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ANALYTIC AND TOPOLOGICAL COMBINATORICS OF PARTITION POSETS AND PERMUTATIONS

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ABSTRACT OF DISSERTATION

JiYoon Jung

The Graduate School
University of Kentucky
2012

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ABSTRACT OF DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
JiYoon Jung
Lexington, Kentucky

Director: Dr. Richard Ehrenborg, Professor of Mathematics
Lexington, Kentucky 2012

ABSTRACT OF DISSERTATION

ANALYTIC AND TOPOLOGICAL COMBINATORICS OF PARTITION POSETS AND PERMUTATIONS

In this dissertation we first study partition posets and their topology. For each composition $\vec{c}$ we show that the order complex of the poset of pointed set partitions is a wedge of spheres of the same dimension with the multiplicity given by the number of permutations with descent composition $\vec{c}$. Furthermore, the action of the symmetric group on the top homology is isomorphic to the Specht module of a border strip associated to the composition. We also study the filter of pointed set partitions generated by knapsack integer partitions. In the second half of this dissertation we study descent avoidance in permutations. We extend the notion of consecutive pattern avoidance to considering sums over all permutations where each term is a product of weights depending on each consecutive pattern of a fixed length. We study the problem of finding the asymptotics of these sums. Our technique is to extend the spectral method of Ehrenborg, Kitaev and Perry. When the weight depends on the descent pattern, we show how to find the equation determining the spectrum. We give two length 4 applications, and a weighted pattern of length 3 where the associated operator only has one non-zero eigenvalue. Using generating functions we show that the error term in the asymptotic expression is the smallest possible.

KEYWORDS: Partition Lattice, Simplicial Complex, Homology, Homotopy, Discrete Morse Theory, Group Action, Specht Module, Pattern Avoidance, Asymptotics, Spectral Method.

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RULES FOR THE USE OF DISSERTATIONS

Name	Date

DISSERTATION

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Dedicated to my parents

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Chapter 1 contains an overview of results that are found in this dissertation and a brief introduction of the mathematical ideas that are used throughout.

In Chapter 2 for each composition $\vec{c}$ we show that the order complex of the poset of pointed set partitions $\Pi^{\bullet}_{\vec{c}}$ is a wedge of spheres of the same dimension with the multiplicity given by the number of permutations with descent composition $\vec{c}$. Furthermore, the action of the symmetric group on the top homology is isomorphic to the Specht module S^B where B is a border strip associated to the composition. We also study the filter of pointed set partitions generated by knapsack integer partitions and show the analogous results on homotopy type and action on the top homology.

Chapter 3 introduces the extended notion of consecutive pattern avoidance to considering sums over all permutations where each term is a product of weights depending on each consecutive pattern of a fixed length. We study the problem of finding the asymptotics of these sums. Our technique is to extend the spectral method of Ehrenborg, Kitaev and Perry. When the weight depends on the descent pattern we show how to find the equation determining the spectrum. We give two length 4 applications. First, we find the asymptotics of the number of permutations with no triple ascents and no triple descents. Second we give the asymptotics of the number of permutations with no isolated ascents or descents. Our next result is a weighted pattern of length 3 where the associated operator only has one nonzero eigenvalue. Using generating functions we show that the error term in the asymptotic expression is the smallest possible.

1.1 Posets

We now provide background for Chapter 2 of this dissertation.

A *partially ordered set* P (or *poset* P, for short) is a set together with a *partial order* $\leq_P$ (or $\leq$, if P is clear), that is, a binary relation satisfying the following three conditions:

1. For all $x \in P$, $x \leq x$ (reflexivity).
2. For all x and $y \in P$ such that if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
3. For all $x, y, z \in P$ such that $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

The minimal element and the maximal element of a poset P, if they exist, are denoted by $\hat{0}$ and $\hat{1}$ respectively. A poset P is said to be bounded if it has both a greatest element $\hat{1}$ and a least element $\hat{0}$. Two elements x and y are called *comparable* if $x \leq y$ or $y \leq x$. Otherwise the two elements are called *incomparable*. We denote by $x \prec y$ if $x < y$ and there does not exist an element z satisfying $x < z < y$ in a poset P, and read x *is covered by* y, or y *covers* x. A poset P can be represented by a *Hasse diagram* in which each element of the poset P is a vertex, and a line segment goes upward from x to y when x is covered by y. See [28, Chapter 3] for a basic introduction to posets.

A *subposet* Q of a poset P is a subset of the poset P having a partial order $x \leq_Q y$ for x and y in the subset Q if and only if $x \leq_P y$. An *interval* $[x, y]$ is a subposet of a poset P, which is a collection of all elements z satisfying $x \leq z \leq y$. A *chain* $c = \{x_0 < x_1 < \cdots < x_n\}$ is a subposet of a poset P such that any pair of elements in the chain is comparable. If x_i covers x_{i-1} for $1 \leq i \leq n$, the chain $c = \{x_0 \prec x_1 \prec \cdots \prec x_n\}$ is called a *maximal chain* in the interval $[x_0, x_n]$. We say two posets P and Q are *isomorphic* if there exist a bijection ϕ from P to Q such that for all x and y in the poset P we have $x \leq_P y$ if and only if $\phi(x) \leq_Q \phi(y)$.

For x and y in a poset P, a *upper bound* of x and y is an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. Then the x *join* y (denoted by $x \vee y$) is the *least upper bound* of two elements x and y, that is, $x \vee y \leq z$ for any upper bound z. The *greatest lower bound* of two elements x and y is dually denoted by $x \wedge y$, read x *meet* y. A *join-semilattice* L is a poset such that every pair of elements of L has the least upper bound. Dually, a *meet-semilattice* L is a poset such that every pair of elements of L has the greatest lower bound. A *lattice* L is a poset such that every pair of elements of L has the least upper bound and the greatest lower bound. Stanley [28, Proposition 3.3.1] states the following.

Proposition 1.1.1. *Let P be a finite meet-semilattice with a maximal element $\hat{1}$. Then P is a lattice. Dually, let P be a finite join-semilattice with a minimal element $\hat{0}$. Then P is again a lattice.*

The *Boolean algebra* B_n on n elements, a collection of subsets of n elements set with an order relation $x \leq y$ by $x \subseteq y$, is a lattice. Note that the join is given by union, that is, $x \vee y = x \cup y$ and the meet by intersection, $x \wedge y = x \cap y$. See the Hasse diagram of the Boolean algebra B_3 in Figure 1.1.

Another classical example of a lattice is the *intersection lattice* of a *hyperplane arrangement*. Let a hyperplane arrangement $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a collection of hyperplanes in $\mathbb{R}^n$, all containing the origin $\vec{0}$. Then the intersection lattice consists of all possible intersections $\mathcal{L}(\mathcal{H}) = \{\bigcap_{i \in I} H_i : I \subseteq \{1, 2, \ldots, m\}\}$. Note that empty intersection (when $I = \emptyset$) is the whole set $\mathbb{R}^n$. The set $\mathcal{L}(\mathcal{H})$ is a poset by reverse inclusion. It is a join-semilattice with the minimal element $\hat{0} = \mathbb{R}^n$ and hence a lattice by Proposition 1.1.1. An example of an intersection lattice is given in Section 1.2.

A *graded poset* P is a poset with the minimal element $\hat{0}$, the maximal element $\hat{1}$, and a *rank function* $\rho : P \longrightarrow \mathbb{Z}$ satisfying:

1. For the minimal element $\hat{0}$ of P, $\rho(\hat{0}) = 0$.

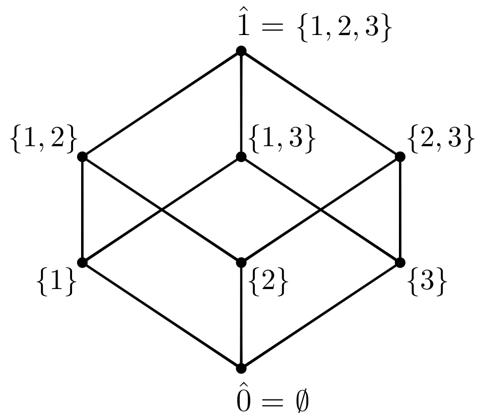


Figure 1.1: The Boolean algebra B_3 of a set $\{1, 2, 3\}$.

2. For any elements $x \leq y$ of P, $\rho(x) \leq \rho(y)$.
3. For all cover relations $x \prec y$ of P, $\rho(x) + 1 = \rho(y)$.

The rank of a graded poset P is defined as the rank of the maximal element $\hat{1}$, that is, $\rho(\hat{1})$. Note that the rank function preserves inequality, that is, $x \leq y$ implies $\rho(x) \leq \rho(y)$. Let P be a graded poset of rank n with a rank function ρ. For any $S \subseteq \{1, \ldots, n-1\}$ let P_S denote the subposet $P_S = \{x \in P \mid \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\} = \rho^{-1}(S \cup \{0, n\})$. Then we call P_S *the rank-selected poset.*

The *Möbius function* μ on a poset P is defined as follows:

1. For all $x \in P$, $\mu(x, x) = 1$.
2. For all $x < y$ in P, $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$.

By Philip Hall's formula the *Möbius function* μ on a poset P is also calculated as follows:

$$\mu(x, y) = \sum_{x = z_0 \prec z_1 \prec \cdots \prec z_i = y} (-1)^i$$

for all $x < y$ in P. For a poset P with minimal and maximal elements $\hat{0}$ and $\hat{1}$, we donote by $\mu(P)$ the value $\mu(\hat{0}, \hat{1})$. In Figure 1.2, the Möbius function $\mu(P_{\{1,3\}})$ is -5.

1.2 Partition Lattices

A *set partition* $\pi = \{B_1, B_2, \ldots, B_k\}$ on a set $[n] = \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets (called *blocks*) of $[n]$ whose union is the whole set $[n]$. As a shorthand, we write a set partition as $126/35/49/78$ instead of the longer $\{\{1, 2, 6\}, \{3, 5\}, \{4, 9\}, \{7, 8\}\}$. The *partition lattice* Π_n is the set of all set partitions of $[n]$ with the order relation $\pi \leq \sigma$

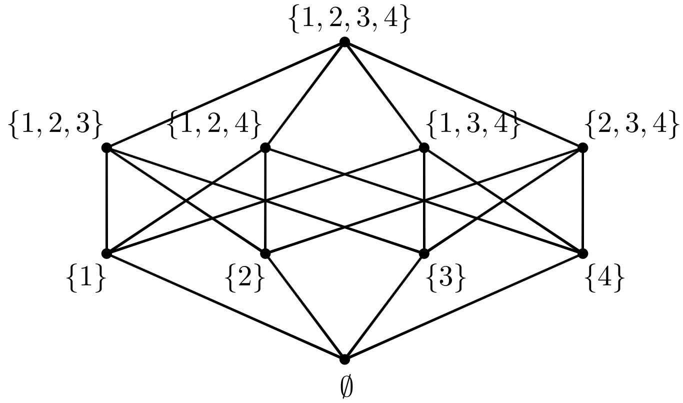


Figure 1.2: The rank selected poset $P_{\{1,3\}}$, where P is the Boolean algebra B_4.

in Π_n if every block of π is contained in a block of σ. In the partition lattice Π_n, the minimal element is the partition consisting of all singleton blocks, that is, $1/2/3/\cdots/n$, and the maximal element is the partition consisting of one block, which is $[n]$. The meet of two partitions π and σ is the partition $\pi \wedge \sigma = \{B \cap C | B \in \pi, C \in \sigma, B \cap C \neq \emptyset\}$. The join exists by Proposition 1.1.1. The partition lattice on 4 elements Π_4 is displayed in Figure 1.3.

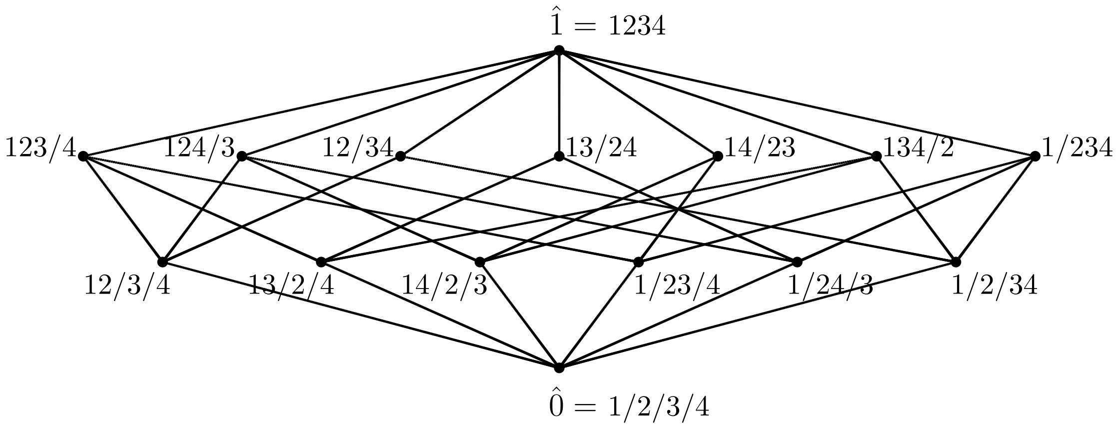


Figure 1.3: The partition lattice Π_4.

A geometric motivation for the partition lattice is given as follows. For $1 \leq i < j \leq n$, let the hyperplane $H_{i,j}$ be the set $\{\vec{x} \in \mathbb{R}^n \mid x_i = x_j\}$, and the hyperplane arrangement $\mathcal{A}_{n-1}$ be defined as $\mathcal{A}_{n-1} = \{H_{i,j} \mid 1 \leq i < j \leq n\}$. Notice that each intersection is nonempty since every $H_{i,j}$ contains the origin $\vec{0} \in \mathbb{R}^n$ for $1 \leq i < j \leq n$. Furthermore, consider the bijection ϕ from the partition lattice Π_n to the intersection lattice $\mathcal{L}(\mathcal{A}_{n-1})$ such that for each set partition $\pi \in \Pi_n$, we denote $\phi(\pi) = \{\vec{x} \in \mathbb{R}^n \mid x_i = x_j \text{ for all } i \text{ and } j \text{ in the same block of } \pi\}$. Under this bijection the intersection lattice $\mathcal{L}(\mathcal{A}_{n-1})$ is isomorphic to the partition lattice

Π_n; see Figure 1.4 for an example.

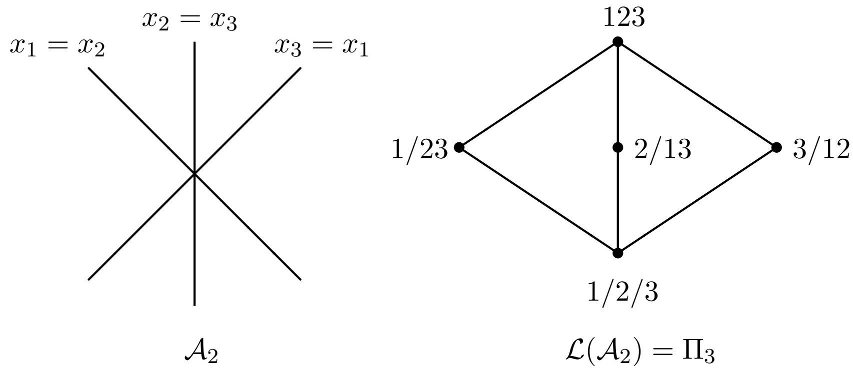


Figure 1.4: An intersection lattices of hyperplane arrangement.

The *d-divisible partition lattice* Π_n^d is the collection of all set partitions of $[n]$ where each block size is divisible by d. Note that the d-divisible partition lattice Π_n^d does not have a minimal element if $d > 1$. As an example, the lattice Π_6^2 consists of 15 set partitions consisting of 3 blocks, 15 set partitions consisting of 2 blocks, and the maximal element 123456. In the lattice Π_6^2, each set partition of 3 blocks is covered by 3 elements, and each set partition of 2 blocks covers 3 elements; see Figure 1.5. Observe that the lattice Π_n^d is a *join-semilattice* since Π_n^d is a filter in the partition lattice Π_n, and inherits the join operation from Π_n. By Proposition 1.1.1, we obtain that $\Pi_n^d \cup \{\hat{0}\}$, that is, Π_n^d joined by an artificial new minimal element $\hat{0}$, is a lattice. Hence, although Π_n^d lacks a minimal element, it is still called a lattice.

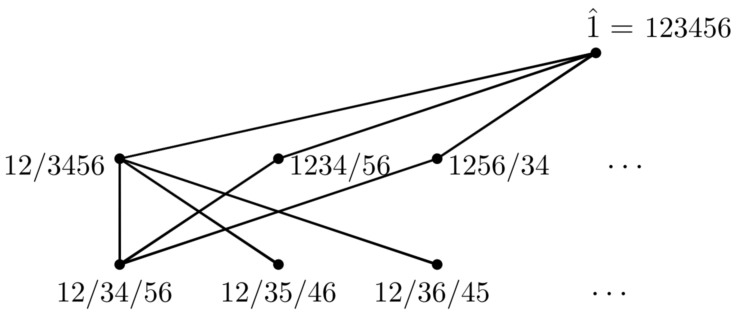


Figure 1.5: A sketch of a part of the even partition lattice Π_6^2.

1.3 Simplicial Complexes

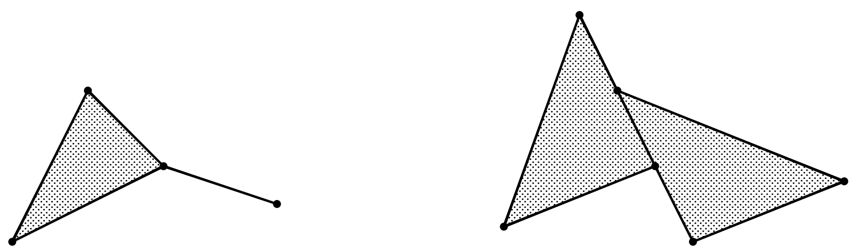
A *simplicial complex* Δ is a collection of subsets of a vertex set V such that singleton sets $\{v\} \in \Delta$ for all $v \in V$, and if $F \subseteq G$ and $G \in \Delta$, then $F \in \Delta$. In geometry, a subset in a simplicial complex Δ is considered as a simplex which is the convex hull of all vertices in the subset. The simplices in a complex Δ are called *faces* of the complex Δ. A *facet* of a complex Δ is a face which is not contained in any other face of the complex Δ. Figure 1.6 shows two sets of simplices where one is a simplicial complex and the other is not.

For a simplicial complex Δ, the *f-vector of* Δ is the integer sequence $(f_{-1}, f_0, f_1, \ldots)$, where f_i is the number of i-faces of the complex Δ for $i \geq 0$ and $f_{-1} = 1$. The *Euler characteristic* $\chi(\Delta)$ of a complex Δ is defined as the sum

$$\chi(\Delta) = f_0 - f_1 + \cdots .$$

The *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ of a complex Δ is the sum

$$\widetilde{\chi}(\Delta) = \chi(\Delta) - 1 = -f_{-1} + f_0 - f_1 + \cdots .$$



a simplicial complex

not a simplicial complex

Figure 1.6: Two sets of simplices.

A *d-dimensional simplicial complex* Δ is a simplicial complex such that the largest dimension of any face in Δ is d. A d-dimensional simplicial complex is called *pure* if all facets of the complex Δ have dimension d. An *abstract simplicial complex* is a finite collection S of nonempty sets such that if A is an element of S then so is every nonempty subset of A. For a simplicial complex Δ of a vertex set V, if K is a collection of all subsets $\{a_0, a_1, \ldots, a_n\}$ of V such that the vertices $a_0, a_1, \ldots a_n$ span a simplex of the complex Δ then the collection K is called the *vertex scheme* of the complex Δ. If an abstract simplicial complex S is isomorphic to the vertex scheme of a simplicial complex Δ then the complex Δ is said to be a *geometric realization* of the complex S.

The *order complex* $\Delta(P)$ of a poset P is a simplicial complex such that the faces of $\Delta(P)$ are the chains of P. Note the vertices of $\Delta(P)$ are the elements (chains consisting of one element) of the poset P. See Figure 1.7 for an example. The *Möbius function* $\mu(P)$ of a poset P is also calculated by the reduced Euler characteristic $\widetilde{\chi}$ of the order complex $\Delta(P - \{\hat{0}, \hat{1}\})$.

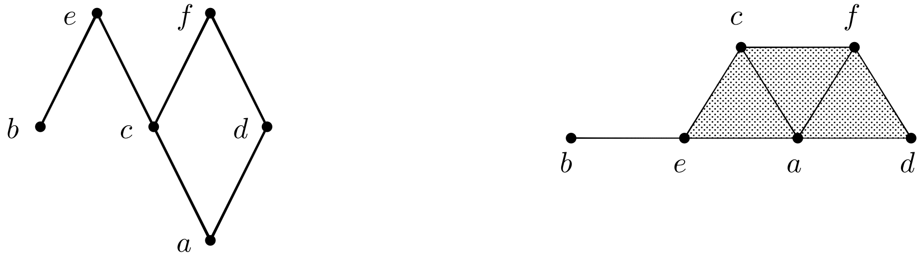


Figure 1.7: A poset P and its order complex $\Delta(P)$.

Given a simplicial complex Δ, the associated *face poset* $P(\Delta)$ is the poset of nonempty faces in the complex Δ ordered by inclusion. The *barycentric subdivision* (or BCS, for short) of a simplicial complex Δ is the order complex of the poset $P(\Delta)$. Geometrically, $\Delta \cong \Delta(P(\Delta))$. See Figure 1.8.

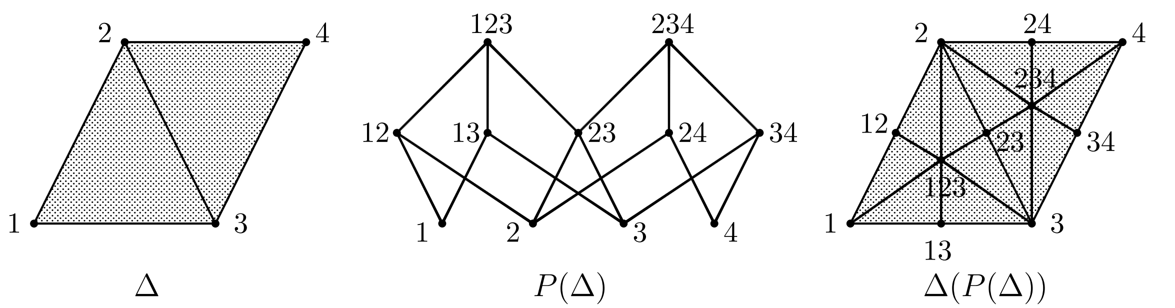


Figure 1.8: A barycentric subdivision.

1.4 Topology

For two topological spaces X and Y, a function f from X to Y is called a *homeomorphism* if it satisfies:

1. f is a bijection.
2. f is continuous.
3. The inverse function f^{-1} is continuous.

If such function exists, we say two topological spaces X and Y are *homeomorphic*.

A topological space X is a *Hausdorff space* if any two distinct points of X can be separated by neighborhoods. A *CW complex* is a Hausdorff space X with a partition of X into open cells satisfying the following two conditions:

1. For each n-dimensional open cell on X, there exists a continuous map f from an n-dimensional closed ball to X such that the image of the interior of the closed ball is homeomorphic to the open cell, and the image of the boundary of the closed ball is contained in the union of a finite number of open cells of dimension less that n.

2. A subset of X is closed if and only if it meets the closure of each cell in a closed set.

For two topological spaces X and Y with basepoints x_0 and y_0 respectively, the *wedge* of two topological spaces X and Y, denoted $X \vee Y$ is the quotient of the disjoint union of X and Y by identifying $x_0 \sim y_0$. Informally, one glues the two spaces X and Y together at the points x_0 and y_0. Observe that when X is connected, the choice of basepoint in X does not matter.

For a non-negative integer n and a non-negative real number r, an *n-dimensional sphere* $\mathbb{S}^n$ of radius r is defined as the set of points $x = (x_1, x_2, \ldots, x_{n+1})$ in $(n+1)$*-dimensional Euclidean space* $\mathbb{R}^{n+1}$ such that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$. An important homotopy type on this dissertation is the k-fold wedge of n-dimensional spheres, that is, $(\mathbb{S}^n)^{\vee k} = \mathbb{S}^n \vee \mathbb{S}^n \vee \cdots \vee \mathbb{S}^n$. We extend this notion to the 0-dimensional sphere $\mathbb{S}^0$, which consists of two points. That is, we let $(\mathbb{S}^0)^{\vee k}$ denote $k+1$ disjoint points.

