{1}{d(x,\partial\Omega)} + \frac{1}{\sqrt{t}}\Big),\ \forall (x,t) \in \Omega \times (0,T]. \tag{2.23}$$

Set $w = u - v$. Then w satisfies

$$\begin{cases} w_t - \Delta w = A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v) & \text{in } \Omega \times (0,T] \\ w = 0 & \text{on } \partial_p(\Omega \times [0,T]). \end{cases} \tag{2.24}$$

Multiplying (2.24) by w and integrating over Ω yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega |w|^2 + 2\int_\Omega |\nabla w|^2 &\le C\int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2)|w|^2 + \int_\Omega (|\nabla u_1| + |\nabla u_2|)|\nabla w||w| \\ &\le \int_\Omega |\nabla w|^2 + C\int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2)|w|^2. \end{aligned}$$

By (2.23), the Poincaré inequality, and the Hardy inequality:

$$\int_\Omega \frac{|f(x)|^2}{d^2(x,\partial\Omega)} \lesssim \int_\Omega |\nabla f|^2,\ \forall f \in H^1_0(\Omega),$$

we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega |w|^2 + \int_\Omega |\nabla w|^2 &\le \frac{C\epsilon_0^2}{R_0^2}\int_\Omega |w|^2 + C\epsilon_0^2\int_\Omega \frac{|w(x)|^2}{d^2(x,\partial\Omega)} + \frac{C\epsilon_0^2}{t}\int_\Omega |w|^2 \\ &\le C\Big(\frac{\epsilon_0^2}{R_0^2} + \epsilon_0^2\Big)\int_\Omega |\nabla w|^2 + \frac{C\epsilon_0^2}{t}\int_\Omega |w|^2. \end{aligned}$$

If we choose $\epsilon_0 \le (2C)^{-\frac{1}{2}}$ and $R_0 \ge \sqrt{2C}\epsilon_0$, then we have $C\Big(\frac{\epsilon_0^2}{R_0^2} + \epsilon_0^2\Big) \le 1$ so that

$$\frac{d}{dt}\int_\Omega |w|^2 \le \frac{C\epsilon_0^2}{t}\int_\Omega |w|^2. \tag{2.25}$$

This yields

$$\begin{aligned} \frac{d}{dt}\Big(t^{-\frac{1}{2}}\int_\Omega |w|^2\Big) &= t^{-\frac{1}{2}}\frac{d}{dt}\int_\Omega |w|^2 - \frac{1}{2}t^{-\frac{3}{2}}\int_\Omega |w|^2 \\ &\le \Big(C\epsilon_0^2 - \frac{1}{2}\Big)t^{-\frac{3}{2}}\int_\Omega |w|^2 \le 0. \end{aligned} \tag{2.26}$$

Thus we obtain that for any $0 < t \le T$,

$$t^{-\frac{1}{2}} \int_\Omega |w(x,t)|^2 \le \lim_{s\downarrow 0^+} s^{-\frac{1}{2}} \int_\Omega |w(x,s)|^2. \tag{2.27}$$

Since $w(\cdot,0)=0$, we have

$$w(x,s) = \int_0^s w_\tau(x,\tau)\,d\tau, \text{ a.e. } x\in\Omega$$

so that by the Hölder inequality,

$$s^{-\frac{1}{2}} \int_\Omega |w(x,s)|^2 \le s^{\frac{1}{2}} \int_0^s \int_\Omega |w_\tau|^2(x,\tau)\,dxd\tau \le Cs^{\frac{1}{2}} \to 0, \text{ as } s\downarrow 0^+.$$

Thus we conclude that $w\equiv 0$ in $\Omega\times[0,T]$.

When (M,g) is either compact or complete non-compact with $\partial M=\emptyset$, observe that we can substitute $d(x,\partial M)=\infty$ into the above proof and obtain the same result without applying the Hardy inequality. This completes the proof. □

2.3 Convexity and uniqueness of weak harmonic maps

A byproduct of the proof of Theorem 2.2.1, is the convexity property on certain weak harmonic maps that yields an alternative, simple proof of the uniqueness theorem on the Dirichlet problem of weak harmonic maps, due to Struwe [85] for $n=3$ and Moser [70] for $n\ge 4$. Furthermore, the statement of the uniqueness theorem for N either a unit sphere or a compact Riemannian homogeneous manifold without boundary is an improvement of that by [85] and [70].

To do it, first recall that the Morrey spaces $M_R^{l,\lambda}(U)$ in $\mathbb{R}^n$ is defined for $1\le l < +\infty$, $0<\lambda\le n$, $0<R\le+\infty$, and $U\subset\mathbb{R}^n$, $f\in M_R^{l,\lambda}(U)$ iff $f\in L^l_{\text{loc}}(U)$ satisfies

$$\|f\|^l_{M_R^{l,\lambda}(U)} := \sup_{x\in U}\ \sup_{0<r\le\min\{R,d(x,\partial U)\}} \left\{ r^{\lambda-n}\int_{B_r(x)} |f|^l \right\} < +\infty.$$

Denote $M^{p,\lambda}(U) = M^{p,\lambda}_\infty(U)$.

For any bounded smooth domain $\Omega\subset\mathbb{R}^n$, the following convexity property of Dirichlet energy holds

Theorem 2.3.1. *For $n\ge 2$, $\delta\in(0,1)$, and $1<p\le 2$, there exist $\epsilon_p=\epsilon(p,\delta)>0$ and $R_p=R(p,\delta)>0$ such that if $u\in H^1(\Omega,N)$ is a weak harmonic map satisfying either*
(i) $\|\nabla u\|_{M^{2,2}_{R_2}(\Omega)}\le\epsilon_2$, when N is a compact Riemannian manifold without boundary, or
(ii) $\|\nabla u\|_{M^{p,p}_{R_p}(\Omega)}\le\epsilon_p$, when $N=S^{k-1}$ or a compact Riemannian homogeneous manifold without boundary. Then

$$\int_\Omega |\nabla v|^2 \ge \int_\Omega |\nabla u|^2 + (1-\delta)\int_\Omega |\nabla(v-u)|^2 \tag{2.28}$$

holds for any $v\in H^1(\Omega,N)$ with $v=u$ on $\partial\Omega$.

Proof. First, as observed by [85] and [70], for an arbitrary manifold N under the condition (i), the small energy regularity theorem on stationary harmonic maps by Bethuel [2] holds. While, for $N = S^{k-1}$ under the condition (ii), the small energy regularity theorem on weak harmonic maps by Moser [65] or Lemma 2.3 is applicable. Thus we have $u \in C^\infty(\Omega, N)$ and, for any $x \in \Omega$, it holds

$$|\nabla u|(x) \le C\epsilon_p\left(\frac{1}{d(x,\partial\Omega)} + \frac{1}{R_p}\right). \tag{2.29}$$

Here $p = 2$ for an arbitrary N.

Now multiplying the equation of u by $(u - v)$ and integrating over Ω, we obtain

$$\int_\Omega \nabla u \cdot \nabla(u - v) = \int_\Omega \langle A(u)(\nabla u, \nabla u), u - v\rangle. \tag{2.30}$$

This, combined with (2.29), the Poincaré inequality, and the Hardy inequality, implies

$$\begin{aligned}\Big|\int_\Omega \langle A(u)(\nabla u, \nabla u), u - v\rangle\Big| &\le C\int_\Omega |\nabla u|^2|u - v|^2 \\ &\le C\epsilon_p^2 \int_\Omega \frac{|u-v|^2}{R_p^2} + \frac{|u-v|^2}{d(x,\partial\Omega)^2} \\ &\le C\epsilon_p^2\left(1 + \frac{1}{R_p^2}\right)\int_\Omega |\nabla(u - v)|^2 \\ &\le \frac{\delta}{2}\int_\Omega |\nabla(u-v)|^2 \end{aligned} \tag{2.31}$$

provided that we have chosen $\epsilon_p \le (\frac{\delta}{4C})^{\frac{1}{2}}$ and R_p such $\frac{C\epsilon_p^2}{R_p^2} \le \frac{\delta}{4}$. Thus, by (2.30) and (2.31) we obtain

$$\begin{aligned}&\int_\Omega |\nabla v|^2 - \int_\Omega |\nabla u|^2 - \int_\Omega |\nabla(v-u)|^2 \\ &= 2\int_\Omega \nabla u \cdot \nabla(v - u) = -2\int_\Omega \langle A(u)(\nabla u, \nabla u), u - v\rangle \\ &\ge -\delta\int_\Omega |\nabla(v-u)|^2.\end{aligned}$$

This clearly implies (2.28), provided that $\epsilon > 0$ is sufficiently small. This proof is complete. □

Corollary 2.3.2. *For $n \ge 2$ and $1 < p \le 2$, there exist $\epsilon_p > 0$ and $R_p > 0$ such that if $u_1, u_2 \in H^1(\Omega, N)$ are two weak harmonic maps satisfying either*
(i) $\max_{i=1}^{2} \|\nabla u_i\|_{M^{2,2}_{R_2}(\Omega)} \le \epsilon_2$, when N is a compact Riemannian manifold without boundary, or
(ii) $\max_{i=1}^{2} \|\nabla u_i\|_{M^{p,p}_{R_p}(\Omega)} \le \epsilon_p$, when $N = S^{k-1}$ or a compact Riemannian homogeneous manifold without boundary.
Then $u_1 \equiv u_2$ in Ω, provided that $u_1 - u_2 \in W^{1,2}_0(\Omega, \mathbb{R}^k)$.

Proof. Choosing $\delta = \frac{1}{2}$, we can apply Theorem 2.3.1 to u_1 and u_2 by choosing sufficiently small $\epsilon_p > 0$ and $R_p > 0$. Thus Theorem 2.3.1 implies

$$\int_\Omega |\nabla u_2|^2 \geq \int_\Omega |\nabla u_1|^2 + \frac{1}{2}\int_\Omega |\nabla(u_2 - u_1)|^2,$$

and

$$\int_\Omega |\nabla u_1|^2 \geq \int_\Omega |\nabla u_2|^2 + \frac{1}{2}\int_\Omega |\nabla(u_1 - u_2)|^2.$$

Adding these two inequalities together yields

$$\int_\Omega |\nabla(u_1 - u_2)|^2 = 0.$$

Therefore, $u_1 \equiv u_2$ in Ω. □

2.4 Convexity and uniqueness of limit at $t = +\infty$ of heat flow of harmonic maps

Motivated by the proof of Theorem 2.2.1 and 2.3.1, the following convexity property on (1.3) holds.

Theorem 2.4.1. *For $n \geq 2$, $1 < p \leq 2$, and $1 \leq T \leq \infty$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$, $R_0 = R_0(M, g, \epsilon_0) > 0$, and $0 < T_0 = T_0(\epsilon_0) < T$ such that if*
(i) (M, g) is a n-dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;
(ii) $(N, h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and
(iii) $u \in H^1(M \times [0, T], N)$ is a weak solution of (1.3), with $u = u_0$ on $\partial_p(M \times [0, T])$ for some $u_0 \in W^{1,2}(M, N)$, that satisfies

$$\|\nabla u\|_{M^{p,p}_{R_0}(M\times(0,T))} + \|\partial_t u\|_{M^{p,2p}_{R_0}(M\times(0,T))} \leq \epsilon_0, \tag{2.32}$$

then
(i) the Dirichlet energy $E(u(t)) := \frac{1}{2}\int_M |\nabla u|^2$ is monotone decreasing for $t \geq T_0$; and
(ii) for any $t_2 \geq t_1 \geq T_0$,

$$\int_M |\nabla(u(t_1) - u(t_2))|^2 \leq C\Big[\int_M |\nabla u(t_1)|^2 - \int_M |\nabla u(t_2)|^2\Big]. \tag{2.33}$$

A direct consequence of Theorem 2.4.1 is the following uniqueness of limit at $t = \infty$ for (1.3).

Corollary 2.4.2. *For $n \geq 2$ and $1 < p \leq 2$, there exist $\epsilon_0 = \epsilon_0(p, n) > 0$, and $R_0 = R_0(M, g, \epsilon_0) > 0$ such that if*
(i) (M, g) is a n-dimensional Riemannian manifold that is either complete noncompact without boundary or compact with or without boundary;

(ii) $(N,h) \subset \mathbb{R}^k$ is either the unit sphere S^{k-1} or a compact Riemannian homogeneous manifold without boundary; and
(iii) $u \in H^1(M\times[0,\infty), N)$ is a weak solution of (1.3), with $u = u_0$ on $\partial_p(M\times[0,\infty])$ for some $u_0 \in W^{1,2}(M,N)$, that satisfies the condition (2.32).
Then there exists a harmonic map $u_\infty \in C^\infty(M,N)\cap W^{1,2}(M,N)$, with $u_\infty = u_0$ on ∂M, such that

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{W^{1,2}(M)} = 0, \tag{2.34}$$

and, for any compact subset $K \subset\subset M$ and $m \geq 1$,

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{C^m(K)} = 0. \tag{2.35}$$

Proof of Theorem 2.4.1. For simplicity, we only consider the difficult case that (M,g) is compact with boundary. First by Theorem 2.1.1, we have that $u \in C^\infty(M\times(0,T))$ and

$$|\nabla u|(x,t) \leq C\epsilon_0\Big(\frac{1}{R_0} + \frac{1}{d(x,\partial M)} + \frac{1}{\sqrt{t}}\Big),\ \forall (x,t)\in M\times(0,T). \tag{2.36}$$

First we need two claims.
Claim 1. There exists $T_0 > 0$ such that $\int_M |\partial_t u(t)|^2$ is monotone decreasing for $t \geq T_0$:

$$\int_M |\partial_t u|^2(t_2) + C\int_{M\times[t_1,t_2]} |\nabla\partial_t u|^2 \leq \int_M |\partial_t u|^2(t_1),\quad T_0\leq t_1\leq t_2 < T. \tag{2.37}$$

To show it, we introduce the finite quotient for u in the t-variable. For sufficiently small $h > 0$, set

$$u^h(x,t) = \frac{u(x,t+h)-u(x,t)}{h},\quad (x,t)\in M\times(0,T-h).$$

Since $u^h = 0$ on ∂M, we see that $u^h \in L^2([0,T-h], H^1_0(M))$, $\partial_t u^h \in L^2([0,T-h], L^2(M))$, and

$$\lim_{h\downarrow 0+} \left\|u^h - u_t\right\|_{L^2(M\times[0,T-h])} = 0.$$

Since u satisfies (1.3), we have

$$u^h_t - \Delta u^h = \frac{1}{h}[A(u(t+h))(\nabla u(t+h),\nabla u(t+h)) - A(u(t))(\nabla u(t),\nabla u(t))]. \tag{2.38}$$

Multiplying (2.38) by u^h, integrating over M, and applying the Hölder inequality and (2.36), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_M |u^h|^2 + \int_M |\nabla u^h|^2 \\
\le\ & C\int_M |u^h|^2(|\nabla u(t+h)|^2 + |\nabla u(t)|^2) + |u^h|(|\nabla u(t+h)| + |\nabla u(t)|)|\nabla u^h| \\
\le\ & \frac{1}{2}\int_M |\nabla u^h|^2 + C\int_M |u^h|^2(|\nabla u(t+h)|^2 + |\nabla u(t)|^2) \\
\le\ & \frac{1}{2}\int_M |\nabla u^h|^2 + C\epsilon_0^2\int_M \Big(\frac{|u^h|^2}{R_0^2} + \frac{|u^h|^2}{d^2(x,\partial M)} + \frac{|u^h|^2}{t}\Big) \\
\le\ & \frac{1}{2}\int_M |\nabla u^h|^2 + C\epsilon_0^2\int_M \Big(\frac{|u^h|^2}{R_0^2} + \frac{|u^h|^2}{d^2(x,\partial M)} + \frac{|u^h|^2}{T_0}\Big) \\
\le\ & \frac{1}{2}\int_M |\nabla u^h|^2 + C\epsilon_0\int_M |\nabla u^h|^2 \le \frac{3}{4}\int_M |\nabla u^h|^2,
\end{aligned}$$

where we have used both the Poincaré inequality and the Hardy inequality in the last step, and chosen $R_0 \ge \sqrt{\epsilon_0}$, $T_0 \ge \epsilon_0$, and $C\epsilon_0 \le \frac{1}{4}$. Integrating this inequality from $T_0 \le t_1 \le t_2$ yields

$$\int_M |u^h|^2(t_2) + C\int_{M\times[t_1,t_2]} |\nabla u^h|^2 \le \int_M |u^h|^2(t_1). \tag{2.39}$$

Sending h to zero in (2.39) yields (2.37).

Next we have

Claim 2. There exists $T_0 > 0$ such that $E(u(t))$ is monotone decreasing for $t \ge T_0$:

$$\int_{M\times[t_1,t_2]} |\partial_t u|^2 + E(u(t_2)) \le E(u(t_1)), \quad T_0 \le t_1 \le t_2 < T. \tag{2.40}$$

For $\delta > 0$, let $\phi_\delta \in C_0^\infty(M)$ be a test function such that

$$0 \le \phi_\delta \le 1,\ \phi_\delta(x) = 1 \text{ for } d(x,\partial M) \ge \delta,\ |\nabla\phi_\delta| \le C\delta^{-1}.$$

Since $u \in C^\infty(M\times(0,T))$, multiplying (1.3) by $\partial_t u\phi_\delta^2$ and integrating over $M\times[t_1,t_2]$, we obtain the following local energy inequality:

$$\int_{M\times[t_1,t_2]} |\partial_t u|^2\phi_\delta^2 + \frac{1}{2}\int_M |\nabla u(t_2)|^2\phi_\delta^2 \le \frac{1}{2}\int_M |\nabla u(t_1)|^2\phi_\delta^2 + 2\int_{M\times[t_1,t_2]} \nabla u\cdot\partial_t u\phi_\delta\nabla\phi_\delta. \tag{2.41}$$

It is clear that (2.40) follows from (2.41), if we can show

$$\lim_{\delta\to 0}\int_{M\times[t_1,t_2]} \nabla u\cdot\partial_t u\phi_\delta\nabla\phi_\delta = 0. \tag{2.42}$$

To see (2.42), observe that (2.37) implies $\partial_t u(t) \in H_0^1(M)$ for $t \in [t_1,t_2]$ so that

$$\begin{aligned}
\int_{M\times[t_1,t_2]} |\partial_t u|^2|\nabla\phi_\delta|^2 \ &\lesssim\ \delta^{-2}\int_{t_1}^{t_2}\int_{\{x\in M: d(x,\partial M)\le\delta\}} |\partial_t u|^2 \\
&\lesssim\ \int_{t_1}^{t_2}\int_{\{x\in M: d(x,\partial M)\le\delta\}} |\nabla\partial_t u|^2 \to 0,\ \text{as } \delta\to 0.
\end{aligned}$$

It is clear that by the Hölder inequality, (2.42) follows from this. Thus (2.40) holds.

Choose $T_0 > 0$ such that both claims hold. Then by (1.3) we can estimate

$$\begin{aligned} & \int_M |\nabla u(t_1)|^2 - \int_M |\nabla u(t_2)|^2 - \int_M |\nabla (u(t_1) - u(t_2))|^2 \\ = \ & 2\int_M \nabla u(t_2) \cdot \nabla (u(t_1) - u(t_2)) \\ = \ & 2\int_M A(u(t_2))(\nabla u(t_2), u(t_2)) \cdot (u(t_1) - u(t_2)) + 2\int_M u_t(t_2) \cdot (u(t_2) - u(t_1)) \\ = \ & I + II. \end{aligned} \tag{2.43}$$

We first estimate I. Recall that for $y \in N$, let $P^{\perp}(y) : \mathbb{R}^k \to (T_yN)^{\perp}$ denote the orthogonal projection from $\mathbb{R}^k$ to the normal space of N at y. Since N is compact, a simple geometric argument implies that there exists $C > 0$ depending on N such that

$$\left| P^{\perp}(y)(z - y) \right| \le C|z - y|^2, \ \forall z \in N. \tag{2.44}$$

Thus

$$\begin{aligned} |I| & \lesssim \int_M |\nabla u(t_2)|^2 |u(t_1) - u(t_2)|^2 \\ & \le C\epsilon_0^2 \int_M \Big(\frac{1}{R_0^2} + \frac{1}{T_0} + \frac{1}{d^2(x, \partial M)} \Big) |u(t_1) - u(t_2)|^2 \\ & \lesssim C\epsilon_0 \int_M |\nabla (u(t_1) - u(t_2))|^2, \end{aligned}$$

where we have used both the Poincaré inequality and the Hardy inequality in the last step.

By (2.37), we have

$$\int_M |\partial_t u(t_2)|^2 \le \frac{1}{t_2 - t_1} \int_{M \times [t_1, t_2]} |\partial_t u|^2.$$

This, combined with the Hölder inequality and (2.40), implies

$$\begin{aligned} |II| & \le \|\partial_t u(t_2)\|_{L^2(M)} \|u(t_1) - u(t_2)\|_{L^2(M)} \\ & \le \sqrt{t_2 - t_1} \|\partial_t u(t_2)\|_{L^2(M)} \|\partial_t u\|_{L^2(M \times [t_1, t_2])} \\ & \le \int_{M \times [t_1, t_2]} |\partial_t u|^2 \\ & \le \frac{1}{2} \Big[\int_M |\nabla u(t_1)|^2 - \int_M |\nabla u(t_2)|^2 \Big]. \end{aligned}$$

Putting the estimates of I, II into (2.43) yields (2.33) so that the conclusions of Theorem 2.4.1 hold. The proof is now complete. □

Proof of Corollary 2.4.2. It follows from Theorem 2.4.1 that $E(u(t))$ is monotone decreasing for $T_0 \le t < +\infty$. Hence

$$\lim_{t\to\infty} E(u(t)) = c < +\infty.$$

Let $\{t_i\}$ be any monotone increasing sequence such that $\lim_{i\to\infty} t_i = +\infty$. Then (2.33) implies that for any $j \ge 1$,

$$\int_M |\nabla(u(t_{i+j}) - u(t_i))|^2 \le C\Big[\int_M |\nabla u(t_i)|^2 - \int_M |\nabla u(t_{i+j})|^2\Big] \to 0$$

as $i \to \infty$. This implies that there exists a map $u_\infty \in H^1(M, N)$, with $u_\infty = u_0$ on ∂M, such that

$$\lim_{t\to\infty} \int_M |\nabla(u(t) - u_\infty)|^2 = 0.$$

Since (2.40) implies there exists $t_i \uparrow \infty$ such that

$$\lim_{i\to\infty} \|\partial_t u(t_i)\|_{L^2(M)} = 0,$$

we see that u_∞ is a weak harmonic map. Moreover, by the gradient estimate (2.36), we have that for any compact set $K \subset\subset M$ and $m \ge 1$, one has that for t sufficiently large,

$$\|\nabla^m u(t)\|_{C^0(K)} \le C(\epsilon_0, m, K),$$

which clearly implies that $u(t) \to u_\infty$ in $C^m(K)$, as $t \to \infty$. This completes the proof. □

2.5 Uniqueness of Serrin's (p, q)-solutions to general Riemannian manifold

The following is the uniqueness result for Serrin's (p, q)-solutions of the heat flow of harmonic maps into a general Riemannian manifold.

