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COEFFICIENT PROBLEMS FOR FUNCTIONS REGULAR IN AN ELLIPSE

BY W. C. ROYSTER

1. Introduction. Let $f(z)$ be an analytic function which is regular in an ellipse with foci at ± 1 . It is known that such a function possesses an expansion in a series of Tchebychef polynomials in the largest ellipse in which $f(z)$ is regular [8]. We shall be concerned in this paper with the problem of obtaining bounds on the coefficients in the expansion whenever $f(z)$ belongs to certain classes of functions which have specified mapping properties. The classes which we shall consider are

- (1) typically real functions (T)
- (2) functions that are univalent and convex in the direction of the imaginary axis (F)
- (3) functions starlike in the direction of the real axis (R)
- (4) starlike functions (S)
- (5) functions having a diametral line (D).

The class of typically real functions was first studied for Taylor series by Rogosinski [6] and later for Laurent series by Nehari and Schwarz [3]. The classes F and R were studied for Taylor series by Robertson [4], [5]. The class S has been studied for Taylor series by many authors and for Laurent series by Nehari and Schwarz [3], while the class D was studied by Umezawa [7] and De Bruijn [2].

2. The class T . The function $f(z)$ is said to be typically real in E if it is real when and only when z is real. The assumption that $f(z) \in T$ is equivalent to the assumption that $\operatorname{Im} f(z) \cdot \operatorname{Im} z$ possesses the same sign for any $z \in E$ for which $\operatorname{Im} z \neq 0$. Without loss of generality suppose $\operatorname{Im} f(z) \cdot \operatorname{Im} z > 0$. Since $f(z)$ is regular in E , it has an expansion in a series of Tchebychef polynomials

$$(2.1) \quad f(z) = \sum_{n=0}^{\infty} a_n T_n(z), \quad z \in E,$$

where $T_n(z) = \cos n (\cos^{-1} z)$, which converges uniformly in E [8].

First let us relate the class of typically real functions in E to functions having positive real part in E . Let the parametric representation of E be given by $z = a_0 \cos t + i b_0 \sin t$, $0 \leq t < 2\pi$. Since $a_0^2 - b_0^2 = 1$, we let $a_0 = \cosh s_0$, $b_0 = \sinh s_0$, $s_0 > 0$. Any ellipse which is confocal with E and interior to E can be represented by $z = a \cos t + i b \sin t$, $0 \leq t < 2\pi$, with $a = \cosh s$, $b = \sinh s$, $0 < s \leq s_0$. Thus we may write $z = \cos(t - is)$.

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THEOREM 2.1. *If $f(z)$ is typically real in E , then*

$$\operatorname{Re} \{ (bz - ia(1 - z^2)^{\frac{1}{2}}) f(z) \} \geq 0.$$

Proof. $\operatorname{Im} f(z) \cdot \operatorname{Im} z > 0$ is equivalent to $\sin t \operatorname{Im} f(z) > 0$. Since $z = \cos(t - is)$ we have $(1 - z^2)^{\frac{1}{2}} = \sin(t - is)$, where the value of the radical is determined by continuity from the condition that at $t = 0$, $z = a$ and $(1 - z^2)^{\frac{1}{2}} = -ib$. A simple calculation shows that $-i \sin t = bz - ia(1 - z^2)^{\frac{1}{2}}$. Since $\sin t \operatorname{Im} f(z) = \operatorname{Re} \{ -i \sin t f(z) \}$ the proof is complete.

THEOREM 2.2. *Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$, $a_0 = 0$, be regular and typically real in E . Then*

$$(2.2) \quad |a_n| \leq n |a_1| \left(\frac{R - R^{-1}}{R^n - R^{-n}} \right), \quad n = 1, 2, \dots$$

where $R = a + b$. The inequality is sharp.