1.5 Homology

For a simplicial complex Δ on m vertices and an integer n, let $C_n(\Delta)$ (or C_n, for short) be the abelian group generated by the n-dimensional simplices (or n-simplices, for short) of the complex Δ over a ring R, and $\mathbf{0}$ denote the trivial group. In this dissertation we take R to be the integer $\mathbb{Z}$. Note that the rank of the group C_n is the number of n-simplices of the complex Δ. Order the vertices of a simplicial complex Δ by $v_1, v_2, \ldots, v_m$. For any $(n-1)$-simplex $\{v_{t_1}, v_{t_2} \ldots, v_{t_n}\}$ of the complex Δ where $1 \leq t_1 < t_2 < \cdots < t_n \leq m$, denote $\{v_{t_1}, \ldots, v_{t_{i-1}}, v_{t_{i+1}}, \ldots, v_{t_n}\}$ by $\{v_{t_1}, \ldots, \widehat{v_{t_i}}, \ldots, v_{t_n}\}$. A *chain complex* $C(\Delta)$ is defined as a sequence of abelian groups $C_0, C_1, \ldots$ with linear operators $\partial_n : C_n \longrightarrow C_{n-1}$ called *boundary operators* such that

$$\partial(\{v_{t_1}, v_{t_2} \ldots, v_{t_n}\}) = \sum_{i=1}^{n} (-1)^i \{v_{t_1}, \ldots, \widehat{v_{t_i}}, \ldots, v_{t_n}\},$$

that is, the chain complex $C(\Delta)$ is

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \mathbf{0}.$$

Observe that the composition of any two consecutive boundary operators is trivial, that is, $\partial_n \circ \partial_{n+1} = 0$. Note that this condition implies $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$ for any integer n. The group $\ker(\partial_n)/\mathrm{im}(\partial_{n+1})$ denoted by $H_n(\Delta)$, is called the *n-th homology group* of the complex Δ. The elements of $\ker(\partial_n)$ are called n-cycles, the elements of $\mathrm{im}(\partial_{n+1})$ are called n-boundaries, and the elements of $H_n(\Delta)$ are called n-homology classes. The n-homology classes in $H_n(\Delta)$ correspond to cosets $c + \mathrm{im}(\partial_{n+1})$ for n-cycles c. The $\dim(H_n(\Delta)) = \mathrm{rank}(\ker(\partial_n)) - \mathrm{rank}(\mathrm{im}(\partial_{n+1}))$ is called the *n-th Betti number* of the complex Δ. Note that the *Euler characteristic* $\chi(\Delta)$ of a complex Δ may be obtained by the sum $\chi(\Delta) = \sum_i (-1)^i \cdot \dim(H_i(\Delta))$.

The *reduced homology groups* $\widetilde{H}_n(\Delta)$ of a complex Δ are isomorphic to the homology groups $H_n(\Delta)$ for $n > 0$. If $n = 0$ then we have $H_0(\Delta) \cong \widetilde{H}_0(\Delta) \oplus \mathbb{Z}$.

There are some important facts that we use in this dissertation. First, the homology groups do not depend on how we triangulate a topological space. Next, as stated in Theorem 1.5.1, the homology is homeomorphism invariant.

Theorem 1.5.1. *If X and Y are homeomorphic simplicial complexes, then their homology groups are isomorphic: $H_i(X) \cong H_i(Y)$ for $i \in \mathbb{Z}$. That is, the homology only depends on the geometric realization.*

More strongly, the homology groups are homotopy type invariants. See Section 1.6 for a discussion of this fact.

1.6 Homotopy

For two continuous maps f and f' of the space X into the space Y, if there exists a continuous map $F : X \times [0, 1] \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each x, then we call the map F a *homotopy* between two maps f and f'. When a homotopy F between f and f' exists, we write $f \simeq f'$ and say that f is *homotopic* to f'. For two continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$, if the map $g \circ f : X \longrightarrow X$ is homotopic to the identity map of the space X, and the map $f \circ g : Y \longrightarrow Y$ is homotopic to the identity map of the space Y, then each of the maps f and g is called a *homotopy equivalence*. If there exists a homotopy equivalence between two spaces X and Y then we say the spaces X and Y are *homotopy equivalent*, and two spaces X and Y are said to have the same *homotopy type*. If two spaces X and Y have the same homotopy type then they have isomorphic homology groups. Hence the Euler characteristic χ is an invariant under homotopy.

We have the following result due to Quillen [26].

Theorem 1.6.1 (Quillen's Fiber Lemma)**.** *Let f be a simplicial map from the simplicial complex Γ to the poset P such that for all elements x in the poset P, the subcomplex $\Delta(f^{-1}(P_{\geq x}))$ is contractible. Then the order complex $\Delta(P)$ and the simplicial complex Γ are homotopy equivalent.*

1.7 Shellability

For a face F of a simplicial complex Δ, let $\langle F\rangle = \{G \in \Delta \mid G \subseteq F\}$. A pure simplicial complex Δ is said to be *shellable* if Δ is 0-dimensional or if there exists an ordering $F_1, F_2, \ldots, F_t$ of the facets of the complex Δ such that the subcomplex $(\bigcup_{i=1}^{k-1}\langle F_i\rangle) \cap \langle F_k\rangle$ is pure and $(\dim(F_k) - 1)$-dimensional for $2 \leq k \leq t$. We call a facet F_k *spanning* if its boundary is contained among previously added faces, that is, $\partial F_k \subseteq \bigcup_{i=1}^{k-1}\langle F_i\rangle$.

Proposition 1.7.1. *A shellable d-dimensional simplicial complex is homotopy equivalent to a wedge of d-dimensional spheres and the number of spheres is given by the number of spanning facets.*

An *edge labeling* λ of a bounded poset P is a map from the set of edges of the Hasse diagram of the poset P to the integer poset $\mathbb{Z}$ with its natural order relation. We say that a maximal chain $c = x_0 \prec x_1 \prec \cdots \prec x_n$ of a poset P is increasing if $\lambda(x_{i-1}, x_i) \leq \lambda(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Dually, the chain c is said to be decreasing if $\lambda(x_{i-1}, x_i) \geq \lambda(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. An edge labeling of a graded poset P is said to be an *R–labeling* of the poset P if for each closed interval $[x, y]$ of P there exists the unique increasing maximal chain of $[x, y]$.

An *edge-lexicographical labeling* (or EL-labeling, for short) of a bounded poset P is an edge labeling of P such that for each closed interval $[x, y]$ of P the lexicographically least maximal chain of $[x, y]$ is the unique increasing maximal chain of $[x, y]$. If there exists an EL-labeling of a graded poset P then the poset P is said to be *edge-lexicographic shellable* (or EL-shellable, for short). As an example the Boolean algebra B_n is edge-lexicographic shellable. In Figure 1.9 an EL-labeling λ of B_3 is defined by letting $\lambda(A, B)$ be the unique element of the singleton set $B - A$ for each pair of sets $A \prec B$ in B_3. The chain $c = \emptyset \prec \{1\} \prec \{1, 2\} \prec \{1, 2, 3\}$ is the lexicographically least maximal chain and the unique increasing maximal chain of $[\emptyset, \{1, 2, 3\}]$.

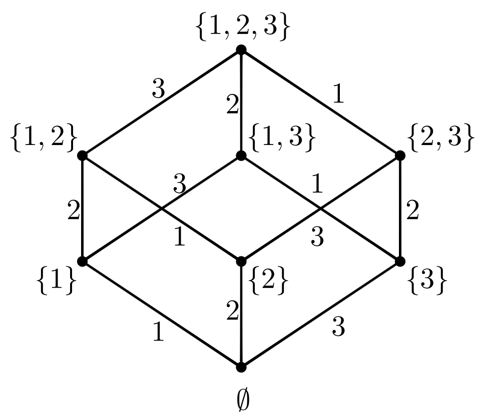


Figure 1.9: An EL-labeling of the Boolean algebra B_3.

Theorem 1.7.2. *(Björner and Wachs [5]) Suppose P is a poset for which $P \cup \{\hat{0}, \hat{1}\}$ admits an EL-labeling. Then the poset P has the homotopy type of a wedge of spheres, where the number of i-spheres is the number of decreasing maximal $(i+2)$-chains of $P \cup \{\hat{0}, \hat{1}\}$. The decreasing maximal $(i+2)$-chains, with $\hat{0}$ and $\hat{1}$ removed, form a basis for the cohomology $\widetilde{H}^i(P; \mathbb{Z})$.*

1.8 Discrete Morse Theory

Morse Theory has been one of the most powerful techniques for the study of the topology of smooth manifolds by studying differentiable functions on the manifolds. R. Forman developed *Discrete Morse Theory*, a combinatorial adaptation of Morse Theory, to analyze the topology of simplicial complexes in [16] and [17]. By discrete Morse theory, the topology of a simplicial complex can be simplified without changing its homotopy type.

For a simplicial complex Δ, let K be the set of simplices of the complex Δ, and K_p be the set of simplices of dimension p of the complex Δ. If a simplex σ lies in the boundary of a simplex τ, we write $\tau > \sigma$. A *discrete Morse function* f on a simplicial complex Δ is a function from K to $\mathbb{R}$ such that for every p-simplex $\sigma \in K_p$

$$\#\{\tau \in K_{(p+1)} \mid \tau > \sigma,\ f(\tau) \leq f(\sigma)\} \leq 1, \text{ and}$$

$$\#\{\tau \in K_{(p-1)} \mid \tau < \sigma,\ f(\tau) \geq f(\sigma)\} \leq 1.$$

In Figure 1.10(a) the function is not a discrete Morse function since the edge $f^{-1}(0)$ and the vertex $f^{-1}(3)$ break the rules. We say the σ is *critical* if

$$\#\{\tau \in K_{(p+1)} \mid \tau > \sigma,\ f(\tau) \leq f(\sigma)\} = 0, \text{ and}$$

$$\#\{\tau \in K_{(p-1)} \mid \tau < \sigma,\ f(\tau) \geq f(\sigma)\} = 0.$$

In Figure 1.10(b) the the edge $f^{-1}(3)$ and the vertex $f^{-1}(0)$ are critical.

(a) Not a Discrete Morse function (b) A Discrete Morse function

Figure 1.10: Functions: $K \longrightarrow \mathbb{R}$.

R. Forman proved a following theorem; see [16] and [17].

Theorem 1.8.1 (Forman)**.** *Suppose a simplicial complex Δ has a discrete Morse function. Then the complex Δ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical simplex of dimension p.*

A *partial matching* M on a poset P is a subset of $P \times P$ such that $x \prec y$ for any $(x, y) \in M$, and each $x \in P$ belongs to at most one element of M. For $(x, y) \in M$, x and y are said to be the *down* of y and the *up* of x, respectively, and we write $x = d(y)$ and $y = u(x)$. A partial matching M on a poset P is *acyclic* if there does not exist a *cycle* in P, that is, distinct elements z_i do not exist in P such that $z_1 \succ d(z_1) \prec z_2 \succ d(z_2) \prec \cdots \prec z_n \succ d(z_n) \prec z_1$. For a partial matching M, the unmatched elements are called *critical*.

For a simplicial complex Δ, let $\mathcal{F}$ denote the face poset of Δ. M. Chari proved the following theorem in [7].

Theorem 1.8.2 (Chari)**.** *For a simplicial complex Δ, if M is an acyclic matching on $\mathcal{F}(\Delta) - \hat{0}$, and k_i denotes the number of critical i-dimensional cells of Δ, then the complex Δ is homotopy equivalent to a CW complex which has k_i cells of dimension i.*

If we study discrete Morse matching on the face poset including the empty set, we have the following corollary.

Theorem 1.8.3. *For a simplicial complex Δ, if M is an acyclic matching on $\mathcal{F}(\Delta)$, and k_i denotes the number of critical i-dimensional cells of Δ for $i \geq 0$, then the complex Δ is homotopy equivalent to a CW complex which has $k_0 + 1$ cells of dimension 0 and k_i cells of dimension i for $i > 0$.*

It is natural question for us to ask when a CW complex is homotopy equivalent to a wedge of spheres. One answer is that for the given complex and acyclic matching, all the critical cells are facets.

Theorem 1.8.4. *Let M be a Morse matching on $\mathcal{F}(\Delta)$ such that all k_i critical cells of dimension i are facets of Δ. Then the complex Δ is homotopy equivalent to a wedge of spheres, that is,*

$$\Delta \simeq \bigvee_i \bigvee_j^{k_i} \mathbb{S}^i.$$

1.9 The symmetric group

The *symmetric group* $\mathfrak{S}_n$ is the set of all bijections from a set $[n]$ to itself with functional composition as an operation. We call the elements α of $\mathfrak{S}_n$ *permutations* and compose permutations from right to left. A *composition* $\vec{c}$ of an integer n is the way of writing n as an ordered sum of positive integers. In this dissertation we use the notation $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ for a permutation $\alpha(1)\alpha(2) \cdots \alpha(n)$ and $\vec{c} = (c_1, c_2, \ldots, c_k)$ for a composition $c_1 + c_2 + \cdots + c_k$.

For a permutation $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$, the *descent set* $\mathrm{Des}(\alpha)$ is the set $\{j \in [n-1] : \alpha_j > \alpha_{j+1}\}$. The *descent composition* $\vec{c} = (c_1, c_2, \ldots, c_k)$ of a permutation α is the composition of n such that $\{c_1, c_1+c_2, \ldots, c_2+c_2+\cdots+c_{k-1}\} = \mathrm{Des}(\alpha)$. In Figure 1.2, observe that the Möbius function $\mu(P_{\{1,3\}}) = -5$, and 5 is the number of permutations in the symmetric group $\mathfrak{S}_4$ with descent set $\{1,3\}$. Namely, these are the permutations 2143, 3142, 3241, 4132 and 4231.

For a permutation α in the symmetric group $\mathfrak{S}_n$, define its *inversion set* to be the set $\mathrm{Inv}(\alpha) = \{(i,j) \mid 1 \leq i < j \leq n, \alpha_i > \alpha_j\}$. The number of inversions is given by $\mathrm{inv}(\alpha) = |\mathrm{Inv}(\alpha)|$. The sign of a permutation α is defined to be $(-1)^\alpha = (-1)^{\mathrm{inv}(\alpha)}$. Note that the sign is a group homomorphism from the symmetric group $\mathfrak{S}_n$ to the two element group $\{-1,1\}$, that is, $(-1)^{\alpha\cdot\beta} = (-1)^\alpha \cdot (-1)^\beta$.

1.10 $\mathfrak{S}_n$-Modules and Group Actions

A *representation* $\Phi : G \times V \longrightarrow V$ is a map of a group G on a vector space V satisfying the following properties:

1. $\Phi(g_1, c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\Phi(g_1, \vec{v}_1) + c_2\Phi(g_1, \vec{v}_2)$ (linearity),
2. $\Phi(e, \vec{v}_1) = \vec{v}_1$ where e is the identity element of G (identity), and
3. $\Phi(g_1, \Phi(g_2, \vec{v}_1)) = \Phi(g_1g_2, \vec{v}_1)$ (associativity)

for any $g_1, g_2 \in G$, $\vec{v}_1, \vec{v}_2 \in V$, and scalars $c_1, c_2 \in \mathbb{C}$. For simplicity, $\Phi(g, \vec{v})$ is denoted by the multiplication $g\vec{v}$ of an element $\vec{v} \in V$ by an element $g \in G$.

A *group homomorphism* $f : G \longrightarrow H$ is a map between two groups G and H such that $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$ for all $g_1, g_2 \in G$ and $f(1) = 1$. The *general linear group* $GL(V)$ is the group of all invertible linear transformations of a vector space V to itself. Another way to view a representation of a group G on a vector space V is that it is a group homomorphism $f : G \longrightarrow GL(V)$. That is $\Phi(g, \vec{v}) = f(g)\vec{v}$. If a representation of a group G on a vector space V exists, V is called a *G-module*.

For any set S if there is a multiplication of elements of S by elements of the symmetric group $\mathfrak{S}_n$ satisfying:

1. $\alpha_1 s \in S$ (closure),
2. $1s = s$ (identity), and
3. $(\alpha_1\alpha_2)s = \alpha_1(\alpha_2 s)$ (associativity)

for all $\alpha_1, \alpha_2 \in \mathfrak{S}_n$ and $s \in S$, then we say $\mathfrak{S}_n$ *acts on* S. Notice that the vector space generated by S over $\mathbb{C}$ is an $\mathfrak{S}_n$-module.

The equivariant version of Quillen's Fiber Lemma is as follows; see [26] and [34, Section 5.2].

Theorem 1.10.1 (Equivariant homology version of Quillen's Fiber Lemma)**.** *Let f be a G-simplicial map from the simplicial complex Γ to the poset P such that for all elements x in the poset P the subcomplex $\Delta(f^{-1}(P_{\geq x}))$ is acyclic. Then the two homology groups $\widetilde{H}_r(\Delta(P))$ and $\widetilde{H}_r(\Gamma)$ are isomorphic as G-modules.*

1.11 Specht Modules

An *integer partition* (or *partition*, for short) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ of a positive integer n is a way of writing n as a sum of positive integers $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$, with $\lambda_m > 0$. For example there are 5 partitions of 4, namely (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, and $(1, 1, 1, 1)$. One may let $\lambda_i = 0$ for $i > m$ as convenient.

A *diagram* D is a finite collection of unit lattice boxes in the plane. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$, the *Young diagram* of a shape λ is a diagram in which boxes are arranged in left-justified rows and the ith-row lengths is λ_i for $1 \leq i \leq m$. For two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l)$ such that $l < m$ and $\mu_i < \lambda_i$ for $1 \leq i \leq m$, the *skew Young diagram* of a shape λ/μ is a diagram which consists of all boxes in the Young diagram of λ and not in the Young diagram of μ. See Figure 1.11.

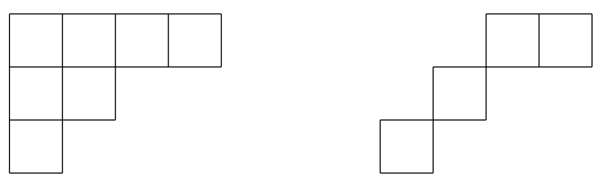


Figure 1.11: A Young diagram of $(4, 2, 1)$ and a skew Young diagram of $(4, 2, 1)/(2, 1)$.

For any diagram D, a *tableau* t of a shape D is the diagram D in which boxes have been filled with positive integers. Similarly, a *(skew) Young tableau* is a (skew) Young diagram filled with positive integers. If a (skew) Young tableau is filled with all the elements of $[n]$ exactly once, and if each row and each column in the (skew) Young tableau is strictly increasing, then the tableau is called a *standard Young tableau.* A standard Young tableaux of shape $(4, 2, 1)$ is given in Figure 1.12.

For a tableau t of any shape D which contains n boxes with labels $1, 2, \ldots, n$, and for the row stabilizer R_t of the tableau t, a tableau t' of the shape D is said to be *row equivalent to* the tableau t if there exists $\alpha \in R_t$ such that $\alpha \cdot t = t'$. A *tabloid* $\{t\}$ of the tableau t is an equivalent class of all tableaux which are row equivalent to the tableau t. For the column

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Figure 1.12: A standard Young tableaux of $(4, 2, 1)$.

stabilizer C_t of the tableau t, the associated *polytabloid* e_t is defined by

$$e_t = \sum_{\alpha \in C_t} (-1)^{\alpha} \cdot \alpha \cdot \{t\}.$$

The *permutation module* M^D is the vector space generated by all the tabloids $\{t\}$ of a shape D over $\mathbb{C}$, and the *Specht module* S^D is the submodule of M^D generated by the polytabloids e_t of the shape D.

For a partition λ of an integer n, if a diagram D is a Young diagram of a shape λ, then the Specht module S^{λ} is an irreducible representation of $\mathfrak{S}_n$, and the dimension of the Specht module S^{λ} is the number of standard Young tableaux of the shape λ. In fact, all irreducible representations of the symmetric group $\mathfrak{S}_n$ are Specht modules of permutations [27]. Specht modules of skew shapes are also understood; see [27]. However for a general shape D, very little is understood [22].

Example 1.11.1. *For a partition $\lambda = (2, 2)$ of 4 we have a Young tableau*

$$t = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}\,.$$

Then a tabloid $\{t\}$ is defined by

$$\{t\} = \begin{array}{|cc|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array} = \left\{ \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array} \right\}.$$

The column stabilizer of the Young tableau t is

$$C_t = \mathfrak{S}_{\{1,3\}} * \mathfrak{S}_{\{2,4\}}.$$

Then the polytabloid e_t is given by

$$e_t = \begin{array}{|cc|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array} - \begin{array}{|cc|}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline\end{array} + \begin{array}{|cc|}\hline 1 & 4 \\ \hline 3 & 2 \\ \hline\end{array} - \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\,.$$

Finally, the Specht module S^λ over $\mathbb{C}$ is

$$S^\lambda = \mathbb{C}[\mathfrak{S}_4] \cdot e_t$$

where $\mathbb{C}[\mathfrak{S}_4]$ denotes the group algebra over the symmetric group $\mathfrak{S}_4$. Note that the Specht module only depends on the shape of a diagram, not a tableau.

1.12 Dissertation results – partition posets and their topology

This dissertation consists of two parts: poset topology of the partition lattice and enumerative combinatorics of pattern avoiding permutations. We begin to give background on the first part.

In his dissertation [31] Sylvester introduced the even partition lattice which is the partition lattice restricted to partitions where each block has an even cardinality. He showed that the Möbius function is given by every other Euler number also known as the tangent numbers. Recall the Euler number enumerates the number of alternating permutations, that is, a linear ordering of the elements $1, 2, \ldots, n$ such that every other step is an ascent, respectively, a descent.

Sylvester's result inspired Richard Stanley, who was on the dissertation committee of Sylvester, to extend the even partition lattice to the d-divisible partition lattice, that is, restricting the partition lattice to partitions where each block size is divisible by d. In fact, the d-divisible partition lattice is a standard example of an exponential structure, the theory of which Stanley introduced in [28]. Stanley showed that the associated Möbius function of the d-divisible partition lattice is given by the number of permutations where each dth step is a descent and the other steps are ascents in [30].

Associated to every poset is its order complex, that is, the collection of chains in the poset. A chain is a totally ordered subset of the poset. The order complex is a simplicial complex, that is, a subset of a chain is also a chain. Since every simplicial complex is a topological space, the order complex forms a topological space. Hence it is natural to ask topological questions about this object. The first question is to determine the Euler characteristic. The reduced Euler characteristic of the order complex is known to be given by the Möbius function of the original poset [34, Philip Hall Theorem].

The next topological question is to understand the homotopy type of the order complex, that is, to understand how the space looks under homotopy equivalence. Michelle Wachs determined the homotopy type of the order complex of the d-divisible partition lattice [32]. She showed that it is a wedge of spheres, that is, a collection of $(n/d-2)$-dimensional spheres glued together at a point. Furthermore, the number of spheres is given by the Möbius function, that is, number of permutations where every d step is a descent and the other steps are ascents.

Wachs' method was to obtain a shelling of the simplicial complex. A *shelling* is an ordering of the maximal faces (facets) of the complex such that when adding the facets one by one at each step the change in the topology is very well-controlled. The type of the shelling she obtained was an edge lexicographic shelling (EL-shelling for short). Namely, Wachs gave a label to each edge in the d-divisible partition lattice such that the each maximal chain had a list of labels. After sorting the maximal chains according to the lexicographic order of their label lists, a shelling order is obtained.

One of consequences of a topological space having the homotopy type of a wedge of spheres of the same dimension is that its reduced homology groups are all trivial except for the top homology group. This group is a vector space whose dimension is given by the number of spheres. Observe that the symmetric group has a natural action on the partition lattice by relabeling the underlying elements of the partitions. This action extends to the d-divisible partition lattice and hence also to the order complex. The final step is that this action also applies to the top homology group.

The action of a group on a vector space is known as a representation. Representations of the symmetric group are a well-studied area of mathematics; see Sagan's book [27]. A classical class of representations is the Specht module of a diagram of boxes. When the diagram is a Ferrers shape, the associated Specht module is irreducible, that is, it cannot be broken up in smaller representations. A larger class is when the diagram is a skew shape. The Specht module of a skew shape is also well-understood in the theory of representations of the symmetric group.

Calderbank, Hanlon and Robinson studied the action of the symmetric group on the top homology of the d-divisible partition lattice. They showed that this representation is isomorphic to a Specht module of a skew shape where each row consists of d boxes, except the last which consists of $d-1$ boxes, and the rows overlap in exactly in one box; see Figure 2.3.

In Chapter 2 we introduce a new subposet of the partition lattice on n elements. Given a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n, that is, a list of positive integers such that $n = c_1 + \cdots + c_k$, let $\beta(\vec{c})$ denote the number of permutations in the symmetric group on n elements where the descents appear at the positions c_1, $c_1 + c_2$, $\ldots$, $c_1 + \cdots + c_{k-1}$.

We define the subposet $\Pi^{\bullet}_{\vec{c}}$ of the partition parameterized by a composition $\vec{c}$. The first result, extending Stanley's result, is that the Möbius function of the partition poset $\Pi^{\bullet}_{\vec{c}}$ is given up to a sign by the permutation statistic $\beta(\vec{c})$. Next, we consider the order complex of the partition poset $\Pi^{\bullet}_{\vec{c}}$. We obtain an extension of Wachs' result, namely, the order complex is a wedge of $\beta(\vec{c})$ spheres of dimension $k-2$.

Our proof method differs from that of Wachs. We do not find a shelling of the order complex. Instead, we lift the homotopy type from another simplicial complex consisting of ordered set partitions. To determine the homotopy type of this simplicial complex is more straightforward. It follows from the classical EL-labeling of the Boolean algebra so that this simplicial complex is shellable. Thus its homotopy type is a wedge of spheres. The lifting method is Quillen's Fiber Lemma, which in this case is straightforward to verify the lemma's condition.

The symmetric group acts on the partition poset $\Pi^{\bullet}_{\vec{c}}$. Hence the symmetric group acts on the top homology of the order complex of the partition poset $\Pi^{\bullet}_{\vec{c}}$. Here we are able to use an equivariant version of Quillen's Fiber Lemma to determine this representation. The result is that the representation is again a Specht module of a skew shape. This shape has c_i boxes in the ith row and every pair of adjacent rows overlap in one square. This extends the results of Calderbank, Hanlon and Robinson.

A different partition poset was introduced in [13]. Namely, we can view the d-divisible partition poset as the filter in the partition lattice generated by all partitions where each block size is d. Recall that a filter is a subset of a poset which is closed under going up in the partial order of the poset. It is also known as an upper order ideal. Recall any partition in the d-divisible partition lattice can be obtained from a partition where each block size is d and then by joining blocks together. Hence it is natural to pick an integer partition of n, that is, a collection of positive integers whose sum is n, and study the filter in the partition lattice generated by partitions whose block sizes are given by the integer partition.