Theorem 2.5.1. *For $n \ge 2$, $0 < T \le +\infty$, let (M, g) be either a compact or complete Riemannian manifold without boundary or a compact Riemannian manifold with boundary, and N be a compact Riemannian manifold without boundary. Let $u_1, u_2 \in H^1(M \times [0, T], N)$ be two weak solutions of (1.3), with $u_1 = u_2 = u_0$ on $\partial_p(M \times [0, T])$ for some $u_0 \in W^{1,2}(M, N)$, such that $\nabla u_1, \nabla u_2 \in L_t^q L_x^l(M \times [0, T])$ for some (l, q) satisfying (1.17) with $l > n, q > 2$. Then $u_1, u_2 \in C^\infty(M \times (0, T))$, and $u_1 \equiv u_2$ on $M \times [0, T]$.*

The following proposition indicate that any Serrin's (l, q)-solution to (1.3), under a suitable initial-boundary data u_0, satisfies the condition (2.1) for some $p > 1$ in Theorem 2.1.1.

Proposition 2.5.2. *For $n \geq 2$, $0 < T < +\infty$, and a compact Riemannian manifold $N \subset \mathbb{R}^k$ without boundary, suppose $u \in H^1(M \times [0,T], N)$ is a weak solution of (1.3), with the initial and boundary value $u_0 : M \to N$ satisfying $\nabla u_0 \in L^r(M)$ for some $n < r < +\infty$, such that $\nabla u \in L_t^q L_x^l(M \times [0,T])$ for some (l,q) satisfying (1.17) with $l > n, q > 2$. Then*
(i) $\partial_t u \in L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(M \times [0,T])$; and
(ii) for any $\epsilon > 0$, there exists $R = R(u,\epsilon) > 0$ such that for any $1 < s < \min\{\frac{l}{2}, \frac{q}{2}\}$,

$$\sup\left\{r^{s-(n+2)} \int_{P_r(x,t)\cap(M\times[0,T])} (|\nabla u|^s + r^s|\partial_t u|^s) \mid (x,t) \in M\times[0,T],\ 0 < r \leq R\right\} \leq \epsilon^s. \tag{2.45}$$

Proof. We consider the case that (M,g) is complete and noncompact, and leave the discussion of the other cases to interested readers. For simplicity, assume $(M,g) = (\mathbb{R}^n, g_0)$.

Let H be the heat kernel in $\mathbb{R}^n$. Then by the Duhamel formula, we have

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} H(x-y,t)u_0(y) \\ &+ \int_0^t \int_{\mathbb{R}^n} H(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s) \\ &= u_1(x,,t) + u_2(x,t). \end{aligned} \tag{2.46}$$

It is easy to see that

$$\nabla^2 u_1(x,t) = \int_{\mathbb{R}^n} \nabla_x H(x-y,t)\nabla_y u_0(y).$$

Hence by the standard integral estimates (see [24] page 234), we have

$$\left\|\nabla^2 u_1\right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n\times[0,T])} \leq CT^{\frac{1}{2}-\frac{n}{2r}} \left\|\nabla u_0\right\|_{L^r(\mathbb{R}^n)}. \tag{2.47}$$

For u_2, since

$$\nabla^2 u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \nabla_x^2 H(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s),$$

we can apply the Calderon-Zgymund's $L_t^s L_x^{s'}$-theorey to obtain

$$\left\|\nabla^2 u_2\right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n\times[0,T])} \leq C\left\||\nabla u|^2\right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n\times[0,T])} \leq C\left\|\nabla u\right\|^2_{L_t^q L_x^l(\mathbb{R}^n\times[0,T])}. \tag{2.48}$$

Substituting (2.47) and (2.48) into (2.46) yields $\nabla^2 u \in L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(\mathbb{R}^n \times [0,T])$. This, combined with the equation (1.3), then implies (i).

To see (ii), observe that by the Hölder inequality, we have that for any $1 < s < \min\{\frac{l}{2}, \frac{q}{2}\}$,

$$\left(r^{s-(n+2)} \int_{P_r(x,t)\cap(M\times[0,T])} |\nabla u|^s\right)^{\frac{1}{s}} \leq \left\|\nabla u\right\|_{L_t^q L_x^l(P_r(x,t)\cap(M\times[0,T]))},$$

and

$$\left(r^{2s-(n+2)} \int_{P_r(x,t)\cap(M\times[0,T])} |\partial_t u|^s\right)^{\frac{1}{s}} \leq \left\|\partial_t u\right\|_{L_t^{\frac{q}{2}} L_x^{\frac{l}{2}}(P_r(x,t)\cap(M\times[0,T]))}.$$

These two inequalities clearly imply (2.45). □

Now the following is the proof of ϵ-regularity of Serrin's solutions to (1.3) for any Riemannian manifold N. This need totally different argument from the cases in former sections.

Lemma 2.5.3. *There is an $\epsilon_0 > 0$ such that if $u \in H^1(P_1, N)$, with $\nabla u \in L_t^q L_x^l(P_1)$ for some $l \geq n$ and $q \geq 2$ satisfying (1.17), is a weak solution to (1.3) and*

$$\|\nabla u\|_{L_t^q L_x^l(P_1))} \leq \epsilon_0, \tag{2.49}$$

then $u \in C^\infty(P_{\frac{1}{2}}, N)$ and

$$\|u\|_{C^m(P_{\frac{1}{2}})} \leq C(m,n,p,q)\|\nabla u\|_{L^2(P_1)} \tag{2.50}$$

for any positive integer m.

The following inequality, due to Serrin ([78] Lemma 1) plays an very important role in the proof.

Lemma 2.5.4. *For any open set $U \subset \mathbb{R}^n$ and any open interval $I \subset \mathbb{R}$, let f, g, $h \in L_t^2 H_x^1(U \times I)$ and $f \in L_t^q L_x^l(U \times I)$ with $l \geq n$ and $q \geq 2$ satisfying (1.17). Then we have*

$$\int_{U\times I} |f||g||\nabla h| \leq C\|\nabla h\|_{L^2(U\times I)}\|g\|^{\frac{n}{l}}_{L_t^2 H_x^1(U\times I)} \left\{\int_I \|f\|^q_{L^l(U)}\|g\|^2_{L^2(U)}\, dt\right\}^{\frac{1}{q}}, \tag{2.51}$$

where $C > 0$ depends only on n.

Proof of Lemma 2.5.3. For any $(x,t) \in P_{\frac{1}{2}}$ and $0 < r \leq \frac{1}{2}$, by (2.49) we have

$$\|\nabla u\|_{L_t^q L_x^l(P_r(x,t)))} \leq \epsilon_0. \tag{2.52}$$

Let $v : P_r(x,t) \to \mathbb{R}^k$ solve

$$\begin{cases} v_t - \Delta v = 0, & \text{in } P_r(x,t) \\ v = u, & \text{on } \partial_p P_r(x,t). \end{cases} \tag{2.53}$$

Denote $w = u - v$. Multiplying (1.3) and (2.53) by w, subtracting the resulting equations and integrating over $P_r(x,t)$, we obtain

$$\sup_{t-r^2\le s\le t}\int_{B_r(x)}|w|^2(\cdot,s)+2\int_{P_r(x,t)}|\nabla w|^2\lesssim\int_{P_r(x,t)}|\nabla u|^2|w|$$
$$\lesssim\begin{cases}\|\nabla u\|_{L^2(P_r(x,t))}\|\nabla w\|^{\frac{n}{l}}_{L^2(P_r(x,t))}\left\{\int_{t-r^2}^t\|\nabla u\|^q_{L^l(B_r(x))}\|w\|^2_{L^2(B_r(x))}\right\}^{\frac{1}{q}}, & q<\infty\\ \|\nabla u\|_{L^2(P_r(x,t))}\|\nabla w\|_{L^2(P_r(x,t))}\|\nabla u\|_{L^\infty L^n(B_r(x))}, & q=\infty\end{cases}\tag{2.54}$$

where we have used (2.51) and the Poincaré inequality in last step. Since $\|\nabla u\|_{L^q_tL^l_x(P_r(z_0))}\le\epsilon_0$, we obtain, by the Young inequality,

$$\begin{aligned}&\sup_{t-r^2\le s\le t}\int_{B_r(x)}|w|^2(\cdot,s)+2\int_{P_r(x,t)}|\nabla w|^2\\&\le\begin{cases}\|\nabla w\|^2_{L^2(P_r(x,t))}+\epsilon_0\|\nabla u\|^2_{L^2(P_r(x,t))}+C\epsilon_0^{\frac{q}{2}}\|w\|^2_{L^\infty_tL^2_x(B_r(x))}, & q<\infty\\ \|\nabla w\|^2_{L^2(P_r(x,t)}+C\epsilon_0^2\|\nabla u\|^2_{L^2(P_r(x,t))}, & q=\infty.\end{cases}\end{aligned}\tag{2.55}$$

Choosing $\epsilon_0>0$ so that

$$\begin{cases}C\epsilon_0^{\frac{q}{2}}\le1, & q<+\infty,\\ C\epsilon_0\le1, & q=\infty,\end{cases}$$

we obtain

$$\int_{P_r(x,t)}|\nabla w|^2\le\epsilon_0\|\nabla u\|^2_{L^2(P_r(x,t))}.\tag{2.56}$$

On the other hand, by the standard estimate on the heat equation, we obtain that for any $0<\theta<1$,

$$(\theta r)^{-n}\int_{P_{\theta r}(x,t)}|\nabla v|^2\le C\theta^2r^{-n}\int_{P_r(x,t)}|\nabla u|^2.\tag{2.57}$$

(2.56) and (2.57) imply that

$$(\theta r)^{-n}\int_{P_{\theta r}(x,t)}|\nabla u|^2\le C\Big(\theta^2+\theta^{-n}\epsilon_0\Big)r^{-n}\int_{P_r(x,t)}|\nabla u|^2.\tag{2.58}$$

For any $0<\alpha<1$, choose first $\theta_0>0$ such that $C\theta_0^2\le\frac12\theta_0^{2\alpha}$ and then

$$\epsilon_0\le\min\left\{\frac{\theta_0^{2\alpha+n}}{2C},\quad\left(\frac{1}{2C}\right)^{\frac{2}{q}}\right\},$$

we obtain that for any $(x,t)\in P_{\frac12}$ and $0<r\le\frac12$, it holds

$$(\theta_0 r)^{-n}\int_{P_{\theta_0 r}(x,t)}|\nabla u|^2\le\theta_0^{2\alpha}r^{-n}\int_{P_{r_0}(x,t)}|\nabla u|^2.\tag{2.59}$$

Iterating (2.59), we obtain for any positive integer l,

$$(\theta_0^l r)^{-n} \int_{P_{\theta_0^l r}(x,t)} |\nabla u|^2 \leq \theta_0^{2l\alpha} r^{-n} \int_{P_r(x,t)} |\nabla u|^2. \tag{2.60}$$

It is standard that (2.60) implies

$$r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \leq C r^{2\alpha} \int_{P_1} |\nabla u|^2, \ \forall (x,t) \in P_{\frac{1}{2}}, \ 0 < r \leq \frac{1}{2}. \tag{2.61}$$

By (2.61), we have that $\nabla u \in M^{2,2-2\alpha}(P_1)$ for any $0 < \alpha < 1$. Now we can apply the regularity theorem by Huang-Wang [45] Theorem 1.5 to conclude that $u \in C^\infty(P_{\frac{1}{2}})$ and the estimate (2.50) holds. This completes the proof. □

By suitable scaling, the possible blow-up rate of $\|\nabla u(t)\|_{L^\infty}$ as t tends to zero can be estimated as below.

Lemma 2.5.5. *For $T > 0$ and a compact or complete manifold (M, g) without boundary, suppose that $u \in H^1(M \times [0,T], N)$ is a weak solution to (1.3), with $\nabla u \in L_t^q L_x^l(M \times [0,T])$ for some $l > n$ and $q > 2$ satisfying (1.17), then $u \in C^\infty(M \times (0,T], N)$ and there exists $t_0 > 0$ such that*

$$\sup_{0<t\leq t_0} \sqrt{t} \Big\| \nabla u(t) \Big\|_{L^\infty(M)} \leq C \Big\| \nabla u \Big\|_{L_t^q L_x^l(M\times[0,t_0])}. \tag{2.62}$$

In particular,

$$\lim_{t \downarrow 0^+} \sqrt{t} \Big\| \nabla u(t) \Big\|_{L^\infty(M)} = 0. \tag{2.63}$$

Proof. For simplicity, we assume that $(M, g) = (\mathbb{R}^n, g_0)$. Since $\nabla u \in L_t^q L_x^l(\mathbb{R}^n \times [0,T])$ for some $l > n$ and $q > 2$ satisfying (1.17), we have that for $\epsilon_0 > 0$ given by Lemma 2.5.3, there exists $\delta_0 > 0$ such that

$$\sup_{(x_0,t_0)\in\mathbb{R}^n\times[0,T]} \Big\| \nabla u \Big\|_{L_t^q L_x^l(P_{\delta_0}(x_0,t_0)\cap\mathbb{R}_+^{n+1})} \leq \epsilon_0,$$

In particular, for any $0 < \tau \leq \delta_0$ and any $x_0 \in \mathbb{R}^n$, we have

$$\Big\| \nabla u \Big\|_{L_t^q L_x^l(B_\tau(x_0)\times[0,\tau^2])} \leq \epsilon_0. \tag{2.64}$$

Define $v(y,s) = u(x_0 + \tau y, \tau^2 + \tau^2 s)$ for $(y,s) \in P_1(0,0)$. Then v solves (1.3) on $P_1(0,0)$, and satisfies

$$\Big\| \nabla v \Big\|_{L_t^q L_x^l(P_1(0,0))} \leq \epsilon_0.$$

Hence Lemma 2.5.3 implies

$$\|\nabla v\|_{L^\infty(P_{\frac{1}{2}}(0,0))} \leq C \|\nabla v\|_{L^2(P_1(0,0))}. \tag{2.65}$$

After rescalings, (2.65) implies that $u \in C^\infty(P_{\frac{\tau}{2}}(x_0, \tau^2))$ and

$$\tau \Big\|\nabla u\Big\|_{L^\infty(P_{\frac{\tau}{2}}(x,\tau^2))} \leq C\tau^{-\frac{n}{2}} \Big\|\nabla u\Big\|_{L^2(P_\tau(x,\tau^2))}. \tag{2.66}$$

By Hölder's inequality and (1.17), we have

$$\tau^{-\frac{n}{2}} \Big\|\nabla u\Big\|_{L^2(P_\tau(x_0,\tau^2))} \leq \Big\|\nabla u\Big\|_{L^q_t L^l_x(P_\tau(x_0,\tau^2))}. \tag{2.67}$$

Putting (2.67) together with (2.66), we obtain

$$\tau \Big\|\nabla u(\tau^2)\Big\|_{L^\infty(\mathbb{R}^n)} \leq C \Big\|\nabla u\Big\|_{L^q_t L^l_x(\mathbb{R}^n \times [0,\tau^2])}. \tag{2.68}$$

After sending $\tau \to 0$, (2.68) clearly implies (2.63). It is not hard to see that (2.62) also follows. This completes the proof. □

The next lemma handles the case that (M, g) is a compact Riemannian manifold with boundary.

Lemma 2.5.6. *For $T > 0$ and a compact manifold (M, g) with boundary, suppose that $u \in H^1(M \times [0, T], N)$ is a weak solution of (1.3), with $\nabla u \in L^q_t L^l_x(M \times [0, T])$ for some $l > n$ and $q > 2$ satisfying (1.17), then $u \in C^\infty(M \times (0, T], N)$. Moreover, for any sufficiently small $\epsilon_0 > 0$, there exists $T_0 > 0$ depending only on ϵ_0 and u such that*

$$|\nabla u(x_0, t_0)| \leq C\epsilon_0 \Big(\frac{1}{d(x_0, \partial M)} + \frac{1}{\sqrt{t_0}}\Big), \ \forall (x_0, t_0) \in M \times (0, T_0]. \tag{2.69}$$

Proof. Let $\epsilon_0 > 0$ be given by Lemma 2.5.3. Since $\nabla u \in L^q_t L^l_x(M \times [0, T])$ with $l > n, q > 2$, there exists $T_0 > 0$ such that

$$\Big\|\nabla u\Big\|_{L^q_t L^l_x(M \times [0,T_0])} \leq \epsilon_0.$$

For any $x_0 \in M$ and $0 < t_0 \leq T_0$, we divide the proof into two cases:
(i) $d(x_0, \partial M) > \sqrt{t_0}$; and
(ii) $d(x_0, \partial M) \leq \sqrt{t_0}$.
For (i), since $P_{\sqrt{t_0}}(z_0) \subset M \times (0, T_0]$, we have $\|\nabla u\|_{L^q_t L^l_x(P_{\sqrt{t_0}}(z_0))} \leq \epsilon_0$. As in Lemma 2.5.5, we conclude that $u \in C^\infty(P_{\frac{\sqrt{t_0}}{2}}(z_0))$ and

$$|\nabla u|(z_0) \leq \frac{C\epsilon_0}{\sqrt{t_0}}.$$

For (ii), set $r_0 = \min\{d(x_0, \partial M), \sqrt{t_0}\}$. Then $P_{r_0}(z_0) \subset M \times (0, T_0]$ and $\|\nabla u\|_{L^q_t L^l_x(P_{r_0}(z_0))} \leq \epsilon_0$. Hence we can conclude that $u \in C^\infty(P_{\frac{r_0}{2}}(z_0))$ and

$$|\nabla u|(z_0) \leq \frac{C\epsilon_0}{r_0} \leq C\epsilon_0 \Big(\frac{1}{d(x_0, \partial M)} + \frac{1}{\sqrt{t_0}}\Big).$$

Thus (2.69) holds. This completes the proof. □

Proof of Theorem 2.5.1. It follows from Lemma 2.5.5 and Lemma 2.5.6 that there exists $T_0 > 0$ such that both the condition (2.22) of Theorem 2.2.1 and the estimate (2.2) of Theorem 2.1.1 hold on $M \times [0, T_0]$. Thus we can apply the same proof of Theorem 2.2.1 to obtain that $u = v$ on $M \times [0, T_0]$. One can repeat the same argument to show that $u = v$ on $M \times [T_0, T]$. □

Chapter 3 Regularity and uniqueness for a class of solutions to the hydrodynamic flow of nematic liquid crystals

In this chapter, the Serrin's (p, q)-solution to hydrodynamic flow of nematic liquid crystals will be studied by similar argument as the study of heat flow of harmonic maps.

3.1 Regularity

Recall the definition of Serrin's (p, q)-solution is as follows:

Definition 3.1.1. *A weak solution $(u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)$ of (1.5) if called a Serrin's (p, q)-solution, if $(u, \nabla d) \in L_t^p L_x^q(\mathbb{R}^n \times [0, T])$ for some (p, q) satisfying (1.6).*

The result concerns an ϵ_0-regularity criterion for Serrin's (p, q)-solutions to (1.5).

Theorem 3.1.2. *There exists $\epsilon_0 > 0$ such that if a weak solution $(u, d) \in H^1(P_r(x_0, t_0), \mathbb{R}^n \times S^2)$ to (1.5) satisfies*

$$\|u\|_{L_t^p L_x^q(P_r(x_0,t_0))} + \|\nabla d\|_{L_t^p L_x^q(P_r(x_0,t_0))} \le \epsilon_0, \tag{3.1}$$

where $p \ge 2$ and $q \ge n$ satisfy (1.6), then $(u, d) \in C^\infty(P_{\frac{r}{16}}(x_0, t_0))$, and

$$r\|u\|_{L^\infty(P_{\frac{r}{16}}(x_0,t_0))} + r\|\nabla d\|_{L^\infty(P_{\frac{r}{16}}(x_0,t_0))} \le C\left(\|u\|_{L_t^p L_x^q(P_r(x_0,t_0))} + \|\nabla d\|_{L_t^p L_x^q(P_r(x_0,t_0))}\right). \tag{3.2}$$

A direct corollary of Theorem 3.1.2 is the following regularity theorem for Serrin's (p, q)-solutions to (1.5).

Corollary 3.1.3. *For some $0 < T < +\infty$, suppose that $(u, d) \in H^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n \times S^2)$ is a weak solution to (1.5) with $(u, \nabla d) \in L_t^p L_x^q(\mathbb{R}^n \times [0, T])$, for some $p > 2$ and $q > n$ satisfying (1.6). Then $(u, d) \in C^\infty(\mathbb{R}^n \times (0, T], \mathbb{R}^n \times S^2)$.*

Remark 3.1.4. *(i) For the heat flow of harmonic maps and the Navier-Stokes equations, Corollary 3.1.3 is valid for the end point case $(p, q) = (+\infty, n)$. It is an interesting open question to investigate the regularity of Serrin's solutions to (1.5) in this end point case.*
(ii) If $(u_0, \nabla d_0) \in L^\gamma(\mathbb{R}^n)$ for some $\gamma > n$, then the local existence of Serrin's solutions in $L_t^p L_x^q$ for some $p > 2$ and $q > n$ can be obtained by the fixed point argument (see e.g., [24] §4).

To prove Theorem 3.1.2 and Corollary 3.1.3 for nematic liquid crystal flows (1.5), the crucial step is to establish an ϵ_0-regularity criterion.

Lemma 3.1.5. *There exists $\epsilon_0 > 0$ such that if $(u, \nabla d) \in L_t^p L_x^q(P_1(0,1))$, for some $p \geq 2$ and $q \geq n$ satisfying (1.6), is a weak solution to (1.5) that satisfies*

$$\|u\|_{L_t^p L_x^q(P_1(0,1))} + \|\nabla d\|_{L_t^p L_x^q(P_1(0,1)))} \leq \epsilon_0, \tag{3.3}$$

then $(u, d) \in C^\infty(P_{\frac{1}{16}}(0,1))$, and

$$\|u\|_{L^\infty(P_{\frac{1}{16}}(0,1))} + \|\nabla d\|_{L^\infty(P_{\frac{1}{16}}(0,1))} \leq C\epsilon_0. \tag{3.4}$$

In the proof of this lemma, the following inequality, due to Serrin [78], plays an important role.