Proof. Consider $G(z) = (bz - ia(1 - z^2)^{\frac{1}{2}}) f(z)$. Then

$$(2.3) \quad G(z) = -i \sin t \sum_{n=1}^{\infty} a_n \cos n(t - is),$$

$$(2.4) \quad \begin{aligned} \operatorname{Re} \{G(z)\} &= \sum_{n=1}^{\infty} a_n \sinh ns \sin nt \sin t \\ &= \frac{1}{2} \left\{ a_1 \sinh s + \sum_{n=1}^{\infty} [a_{n+1} \sinh(n+1)s - a_{n-1} \sinh(n-1)s] \cos nt \right\}. \end{aligned}$$

Since $\operatorname{Re} \{G(z)\} \geq 0$, a well known classical inequality for non-negative trigonometric series, yields

$$(2.5) \quad |a_{n+1} \sinh(n+1)s - a_{n-1} \sinh(n-1)s| \leq 2 |a_1| \sinh s.$$

Using induction and the fact that $a_0 = 0$ we see that

$$(2.6) \quad |a_n| \leq n |a_1| \frac{\sinh s}{\sinh ns} = n |a_1| \left(\frac{R - R^{-1}}{R^n - R^{-n}} \right).$$

In showing that (2.2) is sharp let us assume that $a_1 = 1$ in order to simplify the manipulations. From [9; 535] we obtain the expansion of the Jacobian elliptic function $1/sn^2u$ which, except for an additive real constant, is

$$(2.7) \quad \frac{1}{sn^2\left(\frac{2Ku}{\pi}\right)} = \left(\frac{\pi}{2K}\right)^2 \left[\csc^2 u - 8 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos 2nu \right],$$

valid for $|\operatorname{Im} u| < \log 1/q$, where for our purpose $K = \pi/2$ and $q = R^{-1}$. Let $u = \frac{1}{2}(\cos^{-1}z + i \log R)$, then a simple computation yields

$$\begin{aligned}
 (2.8) \quad \frac{1}{sn^2(u)} &= -\frac{8q}{1-q^2} \sum_{n=1}^{\infty} n \left(\frac{R - R^{-1}}{R^n - R^{-n}} \right) \cos n(\cos^{-1} z) \\
 &= -\frac{8}{R - R^{-1}} \sum_{n=1}^{\infty} n \left(\frac{R - R^{-1}}{R^n - R^{-n}} \right) T_n(z) \\
 &= -\frac{8}{R - R^{-1}} f_0(z).
 \end{aligned}$$

Hence, except for a real additive constant, the extremal function is

$$(2.9) \quad f_0(z) = -\frac{1}{8} \cdot \frac{R - R^{-1}}{sn^2 \frac{1}{2}(\cos^{-1} z + i \log R)}.$$

To show $f_0(z)$ is typically real in E we note, except for a positive multiplicative constant, that (see [1; 38])

$$\begin{aligned}
 \operatorname{Im} \{f_0(z)\} &= (2sn \, t/2 \, cn \, t/2 \, dn \, t/2) \\
 &\quad \cdot sn[(\log R) - s] cn[(\log R) - s] dn[(\log R) - s].
 \end{aligned}$$

However $e^s \leq e^{s_0} = R$, hence $s \leq \log R$ and

$$\begin{aligned}
 \operatorname{Im} f_0(z) &\geq 0, & 0 \leq t \leq \pi \\
 \operatorname{Im} f_0(z) &\leq 0, & \pi \leq t \leq 2\pi
 \end{aligned}$$

which shows that $f_0(z) \in T$.

3. Functions convex in one direction. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$, a_n real, is regular and convex in the direction of the imaginary axis whenever z is in E . By convex in the direction of the imaginary axis we shall mean that the ellipse $z = \cos(t - is)$, $s \leq s_0$, is mapped by $f(z)$ onto a contour which is cut by a straight line parallel to the imaginary axis in at most two points.

Since the coefficients are real and $T_n^*(z) = T_n(z^*)$ (z^* denotes the complex conjugate of z), the contour is symmetrical about the real axis. Let $f(z) = U(s, t) + iV(s, t)$. If $f(z) \in F$, then $\operatorname{Re} \{f(z)\}$ is a monotonically decreasing function of t for $0 < t < \pi$ and a monotonically increasing function of t for $\pi < t < 2\pi$. We now have the following

LEMMA 3.1. *If $f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$, a_n real, belongs to F for z in E , then $(z^2 - 1)^{\frac{1}{2}} f'(z)$ is typically real, and conversely.*

Proof.

$$\frac{df}{dt} = \frac{df}{dz} \frac{\partial z}{\partial t}, \quad z = \cos(t - is),$$

so that

$$(3.1) \quad \frac{\partial U}{\partial t} + i \frac{\partial V}{\partial t} = -(1 - z^2)^{\frac{1}{2}} f'(z).$$

Hence $\text{Im} \{ (z^2 - 1)^{\frac{1}{2}} f'(z) \} = \partial U / \partial t$. Using the known properties of $\partial U / \partial t$ we have that $(z^2 - 1)^{\frac{1}{2}} f'(z)$ is typically real in E .

