DIRECT PRODUCTS AND THE INTERSECTION MAP OF CERTAIN CLASSES OF FINITE GROUPS

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ABSTRACT OF DISSERTATION

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The Graduate School
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ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Julia Chifman
Lexington, Kentucky

Director: Dr. James C. Beidleman, Professor of Mathematics
Lexington, Kentucky 2009

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ABSTRACT OF DISSERTATION

DIRECT PRODUCTS AND THE INTERSECTION MAP OF CERTAIN CLASSES OF FINITE GROUPS.

The main goal of this work is to examine classes of finite groups in which normality, permutability and Sylow-permutability are transitive relations. These classes of groups are called $T$, $PT$ and $PST$, respectively. The main focus is on direct products of $T$, $PT$ and $PST$ groups and the behavior of a collection of cyclic normal, permutable and Sylow-permutable subgroups under the intersection map. In general, a direct product of finitely many groups from one of these classes does not belong to the same class, unless the orders of the direct factors are relatively prime. Examples suggest that for solvable groups it is not required to have relatively prime orders to stay in the class. In addition, the concept of normal, permutable and S-permutable cyclic sensitivity is tied with that of $T_c$, $PT_c$ and $PST_c$ groups, in which cyclic subnormal subgroups are normal, permutable or Sylow-permutable. In the process another way of looking at the Dedekind, Iwasawa and nilpotent groups is provided as well as possible interplay between direct products and the intersection map is observed.

KEYWORDS: S-permutable, permutable, direct product, intersection map, cyclic subgroups.

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DIRECT PRODUCTS AND THE INTERSECTION MAP OF CERTAIN
CLASSES OF FINITE GROUPS.

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DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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Dedicated to my husband, Igor Chifman. Without his unconditional love and support this work would not be possible.
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Groups are the main components of algebraic structures such as rings, fields, modules and vector spaces. They have many applications in chemistry, physics and last but not least, algebraic biology.

The main goal of this work is to examine classes of finite groups with certain transitivity properties, their direct products and the behavior of a collection of subgroups under the intersection map. A direct product is an important mathematical concept that is defined on already known objects, in this case groups, and it provides a way to build new groups as well as to analyze groups from its direct factors. It is not always true that given two groups from one class, the new group formed using a direct product will stay in the same class. Classes of groups with certain transitivity properties have been studied and characterized by many authors, however characterizations of the direct product with respect to these classes have not been provided. Therefore the task is to find necessary and sufficient conditions for a direct product to stay in the same class.

The intersection map of subgroups is another important concept that was tied by numerous authors to the classes of group with certain transitivity properties, but there are no known generalizations of their results in connection to the wider classes of groups introduced in recent years. The latter is the question of interest.

The following sections provide a brief overview of the Chapters 1, 2 and 3.

Certain classes of groups

Over the past fifty years finite solvable groups in which normality, permutability and Sylow-permutability are transitive relations have been studied by many authors with Gaschütz [8], Zacher [17] and Agrawal [1] being the first pioneers. As students, when we take a modern algebra course, we quickly learn that if $H$ is a normal subgroup of $K$ and $K$ is a normal subgroup of the group $G$, then it is not necessarily true that $H$ will be a normal subgroup of $G$. Groups in which normality is a transitive relation are called $T$-groups. In other words, $T$-groups are precisely the groups in which every subnormal subgroup is normal. Gaschütz [8] in 1957 provided a characterization of solvable $T$-groups.

Zacher [17] and Agrawal [1] went on further by asking a similar question about permutability and Sylow-permutability being transitive relations, respectively. A sub-
group $H$ of the group $G$ is called permutable (Sylow-permutable) if $HP = PH$ for all subgroups (Sylow subgroups) $P$ of $G$. $\PT$ will denote the class of groups in which permutability is a transitive relation and $\PST$ will denote the class of groups in which Sylow-permutability is a transitive relation. Ore [11] proved that permutable subgroups are subnormal and Kegel [10] has proved that Sylow-permutable subgroups are subnormal as well. A direct consequence of the latter is that $\PT$ ($\PST$) groups are precisely the groups in which every subnormal subgroup is permutable (Sylow-permutable). Since normal subgroups are permutable and permutable subgroups are Sylow-permutable then it follows that $T \subset \PT \subset \PST$. Also, note that the containment is proper, since a dihedral group of order 8 is a $\PST$-group but not a $\PT$-group and the non-Dedekind modular group of order 16 is a $\PT$-group but not a $T$-group. Zacher [17] in 1964 and Agrawal [1] in 1975 have provided characterizations for solvable $\PT$ and $\PST$ groups, respectively.

**Direct Products of Certain Classes of Finite Groups**

Direct products of finite solvable groups in which normality, permutability and Sylow-permutability are transitive relations was the first topic that I have addressed. Some ideas and results presented in Chapter 2 were accepted for publication in *Communications in Algebra*, [7].

When we study, for example, nilpotent groups we ask the question whether a direct product of finitely many nilpotent groups is nilpotent. A direct product of known objects gives us a new object, and it is preferable that our new object has similar attributes as its direct factors.

It was natural to ask the question about direct products of $T$, $\PT$ and $\PST$-groups. Part of the answer is found in [1]; Agrawal has shown that if $G_1$ and $G_2$ are $\PST$ groups such that their orders are relatively prime, then $G_1 \times G_2$ is a $\PST$-group. However, it is not necessary for solvable groups to have relatively prime orders for the product to remain in the class. For example, consider a dihedral group of order 12, which is a direct product of a symmetric group of order 6, $S_3$, and a cyclic group of order 2, $C_2$. Notice that the orders of $S_3$ and $C_2$ are not relatively prime and yet $S_3 \times C_2$ is a solvable $\PST$-group, while the group $S_3 \times C_3$, where $C_3$ is a cyclic group of order three, is not a $\PST$-group.

It turns out that the key lies in the nilpotent residual of the group. In the following theorem, $L_i$ stands for the nilpotent residual of the group $G_i$, by which we mean $L_i = \bigcap \{H_i | H_i \triangleleft G_i \text{ and } G_i/H_i \text{ is nilpotent} \}$. 

2
Theorem. \(2.1.2\) Let \(G_1\) and \(G_2\) be finite groups. \(G_1 \times G_2\) is a solvable \(\mathcal{PST}\)-group if and only if \(G_1\) and \(G_2\) are solvable \(\mathcal{PST}\) groups and \(|L_i|, |G_j| = 1\) for \(i \neq j\) and \(i, j \in \{1, 2\}\).

Theorem \(2.1.2\) is true for a direct product of finitely many solvable \(\mathcal{PST}\)-groups. To extend Theorem \(2.1.2\) to the classes of solvable \(T\) and \(\mathcal{T}\) groups the focus had to be shifted to their Sylow subgroups. If \(G_1\) and \(G_2\) are solvable \(\mathcal{T}\) (\(T\)) groups and \(|L_i|, |G_j| = 1\) for \(i \neq j\), then Theorem \(2.1.2\) implies that \(G_1 \times G_2\) is a solvable \(\mathcal{PST}\)-group. Agrawal [1] has shown that a \(\mathcal{PST}\) group \(G\) is a \(\mathcal{T}\) (\(T\)) group if all of its Sylow subgroups are Iwasawa (Dedekind) respectively. Thus, the question was: When is the direct product of two or more Iwasawa (Dedekind) \(p\)-groups will be an Iwasawa (Dedekind) group? By an Iwasawa (Dedekind) group we mean a group in which every subgroup is permutable (normal).

It follows directly from the Dedekind-Baer Theorem [13] when the direct product of two Dedekind \(p\)-groups is again a Dedekind group. In the case of Iwasawa \(p\)-groups, there is a nice result in [15] that gives a precise description of such \(p\)-groups, which was first stated in 1941 by Iwasawa himself. But, if \(P_1\) and \(P_2\) are two Iwasawa \(p\)-groups for the same prime \(p\) and \(Q_8\), the Quaternion group of order 8, is not a subgroup of \(P_1\) nor \(P_2\) in case \(p = 2\), then it was not clear when a direct product \(P_1 \times P_2\) will be again an Iwasawa group. For example, let \(G\) be a modular group of order 16 with presentation \(G = \langle x, y|x^8 = y^2 = 1, x^y = x^5\rangle\) and \(C_2\) and \(C_8\) be cyclic groups of order 2 and 8, respectively. Note, \(G\) is a non-abelian Iwasawa group. The direct product \(G \times C_2\) is an Iwasawa group but \(G \times C_8\) is not.

A direct consequence of Iwasawa’s theorem [15] is that if \(P_i\) is non-abelian Iwasawa \(p\)-group and \(Q_8\) is not contained in \(P_i\), then \(P_i = A_i\langle x_i \rangle\) where \(A_i\) is abelian normal subgroup of \(P_i\) and \(P_i/A_i\) is cyclic; furthermore there is a positive integer \(s_i\) such that \(a^{x_i} = a^{1+p^{s_i}}\) for all \(a \in A_i\) with \(s_i \geq 2\) in case \(p = 2\).

Using the latter, I was able to show that if \(P := P_1 \times \cdots \times P_n\) is non-abelian Iwasawa \(p\)-group so that \(Q_8 \not\subseteq P\), then exactly one \(P_i\) will be non abelian Iwasawa \(p\)-group and \(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n\) must be an abelian \(p\)-group such that the exponent of \(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n\) is less than or equal to \(p^{s_i}\), where \(s_i\) is the positive integer as described above. By the exponent of the group \(G\), denoted by \(\text{Exp}(G)\), we mean the smallest positive integer \(n\) such that \(x^n = 1\) for all \(x \in G\). The next theorem will employ the following notation: let \(P := P_1 \times \cdots \times P_n\) for \(n \geq 2\) then \(P/P_i := P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n\).
Theorem. Let \( G := G_1 \times \cdots \times G_n \) be a finite group and \( P := P_1 \times \cdots \times P_n \in \text{Syl}_p(G) \) where \( P_i \in \text{Syl}_p(G_i) \) for some prime \( p \) dividing the order of \( G \) and \( n \geq 2 \). Then \( G \) is a solvable \( \mathcal{PT} \)-group if and only if the following hold:

(i) \( G_1, \ldots, G_n \) are solvable \( \mathcal{PT} \)-groups such that \( (|\gamma_s(G_i)|, |G_j|) = 1 \) for all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \).

(ii) If \( Q_8 \) is contained in \( P \), then it is a subgroup of exactly one \( P_i \) and \( P \setminus P_i \) is elementary 2-abelian or trivial.

(iii) If \( Q_8 \) is not contained in \( P \), in case \( p = 2 \), and \( P \) is not abelian, then exactly one \( P_i \) is non-abelian and \( P \setminus P_i \) is an abelian \( p \)-group such that \( \text{Exp}(P \setminus P_i) \leq p^{s_i} \).

A similar result to that of Theorem 2.3.2 can be obtained for direct products of solvable \( T \)-groups with the difference that if \( Q_8 \) is not contained in \( P \) then \( P \) is an abelian group.

In 2005 D. Robinson in [12] introduced three new classes of groups. He called them \( \mathcal{PST}_c \), \( \mathcal{PT}_c \) and \( \mathcal{T}_c \). A group \( G \) is a \( \mathcal{PST}_c \)-group if every cyclic subnormal subgroup of \( G \) is Sylow-permutable in \( G \). Similarly, classes \( \mathcal{PT}_c \) and \( \mathcal{T}_c \) are defined by requiring cyclic subnormal subgroups to be permutable or normal, respectively. Robinson in [12] provided characterizations for both solvable and insolvable cases. I have extended some of the above results to solvable \( \mathcal{PST}_c \), \( \mathcal{PT}_c \), and \( \mathcal{T}_c \) groups.

The Intersection Map of Subgroups

The intersection map of subgroups in connection to the classes \( \mathcal{PST}_c \), \( \mathcal{PT}_c \) and \( \mathcal{T}_c \) is a collaborative work with my advisor James Beidleman. Parts of this work were published in Ricerche di Matematica [5]. At that time my advisor and Matthew Ragland were locally analyzing the intersection map in connection with \( T \), \( \mathcal{PT} \) and \( \mathcal{PST} \) groups. They have generalized in [6] Theorem 1 of Bauman [4] to \( \mathcal{PT} \) and \( \mathcal{PST} \) groups. Beidleman asked if similar generalizations are possible for the classes \( \mathcal{PST}_c \), \( \mathcal{PT}_c \) and \( \mathcal{T}_c \).

A subgroup \( H \) of the group \( G \) is said to be normal sensitive if whenever \( X \) is a normal subgroup of \( H \) there is a normal subgroup \( Y \) of \( G \) such that \( X = Y \cap H \), that is if the map, known as the intersection map, \( Y \mapsto H \cap Y \) sends the lattice of normal subgroups of \( G \) onto the lattice of normal subgroups of \( H \). Permutable sensitive (S-permutable sensitive) are defined in the similar fashion by requiring \( X \) and \( Y \) to be permutable (Sylow-permutable) subgroups of \( H \) and \( G \) respectively. Note, that the
set of all permutable subgroups need not be a sublattice of the lattice of subnormal subgroups of a group $G$. Hence, in the case of permutability the intersection map is not necessarily a lattice map. An example that addresses the latter can be found in [6] on page 220, Example 1. On the other hand, the collection of S-permutable subgroups is a sublattice of the lattice of subnormal subgroups of $G$. For details the reader may consult [10] and [14].

Bauman [4], Beidleman and Ragland [6], have tied the concept of normal, permutable and S-permutable sensitivity with $T$, $PT$ and $PST$ groups. They have showed that $G$ is a solvable $T$, $PT$, or $PST$ group if and on if every subgroup of $G$ is normal, permutable, or S-permutable sensitive in $G$, respectively. In addition, Beidleman and Ragland in [6] went on further by asking a question whether one can restrict S-permutable, permutable, and normal sensitivity to normal subgroups and deduce that $G$ is still a $PST$, $PT$, or a $T$ group respectively. While they have affirmatively answered the question about $PST$ and $T$ groups in [6], the question about permutable sensitivity restricted to normal subgroups of solvable groups was answered later in [3].

Our interest was in developing similar connections with classes $PST_c$, $PT_c$ and $T_c$. In particular, if we restrict the intersection map to cyclic subgroups, then what can we say about the behavior of a collection of cyclic normal, permutable and S-permutable subgroups under this restricted intersection map. The definition of the intersection map restricted to cyclic subgroups together with two Lemmas by R.Schmidt [15] and P.Schmid [14] is equivalent to the following definition.

**Definition.** A subgroup $H$ of the group $G$ is normal (permutable or S-permutable) cyclic sensitive in $G$ if every normal (permutable or Sylow-permutable) cyclic subgroup of $H$ is normal (permutable or Sylow-permutable) subgroup of $G$.

We have tied the concept of normal, permutable and S-permutable cyclic sensitivity with that of $T_c$, $PT_c$ and $PST_c$ groups [5]. In the process we provided another way of looking at Dedekind, Iwasawa and nilpotent groups.

**Theorem.** (3.2.6) Let $G$ be a finite group.

1. $G$ is a nilpotent group if and only if every subgroup of $G$ is S-permutable cyclic sensitive.

2. $G$ is an Iwasawa group if and only if every subgroup of $G$ is permutable cyclic sensitive.

3. $G$ is a Dedekind group if and only if every subgroup of $G$ is normal cyclic sensitive.
If we replace “every subgroup” in Theorem 3.2.6 by “every subnormal subgroup” then we get that \( G \) is a \( PST_c \), \( PT_c \), or a \( T_c \) group, respectively. Robinson in [12] has proved that if every subgroup of a group \( G \) is \( PST_c \) then \( G \) is a solvable \( PST \) group. The same is true for solvable \( PT_c \) and \( T_c \) groups. Robinson’s results motivate the following theorem that relates cyclic sensitivity to solvable \( PST \), \( PT \) and \( T \) groups.