In general, this is a hard problem for a general integer partition. Ehrenborg and Readdy introduced the notion of a knapsack partition for which they were able to study the Möbius function of the associated filter in the partition lattice [13]. The name knapsack was inspired by the knapsack system appearing in cryptography. An integer partition is *knapsack* if every sum of parts of the partition is distinct. As an example $(4, 2, 1, 1)$ is not knapsack since $4 = 2 + 1 + 1$, but $(5, 2, 1, 1)$ and $(4, 2, 1)$ are knapsack. This notion is similar to the NES partitions studied by Kozlov [21].

Ehrenborg and Readdy were able to determine the Möbius function of the associated filter of a knapsack partition as a sum of descent set statistics $\beta(\vec{c})$ where the composition runs over a given set. In Section 2.8, 2.9 and 2.10, we continue the study of these filters. We show that their topology is what you expect, namely the homotopy type of the order complex of the filter is a wedge of spheres, where each sphere has the same dimension. Furthermore, the number of spheres is given by the Ehrenborg–Readdy Möbius computation. Our proof again relies on Quillen's Fiber Lemma. That is, we translate the problem to that studying of a certain simplicial complex of ordered set partitions. This complex can be viewed as the union of the earlier studied simplicial complexes; see Figures 2.1, 2.2 and 2.4. However, it is harder this time to determine the homotopy type of these complexes, and we use Forman's

Discrete Morse Theory [16, 17].

Next we use the equivariant Quillen's Fiber Lemma to determine the action on the top homology of the order complex of the filter. It is the corresponding direct sum of Specht modules.

We end the chapter with questions and directions for further research.

1.13 Pattern Avoidance

For $x_1, x_2, \ldots, x_k$ distinct real values, define $\Pi(x_1, x_2, \ldots, x_k)$ to be the unique permutation σ in the symmetric group $\mathfrak{S}_k$ such that $x_i < x_j$ if and only if $\sigma_i < \sigma_j$ for all indices $1 \leq i < j \leq k$.

Let wt be a real-valued *weight function* on the symmetric group $\mathfrak{S}_{m+1}$. Similarly, let wt_1 and wt_2 be a real-valued *initial weight function*, respectively, a real-valued *final weight function* on the symmetric group $\mathfrak{S}_m$. We extend these three weight functions to the symmetric group $\mathfrak{S}_n$ for $n \geq m$ by defining

$$\begin{aligned} \mathrm{Wt}(\pi) &= \mathrm{wt}_1(\Pi(\pi_1, \pi_2, \ldots, \pi_m)) \\ &\quad \cdot \prod_{i=1}^{n-m} \mathrm{wt}(\Pi(\pi_i, \pi_{i+1}, \ldots, \pi_{i+m})) \\ &\quad \cdot \mathrm{wt}_2(\Pi(\pi_{n-m+1}, \pi_{n-m+2}, \ldots, \pi_n)). \end{aligned}$$

In other words, the *weight of a permutation* π in $\mathfrak{S}_n$ is the product of the initial weight function wt_1 applied to the m first entries of π with the product of the weight function wt applied to every segment of π of length $m+1$ with the final weight function wt_2 applied to the m last entries of π. We are interested in the quantity

$$\alpha_n = \sum_{\pi \in \mathfrak{S}_n} \mathrm{Wt}(\pi),$$

and the study of α_n is continued in Section 1.14.

For two permutations $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$, we say that π *consecutively avoids* σ if there are no consecutive indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$ such that for all indexes $p < q$ we have $\pi_{i_p} < \pi_{i_q}$ if and only if $\sigma_p < \sigma_q$. Then the number of permutations in $\mathfrak{S}_n$ which consecutively avoid the σ (called the *pattern* σ) is denoted by $\alpha_n(\sigma)$. Furthermore for a subset S of permutations of $\mathfrak{S}_k$, let $\alpha_n(S)$ denote the number of permutations in $\mathfrak{S}_n$ which consecutively avoid all the patterns in S.

If we remove the condition that the indexes $i_1, i_2, \ldots, i_k$ are required to be consecutive, we have classical pattern avoidance, that is, for two permutations $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$, we say that π *avoids* σ if there are no indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$ such that for all indexes $p < q$ we have $\pi_{i_p} < \pi_{i_q}$ if and only if $\sigma_p < \sigma_q$. Here $\alpha_n(S)$ denotes the number of permutations in $\mathfrak{S}_n$ which avoid all the patterns in S. The famous Stanley–Wilf conjecture states that for a non-empty set S there is a positive constant c such that $\alpha_n(S) \leq c^n$. This conjecture was settled by Marcus and Tardos [23].

Returning to the topic of consecutive pattern avoidance, Warlimont [35] conjectured that for consecutive pattern avoidance, the quantity $\alpha_n(\sigma)$ is given by $c \cdot \lambda^n \cdot n!$ where c and λ are positive constants. Elizalde and Noy [15] proved a weaker version of this result, that is,

$$\lim_{n\to\infty} \sqrt[n]{\frac{\alpha_n(\sigma)}{n!}} = \lambda.$$

For all permutations $\sigma \in \mathfrak{S}_m$, let the initial weight function $wt_1(\sigma)$ and the final weight function $wt_2(\sigma)$ be 1. For the forbidden pattern set $S \in \mathfrak{S}_{m+1}$, let the weight function $\mathrm{wt}(\sigma) = 1$ if $\sigma \ni S$ and $\mathrm{wt}(\sigma) = 0$ otherwise. Then a permutation π consecutively avoids the patterns in S if and only if $\mathrm{Wt}(\pi) = 1$. Hence, in this special case, $\alpha_n(S)$ is the quantity α_n which we are interested. The methods of Ehrenborg, Kitaev and Perry [11] to study the asymptotics of consecutive pattern avoidance by considering the spectrum of operators on $L^2([0,1]^m)$ naturally extends to this more general setting of weights on permutations.

Define the function χ on the $(m+1)$-dimensional unit cube $[0,1]^{m+1}$ by $\chi(x) = \mathrm{wt}(\Pi(x))$. Similarly, define the two functions κ and μ on the m-dimensional unit cube $[0,1]^m$ by $\kappa(x) = \mathrm{wt}_1(\Pi(x))$ and $\mu(x) = \mathrm{wt}_2(\Pi(x))$, respectively. In fact, only for a point x with all of its coordinates different, the functions χ, κ and μ are defined. The event that the point x has the same coordinates occurs on a set of measure zero and hence can be ignored.

Next define the operator T on the space $L^2([0,1]^m)$ by

$$T(f(x_1,\ldots,x_m)) = \int_0^1 \chi(t,x_1,\ldots,x_m) \cdot f(t,x_1,\ldots,x_{m-1})\,dt. \tag{1.1}$$

Then the adjoint operator T^* defined by the relation $(f, T^*(g)) = (T(f), g)$ is

$$T^*(f(x_1,\ldots,x_m)) = \int_0^1 \chi(x_1,\ldots,x_m,u) \cdot f(x_2,\ldots,x_m,u)\,du,$$

where the inner product on a Hilbert space $L^2([0,1]^m)$ is defined by

$$(f,g) = \int_{[0,1]^m} f(x_1,\ldots,x_m) \cdot \overline{g(x_1,\ldots,x_m)}\,dx_1 \cdots dx_m.$$

The *spectrum* of an operator T is all the values λ such that $T - \lambda \cdot I$ is not an invertible operator. Then we have the generalization of the main result in [11].

Theorem 1.13.1. *The nonzero spectrum of the associated operator T consists of discrete eigenvalues of finite multiplicity which may accumulate only at 0. Furthermore, let r be a positive real number such that there is no eigenvalue of T with modulus r and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T greater in modulus than r. Assume that $\lambda_1, \ldots, \lambda_k$ are simple eigenvalues with associated eigenfunctions φ_i and that the adjoint operator T^* has eigenfunctions ψ_i corresponding the eigenvalues λ_i. Then we have the expansion*

$$\alpha_n/n! = \big(T^{n-m}(\kappa), \mu\big) = \sum_{i=1}^{k} \frac{(\varphi_i, \mu) \cdot \big(\kappa, \overline{\psi_i}\big)}{\big(\varphi_i, \overline{\psi_i}\big)} \cdot \lambda_i^{n-m} + O(r^n). \tag{1.2}$$

Since the adjoint eigenfunction can be determined from the eigenfunction by the symmetry of the weight function, we need to determine only the eigenfunction φ for each eigenvalue in order to compute the constant in each term.

1.14 Descent Pattern Avoidance

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, define its *descent word* to be $u(\pi) = u_1u_2\cdots u_{n-1}$ where $u_i = \mathbf{a}$ if $\pi_i < \pi_{i+1}$ and $u_i = \mathbf{b}$ if $\pi_i > \pi_{i+1}$, that is, if π has an ascent or descent at position i, then u_π has an $\mathbf{a}$ or $\mathbf{b}$, respectively, at position i.

Let wt be a weight function on $\mathbf{ab}$-words of length m, that is, the set $\{\mathbf{a}, \mathbf{b}\}^m$. Similarly, let wt_1 and wt_2 be weight functions on $\mathbf{ab}$-words of length $m-1$. We extend this weight function to words of length n for $n \geq m$ by letting

$$\mathrm{Wt}(v_1 \cdots v_n) = \mathrm{wt}_1(v_1 \cdots v_{m-1}) \cdot \prod_{i=1}^{n-m+1} \mathrm{wt}(v_i \cdots v_{i+m-1}) \cdot \mathrm{wt}_2(v_{n-m+2} \cdots v_n).$$

The weight function $\mathrm{Wt}(\pi)$ of permutations π can be extended by $\mathrm{Wt}(\pi) = \mathrm{Wt}(u(\pi))$.

If we can write $x = w_1 \cdot w_2 \cdots w_t$, we say that for $1 \leq i \leq t$ the word x has the word w_i as a *factor* where the dot denotes concatenation. Let U be a collection of $\mathbf{ab}$-words of length m, that is, U is a subset of $\{\mathbf{a}, \mathbf{b}\}^m$. Define $S(U)$ by

$$S(U) = \{\sigma \in \mathfrak{S}_{m+1} \ : \ u(\sigma) \in U\}.$$

It is clear that a descent word $u(\pi)$ that avoids the descent patterns in U is equivalent to the permutation π that avoids the consecutive patterns in $S(U)$. Hence descent pattern avoidance is a special case of consecutive pattern avoidance.

For an $\mathbf{ab}$-word u of length $m-1$ define the *descent polytope* P_u to be the subset of the unit cube $[0,1]^m$ such that

$$P_u = \{(x_1, \ldots, x_m) \in [0,1]^m \ : \ x_i \leq x_{i+1} \text{ if } u_i = \mathbf{a} \text{ and } x_i \geq x_{i+1} \text{ if } u_i = \mathbf{b}\}.$$

Observe that the unit cube $[0,1]^m$ is the union of the 2^{m-1} descent polytopes. Similar to [11, Proposition 4.3 and Corollary 4.4] we have the next proposition.

Proposition 1.14.1. *Let T be the operator associated with a weighted descent pattern avoidance and k an integer such that $0 \leq k \leq m-1$. Let u be an* ***ab****-word of length $m-1$ and f a function in $L^2([0,1]^m)$. Then the function $T^k(f)$ restricted to the descent polytope P_u only depends on the variables x_1 through x_{m-k}.*

Hence, the eigenfunctions have a special form:

Corollary 1.14.2. *If φ is an eigenfunction of T associated to a non-zero eigenvalue then the eigenfunction φ restricted to any descent polytope P_u only depends on the variable x_1.*

Let V be the subspace of $L^2([0,1]^m)$ consisting of all functions f such that the restriction $f|_{P_u}$ only depends on the variable x_1 for all words u of length $m-1$. Then for $f \in V$ the function $T(f)$ is described as follows. For an **ab**-word u of length $m-2$ and $y \in \{\mathbf{a}, \mathbf{b}\}$ we have

$$T(f)|_{P_{uy}} = \int_0^{x_1} \mathrm{wt}(\mathbf{a}uy) \cdot f(t)|_{P_{\mathbf{a}u}}\, dt + \int_{x_1}^1 \mathrm{wt}(\mathbf{b}uy) \cdot f(t)|_{P_{\mathbf{b}u}}\, dt.$$

In light of Corollary 3.3.5 to solve the eigenvalue problem for the operator $T : L^2([0,1]^m) \longrightarrow L^2([0,1]^m)$, it is enough to solve the eigenvalue problem for the restricted operator $T|_V : V \longrightarrow V$.

1.15 Dissertation results – descent avoidance in permutations

Pattern avoidance in permutations was defined by D. Knuth. For two permutations $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$, we say that π *avoids* σ if there are no indices $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$ such that for all indices $p < q$ we have $\pi_{i_p} < \pi_{i_q}$ if and only if $\sigma_p < \sigma_q$. Furthermore for S a subset of permutations of $\mathfrak{S}_k$, let $\alpha_n(S)$ denote the number of permutations in $\mathfrak{S}_n$ which avoid all of the patterns in S. The famous Stanley–Wilf conjecture states that for a non-empty set S there is a positive constant c such that $\alpha_n(S) \leq c^n$. This conjecture was settled by Marcus and Tardos [23].

Before this conjecture was settled, it generated a lot of interest. See the end of the introduction of [11] and the references therein. Among this body of work, Babson and Steingrímsson [1] introduced the notion of general pattern avoidance. As a special case is consecutive pattern avoidance. It differs from the classical pattern avoidance in the previous paragraph as it requires the indexes $i_1, i_2, \ldots, i_k$ to be consecutive, that is, $i_j = i_1 + j - 1$.

Warlimont [35] conjectured that for consecutive pattern avoidance the asymptotics of the number of permutations in $\mathfrak{S}_n$ consecutively avoiding a pattern σ is given by $c \cdot \lambda^n \cdot n!$, where c and λ are positive constants. A weaker version of this result was proved by Elizalde and Noy [15], that is,

$$\lim_{n \to \infty} \sqrt[n]{\frac{\alpha_n(\sigma)}{n!}} = \lambda.$$

In their paper, Ehrenborg, Kitaev and Perry [11] settled the Warlimont conjecture using spectral methods of integral operators. Their work was inspired by Ehrenborg, Levin and Readdy [12], where they took the discrete problem of counting permutations and translated the problem into a continuous problem which could be solved.

The basic idea from [12] is the following. Instead of counting the number of permutations in $\mathfrak{S}_n$ satisfying a certain condition, compute the probability that a random permutation satisfies the condition. Here random means uniform distribution, that is, each permutation is equally likely with probability $1/n!$. How can we select a random permutation? Choose a point x in the unit cube $[0,1]^n$, that is, pick n real numbers $x_1, \ldots, x_n$ uniformly from the interval $[0,1]$. Construct a permutation π by letting $\pi = \pi_1\pi_2\cdots\pi_n$ be the unique permutation such that for all indexes $p < q$ we have $x_p < x_q$ if and only if $\pi_p < \pi_q$. This guarantees each permutation to be selected equally likely. In fact, for a point x with all its coordinates different, define $\Pi(x)$ to be the unique permutation corresponding to the point x. Observe that the event that the point x has two equal coordinates has measure zero, and hence can be ignored.

A geometric view is that the unit cube $[0,1]^n$ is subdivided into $n!$ simplicies by intersecting the unit cube with the $\binom{n}{2}$ hyperplanes $x_p = x_q$. By symmetry we know that each simplex has the same volume and hence each permutation is equally likely.

The main idea of Ehrenborg, Kitaev and Perry [11] is for S a subset of $\mathfrak{S}_{m+1}$, define the function χ on $[0,1]^{m+1}$ by

$$\chi(x) = \begin{cases} 1 & \text{if } \Pi(x) \not\in S, \\ 0 & \text{otherwise.} \end{cases}$$

Next define the integral operator T on the space $L^2([0,1]^m)$ by

$$T(f(x_1, \ldots, x_m)) = \int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(t, x_1, \ldots, x_{m-1}) dt.$$

Then the probability that a permutation π consecutive avoids the set S is given by

$$\alpha_n(S)/n! = \left(\mathbf{1}, T^{n-m}(\mathbf{1})\right).$$

Hence to understand the asymptotic behavior of this expression as n grows large, one has to understand the spectrum, that is, the eigenvalues of the operator T. Since T^m is a Hilbert-Schmidt operator, the operator T^m is compact. Even though T might not be compact, by a result in functional analysis, the spectral analysis of T can still be done; see [9, Chapter VII, Theorem 4.6]. In the paper [11] the authors obtain an asymptotic expansion theorem. This result gives better estimates on the quantity $\alpha_n(S)/n!$ as more eigenvalues are known. Especially, the asymptotic behavior is determined by the largest eigenvalue.

In Chapter 3 we introduce the notion of weighted patterns. The standard consecutive pattern avoidance corresponds to the case when all the weights are either zero or one. We show that a similar asymptotic expansion holds. However, as shown in [11] with various examples, it is a non-trivial task to determine the eigenvalues of one of these operators. In fact, it is hard even to determine an equation for the eigenvalues.

We study the special case of weighted descent patterns, that is, we weight each permutation according to the descent set of the permutation. More specifically, the weight is determined by the factors of a given length in the descent word. In this case, we determine the equation of the eigenvalues of the associated operator. The drawback is that this is a transcendental equation, and thus is hard to solve exactly.

We give two examples of this theory. First, we study the number of permutations that have no triple ascents and no triple descents. The associated equation contains four exponential terms and hence is only solvable numerically. Similarly, we also study the number of permutations avoiding the descent factors of **aba** and **bab**, that is, no consecutive length four pattern is allowed to be alternating. Yet again we obtain a transcendental equation with four exponential terms and solve it for the largest eigenvalue numerically.

Next we consider a weighted pattern of length three, where we weight the descent factors **aa** by zero and **bb** by 2. In other words, we are looking permutations where we avoid double ascents and we weight each permutation by 2 to the number of double descents. In this case the associated operator only has one non-zero eigenvalue. This eigenvalue is 1. Hence we obtain that the sum over the weights of these permutations is given by

$$\alpha_n/n! = c + O(r^n),$$

where r is arbitrary small but still positive since there are no other eigenvalues to give a lower bound for r. Furthermore, we determine the constant c to be $e - 2 + 1/e$.

Thus then we have the result $\alpha_n = (e - 2 + 1/e) \cdot n! + O(r^n \cdot n!)$. However, the error term could still be large. By considering generating functions, we are able to determine the error term. The error term comes out to be the smallest possible. Note that α_n is an integer by definition, and $(e - 2 + 1/e) \cdot n!$ is a transcendental number and hence not an integer for a given n. The error term comes out to be the smallest possible, that is, α_n is the closest integer to $(e - 2 + 1/e) \cdot n!$.

We also end the chapter with questions and directions for further research.

1.16 Publications

Chapter 2 has been submitted for the publication in *Journal of Algebraic Combinatorics*. An extended abstract of Chapter 2 was presented of the 23rd International Conference on Formal Power Series and Algebraic Combinatorics in Reykjavik, Iceland, June 2011 [10]. Chapter 3 has been submitted to *Journal of Combinatorial Theory, Series A*.

Chapter 2 The topology of restricted partition posets

2.1 Introduction

The study of partitions with restrictions on their block sizes began in the dissertation by Sylvester [31], who studied the poset Π_n^2 of partitions of $\{1, 2, \cdots, n\}$ where every block has even size. He proved that the Möbius function of this poset is given by $\mu(\Pi_n^2 \cup \{\hat{0}\}) = (-1)^{n/2} \cdot E_{n-1}$, where E_n denotes the nth Euler number. Recall that the nth Euler number enumerates the number of alternating permutations, that is, permutations $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group $\mathfrak{S}_n$ such that $\alpha_1 < \alpha_2 > \alpha_3 < \alpha_4 > \cdots$. Stanley [30] generalized this result to the d-divisible partition lattice Π_n^d, that is, the collection of partitions of $\{1, 2, \ldots, n\}$ where each block size is divisible by d. He found that the Möbius function $\mu(\Pi_n^d \cup \{\hat{0}\})$ is, up to the sign $(-1)^{n/d}$, the number of permutations in $\mathfrak{S}_{n-1}$ with descent set $\{d, 2d, \ldots, n-d\}$, in other words, the number of permutations with descent composition $(d, \ldots, d, d-1)$.

The enumerative results for the d-divisible partition lattice were extended homologically. Calderbank, Hanlon and Robinson [6] considered the action of the symmetric group $\mathfrak{S}_{n-1}$ on the top homology group of the order complex of $\Pi_n^d - \{\hat{1}\}$. They showed this action is the Specht module on the border strip corresponding the composition $(d, \ldots, d, d-1)$. Wachs [32] showed that the other reduced homology groups vanish. She presented an EL-labeling for the d-divisible partition lattice and hence as a corollary obtained that the homotopy type is a wedge of spheres of dimension $n/d - 2$. She then gave a more constructive proof of the representation of the top homology of $\Delta(\Pi_n^d - \{\hat{1}\})$ by exhibiting an explicit isomorphism. She identified cycles in the complex $\Delta(\Pi_n^d - \{\hat{1}\})$ which are the barycentric subdivision of cubes and associated with them polytabloids in the Specht module.

So far we see that the d-divisible partition lattice is closely connected with permutations having the descent composition $(d, \ldots, d, d-1)$. We explain this phenomenon in this chapter by introducing pointed partitions. They are partitions where one block is considered special, called the pointed block. We obtain such a partition by removing the element n from its block and making this block the pointed block. We now extend the family of posets under consideration. For each composition $\vec{c} = (c_1, \ldots, c_k)$ of n we define a poset $\Pi_{\vec{c}}^\bullet$ such that the Möbius function $\mu(\Pi_{\vec{c}}^\bullet \cup \{\hat{0}\})$ is the sign $(-1)^k$ times the number of permutations with descent composition $\vec{c}$. Furthermore, we show the order complex of $\Pi_{\vec{c}}^\bullet - \{\hat{1}\}$ is homotopy equivalent to a wedge of spheres of dimension $k-2$. Finally, we show the action of the symmetric group on the top homology group $\widetilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\}))$ is the Specht module corresponding to the composition $\vec{c}$.

Our techniques differ from Wachs' method for studying the d-divisible partition lattice [32] as we do not obtain an EL-labeling of $\Pi_{\vec{c}}^\bullet \cup \{\hat{0}\}$. Instead we apply Quillen's Fiber

Lemma [26] and transform the question into studying a subcomplex $\Delta_{\vec{c}}$ of the complex of ordered partitions. This subcomplex is in fact the order complex of a rank-selected Boolean algebra. Hence $\Delta_{\vec{c}}$ is shellable and its homotopy type is a wedge of spheres. Furthermore, we use an equivariant version of Quillen's Fiber Lemma to conclude that the reduced homology groups $\widetilde{H}_{k-2}(\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}))$ and $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ are isomorphic as $\mathfrak{S}_n$-modules. Finally, to show that the top homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ is isomorphic to the Specht module S^B, we follow Wachs' footsteps by giving an explicit isomorphism between these two $\mathfrak{S}_n$-modules, that is, given a polytabloid $\mathbf{e}_t$ in the Specht module S^B we give an explicit cycle g_α spanning the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$.

Ehrenborg and Readdy [13] introduced the notion of a knapsack partition, which is an integer partition such that no integer can be written as a sum of the parts of the partition in two different ways. They considered the filter in the pointed partition lattice where the generators of the filter have their type given by a knapsack partition. They obtained the Möbius function of this filter is given by a sum of descent set statistics. We extend their results topologically by showing that the associated order complex is a wedge of spheres. The proof follows the same outline as the previous study except that we use discrete Morse theory to determine the homotopy type of the associated complexes of ordered set partitions. Furthermore we obtain that the action of the symmetric group on the top homology is a direct sum of Specht modules.

We end the chapter with open questions for future research.

2.2 Preliminaries

For basic notions concerning partially ordered sets (posets), see Stanley's book [28]. For topological background, see Björner's article [4] and Kozlov's book [20]. For representation theory of the symmetric group, see Sagan [27]. Finally, a good reference for all three areas is Wachs' article [34].

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and for $i \leq j$ let $[i, j]$ denote the interval $\{i, i+1, \ldots, j\}$. A *pointed set partition* π of the set $[n]$ is a pair (σ, Z), where Z is a subset of $[n]$ and $\sigma = \{B_1, B_2, \ldots, B_k\}$ is a partition of the set difference $[n] - Z$. We will write the pointed partition π as

$$\pi = \{B_1, B_2, \ldots, B_k, \underline{Z}\},$$

where we underline the set Z and we write $1358|4|\underline{267}$ as shorthand for $\{\{1,3,5,8\},\{4\},\underline{\{2,6,7\}}\}$. Let $\Pi^{\bullet}_n$ denote the set of all pointed set partitions on the set $[n]$. The set $\Pi^{\bullet}_n$ has a natural poset structure. The cover relation is given by two relations

$$\begin{aligned} \{B_1, B_2, \ldots, B_k, \underline{Z}\} &< \{B_1 \cup B_2, \ldots, B_k, \underline{Z}\}, \\ \{B_1, B_2, \ldots, B_k, \underline{Z}\} &< \{B_2, \ldots, B_k, \underline{B_1 \cup Z}\}. \end{aligned}$$

Lemma 2.2.1. *The poset $\Pi^\bullet_n$ is the intersection lattice of the hyperplane arrangement*

$$\begin{cases} x_i = x_j & 1 \leq i < j \leq n, \\ x_i = 0 & 1 \leq i \leq n. \end{cases}$$

Proof. For the pointed partition $\pi = \{B_1, B_2, \ldots, B_k, \underline{Z}\}$ construct a subspace satisfying the inequalities $x_{j_1} = x_{j_2}$ if j_1 and j_2 belong to the same block of π, and let $x_j = 0$ if j belongs to the block Z. It is straightforward to see that this is a bijection and thus proving the lemma. □

As a corollary to this claim we have that the poset $\Pi^\bullet_n$ is a lattice. Moreover, we call the set Z the *zero set* or the *pointed block.* The first name is motivated by the fact that the set Z corresponds to the variables set to be zero in an element in the intersection lattice.