THEOREM 3.2. *Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$, $a_0 = 0$, a_n real, be regular and convex in the direction of the imaginary axis for z in E , then*

$$(3.2) \quad |a_n| \leq |a_1| \left(\frac{R + R^{-1}}{R^n + R^{-n}} \right), \quad n = 1, 2, \dots$$

Proof. Let $G(z) = (bz - ia(1 - z^2)^{\frac{1}{2}})(z^2 - 1)^{\frac{1}{2}} f'(z)$. By Lemma 3.1 and Theorem 2.1 we have that $G(z)$ has positive real part in E . Now

$$(3.3) \quad f'(z) = \sum_{n=1}^{\infty} na_n(1 - z^2)^{-\frac{1}{2}} \sin n(\cos^{-1} z),$$

and

$$(3.4) \quad (z^2 - 1)^{\frac{1}{2}} f'(z) = i \sum_{n=1}^{\infty} na_n \sin n(t - is).$$

Therefore

$$\begin{aligned} \text{Re} \{G(z)\} &= \text{Re} \{-i \sin t (z^2 - 1)^{\frac{1}{2}} f'(z)\} \\ (3.5) \quad &= \sum_{n=1}^{\infty} na_n \sin t \sin nt \cosh ns. \\ &= A_1 + \sum_{n=1}^{\infty} (A_{n+1} - A_{n-1}) \cos nt \end{aligned}$$

where $A_n = na_n \cosh ns$.

Since $\text{Re} \{G(z)\} \geq 0$ we have by the aforementioned property of positive trigonometric series that

$$(3.6) \quad |A_{n+1} - A_{n-1}| \leq 2 |A_1|,$$

and since $A_0 = 0$, that $|A_n| \leq n |A_1|$ which establishes (3.2).

In order to investigate the sharpness of (3.2) let us consider the function $F_0(u) = i \csc u / \sinh u - 1$ and let $u = \frac{1}{2} (\cos^{-1} z + i \log R)$. Then according to [10;512]

$$(3.7) \quad F_0(u) = -1 + i \left[\cot u - 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 + q^{2n}} \sin 2nu \right]$$

valid for $|\text{Im } u| < \log R$ where $R = q^{-1}$. A short computation shows that

$$(3.8) \quad F_0(z) = 4 \sum_{n=1}^{\infty} \frac{\cos n(\cos^{-1} z)}{R^n + R^{-n}}.$$

Hence

$$f_0(z) = \frac{R + R^{-1}}{4} F_0(z)$$

yields equality in (3.2). By Lemma 3.1 $f_0(z)$ will be convex in the direction of the imaginary axis if $(z^2 - 1)^{\frac{1}{2}} f'_0(z)$ is typically real in E . However, $f'_0(z)$ is given by

$$(3.9) \quad (z^2 - 1)^{\frac{1}{2}} f'_0(z) = \frac{dn^{\frac{1}{2}}(\cos^{-1} z + i \log R)}{sn^{\frac{3}{2}}(\cos^{-1} z + i \log R)}.$$

Using the fact that $z = \cos(t - is)$ and separating into real and imaginary parts (see [1]) we find

$$(3.10) \quad \begin{aligned} \operatorname{Im} \{f'_0(z)\} = & Msn \, t/2 \, cn \, t/2 \, sn(\log R - s) \{k^2 sn^2 \, t/2 \, dn^2(\log R - s) \\ & + dn^2 \, t/2 \, cn^2(\log R - s)[2cn^2(\log R - s) \\ & + k^2 sn^2(\log R - s)(2 - cn^2 \, t/2)]\} \end{aligned}$$

where M is a positive constant and $0 \leq t \leq 2\pi$. (3.10) shows that $(z^2 - 1)^{\frac{1}{2}} f'_0(z)$ is typically real since the quantity in the braces in the right member of (3.10) is positive.