**Theorem.** (3.2.10) Let \( G \) be a finite group.

1. \( G \) is a solvable \( PST \)-group if and only if every subnormal subgroup of \( H \) is \( S \)-permutable cyclic sensitive in \( H \) for all subgroups \( H \) of \( G \).

2. \( G \) is a solvable \( PT \)-group if and only if every subnormal subgroup of \( H \) is permutable cyclic sensitive in \( H \) for all subgroups \( H \) of \( G \).

3. \( G \) is a solvable \( T \)-group if and only if every subnormal subgroup of \( H \) is normal cyclic sensitive in \( H \) for all subgroups \( H \) of \( G \).

Since cyclic subgroups can be written as a direct product of cyclic \( p \)-groups of relatively prime orders, it was natural to look at the Sylow subgroups. In addition, if \( X \) is any subnormal cyclic subgroup of a group \( G \) then \( X \) is contained in the Fitting subgroup of \( G \) and in particular the Sylow \( p \)-subgroup of \( X \), for some prime dividing the order of \( X \), lies in the Sylow \( p \)-subgroup of the Fitting subgroup. Details on the Sylow subgroups and the intersection map are in Chapter 3.
Notation

$G, H, \ldots$ Sets and groups
$g, h, \ldots$ Elements of a set
$\mathcal{G}, \mathcal{H}, \ldots$ Classes of groups
$H \leq G$ $H$ is a subgroup of $G$
$H < G$ $H$ is a proper subgroup of $G$
$H \trianglelefteq G$ $H$ is a normal subgroup of $G$
$H \lhd G$ $H$ is a subnormal subgroup of $G$
$\mathrm{Syl}(G)$ Set of Sylow subgroups of $G$
$\mathrm{Syl}_p(G)$ Set of Sylow $p$-subgroups of $G$
$HK$ \{ $hk | h \in H, k \in K$ \}
$G/H$ Factor group when $H \unlhd G$
$\langle X \rangle$ Group generated by $X$
$\langle X, g \rangle$ $\langle X \cup \{g\} \rangle$
$|H|$ Order of $H$
$\mathrm{ord}(h)$ Order of $h$
$(G : H)$ Index of $H$ in $G$
$Z(G)$ Center of $G$
$C_G(H)$ Centralizer of $H$ in $G$
$N_G(H)$ Normalizer of $H$ in $G$
$\mathrm{Aut}(G)$ Group of automorphisms of $G$
$H \times K$ Direct product of $H$ with $K$
$H \rtimes K$ Semidirect product of $H$ with $K$
$\text{Fit}(G)$ the Fitting subgroup of $G$
$\gamma_\pi(G)$ Last term of the lower central series of $G$
$O^{\pi_0}(G)$ Group generated by all the $\pi_0$-elements of $G$
$O_\pi(G)$ Product of all normal $\pi$-subgroups of $G$
$g^h$ $h^{-1}gh$
$[g, h]$ $g^{-1}h^{-1}gh$
$[H, K]$ Group generated by all $[h, k]$ with $h \in H$ and $k \in K$
$G' = [G, G]$ Derived subgroup of $G$
$C_n$ Cyclic group of order $n$
$S_n$ Symmetric group of order $n!$
$D_{2n}$ Dihedral group of order $2n$
$Q_8$ Quaternion group of order 8

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Chapter 1 Fundamental concepts and certain classes of finite groups

This chapter provides an overview of the fundamental concepts necessary for the consequent chapters, with the underlying assumption that the reader has some familiarity with group theory. For further details on fundamentals of group theory and any unexplained notation the reader may consult [13], [15] and [16].

1.1 Sylow and Hall subgroups

Let $G$ be a finite group and $p$ a prime number dividing the order of $G$. $G$ is called a $p$-group if the order of $G$ is a power of $p$, that is $|G| = p^n$ for some positive integer $n$. Subgroups of $G$ that are $p$-groups are called $p$-subgroups. A maximal $p$-subgroup of $G$ is called a Sylow $p$-subgroup of $G$ and we will denote the set of Sylow $p$-subgroups of $G$ by $\text{Syl}_p(G)$. The fundamental result of finite $p$-groups is that a nontrivial finite $p$-group has a nontrivial center.

Theorem 1.1.1. (1) (Cauchy’s Theorem) If $G$ is a finite group and $p$ is a prime dividing $|G|$ then $G$ has an element of order $p$.

(2) (Sylow’s Theorem) Let $G$ be a finite group of order $p^\alpha m$, where $p$ is a prime not dividing $m$.

(i) Sylow $p$-subgroups of $G$ exist and they are all conjugate (that is isomorphic).

(ii) If $n_p$ is the number of Sylow $p$-subgroups then $n_p \equiv 1 \mod p$ and $n_p|m$.

The second part of (i) in Sylow’s Theorem states that if $P \in \text{Syl}_p(G)$ then for all $g \in G$, $P^g \in \text{Syl}_p(G)$, that is if $P_1$ and $P_2$ are two Sylow $p$-subgroups of $G$ then there exist an element $g$ in $G$ so that $P_1 = P_2^g$. In addition, Sylow’s theorem implies that Sylow $p$-subgroup $P$ is normal in $G$ if and only if $P$ is the unique Sylow $p$-subgroup. Now, let $\pi$ be a set of primes, and $\pi' = \{\text{primes } p \mid p \notin \pi\}$. A positive integer $n$ is called a $\pi$-number if all prime divisors of $n$ belong to $\pi$. A subgroup $H$ of a finite group $G$ is called a $\pi$-group if the order of $H$ is a $\pi$-number. Thus, a Sylow $\pi$-subgroup of $G$ is defined to be a maximal $\pi$-subgroup. For finite groups Sylow $\pi$-subgroups exist but in general they are not conjugate.

P. Hall in [9] extended Sylow’s theorem to solvable groups (solvable groups are defined shortly) where he proved the existence of a subgroup with order relatively prime to
its index and showed that any two such subgroups are conjugate. This paper of P. Hall [9] and others that followed revolutionized the theory of finite solvable groups. Now we provide a general definition of what is known as a Hall subgroup.

**Definition 1.1.2.** $H$ is called a Hall-subgroup of a group $G$ if $(|H|, (G : H)) = 1$. In particular, a Hall $\pi$-subgroup of $G$ is a subgroup $H$ of $G$ such that $|H|$ is a $\pi$-number and $(G : H)$ is a $\pi'$-number.

If $\pi = \{p\}$ for some prime $p$, then $\pi$-group and $p$-group mean the same, that is Hall $p$-subgroup is the same as a Sylow $p$-subgroup and every Hall $\pi$-subgroup is a Sylow $\pi$-subgroup.

**Definition 1.1.3.** Let $G$ be a group and $N \vartriangleleft G$. A complement of $N$ in $G$ is a subgroup $H$ of $G$ such that $H \cap N = 1$ and $G = HN$.

The following theorem is a fundamental result of finite group theory.

**Theorem 1.1.4 (Schur-Zassenhaus Theorem).** Let $G$ be a finite group and $N$ a normal Hall $\pi'$-subgroup of $G$. Then $N$ has a complement in $G$ and all the complements of $N$ in $G$ are conjugate.

### 1.2 Nilpotent, Solvable and Supersolvable groups

Let $G$ be any group and consider the series for $G$

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

where $G_i$ is a subgroup of $G$ for all $i \in \{1, \ldots, n - 1\}$. If each $G_i \vartriangleleft G$ then \[\square\] is called a normal series. If $G_i \vartriangleleft G_{i+1}$ only then \[\square\] is called a subnormal series. We say that a subgroup $H$ of $G$ is subnormal in $G$ if there exist a subnormal series between $H$ and $G$.

**Definition 1.2.1.**

1. A group $G$ is called nilpotent if it has a central series, that is a normal series such that $G_{i+1}/G_i \in Z(G/G_i)$ for all $i \in \{1, \ldots, n - 1\}$.

2. A group $G$ is called solvable if it has an abelian series, that is a subnormal series such that $G_{i+1}/G_i$ is abelian, for all $i \in \{1, \ldots, n - 1\}$.

3. A group $G$ is called supersolvable if it has a cyclic series, that is a normal series such that $G_{i+1}/G_i$ is cyclic, for all $i \in \{1, \ldots, n - 1\}$. 
Example 1.2.2. Consider the following groups:

\[ D_8 = \langle x, y \mid x^4 = y^2 = 1, \ x^y = x^{-1} \rangle \]
\[ S_3 = \langle x, y \mid x^3 = y^2 = 1, \ (xy)^2 = 1 \rangle \]
\[ S_4 = \langle x, y \mid x^3 = y^2 = 1, \ (xy)^4 = 1 \rangle \]

1. Nilpotent: \( D_8 \)
2. Supersolvable: \( D_8, S_3 \)
3. Solvable: \( D_8, S_3, S_4 \)

Note, \( S_3 \) is not nilpotent and \( S_4 \) is not supersolvable.

From the above definition 1.2.1 and the example 1.2.2, we see that the class of finite nilpotent groups, supersolvable groups and solvable groups are proper subclasses of each other, that is

\{finite nilpotent groups\} \( \subset \) \{finite supersolvable groups\} \( \subset \) \{finite solvable groups\}.

It is important to note that \( p \)-groups are nilpotent. Also, we would like to state some important characterizations of finite nilpotent groups.

Theorem 1.2.3. Let \( G \) be a finite group. Then the following are equivalent:

(i) \( G \) is nilpotent.
(ii) All Sylow subgroups of \( G \) are normal.
(iii) \( G \) is a direct product of its Sylow subgroups.
(iv) All subgroups of \( G \) are subnormal.

Definition 1.2.4. Normal series 1.2.1 are called chief series if for all \( i \in \{1, \ldots, n-1\} \) \( G_{i+1}/G_i \) is a minimal normal subgroup of \( G/G_i \). The factor \( G_{i+1}/G_i \) is called a chief factor. If the order of \( G_{i+1}/G_i \) is a power of a prime for some prime \( p \), then \( G_{i+1}/G_i \) is called a \( p \)-chief factor.

Finite groups always have chief series and any two chief series have the same length with their chief factors pairwise isomorphic. One important property finite solvable groups have is that chief factors are elementary abelian \( p \)-groups for some prime \( p \).
1.3 Lower central series and nilpotent residual

Let $G$ be a group and $x, y \in G$. The *commutator* of $x$ and $y$ is $[x, y] = x^{-1}y^{-1}xy$. If $H$ and $K$ are two subgroups of the group $G$ then define $[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$. The *derived subgroup* of $G$ is defined as $G' = [G, G]$.

$G/G'$ is the largest abelian quotient of $G$, that is if $H$ is a normal subgroup of $G$ and $G/H$ is abelian then $G'$ is a subgroup of $H$.

**Remark 1.3.1.** Let $H, K$ be subgroups of the group $G$.

1. $[H, K] = \langle 1 \rangle$ if and only if $H \leq C_G(K)$ if and only if $K \leq C_G(H)$.
2. $K \leq N_G(H)$ if and only if $[H, K] \leq H$.
3. If $H$ and $K$ are normal subgroups of $G$ then $[H, K]$ is normal subgroup of $G$ and $[H, K] \leq H \cap K$.

Let $G$ be any group and define the following series of subgroups inductively.

\[
\begin{align*}
\gamma_1(G) &= G \\
\gamma_2(G) &= [\gamma_1(G), G] = [G, G] \\
& \vdots \\
\gamma_{i+1}(G) &= [\gamma_i(G), G] = [G, G, \ldots, G].
\end{align*}
\]

The series

\[
G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_i(G) \geq \cdots
\]

are called *lower central series* of $G$. Each $\gamma_i(G)$ is characteristic subgroup of $G$ and $\gamma_i(G)/\gamma_{i+1}(G)$ lies in the center of $G/\gamma_{i+1}(G)$.

**Remark 1.3.2.** For any finite group $G$ there is an integer $n$ so that

\[
\gamma_n(G) = \gamma_{n+1}(G) = \gamma_{n+2}(G) = \cdots.
\]

If $G$ is nilpotent then lower central series reach \langle 1 \rangle. For non-nilpotent groups $\gamma_n(G)$ is a nontrivial subgroup of $G$.

Suppose there exist an $n$ as in Remark 1.3.2. Let $\gamma_*(G) = \gamma_n(G)$, that is $\gamma_*(G)$ is the smallest term of the lower central series. Then $\gamma_*(G)$ is the smallest normal subgroup of $G$ with $G/\gamma_*(G)$ being nilpotent.

**Definition 1.3.3.** $\gamma_*(G)$ is called the *nilpotent residual* of $G$. 

\[11\]
The above definition of nilpotent residual is equivalent to the following:

\[ \gamma_s(G) = \bigcap \{ H \mid H < G \text{ and } G/H \text{ is nilpotent} \} \]

Since \( \gamma_s(G) \) is the smallest normal subgroup of \( G \) such that \( G/\gamma_s(G) \) is nilpotent and \( G/G' \) is the largest abelian quotient, that is \( G/G' \) is nilpotent, then \( \gamma_s(G) \leq G' \).

### 1.4 The Fitting subgroup \( \text{Fit}(G) \) and \( O_p(G) \)

The subgroup generated by all the normal nilpotent subgroups of a group \( G \) is called the \textit{Fitting subgroup} of \( G \), denoted by \( \text{Fit}(G) \). For a finite group \( G \) the \( \text{Fit}(G) \) is the unique largest nilpotent subgroup of \( G \).

**Theorem 1.4.1.** If \( G \) is a finite group then

\[ \text{Fit}(G) = \bigcap \{ C_G(H/K) \mid H/K \text{ is a chief factor of } G \} \]

Let \( G \) be a group and \( p \) a prime. Define

\[ O_p(G) = \langle A \mid A < G \text{ and } A \text{ is a } p\text{-group} \rangle \]

Similarly \( O'_p(G) \) is defined where \( A \) is a \( p' \)-group. \( O_p(G) \) is the largest normal \( p \)-subgroup of \( G \) and we know that \( p \)-groups are nilpotent. Hence, it is clear that \( O_p(G) \) is contained in the \( \text{Fit}(G) \). In fact, \( O_p(G) \) is the intersection of all the Sylow \( p \)-subgroup of \( G \), that is

\[ O_p(G) = \bigcap \{ P \mid P \in \text{Syl}_p(G) \} \]

It is easy to see that \( O_p(G) \) is the Sylow \( p \)-subgroup of the \( \text{Fit}(G) \) and that \( \text{Fit}(G) \) is the direct product of \( O_{p_i}(G) \), where \( p_i \) is the prime dividing \( |\text{Fit}(G)| \).

### 1.5 Permutable and Sylow-permutable subgroups.

**Definition 1.5.1.** Let \( H \) be a subgroup of a finite group \( G \).

1. \( H \) is \textit{permutable} in \( G \), denoted by \( H \text{ per } G \), if \( HK = KH \) for all subgroups \( K \) of \( G \).
2. \( H \) is \textit{Sylow-permutable} in \( G \), denoted by \( H \text{ S-per } G \), if \( HP = PH \) for all Sylow subgroups \( P \) of \( G \).

It is clear that if \( H \) is a normal subgroup of \( G \) then \( H \) is permutable (Sylow-permutable) in \( G \). Subnormal subgroups in general are not permutable.
Example 1.5.2. Consider a Dihedral group of order 8:

\[ D_8 = \langle x, y \mid x^4 = y^2 = 1, \ x^y = x^{-1} \rangle \]

\( D_8 \) is nilpotent, thus every subgroup is subnormal. Let \( H = \langle y \rangle \) and \( K = \langle xy \rangle \). It can be easily verified that \( HK \neq KH \).

Other important facts about permutable and Sylow-permutable subgroups are stated below.

**Theorem 1.5.3.** Let \( G \) be a finite group.

1. (Ore, [11]) If \( H \) is maximal permutable subgroup of \( G \) then \( H \triangleleft G \).
2. (Ore, [11]) If \( H \) per \( G \) then \( H \) is subnormal subgroup of \( G \).
3. (Kegel, [10]) If \( H \leq K \leq G \) and \( H \) S-per \( G \) then \( H \) S-per \( K \).
4. (Kegel, [10]) If \( H \) S-per \( G \) then \( H \) is subnormal subgroup of \( G \).