The lattice $\Pi^\bullet_n$ is isomorphic to the partition lattice Π_{n+1} by the bijection

$$\{B_1, \ldots, B_k, \underline{Z}\} \longmapsto \{B_1, \ldots, B_k, Z \cup \{n+1\}\}.$$

However it is to our advantage to work with pointed set partitions.

For a permutation $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group $\mathfrak{S}_n$ define its *descent set* to be the set

$$\{i \in [n-1] \ : \ \alpha_i > \alpha_{i+1}\}.$$

Subsets of $[n-1]$ are in a natural bijective correspondence with compositions of n. Hence we define the *descent composition* of the permutation α to be the composition

$$\mathrm{Des}(\alpha) = (s_1, s_2 - s_1, s_3 - s_2, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1}),$$

where the descent set of α is the set $\{s_1 < s_2 < \cdots < s_{k-1}\}$. We define a *pointed integer composition* $\vec{c} = (c_1, \ldots, c_k)$ to be a list of positive integers $c_1, \ldots, c_{k-1}$ and a non-negative integer c_k with $c_1 + \cdots + c_k = n$. Note that the only part allowed to be 0 is the last part. When the last part is positive we refer to $\vec{c}$ as a *composition.* Let $\beta(\vec{c})$ denote the number of permutations in $\alpha \in \mathfrak{S}_n$ with descent composition $\vec{c}$ for $c_k > 0$. If $c_k = 0$, let $\beta(\vec{c}) = 0$ for $k \geq 2$ and $\beta(\vec{c}) = 1$ for $k = 1$.

Define the (right) weak Bruhat order on the symmetric group $\mathfrak{S}_n$ by the cover relation

$$\alpha \prec \alpha \circ (i, i+1)$$

if $\alpha_i < \alpha_{i+1}$. Observe that the smallest element is the identity element $12 \cdots n$ and the largest is $n \cdots 21$.

On the set of compositions of n we define an order relation by letting the cover relation be adding adjacent entries, that is,

$$(c_1, \ldots, c_i, c_{i+1}, \ldots, c_k) \prec (c_1, \ldots, c_i + c_{i+1}, \ldots, c_k).$$

Observe that this poset is isomorphic to the Boolean algebra B_n on n elements and the maximal and minimal elements are the two compositions (n) and $(1, \ldots, 1, 0)$.

An *integer partition* λ of a non-negative integer n is a multiset of positive integers whose sum is n. We will indicate multiplicities with a superscript. Thus $\{5, 3, 3, 2, 1, 1, 1\} = \{5, 3^2, 2, 1^3\}$ is a partition of 16. A *pointed integer partition* (λ, m) of n is pair where m is a non-negative integer and λ is a partition of $n - m$. We write this as $\{\lambda_1, \ldots, \lambda_p, \underline{m}\}$ where $\lambda = \{\lambda_1, \ldots, \lambda_p\}$ is the partition and m is the pointed part. This notion of pointed integer partition is related to pointed set partitions by defining the type of a pointed set partition $\pi = \{B_1, B_2, \ldots, B_k, \underline{Z}\}$ to be the pointed integer partition

$$\mathrm{type}(\pi) = \{|B_1|, |B_2|, \ldots, |B_k|, \underline{|Z|}\}.$$

Similarly, the type of a composition $\vec{c} = (c_1, \ldots, c_k)$ is the pointed integer partition

$$\mathrm{type}(\vec{c}) = \{c_1, \ldots, c_{k-1}, \underline{c_k}\}.$$

Recall that the *Möbius function* of a poset P is defined by the initial condition $\mu(x, x) = 1$ and the recursion $\mu(x, z) = -\sum_{x \leq y < z} \mu(x, y)$. When the poset P has a minimal element $\hat{0}$ and a maximal element $\hat{1}$, we call the quantity $\mu(P) = \mu(\hat{0}, \hat{1})$ the Möbius function of the poset P.

For a poset P define its *order complex* to be the simplicial complex $\Delta(P)$ where the vertices of the complex $\Delta(P)$ are the elements of the poset P and the faces are the chains in poset. In other words, the order complex of P is given by

$$\Delta(P) = \{\{x_1, x_2, \ldots, x_k\} \ : \ x_1 < x_2 < \cdots < x_k, \ x_1, \ldots, x_k \in P\}.$$

As a consequence of Hall's Theorem [28, Proposition 3.8.5] we have that the reduced Euler characteristic of the order complex $\Delta(P)$, that is, $\widetilde{\chi}(P)$, is given by the Möbius function $\mu(P \cup \{\hat{0}, \hat{1}\})$, where $P \cup \{\hat{0}, \hat{1}\}$ denotes the poset P with new minimal and maximal elements adjoined.

Shelling and discrete Morse theory are two powerful tools for determining the homotopy type of simplicial complexes. See [16, 17] and [20] for more details. We begin by a review of shellings. A simplicial complex is *pure* if all its facets (maximal faces) have the same dimension. A pure simplicial complex is *shellable* if either it is a collection of disjoint vertices or there is an ordering on the facets $F_1, \ldots, F_m$ such that for $2 \leq i \leq m$ the intersection $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is a pure subcomplex of F_i of dimension $\dim(F_i) - 1$. A facet F_i is called *spanning* if the intersection $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is the boundary of F_i.

Theorem 2.2.2. *A shellable simplicial complex of dimension d is homotopy equivalent to a wedge of b d-dimensional spheres where b is the number of spanning facets in the shelling.*

Next we review discrete Morse theory.

Definition 2.2.3. *A* partial matching *in a poset P is a partial matching in the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ such that $(x, y) \in M$ implies $x \prec y$ and each $x \in P$ belongs to at most one element of M. For a pair (x, y) in the matching M we write $x = d(y)$ and $y = u(x)$, where d and u stand for down and up, respectively.*

Definition 2.2.4. *A partial matching M on P is* acyclic *if there does not exist a cycle*

$$z_1 \prec u(z_1) \succ z_2 \prec u(z_2) \succ \cdots \succ z_n \prec u(z_n) \succ z_1,$$

of elements in P with $n \geq 2$ and all $z_i \in P$ distinct. Given an acyclic matching, the unmatched elements are called critical.

We need the following version of the main theorem of discrete Morse theory.

Theorem 2.2.5. *Let Γ be a simplicial complex with an acyclic matching on its face poset, where the empty face (set) is included. Assume that there are b critical cells and that they all have the same dimension k. Then the simplicial complex Γ is homotopy equivalent to a wedge of b spheres of dimension k.*

For the remainder of this section, we restrict ourselves to considering compositions of n where the last part is positive. This restriction will also hold in Sections 2.6, 2.7, 2.9 and 2.10. Such a composition lies in the interval from $(1, \ldots, 1)$ to (n). This interval is isomorphic to the Boolean algebra B_{n-1} which is a complemented lattice. Hence for such a composition $\vec{c}$ there exists a complementary composition $\vec{c}'$ such that $\vec{c} \wedge \vec{c}' = (1, \ldots, 1)$ and $\vec{c} \vee \vec{c}' = (n)$. As an example, the complement of the composition $(1, 3, 1, 1, 4) = (1, 1+1+1, 1, 1, 1+1+1+1)$ is obtained by exchanging commas and plus signs, that is, $(1+1, 1, 1+1+1+1, 1, 1, 1) = (2, 1, 4, 1, 1, 1)$. Note that the complementary composition has $n - k + 1$ parts.

For a composition $\vec{c} = (c_1, \ldots, c_k)$ define the intervals $R_1, \ldots, R_k$ by $R_i = [c_1 + \cdots + c_{i-1} + 1, c_1 + \cdots + c_i]$. Define the subgroup $\mathfrak{S}_{\vec{c}}$ of the symmetric group $\mathfrak{S}_n$ by

$$\mathfrak{S}_{\vec{c}} = \mathfrak{S}_{R_1} \times \cdots \times \mathfrak{S}_{R_k}.$$

Let $K_1, \ldots, K_{n-k+1}$ be the corresponding intervals for the complementary composition $\vec{c}\,'$. Define the subgroup $\mathfrak{S}'_{\vec{c}}$ by

$$\mathfrak{S}'_{\vec{c}} = \mathfrak{S}_{\vec{c}\,'} = \mathfrak{S}_{K_1} \times \cdots \times \mathfrak{S}_{K_{n-k+1}}.$$

A *border strip* is a connected skew shape which does not contain a 2 by 2 square [29, Section 7.17]. For each composition $\vec{c}$ there is a unique border strip B that has k rows and the ith row from below consists of c_i boxes. If we label the n boxes of the border strip from southwest to northeast, then the intervals $R_1, \ldots, R_k$ correspond to the rows and the intervals $K_1, \ldots, K_{n-k+1}$ correspond to the columns. Furthermore, the group $\mathfrak{S}_{\vec{c}}$ is the row stabilizer and the group $\mathfrak{S}'_{\vec{c}}$ is the column stabilizer of the border strip B.

2.3 Two subposets of the pointed partition lattice

We now define the first poset central to this chapter.

Definition 2.3.1. *For $\vec{c}$ a composition of n, let $\Pi^\bullet_{\vec{c}}$ be the subposet of the pointed partition lattice $\Pi^\bullet_n$ described by*

$$\Pi^\bullet_{\vec{c}} = \left\{\pi \in \Pi^\bullet_n \ : \ \exists \vec{d} \geq \vec{c}, \ \text{type}(\pi) = \text{type}(\vec{d})\right\}.$$

In other words, the poset $\Pi^\bullet_{\vec{c}}$ consists of all pointed set partitions such that their type is the type of some composition $\vec{d}$ which is greater or equal to the composition $\vec{c}$ in the composition order.

Example 2.3.2. *Consider the composition $\vec{c} = (d, \ldots, d, d-1)$ of the integer $n = d \cdot k - 1$. For a composition to be greater than or equal to $\vec{c}$, it is equivalent to all of its parts must be divisible by d except the last part which is congruent to $d-1$ modulo d. Hence $\Pi^\bullet_{\vec{c}}$ consists of all pointed set partitions where the block sizes are divisible by $\vec{d}$ except the zero block whose size is congruent to $d-1$ modulo d. Hence the poset $\Pi^\bullet_{\vec{c}}$ is isomorphic to the d-divisible partition lattice Π^d_{n+1}.*

Example 2.3.3. *We note that $\Pi^\bullet_{\vec{c}} \cup \{\hat{0}\}$ is in general not a lattice. Consider the composition $\vec{c} = (1, 1, 2, 1)$ and the four pointed set partitions*

$$\pi_1 = 1|2|34|\underline{5}, \ \ \pi_2 = 2|5|34|\underline{1}, \ \ \pi_3 = 34|\underline{125} \quad \textit{and} \quad \pi_4 = 2|\underline{1345}$$

in $\Pi^\bullet_{(1,1,2,1)}$. In the pointed partition lattice $\Pi^\bullet_5$ we have that $\pi_1, \pi_2 < 2|34|\underline{15} < \pi_3, \pi_4$. Since the pointed set partition $2|34|\underline{15}$ does not belong to $\Pi^\bullet_{(1,1,2,1)}$, we conclude that $\Pi^\bullet_{(1,1,2,1)} \cup \{\hat{0}\}$ is not a lattice.

We now turn our attention to filters in the pointed partition lattice $\Pi^\bullet_n$ that are generated by a pointed knapsack partition. These filters were introduced in [13].

Recall that we view an integer partition λ as a multiset of positive integers. Let $\lambda = \{\lambda_1^{e_1}, \ldots, \lambda_q^{e_q}\}$ be an integer partition, where we assume that the λ_i's are distinct. If all the $(e_1 + 1) \cdots (e_q + 1)$ integer linear combinations

$$\left\{\sum_{i=1}^{q} f_i \cdot \lambda_i \ : \ 0 \leq f_i \leq e_i\right\}$$

are distinct, we call λ a *knapsack partition.* A pointed integer partition $\{\lambda, \underline{m}\}$ is called a *pointed knapsack partition* if the partition λ is a knapsack partition.

This definition was introduced by Ehrenborg–Readdy [13]. Their motivation was to compute the Möbius function of filters generated by knapsack partitions in the pointed partition lattice; see Corollary 2.8.7. However, earlier Kozlov [21] introduced the same notion under the name no equal-subsets sums (NES). His motivation was the same, except he studied the topology of filters in the partition lattice.

Definition 2.3.4. *For a pointed knapsack partition $\{\lambda, \underline{m}\} = \{\lambda_1, \lambda_2, \dots, \lambda_k, \underline{m}\}$ of n define the subposet $\Pi^\bullet_{\{\lambda,\underline{m}\}}$ to be the filter of $\Pi^\bullet_n$ generated by all pointed set partitions of type $\{\lambda, \underline{m}\}$.*

Example 2.3.5. *Observe that $\lambda = \{d, d, \dots, d\}$ is a knapsack partition. Hence all the block sizes in the filter $\Pi^\bullet_{\{\lambda,\underline{m}\}}$ are divisible by d, except the pointed block. Hence for the pointed knapsack partition $\{d, d, \dots, d, \underline{d-1}\}$ we obtain the d-divisible partition lattice again, as in Example 2.3.2.*

Note that $\Pi^\bullet_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}$ is indeed a lattice since it inherits the join operation from $\Pi^\bullet_n$. The fact the meet exists is due to Proposition 3.3.1 in [28].

2.4 The simplicial complex of ordered set partitions

An *ordered set partition* τ of set S is a list of blocks $(C_1, C_2, \dots, C_m)$ where the blocks are subsets of the set S satisfying:

(i) All blocks except possibly the last block are non-empty, that is, $C_i \neq \emptyset$ for $1 \leq i \leq m-1$.

(ii) The blocks are pairwise disjoint, that is, $C_i \cap C_j = \emptyset$ for $1 \leq i < j \leq m$.

(iii) The union of the blocks is S, that is, $C_1 \cup \cdots \cup C_m = S$.

To distinguish from pointed partitions we write 36-127-8-45 for $(\{3, 6\}, \{1, 2, 7\}, \{8\}, \{4, 5\})$. The *type* of an ordered set partitions, $\text{type}(\tau)$, is the composition $(|C_1|, |C_2|, \dots, |C_m|)$.

Let Δ_n denote the collection of all ordered set partitions of the set $[n]$. We view Δ_n as a simplicial complex. The ordered set partition $\tau = (C_1, C_2, \dots, C_m)$ forms an $(m-2)$-dimensional face. It has $m-1$ facets, which are $\tau = (C_1, \dots, C_i \cup C_{i+1}, \dots, C_m)$ for $1 \leq i \leq m-1$. The empty face corresponds to the ordered partition $([n])$. The complex Δ_n has $2^n - 1$ vertices that are of the form (C_1, C_2) where $C_1 \neq \emptyset$. Moreover there are $n!$ facets corresponding to permutations in the symmetric group $\mathfrak{S}_n$, that is, for a permutation $\alpha = \alpha_1 \cdots \alpha_n$, the associated facet is $(\{\alpha_1\}, \{\alpha_2\}, \dots, \{\alpha_n\}, \emptyset)$.

The permutahedron is the $(n-1)$-dimensional polytope obtained by taking the convex hull of the $n!$ points $(\alpha_1, \dots, \alpha_n)$ where $\alpha = \alpha_1 \cdots \alpha_n$ ranges over all permutations in the symmetric group $\mathfrak{S}_n$. Let P_n denote the boundary complex of the dual of the $(n-1)$-dimensional permutahedron. Since the permutahedron is a simple polytope the complex P_n is a simplicial complex homeomorphic to an $(n-2)$-dimensional sphere. Another view is that P_n is the barycentric subdivision of the boundary of the n-dimensional simplex. Note that the link of the vertex $([n], \emptyset)$ in the complex Δ_n is the complex P_n. In fact, the complex Δ_n is the cone of P_n.

For a permutation $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group $\mathfrak{S}_n$ and a composition $\vec{c} = (c_1, \ldots, c_k)$ of n, define the ordered partition

$$\begin{aligned} \sigma(\alpha, \vec{c}) &= (\{\alpha_j \ : \ j \in R_i\})_{1 \leq i \leq k} \\ &= (\{\alpha_1, \ldots, \alpha_{c_1}\}, \{\alpha_{c_1+1}, \ldots, \alpha_{c_1+c_2}\}, \ldots, \{\alpha_{c_1+\cdots+c_{k-1}+1}, \ldots, \alpha_n\}). \end{aligned}$$

We write $\sigma(\alpha)$ when it is clear from the context what the composition $\vec{c}$ is.

For a composition $\vec{c}$ define the subcomplex $\Delta_{\vec{c}}$ to be

$$\Delta_{\vec{c}} = \{\tau \in \Delta_n \ : \ \vec{c} \leq \text{type}(\tau)\}.$$

This complex has all of its facets of type $\vec{c}$. Especially, each facet has the form $\sigma(\alpha, \vec{c})$ for some permutation α. As an example, note that $\Delta_{(1,1,\ldots,1)}$ is the complex P_n.

Lemma 2.4.1. *If the pointed composition $\vec{c} = (c_1, \ldots, c_k)$ ends with 0, then the simplicial complex $\Delta_{\vec{c}}$ is a cone over the complex $\Delta_{(c_1,\ldots,c_{k-1})}$ with apex $([n], \emptyset)$ and hence contractible.*

However for a facet F in $\Delta_{\vec{c}}$ there are $\vec{c}! = c_1! \cdots c_k!$ permutations that map to it by the function σ. Hence let $\sigma^{-1}(F)$ denote the smallest permutation α with respect to the weak Bruhat order that gets mapped to the facet F. This permutation satisfies the inequalities

$$\alpha_{c_1+\cdots+c_i+1} < \cdots < \alpha_{c_1+\cdots+c_{i+1}}$$

for $0 \leq i \leq k-1$. Furthermore, the descent composition of the permutation $\sigma^{-1}(F)$ is greater than or equal to the composition $\vec{c}$, that is, $\text{Des}(\sigma^{-1}(F)) \geq \vec{c}$.

Theorem 2.4.2. *Let $\vec{c}$ be a composition not ending with a zero. Then the simplicial complex $\Delta_{\vec{c}}$ is shellable. The spanning facets are of the form $\sigma(\alpha)$ where α ranges over all permutations in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{c}$, that is,* $\text{Des}(\alpha) = \vec{c}$. *Hence the complex $\Delta_{\vec{c}}$ is homotopy equivalent to a wedge of $\beta(\vec{c})$ spheres of dimension $k-2$.*

Proof. Let S be the subset of $[n-1]$ associated with the composition $\vec{c}$, that is, $S = \{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\}$. A facet $(C_1, \ldots, C_k)$ of the complex $\Delta_{\vec{c}}$ corresponds to the maximal chain

$$\emptyset \subseteq C_1 \subseteq C_1 \cup C_2 \subseteq \cdots \subseteq C_1 \cup \cdots \cup C_{k-1} \subseteq [n]$$

of the S-rank-selected Boolean algebra $B_n(S) = \{x \in B_n : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. Hence $\Delta_{\vec{c}}$ is the order complex $\Delta(B_n(S) - \{\hat{0}, \hat{1}\})$. The Boolean algebra has an EL-labeling,

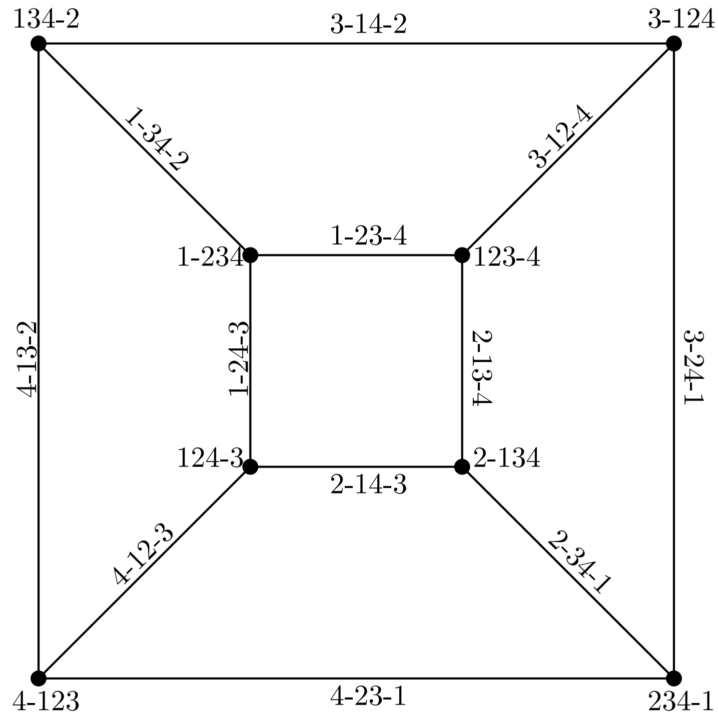


Figure 2.1: The simplicial complex $\Delta_{(1,2,1)}$ of ordered partitions. Note that the ordered partition 1234 corresponds to the empty face.

which implies that the rank-selected poset $B_n(S)$ has a shellable order complex; see [3, Theorem 4.1]. In fact, the rank-selected poset $B_n(S)$ is CL-shellable; see [5, Theorem 8.1] or [34, Theorem 3.4.1]. Furthermore, it follows from Björner's construction that the spanning facet is exactly of the above form. □

Finally, we note the following consequence.

Corollary 2.4.3. *If the pointed composition $\vec{c} = (c_1, \ldots, c_k)$ ends with 0, then the simplicial complex $\Delta_{\vec{c}}$ is shellable.*

Proof. The statement follows from the fact that the complex is the cone over a shellable complex. □

2.5 The homotopy type of the poset $\Pi^\bullet_{\vec{c}}$

We now will use Quillen's Fiber Lemma to show that the chain complex $\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})$ is homotopy equivalent to the simplicial complex $\Delta_{\vec{c}}$. Recall that a *simplicial map* f from a simplicial complex Γ to a poset P sends vertices of Γ to elements of P and faces of the simplicial complex to chains of P. We have the following result due to Quillen [26]. See also [4, Theorem 10.5] and [34, Theorem 5.2.1].

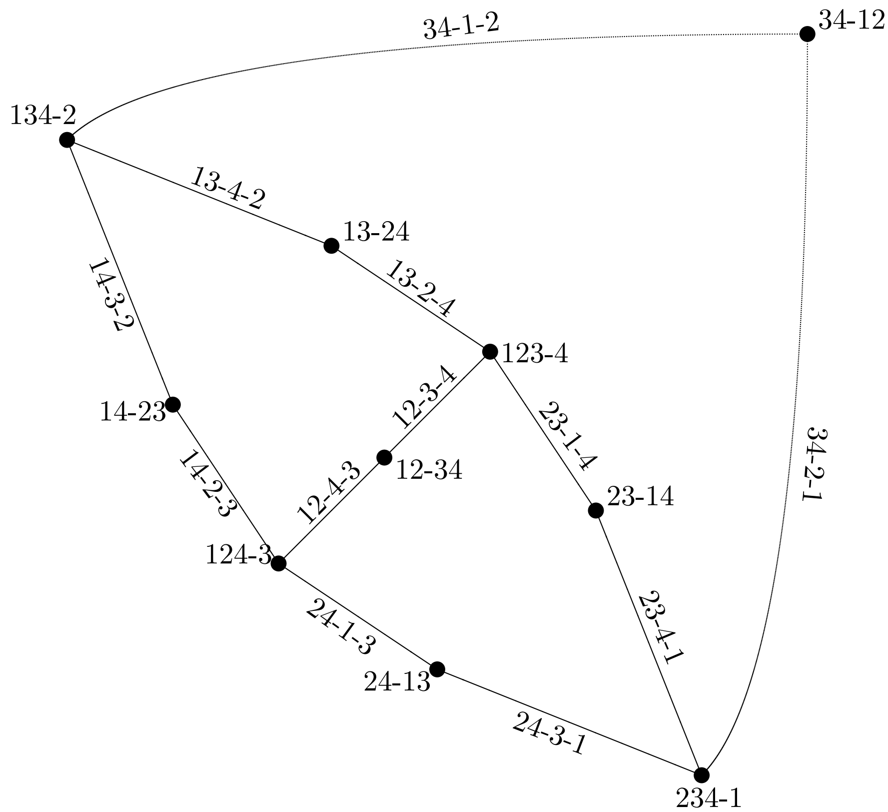


Figure 2.2: The simplicial complex $\Delta_{(2,1,1)}$ of ordered partitions.

Theorem 2.5.1 (Quillen's Fiber Lemma)**.** *Let f be a simplicial map from the simplicial complex Γ to the poset P such that for all x in P, the complex $\Delta(f^{-1}(P_{\geq x}))$, that is, the subcomplex of Γ induced by $f^{-1}(P_{\geq x})$, is contractible. Then the order complex $\Delta(P)$ and the simplicial complex Γ are homotopy equivalent.*

Recall that the *barycentric subdivision* of a simplicial complex Γ is the simplicial complex $\mathrm{sd}(\Gamma)$ whose vertices are the non-empty faces of Γ and whose faces are subsets of chains of faces in Γ ordered by inclusion. It is well-known that Γ and $\mathrm{sd}(\Gamma)$ are homeomorphic since they have the same geometric realization and hence are homotopy equivalent.

Consider the map $\phi : \Delta_n \longrightarrow \Pi^\bullet_n$ that sends an ordered set partition $(C_1, C_2, \ldots, C_k)$ to the pointed partition $\{C_1, C_2, \ldots, C_{k-1}, \underline{C_k}\}$. We call this map the *forgetful* map since it forgets the order between the blocks except it keeps the last part as the pointed block. Observe that the inverse image of the pointed partition $\{C_1, C_2, \ldots, C_{k-1}, \underline{C_k}\}$ consists of $(k-1)!$ ordered set partitions.