4. Functions starlike in the direction of the real axis. Let $W = f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$, be regular and single-valued in E . Suppose $f(z)$ is starlike in the direction of the real axis, i.e. suppose for each s , $0 < s \leq s_0$, $f(z)$ maps the ellipse $z = \cos(t - is)$ onto a curve which is cut by the real axis in exactly two points, or $\operatorname{Im} \{f(\cos(t - is))\}$ changes sign twice.

LEMMA 4.1. *Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$ be regular in E and $\operatorname{Im} \{f(z)\}$ change sign exactly twice, say at $t = t_1, t_2$. There exist real numbers μ and ν such that if*

$$(4.1) \quad g(z) = (az - ib(1 - z^2)^{\frac{1}{2}} - \cos \nu) f(\cos((t + \mu) - is)),$$

then $\operatorname{Im} g(z)$ does not change sign for z on E .

Proof. Since $az - ib(1 - z^2)^{\frac{1}{2}} = \cos t$ we have

$$\operatorname{Im} \{g(z)\} = (\cos t - \cos \nu) \operatorname{Im} \{f(\cos((t + \mu) - is))\}.$$

Take $\mu = (t_1 + t_2)/2$ and $\nu = (t_2 - t_1)/2$. Now $\operatorname{Im} \{f(\cos(t + \mu) - is)\}$ changes sign at $t + \mu = t_1, t_2$, or $t = \pm(t_2 - t_1)/2 = \pm \nu$. But $\cos t - \cos \nu$ changes sign at $t = \pm \nu$, $-\pi \leq t \leq \pi$, hence $\operatorname{Im} \{g(z)\}$ does not change sign at $t = \pm \nu$. Consequently $\operatorname{Re} \{ig(z)\}$ or $\operatorname{Re} \{-ig(z)\}$ is positive for z on E .

THEOREM 4.2. *Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$ be regular and single-valued in E . Further, let $\operatorname{Im} \{f(z)\}$ change sign exactly twice on E . Then*

$$(4.2) \quad |a_n R^n - a_n^* R_n^{-n}| \leq n^2 |a_1 R - a_1^* R^{-1}|.$$

Proof. By Lemma (4.1) there exist μ and ν such that

$$(4.3) \quad g(z) = [az - ib(1 - z^2)^{\frac{1}{2}} - \cos \nu] f(\cos((t + \mu) - is))$$

is regular on E and $\text{Im } \{g(z)\}$ does not change signs on E . Since

$$(4.4) \quad g(z) = (\cos t - \cos \nu) \sum_{n=1}^{\infty} a_n T_n(\cos((t + \mu) - is)),$$

with $a_n = \alpha_n + \beta_n$, then

$$\begin{aligned} & \text{Re } \{-ig(z)\} \\ &= (\cos t - \cos \nu) \sum_{n=1}^{\infty} (\alpha_n \sinh ns \sin n\mu + \beta_n \cosh ns \cos n\mu) \cos nt \\ (4.5) \quad &+ (\alpha_n \sinh ns \cos n\mu - \beta_n \cosh ns \sin n\mu) \sin nt \\ &= \frac{1}{2} \sum_{n=1}^{\infty} A_n [\cos(n-1)t + \cos(n+1)t] - 2A_n \cos \nu \cos nt \\ &+ B_n [\sin(n+1)t + \sin(n-1)t] - 2B_n \cos \nu \sin nt, \end{aligned}$$

where

$$\begin{aligned} A_n &= \alpha_n \sinh ns \sin n\mu + \beta_n \cosh ns \cos n\mu \\ (4.6) \quad B_n &= \alpha_n \sinh ns \cos n\mu - \beta_n \cosh ns \sin n\mu, \quad n = 1, 2, \dots \\ A_0 &= B_0 = 0. \end{aligned}$$

Hence

$$\begin{aligned} (4.7) \quad \text{Re } \{-ig(z)\} &= \frac{1}{2} A_1 + \frac{1}{2} \sum_{n=1}^{\infty} [A_{n+1} + A_{n-1} - 2A_n \cos \nu] \cos nt \\ &+ [B_{n+1} + B_{n-1} - 2B_n \cos \nu] \sin nt \geq 0. \end{aligned}$$