1.6 \( \mathcal{PST} \), \( \mathcal{PT} \) and \( T \) groups

In general, normality, permutability and Sylow-permutability are not transitive relations.

**Definition 1.6.1.** Groups in which normality, permutability and Sylow-permutability are transitive relations are called \( T \), \( \mathcal{PT} \) and \( \mathcal{PST} \) groups, respectively.

It is clear that normality is transitive is the same as saying that a subnormal subgroup is normal. In other words, \( T \)-groups are precisely the groups in which every subnormal subgroup is normal. Also, Theorem 1.5.3 parts (2) and (4) imply that \( \mathcal{PT} \) (\( \mathcal{PST} \)) groups are precisely the groups in which every subnormal subgroup is permutable (Sylow-permutable). Since normal subgroups are permutable and obviously permutable subgroups are Sylow-permutable then it follows that

\[ T \subset \mathcal{PT} \subset \mathcal{PST}. \]

Also, note that the containment is proper, for a dihedral group of order 8 is a \( \mathcal{PST} \)-group but not a \( \mathcal{PT} \)-group and the modular group of order 16 is a \( \mathcal{PT} \)-group but not a \( T \)-group. In addition, nilpotent groups are solvable \( \mathcal{PST} \)-groups.

Solvable \( \mathcal{PST} \), \( \mathcal{PT} \) and \( T \) groups were studied and characterized by Agrawal [1], Zacher [17], and Gaschütz [8]. The following theorem summarizes some of their results.
Theorem 1.6.2. Let $G$ be a finite group and let $L$ be the nilpotent residual of $G$.

(i) (Agrawal [1]) $G$ is a solvable PST group if and only if $L$ is abelian Hall subgroup of $G$ of odd order, and $G$ acts by conjugation as a power automorphism on $L$.

(ii) (Zacher [17]) $G$ is a solvable PT-group if and only if $G$ is a solvable PST-group and $G/L$ is an Iwasawa group.

(iii) (Gaschütz [8]) $G$ is a solvable T-group if and only if $G$ is a solvable PST-group and $G/L$ is a Dedekind group.

Definition 1.6.3.

1. A Dedekind group is a group in which every subgroup is normal.

2. An Iwasawa group is a group in which every subgroup is permutable.

Also, note that if $G$ is a solvable $T$, $PT$ or $PST$ group then every subgroup and every quotient inherits the same properties. The above classes of groups were studied in detail by many authors. Another characterization of solvable PST-groups that will be used later was provided by Alejandre, Ballester-Bolinches, and Pedraza-Aguilera in [2].

Theorem 1.6.4. Let $G$ be a finite group. Then $G$ is a PST solvable group if and only if $G$ is supersolvable and all its $p$-chief factors are isomorphic when regarded as $G$-modules for every prime $p$.

1.7 PST$_c$, PT$_c$ and T$_c$ groups

Robinson in [12] introduced classes of groups in which cyclic subnormal subgroups are S-permutable, permutable, or normal.

Definition 1.7.1. A group $G$ is called a PST$_c$, PT$_c$ or a T$_c$-group if every cyclic subnormal subgroup of $G$ is Sylow-permutable, permutable or normal in $G$, respectively.

Recall, that PST, PT and T groups are defined in terms of transitivity properties. For PT$_c$ (T$_c$) groups it is not enough to have a cyclic subgroup to be permutable (normal) in some permutable (normal) subgroup of $G$. We must have conditions on the Fitting subgroup. An example that demonstrates that conditions on $\text{Fit}(G)$ cannot be omitted can be found in [12] page 174.
Lemma 1.7.2. (Robinson [12]) Let $G$ be a finite group.

(i) $G$ is a $\mathcal{PST}_c$-group if and only if a cyclic subgroup of $G$ is Sylow-permutable in $G$ whenever it is Sylow-permutable in some Sylow-permutable subgroup of $G$.

(ii) $G$ is a $\mathcal{PT}_c$-group if and only if $\text{Fit}(G)$ is an Iwasawa group and a cyclic subgroup of $G$ is permutable in $G$ whenever it is permutable in some permutable subgroup of $G$.

(iii) $G$ is a $\mathcal{T}_c$-group if and only if $\text{Fit}(G)$ is a Dedekind group and a cyclic subgroup of $G$ is normal in $G$ whenever it is subnormal with defect at most 2.

The classes of $\mathcal{T}_c$, $\mathcal{PT}_c$ and $\mathcal{PST}_c$-groups are proper subclasses of each other as demonstrated in Example 2.7.1 in Chapter 2. Also, it is easy to see that a $\mathcal{T}$-group is necessarily a $\mathcal{T}_c$-group. Similarly for $\mathcal{PT}$ and $\mathcal{PST}$-groups.

$$
\begin{array}{ccc}
\mathcal{T} & \rightarrow & \mathcal{PT} & \rightarrow & \mathcal{PST} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{T}_c & \rightarrow & \mathcal{PT}_c & \rightarrow & \mathcal{PST}_c
\end{array}
$$

Robinson in [12] provided characterizations for both solvable and insolvable cases, here we mention only the solvable case.

Theorem 1.7.3. (Robinson [12]) Let $G$ be a finite group and $F = \text{Fit}(G)$.

(1) A group $G$ is a solvable $\mathcal{PST}_c$ - group if and only if there is a normal subgroup $L$ such that,

(i) $L$ is abelian and $G/L$ is nilpotent.

(ii) $p'$-elements of $G$ induce power automorphisms in $L_p$ for all primes $p$.

(iii) $\pi(L) \cap \pi(F/L) = \emptyset$

(2) A group $G$ is a solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$) - group if and only if $G$ is a solvable $\mathcal{PST}_c$-group such that all elements of $G$ induce power automorphisms in $L$ and $F/L$ is Iwasawa (Dedekind) group, where $L$ is the normal subgroup as described in (1).

Note the important distinction between solvable $\mathcal{PST}$ and $\mathcal{PST}_c$ groups is that the nilpotent residual is a Hall-subgroup of the Fitting subgroup whereas the nilpotent residual of a solvable $\mathcal{PST}$-group is a Hall subgroup of the entire group.
Remark. ([7], 2.4.1) Let $G$ be a finite solvable $\mathcal{PST}_c$-group such that the nilpotent residual of $G$ is a Hall-subgroup of $G$. Then $G$ is a solvable $\mathcal{PST}$-group.

Also, note that the class of solvable $\mathcal{PST}_c$-groups is not subgroup closed nor quotient closed, however a normal subgroup of a solvable $\mathcal{PST}_c$-group is also a $\mathcal{PST}_c$-group. For details the reader may consult [12] Theorems 2.5 and 2.6. The following example is a solvable $\mathcal{T}_c$ group that is not a $\mathcal{T}$-group.

Example 1.7.4.

$$G = \left\langle a, b, c, d, f \mid a^7 = b^3 = c^2 = d^3 = f^5 = (ac)^2 = (bc)^2 = a^5a^d = 1, [a, b] = [c, d] = [a, f] = [b, f] = [c, f] = [d, f] = [b, d] = 1 \right\rangle$$

(i) $G \cong ((C_7 \times C_3) \rtimes (C_2 \times C_3)) \times C_5$.

(ii) The Fitting subgroup of $G$ is $\text{Fit}(G) = (C_7 \times C_3) \times C_5$.

(iii) The nilpotent residual of $G$ is $\gamma_s(G) = C_7 \times C_3$.

(iv) $\gamma_s(G)$ is a Hall-subgroup of the $\text{Fit}(G)$ but not a Hall-subgroup of $G$.

(v) $G$ is a solvable $\mathcal{T}_c$-group that is not a $\mathcal{T}$-group.

(vi) Let $H = \langle b, c, d \rangle$ be a subgroup of $G$. $H \cong S_3 \times C_3$. The nilpotent residual $\gamma_s(H) = \langle b \rangle$ is not a Hall subgroup of the $\text{Fit}(H) = \langle b, d \rangle$. Thus, $H$ is not a $\mathcal{T}_c$ group. This demonstrates that solvable $\mathcal{T}_c$ groups are not subgroup closed.
Chapter 2 Direct Products of Certain Classes of Finite Groups

In this chapter we analyze direct products of solvable groups from classes that have been introduced in the previous chapter. In particular, given two groups from one class what are the necessary and sufficient conditions for a direct product to stay in the same class.

Agrawal [1] has shown that if $G_1$ and $G_2$ are PST groups whose orders are relatively prime, then $G_1 \times G_2$ is a PST-group. However, it is not necessary for the solvable groups to have relatively prime orders to remain in the class. It turns out that the key lies in the nilpotent residual of the group.

Before we proceed to the next section we establish a similar result as in [1] for PT and T groups with relatively prime orders.

**Corollary 2.0.5.** Let $G_1, \ldots, G_n$ be finite groups such that $(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Then,

(i) $G_1, \ldots, G_n \in \mathcal{PT}$ if and only if $G_1 \times \cdots \times G_n \in \mathcal{PT}$.

(ii) $G_1, \ldots, G_n \in \mathcal{T}$ if and only if $G_1 \times \cdots \times G_n \in \mathcal{T}$.

**Proof.** (i). Suppose $G_1, \ldots, G_n \in \mathcal{PT}$ and let $G := G_1 \times \cdots \times G_n$. Let $H$ be a subnormal subgroup of $G$ and $K$ any subgroup of $G$. Since $(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ then $H = H_1 \times \cdots \times H_n$ and $K = K_1 \times \cdots \times K_n$, where $H_i$ is subnormal subgroup of $G_i$ and $K_i$ is a subgroup of $G_i$. Since each $G_i$ is a PT-group then $H_i$ and $K_i$ permute. Also, for $i \neq j$, $G_j$ centralizes $G_i$ hence $H_i$ and $K_j$ permute. Therefore, $HK = KH$ for all subgroups $K$ of $G$. Thus, $G$ is a PT-group. As for the other direction, since normal subgroups of PT-groups inherit the same properties then each $G_i$ is a PT group.

(ii). Suppose $G_1, \ldots, G_n \in \mathcal{T}$ and let $G := G_1 \times \cdots \times G_n$. Let $H$ be a subnormal subgroup of $G$. Since $(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ then $H = H_1 \times \cdots \times H_n$, where $H_i$ is subnormal subgroup of $G_i$. Since each $G_i$ is a T-group then $H_i$ is normal subgroup of $G_i$. Therefore $H = H_1 \times \cdots \times H_n$ is normal subgroup of $G$. Thus, $G$ is a T-group. As for the other direction, since normal subgroups of T-groups inherit the same properties then each $G_i$ is a T group.
2.1 Direct products of solvable \( \mathcal{PST} \)-groups

Two \( \mathcal{PST} \), \( \mathcal{PT} \) or \( \mathcal{T} \) groups with relatively prime orders stay in the same class, however there are numerous examples of solvable \( \mathcal{PST} \), \( \mathcal{PT} \) or \( \mathcal{T} \) groups that suggest otherwise. Consider a few of these examples.

**Example 2.1.1.** In this example \( S_3 \) denotes a symmetric group of order 6, \( C_2 \) and \( C_3 \) are cyclic groups of order 2 and 3, respectively, and \( D_{10} \) a dihedral group of order 10.

(i) Let \( G := S_3 \times C_2 \) with the following presentation:

\[
\langle x, y, z \mid x^3 = y^2 = z^2 = 1, \ (xy)^2 = 1, \ [x, z] = [y, z] = 1 \rangle.
\]

Nilpotent residual of \( S_3 \times C_2 \) is \( \gamma(G) = \langle x \rangle \), which is the nilpotent residual of \( S_3 \). Both \( S_3 \) and \( C_2 \) are solvable \( \mathcal{PST} \)-groups.

Clearly, \( \gamma(G) \) is abelian Hall subgroup of \( G \) of odd order. Also, it is easy to see that every subgroup of \( \gamma(G) \) is normal in \( S_3 \times C_2 \). Thus, \( S_3 \times C_2 \) is a \( \mathcal{PST} \)-group. Notice, the orders of direct factors are not relatively prime but the order of \( \gamma(G) \) is relatively prime to the order of \( C_2 \).

(ii) Let \( G := S_3 \times D_{10} \). This group has the following presentation,

\[
\left\langle x, y, z, w \mid x^3 = y^2 = z^5 = w^2 = (xy)^2 = (zw)^2 = 1, \ [x, z] = [x, w] = [y, z] = [y, w] = 1 \right\}.
\]

The nilpotent residual of \( G \) is \( \langle x \rangle \times \langle z \rangle \), where \( \langle x \rangle \) is the nilpotent residual of \( S_3 \) and \( \langle z \rangle \) is the nilpotent residual of \( D_{10} \). Both \( S_3 \) and \( D_{10} \) are solvable \( \mathcal{PST} \)-groups, and by a direct computation one can determine that \( G \) is a solvable \( \mathcal{PST} \)-group.

Notice again, that the orders of a direct factors are not relatively prime, whereas the order of \( \langle x \rangle \) is relatively prime to the order of \( D_{10} \) and the order of \( \langle z \rangle \) is relatively prime to the order of \( S_3 \).

(iii) Let \( H := S_3 \times C_3 \). \( H \) is not a \( \mathcal{PST} \)-group whereas each direct factor is a \( \mathcal{PST} \)-group. The nilpotent residual of \( H \) is just an alternating group of order 3 and its order is not relatively prime to the order of \( C_3 \).

A closer analysis of the characterization of solvable \( \mathcal{PST} \)-groups and the examples motivate the following result.
Theorem 2.1.2. Let $G_1$ and $G_2$ be finite groups. $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group if and only if $G_1$ and $G_2$ are solvable $\mathcal{PST}$ groups and $(|\gamma_*(G_i)|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$.

Proof. First note that $\gamma_*(G_1 \times G_2) = \gamma_*(G_1) \times \gamma_*(G_2)$. Let $L := \gamma_*(G_1 \times G_2) = L_1 \times L_2$, where $L_i = \gamma_*(G_i)$ for $i \in \{1, 2\}$.

Suppose $G_1 \times G_2$ is a solvable $\mathcal{PST}$ group. It is clear that $G_1$ and $G_2$ are solvable $\mathcal{PST}$ groups. Let $L_p \in \text{Syl}_p(L)$. Then $L_p = L_p^1 \times L_p^2$ where $L_p^i \in \text{Syl}_p(L_i)$ for $i \in \{1, 2\}$. Suppose $L_p^1$ and $L_p^2$ are both not trivial and take $x = x_1x_2 \in L_p$ so that $x_i \neq 1$, where $x_i \in L_p^i$ and $|x_i| = p^{a_i}$ for $i \in \{1, 2\}$. Let $g \in G_1$ with $g \neq 1$. Then $g$ acts by conjugation as a power automorphism on $x_1$ and trivially on $x_2$, that is $(x_1x_2)^g = x_1^mx_2$ for some positive integer $m$. But $g$ also acts as a power automorphism on $x$, i.e. $(x_1x_2)^g = (x_1x_2)^n$ for some positive integer $n$. Thus, we get that $x_1^mx_2 = x_1^n x_2^n$. This means that $m \equiv n \mod p^{a_1}$ and $1 \equiv n \mod p^{a_2}$.

If $\alpha_1 \geq \alpha_2$ then $m \equiv 1 \mod p^{a_2}$, that is $x_2^0 = x_2^n$. But $G_1$ centralizes $G_2$ and $x_2 \neq 1$, thus $m = 1$. If $\alpha_1 \leq \alpha_2$ then $1 \equiv m \mod p^{a_1}$. This means that $x_2^0 = x_1$. Since $x_1 \neq 1$ then $m = 1$. In both cases we get that $m = 1$, which implies that $G_1$ acts trivially on $L_p^1$. Now, $L_p^1 = [L_p^1, G_1] = \langle 1 \rangle$, a contradiction. Therefore, either $L_p^1$ is trivial or $L_p^2$ is trivial, that is $(|L_1|, |L_2|) = 1$.