Lemma 2.5.2. *Let π be the pointed partition $\{B_1, \ldots, B_{m-1}, \underline{B_m}\}$ in $\Pi^\bullet_{\vec{c}}$ where $m \geq 2$. Let Ω be the subcomplex of the complex $\Delta_{\vec{c}}$ whose faces are given by the inverse image*

$$\phi^{-1}\left(\left(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}\right)_{\geq \pi}\right).$$

Then the complex Ω is a cone over the apex $([n] - B_m, B_m)$ and hence is contractible.

Proof. Let $\varsigma = (C_1, C_2, \ldots, C_r)$ be an ordered partition in the complex Ω. Observe that the pointed block B_m is contained in the last block C_r. Let Ω' be the subcomplex of Ω consisting of all faces $(C_1, C_2, \ldots, C_r)$ such that the last block C_r strictly contains the pointed block B_m. Then the complex Ω is a cone over the complex Ω' with the apex $([n] - B_m, B_m)$. $\square$

Theorem 2.5.3. *The order complex $\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})$ is homotopy equivalent to the barycentric subdivision $\mathrm{sd}(\Delta_{\vec{c}})$ and hence $\Delta_{\vec{c}}$.*

Proof. Note that

$$\phi^{-1}\left(\left(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}\right)_{\geq \pi}\right) = \mathrm{sd}(\Omega) \simeq \Omega,$$

which is contractible by Lemma 2.5.2. Hence Quillen's Fiber Lemma applies and we conclude that $\Delta\left(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}\right)$ is homotopy equivalent to $\mathrm{sd}(\Delta_{\vec{c}})$. $\square$

By considering the reduced Euler characteristic of the complex $\Delta\left(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}\right)$, we have the following corollary.

Corollary 2.5.4. *The Möbius function of the poset $\Pi^\bullet_{\vec{c}} \cup \{\hat{0}\}$ is given by $(-1)^k \cdot \beta(\vec{c})$.*

A combinatorial proof can be given for this corollary, which avoids the use of Quillen's Fiber Lemma.

2.6 Cycles in the complex $\Delta_{\vec{c}}$

In this section and the next we assume that the last part of the composition $\vec{c}$ is non-zero, since in the case $c_k = 0$ the homology group is the trivial group; see Lemma 2.4.1.

For α a permutation in the symmetric group $\mathfrak{S}_n$, define the subcomplex Σ_α of the complex $\Delta_{\vec{c}}$ to be the simplicial complex whose facets are given by

$$\{\sigma(\alpha \circ \gamma) \ : \ \gamma \in \mathfrak{S}'_{\vec{c}}\},$$

where σ is defined in the beginning of Section 2.4. Recall that the dual permutahedron P_n is also defined in that section.

Lemma 2.6.1. *The subcomplex Σ_α is isomorphic to the join of the duals of the permutahedra*

$$P_{|K_1|} * \cdots * P_{|K_{n-k+1}|}$$

and hence it is sphere of dimension $k - 2$.

Proof. Represent the complex $P_{|K_i|}$ by ordered partitions on the set K_i. A face of the join $P_{|K_1|} * \cdots * P_{|K_{n-k+1}|}$ is then an $(n-k+1)$-tuple $(\tau_1, \ldots, \tau_{n-k+1})$ where $\tau_i = (D_{i,1}, \ldots, D_{i,j_i}) \in P_{|K_i|}$. We obtain a face of Σ_α by gluing these ordered partitions together, that is,

$$(D_{1,1}, \ldots, D_{1,j_1} \cup D_{2,1}, \ldots, D_{2,j_2} \cup D_{3,1}, \ldots, D_{n-k+1,j_{n-k+1}}).$$

It is straightforward to see this is a bijective correspondence proving the first claim. Since the join of a m-sphere and an n-sphere is an $(m+n+1)$-sphere, we obtain a sphere of dimension $(|K_1|-2) + \cdots + (|K_{n-k+1}|-2) + n - k = k-2$. $\square$

Observe that the facets of $\Delta_{\vec{c}}$ are in bijection with permutations α such that $\mathrm{Des}(\alpha) \geq \vec{c}$ in the composition order.

Lemma 2.6.2. *Let α be a permutation in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{c}$ and let γ belong to the column stabilizer $\mathfrak{S}'_{\vec{c}}$. Then we have the inequality $\alpha \circ \gamma \leq \alpha$ in weak Bruhat order.*

Proof. Write γ as $(\gamma_1, \ldots, \gamma_{n-k+1})$ where γ_i belongs to $\mathfrak{S}_{K_i}$. Then γ_i acts on the interval $K_i = [u, v]$. Note that $v(v-1)\cdots u$ is the largest permutation in the Bruhat order on $\mathfrak{S}_{K_i}$. Hence $\alpha \circ \gamma$ restricted to K_i is a smaller element in the Bruhat order. The inequality follows by concatenating these partial permutations into the permutation $\alpha \circ \gamma$. $\square$

We note that this lemma has a dual version.

Lemma 2.6.3. *Let α be a permutation in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{c}$ and let γ belong to the row stabilizer $\mathfrak{S}_{\vec{c}}$. Then we have the inequality $\alpha \circ \gamma \geq \alpha$ in weak Bruhat order.*

Directly we have the next lemma.

Lemma 2.6.4. *Let F be a facet of Σ_α. Then the inequality $\sigma^{-1}(F) \leq \alpha$ holds in the weak Bruhat order.*

Proof. There is an element γ in the column stabilizer $\mathfrak{S}'_{\vec{c}}$ such that $F = \sigma(\alpha \circ \gamma)$. Since $\sigma^{-1}(F)$ is the smallest permutation in the Bruhat order that maps to F, we have that $\sigma^{-1}(F) \leq \alpha \circ \gamma \leq \alpha$. $\square$

Recall that the boundary map of the face $\sigma = (C_1, \ldots, C_r)$ in the chain complex of $\Delta_{\vec{c}}$ is defined by

$$\partial((C_1, \ldots, C_r)) = \sum_{i=1}^{r-1} (-1)^{i-1} \cdot (C_1, \ldots, C_i \cup C_{i+1}, \ldots, C_r).$$

The next lemma follows from Lemma 2.6.1, apart from the signs.

Lemma 2.6.5. *For $\alpha \in \mathfrak{S}_n$, the element*

$$g_\alpha = \sum_{\gamma \in \mathfrak{S}'_{\vec{c}}} (-1)^\gamma \cdot \sigma(\alpha \circ \gamma)$$

in the chain group $C_{k-2}(\Delta_{\vec{c}})$ belongs to the kernel of the boundary map and hence to the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$.

Proof. Apply the boundary map to the above element g_α in the chain group and exchange the order of the two sums. The inner sum is then

$$\sum_{\gamma \in \mathfrak{S}'_{\vec{c}}} (-1)^\gamma \cdot (\ldots, \{\ldots, \alpha_{\gamma_{c_1+\cdots+c_i}}, \alpha_{\gamma_{c_1+\cdots+c_i+1}}, \ldots\}, \ldots).$$

Observe that the term corresponding to γ cancels with the term corresponding to γ composed with the transposition $(c_1+\cdots+c_i, c_1+\cdots+c_i+1)$ which belongs to the column stabilizer $\mathfrak{S}'_{\vec{c}}$ and thus the sum vanishes. $\square$

Theorem 2.6.6. *The cycles g_α, where α ranges over all permutations with descent composition $\vec{c}$, form a basis for the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$.*

Proof. The complex $\Delta_{\vec{c}} - \{\sigma(\alpha) : \mathrm{Des}(\alpha) = \vec{c}\}$ is contractible by the shelling in the proof of Theorem 2.4.2. Contract this complex to a point and then attach the cells $\sigma(\alpha)$ to this point to obtain a wedge of spheres, denoted by X. Call this contraction map f, that is, we have the continuous function $f : \Delta_{\vec{c}} \longrightarrow X$ and hence the homomorphism $f_* : \widetilde{H}_{k-2}(\Delta_{\vec{c}}) \longrightarrow \widetilde{H}_{k-2}(X)$. Since f is a part of a homotopy equivalence, we know that f_* is an isomorphism. Lastly, let h_α denote the cycle corresponding to sphere $\sigma(\alpha)$ in the homology group $\widetilde{H}_{k-2}(X)$.

It is clear that the cycles h_α, where α ranges over all permutations with descent composition $\vec{c}$, form a basis for the homology group $\widetilde{H}_{k-2}(X)$. We now motivate that the images $f_*(g_\alpha)$ also form a basis for the homology group $\widetilde{H}_{k-2}(X)$.

Take two permutations α and α' with descent composition $\vec{c}$. By Lemma 2.6.4 we have that the coefficient of $h_{\alpha'}$ in $f_*(g_\alpha)$ is zero unless $\alpha' \leq \alpha$ in the weak Bruhat order. Furthermore, the coefficient of h_α in $f_*(g_\alpha)$ is 1. Hence the relationship between the basis $\{h_{\alpha'}\}_{\alpha'}$ and the set $\{f_*(g_\alpha)\}_\alpha$ is triangular and hence invertible. Thus the set $\{f_*(g_\alpha)\}_\alpha$ forms a basis for the homology group $\widetilde{H}_{k-2}(X)$. Finally, since f_* is an isomorphism, the cycles g_α form a basis for the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$. $\square$

2.7 Representation of the symmetric group

The symmetric group $\mathfrak{S}_n$ acts naturally on the the poset $\Pi^\bullet_{\vec{c}}$ by relabeling the elements. Hence it also acts on the order complex $\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})$. Lastly, the symmetric group acts on the top homology group $\widetilde{H}_{k-2}(\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}))$. We show in this section that this action is a Specht module of the border strip B corresponding to the composition $\vec{c}$. For an overview on the representation theory of the symmetric group, we refer the reader to Sagan's book [27] and Wachs' article [34].

Let G be a group which acts on the simplicial complex Γ and on the poset P. We call the simplicial map f a *G-simplicial map* if the map f commutes with this action. The equivariant version of Quillen's Fiber Lemma is as follows [26]. See also [34, Section 5.2].

Theorem 2.7.1 (Equivariant homology version of Quillen's Fiber Lemma)**.** *Let f be a G-simplicial map from the simplicial complex Γ to the poset P such that for all x in P, the subcomplex $\Delta(f^{-1}(P_{\geq x}))$ is acyclic. Then the two homology groups $\widetilde{H}_r(\Delta(P))$ and $\widetilde{H}_r(\Gamma)$ are isomorphic as G-modules.*

The forgetful map ϕ from $\mathrm{sd}(\Delta_{\vec{c}})$ to the order complex of the poset $\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}$ commutes with the action of the symmetric group $\mathfrak{S}_n$. Hence we conclude the next result.

Proposition 2.7.2. *The two homology groups $\widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}}))$ and $\widetilde{H}_{k-2}(\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}))$ are isomorphic as $\mathfrak{S}_n$-modules.*

It is clear that $\widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}}))$ and $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ are isomorphic as $\mathfrak{S}_n$-modules. Hence in the remainder of this section we will study the action the symmetric group $\mathfrak{S}_n$ on $\Delta_{\vec{c}}$ and its action on the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$. This is in the spirit of Wachs' work [32].

Let B be the border strip that has k rows where the ith row consists of c_i boxes. Recall that a tableau is a filling of the boxes of the shape B with the integers 1 through n. A standard Young tableau is a tableau where the rows and columns are increasing. A tabloid is an equivalence class of tableaux under the relation of permuting the entries in each row. To distinguish tabloids from tableaux, only the horizontal lines are drawn in a tabloid. See [27, Section 2.1] for details.

Observe that there is a natural bijection between tabloids of shape B and facets of the complex $\Delta_{\vec{c}}$ by letting the elements in each row form a block and letting the order of the blocks go from lowest to highest row. See Figure 2.3 for an example. Let M^B be the permutation module corresponding to shape B, that is, the linear span of all tabloids of shape B. Notice that the above bijection induces a $\mathfrak{S}_n$-module isomorphism between the permutation module M^B and the chain group $C_{k-2}(\Delta_{\vec{c}})$.

Furthermore, there is a bijection between tableaux of shape B and and permutations by reading the elements in the northeast direction from the border strip. Recall that the group $\mathfrak{S}'_{\vec{c}} = \mathfrak{S}_{K_1} \times \cdots \times \mathfrak{S}_{K_{n-k+1}}$ is the column stabilizer of the border strip B. Let t be a tableau and α its associated permutation. Hence the polytabloid $\mathbf{e}_t$ corresponding to the tableau t is the element g_α presented in Lemma 2.6.5; see [27, Definition 2.3.2]. Since the Specht module S^B is the submodule of M^B spanned by all polytabloids, Lemma 2.6.5 proves

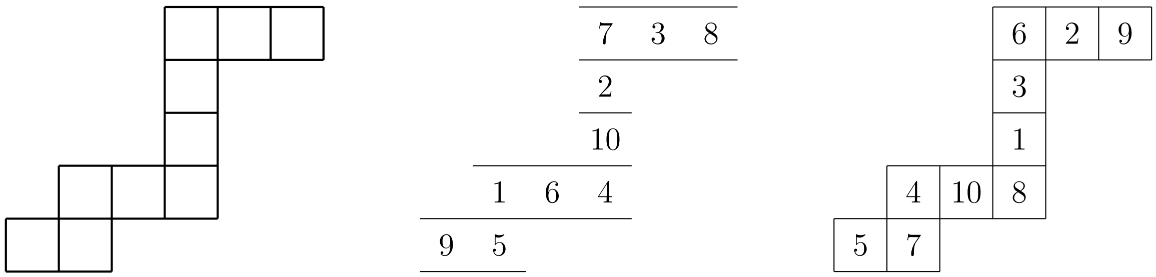


Figure 2.3: (a) Border strip corresponding to composition $(2,3,1,1,3)$. (b) The tabloid corresponding to the face $5\,9 - 1\,4\,6 - 10 - 2 - 3\,7\,8$. (c) The tableau corresponding to the permutation $5\,7\,4\,10\,8\,1\,3\,6\,2\,9$.

that the Specht module S^B is isomorphic to a submodule of the kernel of the boundary map ∂_{k-2}. Since the kernel is the top homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$, and the Specht module S^B and the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ have the same dimension $\beta(\vec{c})$, we conclude that they are isomorphic. To summarize we have:

Proposition 2.7.3. *The top homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ is isomorphic to the Specht module S^B as $\mathfrak{S}_n$-modules.*

By combining Propositions 2.7.2 and 2.7.3, the main result of this section follows.

Theorem 2.7.4. *The top homology group $\widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}))$ is isomorphic to the Specht module S^B as $\mathfrak{S}_n$-modules.*

2.8 Filters generated by knapsack partitions

We now turn our attention to filters in the pointed partition lattice $\Pi^\bullet_n$ that are generated by a pointed knapsack partition, that is, the poset $\Pi^\bullet_{\{\lambda,\underline{m}\}}$, which was introduced in Section 2.3. In order to study this poset we need the corresponding collection of ordered set partitions.

Definition 2.8.1. *For a pointed knapsack partition $\{\lambda,\underline{m}\} = \{\lambda_1,\lambda_2,\ldots,\lambda_k,\underline{m}\}$ of n define the subcomplex $\Lambda_{\{\lambda,\underline{m}\}}$ of the complex of ordered set partitions Δ_n by*

$$\Lambda_{\{\lambda,\underline{m}\}} = \{(C_1,\ldots,C_{r-1},C_r) \in \Delta_n \;\;:\;\; \{C_1,\ldots,C_{r-1},\underline{C_r}\} \in \Pi^\bullet_{\{\lambda,\underline{m}\}}\}.$$

For a pointed knapsack partition $\{\lambda,\underline{m}\}$ of n define F to be the filter in the poset of compositions of n generated by compositions $\vec{c}$ such that $\mathrm{type}(\vec{c}) = \{\lambda,\underline{m}\}$. Now define $V(\lambda,\underline{m})$ to be the collection of all pointed compositions $\vec{c} = (c_1,c_2,\ldots,c_r)$ in the filter F such that each $c_i, 1 \le i \le r-1$, is a sum of distinct parts of the partition λ and $c_r = m$. As an example, for $\lambda = \{1,1,3,7\}$ we have $(4,8,m) \in V(\lambda,\underline{m})$ but $(2,10,m) \notin V(\lambda,\underline{m})$.

For a composition $\vec{d}$ in $V(\lambda, \underline{m})$ define $\epsilon(\vec{d})$ to be the composition of type $\{\lambda, \underline{m}\}$, where each entry d_i of $\vec{d}$ has been replaced with a decreasing list of parts of λ, that is,

$$\epsilon(\vec{d}) = (\lambda_{1,1}, \ldots, \lambda_{1,t_1}, \ldots, \lambda_{s,1}, \ldots, \lambda_{s,t_s}, m),$$

where $\lambda_{i,1} > \lambda_{i,2} > \cdots > \lambda_{i,t_i}$, $\sum_{j=1}^{t_i} \lambda_{i,j} = d_i$ and

$$\{\lambda, \underline{m}\} = \{\lambda_{1,1}, \ldots, \lambda_{1,t_1}, \ldots, \lambda_{s,1}, \ldots, \lambda_{s,t_s}, \underline{m}\}.$$

As an example, for the pointed knapsack partition $\lambda = \{2, 1, \underline{1}\}$ we have $\epsilon((3,1)) = (2,1,1)$, $\epsilon((2,1,1)) = (2,1,1)$ and $\epsilon((1,2,1)) = (1,2,1)$. Also note $\epsilon(\vec{d}) \leq \vec{d}$ in the partial order of compositions.

Similar to Theorem 2.4.2 we have the following topological conclusion. However, this time the tool is not shelling, but discrete Morse theory.

Theorem 2.8.2. *There is a Morse matching on the simplicial complex $\Lambda_{\{\lambda,\underline{m}\}}$ such that the only critical cells are of the form $\sigma(\alpha, \epsilon(\vec{d}))$ where $\vec{d}$ ranges in the set $V(\lambda, \underline{m})$ and α ranges over all permutations in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{d}$. Hence, the simplicial complex $\Lambda_{\{\lambda,\underline{m}\}}$ is homotopy equivalent to wedge of $\sum_{\vec{d} \in V(\lambda,\underline{m})} \beta(\vec{d})$ spheres of dimension $k-1$.*

For a pointed knapsack partition $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \underline{m}\}$ of n, define a function κ on the domain

$$D = \left\{ \sum_{i \in S} \lambda_i \ : \ \emptyset \neq S \subseteq [k] \right\},$$

that is, all non-zero sums of parts of the partition, excluding the pointed part. Now $\kappa : D \longrightarrow \mathbb{P}$ is given by

$$\kappa(\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_p}) = \min(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_p}).$$

Observe that κ is well-defined since λ is a knapsack partition.

For an ordered set partition $\tau = (C_1, C_2, \ldots, C_r)$ in $\Lambda_{\{\lambda,\underline{m}\}}$, consider the following $2r-1$ conditions:

- Let $\mathbf{A}_i$, for $1 \leq i \leq r-2$, be the condition that $\max(C_i) < \min(C_{i+1})$ and $|C_i| \leq \kappa(|C_{i+1}|)$.
- Let $\mathbf{A}_{r-1}$ be the condition that $\max(C_{r-1}) < \min(C_r)$.
- Let $\mathbf{B}_i$, for $1 \leq i \leq r-1$, be the condition that $\kappa(|C_i|) < |C_i|$.
- Let $\mathbf{B}_r$ be the condition that $|C_r| > m$.

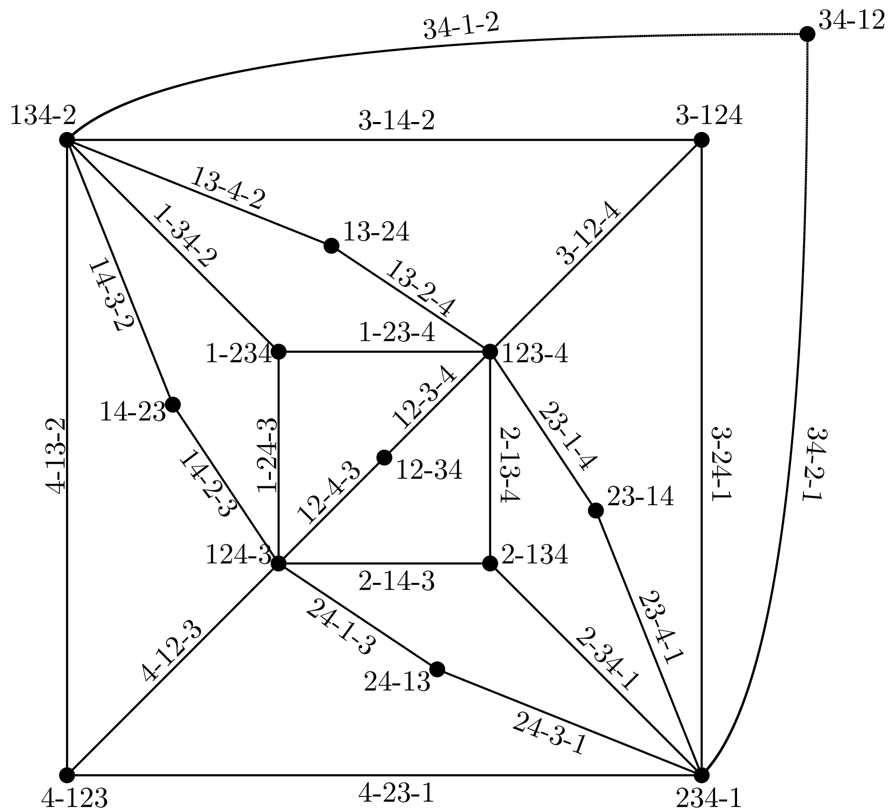


Figure 2.4: The simplicial complex $\Lambda_{\{2,1,\underline{1}\}}$, corresponding to the knapsack partition $\{2,1,\underline{1}\}$. Notice that this complex is the union of two complexes $\Delta_{(1,2,1)}$ and $\Delta_{(2,1,1)}$, appearing in Figures 2.1 and 2.2.

Note that condition $\mathbf{A}_i$ concerns comparing the ith and $(i+1)$st blocks of the set partition τ. We also notice that if $\mathbf{B}_i$ is true and $i \leq r-1$ then the cardinality $|C_i|$ is the sum of at least two parts of the partition λ. Similarly, if $\mathbf{B}_r$ is true then the cardinality $|C_r|$ is the sum of at least two parts of λ and the integer m.

We use these conditions to construct a discrete Morse matching for $\Lambda_{\{\lambda,\underline{m}\}}$. We match the ordered set partition $\tau = (C_1, C_2, \ldots, C_r)$ in $\Lambda_{\{\lambda,\underline{m}\}}$ as follows:

– If conditions $\mathbf{A}_j$ and $\mathbf{B}_j$ are false for $1 \leq j \leq i-1$ and $i \leq r-1$ but condition $\mathbf{B}_i$ is true, then let X be the $\kappa(|C_i|)$ smallest elements of C_i and Y be the $|C_i| - \kappa(|C_i|)$ largest elements of C_i. Let $u(\tau)$ be given by

$$u(\tau) = (C_1, C_2, \ldots, C_{i-1}, X, Y, C_{i+1}, \ldots, C_r)$$

and let the type of the edge $(\tau, u(\tau))$ be i.

– If conditions $\mathbf{A}_j$ and $\mathbf{B}_j$ are false for $1 \leq j \leq r-1$ but condition $\mathbf{B}_r$ is true, then let X be the $\kappa(|C_r| - m)$ smallest elements of C_r and Y be the $|C_r| - \kappa(|C_r| - m)$ largest elements of C_r. Let $u(\tau)$ be given by

$$u(\tau) = (C_1, C_2, \ldots, C_{r-1}, X, Y)$$

and let the type of the edge $(\tau, u(\tau))$ be r.

– If conditions $\mathbf{A}_j$ and $\mathbf{B}_j$ are false for $1 \leq j \leq i-1$ and condition $\mathbf{B}_i$ is false but condition $\mathbf{A}_i$ is true, then let

$$d(\tau) = (C_1, C_2, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_r)$$

and let the type of the edge $(d(\tau), \tau)$ be i.

Lemma 2.8.3. *Let τ and τ' be two different ordered set partitions satisfying the condition $\tau \prec u(\tau) \succ \tau' \prec u(\tau')$. This condition implies the type of $(\tau, u(\tau))$ is greater than the type of $(\tau', u(\tau'))$. Hence the matching is acyclic.*

Proof. Consider the following three ordered set partitions:

$$\begin{array}{rcl} \tau & = & (C_1, C_2, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_r), \\ u(\tau) & = & (C_1, C_2, \ldots, C_r), \\ \tau' & = & (C_1, C_2, \ldots, C_{j-1}, C_j \cup C_{j+1}, C_{j+2}, \ldots, C_r), \end{array}$$

for $i \neq j$. Note that the type of the edge $(\tau, u(\tau))$ is i. If $i < j$, the ordered set partition τ' should be matched to

$$d(\tau') = (C_1, C_2, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_{j-1}, C_j \cup C_{j+1}, C_{j+2}, \ldots, C_r),$$

contradicting the assumption that τ' was matched upwards with $u(\tau')$. Hence, we conclude that $i > j$ and,

$$u(\tau') = (C_1, C_2, \ldots, C_{j-1}, X, Y, C_{j+2}, \ldots, C_r)$$

for $X \cup Y = C_j \cup C_{j+1}$ such that $\max(X) < \min(Y)$. Since $\max(C_j) > \min(C_{j+1})$, we have $u(\tau) \neq u(\tau')$ and the type of $(\tau', u(\tau'))$ is j. Therefore, we have $\mathrm{type}(\tau, u(\tau)) = i > j = \mathrm{type}(\tau', u(\tau'))$.