Lemma 3.1.6. *For any open set $U \subset \mathbb{R}^n$ and any open interval $I \subset \mathbb{R}$, let f, g, $h \in L_t^2 H_x^1(U \times I)$ and $f \in L_t^p L_x^q(U \times I)$ with $3 \leq n \leq q \leq +\infty$ and $2 \leq p \leq +\infty$ satisfying (1.6). Then*

$$\int_{U\times I} |f||g||\nabla h| \leq C\|\nabla h\|_{L^2(U\times I)} \|g\|_{L_t^2 H_x^1(U\times I)}^{\frac{n}{q}} \left\{ \int_I \|f\|_{L^q(\mathbb{R}^n)}^p \|g\|_{L^2(\mathbb{R}^n)}^2 \, dt \right\}^{\frac{1}{p}}, \tag{3.5}$$

where $C > 0$ depends only on n.

Proof of Lemma 3.1.5. For any $(x,t) \in P_{\frac{1}{2}}(0,1)$ and $0 < r < \frac{1}{2}$, we have, by (3.3),

$$\|u\|_{L_t^p L_x^q(P_r(x,t))} + \|\nabla d\|_{L_t^p L_x^q(P_r(x,t))} \leq \epsilon_0. \tag{3.6}$$

The proof will be divided into two claims.

Claim 1. $\nabla d \in L^\gamma(P_{\frac{1}{2}}(0,1))$ for any $1 < \gamma < \infty$, and

$$\|\nabla d\|_{L^\gamma(P_{\frac{1}{4}}(0,1))} \leq C(\gamma) \|\nabla d\|_{L_t^p L_x^q(P_1(0,1))}. \tag{3.7}$$

To show it, let $d_1 : P_r(x,t) \to \mathbb{R}^3$ solve

$$\begin{cases} \partial_t d_1 - \Delta d_1 = 0, & \text{in } P_r(x,t) \\ d_1 = d, & \text{on } \partial_p P_r(x,t). \end{cases} \tag{3.8}$$

Set $d_2 = d - d_1$. Multiplying $(1.5)_3$ and (3.8) by d_2, subtracting the resulting equations and integrating over $P_r(x,t)$, we obtain

$$\begin{aligned} &\sup_{t-r^2 \leq \tau \leq t} \int_{B_r(x)} |d_2|^2(\cdot, \tau) + 2\int_{P_r(x,t)} |\nabla d_2|^2 \\ \leq & C \int_{P_r(x,t)} (|u||d_2||\nabla d| + |\nabla d||d_2||\nabla d|) = J_1 + J_2. \end{aligned} \tag{3.9}$$

By (3.5), the Poincaré inequality and the Young inequality, we have

$$\begin{aligned} |J_1| \lesssim & \begin{cases} \|\nabla d\|_{L^2(P_r(x,t))} \|\nabla d_2\|_{L^2(P_r(x,t))}^{\frac{n}{q}} \\ \cdot \left\{ \int_{t-r^2}^t \|u\|_{L^q(B_r(x))}^p \|d_2\|_{L^2(B_r(x))}^2 \, d\tau \right\}^{\frac{1}{p}}, & p < +\infty \\ \|\nabla d\|_{L^2(P_r(x,t))} \|\nabla d_2\|_{L^2(P_r(x,t))} \|u\|_{L_t^\infty L_x^n(P_r(x,t))}, & p = +\infty, \end{cases} \\ \leq & \begin{cases} \frac{1}{2}\|\nabla d_2\|_{L^2(P_r(x,t))}^2 + C\epsilon_0 \|\nabla d\|_{L^2(P_r(x,t))}^2 + C\epsilon_0^{\frac{p}{2}} \|d_2\|_{L_t^\infty L_x^2(P_r(x,t))}^2, & p < +\infty \\ \frac{1}{2}\|\nabla d_2\|_{L^2(P_r(x,t))}^2 + C\epsilon_0 \|\nabla d\|_{L^2(P_r(x,t))}^2, & p = +\infty. \end{cases} \end{aligned}$$

Similarly, for J_2, we have

$$|J_2| \leq \begin{cases} \frac{1}{2}\|\nabla d_2\|^2_{L^2(P_r(x,t))} + C\epsilon_0\|\nabla d\|^2_{L^2(P_r(x,t))} + C\epsilon_0^{\frac{p}{2}}\|d_2\|^2_{L_t^\infty L_x^2(P_r(x,t))}, & p < +\infty \\ \frac{1}{2}\|\nabla d_2\|^2_{L^2(P_r(x,t))} + C\epsilon_0\|\nabla d\|^2_{L^2(P_r(x,t))}, & p = +\infty. \end{cases}$$

Putting these estimates into (3.9), applying (3.6), and choosing sufficiently small ϵ_0, we have

$$\int_{P_r(x,t)} |\nabla d_2|^2 \leq C\epsilon_0\|\nabla d\|^2_{L^2(P_r(x,t))}. \tag{3.10}$$

This, combined with the standard estimate on d_1, implies that for any $\theta \in (0,1)$,

$$(\theta r)^{-n}\int_{P_{\theta r}(x,t)} |\nabla d|^2 \leq C\Big(\theta^2 + \theta^{-n}\epsilon_0\Big)r^{-n}\int_{P_r(x,t)} |\nabla d|^2. \tag{3.11}$$

By iterations, we obtain for any $(x,t) \in P_{\frac{1}{2}}(0,1)$, $0 < r \leq \frac{1}{2}$ and $0 < \alpha < 1$,

$$r^{-n}\int_{P_r(x,t)} |\nabla d|^2 \leq Cr^{2\alpha}\int_{P_1(0,1)} |\nabla d|^2. \tag{3.12}$$

Hence $\nabla d \in \mathcal{M}^{2,2-2\alpha}(P_{\frac{1}{2}}(0,1))$ and

$$\|\nabla d\|_{\mathcal{M}^{2,2-2\alpha}(P_{\frac{1}{2}}(0,1))} \leq C\|\nabla d\|_{L_t^pL_x^q(P_1(0,1))}. \tag{3.13}$$

Now Claim 1 follows by the same estimate of Riesz potentials between parabolic Morrey spaces as in [46] (Theorem 1.5) and [62] (Lemma 2.1).

Claim 2. $u \in L^\gamma(P_{\frac{1}{4}}(0,1))$ for any $1 < \gamma < \infty$, and

$$\|u\|_{L^\gamma(P_{\frac{1}{4}}(0,1))} \leq C(\gamma)\|u\|_{L_t^pL_x^q(P_1(0,1))}. \tag{3.14}$$

Let $\mathbb{E}^\gamma$ be the closure in $L^\gamma(\mathbb{R}^n,\mathbb{R}^n)$ of all divergence-free vector fields with compact supports. Let $\mathbb{P} : L^2(\mathbb{R}^n,\mathbb{R}^n) \to \mathbb{E}^2$ be the Leray projection operator. It is well-known that $\mathbb{P}$ can be extended to a bounded linear operator from $L^\gamma(\mathbb{R}^n,\mathbb{R}^n)$ to $\mathbb{E}^\gamma$ for all $1 < \gamma < +\infty$. Let $\mathbb{A} = \mathbb{P}\Delta$ denote the Stokes operator.

For any $(x,t) \in P_{\frac{1}{4}}(0,1)$ and $0 < r \leq \frac{1}{4}$, let $\eta \in C_0^\infty(P_{2r}(x,t))$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $P_r(x,t)$, $|\nabla\eta| \leq 4r^{-1}$, and $|\partial_t\eta| \leq 16r^{-2}$. Let $(v,P^1) : \mathbb{R}^n \times (0,1) \to \mathbb{R}^n \times \mathbb{R}$ solve

$$\begin{cases} \partial_t v - \Delta v + \nabla P^1 = -\nabla\cdot\Big(\eta^2(u\otimes u + \nabla d\otimes\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_n)\Big) & \text{in } \mathbb{R}^n\times(0,1) \\ \nabla\cdot v = 0 & \text{in } \mathbb{R}^n\times(0,1) \\ v = 0 & \text{on } \mathbb{R}^n\times\{0\}. \end{cases} \tag{3.15}$$

Define $w : P_r(x,t) \to \mathbb{R}^n$ by $w = u - v$. Then w solves the Stokes equation in $P_r(x,t)$:

$$\begin{cases} \partial_t w - \Delta w + \nabla Q^1 = 0 & \text{in } P_r(x,t) \\ \nabla\cdot w = 0 & \text{in } P_r(x,t). \end{cases} \tag{3.16}$$

By the standard theory of linear Stokes' equations, we have that $w \in C^\infty(P_r(x,t))$ and, for any $\theta \in (0,1)$,

$$\|w\|_{L_t^pL_x^q(P_{\theta r}(x,t))} \le C\theta \ \|w\|_{L_t^pL_x^q(P_r(x,t))}. \tag{3.17}$$

To estimate v, we apply $\mathbb{P}$ to both sides of the equation $(3.15)_1$ to obtain

$$\partial_t v - \mathbb{A}v = -\mathbb{P}\nabla\cdot\Big(\eta^2(u\otimes u+\nabla d\otimes\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_n)\Big) \text{ in } \mathbb{R}^n\times(0,1);\ v=0 \text{ on } \mathbb{R}^n\times\{0\}.$$

By the Duhamel formula, we have

$$v(t) = -\int_0^t e^{-(t-\tau)\mathbb{A}}\mathbb{P}\nabla\cdot\Big(\eta^2(u\otimes u+\nabla d\otimes\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_n)\Big)\,d\tau,\ \ 0<t\le 1. \tag{3.18}$$

Now we can apply Fabes-Jones-Riviere [24] Theorem 3.1 (see also Kato [48] page 474, (2.3′)) to conclude that $v \in L_t^pL_x^q(\mathbb{R}^n\times[0,1])$ and

$$\begin{aligned}\|v\|_{L_t^pL_x^q(\mathbb{R}^n\times[0,1])} &\le C(\|\eta u\|^2_{L_t^pL_x^q(\mathbb{R}^n\times[0,1])} + \|\eta\nabla d\|^2_{L_t^pL_x^q(\mathbb{R}^n\times[0,1])})\\ &\le C\epsilon_0(\|u\|_{L_t^pL_x^q(P_{2r}(x,t)))} + \|\nabla d\|_{L_t^pL_x^q(P_{2r}(x,t))}).\end{aligned} \tag{3.19}$$

Putting (3.17) and (3.19) together, we have that for any $\theta\in(0,1)$,

$$\|u\|_{L_t^pL_x^q(P_{\theta r}(x,t))} \le C(\theta+\epsilon_0)\|u\|_{L_t^pL_x^q(P_{2r}(x,t))} + C\epsilon_0\|\nabla d\|_{L_t^pL_x^q(P_{2r}(x,t))}. \tag{3.20}$$

By Claim 1, we have that for any $\alpha\in(0,1)$, there exists $\epsilon_0>0$ depending on α such that

$$\|\nabla d\|_{L_t^pL_x^q(P_{2r}(x,t))} \le Cr^\alpha\|\nabla d\|_{L_t^pL_x^q(P_1(0,1))}. \tag{3.21}$$

Substituting (3.21) into (3.20) yields

$$\|u\|_{L_t^pL_x^q(P_{\theta r}(x,t))} \le C(\theta+\epsilon_0)\|u\|_{L_t^pL_x^q(P_{2r}(x,t))} + Cr^\alpha\|\nabla d\|_{L_t^pL_x^q(P_1(0,1))}. \tag{3.22}$$

It is standard that by choosing $\theta=\theta_0(\alpha)>0$ and iterating (3.22) finitely many times, we conclude that for any $(x,t)\in P_{\frac14}$, $0<r\le\frac14$ and $0<\alpha<1$,

$$\|u\|_{L_t^pL_x^q(P_r(x,t))} \le C\Big(\|u\|_{L_t^pL_x^q(P_1(0,1))} + \|\nabla d\|_{L_t^pL_x^q(P_1(0,1))}\Big)r^\alpha. \tag{3.23}$$

By Hölder's inequality, (3.23) implies that $u\in\mathcal{M}^{2,2-2\alpha}(P_{\frac38}(0,1))$, and

$$\|u\|_{\mathcal{M}^{2,2-2\alpha}(P_{\frac38}(0,1))} \le C\Big[\|u\|_{L_t^pL_x^q(P_1(0,1))} + \|\nabla d\|_{L_t^pL_x^q(P_1(0,1))}\Big]. \tag{3.24}$$

The higher integrability estimate of u on $P_{\frac14}(0,1)$ can be done by the parabolic Riesz potential estimate in parabolic Morrey spaces. Here we will sketch it. Let $\phi\in C_0^\infty(P_{\frac38}(0,1))$ such that $0\le\phi\le1$, $\phi\equiv1$ on $P_{\frac{5}{16}}(0,1)$, and

$$|\partial_t\phi|+|\nabla\phi|+|\nabla^2\phi|\le C.$$

Define $\widetilde{u} : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ by

$$\widetilde{u}(t) = -\int_0^t e^{-(t-\tau)\mathbb{A}}\mathbb{P}\nabla \cdot \Big(\phi^2(u \otimes u + \nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_n)\Big)\, d\tau,\ 0 < t \le 1. \quad (3.25)$$

Then, as in the proof of Theorem 3.1 (i) of [24], we have that for any $(x,t) \in \mathbb{R}^n \times (0,1]$,

$$|\widetilde{u}(x,t)| \le C \int_0^t \int_{\mathbb{R}^n} \frac{1}{\delta^{n+1}((x,t),(y,s))}(|\phi u|^2 + |\phi \nabla d|^2)(y,s)\, dyds. \quad (3.26)$$

Recall the parabolic Riesz potential of order 1, $I_1(\cdot)$, is defined by

$$I_1(f)(z) := \int_{\mathbb{R}^{n+1}} \frac{|f(w)|}{\delta^{n+1}(z,w)}\, dw,\ \ f \in L^1(\mathbb{R}^{n+1}).$$

Then we have

$$|\widetilde{u}(x,t)| \le CI_1(F)(x,t),\ \ (x,t) \in \mathbb{R}^n \times (0,1], \quad (3.27)$$

where

$$F = \phi^2(|u|^2 + |\nabla d|^2).$$

By Hölder's inequality, (3.13), and (3.24), we have that $F \in \mathcal{M}^{1,2-2\alpha}(\mathbb{R}^{n+1})$ and

$$\|F\|_{\mathcal{M}^{1,2-2\alpha}(\mathbb{R}^{n+1})} \le C\Big(\|\nabla d\|^2_{L^p_tL^q_x(P_1(0,1))} + \|u\|^2_{L^p_tL^q_x(P_1(0,1))}\Big). \quad (3.28)$$

Hence, by [46] Theorem 3.1 (ii), we conclude that $\widetilde{u} \in \mathcal{M}_*^{\frac{2-2\alpha}{1-2\alpha},2-2\alpha}(\mathbb{R}^n \times [0,1])$, and

$$\begin{aligned}\|\widetilde{u}\|_{\mathcal{M}_*^{\frac{2-2\alpha}{1-2\alpha},2-2\alpha}(\mathbb{R}^n\times[0,1])} &\le C\|F\|_{\mathcal{M}^{1,2-2\alpha}(\mathbb{R}^{n+1})} \\ &\le C\left(\|\nabla d\|^2_{L^p_tL^q_x(P_1(0,1))} + \|u\|^2_{L^p_tL^q_x(P_1(0,1))}\right). \quad (3.29)\end{aligned}$$

As $\lim\limits_{\alpha\uparrow\frac{1}{2}} \frac{2-2\alpha}{1-2\alpha} = +\infty$, we have that $\widetilde{u} \in L^\gamma(P_{\frac{5}{16}}(0,1))$ for any $1 < \gamma < +\infty$, and

$$\|\widetilde{u}\|_{L^\gamma(P_{\frac{5}{16}})} \le C(\gamma)\Big(\|\nabla d\|^2_{L^p_tL^q_x(P_1(0,1))} + \|u\|^2_{L^p_tL^q_x(P_1(0,1))}\Big). \quad (3.30)$$

Set $\widetilde{w} = u - \widetilde{u}$ on $P_{\frac{5}{16}}(0.1)$. Then it follows from (1.5) and (3.25) that

$$\partial_t\widetilde{w} - \Delta\widetilde{w} + \nabla\widetilde{Q} = 0;\ \nabla \cdot \widetilde{w} = 0 \quad \text{in} \ \ P_{\frac{5}{16}}(0,1).$$

By the standard theory of linear Stokes' equations, we have that $\widetilde{w} \in L^\infty(P_{\frac{1}{4}}(0,1))$, and

$$\begin{aligned}\|\widetilde{w}\|_{L^\infty(P_{\frac{1}{4}}(0,1))} &\le C\|\widetilde{w}\|_{L^1(P_{\frac{5}{16}}(0,1))} \le C\Big(\|u\|_{L^1(P_{\frac{5}{16}}(0,1))} + \|\widetilde{u}\|_{L^1(P_{\frac{5}{16}}(0,1))}\Big) \\ &\le C\Big(\|\nabla d\|_{L^p_tL^q_x(P_1(0,1))} + \|u\|_{L^p_tL^q_x(P_1(0,1))}\Big). \quad (3.31)\end{aligned}$$

It is clear that (3.14) follows from (3.30) and (3.31). This completes the proof of Claim 2.

Finally, it is not hard to see that by the $W^{2,1}_{\gamma}$-theory for the heat equation and the linear Stokes equation, and the Sobolev embedding theorem, we have that $(u, \nabla d) \in L^\infty(P_{\frac{1}{8}}(0,1))$. Then the Schauder's theory and the bootstrap argument can imply that $(u, d) \in C^\infty(P_{\frac{1}{16}}(0,1))$. Furthermore, the estimate (3.4) holds. This completes the proof. □

Proof of Corollary 3.1.3: It is easy to see that when $p > 2$, $q > n$, for any $(x,t) \in \mathbb{R}^n \times (0,T]$, we can find $R_0 > 0$ such that

$$\|u\|_{L^p_t L^q_x(P_{R_0}(x,t))} + \|\nabla d\|_{L^p_t L^q_x(P_{R_0}(x,t))} \le \epsilon_0, \tag{3.32}$$

where ϵ_0 is given in Lemma 3.1.5. By Theorem 3.1.2, we conclude that $(u, d) \in C^\infty(P_{\frac{R_0}{16}}(x,t))$. This completes the proof of Corollary 3.1.3 □

3.2 Uniqueness

As a corollary of Theorem 3.1.2 and Corollary 3.1.3, the following uniqueness of Serrin's (p,q)-solutions to (1.5) holds.

Theorem 3.2.1. *For $n \ge 2$, $0 < T < +\infty$, and $i = 1,2$, if $(u_i, d_i) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n \times S^2$ are two weak solutions to (1.5) with the same initial data $(u_0, d_0) : \mathbb{R}^n \to \mathbb{R}^n \times S^2$. Suppose, in additions, there exists $p > 2$ and $q > n$ satisfying (1.6) such that $(u_1, \nabla d_1), (u_2, \nabla d_2) \in L^p_t L^q_x(\mathbb{R}^n \times [0,T])$. Then $(u_1, d_1) \equiv (u_2, d_2)$ on $\mathbb{R}^n \times [0,T]$.*

To prove Theorem 3.2.1, one need the following estimate.

Lemma 3.2.2. *For $T > 0$, suppose that (u, d) is a weak solution to (1.5) in $\mathbb{R}^n \times (0,T]$, which satisfies the assumption of Theorem 3.2.1. Then $(u,d) \in C^\infty(\mathbb{R}^n \times (0,T], \mathbb{R}^n \times S^2)$, and there exists $t_0 > 0$ such that for $0 < t \le t_0$, it holds*

$$\sup_{0<\tau\le t} \sqrt{\tau}\Big(\|u(\tau)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d(\tau)\|_{L^\infty(\mathbb{R}^n)}\Big) \le C\Big(\|u\|_{L^p_t L^q_x(\mathbb{R}^n\times[0,t])} + \|\nabla d\|_{L^p_t L^q_x(\mathbb{R}^n\times[0,t])}\Big). \tag{3.33}$$

In particular, we have

$$\lim_{t\downarrow 0^+} \sqrt{t}\Big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d\|_{L^\infty(\mathbb{R}^n)}\Big) = 0. \tag{3.34}$$

Proof. Let ϵ_0 be given by Lemma 3.1.5. Since $p > 2$ and $q > n$ satisfy (1.6), for any $0 < \epsilon \le \epsilon_0$ we can find $t_0 > 0$ such that for any $0 < \tau \le \sqrt{t_0}$

$$\|u\|_{L^p_t L^q_x(\mathbb{R}^n\times[0,\tau^2])} + \|\nabla d\|_{L^p_t L^q_x(\mathbb{R}^n\times[0,\tau^2])} \le \epsilon. \tag{3.35}$$

For any $x_0 \in \mathbb{R}^n$, define

$$\begin{aligned} \bar{u}(y,s) &= \tau u(x_0 + y\tau, s\tau^2) \\ \bar{P}(y,s) &= \tau^2 P(x_0 + y\tau, s\tau^2) \\ \bar{d}(y,s) &= d(x_0 + y\tau, s\tau^2). \end{aligned}$$

Then $(\bar{u}, \bar{P}, \bar{d})$ is a weak solution to (1.5) on $P_1(0,1)$, and by (3.35),

$$\|\bar{u}\|_{L_t^p L_x^q(P_1(0,1))} + \|\nabla \bar{d}\|_{L_t^p L_x^q(P_1(0,1))} \leq \epsilon. \tag{3.36}$$

By Lemma 3.1.5, we conclude that

$$|\bar{u}(0,1)| + |\nabla \bar{d}(0,1)| \leq C \left(\|\bar{u}\|_{L_t^p L_x^q(P_1(0,1))} + \|\nabla \bar{d}\|_{L_t^p L_x^q(P_1(0,1))} \right). \tag{3.37}$$

By rescaling, this implies

$$\tau \left(|u(x_0, \tau^2)| + |\nabla d(x_0, \tau^2)| \right) \leq C \left(\|u\|_{L_t^p L_x^q(\mathbb{R}^n \times [0,\tau^2])} + \|\nabla d\|_{L_t^p L_x^q(\mathbb{R}^n \times [0,\tau^2])} \right) \leq C\epsilon. \tag{3.38}$$

Taking supremum over all $x_0 \in \mathbb{R}^n$ completes the proof. □

Proof of Theorem 3.2.1: By (3.34), we have that for any $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that

$$\begin{aligned} \mathcal{A}(t_0) &= \sum_{i=1}^{2} \Big[\sup_{0 \leq t \leq t_0} \sqrt{t} (\|u_i(t)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla d_i(t)\|_{L^\infty(\mathbb{R}^n)}) \\ &\quad + (\|u_i\|_{L_t^p L_x^q(\mathbb{R}^n \times [0,t_0])} + \|\nabla d_i\|_{L_t^p L_x^q(\mathbb{R}^n \times [0,t_0])}) \Big] \leq \epsilon. \end{aligned} \tag{3.39}$$