Since $L$ is a Hall subgroup of $G_1 \times G_2$, then

$$(|L|, (G_1 \times G_2 : L)) = (|L_1||L_2|, (G_1 : L_1)(G_2 : L_2)) = 1.$$  

Hence, the above implies that

$$(|L_i|, |L_j|)(G_i : L_i)(G_j : L_j)) = 1,$$

which means $(|L_i|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$.

Conversely, suppose that $G_1$ and $G_2$ are solvable $\mathcal{PST}$-groups and $(|L_i|, |G_j|) = 1$ for $i \neq j$ and $i, j \in \{1, 2\}$. It is clear that $G_1 \times G_2$ is solvable. In addition, since each $G_i$, $i \in \{1, 2\}$, is supersolvable it follows that $G_1 \times G_2$ is supersolvable. Next, $(G_1 \times G_2)/L$ is nilpotent, thus every $p$-chief factor of $G_1 \times G_2$ above $L$ is central, where $p$ is some prime dividing the order of $G_1 \times G_2$. Hence, if $H_1/K_1$ and $H_2/K_2$ are two $p$-chief factors of $G_1 \times G_2$ above $L$, then for all $g \in G_1 \times G_2$ and $h_i k_i \in H_i/K_i$, $h_i^g k_i = h_iK_i$, for $i \in \{1, 2\}$, that is $g$ acts in the same way on $h_iK_1$ and $h_2K_2$. Therefore, all $p$-chief factors above $L$ are $(G_1 \times G_2)$-isomorphic. Note, that $(|L_i|, |G_j|) = 1$ for $i \neq j$ implies $(|L_1|, |L_2|) = 1$, then every $p$-chief factor of either $G_1$ or $G_2$. Since $G_i$ is a $\mathcal{PST}$-group then all $p$-chief factors of
$G_i$ are $G_i$-isomorphic, for $i \in \{1, 2\}$. In particular, $p$-chief factors of $G_1 \times G_2$ below $L = L_1 \times L_2$ are $(G_1 \times G_2)$-isomorphic. Therefore, $G_1 \times G_2$ is a $\mathcal{PST}$-group by Theorem 1.2. This concludes the proof.

Theorem 2.1.2 is true for a direct product of finitely many solvable $\mathcal{PST}$-groups. Let $G := G_1 \times G_2 \times \cdots \times G_n$ and suppose that $|\gamma_s(G_i)|, |G_j| = 1$ for $1 \leq i \neq j \leq n$. Then

$$|(\gamma_s(G_i)), |G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n|| = 1 \tag{2.1}$$

and

$$|(\gamma_s(G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_n)), |G_j|| = 1. \tag{2.2}$$

**Corollary 2.1.3.** Let $G_1, \ldots, G_n$ be finite groups. $G_1 \times G_2 \times \cdots \times G_n$ is a solvable $\mathcal{PST}$ group if and only if $G_1, G_2, \ldots, G_n$ are solvable $\mathcal{PST}$ groups, and $|\gamma_s(G_i)|, |G_j| = 1$ for $1 \leq i, j \leq n$, $i \neq j$ and $n \geq 2$.

The proof of Corollary 2.1.3 is by induction on $n$ with a direct application of Theorem 2.1.2.

**Proof.** The solvability is clear. Next, suppose that the statement is true for $n - 1$ and let $G := G_1 \times \cdots \times G_{n-1}$. Then, Theorem 2.1.2 and 2.1.2.2 imply that $G \times G_n$ is a solvable $\mathcal{PST}$-group if and only if $G$ and $G_n$ are solvable $\mathcal{PST}$-groups such that $|\gamma_s(G), |G_n|| = 1$ and $|G, |\gamma_s(G_n)|| = 1$. \qed

Now the question arises whether Theorem 2.1.2 could be applied to the $T$ and $\mathcal{PT}$ groups and what are the additional conditions needed in order to stay in the class.

### 2.2 Direct products of Iwasawa and Dedekind groups

The extension of Theorem 2.1.2 to solvable $\mathcal{PT}$ and $T$ groups will depend on our understanding of direct products of Iwasawa and Dedekind $p$-groups.

**Definition 2.2.1.**

(1) A *Dedekind group* is a group in which every subgroup is normal.

(2) An *Iwasawa group* is a group in which every subgroup is permutable.

The reader should note that Iwasawa (Dedekind) groups are necessarily solvable $\mathcal{PT}$ ($T$) groups, respectively.
Theorem 2.2.2. (Agrawal [1]) Let $G$ be a finite PST-group.

(1) If all Sylow subgroups of $G$ are Iwasawa, then $G$ is a $\mathcal{PT}$-group.

(2) If all Sylow subgroups of $G$ are Dedekind, then $G$ is a $T$-group.

Let $G_1$ and $G_2$ be solvable $\mathcal{PT}$-groups and $(|\gamma_i(G_1)|, |G_2|) = 1$ for $i \neq j$. Since the class of $\mathcal{PT}$-groups is contained in the class of PST-groups then from Theorem 2.1.2 we immediately get that $G_1 \times G_2$ is a solvable PST-group. Also, solvable $\mathcal{PT}$-groups are subgroup-closed, hence every Sylow subgroup of $G_i$ is a $\mathcal{PT}$-group, that is an Iwasawa-group. In particular, if $P$ is a Sylow subgroup of $G_1 \times G_2$ then it is a direct product of two Iwasawa groups. Similar ideas apply to $T$ groups. Therefore, we consider at first when a direct product of Iwasawa (Dedekind) groups is again an Iwasawa (Dedekind) group.

The following theorems will be used in order to establish results of this section.

Theorem 2.2.3 (Iwasawa, 1941 [15], page 55). A finite $p$-group $G$ has modular subgroup lattice if and only if

(a) $G$ is a direct product of a $Q_8$ with an elementary abelian 2-group, or

(b) $G$ contains an abelian normal subgroup $A$ with cyclic factor group $G/A$; further there exists an element $b \in G$ with $G = A\langle b \rangle$ and $s$ positive integer such that $a^b = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ in case $p = 2$.

Note, that a finite $p$-group $P$ has modular subgroup lattice if and only if all subgroups of $P$ are permutable that is if and only if $P$ is an Iwasawa-group. The details can be found in [15].

Remark 2.2.4. The rest of this thesis will assume the notation of Theorem 2.2.3 whenever there is a finite non abelian Iwasawa $p$-group. In particular, if $P_i$ is non abelian Iwasawa $p$-group and $Q_8$ is not contained in $P_i$, in case $p = 2$, then $P_i = A_i \langle x_i \rangle$ where $A_i$ is abelian normal subgroup of $P_i$ and $P_i/A_i$ is cyclic; furthermore there is a positive integer $s_i$ such that $a^{x_i} = a^{1+p^{s_i}}$ for all $a \in A_i$ with $s_i \geq 2$ in case $p = 2$.

Theorem 2.2.5 (Dedekind-Baer, [13], page 139 ). All the subgroups of a group $G$ are normal if and only if

(a) $G$ is abelian, or

(b) $G$ is the direct product of a $Q_8$, an elementary abelian 2-group and an abelian group with all its elements of odd order.
The following remark is a restriction of Corollary 2.0.5 to \( p \)-groups.

**Remark 2.2.6.** Let \( G_1, G_2, \ldots, G_n \) be finite \( p_i \)-groups, where \( p_i \) is a prime and \( p_i \neq p_j \) for all \( i, j \in \{1, \ldots, n\} \). Then,

(i) \( G_1, \ldots, G_n \in \mathcal{P}T \) if and only if \( G_1 \times \cdots \times G_n \in \mathcal{P}T \).

(ii) \( G_1, \ldots, G_n \in T \) if and only if \( G_1 \times \cdots \times G_n \in T \).

In particular, Remark 2.2.6 implies that if \( P \) is a \( p \)-group and \( Q \) is a \( q \)-group for some primes \( p \neq q \) such that \( P \) and \( Q \) are Iwasawa (Dedekind) groups, then direct product, \( P \times Q \), is Iwasawa (Dedekind) group respectively. Direct products of Dedekind \( p \)-groups, for the same prime \( p \), will follow directly from Dedekind-Baer Theorem 2.2.5 and present no difficulty, that is we only have to be careful about Sylow 2-group.

A more interesting case arises, that requires some investigation, is when a \( p \)-group \( P \) is a direct product of two or more Iwasawa \( p \)-groups and \( Q_8 \) is not contained in \( P \).

**Example 2.2.7.** Consider a Modular group of order 16

\[
G_1 = \langle x, y | x^8 = y^2 = 1, xy = x^5 \rangle
\]

Note, that \( G_1 \) is an Iwasawa group.

(i) Let \( G := G_1 \times C_2 \). Then \( G \) is an Iwasawa group.

(ii) Let \( G := G_1 \times C_8 \) where \( C_8 = \langle z \rangle \). Then \( \langle xz \rangle \) is not permutable in \( G \). Thus, \( G \) is not an Iwasawa group. Also, note that \( \text{Exp}(C_8) \) is greater than \( 2^2 \).

(iii) Let \( G := G_1 \times G_1 \). Then \( G \) is not an Iwasawa since \( \langle x^i x^j \rangle \) does not permute with \( \langle y \rangle \).

Example 2.2.7 shows that a direct product of two Iwasawa groups is not necessarily an Iwasawa even if one factor is abelian and the other is not.

The next Theorem assumes the notation of Remark 2.2.4

**Theorem 2.2.8.** Let \( P = P_1 \times P_2 \) be a non abelian finite \( p \)-group and \( Q_8 \not\subset P \), in case \( p = 2 \). Then, \( P \) is an Iwasawa group if and only if \( P_1 \) and \( P_2 \) are Iwasawa and either for \( i = 1 \) or \( i = 2 \) \( P_i \) is abelian such that \( \text{Exp}(P_i) \leq p^s \) and \( s \) is the integer that comes from the non abelian factor \( P_j \) for \( j \neq i \).
Proof. Suppose $P$ is an Iwasawa group but $P_1$ and $P_2$ are both not abelian. It is clear that each $P_i$ is an Iwasawa group, thus $P_i = A_i \langle x_i \rangle$ where $A_i$ is abelian normal subgroup of $P_i$ such that $a^{s_i} = a^{1+p^{s_i}}$ for all $a \in A_i$ with $s_i \geq 2$. Also, notice that $p^{s_i}$ must be strictly less than Exp$(A_i)$, otherwise $a_i^{s_i} = a_i^{1+p^{s_i}} = a_i a_i^{\text{Exp}(A_i)\ell} = a_i$ for some positive integer $\ell$, i.e. $x_i$ will act trivially on all $a_i \in A_i$.

Let $a_i \in A_i$ such that $a_i \neq 1$. Since $P$ is an Iwasawa then all subgroups of $P$ are permutable. This means that $\langle a_1a_2 \rangle$ permutes with $\langle x_1 \rangle$, that is $a_1a_2x_1$ must be an element of $\langle x_1 \rangle \langle a_1a_2 \rangle$. Hence,

$$a_1a_2x_1 = x_1^k(a_1a_2)^\ell = x_1^ka_1^{\ell}a_2^\ell$$

for some positive integers $k$ and $\ell$. But,

$$a_1a_2x_1 = x_1a_1^{-1}a_1x_1^{-1}a_2x_1 = x_1a_1^{1+p^{s_1}}a_2$$

$$x_1^ka_1^{\ell}a_2^\ell = x_1a_1^{1+p^{s_1}}a_2,$$

which implies that either $\ell = 1$ or $\ell = 1 + p^{s_1}$. Since $x_1$ does not act trivially on $A_1$, then it must be that $\ell = 1 + p^{s_1}$. Therefore, $a_2 = a_2^{1+p^{s_1}}$ for all $a_2 \in A_2$, and as a result the Exp$(A_2) \leq p^{s_1}$. Similarly, $\langle a_1a_2 \rangle$ must permute with $\langle x_2 \rangle$, and hence by a similar argument Exp$(A_1) \leq p^{s_2}$.

Now, for $\langle a_1a_2 \rangle$ to permute with $\langle x_1 \rangle$ and $\langle x_2 \rangle$ it must be that Exp$(A_2) \leq p^{s_1}$ and Exp$(A_1) \leq p^{s_2}$. Recall, that $p^{s_i}$ must be strictly less that the exponent of $A_i$. Therefore,

$$p^{s_1} \leq \text{Exp}(A_1) \leq p^{s_2} \leq \text{Exp}(A_2) \leq p^{s_1},$$

in particular $p^{s_i} < p^{s_j}$ which is a contradiction. Hence Exp$(A_2) \leq p^{s_1}$ and Exp$(A_1) \leq p^{s_2}$ cannot happen at the same time, therefore for either $i = 1$ or $i = 2$ the $x_i$ must act trivially on $A_i$, i.e. $P_i = A_i \times \langle x_i \rangle$ is abelian.

The Exp$(P_i) \leq p^{s_i}$ for $i \neq j$ follows by a similar argument. In particular, without loss of generality suppose that $P_2$ is abelian. Let $a_2 \in P_2$ and $a_1 \in A_1$, where $A_1$ as above. Then $\langle a_1a_2 \rangle$ and $\langle x_1 \rangle$ must permute, that is $a_1a_2x_1$ must be an element of $\langle x_1 \rangle (a_1a_2)$. Hence,

$$x_1^ka_1^{\ell}a_2^\ell = x_1a_1^{1+p^{s_1}}a_2,$$

which implies that either $\ell = 1$ or $\ell = 1 + p^{s_1}$. Since $x_1$ does not act trivially on $A_1$, then it must be that $\ell = 1 + p^{s_1}$. Therefore, $a_2 = a_2^{1+p^{s_1}}$ for all $a_2 \in P_2$, and as a result the Exp$(P_2) \leq p^{s_1}$.

Conversely, suppose that $P_1$ is abelian and let $A = P_1 \times A_2$. Then $A$ is abelian normal
subgroup of $P_1 \times P_2$ and $(P_1 \times P_2)/A$ is cyclic. Let $a \in A$ then $a = gh$ where $g \in P_1$ and $h \in A_2$. Then

$$a^x = (gh)^x = gh^x = gh^{1+p^s}.$$  

Since $\text{Exp}(P_1) \leq p^{s_1}$ then $\text{ord}(g) \leq p^{s_1}$ that is $g^{p^{s_1}} = 1$ which implies that $g^{1+p^s} = g$. Therefore,

$$a^x = gh^{1+p^s} = g^{1+p^s}h^{1+p^s} = a^{1+p^s}.$$ 

Hence $P_1 \times P_2$ is an Iwasawa group. This concludes the proof. \(\square\)

Theorem 2.2.8 implies that if $P := P_1 \times \cdots \times P_n$ is a non abelian Iwasawa $p$-group and $Q_8$ is not contained in $P$, then exactly one $P_i$ will be non abelian Iwasawa $p$-group and $P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$ must be abelian $p$-group such that $\text{Exp}(P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n) \leq p^{s_i}$.

Also, note that if $G$ is any Iwasawa group then every subgroup is permutable in $G$ and consequently subnormal which implies that $G$ is nilpotent and thus a direct product of its Sylow subgroups. Hence, a general Iwasawa group $G$ is a direct product of unique maximal Iwasawa $p$-groups.

### 2.3 Direct products of solvable $\mathcal{PT}$ and $\mathcal{T}$ groups

The results up to this point provide conditions needed in order to extend Theorem 2.1.2 to solvable $\mathcal{PT}$ and $\mathcal{T}$ groups.

**Remark 2.3.1.** The next theorems will employ the following notation.

Let $P := P_1 \times \cdots \times P_n$ for $n \geq 2$ then $P \backslash P_i := P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_n$.

**Theorem 2.3.2.** Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \text{Syl}_p(G)$ where $P_i \in \text{Syl}_p(G_i)$ for some prime $p$ dividing the order of $G$ and $n \geq 2$.