It follows by the aforementioned property of positive trigonometric series that

$$(4.8) \quad |(A_{n+1} + A_{n-1} - 2A_n \cos \nu) - i(B_{n+1} + B_{n-1} - 2B_n \cos \nu)| \leq 2 |A_1|.$$

We note that $A_n - iB_n = -ie^{in\mu}/2 (a_n R^n - a_n^* R^{-n})$ and $A_1 = \text{Re } \{-i/2 (a_1 R - a_1^* R^{-1})e^{i\mu}\}$ so that (4.8) becomes

$$\begin{aligned} (4.9) \quad & |(a_{n+1} R^{n+1} - a_{n+1}^* R^{-(n+1)})e^{i(n+1)\mu} + (a_{n-1} R^{n-1} - a_{n-1}^* R^{-(n-1)})e^{i(n-1)\mu} \\ & - 2(a_n R^n - a_n^* R^{-n})e^{in\mu} \cos \nu| \leq 2 |a_1 R - a_1^* R^{-1}|. \end{aligned}$$

Let $c_n = (a_n R^n - a_n^* R^{-n})e^{in\mu}$; then the left hand member of (4.9) may be written as

$$(4.10) \quad c_{n+1} - 2c_n \cos \nu + c_{n-1} = b_n, \quad n = 1, 2, \dots, \text{ with } b_0 = c_1$$

which by (4.9) gives $|b_n| \leq 2 |b_0|$. One can show from (4.10) that

$$(4.11) \quad c_n = \sum_{p=0}^{n-1} b_p \frac{\sin(n-p)\nu}{\sin \nu}.$$

Substituting for b_0 we have

$$(4.12) \quad c_n = c_1 \frac{\sin n\nu}{\sin \nu} + \sum_{p=1}^{n-1} b_p \frac{\sin (n-p)\nu}{\sin \nu}$$

and

$$(4.13) \quad |c_n| \leq n |c_1| + \sum_{p=1}^{n-1} (n-p) |b_p|$$

$$\leq n |c_1| + 2 |c_1| \sum_{p=1}^{n-1} (n-p) = n^2 |c_1|.$$

Hence

$$(4.14) \quad |a_n R^n - a_n^* R^{-1}| \leq n^2 |a_1 R - a_1^* R^{-1}|.$$

It has been tacitly assumed that $f(z)$ is regular on E , however it is sufficient to assume that $f(z) = f(\cos(t - is))$ is regular in and on the ellipse E_1 given by $z = \cos(t - is_1)$, $s_1 < s_0$. Upon proving the result for $f(z)$ regular in the closure of E_1 the corresponding results follow for $f(z)$ regular in E by letting $s_1 \rightarrow s_0$.

Inequality (4.14) has greater geometric significance than is apparent in its present form. Upon separating into real and imaginary parts and squaring, we have

$$(4.15) \quad \alpha_n^2 (R^n - R^{-n})^2 + \beta_n^2 (R^n + R^{-n})^2 \leq n^4 \lambda^2,$$

where $\lambda^2 = \alpha_1^2 (R - R^{-1})^2 + \beta_1^2 (R + R^{-1})^2$. Inequality (4.15) implies that the coefficients in the expansion of $f(z)$ in Theorem 4.2 lie in the closed ellipse with center at the origin and semi-axes $n^2 \lambda / (R^n - R^{-n})$ and $n^2 \lambda / (R^n + R^{-n})$.

To study the question of sharpness of (4.15) let us consider

$$(4.16) \quad F(u) = \frac{1}{16} \frac{d}{du} \left(\frac{1}{\sin^2(u + i \log R + \pi/4)} - \frac{1}{\sin^2(u - \pi/4)} \right)$$

where $1/\sin^2 u$ is given by (2.7). With $K = \pi/2$, $K' = \log R$ we have $q = R^{-2}$ and

$$(4.17) \quad \frac{d}{du} \left(\frac{1}{\sin^2 u} \right) = -8i \sum_{n=1}^{\infty} \frac{n^2}{R^{2n} - R^{-2n}} (R^{2n} e^{2inu} - R^{-2n} e^{-2inu})$$

so that

$$(4.18) \quad F(u) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{in^2}{R^{2n} - R^{-2n}} (R^n e^{-in\pi/2} - R^{-n} e^{in\pi/2}) (R^n e^{2inu} + R^{-n} e^{-2inu}).$$