It suffices to show $(u_1, d_1) = (u_2, d_2)$ on $\mathbb{R}^n \times [0, t_0]$. To do so, let $u = u_1 - u_2$ and $d = d_1 - d_2$. Applying $\mathbb{P}$ to both $(1.5)_1$ for u_1 and u_2 and taking the difference of resulting equations, we have that

$$\begin{cases} u_t - \mathbb{A}u \\ = -\mathbb{P}\nabla \cdot \left(u \otimes u_1 + u_2 \otimes u + \nabla d \otimes \nabla d_1 + \nabla d_2 \otimes \nabla d + (|\nabla d_1| + |\nabla d_2|)|\nabla d| \mathbb{I}_n \right), \\ \nabla \cdot u = 0, \\ d_t - \Delta d = [(\nabla d_1 + \nabla d_2) \cdot \nabla d \; d_2 + |\nabla d_1|^2 d] - [u \cdot \nabla d_1 + u_2 \cdot \nabla d], \\ (u,d)\Big|_{t=0} = (0,0). \end{cases} \tag{3.40}$$

By the Duhamel formula, we have that for any $0 < t \leq t_0$,

$$\begin{aligned} u(t) = -\int_0^t \; & e^{-(t-\tau)\mathbb{A}} \mathbb{P}\nabla \cdot \Big(u \otimes u_1 + u_2 \otimes u \\ & + \nabla d \otimes \nabla d_1 + \nabla d_2 \otimes \nabla d + (|\nabla d_1| + |\nabla d_2|)|\nabla d| \mathbb{I}_n \Big) \, d\tau, \end{aligned}$$

$$d(t) = \int_0^t e^{-(t-\tau)\Delta} \Big((\nabla d_1 + \nabla d_2) \cdot \nabla d \; d_2 + |\nabla d_1|^2 d - u \cdot \nabla d_1 - u_2 \cdot \nabla d \Big) \, d\tau. \tag{3.41}$$

For $0 < t \leq t_0$, set

$$\Phi(t) = \|u\|_{L_t^p L_x^q(\mathbb{R}^n \times [0,t])} + \|\nabla d\|_{L_t^p L_x^q(\mathbb{R}^n \times [0,t])} + \sup_{0 \leq \tau \leq t} \|d(\cdot, \tau)\|_{L^\infty(\mathbb{R}^n)}.$$

By (3.41) and the standard estimate on the heat kernel, we obtain that

$$
\begin{aligned}
\Big\|\nabla d(t)\Big\|_{L^q(\mathbb{R}^n)} \le & C\Big[\sum_{i=1}^{2}\int_0^t (t-\tau)^{\frac{1}{p}-1}\|\nabla d_i\|_{L^q(\mathbb{R}^n)}\|\nabla d\|_{L^q(\mathbb{R}^n)}\,d\tau \\
&+\|d\|_{L^\infty(\mathbb{R}^n)}\int_0^t (t-\tau)^{\frac{1}{p}-1}\|\nabla d_1\|^2_{L^q(\mathbb{R}^n)}\,d\tau \\
&+\int_0^t (t-\tau)^{\frac{1}{p}-1}\|\nabla d_1\|_{L^q(\mathbb{R}^n)}\|u\|_{L^q(\mathbb{R}^n)}\,d\tau \\
&+\int_0^t (t-\tau)^{\frac{1}{p}-1}\|u_2\|_{L^q(\mathbb{R}^n)}\|\nabla d\|_{L^q(\mathbb{R}^n)}\,d\tau\Big].
\end{aligned}
\tag{3.42}
$$

By the standard Riesz potential estimate in L^p-spaces (see [24] Theorem 3.0), we see that $\nabla d \in L^p_t L^q_x(\mathbb{R}^n \times [0,t_0])$, and

$$
\begin{aligned}
\Big\|\nabla d\Big\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])} \le & C\Big[\sum_{i=1}^{2}\|\nabla d_i\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0]))}\|\nabla d\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])} \\
&+\|d\|_{L^\infty(\mathbb{R}^n\times[0,t_0])}\|\nabla d_1\|^2_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])} \\
&+\|\nabla d_1\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])}\|u\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])} \\
&+\|u_2\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])}\|\nabla d\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])}\Big] \\
\le & C\mathcal{A}(t_0)\Phi(t_0).
\end{aligned}
\tag{3.43}
$$

Similarly, by using the estimate of Theorem 3.1 (i) of [24], we have that $u \in L^p_t L^q_x(\mathbb{R}^n \times [0,t_0])$, and

$$
\|u\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t_0])} \le C\mathcal{A}(t_0)\Phi(t_0). \tag{3.44}
$$

Now we need to estimate $\sup\limits_{0\le\tau\le t_0}\|d(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}$. We claim

$$
\|d\|_{L^\infty(\mathbb{R}^n\times[0,t_0])} \le C\mathcal{A}(t_0)\Phi(t_0). \tag{3.45}
$$

To show (3.45), let $H(x,t)$ be the heat kernel of $\mathbb{R}^n$. By (3.41), we have

$$
\begin{aligned}
|d(x,t)| = & \Big|\int_0^t\int_{\mathbb{R}^n} H(x-y,t-\tau)\left((\nabla d_1+\nabla d_2)\cdot\nabla d\ d_2+|\nabla d_1|^2 d\right)(y,\tau)\,dyd\tau \\
&-\int_0^t\int_{\mathbb{R}^n} H(x-y,t-\tau)\left(u\cdot\nabla d_1+u_2\cdot\nabla d\right)(y,\tau)\,dyd\tau\Big| \\
\le & C\Big[\int_0^t\int_{\mathbb{R}^n} H(x-y,t-\tau)K(y,\tau)\,dyd\tau \\
&+\int_0^t\int_{\mathbb{R}^n} H(x-y,t-\tau)|\nabla d_1|^2(y,\tau)\,dyd\tau\cdot\sup_{0\le\tau\le t}\|d(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\Big],
\end{aligned}
\tag{3.46}
$$

where

$$K(y,\tau) := \sum_{i=1}^{2}(|u_i| + |\nabla d_i|)(|u| + |\nabla d|)(y,\tau).$$

By (3.39), we have that for any $0 < t \le t_0$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} H(x-y,t-\tau)K(y,\tau)\,dyd\tau \\
\le&\mathcal{A}(t_0)\int_0^t (t-\tau)^{-\frac{n}{2}}\tau^{-\frac{1}{2}}\int_{\mathbb{R}^n}(|u|+|\nabla d|)\exp\Big(-\frac{|x-y|^2}{4(t-\tau)}\Big)\,dyd\tau \\
\le&\mathcal{A}(t_0)\Big\|(t-\tau)^{-\frac{n}{2q}}\tau^{-\frac{1}{2}}\Big\|_{L^{\frac{p}{p-1}}([0,t])}\Big\||u|+|\nabla d|\Big\|_{L^p_tL^q_x(\mathbb{R}^n\times[0,t])} \\
\le&C\mathcal{A}(t_0)\Phi(t_0),
\end{aligned} \tag{3.47}$$

where we have used Hölder inequality and

$$\begin{aligned}
\Big\|(t-\tau)^{-\frac{n}{2q}}\tau^{-\frac{1}{2}}\Big\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}([0,t])} &= t^{(\frac{1}{2}-(\frac{n}{2q}+\frac{1}{p}))\frac{p}{p-1}}\int_0^1(1-\tau)^{-\frac{np}{2(p-1)q}}\tau^{-\frac{p}{2(p-1)}}\,d\tau \\
&= \int_0^1(1-\tau)^{-\frac{p-2}{2(p-1)}}\tau^{-\frac{p}{2(p-1)}}\,d\tau < +\infty,
\end{aligned}$$

as (i) $\frac{n}{2q}+\frac{1}{p}=\frac{1}{2}$, and (ii) $2<p<+\infty$ yields $\frac{p}{2(p-1)}<1$ and $\frac{p-2}{2(p-1)}<1$.

Similarly, we can obtain that for $0 \le t \le t_0$,

$$\int_0^t\int_{\mathbb{R}^n} H(x-y,t-\tau)|\nabla d_1|^2(y,\tau)\,dyd\tau \le C\mathcal{A}^2(t_0). \tag{3.48}$$

Putting (3.47) and (3.48) into (3.46) and taking supremum over $(x,t)\in\mathbb{R}^n\times[0,t_0]$, we have

$$\sup_{0\le t\le t_0}\|d\|_{L^\infty(\mathbb{R}^n)} \le C\mathcal{A}(t_0)\Phi(t_0) + C\mathcal{A}^2(t_0)\sup_{0\le t\le t_0}\|d\|_{L^\infty(\mathbb{R}^n)}. \tag{3.49}$$

Therefore, if we choose $\epsilon \le \sqrt{\frac{1}{2C}}$ so that $C\mathcal{A}^2(t_0) \le C\epsilon^2 \le \frac{1}{2}$, then we obtain (3.45).

Putting (3.43), (3.44) and (3.45) together, and choosing $\epsilon \le \frac{1}{2C}$, we obtain

$$\Phi(t_0) \le C\mathcal{A}(t_0)\Phi(t_0) \le \frac{1}{2}\Phi(t_0).$$

This implies that $\Phi(t_0)=0$ and hence $(u_1,d_1)\equiv(u_2,d_2)$ on $\mathbb{R}^n\times[0,t_0]$. If $t_0<T$, then we can repeat the argument for $t\in[t_0,T]$ and eventually show that $(u_1,d_1)\equiv(u_2,d_2)$ on $\mathbb{R}^n\times[0,T]$. This completes the proof. □

Motivated by the study of uniqueness of heat flow of harmonic maps in Chapter 2, in this chapter, the regularity and uniqueness of heat flow of biharmonic maps (1.9) will be discussed by similar argument based on the Riesz potential estimates between Morrey spaces, the higher order Hardy inequality and the general Gronwall argument. As applications, the uniqueness of weakly biharmonic maps, uniqueness of limit at $t = +\infty$ of heat flow of biharmonic maps and uniqueness of Serrin's-(p, q) solution will also be considered.

4.1 ϵ-regularity

This section is devoted to the proof of Theorem 4.1.1, i.e., the regularity of heat flow of biharmonic maps to $\mathbb{S}^L$ under the smallness condition (4.1). The idea is motivated by [10] on the regularity of stationary biharmonic maps to $\mathbb{S}^L$.

The first theorem concerns the regularity of (1.9).

Theorem 4.1.1. *For $\frac{3}{2} < p \leq 2$ and $0 < T < +\infty$, there exists $\epsilon_p > 0$ such that if $u \in W_2^{1,2}(\Omega \times [0,T], \mathbb{S}^L)$ is a weak solution of (1.9) and satisfies that, for $z_0 = (x_0, t_0) \in \Omega \times (0, T]$ and $0 < R_0 \leq \frac{1}{2}\min\{d(x_0, \partial\Omega), \sqrt{t_0}\}$,*

$$\|\nabla^2 u\|_{M_{R_0}^{p,2p}(P_{R_0}(z_0))} + \|\partial_t u\|_{M_{R_0}^{p,4p}(P_{R_0}(z_0))} \leq \epsilon_p, \tag{4.1}$$

then $u \in C^\infty\left(P_{\frac{R_0}{16}}(z_0), \mathbb{S}^L\right)$, and

$$\left|\nabla^m u(z_0)\right| \leq \frac{C\epsilon_p}{R_0^m}, \ \forall\ m \geq 1. \tag{4.2}$$

Remark 4.1.2. *It is an open question whether Theorem 4.1.1 holds for any compact Riemannian manifold N without boundary (with $p = 2$).*

The first step is to rewrite (1.9) into the form where nonlinear terms are of divergence structures, analogous to [10] on the equation of biharmonic maps to $\mathbb{S}^L$. As in [10], the nonlinearities in (1.9) can be divided into four different types: for $1 \leq \alpha \leq L+1$,

$$\begin{aligned}
&T_{11}^\alpha = \left(u_j^\alpha \Delta u^\beta (u^\beta - c^\beta)\right)_j \text{ or } \left(u_j^\beta \Delta u^\alpha (u^\beta - c^\beta)\right)_j, \ T_{12}^\alpha = \left((u^\alpha - c^\alpha) u_i^\beta u_{ij}^\beta\right)_j, \\
&T_{21}^\alpha = \Delta\left((u^\alpha - c^\alpha)|\nabla u|^2\right), \ T_{22} = \Delta\left((u^\beta - c^\beta)\Delta u^\beta\right), \\
&T_{23}^\alpha = \Delta\left(u^\alpha (u^\beta - c^\beta)\Delta u^\beta\right) \text{ or } \Delta\left(u^\beta (u^\beta - c^\beta)\Delta u^\alpha\right), \\
&T_{33} = \left((u^\beta - c^\beta) u_j^\beta\right)_{jii}, \ T_{41}^\alpha = \left(u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha\right)\left(u^\beta - c^\beta\right),
\end{aligned} \tag{4.3}$$

where the upper index $\alpha,\ \beta$ denotes the component of a vector, the lower index $i,\ j$ denotes the differentiation in the direction $x_i,\ x_j$, $c^\alpha \in \mathbb{R}^{L+1}$ is a constant, and the Einstein convention of summation is used.

Lemma 4.1.3. *The equation (1.9) is equivalent to*

$$\partial_t u^\alpha + \Delta^2 u^\alpha = \mathcal{F}_\alpha(T_{11}^\alpha,\ T_{12}^\alpha,\ T_{21}^\alpha,\ T_{22},\ T_{23}^\alpha,\ T_{33},\ T_{41}^\alpha),\ 1 \le \alpha \le L+1, \tag{4.4}$$

where $\mathcal{F}_\alpha$ denotes a linear function of its arguments such that the coefficients can be bounded independent of u.

Proof. We follow [10] Proposition 1.2 closely. First, by Lemma 1.3 of [10], for every fixed α,

$$c^\alpha \Delta\left(|\nabla u|^2\right) \text{ and } \left(u_j^\alpha |\nabla u|^2\right)_j \text{ are linear functions of } T_{11}^\alpha, T_{12}^\alpha, T_{21}^\alpha, T_{22}, T_{23}^\alpha, T_{33}, \tag{4.5}$$

whose coefficients can be bounded independent of u. For $1 \le \alpha \le L+1$, set

$$S_1^\alpha = u^\alpha |\Delta u|^2,\ S_2^\alpha = 2u^\alpha u_j^\beta \left(\Delta u^\beta\right)_j,\ S_3^\alpha = u^\alpha \Delta\left(|\nabla u|^2\right). \tag{4.6}$$

Differentiation of $|u| = 1$ gives

$$u^\gamma u_j^\gamma = 0,\ u^\gamma \Delta u^\gamma + |\nabla u|^2 = 0. \tag{4.7}$$

By the equation (1.8), we have

$$u^\alpha \Delta^2 u^\beta + u^\alpha \partial_t u^\beta = u^\beta \Delta^2 u^\alpha + u^\beta \partial_t u^\alpha,\ 1 \le \alpha, \beta \le L+1. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\begin{aligned}
\frac{S_2^\alpha}{2} =& u^\alpha u_j^\beta (\Delta u^\beta)_j = u_j^\beta \left(u^\alpha (\Delta u^\beta)_j - u^\beta \left(\Delta u^\alpha\right)_j\right) \\
=& u_j^\beta \left(u^\alpha (\Delta u^\beta)_j - u^\beta \left(\Delta u^\alpha\right)_j - u_j^\alpha \Delta u^\beta + u_j^\beta \Delta u^\alpha\right) + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) \\
=& \left\{\left(u^\beta - c^\beta\right)\left(u^\alpha (\Delta u^\beta)_j - u^\beta \left(\Delta u^\alpha\right)_j - u_j^\alpha \Delta u^\beta + u_j^\beta \Delta u^\alpha\right)\right\}_j \\
&+ \left(u^\beta - c^\beta\right)\left(u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha\right) + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) \\
=& \left\{\left(u^\beta - c^\beta\right)\left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right)\right\}_{jj} - \left\{u_j^\beta \left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right)\right\}_j \\
&- 2\left\{\left(u^\beta - c^\beta\right)\left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right)\right\}_j + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) + T_{41}^\alpha \\
=& -\left\{u_j^\beta \left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right)\right\}_j + u_j^\beta \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) + G_\alpha(T_{11}^\alpha, T_{21}^\alpha, T_{23}^\alpha, T_{41}^\alpha),
\end{aligned} \tag{4.9}$$

where G_α is a linear function of its arguments whose coefficients can be bounded independent of u. By (4.5) and (4.7), we have

$$\begin{aligned}
S_3^\alpha =& \left(u^\alpha - c^\alpha\right)\Delta\left(|\nabla u|^2\right) + c^\alpha \Delta\left(|\nabla u|^2\right) \\
=& \Delta\left(\left(u^\alpha - c^\alpha\right)|\nabla u|^2\right) - 2\left(u_j^\alpha (|\nabla u|^2\right)_j - \Delta u^\alpha u^\beta \Delta u^\beta + c^\alpha \Delta\left(|\nabla u|^2\right) \\
=& -\Delta u^\alpha u^\beta \Delta u^\beta + H_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{21}^\alpha, T_{22}, T_{23}^\alpha, T_{33}),
\end{aligned} \tag{4.10}$$

where H_α is a linear function of its arguments whose coefficients can be bounded independent of u. By (4.10), the definition of S_1^α, and (4.9), we have

$$\begin{aligned} S_1^\alpha + S_3^\alpha &= \left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right) \Delta u^\beta + \mathcal{H}_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{21}^\alpha, T_{22}, T_{23}^\alpha, T_{31}), \\ &= \left\{\left(u^\alpha \Delta u^\beta - u^\beta \Delta u^\alpha\right) u_j^\beta\right\}_j - \left(u_j^\alpha \Delta u^\beta - u_j^\beta \Delta u^\alpha\right) u_j^\beta \\ &\quad - \left(u^\alpha \Delta u_j^\beta - u^\beta \Delta u_j^\alpha\right) u_j^\beta + \mathcal{H}_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{21}^\alpha, T_{22}, T_{23}^\alpha, T_{31}), \\ &= -\frac{S_2^\alpha}{2} - \frac{S_2^\alpha}{2} + L_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{21}^\alpha, T_{22}, T_{23}^\alpha, T_{33}, T_{41}^\alpha), \end{aligned} \tag{4.11}$$

where L_α is a linear function of its arguments whose coefficients can be bounded independent of u. Therefore we obtain

$$S_1^\alpha + S_2^\alpha + S_3^\alpha = L_\alpha(T_{11}^\alpha, T_{12}^\alpha, T_{21}^\alpha, T_{22}, T_{23}^\alpha, T_{33}, T_{41}^\alpha).$$

This completes the proof. $\square$

Next we recall some basic properties of the heat kernel for Δ^2 in $\mathbb{R}^n$, and the definition of Riesz potentials on $\mathbb{R}^{n+1}$, and the definition of BMO space and John-Nirenberg's inequality (see [47]). Let $b(x,t)$ be the fundamental solution of

$$(\partial_t + \Delta^2)v = 0 \text{ in } \mathbb{R}_+^{n+1}.$$

Then we have (see [50] §2.2):

$$b(x,t) = t^{-\frac{n}{4}} g\left(\frac{x}{t^{\frac{1}{4}}}\right), \text{ with } g(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi\eta - |\eta|^4}, \ \xi \in \mathbb{R}^n,$$

and the estimate

$$\left|\nabla^m b(x,t)\right| \le C\left(|t|^{\frac{1}{4}} + |x|\right)^{-n-m}, \ \forall\ (x,t) \in \mathbb{R}_+^{n+1}, \forall\ m \ge 1. \tag{4.12}$$

We equip $\mathbb{R}^{n+1}$ with the parabolic distance δ:

$$\delta((x,t),(y,s)) = |t-s|^{\frac{1}{4}} + |x-y|, \ (x,t), \ (y,s) \in \mathbb{R}^{n+1}.$$

For $0 \le \alpha \le n+4$, define the Riesz potential of order α on $(\mathbb{R}^{n+1}, \delta)$ by

$$I_\alpha(f)(x,t) = \int_{\mathbb{R}^{n+1}} \left(|t-s|^{\frac{1}{4}} + |x-y|\right)^{\alpha-n-4} |f|(y,s), \ (x,t) \in \mathbb{R}^{n+1}. \tag{4.13}$$

For any open set $U \subset \mathbb{R}^{n+1}$, let $\mathrm{BMO}(U)$ denote the space of functions of bounded mean oscillations: $f \in \mathrm{BMO}(U)$ if

$$[f]_{\mathrm{BMO}(U)} := \sup\left\{ ⨍_{P_r(z)} |f - f_{P_r(z)}| : \ P_r(z) \subset U \right\} < +\infty, \tag{4.14}$$

where $\fint_{P_r(z)} = \frac{1}{|P_r(z)|}\int_{P_r(z)}$ and $f_{P_r(z)} = \fint_{P_r(z)} f$ denotes the average of f over $P_r(z)$. By the celebrated John-Nirenberg inequality (see [47]), we have that if $f \in \mathrm{BMO}(U)$, then for any $1 < q < +\infty$ it holds

$$\sup\left\{ \left(\fint_{P_r(z)} |f - f_{P_r(z)}|^q\right)^{\frac{1}{q}} : \quad P_r(z) \subset U\right\} \leq C(q)\Big[f\Big]_{\mathrm{BMO}(U)}. \tag{4.15}$$

Now it is ready to prove the ϵ-regularity for the heat flow of biharmonic maps to $\mathbb{S}^L$.

Proposition 4.1.4. *For any $\frac{3}{2} < p \leq 2$, there exists $\epsilon_p > 0$ such that if $u : P_4 \to \mathbb{S}^L$ is a weak solution of (1.9) and satisfies*

$$\sup_{(x,t)\in P_3, 0<r\leq 1} r^{2p-n-4}\int_{P_r(x,t)} \left(|\nabla^2 u|^p + r^{2p}|\partial_t u|^p\right) \leq \epsilon_p^p, \tag{4.16}$$

then $u \in C^\infty(P_{\frac{1}{2}}, \mathbb{S}^L)$, and

$$\Big\|\nabla^m u\Big\|_{C^0(P_{\frac{1}{2}})} \leq C(p,n,m), \ \forall\ m \geq 1. \tag{4.17}$$

Proof. We first establish Hölder continuity of u in $P_{\frac{3}{4}}$. It is based on the decay estimate.
Claim. There exist $\epsilon_p > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that

$$\Big[u\Big]_{\mathrm{BMO}(P_{\theta_0})} \leq \frac{1}{2}\Big[u\Big]_{\mathrm{BMO}(P_2)}. \tag{4.18}$$

In order to establish (4.18), we first want to prove that there exists $q > 1$ such that

$$\fint_{P_{\theta r}(z_0)} |u - u_{P_{\theta r}(z_0)}| \leq C\left(\theta^{-(n+4)}\epsilon_p + \theta\right)\left(\fint_{P_r(z_0)} |u - u_{P_r(z_0)}|^q\right)^{\frac{1}{q}} \tag{4.19}$$

holds for any $0 < \theta \leq \frac{1}{2}$, $z_0 \in P_1$, and $0 < r \leq 2$.