If there was a cyclic matching, we would reach a contradiction by following the inequalities of types around the cycle. □

Lemma 2.8.4. *The critical cells of the above matching of the face poset of $\Lambda_{\{\lambda,\underline{m}\}}$ are of the form $\sigma(\alpha, \epsilon(\vec{d}))$ where $\vec{d}$ ranges in the set $V(\lambda, \underline{m})$ and α ranges over all permutations in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{d}$.*

Proof. The unmatched cells of the matching presented on the face poset of $\Lambda_{\{\lambda,\underline{m}\}}$ have the form $(C_1, C_2, \ldots, C_r)$ such that all the conditions $\mathbf{A}_i$ and $\mathbf{B}_i$ are false for $1 \leq i \leq r-1$ and

$\mathbf{B}_r$ is false.

Since the condition $\mathbf{B}_i$ is false for $i \leq r-1$ we have that $|C_i| = \kappa(|C_i|)$, that is, $|C_i|$ is a part of the partition λ for $i \leq r-1$. Also $\mathbf{B}_r$ is false implies that $|C_r| = m$. Hence $\{|C_1|, \ldots, |C_{r-1}|, \underline{|C_r|}\}$ is the pointed partition $\{\lambda, \underline{m}\}$. Since the condition $\mathbf{A}_i$ is false, we have $\max(C_i) > \min(C_{i+1})$ or $|C_i| > \kappa(|C_{i+1}|) = |C_{i+1}|$ for $1 \leq i \leq r-2$. Finally, $\mathbf{A}_{r-1}$ is false, implying $\max(C_{r-1}) > \min(C_r)$.

For the unmatched cell $(C_1, \ldots, C_r)$, let α be the permutation obtained by writing each block of the critical cell in increasing order. Furthermore, let $\vec{d}$ be the descent composition of the permutations α. Observe that the composition $\vec{d}$ belongs to the set $V(\lambda, \underline{m})$ since an entry in $\vec{d}$ is a sum of distinct parts of λ. Furthermore the composition $(|C_1|, \ldots, |C_{r-1}|, |C_r|)$ is the composition $\epsilon(\vec{d})$. Hence the unmatched cell is $\sigma(\alpha, \epsilon(\vec{d}))$. □

Example 2.8.5. *Consider the pointed knapsack partition $\{\lambda, \underline{m}\} = \{2, 1, \underline{1}\}$ whose associated complex $\Lambda_{\{2,1,\underline{1}\}}$ is shown in Figure 2.4. Note that $V(\lambda, \underline{m}) = \{(1,2,1), (2,1,1), (3,1)\}$. The critical cells of the complex $\Lambda_{\{2,1,\underline{1}\}}$ are as follows:*

$\vec{d}$	$\beta(\vec{d})$	$\epsilon(\vec{d})$	$W(\vec{d})$	*critical cells*
$(1,2,1)$	5	$(1,2,1)$	$\{(1,2,1)\}$	*2-14-3, 3-14-2, 3-24-1, 4-13-2, 4-23-1*
$(2,1,1)$	3	$(2,1,1)$	$\{(2,1,1)\}$	*14-3-2, 24-3-1, 34-2-1*
$(3,1)$	3	$(2,1,1)$	$\{(1,2,1), (2,1,1)\}$	*12-4-3, 13-4-2, 23-4-1*

Note that $\Lambda_{\{2,1,\underline{1}\}}$ is homotopy equivalent to a wedge of 11 *circles. The notion $W(\vec{d})$ will be defined in the beginning of the next section.*

Proof of Theorem 2.8.2. By Lemma 2.8.3 the matching presented is a Morse matching and Lemma 2.8.4 describes the critical cells, proving the theorem. □

Now by the same reasoning as in Section 2.5, that is, using the forgetful map ϕ and Quillen's Fiber Lemma, we obtain a homotopy equivalence between the order complex of pointed partitions $\Pi^{\bullet}_{\{\lambda,\underline{m}\}} - \{\hat{1}\}$ and the simplicial complex of ordered set partitions $\Lambda_{\{\lambda,\underline{m}\}}$. Since the proof follows the same outline, it is omitted.

Theorem 2.8.6. *The order complex $\Delta\left(\Pi^{\bullet}_{\{\lambda,\underline{m}\}} - \{\hat{1}\}\right)$ is homotopy equivalent to the barycentric subdivision $\mathrm{sd}(\Lambda_{\{\lambda,\underline{m}\}})$ and hence the simplicial complex $\Lambda_{\{\lambda,\underline{m}\}}$.*

As a corollary we obtain the Möbius function of the poset $\Pi^{\bullet}_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}$; see [13].

Corollary 2.8.7 (Ehrenborg–Readdy)**.** *The Möbius function of the poset $\Pi^{\bullet}_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}$ is given by*

$$\mu\left(\Pi^{\bullet}_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}\right) = (-1)^k \cdot \sum_{\vec{d} \in V(\lambda,\underline{m})} \beta(\vec{d}).$$

2.9 Cycles in the complex $\Lambda_{\{\lambda,\underline{m}\}}$

Observe that when the pointed part m is equal to zero, the complex $\Lambda_{\{\lambda,\underline{m}\}}$ is contractible. Hence we tacitly assume that m is positive in this and the next section.

For a pointed knapsack partition $\{\lambda, \underline{m}\}$ of n and $\vec{d} \in V(\lambda, \underline{m})$, let $W(\vec{d})$ be the set

$$W(\vec{d}) = \{\vec{c} \in V(\lambda, \underline{m}) : \vec{c} \leq \vec{d}, \text{type}(\vec{c}) = \{\lambda, \underline{m}\}\}.$$

Especially we have $\epsilon(\vec{d}) \in W(\vec{d})$. For the case when the pointed knapsack partition is $\{2, 1, \underline{1}\}$, see Example 2.8.5.

For α a permutation in the symmetric group $\mathfrak{S}_n$ and $\vec{d}$ a composition of $V(\lambda, \underline{m})$, define subcomplex $\Sigma_{\alpha,\vec{d}}$ of the complex $\Lambda_{\{\lambda,\underline{m}\}}$ to be the simplicial complex whose facets are given by

$$\{\sigma(\alpha \circ \gamma, \vec{c}) : \vec{c} \in W(\vec{d}),\ \gamma \in \mathfrak{S}'_{\vec{d}}\}.$$

This means the types of facets in $\Sigma_{\alpha,\vec{d}}$ belong to the set $W(\vec{d})$.

For $\vec{d} = (d_1, \ldots, d_r)$, if d_i splits into t_i parts in a composition in $W(\vec{d})$, then the group $\mathfrak{S}_{t_1} \times \cdots \times \mathfrak{S}_{t_r}$ acts on $W(\vec{d})$ by permuting the t_i parts d_i splits into. Given $\vec{c} \in W(\vec{d})$ there exists a permutation $\rho \in \mathfrak{S}_{t_1} \times \cdots \times \mathfrak{S}_{t_r}$ so $\rho(\epsilon(\vec{d})) = \vec{c}$. Define the sign of $\vec{c}$, that is, $(-1)^{\vec{c}}$ to be the sign $(-1)^{\rho}$. Especially, we have $(-1)^{\epsilon(\vec{d})} = 1$.

Similar to Lemma 2.6.1 we have the next result.

Lemma 2.9.1. *The subcomplex $\Sigma_{\alpha,\vec{d}}$ is isomorphic to the join of the duals of the permutahedra*

$$P_{|K_1|} * \cdots * P_{|K_{n-r+1}|} * P_{t_1} * \cdots * P_{t_r}$$

and hence it is a sphere of dimension $k-1$, where k is the number of parts in the partition λ.

Since this lemma is not necessary for the remainder of this chapter, the proof is omitted. However, let us verify the dimension of the sphere. First the dimension of the $(n-r)$-fold join is $P_{|K_1|} * \cdots * P_{|K_{n-r+1}|}$ is $(|K_1|-2) + \cdots + (|K_{n-r+1}|-2) + (n-r) = n - 2\cdot(n-r+1) + n - r = r - 2$. Similarly, the dimension of $P_{t_1} * \cdots * P_{t_r}$ is given by $(t_1 - 2) + \cdots + (t_r - 2) + (r - 1) = (k + 1) - 2 \cdot r + r - 1 = k - r$. Thus the dimension of the sphere in the lemma is $(r - 2) + (k - r) + 1 = k - 1$.

Define the element $g_{\alpha,\vec{d}}$ in the chain group $C_{k-1}(\Lambda_{\{\lambda,\underline{m}\}})$ to be

$$g_{\alpha,\vec{d}} = \sum_{\gamma \in \mathfrak{S}'_{\vec{d}}} \sum_{\vec{c} \in W(\vec{d})} (-1)^{\gamma} \cdot (-1)^{\vec{c}} \cdot \sigma(\alpha \circ \gamma, \vec{c}).$$

When α has descent composition $\vec{d}$, note the critical cell $\sigma(\alpha, \epsilon(\vec{d}))$ has sign 1 in $g_{\alpha,\vec{d}}$.

Lemma 2.9.2. *For $\alpha \in \mathfrak{S}_n$, the element $g_{\alpha,\vec{d}}$ in the chain group $C_{k-1}(\Lambda_{\{\lambda,\underline{m}\}})$ belongs to the kernel of the boundary map and hence the homology group $\widetilde{H}_{k-1}(\Lambda_{\{\lambda,\underline{m}\}})$.*

Proof. Apply the boundary map ∂ to $g_{\alpha,\vec{d}}$ in $C_{k-1}(\Lambda_{\{\lambda,\underline{m}\}})$ and exchange the order of the three sums to obtain

$$\partial(g_{\alpha,\vec{d}}) = \sum_{i=1}^{k} (-1)^{i-1} \cdot \sum_{\gamma \in \mathfrak{S}'_{\vec{d}}} \sum_{\vec{c} \in W(\vec{d})} (-1)^{\gamma} \cdot (-1)^{\vec{c}} \cdot (\ldots, \{\ldots, \alpha_{\gamma_{c_1+\cdots+c_i}}, \alpha_{\gamma_{c_1+\cdots+c_i+1}}, \ldots\}, \ldots).$$

If $\sum_{s=1}^{i} c_s = \sum_{s=1}^{j} d_s$ for some j, the term corresponding to γ cancels with the term corresponding to $\gamma \circ (c_1 + \cdots + c_i, c_1 + \cdots + c_i + 1)$. If there does not exist such an index j, we find the smallest integer ℓ such that $\sum_{s=1}^{i} c_s < \sum_{s=1}^{\ell} d_s$. Note that d_ℓ splits into $t_\ell \geq 2$ parts. Now consider the composition $\vec{c}\,' = (\ldots, c_{i-1}, c_{i+1}, c_i, c_{i+2}, \ldots)$ where we switched the ith and the $(i+1)$st parts of $\vec{c}$. Note that $\vec{c}\,'$ also belongs to $W(\vec{d})$ and the sign differs from the sign of $\vec{c}$, that is, $(-1)^{\vec{c}\,'} = -(-1)^{\vec{c}}$. Then the term corresponding to $\vec{c}$ cancels with the term corresponding to $\vec{c}\,'$. Hence, the sum vanishes. $\square$

Lemma 2.9.3. *If the critical cell $\sigma(\alpha', \epsilon(\vec{d}\,'))$ belongs to the cycle $g_{\alpha,\vec{d}}$, then we have the inequality $\alpha' \leq \alpha$ in the weak Bruhat order. Furthermore, if the two permutations α and α' are equal, then two compositions $\vec{d}$ and $\vec{d}\,'$ are equal.*

Proof. Since $\sigma(\alpha', \epsilon(\vec{d}\,'))$ belongs to $g_{\alpha,\vec{d}}$, we have $\sigma(\alpha', \epsilon(\vec{d}\,')) = \sigma(\alpha \circ \gamma, \vec{c})$ for some $\gamma \in \mathfrak{S}'_{\vec{d}}$ and $\vec{c} \in W(\vec{d})$. Note that $\epsilon(\vec{d}\,')) = \vec{c}$. Hence a permutation α' corresponding to the critical cell $\sigma(\alpha', \epsilon(\vec{d}\,'))$ is obtained by writing each block of $\sigma(\alpha \circ \gamma, \vec{c})$ in increasing order and this implies $\alpha' \leq \alpha \circ \gamma$. Furthermore, $\alpha \circ \gamma \leq \alpha$ in weak Bruhat order by Lemma 2.6.2. Hence, we have $\alpha' \leq \alpha$.

Finally, if the two permutations α and α' are equal, we have $\vec{d} = \mathrm{Des}(\alpha) = \mathrm{Des}(\alpha') = \vec{d'}$. $\square$

Using Lemma 2.9.3 we have next result. Observe that the proof is similar to that of Theorem 2.6.6 and hence omitted.

Theorem 2.9.4. *For a pointed knapsack partition $\{\lambda, \underline{m}\}$ of n, the cycles $g_{\alpha,\vec{d}}$ where the composition $\vec{d}$ ranges over the set $V(\lambda, \underline{m})$ and α ranges over all permutations with descent composition $\vec{d}$, form a basis for the homology group $\widetilde{H}_{k-1}(\Lambda_{\{\lambda,\underline{m}\}})$.*

2.10 The group action on the top homology

By the equivariant homology version of Quillen's Fiber Lemma, Theorem 2.7.1, we have the following isomorphism.

Theorem 2.10.1. *The two homology groups $\widetilde{H}_{k-1}\left(\Delta\left(\Pi^{\bullet}_{\{\lambda,\underline{m}\}} - \{\hat{1}\}\right)\right)$ and $\widetilde{H}_{k-1}\left(\Lambda_{\{\lambda,\underline{m}\}}\right)$ are isomorphic as $\mathfrak{S}_n$-modules.*

Hence it remains to make the connection between the action on ordered set partitions and Specht modules.

Theorem 2.10.2. *The direct sum of Specht modules*

$$\bigoplus_{\vec{d}\in V(\lambda,\underline{m})} S^{B(\vec{d})}$$

is isomorphic to the top homology group $\widetilde{H}_{k-1}\left(\Lambda_{\{\lambda,\underline{m}\}}\right)$ *as* $\mathfrak{S}_n$*-modules.*

Proof. We define an homomorphism

$$\Psi : \bigoplus_{\vec{d}\in V(\lambda,\underline{m})} M^{B(\vec{d})} \longrightarrow C_{k-1}\left(\Lambda_{\{\lambda,\underline{m}\}}\right),$$

by sending a tabloid s of shape $B(\vec{d})$ to the element $\sum_{\vec{c}\in W(\vec{d})}(-1)^{\vec{c}}\sigma(\alpha,\vec{c})$ in the chain group, where α is the permutation obtained from the tabloid s by reading the elements in each row in increasing order. Observe that Ψ is a $\mathfrak{S}_n$-module homomorphism.

Now consider the restriction of the homomorphism Ψ to the direct sum of Specht modules. Note that the group $\mathfrak{S}'_{\vec{d}}$ is the column stabilizer of the border strip $B(\vec{d})$. Let t be a tableau and α its association permutation. The homomorphism Ψ applied to the polytabloid $\mathbf{e}_t$ (see reference [27, Definition 2.3.2]) is as follows:

$$\Psi(\mathbf{e}_t) = \sum_{\gamma\in\mathfrak{S}'_{\vec{d}}}(-1)^{\gamma}\cdot\sum_{\vec{c}\in W(\vec{d})}(-1)^{\vec{c}}\cdot\sigma(\alpha\circ\gamma,\vec{c}) = g_{\alpha,\vec{d}},$$

which belongs to kernel of the boundary map. Hence Ψ maps the directed sum of the Specht modules to the homology group $\widetilde{H}_{k-1}\left(\Lambda_{\{\lambda,\underline{m}\}}\right)$.

Since $g_{\alpha,\vec{d}}$ lies in the image of the restriction of Ψ and the elements $g_{\alpha,\vec{d}}$ span the homology group, the restriction is surjective. Furthermore, since the two $\mathfrak{S}_n$-modules have the same dimension, we conclude that they are isomorphic. □

Hence we conclude

Theorem 2.10.3. *The top homology group* $\widetilde{H}_{k-1}\left(\Delta\left(\Pi^{\bullet}_{\{\lambda,\underline{m}\}} - \{\hat{1}\}\right)\right)$ *and the direct sum of Specht modules*

$$\bigoplus_{\vec{d}\in V(\lambda,\underline{m})} S^{B(\vec{d})}$$

are isomorphic as $\mathfrak{S}_n$*-modules.*

2.11 Concluding remarks

We have not dealt with the question whether the poset $\Pi^{\bullet}_{\vec{c}}$ is EL-shellable. Recall that Wachs proved that the d-divisible partition lattice $\Pi^{d}_{n} \cup \{\hat{0}\}$ has an EL-labeling. Ehrenborg and Readdy gave an extension of this labeling to prove that $\Pi^{\bullet}_{(d,\ldots,d,m)}$ is EL-shellable [14]. Woodroofe [36] has shown that the order complex $\Delta(\Pi^{d}_{n} - \{\hat{1}\})$ has a convex ear decomposition. This is not true in general for $\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\})$. See for instance $\Delta(\Pi^{\bullet}_{(1,1,1)} - \{\hat{1}\})$ in Figure 2.5.

1. Can the order complex $\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\})$ be shown to be shellable, using the fact that the complex $\Delta_{\vec{c}}$ is shellable? That is, can a shelling of $\Delta_{\vec{c}}$ be lifted, similar to using Quillen's Fiber Lemma? Would this shelling relationship also extend to the two complexes $\Lambda_{\{\lambda,\underline{m}\}}$ and $\Delta\left(\Pi_{\{\lambda,\underline{m}\}} - \{\hat{1}\}\right)$?

2. Kozlov [21] introduced the notion of a poset to be EC-shellable. He showed that the filter Π_{λ} in the partition lattice generated by a knapsack partition λ is EC-shellable [21, Theorem 4.1]. Is is possible to show that $\Pi^{\bullet}_{\vec{c}}$ and $\Pi^{\bullet}_{\{\lambda,\underline{m}\}}$ are EC-shellable? Furthermore, Kozlov computes the Möbius function of Π_{λ}. See [21, Corollary 8.5]. Can his answer be expressed in terms of permutation statitics, such as the descent set statistic?

3. Can these techniques be used for studying other subposets of the partition lattice? One such subposet is the odd partition lattice, that is, the collection of all partition where each block size is odd. More generally, what can be said about the case when all the block sizes are congruent to r modulo d? These posets have been studied in [6] and [33]. Moreover, what can be said about the poset $\Pi^{\bullet}_{\{\lambda,\underline{m}\}}$ when $\{\lambda, \underline{m}\}$ is not a pointed knapsack partition?

4. Another analogue of the partition lattice is the Dowling lattice. Subposets of the Dowling lattice have been studied in [14] and [18, 19]. Here the first question to ask is: what is the right analogue of ordered set partitions?

Wachs gave a basis for the top homology of the order complex of the d-divisible partition lattice [32, Section 2]. Each cycle in her basis is the barycentric subdivision of the boundary of a cube. We can similarly describe a basis for the order complex of $\Pi^{\bullet}_{\vec{c}}$. The major difference is that the cycles are the barycentric subdivision of a different polytope depending on the composition $\vec{c}$. In order to describe this polytope, recall that the n-dimensional root polytope R_n (of type A) is the defined as the intersection of the $(n+1)$-dimensional cross-polytope $\mathrm{conv}(\{\pm 2\mathbf{e}_i\}_{1 \leq i \leq n+1})$ and the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$. Equivalently, the root polytope R_n can be defined as the convex hull of the set $\{\mathbf{e}_i - \mathbf{e}_j\}_{1 \leq i,j \leq n+1}$. Lastly, let S_n denote the n-dimensional simplex.

We state the following theorem without proof.

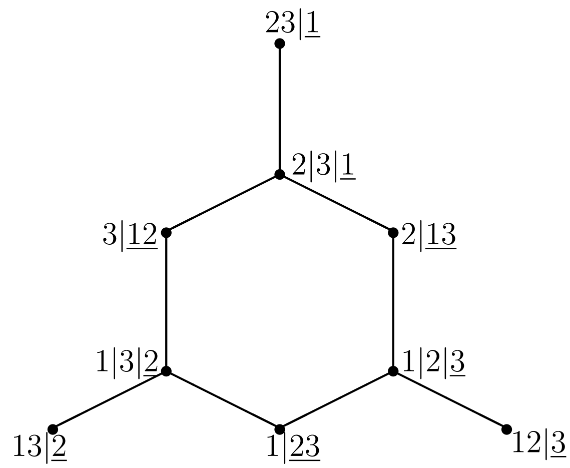


Figure 2.5: The order complex of the poset $\Pi^{\bullet}_{(1,1,1)} - \{\hat{1}\}$.

Theorem 2.11.1. *Given a composition $\vec{c}$, there is a basis for $\widetilde{H}_{k-2}(\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}))$ where each basis element is the barycentric subdivision of the boundary of the Cartesian product:*

$$\begin{array}{ll} R_{|K_1|-1} \times R_{|K_2|-1} \times \cdots \times R_{|K_{n-k}|-1} \times S_{|K_{n-k+1}|-1} & \text{if } c_1 \neq 1, \\ S_{|K_1|-1} \times R_{|K_2|-1} \times \cdots \times R_{|K_{n-k}|-1} \times S_{|K_{n-k+1}|-1} & \text{if } c_1 = 1. \end{array}$$

Note that when all the parts of the composition $\vec{c}$ is greater than 1, this polytope reduces to the $(k-1)$-dimensional cube.

In recent preprint Miller [24] has extended the definition of the partition poset $\Pi^{\bullet}_{\vec{c}}$ from type A to all real reflection groups and the complex reflection groups known as Shephard groups. Can his techniques also extend our results for the filter $\Pi^{\bullet}_{\{\lambda,\underline{m}\}}$ where $\{\lambda, \underline{m}\}$ is a knapsack partition?

Chapter 3 Descent pattern avoidance

3.1 Introduction

Ehrenborg, Kitaev and Perry [11] used the spectrum of linear operators on the space $L^2([0,1]^m)$ to study the asymptotics of consecutive pattern avoidance. We extend their techniques to study asymptotics of sums over all permutations where each term is a product of weights which depend on the consecutive patterns of a fixed length $m+1$. When the weights are all zero or one, this reduces to studying consecutive pattern avoidance. Furthermore, when the weights depend on the descent pattern, we show how to obtain the equation whose roots are the spectrum of the associated linear operator. In general this is a transcendental equation and hard to solve.

We give two length 4 examples. First we study the number of permutations with no triple ascents and no triple descents. This is equivalent to $\{1234, 4321\}$-avoiding permutations. We determine the transcendental eigenvalue equation and a numerical approximation to the largest root, which gives the asymptotics of the number such permutations. Unfortunately, the computation of the eigenvalue equation is too long to do by hand and a computer algebra system such as Maple or Mathematica is needed to carry out the calculation.

The second example is permutations that avoid the ten alternating patterns 1324, 1423, 2314, 2413, 3412 and 2143, 3142, 3241, 4132, 4231. This is the class of permutations with no isolated ascents or descents. Yet again, we obtain the transcendental eigenvalue equation satisfied by the spectrum and give a numerical approximation to its largest root.

We next turn to a weighted length 3 example. We are interested in the sum over all 123-avoiding permutations where the term is 2 to the power of the number of double descents. Here we also consider the extra conditions if the permutation begins/ends with an ascent or a descent. The associated operator only has one non-zero eigenvalue, namely 1. Hence the asymptotics is a constant c times n factorial and the error term is bounded by $n! \cdot r^n$ where r is an arbitrary small positive number.

It remains to understand the error. We are able to find their generating functions and give the explicit recurrences for the sums. Furthermore, we show that the error is the smallest possible! The asymptotics is $c \cdot n!$ (where the constant c is irrational, in fact, transcendental) and the explicit expression is the nearest integer to $c \cdot n!$ for large enough n. This behavior occurs with the derangement numbers. This classical sequence makes its appearance as one of the sequences that we study.

We end the chapter with concluding remarks and open problems.

3.2 Weighted consecutive pattern avoidance

For $x_1, x_2, \ldots, x_k$ distinct real values, define $\Pi(x_1, x_2, \ldots, x_k)$ to be the unique permutation σ in the symmetric group $\mathfrak{S}_k$ such that $x_i < x_j$ if and only if $\sigma_i < \sigma_j$ for all indices $1 \leq i < j \leq k$.

Let wt be a real-valued weight function on the symmetric group $\mathfrak{S}_{m+1}$. Similarly, let $\mathrm{wt}_1, \mathrm{wt}_2$ be two real-valued weight functions on the symmetric group $\mathfrak{S}_m$. We call wt_1 and wt_2 the initial, respectively, the final weight function. We extend these three weight functions to the symmetric group $\mathfrak{S}_n$ for $n \geq m$ by defining

$$\begin{aligned} \mathrm{Wt}(\pi) &= \mathrm{wt}_1(\Pi(\pi_1, \pi_2, \ldots, \pi_m)) \\ &\quad \cdot \prod_{i=1}^{n-m} \mathrm{wt}(\Pi(\pi_i, \pi_{i+1}, \ldots, \pi_{i+m})) \\ &\quad \cdot \mathrm{wt}_2(\Pi(\pi_{n-m+1}, \pi_{n-m+2}, \ldots, \pi_n)). \end{aligned}$$

In other words, the weight of a permutation π in $\mathfrak{S}_n$ is the product of the initial weight function wt_1 applied to the m first entries of π with the product of the weight function wt applied to every segment of π of length $m+1$ with the final weight function wt_2 applied to the m last entries of π. The question is what can one say about the quantity

$$\alpha_n = \sum_{\pi \in \mathfrak{S}_n} \mathrm{Wt}(\pi).$$

Consecutive pattern avoidance can be studied this way by using the weight functions $\mathrm{wt}_1(\sigma) = \mathrm{wt}_2(\sigma) = 1$ for all σ in $\mathfrak{S}_m$ and $\mathrm{wt}(\sigma) = 1$ if $\sigma \notin S$ and $\mathrm{wt}(\sigma) = 0$ otherwise, where $S \subseteq \mathfrak{S}_{m+1}$ is the set of forbidden patterns. Observe then that a permutation $\pi \in \mathfrak{S}_n$ avoids the patterns in S if and only if $\mathrm{Wt}(\pi) = 1$. Note that by letting the initial weight function wt_1 and the final weight function wt_2 be $0,1$-functions, we are studying consecutive pattern avoidance with forbidden initial and final configurations.