Further, let $u = \frac{1}{2}(\cos^{-1} z + i \log R)$; then $T_n(z) = (R^n e^{2inu} - R^{-n} e^{-2inu})/2$ which yields

$$(4.19) \quad F(u) = F_1(z) = \sum_{n=1}^{\infty} \frac{in^2}{R^{2n} - R^{-2n}} (R^n e^{-in\pi/2} - R^{-n} e^{in\pi/2}) T_n(z).$$

It is easy to verify that the coefficients in (4.19) yield equality in (4.15), and (4.9) with $\nu = 0$. From reduction formulae [1] it follows that

$$(4.20) \quad F(u) = -\frac{1}{8} cn(u - \pi/4) \left(\frac{k'^2 k^2 sn(u - \pi/4)}{dn^3(u - \pi/4)} - \frac{dn(u - \pi/4)}{sn^3(u - \pi/4)} \right)$$

where k is a parameter called the modulus associated with these elliptic functions and $k' = (1 - k^2)^{1/2}$ is the complementary modulus.

We shall exhibit the existence of an elliptic annulus which we shall call a boundary strip G , formed by E and a confocal ellipse lying inside E , such that on any ellipse E_1 confocal with E and with boundary in G , $\text{Im} \{F_1(z)\}$ changes sign exactly twice, provided $k' > 2^{-1/2}$. Let E_1 be given parametrically by $z = \cos(t - is_1)$, where $s_1 = \log R_1$, $R_1 < R_0$, so that on E_1 , $u = \frac{1}{2}(t + i \log R_0/R_1) = \frac{1}{2}t + i\epsilon$, $\epsilon = \frac{1}{2} \log R_0/R_1$. We shall first show that $\text{Im} \{F_1(z)\}$ changes sign at least twice on E_1 . We shall then show that $d/dt (\text{Im} \{F_1(z)\})$ vanishes exactly twice on E_1 from which it will follow that $\text{Im} \{F_1(z)\}$ changes sign exactly twice on E_1 .

To this end let $t = \pi/2$, then $u - \pi/4 = i\epsilon$. Substituting this value of $u - \pi/4$ into (4.20) yields

$$(4.21) \quad \text{Im} \{F(u)\} = -\frac{1}{8} \frac{cn\epsilon}{sn^3\epsilon dn^3\epsilon} (k^2 k'^2 sn^4\epsilon - dn^4\epsilon).$$

Since $0 < k < 1$, $0 < k' < 1$, $k' > 2^{-1/2}$, $sn\epsilon \rightarrow 0$, $cn\epsilon \rightarrow 1$, $dn\epsilon \rightarrow 1$ with ϵ tending to zero, it follows that there is a $\delta_1 > 0$ such that $\text{Im} \{F(u)\} > 0$ for each ϵ satisfying the relation $0 < \epsilon < \delta_1$, whenever $t = \pi/2$.

Next let $t = 3\pi/2$, then $u - \pi/4 = K + i\epsilon$ in which case

$$(4.22) \quad \text{Im} \{F(u)\} = \frac{1}{8} \frac{sn\epsilon dn\epsilon}{cn^3\epsilon} (k^2 - k'^2 cn^4\epsilon).$$

Let $\eta > 1/\sqrt{2}$ be given, then for $k' > \eta > 2^{-1/2}$ there exists a $\delta_2 > 0$ such that $\text{Im} \{F(u)\} < 0$ for each ϵ with $0 < \epsilon < \delta_2$ whenever $t = 3\pi/2$. Hence there is a $\delta > 0$ such that $\text{Im} \{F_1(z)\}$ changes sign at least once on each ellipse E_ϵ confocal with E situated in the boundary strip $0 < \epsilon < \delta$ whenever $\pi/2 < t < 3\pi/2$. Since $F(\frac{1}{2}t - \pi/4 + i\epsilon + \pi) = F(\frac{1}{2}t - \pi/4 + i\epsilon)$ it follows that $\text{Im} \{F_1(z)\}$ changes sign at least twice on each ellipse E_ϵ in the boundary strip.