By translation and scaling, it suffices to show (4.19) for $z_0 = (0,0)$ and $r = 2$. First, we need to extend u from P_1 to $\mathbb{R}^{n+1}$. Let the extension, still denoted by u, be such that

$$|u| \leq 2 \text{ in } \mathbb{R}^{n+1}, \quad u = 0 \text{ outisde } P_2,$$

and

$$\int_{\mathbb{R}^{n+1}} |\nabla^2 u|^p + |\partial_t u|^p \lesssim \int_{P_2} |\nabla^2 u|^p + |\partial_t u|^p.$$

For $1 \leq \alpha \leq L+1$, let $w_{ij}^\alpha : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be solutions of

$$\partial_t w_{ij}^\alpha + \Delta^2 w_{ij}^\alpha = T_{ij}^\alpha \quad \text{in } \ \mathbb{R}^{n+1}_+; \ w_{ij}^\alpha = 0 \quad \text{on } \ \mathbb{R}^n \times \{0\} \tag{4.20}$$

for $ij \in \{11, 12, 21, 23, 41\}$, and and $w_{kk} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ be solutions of

$$\partial_t w_{kk} + \Delta^2 w_{kk} = T_{kk} \quad \text{in} \ \ \mathbb{R}^{n+1}_+; \ w_{kk} = 0 \quad \text{on} \ \ \mathbb{R}^n \times \{0\} \tag{4.21}$$

for $k \in \{2, 3\}$. Define $v : P_1 \to \mathbb{R}^{L+1}$ by letting

$$v^\alpha = u^\alpha - \mathcal{F}_\alpha(w^\alpha_{11}, w^\alpha_{12}, w^\alpha_{21}, w_{22}, w^\alpha_{23}, w_{33}, w^\alpha_{41}), \ 1 \le \alpha \le L + 1.$$

Here $\mathcal{F}_\alpha$ is the linear function given by Lemma 4.1.3. By (4.4), we have

$$\partial_t v + \Delta^2 v = 0 \ \text{ in } \ P_1. \tag{4.22}$$

It follows from (4.21) and the Duhamel formula that for $1 \le \alpha \le L + 1$,

$$\begin{cases} w^\alpha_{ij}(x, t) = \int_{\mathbb{R}^n \times [0,t]} b(x - y, t - s) T^\alpha_{ij}(y, s), & ij \in \{11, 12, 21, 23, 41\}, \\ w_{kk}(x, t) = \int_{\mathbb{R}^n \times [0,t]} b(x - y, t - s) T_{kk}(y, s), & k \in \{2, 3\}. \end{cases} \tag{4.23}$$

Set $c^\alpha = u^\alpha_{P_2}$ in (4.3). Then it is easy to see $|c^\alpha| \le 1$. Now we can estimate w^α_{12} by (w^α_{11} can be estimated similarly):

$$\begin{aligned} |w^\alpha_{12}(x, t)| = & \left| \int_{\mathbb{R}^n \times [0,t]} \nabla_j b(x - y, t - s)(u^\alpha - u^\alpha_{P_2}) u^\beta_i u^\beta_{ij}(y, s) \right| \\ \lesssim & \int_{\mathbb{R}^{n+1}} \left(|t - s|^{\frac{1}{4}} + |x - y| \right)^{-n-1} |u - u_{P_2}||\nabla u||\nabla^2 u|(y, s) \\ \lesssim & I_3 \left(\chi_{P_2} |u - u_{P_2}||\nabla u||\nabla^2 u| \right)(x, t), \end{aligned} \tag{4.24}$$

where χ_{P_2} is the characteristic function of P_2.

By the estimate of Riesz potentials in L^q-spaces (see also Proposition 4.5.5 below), we have that for any $f \in L^q$, $1 < q < +\infty$, $I_\alpha(f) \in L^{\tilde{q}}$, where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{\alpha}{n+4}$. As $p > \frac{3}{2}$, we can check that for sufficiently large $q_1 > 1$, there exists $\widetilde{q_1} > 1$ such that

$$\frac{1}{\widetilde{q_1}} = \frac{1}{p} + \frac{1}{2p} + \frac{1}{q_1} - \frac{3}{n + 4}.$$

Hence we obtain

$$\left\| w^\alpha_{12} \right\|_{L^{\widetilde{q_1}}(P_2)} \le C \left\| u - u_{P_2} \right\|_{L^{q_1}(P_2)} \left\| \nabla u \right\|_{L^{2p}(P_2)} \left\| \nabla^2 u \right\|_{L^p(P_2)} \le C \epsilon_p \left\| u - u_{P_2} \right\|_{L^{q_1}(P_2)}. \tag{4.25}$$

Next we can estimate w^α_{21} by (w_{22} and w^α_{23} can be estimated similarly):

$$\begin{aligned} |w^\alpha_{21}(x, t)| = & \left| \int_{\mathbb{R}^n \times [0,t]} \Delta b(x - y, t - s)(u^\alpha - u^\alpha_{P_2}) |\nabla u|^2(y, s) \right| \\ \lesssim & \int_{\mathbb{R}^{n+1}} \left(|t - s|^{\frac{1}{4}} + |x - y| \right)^{-n-2} |u - u_{P_2}||\nabla u|^2(y, s) \\ \lesssim & I_2 \left(\chi_{P_2} |u - u_{P_2}||\nabla u|^2 \right)(x, t). \end{aligned} \tag{4.26}$$

For $q_2 > 1$ sufficiently large, there exists $\widetilde{q}_2 > 1$ be such that

$$\frac{1}{\widetilde{q}_2} = \frac{1}{p} + \frac{1}{q_2} - \frac{2}{n+4}.$$

Hence we obtain

$$\Big\|w_{21}^\alpha\Big\|_{L^{\widetilde{q}_2}(P_2)} \le C\Big\|u - u_{P_2}\Big\|_{L^{q_2}(P_2)}\Big\||\nabla u|^2\Big\|_{L^p(P_2)} \le C\epsilon_p\Big\|u - u_{P_2}\Big\|_{L^{q_2}(P_2)}. \tag{4.27}$$

For w_{33}, we have

$$\begin{aligned} |w_{33}(x,t)| =& \Big|\int_{\mathbb{R}^n\times[0,t]} \Delta b_j(x-y,t-s)(u^\beta - u^\beta_{P_2})u^\beta_j(y,s)\Big| \\ \lesssim& \int_{\mathbb{R}^{n+1}} \Big(|t-s|^{\frac{1}{4}} + |x-y|\Big)^{-n-3} |u - u_{P_2}||\nabla u|(y,s) \\ \lesssim& I_1\left(\chi_{P_2}|u - u_{P_2}||\nabla u|\right). \end{aligned} \tag{4.28}$$

For $q_3 > 1$ sufficiently large, there exists $\widetilde{q}_3 > 1$ such that

$$\frac{1}{\widetilde{q}_3} = \frac{1}{2p} + \frac{1}{q_3} - \frac{1}{n+4}.$$

Hence we obtain

$$\Big\|w_{33}\Big\|_{L^{\widetilde{q}_3}(P_2)} \le C\Big\|u - u_{P_2}\Big\|_{L^{q_3}(P_2)}\Big\|\nabla u\Big\|_{L^{2p}(P_2)} \le C\epsilon_p\Big\|u - u_{P_2}\Big\|_{L^{q_3}(P_2)}. \tag{4.29}$$

For w_{41}^α, we have

$$\partial_t w_{41}^\alpha + \Delta^2 w_{41}^\alpha = \left(u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha\right)\left(u^\beta - u^\beta_{P_2}\right). \tag{4.30}$$

By the Duhamel formular, we have

$$w_{41}^\alpha(x,t) = \sum_\beta \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s)\left(u^\alpha \partial_t u^\beta - u^\beta \partial_t u^\alpha\right)\left(u^\beta - u^\beta_{P_2}\right)(y,s),$$

so that by applying the Young inequality we obtain

$$\begin{aligned} \|w_{41}\|_{L^{\widetilde{q}_4}(\mathbb{R}^n\times[0,2])} \lesssim& \|b\|_{L^1(\mathbb{R}^n\times[0,2])}\left(\sum_{\alpha,\beta}\Big\|(u^\alpha\partial_t u^\beta - u^\beta\partial_t u^\alpha)(u^\beta - u^\beta_{P_2})\Big\|_{L^{\widetilde{q}_4}(\mathbb{R}^n\times[0,2])}\right) \\ \lesssim& \|\partial_t u\|_{L^p(P_2)}\|u - u_{P_2}\|_{L^{q_4}(P_2)}, \end{aligned} \tag{4.31}$$

where $q_4 > \frac{p}{p-1}$ and $1 < \widetilde{q}_4 < p$ satisfy

$$\frac{1}{\widetilde{q}_4} = \frac{1}{p} + \frac{1}{q_4}.$$

Set

$$q = \max\{q_1, q_2, q_3, q_4\} > 1 \text{ and } \widetilde{q} = \min\{\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3, \widetilde{q}_4\} > 1.$$

By (4.25), (4.27), (4.29) and (4.31), we have

$$\sum_{ij=11,12,21,23,41}^{1\le\alpha\le L+1} \|w_{ij}^{\alpha}\|_{L^{\tilde q}(P_2)} + \sum_{k=2}^{3} \|w_{kk}\|_{L^{\tilde q}(P_2)} \le C\epsilon_p \Big\|u-u_{P_2}\Big\|_{L^q(P_2)}. \tag{4.32}$$

On the other hand, by the standard estimate on v, we have that for any $0<\theta<1$,

$$\left(⨍_{P_\theta} |v-v_{P_\theta}|^{\tilde q}\right)^{\frac{1}{\tilde q}} \le C\theta \left(⨍_{P_1} |v-v_{P_1}|^{q}\right)^{\frac{1}{q}} \le C\theta \Big\|u-u_{P_2}\Big\|_{L^q(P_2)}. \tag{4.33}$$

Adding (4.32) and (4.33) together and applying the Hölder inequality, we obtain

$$⨍_{P_\theta} |u-u_{P_\theta}| \le \left(⨍_{P_\theta} |u-u_{P_\theta}|^{\tilde q}\right)^{\frac{1}{\tilde q}} \le C\left(\theta^{-(n+4)}\epsilon_p+\theta\right)\left(⨍_{P_2} |u-u_{P_2}|^{q}\right)^{\frac{1}{q}}. \tag{4.34}$$

This implies (4.19).

Now we indicate how (4.18) follows from (4.19). It follows from the Poincaré inequality and (4.16) that $u\in \mathrm{BMO}(P_3)$, and hence by (4.15) we have

$$⨍_{P_{\theta r}(z_0)} |u-u_{P_{\theta r}(z_0)}| \le C\left(\theta^{-(n+4)}\epsilon_p+\theta\right)\Big[u\Big]_{\mathrm{BMO}(P_2)} \tag{4.35}$$

holds for any $0<\theta\le\frac{1}{2}$, $z_0\in P_1$, and $0<r\le 1$. Taking supremum of (4.35) over all $z_0\in P_\theta$ and $0<r\le 1$, we obtain

$$\Big[u\Big]_{\mathrm{BMO}(P_\theta)} \le C\left(\theta^{-(n+4)}\epsilon_p+\theta\right)\Big[u\Big]_{\mathrm{BMO}(P_2)}. \tag{4.36}$$

If we choose $\theta=\theta_0\in(0,\frac{1}{2})$ and ϵ_p small enough so that

$$C\left(\theta_0^{-(n+4)}\epsilon_p+\theta_0\right)\le\frac{1}{2},$$

then (4.36) implies (4.18).

It is standard that iterating (4.18) yields the Hölder continuity of u by using the Campanato theory [4]. To prove the higher-order regularity, we need the following proposition

Proposition 4.1.5. *For $0<\alpha<1$, if $u\in W_2^{1,2}\cap C^\alpha(P_2,N)$ is a weak solution of (1.8), then $u\in C^\infty(P_1,N)$, and*

$$\Big\|\nabla^m u\Big\|_{C^0(P_1)} \lesssim \Big[u\Big]_{C^\alpha(P_2)} + \Big\|u\Big\|_{L_t^2W_x^{2,2}(P_2)},\ \forall\ m\ge 1. \tag{4.37}$$

Proof. By *Claim* 2 and *Claim* 3 in the proof of Theorem 4.5.3 below, it suffices to establish that $\nabla^2 u\in M^{2,4-4\tilde\alpha}(P_{\frac{3}{2}})$ for some $\frac{2}{3}<\tilde\alpha<1$, and

$$\Big\|\nabla^2 u\Big\|_{M^{2,4-4\tilde\alpha}(P_{\frac{3}{2}})} \lesssim \Big[u\Big]_{C^\alpha(P_2)} + \Big\|\nabla^2 u\Big\|_{L^2(P_2)}. \tag{4.38}$$

This will be achieved by the hole-filling type argument. For any fixed $z_0=(x_0,t_0)\in P_{\frac{3}{2}}$ and $0<r\le\frac{1}{4}$, let $\phi\in C_0^\infty(\mathbb{R}^n)$ be a cut-off function of $B_r(x_0)$, i.e.,

$$0\le\phi\le 1,\ \phi\equiv 1 \text{ in } B_r(x_0),\ \phi\equiv 0 \text{ outside } B_{2r}(x_0),\ |\nabla^m\phi|\le Cr^{-m},\ \forall\, m\ge 1.$$

Set $c:=\fint_{P_r(z_0)} u\in\mathbb{R}^{L+1}$. Multiplying (1.8) by $(u-c)\phi^4$ and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}|u-c|^2\phi^4+2\int_{\mathbb{R}^n}\Delta(u-c)\cdot\Delta((u-c)\phi^4)=2\int_{\mathbb{R}^n}\mathcal{N}_{\rm bh}[u]\cdot(u-c)\phi^4\\ \lesssim&\int_{\mathbb{R}^n}|\nabla^2u|^2|u-c|\phi^4+\int_{\mathbb{R}^n}|\nabla u||\nabla^2u||\nabla((u-c)\phi^4)|.\end{aligned}\tag{4.39}$$

For the second term in the left hand side of (4.39), we have

$$\begin{aligned}&2\int_{\mathbb{R}^n}\Delta(u-c)\cdot\Delta((u-c)\phi^4)=2\int_{\mathbb{R}^n}\nabla^2(u-c)\cdot\nabla^2((u-c)\phi^4)\\ \ge&\ 2\int_{B_r(z_0)}|\nabla^2u|^2-C\int_{\mathbb{R}^n}|u-c|^2(|\nabla^2\phi|^2+|\nabla\phi|^4)+\phi^2|\nabla\phi|^2|\nabla u|^2.\end{aligned}\tag{4.40}$$

Substituting (4.40) into (4.39) and integrating over $t\in[t_0-r^4,t_0]$, we obtain

$$\begin{aligned}\int_{P_r(z_0)}|\nabla^2u|^2\le&\int_{B_{2r}(x_0)\times\{t_0-r^4\}}|u-c|^2+\left(2^{-(n+4)}+C\mathrm{osc}_{P_{2r}(z_0)}u\right)\int_{P_{2r}(z_0)}|\nabla^2u|^2\\ &+Cr^n\left(\mathrm{osc}_{P_{2r}(z_0)}u\right)^2+C\left[1+(\mathrm{osc}_{P_{2r}(z_0)}u)^2\right]r^{-2}\int_{P_{2r}(z_0)}\phi^2|\nabla u|^2\\ &+C\int_{P_{2r}(z_0)}|\nabla u|^4\phi^4\end{aligned}\tag{4.41}$$

By integration by parts and the Hölder inequality, we have

$$r^{-2}\int_{P_{2r}(z_0)}\phi^2|\nabla u|^2\le Cr^{-2}\left(\mathrm{osc}_{P_{2r}(z_0)}u\right)\int_{P_{2r}(z_0)}|\nabla^2u|+Cr^n\left(\mathrm{osc}_{P_{2r}(z_0)}u\right)^2,$$

and

$$\begin{aligned}&\int_{P_{2r}(z_0)}\phi^4|\nabla u|^4\\ \le&\ 2^{-(n+4)}\int_{P_{2r}(z_0)}|\nabla^2u|^2+Cr^n\left(\mathrm{osc}_{P_{2r}(z_0)}u\right)^4+C\left(\mathrm{osc}_{P_{2r}(z_0)}u\right)^2\int_{P_{2r}(z_0)}|\nabla^2u|^2.\end{aligned}$$

Putting these two inequalities into (4.41) and using $\mathrm{osc}_{P_{2r}(z_0)}u\le Cr^\alpha$, we get

$$\begin{aligned}&\int_{P_r(z_0)}|\nabla^2u|^2\\ \le&\ \left(2^{-(n+3)}+Cr^\alpha\right)\int_{P_{2r}(z_0)}|\nabla^2u|^2+Cr^{n+2\alpha}+C(1+r^{2\alpha})r^{\alpha-2}\int_{P_{2r}(z_0)}|\nabla^2u|\\ \le&\ \left(2^{-(n+2)}+Cr^\alpha\right)\int_{P_{2r}(z_0)}|\nabla^2u|^2+Cr^{n+2\alpha},\end{aligned}\tag{4.42}$$

where we have used the following inequality in the last step:

$$C(1+r^{2\alpha})r^{\alpha-2}\int_{P_{2r}(z_0)}|\nabla^2 u| \le 2^{-(n+3)}\int_{P_{2r}(z_0)}|\nabla^2 u|^2 + Cr^{n+2\alpha}.$$

Choosing $r>0$ so small that $Cr^\alpha \le 2^{-(n+3)}$, we see that (4.42) implies

$$r^{-n}\int_{P_r(z_0)}|\nabla^2 u|^2 \le \frac{1}{2}(2r)^{-n}\int_{P_{2r}(z_0)}|\nabla^2 u|^2 + Cr^{2\alpha}. \tag{4.43}$$

It is clear that iterating (4.43) implies that there is $\alpha_0 \in (0,1)$ such that $\nabla^2 u \in M^{2,4-2\alpha_0}(P_{\frac{3}{2}})$ and

$$\left\|\nabla^2 u\right\|_{M^{2,4-2\alpha_0}(P_{\frac{3}{2}})} \lesssim \left[u\right]_{C^\alpha(P_2)} + \left\|\nabla^2 u\right\|_{L^2(P_2)}. \tag{4.44}$$

We can apply the estimate (4.44) and repeat the above argument to show that $\nabla^2 u \in M^{2,4-4\alpha_0}(P_{\frac{3}{2}})$ and the estimate (4.44) holds with α_0 replaced by $2\alpha_0$. Repeating these argument again and again until there exists $\tilde{\alpha} \in (\frac{2}{3},1)$ such that $\nabla^2 u \in M^{2,4-4\tilde{\alpha}}(P_{\frac{3}{2}})$ and the estimate (4.38) holds. The remaining parts of the proof can be done by following the same arguments as in *Claim* 2 and *Claim* 3 of the proof of Theorem 4.5.3 below. This completes the proof. □

Continued proof of Proposition 4.1.4. The higher-order regularity now follows from the Proposition 4.1.5. After this, we have that $u \in C^\infty(P_{\frac{1}{2}}, \mathbb{S}^L)$ and the estimate (4.17) holds. □

Proof of Theorem 4.1.1. By the definition of Morrey spaces, for $z_0 = (x_0,t_0) \in \Omega \times (0,T)$ and $R_0 \le \frac{1}{2}\min\{d(x_0,\partial\Omega), \sqrt{t_0}\}$, we have

$$\sup_{z\in P_{\frac{R_0}{2}}(z_0),\ r\le\frac{R_0}{2}} r^{2p-(n+4)}\int_{P_r(z)}(|\nabla^2 u|^p + r^{2p}|\partial_t u|^p) \le \epsilon_p^p. \tag{4.45}$$

Consider $v(x,t) = u(x_0 + \frac{R_0}{8}x, t_0 + (\frac{R_0}{8})^4 t) : P_4 \to \mathbb{S}^L$. It is easy to check that v is a weak solution of (1.9) and satisfies (4.16). Hence Proposition 4.1.4 implies that $v \in C^\infty(P_{\frac{1}{2}}, \mathbb{S}^L)$ and satisfies (4.17). After rescaling, we see that $u \in C^\infty(P_{\frac{R_0}{16}}(z_0), \mathbb{S}^L)$ and the estimate (4.2) holds. □

Since biharmonic maps are steady solutions of the heat flow of biharmonic maps, as a direct consequence of Theorem 4.1.1 we have the following ϵ-regularity for biharmonic maps to $\mathbb{S}^L$.

Corollary 4.1.6. *For $\frac{3}{2} < p \le 2$, there exist $\epsilon_p > 0$ and $r_0 > 0$ such that if $u \in W^{2,p}(\Omega, \mathbb{S}^L)$ is a weak solution of (1.7) and satisfies*

$$\sup_{x\in\Omega}\ \sup_{0<r\le\min\{r_0,d(x,\partial\Omega)\}} r^{2p-n}\int_{B_r(x)}|\nabla^2 u|^p \le \epsilon_p^p, \tag{4.46}$$

then $u \in C^\infty(\Omega, \mathbb{S}^L)$, *and*

$$|\nabla^m u(x)| \le C\epsilon_p \Big(\frac{1}{r_0^m} + \frac{1}{d^m(x, \partial\Omega)} \Big), \ \forall\ m \ge 1. \tag{4.47}$$

Remark 4.1.7. *For* $p = 2$, *Corollary 4.1.6 was first proved by Chang-Wang-Yang [10]. For biharmonic maps into any compact Riemannian manifold* N *without boundary, Corollary 4.1.6 was proved by [89, 92] for* $p = 2$.

4.2 Uniqueness and convexity of heat flow of biharmonic maps

Utilizing Theorem 4.1.1, the following uniqueness theorem is easy to be proved similar to the case in heat flow of harmonic maps.

Theorem 4.2.1. *For* $n \ge 4$ *and* $\frac{3}{2} < p \le 2$, *there exist* $\epsilon_0 = \epsilon_0(p, n) > 0$ *and* $R_0 = R_0(\Omega, \epsilon_0) > 0$ *such that if* $u_1, u_2 \in W_2^{1,2}(\Omega \times [0, T], \mathbb{S}^L)$ *are weak solutions of (1.8), with the same initial and boundary value* $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$, *that satisfy*

$$\max_{i=1,2} \Big[\|\nabla^2 u_i\|_{M_{R_0}^{p,2p}(\Omega\times(0,T))} + \|\partial_t u_i\|_{M_{R_0}^{p,4p}(\Omega\times(0,T))} \Big] \le \epsilon_0, \tag{4.48}$$

then $u_1 \equiv u_2$ *on* $\Omega \times [0, T]$.