Then $G$ is a solvable $\mathcal{PT}$-group if and only if the following hold:

(i) $G_1, \ldots, G_n$ are solvable $\mathcal{PT}$-groups such that $(|\gamma_i(G_i)|, |G_j|) = 1$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

(ii) If $Q_8$ is contained in $P$, then it is a subgroup of exactly one $P_i$ and $P \backslash P_i$ is elementary 2-abelian or trivial.

(iii) If $Q_8$ is not contained in $P$, in case $p = 2$, and $P$ is not abelian, then exactly one $P_i$ is non abelian and $P \backslash P_i$ is an abelian $p$-group such that $\text{Exp}(P \backslash P_i) \leq p^{s_i}$. 

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Proof. First suppose that $G$ is a solvable $PT$-group. Since each $G_i$ is normal in $G$ then $G_1,\ldots,G_n$ are solvable $PT$-groups. Also, note that $G$ is a solvable $PST$-group, thus by Corollary 2.1.3 ($|g_i(G_i)|, |G_j|$) = 1 for all $i, j \in \{1,\ldots,n\}$ with $i \neq j$. This proves (i).

Since solvable $PT$-groups are subgroup closed then $P$ is a solvable $PT$-group, i.e. $P$ is an Iwasawa group and consequently each $P_i$ is an Iwasawa-group. If $Q_8$ is contained in $P$ then by Theorem 2.2.3 part (a) $P$ is a direct product of $Q_8$ and an abelian 2-group, which shows (ii). If $Q_8$ is not contained in $P$ and $P$ is not abelian then Theorem 2.2.8 implies (iii).

Conversely, suppose (i)-(iii). The solvability of $G$ is clear. Note that $G_i$ are solvable $PST$-groups for all $i$, hence (i) implies that $G$ is a solvable $PST$-group. Also, since $G_i$ are solvable $PT$-groups then each $P_i$ is an Iwasawa group, in particular $P$ is a direct product of Iwasawa groups. If $P$ is abelian then $P$ is an Iwasawa. If $Q_8$ is contained in $P$ then (ii) satisfies the hypothesis of Theorem 2.2.3 (a), i.e. $P$ is an Iwasawa. If $Q_8$ is not contained in $P$ and $P$ is not abelian then (iii) and Theorem 2.2.8 imply that $P$ is an Iwasawa. This means that all Sylow subgroups of $G$ are Iwasawa and since $G$ is a $PST$-group then by $G$ is a $PT$-group. This concludes the proof.

Theorem 2.3.3. Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in Syl_p(G)$ where $P_i \in Syl_p(G_i)$ for some prime $p$ dividing the order of $G$ and $n \geq 2$. Then $G$ is a solvable $T$-group if and only if the following hold:

(i) $G_1,\ldots,G_n$ are solvable $T$-groups such that ($|g_i(G_i)|, |G_j|$) = 1 for all $i, j \in \{1,\ldots,n\}$ with $i \neq j$.

(ii) if $Q_8$ is contained in $P$, then it is a subgroup of exactly one $P_i$ and $P\setminus P_i$ is elementary 2-abelian or trivial; otherwise $P$ is abelian.

Proof. First suppose that $G$ is a solvable $T$-group. It is immediate from the proof of Theorem 2.3.2 that (i) is satisfied and $P$ is Dedekind group and consequently each $P_i$ is Dedekind-group. If $Q_8$ is contained in $P$ then by Theorem 2.2.5 part (b) $P$ is a direct product of $Q_8$ and an abelian 2-group. If $Q_8$ is not contained in $P$ then $P$ must be abelian. Thus (ii) is proved.

Conversely, suppose (i) and (ii). The solvability of $G$ is clear and since the $G_i$ are solvable $PST$-groups for all $i$ then (i) implies that $G$ is a solvable $PST$-group. Also, since each $G_i$ is a solvable $T$-group then each $P_i$ is Dedekind group, in particular $P$ is a direct product of Dedekind groups. If $p$ is odd prime, then each $P_i$ is abelian,
hence $P$ is abelian. If $p = 2$ then (ii) implies that either $P$ is abelian or a direct product of $Q_8$ and an elementary abelian 2-group. Hence, all Sylow subgroups of $G$ are Dedekind and since $G$ is a $\mathcal{PST}$-group then by $[1]$ $G$ is a $T$-group. This concludes the proof.

2.4 Direct products of $\mathcal{PST}_e$ groups

We begin this section by revisiting the important distinction between solvable $\mathcal{PST}$ and $\mathcal{PST}_e$. Recall, the nilpotent residual of a solvable $\mathcal{PST}_e$-group is a Hall-subgroup of the Fitting subgroup whereas the nilpotent residual of solvable $\mathcal{PST}$-groups is a Hall subgroup of the entire group. Also, bear in mind that the class of solvable $\mathcal{PST}_e$-groups is not subgroup closed nor quotient closed class, however a normal subgroup of a solvable $\mathcal{PST}_e$-group is also a $\mathcal{PST}_e$-group.

Remark 2.4.1. Let $G$ be a finite solvable $\mathcal{PST}_e$-group such that the nilpotent residual of $G$ is a Hall-subgroup of $G$. Then $G$ is a solvable $\mathcal{PST}$-group.

Proof. Let $L$ be a nilpotent residual of $G$. By Theorem 1.1 in $[12]$ $L$ is abelian of odd order. Thus, all it remains to show that $G$ acts by conjugation as a power automorphism on $L$.

Let $x \in L$ such that $x$ is not trivial. Since $L$ is abelian then $\langle x \rangle$ is normal subgroup of $L$, i.e. $\langle x \rangle$ is subnormal subgroup of $G$. But $G$ is a $\mathcal{PST}_e$, hence $\langle x \rangle$ is $S$-permutable in $G$. Let $\pi := \pi(L)$ be the set of primes that divide the order of $L$ and let $q \in \pi'$. Consider a Sylow $q$ subgroup of $G$, $G_q$. Then $\langle x \rangle G_q = G_q \langle x \rangle$. Also, $\langle x \rangle$ is a subnormal Hall $\pi$-subgroup of $G_q \langle x \rangle$, thus $\langle x \rangle$ is normal subgroup of $G_q \langle x \rangle$, that is $G_q$ normalizes $\langle x \rangle$ and hence $O^\pi(G)$ normalizes $\langle x \rangle$, where $O^\pi(G)$ is the subgroup of $G$ generated by all Sylow $q$ subgroups of $G$, $q \in \pi'$. Since, $L$ is a Hall $\pi$-subgroup then $G = LO^\pi(G)$. But $L$ is nilpotent and so $G/O^\pi(G)$ is nilpotent. Also, $L$ is the smallest normal subgroup of $G$ such that $G/L$ is nilpotent, then $L \leq O^\pi(G)$. Thus, $G = O^\pi(G)$ normalizes $\langle x \rangle$, i.e. $G$ acts by conjugation on $L$ as a power automorphism. Therefore, $G$ is a solvable $\mathcal{PST}$-group.

We now extend Theorem 2.1.2 to a direct product of $\mathcal{PST}_e$-groups.

Theorem 2.4.2. Let $G_1$ and $G_2$ be finite groups. $G_1 \times G_2$ is a solvable $\mathcal{PST}_e$ group if and only if $G_1$ and $G_2$ are solvable $\mathcal{PST}_e$ groups such that $(|\gamma_s(G_i)|,|\text{Fit}(G_j)|) = 1$ for $i \neq j$ and $i,j \in \{1,2\}$.
Proof. Let \( L := \gamma_s(G_1 \times G_2) \) and \( F := \text{Fit}(G_1 \times G_2) \). Then \( L = L_1 \times L_2 \) and \( F = F_1 \times F_2 \) where \( L_i \) is nilpotent residual of \( G_i \), and \( F_i \) is Fitting subgroup of \( G_i \).

First suppose that \( G_1 \times G_2 \) is a solvable \( \mathcal{PST}_c \) group. Since \( G_i \) is normal in \( G_1 \times G_2 \) then each \( G_i \) is a solvable \( \mathcal{PST}_c \) group. Let \( L_p \in \text{Syl}_p(L) \), then \( L_p = L_p^1 \times L_p^2 \). Following the same argument as in Theorem 2.1.2 either \( L_p^1 \) or \( L_p^2 \) is trivial. Hence \( (|L_1|, |L_2|) = 1 \), which implies that \( (|L_i|, |F_j|) = 1 \) for \( i \neq j \).

As for the other direction, suppose that \( G_1 \) and \( G_2 \) are solvable \( \mathcal{PST}_c \) groups such that \( (|L_i|, |F_j|) = 1 \) for \( i \neq j \). Trivially, \( G_1 \times G_2 \) is solvable, \( L \) is abelian and \( G_1 \times G_2/L \) is nilpotent.

Since \( L_i \) is a Hall subgroup of \( F_i \) and \( F_i \) is a solvable \( \mathcal{PST} \)-group, then using similar ideas as in the proof of Theorem 2.1.2 we get that \( L \) is a Hall subgroup of \( F \) and \( (|L_1|, |L_2|) = 1 \). Thus, all we need to show that \( p' \)-elements of \( G_1 \times G_2 \) induce power automorphisms in \( L_p \in \text{Syl}_p(L) \) for all primes \( p \).

Let \( L_p \in \text{Syl}_p(L) \) and \( Q \in \text{Syl}_q(G_1 \times G_2) \) for \( p \neq q \). Then \( Q = Q_1 \times Q_2 \) where \( Q_i \in \text{Syl}_q(G_i) \). Since \( (|L_1|, |L_2|) = 1 \) then \( L_p \leq L_1 \) or \( L_p \leq L_2 \). Without loss of generality suppose \( L_p \leq L_1 \).

Let \( a \in L_p \) not trivial, then \( \langle a \rangle \) is subnormal subgroup of \( G_1 \) and \( G_1 \) is \( \mathcal{PST}_c \). This implies that \( \langle a \rangle \) S-per \( G_1 \), in particular \( \langle a \rangle Q_1 = Q_1 \langle a \rangle \). Also, \( G_1 \) centralizes \( G_2 \), then \( \langle a \rangle \) permutes with \( Q_2 \). Thus, \( \langle a \rangle \) permutes with \( Q = Q_1 \times Q_2 \).

Hence

\[
a^Q = a^Q \cap (\langle a \rangle Q) = \langle a \rangle (a^Q \cap Q)
\]

But \( a^Q \) is a \( p \)-group implies that \( a^Q \cap Q \) is trivial. Therefore,

\[
a^Q = \langle a \rangle
\]

Thus we conclude that \( G_1 \times G_2 \) is a solvable \( \mathcal{PST}_c \)-group and the proof is complete. \( \square \)

Using ideas similar to those in Corollary 2.1.3 one can show by induction that Theorem 2.4.2 is true for a direct product of finitely many solvable \( \mathcal{PST}_c \)-groups.

### 2.5 Direct products of \( \mathcal{PT}_c \) and \( T_c \) groups

Recall, Agrawal[1, 2.2.2] has shown that a \( \mathcal{PST} \) group \( G \) is a \( \mathcal{PT} (T) \) group if all of its Sylow subgroups are Iwasawa (Dedekind) respectively. A similar result can be proved for \( \mathcal{PST}_c \)-groups.
Theorem 2.5.1. Let $G$ be a finite $\mathcal{PST}_c$-group.

(1) If all Sylow subgroups of $G$ are Iwasawa, then $G$ is a $\mathcal{PT}_c$-group.

(2) If all Sylow subgroups of $G$ are Dedekind, then $G$ is a $\mathcal{T}_c$-group.

Proof. (1). Let $H$ be any cyclic subnormal subgroup of the group $G$ and $H_p$ be a direct factor of $H$ for some prime $p$ dividing the order of $H$. Then $H_p$ is a subgroup of the $F_p \in \text{Syl}_p(\text{Fit}(G))$. Since $F_p = \cap \{G_p \mid G_p \in \text{Syl}_p(G)\}$ then $H_p$ is contained in every Sylow $p$-subgroup of $G$. Let $g \in G$ be non trivial, then $\langle g \rangle = \langle g_1 \rangle \times \langle g_2 \rangle$ where $g_1$ is a $p$-element and $g_2$ is a $p'$-element. Since $H_p$ is subnormal subgroup of $G$ and $G$ is a $\mathcal{PST}_c$-group then by Lemma 2.2 in [12] $p'$-elements induce power automorphisms in $H_p$. In particular, $g_2$ and $H_p$ permute. Also, by hypothesis each $G_p$ is Iwasawa and $\langle g_1 \rangle$ is contained in some Sylow $p$-subgroup of $G$, then $H_p$ and $\langle g_1 \rangle$ permute. Hence, $H_p$ and $\langle g \rangle$ permute. Since, $H$ is a direct product of cyclic subgroups of prime power order then $H$ and $\langle g \rangle$ permute, which implies that $H$ is permutable in $G$, that is $G$ is a $\mathcal{PT}_c$-group.

(2). Let $H$ be any cyclic subnormal subgroup of the group $G$ and $H_p$ be a direct factor of $H$ for some prime $p$ dividing the order of $H$. Then $H_p$ is a subgroup of the $F_p \in \text{Syl}_p(\text{Fit}(G))$. In particular, $H_p$ is contained in every Sylow $p$-subgroup of $G$. Since, every Sylow subgroup of $G$ is Dedekind then $H_p$ is normal subgroup of every Sylow $p$-subgroup. Also, $p'$-elements induce power automorphisms in $H_p$, that is $H_p$ is normalized by every $p'$-element. Thus, we conclude that $H_p$ is normal subgroup of $G$, which implies that $H$ is normal in $G$, that is $G$ is a $\mathcal{T}_c$-group. \hfill \Box

Theorem 2.5.1 suggests that Theorems 2.3.2 and 2.3.3 might be extended to $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups, however these classes of groups are not subgroup closed. In particular, it is not clear if a Sylow subgroup of the solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$)-group will inherit the same properties as the group, that is Sylow subgroups are not necessarily Iwasawa (Dedekind).

Open Question. Given a finite solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$)-group, identify the structure of the Sylow subgroups.

There are countless examples of solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$-groups in which Sylow subgroups are all Iwasawa and Dedekind, respectively. Thus, it makes sense to make a restriction to such groups, that is we restrict to solvable $\mathcal{PT}_c$ ($\mathcal{T}_c$) groups in which every Sylow subgroup is Iwasawa (Dedekind).
Corollary 2.5.2. Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \Syl_p(G)$ where $P_i \in \Syl_p(G_i)$ for some prime $p$ dividing the order of $G$ and $n \geq 2$. Suppose every Sylow subgroup of $G_i$ is an Iwasawa group, for all $i \in \{1, \ldots, n\}$.

Then $G$ is a solvable $\mathcal{PT}_c$-group if the following hold:

(i) $G_1, \ldots, G_n$ are solvable $\mathcal{PT}_c$-groups such that $(|\gamma_s(G_i)|, |\Fit(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.

(ii) If $Q_8$ is contained in $P$, then it is a subgroup of exactly one $P_i$ and $P\backslash P_i$ is elementary 2-abelian or trivial.

(iii) If $Q_8$ is not contained in $P$ and $P$ is not abelian, then exactly one $P_i$ is non-abelian and $P\backslash P_i$ is abelian $p$-group such that $\Exp(P\backslash P_i) \leq p^{\alpha_i}$. 

Proof. Suppose (i)-(iii). The solvability of $G$ is clear. Since the class of $\mathcal{PT}_c$-groups is a subclass of $\mathcal{PST}_c$-groups then each $G_i$ is a solvable $\mathcal{PST}_c$-group for all $i$. Consequently, the assumption in (i) and Theorem 2.4.2 imply that $G$ is a solvable $\mathcal{PST}_c$-group. Also, by hypothesis each $P_i \in \Syl_p(G_i)$ is an Iwasawa group for all primes $p$ dividing the order of $G_i$. In particular, $P$ is a direct product of Iwasawa groups.