Upon differentiating $F(u)$ we find

$$(4.23) \quad \frac{d}{du} (F(u)) = -\frac{1}{8} \left[\frac{2(1 + k^2)sn^2(u - \pi/4 + i \log R) - 3}{sn^4(u - \pi/4 + i \log R)} \right. \\ \left. \cdot \frac{2(1 + k^2)sn^2(u - \pi/4) - 3}{sn^4(u - \pi/4)} \right].$$

Employing the reduction formulae together with a lengthy computation reveals that

$$(4.24) \quad \frac{d}{dt} \text{Im} \{F(u)\} = -\frac{1}{16} snv cnv dnv sn\epsilon cn\epsilon dn\epsilon g(x, \epsilon),$$

where $v = \frac{1}{2}t - \pi/4$, $x = snv$, $g(x, \epsilon)$ is a continuous function of x , except for $x = \epsilon = 0$ and such that

$$\lim_{\epsilon \rightarrow 0} g(x, \epsilon) = x^{-6} z^{-6} \{4x^6 k^2 (k^2 - 1) [(2k^2 - k^4)x^2 + (1 - 2k^2)] \\ + 6[(1 + k^2)x^2 - 2](1 - k^2 x^2)^3\}$$

which is negative since $0 < x^2 \leq 1$, $k' > 2^{-\frac{1}{2}}$. Under these conditions there is a boundary strip given by $0 < \epsilon < \delta_3$ in which $g(x, \epsilon) < 0$ on each ellipse E_ϵ in the strip. On E_ϵ $d/dt (\operatorname{Im} \{F(u)\})$ vanishes at $v = 0$, $v = \pi/2$ which is equivalent to $t = \pi/2$, $t = 3\pi/2$. Hence we conclude that there is a boundary strip G such that $\operatorname{Im} \{F_1(z)\}$ changes sign exactly twice on each ellipse interior to G and confocal with E , provided $k' > 2^{-\frac{1}{2}}$, that is $F_1(z)$ belongs to R .

5. Starlike functions. Let us now consider the class of functions which are regular in E and map E onto a domain D starlike with respect to the origin. By starlikeness with respect to the origin we shall mean the $\arg \{f(\cos(t - is))\}$, $0 < s < s_0$, monotonically increases as t increases from 0 to 2π . From this condition it follows that there exists a real number t_1 such that the variation of $\arg \{f(\cos(t - is))\}$ between t_1 and $t_1 + \pi$ is exactly π . For since $|\Delta \arg f|$ between t_1 and $t_1 + 2\pi$ is exactly 2π , suppose that $G(t) = \arg \{f(\cos(t + \pi - is))\} - \arg \{f(\cos(t - is))\}$ and $G(t_0) = \alpha \leq \pi$, $G(t_0 + \pi) = \beta \geq \pi$, $0 \leq t_0 \leq \pi$. Then since $G(t)$ is a continuous function of t for fixed s , it follows that there exists a t_1 , $t_0 \leq t_1 \leq t_0 + \pi$ such that $G(t_1) = \pi$. Hence there exist real numbers μ and ν such that $g(z) = e^{i\mu} f(\cos(t - \nu - is))$ takes on real values when z is real and on the ellipse. Also $g(z)$ is starlike and $\arg \{g(z)\}$ increases monotonically with t . We see then that $\operatorname{Im} \{g(z)\} > 0$, $0 < t < \pi$, and $\operatorname{Im} g(z) < 0$, $\pi < t < 2\pi$. We should note here that μ and ν are functions of s .

Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$, $a_0 = 0$ and $f(0) = 0$, be regular in E and starlike with respect to the origin. Let $g(z) = \sum_{n=1}^{\infty} b_n T_n(\cos(t - \nu - is))$ where $b_n = e^{i\mu} a_n$. As in Theorem 2.1 we have that

$$(5.1) \quad \operatorname{Re} \{\varphi(z)\} = \operatorname{Re} \{-i \sin t g(z)\} \geq 0.$$

A short computation yields

$$(5.2) \quad \operatorname{Re} \{\varphi(z)\} = \sum_{n=1}^{\infty} [\alpha_n \sin n(t - \nu) \sinh ns + \beta_n \cos n(t - \nu) \cosh ns] \sin t \\ = \sum_{n=1}^{\infty} A_n \sin nt \sin t - B_n \cos nt \sin t$$

where $\alpha_n + i\beta_n = b_n$ and $A_n = \alpha_n \cos n\nu \sinh ns + \beta_n \sin n\nu \cosh ns$, $B_n = \alpha_n \sin n\nu \sinh ns - \beta_n \cos n\nu \cosh ns$, $n = 1, 2, \dots$.