To prove the theorem, we first recall the second order Hardy inequality.

Lemma 4.2.2. *There is* $C > 0$ *depending only on* n *and* Ω *such that if* $f \in W_0^{2,2}(\Omega)$, *then*

$$\int_\Omega \frac{|f(x)|^2}{d^4(x, \partial\Omega)} \le C \int_\Omega |\nabla^2 f(x)|^2. \tag{4.49}$$

Proof. For simplicity, we only indicate a proof for the case $\Omega = B_1$ – the unit ball in $\mathbb{R}^n$. The readers can refer to [18] for a proof of general domains. By approximation, we may assume $f \in C_0^\infty(B_1)$. Writing the left hand side of (4.49) in spherical coordinates, integrating over B_1, and using the Hölder inequality, we obtain

$$\begin{aligned}
\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} &= \int_0^1 \int_{\mathbb{S}^{n-1}} \frac{|f|^2(r,\theta)}{(1-r)^4} r^{n-1}\, dH^{n-1}(\theta) dr \\
&= -\int_0^1 \int_{\mathbb{S}^{n-1}} \frac{1}{3(1-r)^3} \left(2ff_r r^{n-1} + |f|^2(n-1)r^{n-2}\right) dH^{n-1}(\theta) dr \\
&\le -\int_0^1 \int_{\mathbb{S}^{n-1}} \frac{2}{3(1-r)^3} ff_r r^{n-1}\, dH^{n-1}(\theta) dr \\
&\le C \int_0^1 \int_{\mathbb{S}^{n-1}} \frac{|f||f_r| r^{n-1}}{(1-r)^3}\, dH^{n-1}(\theta) dr \\
&\le C \int_{B_1} \frac{|f(x)||\nabla f(x)|}{(1-|x|)^3} \\
&\le C \left(\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} \right)^{\frac{1}{2}} \left(\int_{B_1} \frac{|\nabla f(x)|^2}{(1-|x|)^2} \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.50}$$

Thus, by using the first-order Hardy inequality, we obtain

$$\int_{B_1} \frac{|f(x)|^2}{(1-|x|)^4} \le C \int_{B_1} \frac{|\nabla f(x)|^2}{(1-|x|)^2} \le C \int_{B_1} |\nabla^2 f(x)|^2. \tag{4.51}$$

This yields (4.49). □

Proof of Theorem 4.2.1. First, by Theorem 4.1.1, we have that for $i = 1, 2$, $u_i \in C^\infty(\Omega \times (0,T), \mathbb{S}^L)$, and

$$\left|\nabla^m u_i(x,t)\right| \le C\epsilon_p \left(\frac{1}{R_p^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right),\ \forall (x,t) \in \Omega \times (0,T),\ \forall\, m \ge 1. \tag{4.52}$$

Set $w = u_1 - u_2$. Then w satisfies

$$\begin{cases} \partial_t w + \Delta^2 w = \mathcal{N}_{\text{bh}}[u_1] - \mathcal{N}_{\text{bh}}[u_2] & \text{in } \Omega \times (0,T) \\ w = 0 & \text{on } \partial_p(\Omega \times (0,T)) \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.53}$$

Multiplying (4.53) by w and integrating over Ω, by (4.60), (4.52), the Poincaré inequality and the Hardy inequality (4.49), we obtain that

$$\begin{aligned} \frac{d}{dt}\int_\Omega |w|^2 + 2\int_\Omega |\nabla^2 w|^2 &= 2\int_\Omega (\mathcal{N}_{\text{bh}}[u_1] - \mathcal{N}_{\text{bh}}[u_2]) \cdot w \\ &\lesssim \sum_{i=1}^2 \int_\Omega (|\nabla u_i|^2 |\nabla^2 u_i| + |\nabla^2 u_i|^2 + |\nabla u_i||\nabla^3 u_i|)|w|^2 \\ &\lesssim \epsilon_p^4 \int_\Omega \frac{|w(x,t)|^2}{R_p^4} + \frac{|w(x,t)|^2}{d^4(x,\partial\Omega)} + \frac{|w(x,t)|^2}{t} \\ &\lesssim \epsilon_p \int_\Omega |\nabla^2 w|^2 + \frac{\epsilon_p}{t}\int_\Omega |w|^2. \end{aligned}$$

If we choose $\epsilon_p > 0$ sufficiently small and $R_p \ge \epsilon_p$, then it holds

$$\frac{d}{dt}\int_\Omega |w|^2 \le \frac{C\epsilon_p}{t}\int_\Omega |w|^2. \tag{4.54}$$

It follows from (4.54) that

$$\begin{aligned} \frac{d}{dt}\Big(t^{-\frac{1}{2}}\int_\Omega |w|^2\Big) &= t^{-\frac{1}{2}}\frac{d}{dt}\int_\Omega |w|^2 - \frac{1}{2}t^{-\frac{3}{2}}\int_\Omega |w|^2 \\ &\le (C\epsilon - \frac{1}{2})t^{-\frac{3}{2}}\int_\Omega |w|^2 \le 0. \end{aligned} \tag{4.55}$$

Integrating this inequality from 0 to t yields

$$t^{-\frac{1}{2}}\int_\Omega |w|^2 \le \lim_{t\downarrow 0^+} t^{-\frac{1}{2}}\int_\Omega |w|^2. \tag{4.56}$$

Since $w(\cdot,0)=0$, we have

$$w(x,t)=\int_0^t w_t(x,\tau)\,d\tau,\ \text{ a.e. } x\in\Omega,$$

so that, by the Hölder inequality,

$$t^{-\frac{1}{2}}\int_\Omega |w(x,t)|^2 \le t^{\frac{1}{2}}\int_0^t\int_\Omega |w_t|^2(x,\tau)\,dxd\tau \le Ct^{\frac{1}{2}} \to 0,\ \text{ as } t\downarrow 0^+.$$

This, combined with (4.56), implies $w\equiv 0$ in $\Omega\times[0,T]$. The proof is complete. □

4.3 Convexity and uniqueness of biharmonic maps

In this section, it will be shown that the convexity and uniqueness properties for biharmonic maps with small energy holds, which are the second-order extensions of the theorems on harmonic maps with small energy by Struwe [85], Moser [70], and in Chapter 2.

Recall that the Dirichlet problem for a biharmonic map $u\in W^{2,2}(\Omega,N)$ is defined by:

$$\begin{cases} \Delta^2 u = \mathcal{N}_{\text{bh}}[u] & \text{in } \Omega \\ \left(u,\dfrac{\partial u}{\partial\nu}\right) = \left(u_0,\dfrac{\partial u_0}{\partial\nu}\right) & \text{on } \partial\Omega. \end{cases} \tag{4.57}$$

where $u_0\in W^{2,2}(\Omega,N)$ is given.

Now we introduce the Morrey spaces in $\mathbb{R}^n$. For $1\le l<+\infty$, $0<\lambda\le n$, and $0<R\le+\infty$, $f\in M_R^{l,\lambda}(\Omega)$ if $f\in L^l_{\text{loc}}(\Omega)$ satisfies

$$\|f\|^l_{M_R^{l,\lambda}(\Omega)} := \sup_{x\in\Omega}\ \sup_{0<r\le\min\{R,d(x,\partial\Omega)\}}\left\{r^{\lambda-n}\int_{B_r(x)}|f|^l\right\}<+\infty.$$

The following convexity property of biharmonic maps with small energy can be proved.

Theorem 4.3.1. *For $n\ge 4$, $\delta\in(0,1)$, and $\frac{3}{2}<p\le 2$, there exist $\epsilon_p=\epsilon(p,\delta)>0$ and $R_p=R(p,\delta)>0$ such that if $u\in W^{2,2}(\Omega,N)$ is a biharmonic map satisfying either*
(i) $\|\nabla^2 u\|_{M^{2,4}_{R_2}(\Omega)}\le\epsilon_2$, when N is a compact Riemannian manifold without boundary, or
(ii) $\|\nabla^2 u\|_{M^{p,2p}_{R_p}(\Omega)}\le\epsilon_p$, when $N=\mathbb{S}^L$,
then

$$\int_\Omega|\Delta v|^2\ge\int_\Omega|\Delta u|^2+(1-\delta)\int_\Omega|\nabla^2(v-u)|^2 \tag{4.58}$$

holds for any $v\in W^{2,2}(\Omega,N)$ with $\left(v,\dfrac{\partial v}{\partial\nu}\right)=\left(u,\dfrac{\partial u}{\partial\nu}\right)$ on $\partial\Omega$.

Proof. First, it follows from Corollary 4.1.6 for $N = \mathbb{S}^L$ or Wang [92] that if $\epsilon_p > 0$ is sufficiently small then $u \in C^\infty(\Omega, N)$, and

$$|\nabla^m u(x)| \le C\epsilon_p \left(\frac{1}{R_p^m} + \frac{1}{d^m(x, \partial\Omega)}\right), \ \forall\, x \in \Omega, \ \forall\, m \ge 1. \tag{4.59}$$

For $y \in N$, let $P^\perp(y) : \mathbb{R}^{L+1} \to (T_y N)^\perp$ denote the orthogonal projection from $\mathbb{R}^{L+1}$ to the normal space of N at y. Since N is compact, a simple geometric argument implies that there exists $C > 0$ depending on N such that

$$\left|P^\perp(y)(z - y)\right| \le C|z - y|^2, \ \forall z \in N. \tag{4.60}$$

Since

$$\mathcal{N}_{\mathrm{bh}}[u] \perp T_u N,$$

it follows from (4.60) that multiplying (1.7) by $(u - v)$ and integrating over Ω yields

$$\begin{aligned}
\int_\Omega \Delta u \cdot \Delta(u - v) &= \int_\Omega \mathcal{N}_{\mathrm{bh}}[u] \cdot (u - v) \\
&\lesssim \int_\Omega [|\nabla u|^2 |\nabla^2 u| + |\nabla^2 u|^2 + |\nabla u||\nabla^3 u|]|u - v|^2 \\
&\lesssim \epsilon_p^4 \int_\Omega \frac{|u - v|^2}{R_p^4} + \frac{|u - v|^2}{d^4(x, \partial\Omega)} \\
&\lesssim \epsilon_p \int_\Omega |\nabla^2(u - v)|^2,
\end{aligned} \tag{4.61}$$

where we choose $R_p \ge \epsilon_p$, apply (4.59) and the Poincaré inequality and the Hardy inequality (4.49) during the last two steps.

It follows from (4.61) that

$$\int_\Omega |\Delta v|^2 - \int_\Omega |\Delta u|^2 - \int_\Omega |\Delta u - \Delta v|^2 = 2\int_\Omega \Delta u \cdot \Delta(v - u) \ge -C\epsilon_p \int_\Omega |\nabla^2(u - v)|^2. \tag{4.62}$$

Since $(u - v) \in W_0^{2,2}(\Omega)$, we have that

$$\int_\Omega |\Delta u - \Delta v|^2 = \int_\Omega |\nabla^2(u - v)|^2,$$

so that

$$\int_\Omega |\Delta v|^2 - \int_\Omega |\Delta u|^2 \ge (1 - C\epsilon_p) \int_\Omega |\nabla^2(u - v)|^2.$$

This yields (4.58), if $\epsilon_p > 0$ is chosen so that $C\epsilon_p \le \delta$. □

Corollary 4.3.2. *For $n \ge 2$ and $\frac{3}{2} < p \le 2$, there exist $\epsilon_p > 0$ and $R_p > 0$ such that if $u_1, u_2 \in W^{2,2}(\Omega, N)$ are biharmonic maps, with $u_1 - u_2 \in W_0^{2,2}(\Omega, \mathbb{R}^{L+1})$, satisfying either*

(i) $\max_{i=1,2} \|\nabla^2 u_i\|_{M^{2,4}_{R_2}(\Omega)} \le \epsilon_2$, when N is a compact Riemannian manifold without

boundary, or
(ii) $\max_{i=1,2} \|\nabla^2 u_i\|_{M^{p,2p}_{R_p}(\Omega)} \leq \epsilon_p$*, when* $N = \mathbb{S}^L$*,*
then $u_1 \equiv u_2$ *in* Ω*.*

Proof. Choose $\delta = \frac{1}{2}$, apply Theorem 4.3.1 to u_1 and u_2 by choosing sufficiently small $\epsilon_p > 0$ and $R_p > 0$. We have

$$\int_\Omega |\Delta u_2|^2 \geq \int_\Omega |\Delta u_1|^2 + \frac{1}{2}\int_\Omega |\nabla^2(u_2 - u_1)|^2,$$

and

$$\int_\Omega |\Delta u_1|^2 \geq \int_\Omega |\Delta u_2|^2 + \frac{1}{2}\int_\Omega |\nabla^2(u_1 - u_2)|^2.$$

Adding these two inequalities together yields $\int_\Omega |\nabla^2(u_1 - u_2)|^2 = 0$. This, combined with $u_1 - u_2 \in W^{2,2}_0(\Omega)$, implies $u_1 \equiv u_2$ in Ω. □

4.4 Convexity and uniqueness of limit at $t = +\infty$ of heat flow of biharmonic maps

Prompted by the ideas of proof of Theorem 4.2.1, the convexity property of the E_2-energy along the heat flow of biharmonic maps to $\mathbb{S}^L$ can be obtained.

Theorem 4.4.1. *For* $n \geq 4$, $\frac{3}{2} < p \leq 2$*, and* $1 \leq T \leq \infty$*, there exist* $\epsilon_0 = \epsilon_0(p, n) > 0$*,* $R_0 = R_0(\Omega, \epsilon_0) > 0$*, and* $0 < T_0 = T_0(\epsilon_0) < T$ *such that if* $u \in W^{1,2}_2(\Omega \times [0, T], \mathbb{S}^L)$ *is a weak solution of (1.8), with the initial and boundary value* $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$*, satisfying*

$$\|\nabla^2 u\|_{M^{p,2p}_{R_0}(\Omega\times(0,T))} + \|\partial_t u\|_{M^{p,4p}_{R_0}(\Omega\times(0,T))} \leq \epsilon_0, \tag{4.63}$$

then
(i) $E_2(u(t))$ *is monotone decreasing for* $t \geq T_0$*; and*
(ii) for any $t_2 \geq t_1 \geq T_0$*,*

$$\int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2 \leq C\Big[\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2\Big] \tag{4.64}$$

for some $C = C(n, \epsilon_0) > 0$*.*

A direct consequence of the convexity property of E_2-energy is the unique limit at $t = \infty$ of (1.8).

Corollary 4.4.2. *For* $n \geq 4$ *and* $\frac{3}{2} < p \leq 2$*, there exist* $\epsilon_0 = \epsilon_0(p, n) > 0$*, and* $R_0 = R_0(\Omega, \epsilon_0) > 0$ *such that if* $u \in W^{1,2}_2(\Omega \times [0, \infty), \mathbb{S}^L)$ *is a weak solution of (1.8), with the initial and boundary value* $u_0 \in W^{2,2}(\Omega, \mathbb{S}^L)$*, satisfying the condition (4.63), then there exists a biharmonic map* $u_\infty \in C^\infty \cap W^{2,2}(\Omega, \mathbb{S}^L)$*, with* $(u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu})$ *on* $\partial\Omega$*, such that*

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{W^{2,2}(\Omega)} = 0, \tag{4.65}$$

and, for any compact subset $K \subset\subset \Omega$ and $m \geq 1$,

$$\lim_{t\uparrow\infty} \|u(t) - u_\infty\|_{C^m(K)} = 0. \tag{4.66}$$

To prove Theorem 4.4.1 and Corollary 4.4.2, we need the following two lemmas.

Lemma 4.4.3. *Under the same assumptions as in Theorem 4.4.1, there exists $T_0 > 0$ such that $\int_\Omega |\partial_t u(t)|^2$ is monotone decreasing for $t \geq T_0$:*

$$\int_\Omega |\partial_t u|^2(t_2) + C \int_{\Omega\times[t_1,t_2]} |\nabla^2 \partial_t u|^2 \leq \int_\Omega |\partial_t u|^2(t_1), \ T_0 \leq t_1 \leq t_2 \leq T. \tag{4.67}$$

Proof. For any sufficiently small $h > 0$, set

$$u^h(x,t) = \frac{u(x,t+h) - u(x,t)}{h}, \ (x,t) \in \Omega \times (0, T-h).$$

Then $u^h \in L^2([0,T-h], W_0^{2,2}(\Omega))$, $\partial_t u \in L^2(\Omega\times[0,T-h])$, and $\lim_{h\downarrow 0^+} \|u^h - \partial_t u\|_{L^2(\Omega\times[0,T-h])} = 0$. Since u satisfies (1.8), we obtain

$$\partial_t u^h + \Delta^2 u^h = \frac{1}{h}\Big(\mathcal{N}_{\text{bh}}[u(t+h)] - \mathcal{N}_{\text{bh}}[u(t)]\Big). \tag{4.68}$$

Multiplying (4.68) by u^h, integrating over Ω, and applying (4.60) and (4.52), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |u^h|^2 + 2\int_\Omega |\Delta u^h|^2 \ &\lesssim \ \int_\Omega \left(|\mathcal{N}_{\text{bh}}[u(t+h)]| + |\mathcal{N}_{\text{bh}}[u(t)]|\right)|u^h|^2 \\
&\lesssim \ \int_\Omega \left(|\nabla^2 u|^2 + |\nabla u||\nabla^3 u| + |\nabla u|^2|\nabla^2 u|\right)(t+h)|u^h|^2 \\
&\quad + \int_\Omega \left(|\nabla^2 u|^2 + |\nabla u||\nabla^3 u| + |\nabla u|^2|\nabla^2 u|\right)(t)|u^h|^2 \\
&\lesssim \ \epsilon_p^4 \int_\Omega \frac{|u^h|^2}{R_p^4} + \frac{|u^h|^2}{d^4(x,\partial\Omega)} + \frac{|u^h|^2}{T_0} \\
&\lesssim \ \epsilon_p \int_\Omega |\nabla^2 u^h|^2
\end{aligned}$$

provided that we choose $R_p \geq \epsilon_p$ and $T_0 \geq \epsilon_p$. Since

$$\int_\Omega |\nabla^2 u^h|^2 = \int_\Omega |\Delta u^h|^2,$$

this implies

$$\frac{d}{dt}\int_\Omega |u^h|^2 + 2\int_\Omega |\nabla^2 u^h|^2 \leq \left(\frac{1}{2} + C\epsilon_p\right)\int_\Omega |\nabla^2 u^h|^2. \tag{4.69}$$

Choosing $\epsilon_p > 0$ so that $C\epsilon_p \leq \frac{1}{2}$, integrating over $T_0 \leq t_1 \leq t_2 \leq T$, we have

$$\int_\Omega |u^h|^2(t_2) + C\int_{t_1}^{t_2}\int_\Omega |\nabla^2 u^h|^2 \leq \int_\Omega |u^h|^2(t_1). \tag{4.70}$$

Sending $h \to 0$, (4.70) yields (4.67). □

Now we can show the monotonicity of E_2-energy for heat flow of biharmonic maps for $t \geq T_0$.

Lemma 4.4.4. *Under the same assumptions as in Theorem 4.4.1, there is $T_0 > 0$ such that $\int_\Omega |\Delta u(t)|^2$ is monotone decreasing for $t \geq T_0$:*

$$\int_\Omega |\Delta u|^2(t_2) + 2\int_{\Omega\times[t_1,t_2]} |\partial_t u|^2 \leq \int_\Omega |\Delta u|^2(t_1),\ T_0 \leq t_1 \leq t_2 \leq T. \tag{4.71}$$

Proof. For $\delta > 0$, let $\eta_\delta \in C_0^\infty(\Omega)$ be such that

$$0 \leq \eta_\delta \leq 1,\ \eta_\delta \equiv 1 \text{ for } x \in \Omega \setminus \Omega_\delta,\ \text{and } |\nabla^m \eta_\delta| \leq C\delta^{-m},$$

where $\Omega_\delta = \{x \in \Omega :\ d(x, \partial\Omega) \leq \delta\}$. Multiplying (1.8) by $\partial_t u \eta_\delta^2$ and integrating over $\Omega \times [t_1, t_2]$, we obtain

$$\begin{aligned}
&\int_\Omega |\Delta u(t_2)|^2\eta_\delta^2 - \int_\Omega |\Delta u(t_1)|^2\eta_\delta^2 + 2\int_{t_1}^{t_2}\int_\Omega |\partial_t u|^2\eta_\delta^2 \\
=& -4\int_{t_1}^{t_2}\int_\Omega \Delta u \cdot \partial_t u \left(|\nabla\eta_\delta|^2 + \eta_\delta\Delta\eta_\delta\right) - 8\int_{t_1}^{t_2}\int_\Omega \Delta u \cdot \nabla\partial_t u \eta_\delta \nabla\eta_\delta.
\end{aligned} \tag{4.72}$$

It suffices to show the right hand side of the above identity tends to 0 as $\delta \to 0^+$. By Lemma 4.4.3, we have that $\partial_t u \in L^2([T_0, T], W_0^{2,2}(\Omega))$ so that

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega |\nabla\partial_t u|^2|\nabla\eta_\delta|^2 + |\partial_t u|^2\left(|\nabla\eta_\delta|^4 + |\Delta\eta_\delta|^2\right) \\
\lesssim& \delta^{-2}\int_{t_1}^{t_2}\int_{\Omega_\delta} |\nabla\partial_t u|^2 + \delta^{-2}|\partial_t u|^2 \\
\lesssim& \int_{t_1}^{t_2}\int_{\Omega_\delta} |\nabla^2\partial_t u|^2 \to 0,\ \text{as } \delta \to 0.
\end{aligned} \tag{4.73}$$

This, combined with the Hölder inequality, implies that for $t_2 \geq t_1 \geq T_0$,

$$-4\int_{t_1}^{t_2}\int_\Omega \Delta u \cdot \partial_t u \left(|\nabla\eta_\delta|^2 + \eta_\delta\Delta\eta_\delta\right) - 8\int_{t_1}^{t_2}\int_\Omega \Delta u \cdot \nabla\partial_t u \eta_\delta\nabla\eta_\delta \to 0,\ \text{as } \delta \to 0^+.$$

Thus (4.71) follows. □

Proof of Theorem 4.4.1. First, by Theorem 4.1.1, we have that $u \in C^\infty(\Omega \times (0, T], \mathbb{S}^L)$, and

$$\left|\nabla^m u(x,t)\right| \leq C\epsilon_p \left(\frac{1}{R_p^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right),\ \forall\ (x,t) \in \Omega\times(0,T),\ \forall\ m \geq 1. \tag{4.74}$$