If $P$ is abelian then $P$ is an Iwasawa. If $Q_8$ is contained in $P$ then (ii) satisfies the hypothesis of Theorem 2.2.3 (a), i.e. $P$ is an Iwasawa. If $Q_8$ is not contained in $P$ and $P$ is not abelian then (iii) and Theorem 2.2.8 imply that $P$ is an Iwasawa. This means that all Sylow subgroups of $G$ are Iwasawa and since $G$ is a $\mathcal{PST}_c$-group then by Lemma 2.5.1 $G$ is a $\mathcal{PT}_c$-group. This concludes the proof.

Notice that in Theorem 2.3.2 we had a biconditional statement whereas in Corollary 2.5.2 if we suppose that $G := G_1 \times \cdots \times G_n$ is a $\mathcal{PT}_c$-group then we do not know if $P := P_1 \times \cdots \times P_n \in \Syl_p(G)$ is an Iwasawa group since we only supposed that each $P_i$ was an Iwasawa-group.

Remark 2.5.3. Let $G := G_1 \times \cdots \times G_n$ be a finite group. Suppose that for all primes $p$ dividing the order of $G$ every Sylow $p$-subgroup of $G$ is an Iwasawa group. Then $G$ is a solvable $\mathcal{PT}_c$-group if and only if $G_1, \ldots, G_n$ are solvable $\mathcal{PT}_c$-groups such that $(|\gamma_s(G_i)|, |\Fit(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.

Proof. Suppose that $G$ is a solvable $\mathcal{PT}_c$-group. Then each $G_i$ is a solvable $\mathcal{PT}_c$-group for all $i \in \{1, \ldots, n\}$. Also, $G \in \mathcal{PT}_c$ implies that $G \in \mathcal{PST}_c$. Hence by Theorem 2.4.2 we get that $(|\gamma_s(G_i)|, |\Fit(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$. Conversely, suppose that $G_1, \ldots, G_n$ are solvable $\mathcal{PT}_c$-groups such that $(|\gamma_s(G_i)|, |\Fit(G_j)|) = 1$ for all $i \neq j \in \{1, \ldots, n\}$.
Corollary 2.5.4. Let $G := G_1 \times \cdots \times G_n$ be a finite group and $P := P_1 \times \cdots \times P_n \in \text{Syl}_p(G)$ where $P_i \in \text{Syl}_p(G_i)$ for some prime $p$ dividing the order of $G$ and $n \geq 2$. Suppose every Sylow subgroup of $G_i$ is Dedekind, for all $i \in \{1, \ldots, n\}$. Then $G$ is a solvable $T_c$-group if the following hold:

(i) $G_1, \ldots, G_n$ are solvable $T_c$-groups such that $|\gamma_*(G_i)| \cdot |\text{Fit}(G_j)| = 1$ for all $i \neq j \in \{1, \ldots, n\}$.

(ii) if $Q_8$ is contained in $P$, then it is a subgroup of exactly one $P_i$ and $P \backslash P_i$ is elementary 2-abelian or trivial; otherwise $P$ is abelian.

Proof. Suppose (i) and (ii). The solvability of $G$ is clear. Since the class of $T_c$-groups is a subclass of PST$_c$-groups then each $G_i$ is a solvable PST$_c$-group for all $i$. Consequently, the assumption in (i) and Theorem 2.4.2 imply that $G$ is a solvable PST$_c$-group. Also, by hypothesis each $P_i \in \text{Syl}_p(G_i)$ is a Dedekind group for all primes $p$ dividing the order of $G_i$. In particular $P$ is a direct product of Dedekind groups. If $p$ is an odd prime, then each $P_i$ is abelian, hence $P$ is abelian. If $p = 2$ then (ii) implies that either $P$ is abelian or a direct product of $Q_8$ and an elementary abelian 2-group. Hence, all Sylow subgroups of $G$ are Dedekind and since $G$ is a PST$_c$-group then by Lemma 2.5.1 $G$ is a $T_c$-group. This concludes the proof.

Remark 2.5.5. Let $G := G_1 \times \cdots \times G_n$ be a finite group. Suppose that for all primes $p$ dividing the order of $G$ every Sylow $p$-subgroup of $G$ is a Dedekind group. Then $G$ is a solvable $T_c$-group if and only if $G_1, \ldots, G_n$ are solvable $T_c$-groups such that $|\gamma_*(G_i)| \cdot |\text{Fit}(G_j)| = 1$ for all $i \neq j \in \{1, \ldots, n\}$.

Proof. Suppose that $G$ is a solvable $T_c$-group. Then each $G_i$ is a solvable $T_c$-group for all $i \in \{1, \ldots, n\}$. Also, $G \in T_c$ implies that $G \in \text{PST}_c$. Hence by Theorem 2.4.2 we get that $|\gamma_*(G_i)| \cdot |\text{Fit}(G_j)| = 1$ for all $i \neq j \in \{1, \ldots, n\}$. Conversely, suppose that $G_1, \ldots, G_n$ are solvable $T_c$-groups such that $|\gamma_*(G_i)| \cdot |\text{Fit}(G_j)| = 1$ for all $i \neq j \in \{1, \ldots, n\}$. Then by Theorem 2.4.2 $G$ is a solvable PST$_c$-group. But now, hypothesis and Lemma 2.5.1 imply that $G$ is a $T_c$-group.

If $G$ is a $PT_c$ or $T_c$ group then the Fitting subgroup of $G$ is an Iwasawa group or a Dedekind group, by Lemma 1.7.2, Subgroups of the Iwasawa (Dedekind) groups.

for all $i \neq j \in \{1, \ldots, n\}$. Then by Theorem 2.4.2 $G$ is a solvable PST$_c$-group. But now, hypothesis and Lemma 2.5.1 imply that $G$ is a $PT_c$-group. □

Similar extensions can be made for solvable $T_c$-groups.
inherit the same properties, thus Sylow subgroups of the Fitting subgroup of the $\mathcal{PT}_c$ ($T_c$) groups are Iwasawa (Dedekind) groups.

**Open Question.** Restricting Theorems 2.3.2 and 2.3.3 to Sylow subgroups of the Fitting subgroup, what are the additional conditions (if any) needed in order to extend these theorems to solvable $\mathcal{PT}_c$ and $T_c$-groups?

2.6 The Fitting subgroup and direct products

Recall that if orders of groups are relatively prime then a direct product of $\mathcal{PST}$, $\mathcal{PT}$ or $T$ groups was again a $\mathcal{PST}$, $\mathcal{PT}$ or $T$-group, 2.0.5 and [1]. Trivially, it works the same way for the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $T_c$. On the other hand, if $H$ is any subnormal cyclic subgroup of a group $G$ then $H$ is contained in the Fitting subgroup of $G$ and hence the question arises whether an assumption can be made that involves only the orders of the Fitting subgroups.

**Theorem 2.6.1.** Let $G_1$ and $G_2$ be finite groups so that $(|\text{Fit}(G_1)|, |\text{Fit}(G_2)|) = 1$.

(i) $G_1$ and $G_2$ are $\mathcal{PST}_c$-groups if and only if $G_1 \times G_2 \in \mathcal{PST}_c$.

(ii) $G_1$ and $G_2$ are $\mathcal{T}_c$-groups if and only if $G_1 \times G_2 \in \mathcal{T}_c$.

**Proof.** First we note that normal subgroups of $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $T_c$ groups inherit the same properties. In particular if $G_1 \times G_2$ is a $\mathcal{PST}_c$, $\mathcal{PT}_c$, or $T_c$ group then each $G_i$ is a $\mathcal{PST}_c$, $\mathcal{PT}_c$, or $T_c$ group.

Let $H$ be a subnormal cyclic subgroup of $G_1 \times G_2$. Then $H$ is contained in the $\text{Fit}(G_1 \times G_2)$. Since $(|\text{Fit}(G_1)|, |\text{Fit}(G_2)|) = 1$ and $\text{Fit}(G_1 \times G_2) = \text{Fit}(G_1) \times \text{Fit}(G_2)$ then $H = H_1 \times H_2$ where each $H_i$ is a subgroup of $\text{Fit}(G_i)$. In particular $H_i$ is subnormal subgroup of $G_i$.

Let $G_p = G_p^1 \times G_p^2 \in \text{Syl}_p(G_1 \times G_2)$ where $G_p^i \in \text{Syl}_p(G_i)$. If $G_i$ is a $\mathcal{PST}_c$ then $H_i$ is Sylow-permutable in $G_i$, i.e. $H_i$ permutes with $G_p^i$. But $G_j$ centralizes $G_i$, hence $H_i$ permutes with $G_p^j$. Thus $H$ and $G_p$ permute, that is $G_1 \times G_2$ is $\mathcal{PST}_c$ group.

If $G_i$ is a $\mathcal{T}_c$ group then $H_i$ is normal subgroup in $G_i$, that is $H_1 \times H_2$ is normal subgroup of $G_1 \times G_2$. Hence, $G_1 \times G_2$ is $\mathcal{T}_c$ group. This proves (i) and (ii). 

To prove a similar result for $\mathcal{PT}_c$-groups, one has to be concerned about subgroups that cannot be written as a direct product, that is subgroups that are not of the form $K_1 \times K_2$, $K_i \in G_i$. In particular, make the same assumptions as in Theorem 2.6.1 and consider a cyclic subnormal subgroup $H$ of the group $G_1 \times G_2$, where each $G_i \in \mathcal{PT}_c$. 

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Let $H_p$ be a Sylow $p$-subgroup of $H$, for some prime dividing the order of $H$. Then $H_p$ is contained in $O_p(G)$ and Fitting subgroups of $G_1$ and $G_2$ have relatively prime orders, so $H_p$ will be contained in $G_1$ or $G_2$. Now, $p'$-elements of $G_1 \times G_2$ present no problem, but if $g_1 g_2$ is any $p$-element of $G_1 \times G_2$, where $g_i \in G_i$, then it is not clear whether $H_p$ will permute with $\langle g_1 g_2 \rangle$, even though $H_p$ permutes with each $\langle g_i \rangle$.

**Remark 2.6.2.** Let $g_1 g_2$ be any $p$-element of $G_1 \times G_2$ for some prime divisor $p$ of the order of $G_1 \times G_2$. Then $\text{ord}(g_1 g_2) = p^\alpha$ for some integer $\alpha$. The reader should keep in mind that $g_1 g_2 = (g_1, g_2)$. Now, $\text{ord}(g_1 g_2) = \text{lcm}(\text{ord}(g_1), \text{ord}(g_2)) = p^\alpha$. This means that $\text{ord}(g_1)|p^\alpha$ and $\text{ord}(g_2)|p^\alpha$. Therefore, $\text{ord}(g_i) = p^{\alpha_i}$, for some integer $\alpha_i \leq \alpha$. Thus, $g_i$ is a $p$-element of $G_i$.

**Lemma 2.6.3.** Let $G_1 \times G_2$ be a finite group and let $g_1 g_2 \in G_1 \times G_2$ be any $p$-element for some prime $p$ dividing the order of $G_1 \times G_2$, where $g_i \in G_i$. Then, $g_1^n g_2 \in \langle g_1 g_2 \rangle$ if and only if for some integer $n$,

$$m \equiv (n \cdot \text{ord}(g_2) + 1) \mod \text{ord}(g_1),$$

where $\text{ord}(g_i)$ is the order of $g_i$.

**Proof.** Let $G_1 \times G_2$ be a finite group and let $g_1 g_2 \in G_1 \times G_2$ be any $p$-element. Let $\text{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Suppose that $g_1^n g_2 \in \langle g_1 g_2 \rangle$. Then for some integer $j$, $g_1^m g_2 = (g_1 g_2)^j = g_1^j g_2^j$, that is

$$m \equiv j \mod p^{\alpha_1} \text{ and } 1 \equiv j \mod p^{\alpha_2}.$$ 

Now, for some integers $k_1$ and $k_2$ we have,

$$j - m = k_1 \cdot p^{\alpha_1} \text{ and } j - 1 = k_2 \cdot p^{\alpha_2}$$

$$m + k_1 \cdot p^{\alpha_1} = 1 + k_2 \cdot p^{\alpha_2}$$

$$k_1 \cdot p^{\alpha_1} = (1 + k_2 \cdot p^{\alpha_2}) - m.$$ 

Therefore, $m \equiv (1 + k_2 \cdot p^{\alpha_2}) \mod p^{\alpha_1}$. Conversely, if $m \equiv (1 + n \cdot p^{\alpha_2}) \mod p^{\alpha_1}$ for some integer $n$, then

$$g_1^m = g_1^{1 + n \cdot p^{\alpha_2}}.$$

But, $g_2^{1 + n \cdot p^{\alpha_2}} = g_2$, since the order of $g_2$ is $p^{\alpha_2}$. Hence,

$$g_1^m g_2 = g_1^{1 + n \cdot p^{\alpha_2}} g_2^{1 + n \cdot p^{\alpha_2}} = (g_1 g_2)^{1 + n \cdot p^{\alpha_2}} \in \langle g_1 g_2 \rangle.$$ 

$\square$

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Theorem 2.6.4. Let $G_1$ and $G_2$ be finite groups so that $(|\text{Fit}(G_1)|, |\text{Fit}(G_2)|) = 1$. Let $H_p$ be a subnormal cyclic $p$-subgroup of $G_i$ and $g_i, g_2$ be any $p$-element of $G_1 \times G_2$, where $g_i \in G_i$. Then, $G_1 \times G_2 \in \mathcal{PT}_c$ if and only if each $G_i$ is a $\mathcal{PT}_c$-group so that

$$h^{g_i} = g_i^{n \cdot \text{ord}(g_i)} h^k$$

for all $h \in H_p$ and every prime divisor $p$ of the order of $G_i$ and some integers $n$, $k$ and $i \neq j \in \{1, 2\}$.

Proof. First we note that normal subgroups of $\mathcal{PT}_c$-groups inherit the same properties. In particular if $G_1 \times G_2$ is a $\mathcal{PT}_c$-group then each $G_i$ is a $\mathcal{PT}_c$-group. Without loss of generality, let $H_p$ be any subnormal cyclic $p$-subgroup of $G_1$ and $g_1, g_2$ a $p$-element of $G_1 \times G_2$, where $g_i \in G_i$. Let $\text{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Now, $H_p$ is permutably subgroup of $G_1$ and of $G_1 \times G_2$. Thus, $H_p$ permutes with $\langle g_1 \rangle$ and with $\langle g_1, g_2 \rangle$. In particular, $h g_1 = g_1^m h^k$, where $h, h^k \in H_p$ and $g_1^m \in \langle g_1 \rangle$, for some integers $m$ and $k$. Now, consider $h(g_1 g_2) \in H_p(g_1 g_2)$. Then

$$h(g_1 g_2) = h g_1 g_2 = g_1^m h^k g_2 = (g_1^m g_2) h^k.$$  

Since $H_p$ and $\langle g_1 g_2 \rangle$ permute then $g_1^m g_2 \in \langle g_1 g_2 \rangle$. Now, Lemma 2.6.3 implies that $m \equiv (1 + n \cdot p^{\alpha_2}) \text{ mod } p^{\alpha_1}$, for some integer $n$. This means that $g_1^m = g_1^{1 + n \cdot p^{\alpha_2}}$. Hence,

$$hg_1 = g_1^m h^k = g_1^{1 + n \cdot p^{\alpha_2}} h^k = g_1 g_1^{n \cdot p^{\alpha_2}} h^k$$

$$g_1^{-1} h g_1 = g_1^{n \cdot p^{\alpha_2}} h^k.$$  

Conversely, suppose that $G_i$ is a $\mathcal{PT}_c$ group and let $H$ be a subnormal cyclic subgroup of $G_1 \times G_2$. Then $H$ is contained in the $\text{Fit}(G_1 \times G_2)$. Let $H_p$ be a direct factor of $H$ for some prime divisor $p$ of the order of $H$. Since $(|\text{Fit}(G_1)|, |\text{Fit}(G_2)|) = 1$ then $H_p$ is contained in $F_p^1 \in \text{Syl}_p(\text{Fit}(G_1))$ or $F_p^2 \in \text{Syl}_p(\text{Fit}(G_2))$. Without loss of generality suppose that $H_p \in F_p^1$. Since $\mathcal{PT}_c$ is a subclass of $\mathcal{PST}_c$ then $G_1 \times G_2$ is a $\mathcal{PST}_c$ by part (i) of Theorem 2.6.1. Let $g \in G_1 \times G_2$ and consider a cyclic group generated by $g$. Then $\langle g \rangle = \langle g_p \rangle \times \langle g_p' \rangle$ where $g_p$ is a $p$-element and $g_p'$ is a $p'$-element of $G_1 \times G_2$. Since $G_1 \times G_2$ is a $\mathcal{PST}_c$ group and $H_p$ is subnormal in $G_1$, i.e. subnormal subgroup of $G_1 \times G_2$, then $p'$-elements of $G_1 \times G_2$ induce power automorphism in $H_p$. That is, $H_p$ permutes with $\langle g_p \rangle$. 