The methods of Ehrenborg, Kitaev and Perry [11] to study the asymptotics of consecutive pattern avoidance by considering the spectrum of operators on $L^2([0,1]^m)$ naturally extends to this more general setting of weights on permutations.

Define the function χ on the $(m+1)$-dimensional unit cube $[0,1]^{m+1}$ by $\chi(x) = \mathrm{wt}(\Pi(x))$. Similarly, define the two functions κ and μ on the m-dimensional unit cube $[0,1]^m$ by $\kappa(x) = \mathrm{wt}_1(\Pi(x))$ and $\mu(x) = \mathrm{wt}_2(\Pi(x))$. Note that χ, κ and μ are undefined on a point with two equal coordinates. However, this situation occurs on a set of measure zero and hence can be ignored. Next define the operator T on the space $L^2([0,1]^m)$ by

$$T(f(x_1, \ldots, x_m)) = \int_0^1 \chi(t, x_1, \ldots, x_m) \cdot f(t, x_1, \ldots, x_{m-1}) \, dt. \tag{3.1}$$

Note that $L^2([0,1]^m)$ is a Hilbert space with the inner product defined by

$$(f,g) = \int_{[0,1]^m} f(x_1,\ldots,x_m) \cdot \overline{g(x_1,\ldots,x_m)}\, dx_1 \cdots dx_m.$$

The adjoint operator T^* is defined by the relation $(f, T^*(g)) = (T(f), g)$. For the operator T defined in equation (3.1) we have that

$$T^*(f(x_1,\ldots,x_m)) = \int_0^1 \chi(x_1,\ldots,x_m,u) \cdot f(x_2,\ldots,x_m,u)\, du$$

Finally, the spectrum of an operator T is all the values λ such that $T - \lambda \cdot I$ is not an invertible operator.

Proposition 3.2.1. *The sum of the weights of the permutations in $\mathfrak{S}_n$ for $n \geq m$ is given by*

$$\alpha_n = n! \cdot \left(T^{n-m}(\kappa), \mu\right).$$

Proof. We have

$$\begin{aligned}
\alpha_n/n! &= 1/n! \cdot \sum_{\Pi \in \mathfrak{S}_n} \mathrm{Wt}(\Pi) \\
&= \int_{[0,1]^n} \kappa(x_1,\ldots,x_m) \cdot \prod_{i=1}^{n-m} \chi(x_i,\ldots,x_{i+m}) \cdot \mu(x_{n-m+1},\ldots,x_n)\, dx.
\end{aligned}$$

By induction on j we have that

$$T^j(\kappa)(x_{j+1},\ldots,x_{j+m}) = \int_{[0,1]^j} \kappa(x_1,\ldots,x_m) \cdot \prod_{i=1}^{j} \chi(x_i,\ldots,x_{i+m})\, dx_1 \cdots dx_j.$$

Substitute this into the previous identity we obtain

$$\begin{aligned}
\alpha_n/n! &= \int_{[0,1]^m} \left(\int_{[0,1]^{n-m}} \kappa(x_1,\ldots,x_m) \cdot \prod_{i=1}^{n-m} \chi(x_i,\ldots,x_{i+m})\, dx_1 \cdots dx_{n-m} \right) \\
&\qquad \cdot \mu(x_{n-m+1},\ldots,x_n)\, dx_{n-m+1} \cdots dx_n \\
&= \int_{[0,1]^m} T^{n-m}(\kappa)(x_{n-m+1},\ldots,x_n) \cdot \mu(x_{n-m+1},\ldots,x_n)\, dx_{n-m+1} \cdots dx_n \\
&= \left(T^{n-m}(\kappa), \mu\right).
\end{aligned}$$

□

Generalizing the main result in [11], we have the following theorem.

Theorem 3.2.2. *The nonzero spectrum of the associated operator T consists of discrete eigenvalues of finite multiplicity which may accumulate only at 0. Furthermore, let r be a positive real number such that there is no eigenvalue of T with modulus r and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T greater in modulus than r. Assume that $\lambda_1, \ldots, \lambda_k$ are simple eigenvalues with associated eigenfunctions φ_i and that the adjoint operator T^* has eigenfunctions ψ_i corresponding the eigenvalues λ_i. Then we have the expansion*

$$\alpha_n/n! = \big(T^{n-m}(\kappa), \mu\big) = \sum_{i=1}^{k} \frac{(\varphi_i, \mu) \cdot \big(\kappa, \overline{\psi_i}\big)}{\big(\varphi_i, \overline{\psi_i}\big)} \cdot \lambda_i^{n-m} + O(r^n). \tag{3.2}$$

Proof. By analytic functional calculus we can evaluate the operator T^{n-m} by integrating in the complex plane; see Theorem 6(c) in [9, Section VII.3]. We have

$$T^{n-m} = \frac{1}{2\pi i} \cdot \oint_{|z|=R} \frac{z^{n-m}}{zI - T} dz,$$

where R is greater than the spectral radius of T and we orient the circle in positive orientation.

Let σ be the set $\{\lambda_1, \ldots, \lambda_k\}$ and let $E(\sigma)$ denotes the sum of the projections $E(\lambda_1) + \cdots + E(\lambda_k)$. By Theorem 22 in [9, Section VII.3] and that the eigenvalues $\lambda_1, \ldots, \lambda_k$ are simple, we have that

$$T^{n-m} \cdot E(\sigma) = \sum_{i=1}^{k} E(\lambda_i) \cdot \lambda_i^{n-m}.$$

We can estimate the operator $T^{n-m} \cdot (I - E(\sigma))$ by shrinking the path of integration to a circle of radius r

$$T^{n-m} \cdot (I - E(\sigma)) = \frac{1}{2\pi i} \cdot \oint_{|z|=r} \frac{z^{n-m}}{zI - T} dz.$$

We bound this integral by

$$\begin{aligned} \big\|T^{n-m} \cdot (I - E(\sigma))\big\| &= \left\| \frac{1}{2\pi i} \cdot \oint_{|z|=r} \frac{z^{n-m}}{zI - T} dz \right\| \\ &\leq \frac{1}{2\pi} \cdot \oint_{|z|=r} \left\| \frac{1}{zI - T} \right\| dz \cdot r^{n-m} \\ &\leq \sup_{|z|=r} \big\|(zI - T)^{-1}\big\| \cdot r^{n-m} \\ &= O\left(r^n\right), \end{aligned}$$

where the last equality follows from that the supremum does not depend on n. Hence the inner product $(T^{n-m} \cdot (I - E(\sigma))\kappa, \mu)$ is also bounded by $O(r^n)$. Thus we conclude that

$$\begin{aligned}
\left(T^{n-m}(\kappa), \mu\right) &= \left(T^{n-m} E(\sigma)\kappa, \mu\right) + \left(T^{n-m} \cdot (I - E(\sigma))\kappa, \mu\right) \\
&= \sum_{i=1}^{k} (E(\lambda_i)\kappa, \mu) \cdot \lambda_i^{n-m} + O(r^n) \\
&= \sum_{i=1}^{k} \frac{(\varphi_i, \mu) \cdot \left(\kappa, \overline{\psi_i}\right)}{\left(\varphi_i, \overline{\psi_i}\right)} \cdot \lambda_i^{n-m} + O(r^n).
\end{aligned}$$

□

Theorem 3.2.2 requires us to determine both the eigenfunction φ and the adjoint eigenfunction ψ for each eigenvalue in order to compute the constant in each term. However, when the weight function has symmetry in the sense described below then the adjoint eigenfunction can be determined from the eigenfunction.

Let J be the involution on $L^2([0,1]^m)$ given by $J(f(x_1, x_2, \ldots, x_m)) = f(1 - x_m, \ldots, 1 - x_2, 1 - x_1)$. Note that J is a self-adjoint operator on $L^2([0,1]^m)$, that is, $(Jf, g) = (f, Jg)$. Similar to [11, Lemma 4.7] we have that

Lemma 3.2.3. *Assume that the weight function* wt *is real valued and satisfies the symmetry*

$$\mathrm{wt}(\sigma) = \mathrm{wt}(m + 2 - \sigma_{m+1}, m + 2 - \sigma_m, \ldots, m + 2 - \sigma_1)$$

for all $\sigma \in \mathfrak{S}_{m+1}$. If φ is an eigenfunction of the operator T with eigenvalue λ then $\psi = J\varphi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ. Furthermore, we have the equality $\left(f, \overline{\psi}\right) = (\varphi, Jf)$ for a real valued function f

To prove Lemma 3.2.3, the only part that differs from the proof in [11, Lemma 4.7] is the line $\left(f, \overline{\psi}\right) = \left(f, \overline{J\varphi}\right) = (f, J\overline{\varphi}) = (Jf, \overline{\varphi}) = \left(\varphi, \overline{Jf}\right) = (\varphi, Jf)$.

3.3 Weighted descent pattern avoidance

We now introduce weighted descent pattern avoidance and the connection with consecutive pattern avoidance. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ define its *descent word* to be $u(\pi) = u_1u_2\cdots u_{n-1}$ where $u_i = \mathbf{a}$ if $\pi_i < \pi_{i+1}$ and $u_i = \mathbf{b}$ if $\pi_i > \pi_{i+1}$, that is, an $\mathbf{a}$ at position i encodes that π has an ascent at position i and a $\mathbf{b}$ encodes a descent.

Let wt be a weight function on **ab**-words of length m, that is, the set $\{\mathbf{a}, \mathbf{b}\}^m$. Similarly, let wt_1 and wt_2 be weight functions on **ab**-words of length $m - 1$. We extend this weight function to words of length n greater than $m - 1$ by letting

$$\mathrm{Wt}(v_1 \cdots v_n) = \mathrm{wt}_1(v_1 \cdots v_{m-1}) \cdot \prod_{i=1}^{n-m+1} \mathrm{wt}(v_i \cdots v_{i+m-1}) \cdot \mathrm{wt}_2(v_{n-m+2} \cdots v_n).$$

Finally, we extend the weight to permutations by letting $\mathrm{Wt}(\pi) = \mathrm{Wt}(u(\pi))$.

Recall that the word x has the word w as a *factor* if we can write $x = v \cdot w \cdot z$, where v and z are also words and the dot denotes concatenation. Let U be a collection of **ab**-words of length m, that is, U is a subset of $\{\mathbf{a}, \mathbf{b}\}^m$. Define $S(U)$ by

$$S(U) = \{\sigma \in \mathfrak{S}_{m+1} \; : \; u(\sigma) \in U\}.$$

It is clear that a permutation π that avoids the descent patterns in U is equivalent to that the permutation avoids the consecutive patterns in $S(U)$. Hence descent pattern avoidance is a special case of consecutive pattern avoidance.

A few examples are in order.

Example 3.3.1. *$m = 1$ and $U = \{\boldsymbol{b}\}$. There is only one permutation without any descents, namely $12 \cdots n$, and hence $\alpha_n = 1$.*

Example 3.3.2. *$m = 2$ and $U = \{\boldsymbol{ab}\}$. This forces the permutation to have no peaks. Hence $\alpha_n = 2^{n-1}$ for $n \geq 1$.*

Example 3.3.3. *$m = 2$ and $U = \{\boldsymbol{aa}, \boldsymbol{bb}\}$. This forces the permutation to be alternating. Alternating permutations are enumerated by the Euler numbers, that is, $\alpha_n = 2 \cdot E_n$ for $n \geq 2$ and $\alpha_n = 1$ for $n \leq 1$; see [28, Section 3.16] or [11, Example 1.11].*

For an **ab**-word u of length $m-1$ define the descent polytope P_u to be the subset of the unit cube $[0,1]^m$ corresponding to all vectors with descent word u, that is,

$$P_u = \{(x_1, \ldots, x_m) \in [0,1]^m \; : \; x_i \leq x_{i+1} \text{ if } u_i = \mathbf{a} \text{ and } x_i \geq x_{i+1} \text{ if } u_i = \mathbf{b}\}.$$

Observe that the unit cube $[0,1]^m$ is the union of the 2^{m-1} descent polytopes. Similar to [11, Proposition 4.3 and Corollary 4.4] we have the next proposition. Furthermore, the proof is also similar and hence omitted.

Proposition 3.3.4. *Let T be the operator associated with a weighted descent pattern avoidance and k an integer such that $0 \leq k \leq m-1$. Let u be an **ab**-word of length $m-1$ and f a function in $L^2([0,1]^m)$. Then the function $T^k(f)$ restricted to the descent polytope P_u only depends on the variables x_1 through x_{m-k}.*

A direct consequence of Proposition 3.3.4 is that the eigenfunctions have a special form:

Corollary 3.3.5. *If φ is an eigenfunction of T associated to a non-zero eigenvalue, then the eigenfunction φ restricted to any descent polytope P_u only depends on the variable x_1.*

Let V be the subspace of $L^2([0,1]^m)$ consisting of all functions f such that the restriction $f|_{P_u}$ only depends on the variable x_1 for all words u of length $m-1$. Let f be a function in the subspace V. Then the function $T(f)$ is described as follows. For an **ab**-word u of length $m-2$ and $y \in \{\mathbf{a}, \mathbf{b}\}$ we have

$$T(f)|_{P_{uy}} = \int_0^{x_1} \mathrm{wt}(\mathbf{a}uy) \cdot f(t)|_{P_{\mathbf{a}u}}\, dt + \int_{x_1}^{1} \mathrm{wt}(\mathbf{b}uy) \cdot f(t)|_{P_{\mathbf{b}u}}\, dt. \tag{3.3}$$

In light of Corollary 3.3.5 to solve the eigenvalue problem for the operator $T : L^2([0,1]^m) \longrightarrow L^2([0,1]^m)$, it is enough to solve the eigenvalue problem for the restricted operator $T|_V : V \longrightarrow V$. The restricted operator is of a particular form, which we describe in the next section.

3.4 A general operator and its spectrum

Recall that for a square matrix M the exponential matrix M is defined by the converging power series

$$e^M = \sum_{k \geq 0} M^k/k! = I + M + M^2/2 + M^3/3! + \cdots .$$

The general solution of the system of first order linear equations $\frac{d}{dx}\vec{p}(x) = M \cdot \vec{p}(x)$ is given by $\vec{p}(x) = e^{M \cdot x} \cdot \vec{c}$ where $\vec{c}$ is the initial condition $\vec{p}(0)$.

To help notation, let $\gamma(M)$ denote the matrix

$$\gamma(M) = \int_0^1 e^{M \cdot t} dt,$$

where the integration is entrywise. Observe that

$$M \cdot \gamma(M) = \int_0^1 M \cdot e^{M \cdot t} dt = \left[e^{M \cdot t}\right]_0^1 = e^M - I. \tag{3.4}$$

Hence when M is non-singular we can write $\gamma(M) = M^{-1} \cdot (e^M - I)$. Also note that by integrating the power series of $e^{M \cdot t}$ term by term we obtain that

$$\gamma(M) = \sum_{k \geq 0} M^k/(k+1)! = I + M/2 + M^2/3! + M^3/4! + \cdots .$$

The matrix $\gamma(M)$ appear in the following two integrals.

Lemma 3.4.1. *The two following indefinite integrals hold:*

$$\begin{aligned} \int e^{M \cdot t}\, dt &= \gamma(M \cdot t) \cdot t + \vec{C}, \\ \int M \cdot t \cdot e^{M \cdot t}\, dt &= t \cdot e^{M \cdot t} - \gamma(M \cdot t) \cdot t + \vec{C}. \end{aligned}$$

Proof. The first integral follows by integrating the power series termwise. The second integral follows from integrating the identity $M \cdot t \cdot e^{M \cdot t} + e^{M \cdot t} = \frac{d}{dt}\left(t \cdot e^{M \cdot t}\right)$. $\square$

Let A and B be two $k \times k$ matrices. Consider the integral operator T defined on vector-valued functions by

$$T(\vec{p}(x)) = A \cdot \int_0^x \vec{p}(t)\, dt + B \cdot \int_x^1 \vec{p}(t)\, dt, \tag{3.5}$$

where the integration is componentwise.

Observe that the restricted operator described in equation (3.3) is of the form (3.5) by letting A and B be matrices indexed by **ab**-words of length $m-1$ and the entries be given by

$$A_{uy,\mathbf{a}u} = \mathrm{wt}(\mathbf{a}uy) \quad \text{and} \quad B_{uy,\mathbf{b}u} = \mathrm{wt}(\mathbf{b}uy)$$

where $y \in \{\mathbf{a}, \mathbf{b}\}$ and u in an **ab**-word of length $m-2$, and the remaining entries of the matrices are 0.

The following theorem concerns the eigenvalues and eigenfunctions of the operator in (3.5).

Theorem 3.4.2. *The non-zero spectrum of the operator T is given by the non-zero roots of the equation* $\det(P) = 0$*, where the matrix P is given by*

$$P = -\lambda \cdot I + B \cdot \gamma((A-B)/\lambda), \tag{3.6}$$

and the eigenfunctions are of the form $\vec{p}(x) = e^{(A-B)/\lambda \cdot x} \cdot \vec{c}$, where the vector $\vec{c}$ satisfies the equation $P \cdot \vec{c} = 0$.

Proof. Differentiate the eigenfunction equation $\lambda \cdot \vec{p} = T(\vec{p})$ with respect to x to obtain the differential equation

$$\frac{d}{dx}\vec{p}(x) = M \cdot \vec{p}(x),$$

where we let M denote the matrix $1/\lambda \cdot (A-B)$. This equation has the solution

$$\vec{p}(x) = e^{M \cdot x} \cdot \vec{c},$$

where $\vec{c}$ is the initial condition. Substituting the solution for the differential equation back into the eigenfunction equation, we obtain

$$\begin{aligned}
\lambda \cdot e^{M \cdot x} \cdot \vec{c} &= A \cdot \int_0^x e^{M \cdot t} \cdot \vec{c}\, dt + B \cdot \int_x^1 e^{M \cdot t} \cdot \vec{c}\, dt \\
&= A \cdot [\gamma(M \cdot t) \cdot t]_0^x \cdot \vec{c} + B \cdot [\gamma(M \cdot t) \cdot t]_x^1 \cdot \vec{c} \\
&= ((A-B) \cdot \gamma(M \cdot x) \cdot x + B \cdot \gamma(M)) \cdot \vec{c} \\
&= \left(\lambda \cdot \left(e^{M \cdot x} - I\right) + B \cdot \gamma(M)\right) \cdot \vec{c}.
\end{aligned}$$

Canceling terms we obtain $P \cdot \vec{c} = 0$. We can only find the non-zero vector $\vec{c}$ if the matrix P is singular, that is, has a zero determinant. $\square$

In the case when $A - B$ is non-singular the condition in Theorem 3.4.2 can be expressed as

$$\begin{aligned} 0 &= \det(P) \cdot \det(M) \\ &= \det\left(-A + B \cdot e^{(A-B)/\lambda}\right). \end{aligned}$$

Theorem 3.4.3. *An eigenvalue λ of the operator T is simple if its associated eigenfunction $\vec{p}(x)$ satisfies the vector identity*

$$B \cdot e^{(A-B)/\lambda} \cdot \vec{p}(0) \neq 0. \tag{3.7}$$

Proof. Assume that the eigenvalue λ is not simple, that is, it satisfies the generalized eigenvalue equation $\lambda \cdot \vec{q} = T(\vec{q}) + \vec{p}$. Differentiate this equation to obtain

$$\lambda \cdot \frac{d}{dx}\vec{q}(x) = (A - B) \cdot \vec{q}(x) + \frac{d}{dx}\vec{p}(x).$$

Again let $M = (A - B)/\lambda$. Multiply both sides with $1/\lambda \cdot e^{-M \cdot x}$ to obtain

$$e^{-M \cdot x} \cdot \frac{d}{dx}\vec{q}(x) - M \cdot e^{-M \cdot x} \cdot \vec{q}(x) = 1/\lambda \cdot e^{-M \cdot x} \cdot \frac{d}{dx}\vec{p}(x).$$

This equation is equivalent to

$$\frac{d}{dx}\left(e^{-M \cdot x} \cdot \vec{q}(x)\right) = 1/\lambda \cdot M \cdot \vec{c}.$$

Hence we have the general solution

$$\vec{q}(x) = 1/\lambda \cdot e^{M \cdot x} \cdot M \cdot \vec{c} \cdot x + e^{M \cdot x} \cdot \vec{d},$$

where $\vec{d}$ is a constant vector. Without loss of generality we can set $\vec{d} = 0$ since we are looking for a particular solution. Inserting the particular solution $1/\lambda \cdot e^{M \cdot x} \cdot M \cdot \vec{c} \cdot x$ into the generalized eigenvalue equation, we obtain

$$\begin{aligned} M \cdot x \cdot e^{M \cdot x} \cdot \vec{c} &= A/\lambda \cdot \int_0^x M \cdot t \cdot e^{M \cdot t} dt \cdot \vec{c} + B/\lambda \cdot \int_x^1 M \cdot t \cdot e^{M \cdot t} dt \cdot \vec{c} + e^{M \cdot x} \cdot \vec{c} \\ &= A/\lambda \cdot \left[t \cdot e^{M \cdot t} - \gamma(M \cdot t) \cdot t\right]_0^x \cdot \vec{c} + B/\lambda \cdot \left[t \cdot e^{M \cdot t} - \gamma(M \cdot t) \cdot t\right]_x^1 \cdot \vec{c} + e^{M \cdot x} \cdot \vec{c} \\ &= M \cdot \left(x \cdot e^{M \cdot x} - \gamma(M \cdot x) \cdot x\right) \cdot \vec{c} + B/\lambda \cdot \left(e^M - \gamma(M)\right) \cdot \vec{c} + e^{M \cdot x} \cdot \vec{c}. \end{aligned}$$

Cancelling terms using the identity (3.4) and multiplying with λ we have

$$0 = B \cdot \left(e^M - \gamma(M)\right) \cdot \vec{c} + \lambda \cdot \vec{c}.$$

Adding the equation $P \cdot \vec{c} = 0$ to this identity gives us the conclusion of the theorem. $\square$

3.5 Two length four examples

No triple ascents, no triple descents

Let us consider the case when we avoid the two words **aaa** and **bbb**. This is equivalent to avoiding the consecutive patterns 1234 and 4321. In this case we have the two matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note the matrix $A - B$ is invertible and diagonalizable. To simplify calculations let

$$\tau = \sqrt{\frac{1+\sqrt{5}}{2}} \quad \text{and} \quad \sigma = \sqrt{\frac{-1+\sqrt{5}}{2}}.$$

That is, the four eigenvalues of the matrix $A - B$ are $\pm\sigma$ and $\pm\tau \cdot i$.

The determinant P from Theorem 3.4.2 expands as

$$\begin{aligned} \frac{20}{\lambda^4} \cdot \det(P) \quad = \quad & 8 + \Big(3 + i + \sqrt{5} \cdot (\tau + \sigma \cdot i)\Big) \cdot e^{(\sigma+\tau\cdot i)/\lambda} \\ & + \Big(3 - i + \sqrt{5} \cdot (\tau - \sigma \cdot i)\Big) \cdot e^{(\sigma-\tau\cdot i)/\lambda} \\ & + \Big(3 - i + \sqrt{5} \cdot (-\tau + \sigma \cdot i)\Big) \cdot e^{(-\sigma+\tau\cdot i)/\lambda} \\ & + \Big(3 + i + \sqrt{5} \cdot (-\tau - \sigma \cdot i)\Big) \cdot e^{(-\sigma-\tau\cdot i)/\lambda}. \end{aligned}$$

Thus we obtain

Proposition 3.5.1. *Let λ_0 be the largest real positive root of the equation*

$$\begin{aligned} -8 \quad = \quad & \Big(3 + i + \sqrt{5} \cdot (\tau + \sigma \cdot i)\Big) \cdot e^{(\sigma+\tau\cdot i)/\lambda} \\ & + \Big(3 - i + \sqrt{5} \cdot (\tau - \sigma \cdot i)\Big) \cdot e^{(\sigma-\tau\cdot i)/\lambda} \\ & + \Big(3 - i + \sqrt{5} \cdot (-\tau + \sigma \cdot i)\Big) \cdot e^{(-\sigma+\tau\cdot i)/\lambda} \\ & + \Big(3 + i + \sqrt{5} \cdot (-\tau - \sigma \cdot i)\Big) \cdot e^{(-\sigma-\tau\cdot i)/\lambda}. \end{aligned} \tag{3.8}$$

Then λ_0 is the largest eigenvalue (in modulus) of the associated operator T and the asymptotics of the number of permutations without triple ascents and triple descents is given by

$$\alpha_n/n! = c \cdot \lambda_0^{n-3} + O(r^n),$$

where c and r are two positive constants such that $r < \lambda_0$.

Proof. It remains to show the eigenvalue λ_0 is simple. Observe that the de Bruijn graph with the two directed edges $\mathbf{aa} \xrightarrow{\mathbf{aaa}} \mathbf{aa}$ and $\mathbf{bb} \xrightarrow{\mathbf{bbb}} \mathbf{bb}$ removed is ergodic. Now the conclusion follows from combining Theorems 1.7 and 4.2 in [11]. $\square$

Using Maple we obtain the following decimal expansion of the three largest roots of equation (3.8)

$$\begin{aligned} \lambda_0 &= 0.9240358576\ldots \\ \lambda_{1,2} &= -0.2875224461\ldots \pm 0.4015233122\ldots \cdot i. \end{aligned}$$

Hence we have that r is bounded below by $|\lambda_{1,2}| = 0.4938523335\ldots$.