For $t_2 > t_1 \geq T_0$, we have

$$\begin{aligned}
&\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2 - \int_\Omega |\Delta u(t_1) - \Delta u(t_2)|^2 \\
=&2 \int_\Omega (\Delta u(t_1) - \Delta u(t_2))\, \Delta u(t_2) \\
=&-2 \int_\Omega (u(t_1) - u(t_2))\, u_t(t_2) \\
&+ \int_\Omega \mathcal{N}_{\text{bh}}[u(t_2)] \cdot (u(t_1) - u(t_2)) \\
=&I + II.
\end{aligned} \tag{4.75}$$

For II, applying (4.60), we obtain

$$|\mathcal{N}_{\text{bh}}[u(t_2)] \cdot (u(t_1) - u(t_2))| \lesssim |\mathcal{N}_{\text{bh}}[u(t_2)]||u(t_1) - u(t_2)|^2.$$

Hence, by (4.74), the Hardy inequality and the Poincaré inequality, we have

$$\begin{aligned}
|II| \lesssim& \epsilon_p^4 \int_\Omega \left(\frac{1}{R_p^4} + \frac{1}{d^4(x, \partial\Omega)} + \frac{1}{T_0} \right) |u(t_1) - u(t_2)|^2 \\
\leq& C\epsilon_p \int_\Omega |\nabla^2 (u(t_1) - u(t_2))|^2.
\end{aligned} \tag{4.76}$$

For I, by Lemma 4.4.3, we have

$$\left\| \partial_t u(t_2) \right\|^2_{L^2(\Omega)} \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^2. \tag{4.77}$$

By the Hölder inequality and (4.71), this implies

$$\begin{aligned}
|I| \lesssim& \int_\Omega |\partial_t u(t_2)||u(t_1) - u(t_2)| \\
\lesssim& \|\partial_t u(t_2)\|_{L^2(\Omega)} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} \\
\leq& \sqrt{t_2 - t_1}\, \|\partial_t u(t_2)\|_{L^2(\Omega)} \left(\int_{\Omega \times [t_1, t_2]} |\partial_t u|^2 \right)^{\frac{1}{2}} \\
\leq& \int_{\Omega \times [t_1, t_2]} |\partial_t u|^2 \leq \frac{1}{2} \left[\int_\Omega |\Delta u(t_1)|^2 - \int_\Omega |\Delta u(t_2)|^2 \right].
\end{aligned} \tag{4.78}$$

Putting (4.78) and (4.76) into (4.75) implies (4.64). This completes the proof. □

Proof of Corollary 4.4.2. It follows from Lemma 4.4.4 that $\int_\Omega |\Delta u(t)|^2$ is monotone decreasing for $t \geq T_0$. Hence

$$c = \lim_{t \to \infty} \int_\Omega |\Delta u(t)|^2$$

exists and is finite. Let $\{t_i\}$ be any increasing sequence such that $\lim_{i\to\infty} t_i = +\infty$. Then (4.64) implies that

$$\int_\Omega \Big|\nabla^2(u(t_{i+j}) - u(t_i))\Big|^2 \le C\Big[\int_\Omega |\Delta u(t_{i+j})|^2 - \int_\Omega |\Delta u(t_i)|^2\Big] \to 0, \text{ as } i \to \infty,$$

for all $j \ge 1$. Thus there exists $u_\infty \in W^{2,2}(\Omega, \mathbb{S}^L)$, with $(u_\infty, \frac{\partial u_\infty}{\partial \nu}) = (u_0, \frac{\partial u_0}{\partial \nu})$ on $\partial\Omega$, such that

$$\lim_{t\to\infty} \Big\|u(t) - u_\infty\Big\|_{W^{2,2}(\Omega)} = 0.$$

Since (4.71) implies that there exists a sequence $t_i \to \infty$, such that

$$\lim_{i\to\infty} \Big\|\partial_t u(t_i)\Big\|_{W^{2,2}(\Omega)} = 0.$$

Thus $u_\infty \in W^{2,2}(\Omega, \mathbb{S}^L)$ is a biharmonic map. For any $m \ge 1$, and any compact subset $K \subset\subset \Omega$, since

$$\Big\|u(t)\Big\|_{C^m(K)} \le C(n, m, K), \ \forall t \ge 1,$$

we conclude that

$$\lim_{t\to\infty} \Big\|u(t) - u_\infty\Big\|_{C^m(K)} = 0,$$

and $u_\infty \in C^\infty(\Omega, \mathbb{S}^L)$. This completes the proof. □

4.5 Uniqueness of Serrin's (p, q)-solution to general Riemannian manifold

This section will be devoted to prove the uniqueness of Serrin's (p, q)-solution to heat flow of biharmonic maps (1.8).

For $N = \mathbb{S}^L$, the regularity and uniqueness for such solutions of (1.8) follow from Theorem 4.1.1 and Theorem 4.2.1. However, for a compact Riemannian manifold N without boundary, the regularity and uniqueness for such a class of weak solutions of (1.8) require different arguments. More precisely, we have

Theorem 4.5.1. *For $n \ge 4$ and $0 < T \le \infty$, let $u_1, u_2 \in W_2^{1,2}(\Omega \times [0, T], N)$ be weak solutions of (1.8), with the same initial and boundary value $u_0 \in W^{2,2}(\Omega, N)$. If, in additions, $\nabla^2 u_1, \nabla^2 u_2 \in L_t^q L_x^p(\Omega \times [0, T])$ for some $p > \frac{n}{2}$ and $q < \infty$ satisfying (1.20), then $u_1, u_2 \in C^\infty(\Omega \times (0, T), N)$, and $u_1 \equiv u_2$ in $\Omega \times [0, T]$.*

First, one can verify that

Proposition 4.5.2. *For $n \ge 4$, $0 < T < +\infty$, suppose $u \in W_2^{1,2}(\Omega \times [0, T], N)$ is a weak solution of (1.8), with the initial and boundary value $u_0 \in W^{2,r}(\Omega, N)$ for some $\frac{n}{2} < r < +\infty$, such that $\nabla^2 u \in L_t^q L_x^p(M \times [0, T])$ for some $p > \frac{n}{2}$ and $q < \infty$ satisfying (1.20). Then*

(i) $\partial_t u \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\Omega \times [0,T])$*; and*
(ii) for any $\epsilon > 0$*, there exists* $R = R(u, \epsilon) > 0$ *such that for any* $1 < s < \min\{\frac{p}{2}, \frac{q}{2}\}$,

$$r^{2s-(n+4)} \int_{P_r(x,t)\cap(\Omega\times[0,T])} (|\nabla^2 u|^s + r^{2s}|\partial_t u|^s) \le \epsilon^s, \tag{4.79}$$

for any $(x,t) \in \Omega \times [0,T]$*, and* $0 < r \le R$*.*

Proof. For simplicity, we will sketch the proof for $\Omega = \mathbb{R}^n$. By the Duhamel formula, we have that $u(x,t) = u_1(x,t) + u_2(x,t)$, where

$$u_1(x,t) = \int_{\mathbb{R}^n} b(x-y,t)u_0(y), \tag{4.80}$$

$$u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s)\mathcal{N}_{\text{bh}}[u](y,s). \tag{4.81}$$

We proceed with two claims.
Claim 1. $\nabla^3 u \in L_t^{\frac{2q}{3}} L_x^{\frac{2p}{3}}(\mathbb{R}^n \times [0,T])$. For u_1, we have

$$\nabla^3 u_1(x,t) = \int_{\mathbb{R}^n} \nabla_x b(x-y,t)\nabla^2 u_0(y). \tag{4.82}$$

Direct calculations, using the property of the kernel function b, yield

$$\left\|\nabla^3 u\right\|_{L_t^{\frac{2q}{3}} L_x^{\frac{2p}{3}}(\mathbb{R}^n\times[0,T])} \lesssim T^{\frac{1}{4}(2-\frac{n}{r})} \left\|\nabla^2 u_0\right\|_{L^r(\mathbb{R}^n)}. \tag{4.83}$$

For u_2, we have

$$\begin{aligned} \nabla^3 u_2(x,t) &= \int_0^t \int_{\mathbb{R}^n} \nabla_x^4 b(x-y,t-s)\Big[\nabla(A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla(P(u))\Big] \\ &\quad - \int_0^t \int_{\mathbb{R}^n} \nabla_x^3 b(x-y,t-s)\Delta u \cdot \Delta(P(u))(y,s) \\ &= M_1 + M_2. \end{aligned} \tag{4.84}$$

By the Nirenberg interpolation inequality, we have $\nabla u \in L_t^{2q} L_x^{2p}(\mathbb{R}^n \times [0,T])$. By the Hölder inequality, we then have $\nabla(A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla(P(u))) \in L_t^{\frac{3q}{2}} L_x^{\frac{3p}{2}}(\mathbb{R}^n \times [0,T])$. Hence, by the Calderon-Zygmund $L_t^{\tilde{q}} L_x^{\tilde{p}}$-theory, we have

$$\begin{aligned} \left\|M_1\right\|_{L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n\times[0,T])} &\lesssim \left\|\nabla(A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla(P(u))\right\|_{L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n\times[0,T])} \\ &\lesssim \left\|\nabla u\right\|_{L_t^{2p} L_x^{2q}(\mathbb{R}^n\times[0,T])} \left\|\nabla^2 u\right\|_{L_t^p L_x^q(\mathbb{R}^n\times[0,T])} \\ &\lesssim 1 + \left\|\nabla^2 u\right\|^2_{L_t^p L_x^q(\mathbb{R}^n\times[0,T])}. \end{aligned} \tag{4.85}$$

For M_2, we have

$$|M_2|(x,t) \lesssim I_1\Big(|\nabla^2 u|^2 + |\nabla u|^4\Big)(x,t),\ (x,t) \in \mathbb{R}^n \times [0,T].$$

Recall the following estimate of $I_1(\cdot)$ (see, for example, [24] §4):

$$\Big\|I_1(f)\Big\|_{L_t^{s_2} L_x^{r_2}(\mathbb{R}^n\times[0,T])} \lesssim \Big\|f\Big\|_{L_t^{s_1} L_x^{r_1}(\mathbb{R}^n\times[0,T])}, \tag{4.86}$$

where $s_2 \geq s_1$ and $r_2 \geq r_1$ satisfy

$$\frac{n}{r_1} + \frac{4}{s_1} \leq \frac{n}{r_2} + \frac{4}{s_2} + 1. \tag{4.87}$$

Applying (4.86) to M_2, we see that $M_2 \in L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n \times [0,T])$, and

$$\Big\|M_2\Big\|_{L_t^{\frac{2p}{3}} L_x^{\frac{2q}{3}}(\mathbb{R}^n\times[0,T])} \lesssim 1 + \Big\|\nabla^2 u\Big\|^2_{L_t^p L_x^q(\mathbb{R}^n\times[0,T])}. \tag{4.88}$$

Combining these estimates of $\nabla^3 u_1, M_1$, and M_2 yields *Claim* 1.

Claim 2. $\nabla^4 u \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])$. It follows from *Claim* 1 that

$$\mathcal{N}_{\text{bh}}[u] = [\Delta(A(u)(\nabla u, \nabla u)) + 2\Delta u \cdot \nabla(P(u))) - \Delta u \cdot \Delta(P(u))] \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T]).$$

Since

$$\nabla^4 u_2(x,t) = \int_0^t \int_{\mathbb{R}^n} \nabla_x^4 b(x-y,t-s)\mathcal{N}_{\text{bh}}[u](y,s),$$

we can apply the Calderon-Zygmund $L_t^{\tilde{q}} L_x^{\tilde{p}}$-theory again to conclude that $\nabla^4 u_2 \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])$. For u_1, we have

$$\nabla^4 u_1(x,t) = \int_{\mathbb{R}^n} \nabla_x^2 b(x-y,t)\nabla^2 u_0(y).$$

Hence, by direct calculations, we have

$$\Big\|\nabla^4 u_1\Big\|_{L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n\times[0,T])} \lesssim T^{\frac{1}{4}(2-\frac{n}{r})}\Big\|\nabla^2 u_0\Big\|_{L^r(\mathbb{R}^n)}.$$

Combining these two estimates yields *Claim* 2.

By (1.8), it is easy to see that $\partial_t u \in L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(\mathbb{R}^n \times [0,T])$. In fact, we have

$$\begin{aligned} \Big\|\partial_t u\Big\|_{L_t^{\frac{p}{2}} L_x^{\frac{q}{2}}(\mathbb{R}^n\times[0,T])} &\lesssim \Big\|\mathcal{N}_{\text{bh}}[u] - \Delta^2 u\Big\|_{L_t^{\frac{p}{2}} L_x^{\frac{q}{2}}(\mathbb{R}^n\times[0,T])} \\ &\lesssim 1 + \Big\|\nabla^2 u\Big\|^2_{L_t^p L_x^q(\mathbb{R}^n\times[0,T])} + T^{\frac{1}{4}(2-\frac{n}{r})}\Big\|\nabla^2 u_0\Big\|_{L^r(\mathbb{R}^n)}. \end{aligned} \tag{4.89}$$

This implies (i).

(ii) follows from (i) and the Hölder inequality. In fact, for any $1 < s < \min\{\frac{p}{2}, \frac{q}{2}\}$, it holds

$$\left(r^{2s-(n+4)} \int_{P_r(x,t)\cap(\Omega\times[0,T])} |\nabla^2 u|^s\right)^{\frac{1}{s}} \le \left\|\nabla^2 u\right\|_{L^q_t L^p_x(P_r(x,t)\cap(\Omega\times[0,T]))},$$

and

$$\left(r^{4s-(n+4)} \int_{P_r(x,t)\cap(M\times[0,T])} |\partial_t u|^s\right)^{\frac{1}{s}} \le \left\|\partial_t u\right\|_{L_t^{\frac{q}{2}} L_x^{\frac{p}{2}}(P_r(x,t)\cap(\Omega\times[0,T]))}.$$

These two inequalities clearly imply (4.79), provided that $R = R(u, \epsilon) > 0$ is chosen sufficiently small. □

Now we proceed to prove an ϵ-regularity property for certain solutions of (1.8).

Theorem 4.5.3. *There exists $\epsilon_0 > 0$ such that if $u \in W_2^{1,2}(P_1, N)$, with $\nabla^2 u \in L^q_t L^p_x(P_1)$ for some $q \ge \frac{n}{2}$ and $p \le \infty$ satisfying (1.20), is a weak solution of (1.8) and satisfies*

$$\left\|\nabla^2 u\right\|_{L^q_t L^p_x(P_1)} \le \epsilon_0, \tag{4.90}$$

then $u \in C^\infty(P_{\frac{1}{2}}, N)$ and

$$\|\nabla^m u\|_{C^0(P_{\frac{1}{2}})} \le C(m, p, q, n)\|\nabla^2 u\|_{L^q_t L^p_x(P_1)}, \ \forall\ m \ge 1. \tag{4.91}$$

Before proving this theorem, we recall the Serrin type inequalities (see [78]) and Adams' estimates of Riesz potential between Morrey spaces in $(\mathbb{R}^{n+1}, \delta)$.

Lemma 4.5.4. *Assume $p \ge \frac{n}{2}$ and $q \le \infty$ satisfy (1.20). For any $f \in L^q_t L^p_x(\Omega \times [0,T])$, $g \in L^2_t W^{2,2}_x(\Omega \times [0,T])$, and $h \in L^2_t W^{1,2}_x(\Omega \times [0,T])$, we have*

$$\int_{\Omega\times[0,T]} |f||g||h| \lesssim \|h\|_{L^2(\Omega\times[0,T])}\|g\|^{\frac{n}{2p}}_{L^2_t W^{2,2}_x(\Omega\times[0,T])} \left(\int_0^T \|f\|^q_{L^p(\Omega)}\|g\|^2_{L^2(\Omega)}\right)^{\frac{1}{q}}, \tag{4.92}$$

and

$$\int_{\Omega\times[0,T]} |f||\nabla g||h| \lesssim \|h\|_{L^2_t W^{1,2}_x(\Omega\times[0,T])}\|g\|^{\frac{n}{2p}}_{L^2_t W^{2,2}_x(\Omega\times[0,T])} \left(\int_0^T \|f\|^q_{L^p(\Omega)}\|g\|^2_{L^2(\Omega)}\right)^{\frac{1}{q}}. \tag{4.93}$$

Proof. For convenience, we sketch the proof here. By the Hölder inequality, we have

$$\int_\Omega |f||g||h| \le \|f\|_{L^p(\Omega)}\|g\|_{L^r(\Omega)}\|h\|_{L^2(\Omega)}, \tag{4.94}$$

where $\dfrac{1}{p} + \dfrac{1}{r} = \dfrac{1}{2}$. It follows from (1.20) that $2 \le r \le \frac{2n}{n-4}$. Hence by the Sobolev inequality we have

$$\|g\|_{L^r(\Omega)} \le \|g\|^{\frac{2}{q}}_{L^2(\Omega)}\|g\|^{\frac{2n}{p}}_{L^{\frac{2n}{n-4}}(\Omega)} \lesssim \|g\|^{\frac{2}{q}}_{L^2(\Omega)}\|g\|^{\frac{n}{2p}}_{W^{2,2}(\Omega)}. \tag{4.95}$$

Putting (4.95) into (4.94) yields

$$\int_{\Omega} |f||g||h| \lesssim \|f\|_{L^p(\Omega)} \|g\|_{L^2(\Omega)}^{\frac{2}{q}} \|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}} \|h\|_{L^2(\Omega)}. \tag{4.96}$$

Since $\frac{1}{q}+\frac{n}{4p}+\frac{1}{2}=1$, (4.92) follows by integrating over $[0,T]$ and the Hölder inequality.

To see (4.93), note that the Hölder inequality implies

$$\int_{\Omega} |f||\nabla g||h| \le \|f\|_{L^p(\Omega)} \|\nabla g\|_{L^s(\Omega)} \|h\|_{L^{\frac{2n}{n-2}}(\Omega)} \tag{4.97}$$

where $\frac{1}{p}+\frac{1}{s}+\frac{n-2}{2n}=1$. Since

$$\frac{1}{s}=\frac{1}{n}+\frac{n}{2p}\left(\frac{1}{2}-\frac{2}{n}\right)+\left(1-\frac{n}{2p}\right)\frac{1}{2},$$

the Nirenberg interpolation inequality implies

$$\|\nabla g\|_{L^s(\Omega)} \lesssim \|g\|_{L^2(\Omega)}^{\frac{2}{q}} \|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}}. \tag{4.98}$$

Putting (4.98) into (4.97) and using the Sobolev inequality, we obtain

$$\int_{\Omega} |f||\nabla g||h| \lesssim \|f\|_{L^p(\Omega)} \|g\|_{L^2(\Omega)}^{\frac{2}{q}} \|g\|_{W^{2,2}(\Omega)}^{\frac{n}{2p}} \|h\|_{W^{1,2}(\Omega)}. \tag{4.99}$$

Since $\frac{1}{q}+\frac{n}{4p}+\frac{1}{2}=1$, (4.93) follows by integration on $[0,T]$ and the Hölder inequality. □

Now we state Adams' estimate for the Riesz potentials on $(\mathbb{R}^{n+1},\delta)$. Since its proof is exactly the same argument as in Huang-Wang ([45] Theorem 3.1), we skip it here.

Proposition 4.5.5. *(i) For any $\beta>0$, $0<\lambda\le n+4$, $1<p<\frac{\lambda}{\beta}$, if $f\in L^p(\mathbb{R}^{n+1})\cap M^{p,\lambda}(\mathbb{R}^{n+1})$, then $I_\beta(f)\in L^{\tilde p}(\mathbb{R}^{n+1})\cap M^{\tilde p,\lambda}(\mathbb{R}^{n+1})$, where $\tilde p=\frac{p\lambda}{\lambda-p\beta}$. Moreover,*

$$\|I_\beta(f)\|_{L^{\tilde p}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{p,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta p}{\lambda}} \|f\|_{L^p(\mathbb{R}^{n+1})}^{1-\frac{\beta p}{\lambda}} \tag{4.100}$$

$$\|I_\beta(f)\|_{M^{\tilde p,\lambda}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{p,\lambda}(\mathbb{R}^{n+1})}. \tag{4.101}$$

(ii) For any $0<\beta<\lambda\le n+4$, if $f\in L^1(\mathbb{R}^{n+1})\cap M^{1,\lambda}(\mathbb{R}^{n+1})$, then $f\in L^{\frac{\lambda}{\lambda-\beta},*}(\mathbb{R}^{n+1})\cap M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^{n+1})$. Moreover,

$$\|I_\beta(f)\|_{L^{\frac{\lambda}{\lambda-\beta},*}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{1,\lambda}(\mathbb{R}^{n+1})}^{\frac{\beta}{\lambda}} \|f\|_{L^1(\mathbb{R}^{n+1})}^{1-\frac{\beta}{\lambda}} \tag{4.102}$$

$$\|I_\beta(f)\|_{M_*^{\frac{\lambda}{\lambda-\beta},\lambda}(\mathbb{R}^{n+1})} \le C\|f\|_{M^{1,\lambda}(\mathbb{R}^{n+1})}. \tag{4.103}$$

Proof of Theorem 4.5.3. The proof is based on three claims.