Now let $g_p = g_1 g_2$ where $g_i \in G_i$ and let $\text{ord}(g_i) = p^{\alpha_i}$ for $i \in \{1, 2\}$. Since $G_1$ is a $\mathcal{PT}_c$ then $H_p$ and $\langle g_1 \rangle$ permute. In particular, $h g_1 = g_1^m h^k$, where $h, h^k \in H_p$ and $g_1, g_1^m \in \langle g_1 \rangle$, for some integers $m$ and $k$. But then we have that $h(g_1 g_2) = h g_1 g_2 = g_1^m h^k g_2 = (g_1^m g_2) h^k$. By hypothesis, $h g_1 = g_1^{n \cdot p^{\alpha_2}} h^k$, i.e.

$$hg_1 = g_1^m h^k = g_1^{1 + n \cdot p^{\alpha_2}} h^k.$$  


Now, Lemma 2.6.3 implies that \( g_1^m g_2 \in \langle g_1 g_2 \rangle \). Hence, \( H_p \) and \( \langle g_p \rangle \) permute, that is \( H_p \) and \( \langle g \rangle \) permute for all \( g \in G_1 \times G_2 \). Since every \( p \)-factor of \( H = H_1 \times H_2 \) is permutable in \( G_1 \times G_2 \) then we conclude that \( H \) is permutable in \( G_1 \times G_2 \) and \( G_1 \times G_2 \) is a \( \mathcal{PT}_c \) group.

From Theorem 2.6.4 we immediately get the following result.

**Lemma 2.6.5.** Let \( G_1 \times G_2 \) be a finite \( \mathcal{PT}_c \) group, \( H_p \) be a subnormal cyclic \( p \)-subgroup of \( G_1 \) and \( g_1 g_2 \) be any \( p \)-element of \( G_1 \times G_2 \), where \( g_i \in G_i \).

If \( \text{ord}(g_1) \leq \text{Exp}(P_2) \) for any Sylow \( p \)-subgroup of \( G_2 \), then elements of \( \langle g_1 \rangle \) induce power automorphisms in \( H_p \).

**Proof.** Let \( \text{ord}(g_i) = p^{\alpha_i} \) for \( i \in \{1, 2\} \). Since \( G_1 \times G_2 \) is a \( \mathcal{PT}_c \)-group and each \( G_i \) is normal in \( G_1 \times G_2 \) then \( G_i \) is a \( \mathcal{PT}_c \)-group. Hence, \( H_p \) is permutable subgroup of \( G_1 \) and of \( G_1 \times G_2 \). Thus, \( H_p \) permutes with \( \langle g_1 \rangle \) and with \( \langle g_1 g_2 \rangle \). In particular, \( h g_1 = g_1^m h^k \), where \( h, h^k \in H_p \) and \( g_1, g_1^m \in \langle g_1 \rangle \), for some integers \( m \) and \( k \). Now, consider \( h(g_1 g_2) \in H_p \langle g_1 g_2 \rangle \). Then

\[
h(g_1 g_2) = h g_1 g_2 = g_1^m h^k g_2 = (g_1^m g_2) h^k.
\]

Since \( H_p \) and \( \langle g_1 g_2 \rangle \) permute then \( g_1^m g_2 \in \langle g_1 g_2 \rangle \). Now, Lemma 2.6.3 implies that

\[
m \equiv (1 + n \cdot p^{\alpha_2}) \mod p^{\alpha_1}.
\]

This means that \( g_1^m = g_1^{1+n \cdot p^{\alpha_2}} \). But \( \text{ord}(g_1) \leq \text{Exp}(P_2) \), which implies that \( g_1^{1+n \cdot p^{\alpha_2}} = g_1 \). Now, \( h g_1 = g_1^m h^k = g_1 h^k \), i.e. \( h g_1 = h^k \).

In order to find necessary and sufficient conditions for a direct product of solvable \( \mathcal{PT}_c \) and \( \mathcal{T}_c \) groups to stay in the class further analysis and a better understanding of these classes of groups is required.

It would be interesting to examine which subgroups inherit the properties of the group and identify the structure of the Sylow subgroups. A lot of examples suggest that Sylow subgroups of solvable \( \mathcal{PT}_c \) and \( \mathcal{T}_c \)-groups are Iwasawa or Dedekind groups, respectively. Nonetheless, more examples of solvable \( \mathcal{PT}_c \) or a \( \mathcal{T}_c \) groups that contain a non Iwasawa or a non Dedekind Sylow subgroups must be constructed and investigated.

### 2.7 Examples

This section provides examples of direct products of \( \mathcal{PST}_c \), \( \mathcal{PT}_c \) and \( \mathcal{T}_c \) groups. To build a solvable \( \mathcal{T}_c \) group we use Robinson’s [12] construction as described on page 176. Then we use Theorem 2.4.2 and Lemma 2.5.1 to build a \( \mathcal{PST}_c \) group that is not
a $\mathcal{P}T_c$ and $\mathcal{PT}_c$ group that is not a $T_c$. This illustrates that classes of $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $T_c$ groups are proper subclasses of each other. Also, we provide an example of a non $\mathcal{PST}_c$ group.

We hope that examples will help reader to see the fundamental differences between the classes of solvable $T$, $\mathcal{PT}$, $\mathcal{PST}$ groups and the classes of solvable $T_c$, $\mathcal{PT}_c$, $\mathcal{PST}_c$ groups. In particular, notice that the nilpotent residual is no longer a Hall-subgroup of the group $G$ and that subgroup closure fails as well.

**Example 2.7.1.**

$$G = \left\langle a, b, c, d, f \mid \begin{array}{l}
a^7 = b^3 = c^2 = d^3 = f^5 = (ac)^2 = (bc)^2 = a^5a^d = 1 \\
[a, b] = [c, d] = [a, f] = [b, f] = [c, f] = [d, f] = [b, d] = 1 
\end{array} \right\rangle$$

(i) $G \cong ((C_7 \times C_3) \rtimes (C_2 \times C_3)) \rtimes C_5$.

(ii) The Fitting subgroup of $G$ is $\text{Fit}(G) = (C_7 \times C_3) \rtimes C_5$.

(iii) The nilpotent residual of $G$ is $\gamma_*(G) = C_7 \times C_3$.

(iv) $\gamma_*(G)$ is a Hall-subgroup of the $\text{Fit}(G)$ but not a Hall-subgroup of $G$.

(v) $G$ is a solvable $T_c$-group that is not a $T$-group.

(vi) Let $H = \langle b, c, d \rangle$ be a subgroup of $G$. $H \cong S_3 \times C_3$. The nilpotent residual $\gamma_*(H) = \langle b \rangle$ is not a Hall subgroup of the $\text{Fit}(H) = \langle b, d \rangle$. Thus, $H$ is not a $T_c$ group.

1. Consider $G \times D_8$, where $D_8 = \langle x, y \mid x^4 = y^2 = 1, x^y = x^{-1} \rangle$

   (i) $D_8$ is a $\mathcal{PST}_c$-group and $(|\text{Fit}(G)|, |\gamma_*(D_8)|) = (|\gamma_*(G)|, |\text{Fit}(D_8)|) = 1$. Therefore, $G \times D_8$ is a $\mathcal{PST}_c$-group.

   (ii) $G \times D_8$ is not a $\mathcal{PT}_c$-group since the cyclic subnormal subgroup $\langle y \rangle$ does not permute with $\langle xy \rangle$.

2. Consider $G \times M$, where $M = \langle x, y \mid x^8 = y^2 = 1, x^y = x^5 \rangle$

   (i) $M$ is a $\mathcal{PST}_c$-group and $(|\text{Fit}(G)|, |\gamma_*(M)|) = (|\gamma_*(G)|, |\text{Fit}(M)|) = 1$, hence $G \times M$ is a $\mathcal{PST}_c$-group.

   (ii) Since every Sylow subgroup of $G \times M$ is Iwasawa then $G \times M$ is a $\mathcal{PT}_c$-group, but $G \times M$ is not a $T_c$-group since the cyclic subnormal subgroup $\langle y \rangle$ is not normal.
3. Consider $G \times Q_8$

(i) $Q_8$ is a PST$_c$-group and $(|\text{Fit}(G)|, |\gamma_*(Q_8)|) = (|\gamma_*(G)|, |\text{Fit}(Q_8)|) = 1$, hence $G_1 \times Q_8$ is a PST$_c$-group.

(ii) Since every Sylow subgroup of $G \times Q_8$ is Dedekind then $G \times Q_8$ is a $T_c$-group.

4. Let $G_1$ be a dihedral group of order 12. We will consider the following presentation of this group: $G_1 := \langle x, y \mid x^6 = y^2 = 1, xy = x^{-1} \rangle$. Clearly, $G_1 \cong (C_3 \rtimes C_2) \times C_2$ and it is a solvable $T_c$-group.

Now, consider $G \times G_1$.

(i) $G$ and $G_1$ are each solvable $T_c$-groups with

$\gamma_*(G \times G_1) = (C_7 \times C_3) \times C_3$ being a Hall subgroup of the $\text{Fit}(G \times G_1) = ((C_7 \times C_3) \times C_5) \times (C_3 \times C_1)$.

(ii) But the order of the $\text{Fit}(G_1) = C_3 \times C_2$ and the order of $\gamma_*(G) = C_7 \times C_3$ are not relatively prime, therefore $G \times G_1$ is not a PST$_c$-group.
Chapter 3 The Intersection map of subgroups

In this chapter we analyze the behavior of a collection of cyclic normal, permutable and Sylow-permutable subgroups under the intersection map into a fixed subgroup of a group. In particular, we tie the concept of normal, permutable and Sylow-permutable cyclic sensitivity with that of $T_c$, $PT_c$ and $PST_c$ groups. In the process we provide another way of looking at Dedekind, Iwasawa and nilpotent groups. The intersection map of subgroups in connection to the classes $PST_c$, $PT_c$ and $T_c$ is a collaborative work with my advisor James Beidleman.

3.1 Background

Definition 3.1.1. A subgroup $H$ of the group $G$ is said to be normal sensitive if whenever $X$ is a normal subgroup of $H$ there is a normal subgroup $Y$ of $G$ such that $X = Y \cap H$, that is if the map $Y \mapsto H \cap Y$ sends the lattice of normal subgroups of $G$ onto the lattice of normal subgroups of $H$.

Permutable sensitive (S-permutable sensitive) are defined in the similar fashion by requiring $X$ and $Y$ to be permutable (S-permutable) subgroups of $H$ and $G$ respectively.

Recall, that a permutable (S-permutable) subgroup of the group $G$ is subnormal. We would like to note, that the set of all permutable subgroups need not be a sublattice of the lattice of subnormal subgroups of a group $G$. Hence, in the case of permutability the intersection map is not necessarily a lattice map. An example that addresses the latter can be found in [6] on page 220, Example 1. On the other hand, the collection of S-permutable subgroups is a sublattice of the lattice of subnormal subgroups of $G$. For details reader may consult [10] and [14].

Several authors, Bauman [4], Beidleman and Ragland [6], have tied the concept of normal, permutable and S-permutable sensitivity with $T$, $PT$ and $PST$ groups.

Theorem 3.1.2. Let $G$ be a finite group.

1. (Beidleman, Ragland [6]) $G$ is a solvable $PST$ ($PT$)-group if and only if every subgroup of $G$ is S-permutable (permutable) sensitive in $G$.

2. (Bauman [3]) $G$ is a solvable $T$-group if and only if every subgroup of $G$ is normal sensitive in $G$. 

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Ragland and Beidleman in [6] go on further by asking a question whether one can restrict S-permutable, permutable, and normal sensitivity to normal subgroups and deduce that $G$ is still a $\mathcal{PST}$, $\mathcal{PT}$, or a $T$ group respectively. While they have affirmatively answered the question about $\mathcal{PST}$ and $T$ groups in [6], the question about permutable sensitivity restricted to normal subgroups was answered later in [3].

**Theorem 3.1.3.** Let $G$ be a group.

1. (Beidleman, Ragland [6]) $G$ is a $\mathcal{PST}$ ($T$)-group if and only if every normal subgroup of $G$ is S-permutable (normal) sensitive in $G$.

2. (Ballester-Bolíinces, A., Beidleman, J.C., Cossey, J., Esteban-Romero, R., Ragland, M.F., Schmidt, J. [3]) $G$ is a solvable $\mathcal{PT}$-group if and only if every normal subgroup of $G$ is permutable sensitive in $G$.

Our interest lies in developing similar connections as in Theorems 3.1.2 and 3.1.3 with classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $T_c$. In particular, if we restrict the intersection map to cyclic subgroups, then what can we say about the behavior of a collection of cyclic normal, permutable and S-permutable subgroups under this restricted intersection map.

### 3.2 The intersection map of subgroups and the classes $\mathcal{PST}_c$, $\mathcal{PT}_c$ and $T_c$.

**Definition 3.2.1.** Let $H$ be a subgroup of the group $G$.

1. $H$ is normal cyclic sensitive if whenever $X$ is a normal cyclic subgroup of $H$ there is a normal cyclic subgroup $Y$ of $G$ such that $X = Y \cap H$.

2. $H$ is permutable cyclic sensitive if whenever $X$ is a permutable cyclic subgroup of $H$ there is a permutable cyclic subgroup $Y$ of $G$ such that $X = Y \cap H$.

3. $H$ is S-permutable cyclic sensitive if whenever $X$ is a S-permutable cyclic subgroup of $H$ there is a S-permutable cyclic subgroup $Y$ of $G$ such that $X = Y \cap H$.

Definition 3.2.1 is an analog of the normal, permutable and S-permutable sensitivity, but the following two lemmas will provide a much simpler and more natural way of looking at the cyclic sensitivity.

**Lemma 3.2.2.** (R. Schmidt [15], Lemma 5.2.11 page 224) If $M$ is a cyclic permutable subgroup of the group $G$, then every subgroup of $M$ is permutable in $G$. 

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Lemma 3.2.3. (P. Schmid [14]) Let $X$ be a cyclic subgroup of a finite group $G$ and let $Y$ be a subgroup of $X$. If $X$ is S-permutable in $G$, then $Y$ is also S-permutable in $G$.

Let $H$ be normal cyclic sensitive subgroup of $G$ and $X$ normal cyclic subgroup of $H$. Then there exist a normal cyclic subgroup $Y$ of $G$ such that $X = Y \cap H$. But now $X$ is the unique cyclic subgroup of $Y$, i.e $X$ is characteristic in $Y$ and $Y$ is normal subgroup of $G$. Thus, $X$ is normal subgroup of $G$.

Similarly, if $H$ is permutable (S-permutable) cyclic sensitive then and $X$ is permutable (S-permutable) cyclic subgroup of $H$ then there is a permutable (S-permutable) cyclic subgroup $Y$ of $G$ such that $X = H \cap Y$. Then by Lemmas 3.2.2 (3.2.3) $X$ is permutable (S-permutable) subgroup of $G$. Hence, normal, permutable and S-permutable cyclic sensitivity is equivalent to the following.

Remark 3.2.4. Let $G$ be a finite group and $H$ a subgroup of $G$.

1. $H$ is normal cyclic sensitive in $G$ if every normal cyclic subgroup of $H$ is normal subgroup of $G$.