For the eigenvalue $\lambda = 0.9240358576\ldots$ we can solve for the vector $\vec{c}$ and we have

$$\vec{c} = \begin{pmatrix} 0.6536190979\ldots \\ 0.6536190979\ldots \\ 0.3815287011\ldots \\ 0 \end{pmatrix}.$$

Thus we have the eigenfunction $\varphi = e^{(A-B)/\lambda \cdot x} \cdot \vec{c}$ and adjoint eigenfunction $\psi = J\varphi$. Note that when we restrict the adjoint eigenfunction ψ to a descent polytope we obtain a function only depending on the last variable x_3. For these two functions we calculate

$$\begin{aligned} (\varphi, \mathbf{1}) = \left(\mathbf{1}, \overline{\psi}\right) &= 0.6020376937\ldots, \\ \left(\varphi, \overline{\psi}\right) &= 0.3647767214\ldots. \end{aligned}$$

Combining this we have the constant

$$\frac{(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} = 0.9936198319\ldots.$$

Thus in numerical terms we have that the asymptotics for the number of permutations with no triple ascents and triple descent is given by

$$0.9936198319\ldots \cdot (0.9240358576\ldots)^{n-3} \cdot n!.$$

Avoiding isolated ascents and descents

We next consider the case when we avoid the two words **aba** and **bab**. This is equivalent to avoiding the ten alternating permutations 1324, 1423, 2314, 2413, 3412 and 2143, 3142, 3241, 4132, 4231. In this case we have the two matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Yet again the matrix $A - B$ is invertible and diagonalizable. The eigenvalues are $\pm\tau$ and $\pm\sigma \cdot i$. Similar to Proposition 3.5.1 we have:

Proposition 3.5.2. *Let λ_0 be the largest real positive root of the equation*

$$\begin{aligned} -8 &= \left(3 - i + \sqrt{5} \cdot (-\tau + \sigma \cdot i)\right) \cdot e^{(\tau+\sigma\cdot i)/\lambda} \\ &\quad + \left(3 + i + \sqrt{5} \cdot (-\tau - \sigma \cdot i)\right) \cdot e^{(\tau-\sigma\cdot i)/\lambda} \\ &\quad + \left(3 + i + \sqrt{5} \cdot (\tau + \sigma \cdot i)\right) \cdot e^{(-\tau+\sigma\cdot i)/\lambda} \\ &\quad + \left(3 - i + \sqrt{5} \cdot (\tau - \sigma \cdot i)\right) \cdot e^{(-\tau-\sigma\cdot i)/\lambda}. \end{aligned} \tag{3.9}$$

Then λ_0 is the largest eigenvalue (in modulus) of the associated operator T and the asymptotics of the number of permutations not having any isolated ascents or descents is given by

$$\alpha_n/n! = c \cdot \lambda_0^{n-3} + O(r^n),$$

where c and r are two positive constants such that $r < \lambda_0$.

The same argument as in Proposition 3.5.1 yields that the largest eigenvalue λ is simple. The only difference is that we consider the de Bruijn graph with the two edges $\mathbf{ab} \xrightarrow{\mathbf{aba}} \mathbf{ba}$ and $\mathbf{ba} \xrightarrow{\mathbf{bab}} \mathbf{ab}$ removed.

Numerically, we find the following three largest roots to equation (3.9):

$$\begin{aligned} \lambda &= 0.6869765032\ldots, \\ \lambda_{1,2} &= 0.1559951131\ldots \pm 0.5317098371\ldots \cdot i. \end{aligned}$$

The next largest root $\lambda_{1,2}$ bounds r from below by $|\lambda_{1,2}| = 0.5541207686\ldots$.

Similar to Subsection 3.5 we can obtain the numerical asymptotic expression for the quantity α_n. The numerical data is as follows:

$$\vec{c} = \begin{pmatrix} 0.4315640876\ldots \\ 0 \\ 0.6378684967\ldots \\ 0.6378684967\ldots \end{pmatrix},$$

and

$$\begin{aligned} (\varphi, \mathbf{1}) = \left(\mathbf{1}, \overline{\psi}\right) &= 0.2798342976\ldots, \\ \left(\varphi, \overline{\psi}\right) &= 0.0878970625\ldots. \end{aligned}$$

Combining this we have the constant

$$\frac{(\varphi, \mathbf{1}) \cdot \left(\mathbf{1}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} = 0.8908970548\ldots.$$

Finally, we conclude that the asymptotics for the number of permutations with no isolated ascents and no isolated descents is given by

$$0.8908970548\ldots \cdot (0.6869765032\ldots)^{n-3} \cdot n!.$$

3.6 A weighted example of length three

Define a weight function on the set of **ab**-words of length 2 such that $\mathrm{wt}(\mathbf{aa}) = 0$, $\mathrm{wt}(\mathbf{bb}) = 2$ and $\mathrm{wt}(\mathbf{ab}) = \mathrm{wt}(\mathbf{ba}) = 1$ and the initial and final weight functions wt_1 and wt_2 are identical to 1. We are interested in understanding the sum

$$\alpha_n = \sum_{\pi \in \mathfrak{S}_n} \mathrm{Wt}(\pi).$$

A more explicit way to write this sum is as follows

$$\alpha_n = \sum_{\pi} 2^{\mathbf{bb}(\pi)},$$

where the sum is over all 123-avoiding permutations of length n and $\mathbf{bb}(\pi)$ denotes the number of double descents of π.

Let us refine the number α_n by considering if the permutation begin with an ascent or a descent, and similarly how the permutation ends, that is, we define $\alpha_n(\mathbf{a}, \mathbf{a})$, $\alpha_n(\mathbf{a}, \mathbf{b})$, $\alpha_n(\mathbf{b}, \mathbf{a})$ and $\alpha_n(\mathbf{b}, \mathbf{b})$ for $n \geq 2$ by

$$\alpha_n(x, y) = \sum \mathrm{Wt}(\pi),$$

where the sum is over all permutations π in $\mathfrak{S}_n$ whose descent word $u(\pi)$ begins with the letter x and ends with the letter y. Note that $\alpha_2(x, y)$ is given by the Kronecker delta $\delta_{x,y}$. These quantities can also be expressed by changing the initial and final weight functions.

By symmetry $\pi_1, \pi_2, \ldots, \pi_n \longmapsto n + 1 - \pi_n, \ldots, n + 1 - \pi_2, n + 1 - \pi_1$ we have that $\alpha_n(\mathbf{a}, \mathbf{b}) = \alpha_n(\mathbf{b}, \mathbf{a})$.

First we consider the spectrum of the associated operator.

Theorem 3.6.1. *The only non-zero eigenvalue of the operator T is $\lambda = 1$. This is a simple eigenvalue. Furthermore, the eigenfunction φ and the adjoint eigenfunction ψ associated with this eigenvalue are given by*

$$\varphi = e^{-x} \cdot \begin{cases} 1 - x & \text{if } 0 \leq x \leq y \leq 1, \\ 2 - x & \text{if } 0 \leq y \leq x \leq 1, \end{cases} \quad \text{and} \quad \psi = e^{y-1} \cdot \begin{cases} y & \text{if } 0 \leq x \leq y \leq 1, \\ y + 1 & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Proof. The associated operator T can be written in the form (3.5) using the matrices

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Note that $A - B$ has eigenvalue -1 of algebraic multiplicity 2, but geometric multiplicity 1, that is, the Jordan form of $A - B$ consists of one Jordan block of size 2. Computing the matrix P we obtain

$$0 = \det(P) = \exp\left(-1/\lambda\right) \cdot \lambda \cdot (\lambda - 1),$$

which only has the non-zero root $\lambda = 1$. Furthermore for this root, the null space of the matrix P is spanned by the vector

$$\vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Finally, it is straightforward to verify $B \cdot e^M \cdot \vec{c} \neq \vec{0}$, hence $\lambda = 1$ is a simple eigenvalue by Theorem 3.4.3. Moreover the eigenfunction φ is given by

$$\varphi = \exp\left(\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \cdot x\right) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{-x} \cdot \begin{pmatrix} 1 - x \\ 2 - x \end{pmatrix}.$$

Since the weight function wt satisfies the symmetry in Lemma 3.2.3, we obtain that the adjoint eigenfunction is given by $\psi = J(\varphi)$. $\square$

The asymptotics can now be calculated:

Theorem 3.6.2. *The asymptotics of the sequences $\alpha_n(\boldsymbol{a}, \boldsymbol{a})$, $\alpha_n(\boldsymbol{a}, \boldsymbol{b})$, $\alpha_n(\boldsymbol{b}, \boldsymbol{b})$ and α_n are given by*

$$\begin{aligned} \alpha_n(\boldsymbol{a}, \boldsymbol{a})/n! &= e - 4 + 4/e + O(r^n), \\ \alpha_n(\boldsymbol{a}, \boldsymbol{b})/n! &= 1 - 2/e + O(r^n), \\ \alpha_n(\boldsymbol{b}, \boldsymbol{b})/n! &= 1/e + O(r^n), \\ \alpha_n/n! &= e - 2 + 1/e + O(r^n), \end{aligned}$$

where r is an arbitrary small positive real number.

Proof. Let $\mathbf{1_a}$ denote the function encoding an ascent, that is, $\mathbf{1_a}(x, y) = 1$ if $x < y$ and 0 otherwise. Similarly, let $\mathbf{1_b}$ be the function encoding a descent, that is, $\mathbf{1_b}(x, y) = 1$ if $x > y$ and 0 otherwise. Note that we have that $J\mathbf{1_a} = \mathbf{1_a}$ and $J\mathbf{1_b} = \mathbf{1_b}$. By letting the initial function κ and the final function μ vary over the two functions $\mathbf{1_a}$ and $\mathbf{1_b}$, we obtain the constant term in the asymptotic expression in Theorem 3.2.2. First we compute the inner products

$$\begin{aligned} (\varphi, \mathbf{1_a}) = \left(\mathbf{1_a}, \overline{\psi}\right) &= 1 - 2 \cdot 1/e, \\ (\varphi, \mathbf{1_b}) = \left(\mathbf{1_b}, \overline{\psi}\right) &= 1/e, \\ \left(\varphi, \overline{\psi}\right) &= 1/e, \end{aligned}$$

where we used Lemma 3.2.3 for two of the five equalities. Hence the constants are:

$$\begin{aligned}
\frac{(\varphi, \mathbf{1_a}) \cdot \left(\mathbf{1_a}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} &= e - 4 + 4 \cdot 1/e, \\
\frac{(\varphi, \mathbf{1_b}) \cdot \left(\mathbf{1_a}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} &= 1 - 2 \cdot 1/e, \\
\frac{(\varphi, \mathbf{1_b}) \cdot \left(\mathbf{1_b}, \overline{\psi}\right)}{\left(\varphi, \overline{\psi}\right)} &= 1/e.
\end{aligned}$$

This proves the three first results of the theorem. The fourth result is obtained by adding the asymptotic expressions for $\alpha_n(\mathbf{a}, \mathbf{a})$, $\alpha_n(\mathbf{a}, \mathbf{b})$, $\alpha_n(\mathbf{b}, \mathbf{a})$ and $\alpha_n(\mathbf{b}, \mathbf{b})$. □

In order to study these sequences further, we introduce the associated exponential generating functions. Let $F_{x,y}(z)$ denote the generating function

$$F_{x,y}(z) = \sum_{n \geq 2} \alpha_n(x, y) \cdot \frac{z^n}{n!}.$$

Similarly, let $F(z)$ be the generating function for the sequence α_n.

Proposition 3.6.3. *The generating functions $F_{x,y}(z)$ satisfies the following equation:*

$$\begin{aligned}
F_{x,y}(z) &= \delta_{x,y} \cdot \frac{z^2}{2!} + \delta_{x,\boldsymbol{b}} \cdot \delta_{y,\boldsymbol{a}} \cdot 2 \cdot \frac{z^3}{3!} \\
&+ \int_0^z \left(F_{x,\boldsymbol{a}}(w) + 2 \cdot F_{x,\boldsymbol{b}}(w)\right) \cdot F_{\boldsymbol{b},y}(w)\, dw \\
&+ \delta_{x,\boldsymbol{a}} \cdot \int_0^z F_{\boldsymbol{b},y}(w)\, dw \\
&+ \delta_{x,\boldsymbol{b}} \cdot \int_0^z w \cdot F_{\boldsymbol{b},y}(w)\, dw \\
&+ \delta_{y,\boldsymbol{b}} \cdot \int_0^z \left(F_{x,\boldsymbol{a}}(w) + 2 \cdot F_{x,\boldsymbol{b}}(w)\right) dw \\
&+ \delta_{y,\boldsymbol{a}} \cdot \int_0^z \left(F_{x,\boldsymbol{a}}(w) + 2 \cdot F_{x,\boldsymbol{b}}(w)\right) \cdot w\, dw. \qquad (3.10)
\end{aligned}$$

Proof. We demonstrate that all the terms on the right-hand side are in fact counting permutations. The first term corresponds to permutations of length 2. The second term corresponds to permutations of length 3 with the element 1 in the middle position, that is, the two permutations 213 and 312.

For the remaining permutations we break a permutation at the position where the element 1 occurs. We obtain two smaller permutations σ and τ of lengths k, respectively, r, where $k+r=n-1$. The elements are distributed in $\binom{n-1}{k}$ ways between these two permutations. This is encoded by multiplication of exponential generating functions. Finally, the integral shifts the coefficient from $w^{n-1}/(n-1)!$ to $z^n/n!$.

We continue to describe the terms. The third term corresponds to $2 \leq k, r$, that is, at least two elements precede the element 1 and at least two elements follow the element 1. Note that τ must begin with a descent to avoid creating a double ascent. Also when σ ends with a descent, we create a double descent when concatenating σ with the element 1. This explains the factor 2 in front of the term $F_{x,b}$.

The fourth term corresponds to $k=0$ and $r \geq 2$. The Kronecker delta states that the permutation starts with an ascent. The fifth term corresponds to $k=1$ and $r \geq 2$, in which the permutation starts with a consecutive descent and ascent. Similarly, the sixth and seventh terms correspond to the two cases $r=0$ and $k \geq 2$, respectively, $r=1$ and $k \geq 2$.

Since each permutation has been accounted for, the equality holds. $\square$

Note that Proposition 3.6.3 is similar in spirit to the equations obtained by Elizalde and Noy [15] for the generating functions for certain classes of pattern avoidance permutations.

Theorem 3.6.4. *The generating functions $F_{x,y}(z)$ and $F(z)$ are given by*

$$\begin{aligned}
F_{\boldsymbol{a},\boldsymbol{a}}(z) &= \frac{1}{1-z} \cdot \left(e^z - 4 + 4 \cdot e^{-z}\right) - 1 + 2 \cdot z, \\
F_{\boldsymbol{a},\boldsymbol{b}}(z) &= \frac{1}{1-z} \cdot \left(1 - 2 \cdot e^{-z}\right) + 1 - z, \\
F_{\boldsymbol{b},\boldsymbol{b}}(z) &= \frac{1}{1-z} \cdot e^{-z} - 1, \\
F(z) &= \frac{1}{1-z} \cdot \left(e^z - 2 + e^{-z}\right).
\end{aligned}$$

Proof. Proposition 3.6.3 can be viewed as a recursion for the coefficient $\alpha_n(x,y)$. Hence the equation in this proposition has a unique solution and it is enough to verify the theorem by showing that the proposed generating functions satisfy equation (3.10).

Finally, the generating function $F(z)$ is obtained by adding the four generating functions $F_{\mathbf{a},\mathbf{a}}(z)$, $F_{\mathbf{a},\mathbf{b}}(z)$, $F_{\mathbf{b},\mathbf{a}}(z)$ and $F_{\mathbf{b},\mathbf{b}}(z)$. $\square$

Since $e^{-z}/(1-z)$ is the generating function for the number of derangements, we obtain

Corollary 3.6.5. *For $n \geq 2$, the number of derangements on n elements, D_n, is given by $\alpha_n(\boldsymbol{b}, \boldsymbol{b})$, that is,*

$$D_n = \sum_{\pi} 2^{\boldsymbol{bb}(\pi)},$$

where the sum is over all permutations π on n elements with no double ascents and starting and ending with a descent.

As corollary to Theorem 3.6.4 we have the following recursions:

Corollary 3.6.6. *Recursions for the sequences $\alpha_n(\boldsymbol{a}, \boldsymbol{a})$, $\alpha_n(\boldsymbol{a}, \boldsymbol{b})$, $\alpha_n(\boldsymbol{b}, \boldsymbol{b})$ and α_n are given by, where $n \geq 3$,*

$$\begin{aligned}
\alpha_n(\boldsymbol{a}, \boldsymbol{a}) &= n \cdot \alpha_{n-1}(\boldsymbol{a}, \boldsymbol{a}) + 1 + 4 \cdot (-1)^n, \\
\alpha_n(\boldsymbol{a}, \boldsymbol{b}) &= n \cdot \alpha_{n-1}(\boldsymbol{a}, \boldsymbol{b}) - 2 \cdot (-1)^n, \\
\alpha_n(\boldsymbol{b}, \boldsymbol{b}) &= n \cdot \alpha_{n-1}(\boldsymbol{b}, \boldsymbol{b}) + (-1)^n, \\
\alpha_n &= n \cdot \alpha_{n-1} + 1 + (-1)^n.
\end{aligned}$$

Using the generating functions in Theorem 3.6.4 we now obtain that the error terms are the smallest possible. We express the result as explicit expressions using the nearest integer function, which we denote by $\lfloor x \rceil$.

Theorem 3.6.7. *The quantities $\alpha_n(\boldsymbol{a}, \boldsymbol{a})$, $\alpha_n(\boldsymbol{a}, \boldsymbol{b})$, $\alpha_n(\boldsymbol{b}, \boldsymbol{b})$ and α_n are given by the explicit expressions*

$$\begin{aligned}
\alpha_n(\boldsymbol{a}, \boldsymbol{a}) &= \lfloor (e - 4 + 4/e) \cdot n! \rceil && \text{for } n \geq 8, \\
\alpha_n(\boldsymbol{a}, \boldsymbol{b}) &= \lfloor (1 - 2/e) \cdot n! \rceil && \text{for } n \geq 3, \\
\alpha_n(\boldsymbol{b}, \boldsymbol{b}) &= \lfloor 1/e \cdot n! \rceil && \text{for } n \geq 2, \\
\alpha_n &= \lfloor (e - 2 + 1/e) \cdot n! \rceil && \text{for } n \geq 4.
\end{aligned}$$

Proof. The third equality is classical. We show the first equality. The coefficient of $z^n/n!$ in the generating function $F_{\mathbf{a},\mathbf{a}}(z)$, for $n \geq 2$, is given by

$$\alpha_n(\mathbf{a}, \mathbf{a}) = n! \cdot \sum_{k=0}^{n} \frac{1^k - 4 \cdot 0^k + 4 \cdot (-1)^k}{k!}.$$

Hence the difference

$$n! \cdot (e - 4 + 4/e) - \alpha_n(\mathbf{a}, \mathbf{a}) = n! \cdot \sum_{k \geq n+1} \frac{1^k + 4 \cdot (-1)^k}{k!},$$

is bounded in absolute value by

$$n! \cdot \sum_{k \geq n+1} \frac{5}{k!} = \frac{5}{n+1} + \frac{5}{(n+1) \cdot (n+2)} + \cdots .$$

Note that this is a decreasing function in n. For $n = 10$ this function dips below $1/2$, showing the first equality for $n \geq 10$. The two cases $n = 8, 9$ can be done by hand. The second and fourth equalities follow by similar arguments. $\square$

3.7 Concluding remarks

Are there other operators of the form (3.1) which only have a finite number of non-zero eigenvalues? Furthermore, if the associated sequences are integer sequences would the corresponding error term be the smallest possible, as in Theorem 3.6.7?

The operators of the form (3.1) have so far yielded four types of behavior:

(i) The operator has an infinite number of eigenvalues and the asymptotic expansion converges. An example of this is alternating permutations. See [11, Example 1.11] and [12]. A second example is $\{123, 231, 312\}$-avoiding permutations. See [11, Section 7].

(ii) The operator has an infinite number of eigenvalues and the asymptotic expansion does not give an expression that converges. This occurs with 123-avoiding permutations and 213-avoiding permutations. See [11, Sections 5 and 6].

(iii) The operator has a finite, but positive, number of non-zero eigenvalues. See Section 3.6. For instance, is there a such operator with exactly two non-zero eigenvalues? What behavior does the error of the asymptotic expansion have? Are there other examples with the smallest possible error?

(iv) The operator has no non-zero eigenvalues. Here the behavior can vary a lot. Compare **ba**-avoiding permutations in Example 3.3.2 with $\{312, 321\}$-avoiding permutations in [8]. Also see [11, Example 3.9].

The two equations (3.8) and (3.9) in Section 3.5 have an interesting pattern in their roots. Consider the two equations in terms of the variable $z = 1/\lambda$. Then the roots lie on the real axis and close to a vertical line in the complex plane. Is there an explanation for this behavior? Switching back to the variable λ it says that the roots lie on the real axis and close to a circle in the complex plane.

Baxter, Nakamura and Zeilberger [2] have developed efficient methods to compute the number of permutations avoiding certain patterns. Their methods using umbral techniques and has been implemented in Maple. Their techniques can be extended to compute the weighted problem introduced in this chapter.

In Section 3.6 we obtained that the number of derangements D_n is given by the sum over all permutations with no double ascents, where each term is 2 to the power of the number of double descents. It is natural to ask for a bijective proof of this fact. In fact, Muldoon Brown and Readdy [25] gave essentially such a bijection.

A *descent run* I in a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is an interval $I = [i, j]$ such that $\pi_i > \pi_{i+1} > \cdots > \pi_j$. Observe that we do not require the interval to be maximal.

Given a permutation π on n elements, write the interval $[1, n]$ as a disjoint union $\bigcup_{r=1}^{k} I_r$ of descent runs of the permutation π. A Muldoon–Readdy barred permutation σ from π is obtained by the following procedure. For each descent run $I_r = [i, j]$ pick an element h_r in the half-open interval $(i, j]$ and bar the elements π_{h_r} through π_j. Observe that for a descent run I_r of cardinality c there are $c - 1$ possibilities to pick the element h_r. Hence to obtain Muldoon–Readdy barred permutation each descent run needs to have cardinality at least 2.

Finally given a permutation π, how many Muldoon–Readdy barred permutations can be obtained from it? If the permutation π has a double ascent there is a maximal descent run of size 1 and hence there is no way to obtain a Muldoon–Readdy permutation. Partition the interval $[1, n]$ into maximal descent runs of π, that is, $[1, n] = \bigcup_s J_s$. Each maximal descent run J_s can be further partitioned into descent runs and be barred. Let the maximal descent run J_s have cardinality c. Then the number of ways of partitioning J_s and barring each descent run is

$$\sum_{(c_1, c_2, \ldots, c_k)} (c_1 - 1) \cdot (c_2 - 1) \cdots (c_k - 1),$$

where the sum ranges over all compositions of the integer c. By the basic generating function argument $1/(1 - x^2/(1 - x)^2) = 1 + x^2/(1 - 2x)$, the above sum is 2^{c-2}. Finally, note that $c - 2$ is the number of double descents in the descent run J_r. Hence there are 2 to the number of double descents in π ways to obtain a Muldoon–Readdy barred permutation from the permutation π.

Finally, Muldoon Brown and Readdy give a bijection between derangements and Muldoon–Readdy barred permutations. See Theorem 6.4 in [25]. This gives a bijective proof of Corollary 3.6.5.

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Vita

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• Research Interests

– Enumerative, Algebraic and Topological Combinatorics.

• Education

– 2012: Ph.D. University of Kentucky, expected.
– 2008: M.A. University of Kentucky.
– 2004: M.A. Yonsei University, South Korea.
– 2001: B.A. Yonsei University.

• Professional Experience

– Teaching Assistant, University of Kentucky, 2006–2012.
– Visiting Scholar, Brandeis University, 2005.
– Teaching Assistant, Yonsei University, 2001–2004.

• Publications

– The topology of restricted partition posets, submitted to Journal of Algebraic Combinatorics, with R. Ehrenborg.
– Descent pattern avoidance, submitted to Journal of Combinatorial Theory, Series A, with R. Ehrenborg.
– Chain-selected complexes, in preparation, with R. Ehrenborg.

• Selected Presentations

– 'The topology of restricted partition posets', poster presentation, Association for Women in Mathematics Workshop in the 2012 Joint Mathematics Meeting, Boston, Massachusetts, January 7, 2012.
– 'Descent pattern avoidance', American Mathematical Society Session on Combinatorics and Graph Theory in the 2012 Joint Mathematics Meeting, Boston, Massachusetts, January 4, 2012.
– 'Descent pattern avoidance', The Early Career and Graduate Student PosterFest, Mathematical Association of America MathFest, Lexington, Kentucky, August 5, 2011.
– 'The topology of restricted partition posets', poster presentation, International Conference

on Formal Power Series and Algebraic Combinatorics, Rejkjavik, Iceland, June 13, 2011.
– 'Descent pattern avoidance', Graduate Student Combinatorics Conference, Pennsylvania State University, April 9, 2011.
– 'The topology of restricted partition posets', Discrete CATS Seminar, University of Kentucky, February 2, 2011.
– 'Extension of the d-divisible partition lattice', Graduate Student Combinatorics Conference, Auburn University, April 4, 2010.

• Awards

– Research Assistantship, University of Kentucky, Summer 2007, 2008, 2010 and 2011.
– University of Kentucky Math Department 2008 Summer Research Award, May 2008.
– National Merit Scholar fellowship from the Korea Research Foundation, January 2005.
– Yonsei University BK Scholarship Student, March 2002–February 2005.
– Yonsei University Scholarship for Students with Excellent Academic Achievements, March 1999.