Claim 1. For any $0 < \alpha < 1$, we have that $\nabla^2 u \in M^{2,4-4\alpha}(P_{\frac{3}{4}})$, and

$$\left\|\nabla^2 u\right\|_{M^{2,4-4\alpha}(P_{\frac{3}{4}})} \leq C\left\|\nabla^2 u\right\|_{L_t^q L_x^p(P_1)}. \tag{4.104}$$

For any $0 < r \leq \frac{1}{4}$ and $z_0 = (x_0, t_0) \in P_{\frac{3}{4}}$, by (4.90) we have

$$\|\nabla^2 u\|_{L_t^q L_x^p(P_r(z_0))} \leq \epsilon. \tag{4.105}$$

Let $v : P_r(z_0) \to \mathbb{R}^{L+1}$ solve

$$\begin{cases} \partial_t v + \Delta^2 v = 0 & \text{in } P_r(z_0) \\ v = u & \text{on } \partial_p P_r(z_0) \\ \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} & \text{on } \partial B_r(x_0) \times (t_0 - r^4, t_0]. \end{cases} \tag{4.106}$$

Set $w = u - v$. Multiplying (4.106) and (1.8) by w, subtracting the resulting equations and integrating over $P_r(z_0)$, we obtain

$$\begin{aligned} &\sup_{t_0 - r^4 \leq t \leq t_0} \int_{B_r(x_0)} |w|^2(t) + 2\int_{P_r(z_0)} |\nabla^2 w|^2 \\ =& \left| \int_{P_r(z_0)} \mathcal{N}_{\text{bh}}[u] \cdot w \right| \\ =& \left| \int_{P_r(z_0)} -\nabla(A(u)(\nabla u, \nabla u))\nabla w - \langle \Delta u, \Delta(P(u)) \rangle w - 2\langle \Delta u, \nabla(P(u)) \rangle \nabla w \right| \\ \lesssim& \int_{P_r(z_0)} |\nabla^2 u|^2 |w| + \int_{P_r(z_0)} |\nabla u||\nabla^2 u||\nabla w| \\ =& I + II. \end{aligned} \tag{4.107}$$

For I, we can apply (4.92) to get

$$|I| \lesssim \|\nabla^2 u\|_{L^2(P_r(z_0))} \|w\|^{\frac{n}{2p}}_{L_t^2 W_x^{2,2}(P_r(z_0))} \left(\int_{t_0 - r^4}^{t_0} \|\nabla^2 u\|^q_{L^p(B_r(x_0))} \|w\|^2_{L^2(B_r(x_0))} \right)^{\frac{1}{q}}. \tag{4.108}$$

For II, by (4.93), we have

$$|II| \lesssim \|\nabla u\|_{L_t^2 W_x^{1,2}(P_r(z_0))} \|w\|^{\frac{n}{2p}}_{L_t^2 W_x^{2,2}(P_r(z_0))} \left(\int_{t_0 - r^4}^{t_0} \|\nabla^2 u\|^q_{L^p(B_r(x))} \|w\|^2_{L^2(B_r(x_0))} \right)^{\frac{1}{q}}. \tag{4.109}$$

Putting (4.108) and (4.109) into (4.107) and applying the Poincaré inequality, we obtain

$$\sup_{t_0-r^4\le t\le t_0}\int_{B_r(x_0)}|w|^2(t)+2\int_{P_r(z_0)}|\nabla^2 w|^2$$
$$\lesssim\begin{cases}\|\nabla u\|_{L_t^2W_x^{1,2}(P_r(z_0))}\|\nabla^2 w\|_{L^2(P_r(z_0))}^{\frac{n}{2p}} \\ \cdot\left(\int_{t_0-r^4}^{t_0}\|\nabla^2 u\|_{L^p(B_r(x_0))}^q\|w\|_{L^2(B_r(x_0))}^2\right)^{\frac{1}{q}}, & q<\infty,\\ \|\nabla u\|_{L_t^2W_x^{1,2}(P_r(z_0))}\|\nabla^2 w\|_{L^2(P_r(z_0))}\|\nabla^2 u\|_{L_t^\infty L_x^{\frac{n}{2}}(B_r(x_0))}, & q=\infty.\end{cases}\tag{4.110}$$

Since $\|\nabla^2 u\|_{L_t^qL_x^p(P_r(z_0))}\le\epsilon$, we obtain, by the Young inequality,

$$\sup_{t_0-r^4\le t\le t_0}\int_{B_r(x_0)}|w|^2(t)+2\int_{P_r(z_0)}|\nabla^2 w|^2$$
$$\le\begin{cases}\|\nabla^2 w\|_{L^2(P_r(z_0))}^2+\epsilon\|\nabla u\|_{L_t^2W_x^{1,2}(P_r(z_0))}^2+C\epsilon^{\frac{p}{2}}\sup_{t_0-r^4\le t\le t_0}\|w\|_{L^2(B_r(x_0))}^2, & q<\infty,\\ \|\nabla^2 w\|_{L^2(P_r(z_0))}^2+C\|\nabla^2 u\|_{L_t^\infty L_x^{\frac{n}{2}}(B_r(x_0))}^2\|\nabla u\|_{L_t^2W_x^{1,2}(P_r(z_0))}^2, & q=\infty.\end{cases}$$

By choosing $\epsilon>0$ sufficiently small, this implies

$$\int_{P_r(z_0)}|\nabla^2 w|^2\lesssim\epsilon\int_{P_r(z_0)}|\nabla u|^2+|\nabla^2 u|^2.\tag{4.111}$$

Since N is compact and u maps into N, $|u|\le C_N$. Hence, by the Nirenberg interpolation inequality, we have

$$\int_{P_r(z_0)}|\nabla u|^2\lesssim\int_{P_r(z_0)}|\nabla^2 u|^2+r^{n+4}.\tag{4.112}$$

Combining (4.112) with (4.111), we have

$$\int_{P_r(z_0)}|\nabla^2 w|^2\lesssim\epsilon\int_{P_r(z_0)}|\nabla^2 u|^2+\epsilon r^{n+4}.\tag{4.113}$$

By the standard estimate on v, we have

$$(\theta r)^{-n}\int_{P_{\theta r}(z_0)}|\nabla^2 v|^2\lesssim\theta^4 r^{-n}\int_{P_r(z_0)}|\nabla^2 v|^2,\ \forall\,\theta\in(0,1).\tag{4.114}$$

Combining (4.113) with (4.114), we obtain

$$(\theta r)^{-n}\int_{P_{\theta r}(z_0)}|\nabla^2 u|^2\le C\left(\theta^4+\theta^{-n}\epsilon\right)r^{-n}\int_{P_r(z_0}|\nabla^2 u|^2+C\epsilon\theta^{-n}r^4,\ \forall\,\theta\in(0,1).\tag{4.115}$$

For any $0<\alpha<1$, choose $0<\theta<1$ and ϵ such that

$$C\theta^4\le\frac{1}{2}\theta^{4\alpha}\text{ and }\epsilon\le\min\left\{\left(\frac{1}{2C}\right)^{\frac{2}{p}},\frac{\theta^{4\alpha+n}}{2C}\right\}.$$

Therefore, for any $(z_0) \in P_{\frac{3}{4}}$ and $0 < r \le \frac{1}{4}$, it holds

$$(\theta r)^{-n} \int_{P_{\theta r}(x,t)} |\nabla^2 u|^2 \le \theta^{4\alpha} r^{-n} \int_{P_r(x,t)} |\nabla^2 u|^2 + \theta^{4\alpha} r^4. \tag{4.116}$$

It is standard that iterating (4.116) implies

$$r^{-n} \int_{P_r(z_0)} |\nabla^2 u|^2 \le C r^{4\alpha} \left(\int_{P_1} |\nabla^2 u|^2 + 1 \right) \tag{4.117}$$

for any $z_0 \in P_{\frac{3}{4}}$ and $0 < r \le \frac{1}{4}$. (4.117) implies that $\nabla^2 u \in M^{2,4-4\alpha}(P_{\frac{3}{4}})$, and the estimate (4.104) holds. This proves *Claim* 1.

Claim 2. For any $1 < \beta < +\infty$, $\nabla^2 u \in L^\beta(P_{\frac{9}{16}})$, and

$$\left\| \nabla^2 u \right\|_{L^\beta(P_{\frac{9}{16}})} \lesssim \left\| \nabla^2 u \right\|^2_{L^q_t L^p_x(P_1)}. \tag{4.118}$$

This can be proven by estimates of Riesz potentials between Morrey spaces. To do so, let $\eta \in C_0^\infty(P_1)$ be such that

$$0 \le \eta \le 1,\ \eta \equiv 1 \text{ in } P_{\frac{5}{8}},\ |\eta_t| + \sum_{m=1}^{4} |\nabla^m \eta| \le C.$$

Let $Q : \mathbb{R}^n \times [-1, \infty] \to \mathbb{R}^{L+1}$ solve

$$\begin{aligned} \partial_t Q + \Delta^2 Q &= \nabla \cdot \Big(\eta^2 \nabla (A(u)(\nabla u, \nabla u)) + 2\eta^2 \langle \Delta u, \nabla (P(u)) \rangle \Big) - \eta^2 \langle \Delta u, \Delta (P(u)) \rangle \\ Q\Big|_{t=-1} &= 0. \end{aligned}$$

Set

$$J_1 = \nabla \cdot \Big(\eta^2 \nabla (A(u)(\nabla u, \nabla u)) + 2\eta^2 \langle \Delta u, \nabla (P(u)) \rangle \Big) \ \text{ and } \ J_2 = -\eta^2 \langle \Delta u, \Delta (P(u)) \rangle.$$

By the Duhamel formula, we have, for $(x,t) \in \mathbb{R}^n \times (-1, \infty)$,

$$\begin{aligned} \nabla^2 Q(x,t) &= \int_{\mathbb{R}^n \times [-1,t]} \nabla_x^2 b(x-y, t-s)\, (J_1 + J_2)(y,s) \\ &= \int_{\mathbb{R}^n \times [-1,t]} \nabla_x^3 b(x-y,t-s) \Big(\eta^2 \nabla (A(u)(\nabla u, \nabla u)) + 2\eta^2 \langle \Delta u, \nabla (P(u)) \rangle \Big)(y,s) \\ &\quad - \int_{\mathbb{R}^n \times [-1,t]} \nabla_x^2 b(x-y,t-s) \eta^2 \langle \Delta u, \Delta (P(u)) \rangle (y,s) \\ &= K_1(x,t) + K_2(x,t). \end{aligned} \tag{4.119}$$

It is clear that for $(x,t) \in \mathbb{R}^n \times (-1, \infty)$,

$$|K_1|(x,t) \lesssim I_1 \Big(\eta^2 (|\nabla u|^3 + |\nabla u||\nabla^2 u|) \Big)(x,t),\ |K_2|(x,t) \le I_2 \Big(\eta^2 (|\nabla^2 u|^2 + |\nabla u|^4) \Big)(x,t).$$

It follows from (4.104) and the Nirenberg interpolation inequality that $\nabla u \in M^{4,4-4\alpha}(P_{\frac{3}{4}})$ and

$$\Big\|\nabla u\Big\|_{M^{4,4-4\alpha}(P_{\frac{3}{4}})} \lesssim \Big\|\nabla^2 u\Big\|_{L^q_t L^p_x(P_1)}. \tag{4.120}$$

Hence, by the Hölder inequality, we have that for any $0 < \alpha_1, \alpha_2 < 1$,

$$\eta^2(|\nabla u|^3 + |\nabla u||\nabla^2 u|) \in M^{\frac{4}{3},4-4\alpha_1}(\mathbb{R}^{n+1}) \text{ and } \eta^2(|\nabla^2 u|^2 + |\nabla u|^4) \in M^{1,4-4\alpha_2}(\mathbb{R}^{n+1}),$$

and

$$\begin{aligned}\Big\|\eta^2(|\nabla u|^3 + |\nabla u||\nabla^2 u|)\Big\|_{M^{\frac{4}{3},4-4\alpha_1}(\mathbb{R}^{n+1})} &\lesssim \Big\|\nabla u\Big\|_{M^{4,4-4\alpha_1}(P_{\frac{3}{4}})}\Big\|\nabla^2 u\Big\|_{M^{2,4-4\alpha_1}(P_{\frac{3}{4}})} \\ &\lesssim \Big\|\nabla^2 u\Big\|^2_{L^q_t L^p_x(P_1)}, \end{aligned} \tag{4.121}$$

$$\begin{aligned}\Big\|\eta^2(|\nabla^2 u|^2 + |\nabla u|^4)\Big\|_{M^{1,4-4\alpha_2}(\mathbb{R}^{n+1})} &\lesssim \Big\|\nabla u\Big\|_{M^{4,4-4\alpha_2}(P_{\frac{3}{4}})} + \Big\|\nabla^2 u\Big\|_{M^{2,4-4\alpha_2}(P_{\frac{3}{4}})} \\ &\lesssim \Big\|\nabla^2 u\Big\|^2_{L^q_t L^p_x(P_1)}. \end{aligned} \tag{4.122}$$

Now applying Proposition 4.5.5, we conclude that

$$K_1 \in M^{\frac{4-4\alpha_1}{2-3\alpha_1},4-4\alpha_1} \cap L^{\frac{4-4\alpha_1}{2-3\alpha_1}}(\mathbb{R}^{n+1}), \quad K_2 \in M_*^{\frac{2-2\alpha_2}{1-2\alpha_2},4-4\alpha_2} \cap L^{\frac{2-2\alpha_2}{1-2\alpha_2},*}(\mathbb{R}^{n+1}),$$

and

$$\Big\|K_1\Big\|_{M^{\frac{4-4\alpha_1}{2-3\alpha_1},4-4\alpha_1}(\mathbb{R}^{n+1})} + \Big\|K_2\Big\|_{M_*^{\frac{2-2\alpha_2}{1-2\alpha_2},4-4\alpha_2}(\mathbb{R}^{n+1})} \lesssim \Big\|\nabla^2 u\Big\|^2_{L^q_t L^p_x(P_1)}. \tag{4.123}$$

Sending $\alpha_1 \uparrow \frac{2}{3}$ and $\alpha_2 \uparrow \frac{1}{2}$, we obtain that for any $1 < \beta < +\infty$, $K_1, K_2 \in L^\beta(\mathbb{R}^{n+1})$, and

$$\|K_1\|_{L^\beta(\mathbb{R}^{n+1})} + \|K_2\|_{L^\beta(\mathbb{R}^{n+1})} \lesssim \Big\|\nabla^2 u\Big\|^2_{L^q_t L^p_x(P_1)}. \tag{4.124}$$

This implies that for any $1 < \beta < +\infty$, $\nabla^2 Q \in L^\beta(\mathbb{R}^{n+1})$, and

$$\Big\|\nabla^2 Q\Big\|_{L^\beta(\mathbb{R}^{n+1})} \lesssim \Big\|\nabla^2 u\Big\|^2_{L^q_t L^p_x(P_1)}. \tag{4.125}$$

Since $(u - Q)$ solves

$$\Big(\partial_t + \Delta^2\Big)(u - Q) = 0 \text{ in } P_{\frac{5}{8}},$$

it follows that for any $1 < \beta < +\infty$, $\nabla^2 u \in L^\beta(P_{\frac{9}{16}})$, and

$$\Big\|\nabla^2 u\Big\|_{L^\beta(P_{\frac{9}{16}})} \lesssim \Big\|\nabla^2 u\Big\|^2_{L^q_t L^p_x(P_1)}. \tag{4.126}$$

This implies (4.125). Hence *Claim* 2 is proven.

Claim 3. $u \in C^\infty(P_{\frac{1}{2}}, N)$ and (4.91) holds. It follows from (4.125) that for any $1 < \beta < +\infty$, there exist $f, g \in L^\beta(P_{\frac{9}{16}})$ such that (1.8) can be written as

$$(\partial_t + \Delta^2)u = \nabla \cdot f + g.$$

Thus, by the L^p-theory of higher-order parabolic equations, we conclude that $\nabla^3 u \in L^\beta(P_{\frac{17}{32}})$. Applying the L^p-theory again, we would obtain that $\partial_t u, \nabla^4 u \in L^\beta(P_{\frac{33}{64}})$. Taking derivatives of the equation (1.8) and repeating this argument, we can conclude that $u \in C^\infty(P_{\frac{1}{2}}, N)$, and the estimate (4.91) holds. Putting together these three claims completes the proof. □

Proof of Theorem 4.5.1. Let $\epsilon_0 > 0$ be given by Theorem 4.5.3. Since $p > \frac{n}{2}$ and $q < \infty$, there exists $T_0 > 0$ such that

$$\max_{i=1,2} \|\nabla^2 u_i\|_{L^q_t L^p_x(\Omega \times [0,T_0])} \le \epsilon_0. \tag{4.127}$$

This implies that for any $x_0 \in \Omega$ and $0 < t_0 \le T_0$, if $R_0 = \min\{d(x_0, \partial\Omega), t_0^{\frac{1}{4}}\} > 0$, then

$$\max_{i=1,2} \|\nabla^2 u_i\|_{L^q_t L^p_x(P_{R_0}(z_0))} \le \epsilon_0. \tag{4.128}$$

Hence by suitable scalings of the estimate of Theorem 4.5.3, we have that for $i = 1, 2$, $u_i \in C^\infty(P_{\frac{R_0}{2}}(z_0), N)$ and

$$\Big|\nabla^m u_i\Big|(x_0, t_0) \lesssim \epsilon_0 \left(\frac{1}{d^m(x_0, \partial\Omega)} + \frac{1}{t_0^{\frac{m}{4}}} \right). \tag{4.129}$$

Using (4.129), the same proof of Theorem 4.2.1 implies that $u_1 \equiv u_2$ in $\Omega \times [0, T_0]$. Repeating this argument on the interval $[T_0, T]$ yields $u_1 \equiv u_2$ in $\Omega \times [0, T]$. □

4.6 Uniqueness of biharmonic maps in dimension four

For dimension $n = 4$, by applying Theorem 4.5.3 (with $p = 2$ $(= \frac{n}{2})$ and $q = \infty$) and the second half of the proof of Theorem 4.2.1, the following uniqueness result holds.

Corollary 4.6.1. *For $n = 4$ and $0 < T \le \infty$, there exists $\epsilon_1 > 0$ such that if u_1 and $u_2 \in W_2^{1,2}(\Omega \times [0, T], N)$ are weak solutions of (1.8), under the same initial and boundary value $u_0 \in W^{2,2}(\Omega, N)$, satisfying*

$$\limsup_{t \downarrow t_0^+} E_2(u_i(t)) \le E_2(u_i(t_0)) + \epsilon_1, \ \forall\ t_0 \in [0, T), \tag{4.130}$$

for $i = 1, 2$. Then $u_1 \equiv u_2$ in $\Omega \times [0, T)$. In particular, the uniqueness holds among weak solutions of (1.8), whose E_2-energy is monotone decreasing for $t \ge 0$.

For the Cauchy problem (1.10) of heat flow of biharmonic maps on a compact 4-dimensional Riemannian manifold M without boundary, Corollary 4.6.1 has been recently proven by Rupflin [74] through a different argument.

Proof of Corollary 4.6.1. Let $\epsilon_0 > 0$ be given by Theorem 4.5.3. Since $u_0 \in W^{2,2}(\Omega, N)$, by the absolute continuity of $\int |\nabla^2 u_0|^2$ there exists $r_0 > 0$ such that

$$\max_{x\in\Omega} \int_{B_{r_0}(x)\cap\Omega} |\nabla^2 u_0|^2 \leq \frac{\epsilon_0^2}{2}. \tag{4.131}$$

Choosing $\epsilon_1 \leq \frac{\epsilon_0^2}{2}$ and applying (4.130), we conclude that there exists $0 < t_0 \leq r_0^4$ such that

$$\max_{x\in\Omega, 0\leq t\leq t_0} \int_{B_{r_0}(x)\cap\Omega} |\nabla^2 u_i(t)|^2 \leq \epsilon_0^2, \quad \text{for} \ \ i = 1, 2. \tag{4.132}$$

Set $R_0 = \min\{r_0, t_0^{\frac{1}{4}}\} = t_0^{\frac{1}{4}} > 0$. Then (4.132) implies

$$\max_{z=(x,t)\in\Omega\times[0,t_0]} \left\|\nabla^2 u_i\right\|_{L_t^\infty L_x^2(P_{R_0}(z)\cap(\Omega\times[0,t_0]))} \leq \epsilon_0, \quad \text{for} \ \ i = 1, 2. \tag{4.133}$$

Hence u_1 and u_2 satisfy (4.90) of Theorem 4.5.3 (with $p = 2$ and $q = \infty$) on $P_r(z)$, for any $z \in \Omega \times [0, t_0]$ and $r = \min\{R_0, d(x, \partial\Omega), t^{\frac{1}{4}}\} > 0$. Hence by suitable scalings of the estimate of Theorem 4.5.3, we have

$$\max_{i,2} \left|\nabla^m u_i(x,t)\right| \lesssim \epsilon_0 \left(\frac{1}{R_0^m} + \frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right) \lesssim \epsilon_0 \left(\frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right), \ \forall\, m \geq 1, \tag{4.134}$$

for any $(x,t) \in \Omega\times[0, t_0]$. Here we have used $R_0 \geq t^{\frac{1}{4}}$ in the last inequality. Applying (4.134) and the proof of Theorem 4.2.1, we can conclude that $u_1 \equiv u_2$ in $\Omega \times [0, t_0]$. Continuing this argument on the interval $[t_0, T]$ shows $u_1 \equiv u_2$ in $\Omega \times [0, T]$. □

Concerning the convexity and unique limit of (1.8) at $t = \infty$ in dimension $n = 4$, it holds

Corollary 4.6.2. *For $n = 4$, there exist $\epsilon_2 > 0$ and $T_1 > 0$ such that if $u \in W_2^{1,2}(\Omega \times (0, +\infty), N)$ is a weak solution of (1.8), with the initial-boundary value $u_0 \in W^{2,2}(\Omega, N)$, satisfying*

$$E_2(u(t)) \leq \epsilon_2^2, \ \forall\, t \geq 0, \tag{4.135}$$

then (i) $E_2(u(t))$ is monotone decreasing for $t \geq T_1$;
(ii) for $t_2 \geq t_1 \geq T_2$, it holds

$$\int_\Omega |\nabla^2(u(t_1) - u(t_2))|^2 \leq C\left(E_2(u(t_1)) - E_2(u(t_2))\right)$$

for some $C = C(\epsilon_2) > 0$; and
(iii) there exists a biharmonic map $u_\infty \in C^\infty \cap W^{2,2}(\Omega, N)$, with $(u_\infty, \frac{\partial u_\infty}{\partial\nu}) = (u_0, \frac{\partial u_0}{\partial\nu})$ on $\partial\Omega$, such that $\lim_{t\to\infty} \|u(t) - u_\infty\|_{W^{2,2}(\Omega)} = 0$, and for any $m \geq 1$, $K \subset\subset \Omega$, $\lim_{t\to\infty} \|u(t) - u_\infty\|_{C^m(K)} = 0$.

It is easy to see that the condition (4.135) holds for any solution $u \in W_2^{1,2}(\Omega \times [0,\infty), N)$ of (1.8) that satisfies $E_2(u(t)) \leq E_2(u_0)$ for $t \geq 0$ (e.g., the solution by [27] and [93]) and $E_2(u_0) \leq \epsilon_2^2$.

Proof of Corollary 4.6.2. Let $\epsilon_2 > 0$ be given by Theorem 4.5.3. Then (4.135) yields

$$\left\|\nabla^2 u\right\|_{L_t^\infty L_x^2(\Omega\times[0,\infty))} \leq \epsilon_2. \tag{4.136}$$

Hence by suitable scalings of the estimate of Theorem 4.5.3, we have $u \in C^\infty(\Omega \times (0,\infty), N)$ and there exists $T_1 > 0$ such that

$$\left|\nabla^m u(x,t)\right| \lesssim \epsilon_2 \left(\frac{1}{d^m(x,\partial\Omega)} + \frac{1}{t^{\frac{m}{4}}}\right), \ \forall\ m \geq 1, \tag{4.137}$$

holds for all $x \in \Omega$ and $t \geq T_1$. Now we can apply the same arguments as in the proof of Theorem 4.4.1 and Corollary 4.4.2 to prove the conclusions of Corollary 4.6.2. □

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