Rewriting (5.2) we have

$$(5.3) \quad \operatorname{Re} \{\varphi(z)\} = \frac{1}{2} \left[A_1 + \sum_{n=1}^{\infty} (A_{n+1} - A_{n-1}) \cos nt + (B_{n+1} - B_{n-1}) \sin nt. \right]$$

Since by (5.1) the trigonometric series which is the right member of (5.3) is positive, we have as before

$$(5.4) \quad |(A_{n+1} - A_{n-1}) - i(B_{n+1} - B_{n-1})| \leq 2 |A_1|.$$

Now $A^n - iB_n = \frac{1}{2} (b_n^* R^n - b_n R^{-n}) e^{in\nu}$ so that (5.4) becomes

$$(5.5) \quad |(b_{n+1}^* R^{n+1} - b_{n+1} R^{-(n+1)}) e^{i(n+1)\nu} - (b_{n-1}^* R^{n-1} - b_{n-1} R^{-(n-1)}) e^{i(n-1)\nu}| \leq 2 |b_1^* R - b_1 R^{-1}|.$$

In view of the fact that $A_0 = B_0 = 0$ and employing induction we get

$$(5.6) \quad |b_n^* R^n - b_n R^{-n}| \leq n |b_1^* R - b_1 R^{-1}|$$

or

$$|b_n| \leq \frac{n}{R^n - R^{-n}} |b_1^* R - b_1 R^{-1}|.$$

Returning to the original coefficients a_n we have

$$(5.7) \quad |a_n| \leq \frac{n}{R^n - R^{-n}} |e^{-i\nu} a_1^* R - e^{i\nu} a_1 R^{-1}|$$

$$\leq \frac{n |a_1| (R + R^{-1})}{R^n - R^{-n}}.$$

We state the results in the following

THEOREM 5.1. *Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$, $z \in E$, $a_0 = 0$ map E onto a domain starlike with respect to the origin; then*

$$(5.7) \quad |a_n| \leq n |a_1| \frac{(R + R^{-1})}{R^n - R^{-n}}, \quad n = 1, 2, \dots$$

It is unlikely that (5.7) is sharp. The function (2.8) belongs to the class of starlike functions and more than likely plays the role of the extremal function in the above theorem.

6. The class D. Let $W = f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$ be regular in E and have one zero interior to E . Following Umezawa [7] we say that $f(z)$ has a diametral line if there exists a point ζ on E such that the intersection of the straight line in the W plane passing through the points $f(\zeta)$, 0 , $f(-\zeta)$ with the map of the ellipse is connected. If the direction of starlikeness of $f(z)$ is that of the diametral line we say $f(z)$ belongs to the class D . This implies that there exist real numbers μ and ν such that

$$(6.1) \quad \operatorname{Im} \{f(z)e^{-i\mu}\} \geq 0 \quad \text{for} \quad \operatorname{Im} \{z \cos \nu + (1 - z^2)^{\frac{1}{2}} \sin \nu\} \geq 0$$

$$\operatorname{Im} \{f(z)e^{-i\mu}\} \leq 0 \quad \text{for} \quad \operatorname{Im} \{z \cos \nu + (1 - z^2)^{\frac{1}{2}} \sin \nu\} \leq 0$$

for z on E , where we note that $z \cos \nu + (1 - z^2)^{\frac{1}{2}} \sin \nu = \cos (t - \nu - is)$. Following a procedure analogous to that in §2 we obtain a theorem similar to Theorem 5.1.

THEOREM 6.1. *Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$ belong to the class D ; then*

$$(6.2) \quad |a_n| \leq \frac{n |a_1 R - a_1^* R^{-1}|}{R^n - R^{-n}}, \quad n = 1, 2, \dots$$

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