2. $H$ is permutable cyclic sensitive in $G$ if every permutable cyclic subgroup of $H$ is permutable subgroup of $G$.

3. $H$ is S-permutable cyclic sensitive in $G$ if every S-permutable cyclic subgroup of $H$ is S-permutable subgroup of $G$.

From now on, we will use Remark 3.2.4 in place of Definition 3.2.1.

In general, normal subgroups of normal, permutable, S-permutable cyclic sensitive groups do not inherit the same properties. That is, if $H$ is normal cyclic sensitive in $G$ and $K$ is normal subgroup of $H$ then $K$ need not be normal cyclic sensitive as the following example shows.

Example 3.2.5. Consider the following group

$$G = \langle x, y, z \mid x^6 = y^2 = z^3 = (xy)^2 = [x, z] = [y, z] = 1 \rangle$$

$G \cong D_{12} \times C_3$. Let $H = \langle x^2, y, z \rangle$ and $K = \langle x^2, z \rangle$. $H$ is normal cyclic sensitive in $G$ since the only normal cyclic subgroups of $H$ are $\langle x^2 \rangle$ and $\langle z \rangle$ and both of them are normal in $G$. Now, $K$ is normal subgroup of $H$ but $K$ has a normal cyclic subgroup $\langle x^2 z \rangle$ that is not normal in $H$ nor $G$. Thus, $K$ is not normal cyclic sensitive.
Recall that Theorem 3.1.2 by Beidleman, Ragland [6] and Bauman [4] relates solvable $\mathcal{PST}$, $\mathcal{PT}$ and $\mathcal{T}$ groups to S-permutable, permutable and normal sensitivity. Our first result addresses a similar question in connection to cyclic sensitivity.

**Theorem 3.2.6.** Let $G$ be a finite group.

(1) $G$ is a nilpotent group if and only if every subgroup of $G$ is S-permutable cyclic sensitive.

(2) $G$ is an Iwasawa group if and only if every subgroup of $G$ is permutable cyclic sensitive.

(3) $G$ is a Dedekind group if and only if every subgroup of $G$ is normal cyclic sensitive.

*Proof.* (1) If $G$ is nilpotent then every subgroup of $G$ is subnormal and every Sylow subgroup of $G$ is normal. Hence, every subgroup is S-permutable cyclic sensitive. Conversely, suppose that every subgroup of $G$ is S-permutable cyclic sensitive. Let $X$ be any cyclic subgroup of $G$. Then $X$ is S-permutable cyclic sensitive, i.e. $X$ is S-permutable in $G$. Kegel [10] has shown that S-permutable subgroups are subnormal, then $X$ is subnormal subgroup of $G$. Hence, we conclude that every subgroup of $G$ is subnormal, in particular $G$ is nilpotent.

(2) If $G$ is an Iwasawa group then every subgroup is permutable in $G$ and consequently permutable cyclic sensitive. Conversely, let $X$ be any cyclic subgroup of $G$, then $X$ is permutable cyclic sensitive. Thus, $X$ is permutable subgroup of $G$, and $G$ is an Iwasawa group.

(3) If $G$ is a Dedekind group then every subgroup is normal cyclic sensitive. Conversely, if $X$ is any cyclic subgroup of $G$ then $X$ is normal cyclic sensitive and hence normal in $G$. Thus, $G$ is a Dedekind group. 

**Theorem 3.2.7.** Let $G$ be a finite group.

(1) $G$ is a $\mathcal{PST}_c$-group if and only if every subnormal subgroup of $G$ is S-permutable cyclic sensitive.

(2) $G$ is a $\mathcal{PT}_c$-group if and only if every subnormal subgroup of $G$ is permutable cyclic sensitive.

(3) $G$ is a $\mathcal{T}_c$-group if and only if every subnormal subgroup of $G$ is normal cyclic sensitive.
Proof. (1) Suppose that $G$ is a $\mathcal{PST}_c$-group. Let $H$ be any subnormal subgroup of $G$ and $X$ an S-permutable cyclic subgroup of $H$. Then $X$ is subnormal subgroup of $G$ and hence S-permutable. Conversely, if $X$ is subnormal cyclic subgroup of $G$ then hypothesis implies that $X$ is S-permutable.

(2) Suppose that $G$ is a $\mathcal{PT}_c$-group. Let $H$ be any subnormal subgroup of $G$ and $X$ a permutable cyclic subgroup of $H$. Then $X$ is subnormal subgroup of $G$ and hence permutable. Conversely, if $X$ is subnormal cyclic subgroup of $G$ then hypothesis implies that $X$ is permutable.

(3) Suppose that $G$ is a $\mathcal{T}_c$-group. Let $H$ be any subnormal subgroup of $G$ and $X$ a normal cyclic subgroup of $H$. Then $X$ is subnormal subgroup of $G$ and hence normal in $G$. Conversely, if $X$ is subnormal cyclic subgroup of $G$ then hypothesis implies that $X$ is normal in $G$.

One would hope that it is possible to replace “subnormal” with “normal” in Theorem 3.2.7 so that one can have an analog to Theorem 3.1.3 but it is not the case as the following example shows.

Example 3.2.8. Consider the dihedral group of order 16.

$$D_{16} = \langle x, y, \mid x^8 = y^2 = (xy)^2 = 1 \rangle$$

One can check that normal subgroups of $D_{16}$ are normal cyclic sensitive but $D_{16}$ is not a $\mathcal{T}_c$-group since $\langle y \rangle$ is subnormal cyclic subgroup of $D_{16}$ which is not normal.

Also note, that if $H$ is any subgroup of $G$ such that every subgroup of $H$ is normal cyclic sensitive in $H$ then in general $H$ is not normal cyclic sensitive in $G$.

Example 3.2.9. Consider again a dihedral group of order 16 with the same generators as in the previous example. Let $H = \langle x^4, y \rangle$. $H$ is isomorphic to $K_4$ and hence every subgroup of $H$ is normal cyclic sensitive in $H$, but $H$ is not normal cyclic sensitive in $G$ since $\langle y \rangle$ is subnormal cyclic subgroup of $D_{16}$ which is not normal.

Robinson in [12] has proved that if every subgroup of a group $G$ is $\mathcal{PST}_c$ then $G$ is a solvable $\mathcal{PST}$ group. Same is true for solvable $\mathcal{PT}_c$ and $\mathcal{T}_c$ groups. Robinson’s results and Theorem 3.2.7 motivate the following theorem.
Theorem 3.2.10. Let $G$ be a finite group.

(1) $G$ is a solvable $\mathcal{PST}$-group if and only if every subnormal subgroup of $H$ is $S$-permutable cyclic sensitive in $H$ for all subgroups $H$ of $G$.

(2) $G$ is a solvable $\mathcal{PT}$-group if and only if every subnormal subgroup of $H$ is permutable cyclic sensitive in $H$ for all subgroups $H$ of $G$.

(3) $G$ is a solvable $\mathcal{T}$-group if and only if every subnormal subgroup of $H$ is normal cyclic sensitive in $H$ for all subgroups $H$ of $G$.

Proof. (1) Suppose $G$ is a solvable $\mathcal{PST}$-group and let $K$ be any subnormal subgroup of $H$. Let $X$ be an $S$-permutable cyclic subgroup of $K$. Since $S$-permutable subgroups are subnormal [1.5.3] part (iv), then $X$ is a subnormal subgroup of $H$. Since solvable $\mathcal{PST}$ groups are subgroup closed, then $H$ is a solvable $\mathcal{PST}$ group. Thus, $X$ is $S$-permutable in $H$, i.e. every subnormal subgroup of $H$ is $S$-permutable cyclic sensitive in $H$.

Conversely, if every subnormal subgroup of $H$ is $S$-permutable cyclic sensitive in $H$, then by Theorem 3.2.7 $H$ is a $\mathcal{PST}_c$-group. Now, every subgroup of $G$ is a $\mathcal{PST}_c$ group, thus Theorem 2.5 in [12] implies that $G$ is a solvable $\mathcal{PST}$ group.

(2) Suppose $G$ is a solvable $\mathcal{PT}$-group and let $K$ be any subnormal subgroup of $H$. Let $X$ be a permutable cyclic subgroup of $K$. Since permutable subgroups are subnormal [1.5.3] part (ii), then $X$ is a subnormal subgroup of $H$. Since solvable $\mathcal{PT}$ groups are subgroup closed, then $H$ is a solvable $\mathcal{PT}$ group. Thus, $X$ is permutable in $H$, i.e. every subnormal subgroup of $H$ is permutable cyclic sensitive in $H$.

Conversely, if every subnormal subgroup of $H$ is permutable cyclic sensitive in $H$, then by Theorem 3.2.7 $H$ is a $\mathcal{PT}_c$-group. Now, every subgroup of $G$ is a $\mathcal{PT}_c$ group, thus Theorem 2.5 in [12] implies that $G$ is a solvable $\mathcal{PT}$ group.

(3) Suppose $G$ is a solvable $\mathcal{T}$-group and let $K$ be any subnormal subgroup of $H$. Let $X$ be a normal cyclic subgroup of $K$. Then $X$ is a subnormal subgroup of $H$. Since solvable $\mathcal{T}$ groups are subgroup closed, then $H$ is a solvable $\mathcal{T}$ group. Thus, $X$ is normal in $H$, i.e. every subnormal subgroup of $H$ is normal cyclic sensitive in $H$.

Conversely, if every subnormal subgroup of $H$ is normal cyclic sensitive in $H$, then by Theorem 3.2.7 $H$ is a $\mathcal{T}_c$-group. Now, every subgroup of $G$ is a $\mathcal{PT}_c$ group, thus Theorem 2.5 in [12] implies that $G$ is a solvable $\mathcal{T}$ group. 

\[\square\]
3.3 Sylow Subgroups and the Intersection Map

We switch our attention to Sylow subgroups of a group $G$. Since cyclic subgroups can be written as a direct product of cyclic $p$-groups of relatively prime orders, it is natural to look at the Sylow subgroups.

**Theorem 3.3.1.** Let $G$ be a finite group and let $G_p$ be a Sylow $p$-subgroup of $G$.

1. $G$ is a Dedekind group if and only if every subgroup of $G_p$ is normal cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

2. $G$ is an Iwasawa group if and only if every subgroup of $G_p$ is permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

3. $G$ is nilpotent group if and only if $G_p$ is $S$-permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

**Proof.** (1) If $G$ is a Dedekind group then every subgroup is normal and hence every subgroup of a Sylow subgroup $G_p$ is normal cyclic sensitive. Conversely, suppose that every subgroup of $G_p$ is normal cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$. First, we would like to note, that hypothesis implies that $G_p$ is normal cyclic sensitive in $G$ since if $X$ is any normal cyclic subgroup of $G_p$ then it is normal cyclic sensitive in $G$, that is $X$ is normal subgroup of $G$. Second, if $H$ is any subgroup of $G_p$ and $X$ is normal cyclic subgroup of $H$ then hypothesis implies that $X$ is normal in $G$ and consequently normal in $G_p$. This means that every subgroup of $G_p$ is normal cyclic sensitive in $G_p$. Now, Theorem 3.2.6 implies that $G_p$ is a Dedekind group. Hence, if $X$ is any cyclic subgroup of $G$ and $X_p$ is a Sylow $p$-subgroup of $X$, for some prime $p$ dividing the order of $X$, then $X_p$ is normal cyclic subgroup of $G_p$ and $G_p$ is normal cyclic sensitive in $G$ implies $X_p$ is normal in $G$. Since $X$ is a direct product of its Sylow subgroups, then $X$ is normal in $G$. Thus, $G$ is a Dedekind group.

(2) If $G$ is an Iwasawa group then every subgroup is permutable in $G$ and consequently permutable cyclic sensitive. That is every subgroup of a Sylow subgroup $G_p$ is permutable cyclic sensitive. Conversely, suppose that every subgroup of $G_p$ is permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$. Note that hypothesis implies that $G_p$ is permutable cyclic sensitive in $G$ since if $X$ is any permutable cyclic subgroup of $G_p$ then it is permutable cyclic sensitive in $G$, that is $X$ is permutable subgroup of $G$. Second, if $H$ is any subgroup of $G_p$ and $X$ is permutable cyclic subgroup of $H$ then hypothesis
implies that $X$ is permutable in $G$ and consequently permutable in $G_p$. This means that every subgroup of $G_p$ is permutable cyclic sensitive in $G_p$. Now, Theorem 3.2.6 implies that $G_p$ is an Iwasawa group. Hence, if $X$ is any cyclic subgroup of $G$ and $X_p$ is a Sylow $p$-subgroup of $X$, for some prime $p$ dividing the order of $X$, then $X_p$ is permutable cyclic subgroup of $G_p$ and $G_p$ is permutable cyclic sensitive in $G$ implies $X_p$ is permutable in $G$. Since $X$ is a direct product of its Sylow subgroups, then $X$ is permutable in $G$. Thus, $G$ is an Iwasawa group.

(3) Suppose $G$ is nilpotent group, then every subgroup of $G$ is S-permutable in $G$. Thus, every Sylow subgroup is S-permutable cyclic sensitive.

Conversely, suppose that every Sylow subgroup of $G$ is S-permutable cyclic sensitive. Let $H$ be any cyclic subgroup of $G$ and let $H_p$ be a Sylow $p$-subgroup of $H$, for some prime $p$ dividing the order of $H$. Now, $H_p$ is contained in $G_p$. Since $G_p$ is S-permutable cyclic sensitive then $H_p$ is S-permutable in $G$. But, Kegel [10] has shown that S-permutable subgroups are subnormal, hence $H_p$ is subnormal subgroup of $G$. Thus, we conclude $H$ is subnormal subgroup of $G$. Since, every cyclic subgroup of $G$ is subnormal in $G$ then every subgroup is subnormal in $G$, that is $G$ is nilpotent.

Note, that in Theorem 3.3.1 parts (1) and (2) can be restated as follows.

**Corollary 3.3.2.** Let $G$ be a finite group and $G_p$ be a Sylow $p$-subgroup of $G$.

(1) $G$ is a Dedekind group if and only if $G_p$ is a Dedekind group and normal cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

(2) $G$ is an Iwasawa group if and only if $G_p$ is an Iwasawa group and permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

**Proof.** (1) If $G$ is a Dedekind group then every subgroup of $G$ is a Dedekind group and normal in $G$. In particular, $G_p$ is a Dedekind group and normal cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$. Conversely, suppose that $G_p$ is a Dedekind group and normal cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$. Then, if $K$ is any subgroup of $G_p$ and $X$ is normal cyclic subgroup of $K$ then $X$ is normal cyclic subgroup of $G_p$ and hence normal subgroup of $G$. That is, every subgroup of $G_p$ is normal cyclic sensitive in $G$. Hence, by theorem 3.3.1 $G$ is a Dedekind group.

(2) If $G$ is an Iwasawa group then every subgroup of $G$ is an Iwasawa group and permutable in $G$. In particular, $G_p$ is an Iwasawa group and permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$. Conversely, $G_p$ is an Iwasawa group and permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$. So, if $K$ is any subgroup of $G_p$ and $X$ is a permutable cyclic subgroup of $K$ then $X$ is a
permutable cyclic subgroup of $G_p$ and hence a permutable subgroup of $G$. That is, every subgroup of $G_p$ is permutable cyclic sensitive in $G$. Hence, by Theorem 3.3.1 $G$ is an Iwasawa group.

If $X$ is any subnormal cyclic subgroup of a group $G$ then $X$ is contained in the Fitting subgroup of $G$ and in particular the Sylow $p$-subgroup $X_p$ of $X$, for some prime dividing the order of $X$, lies in the Sylow $p$-subgroup of the Fitting, that is $X_p$ is contained in $O_p(G)$.

**Theorem 3.3.3.** Let $G$ be a finite group.

(1) $G$ is a $T_c$ group if and only if every subgroup of $O_p(G)$ is normal cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

(2) $G$ is a $PT_c$ group if and only if every subgroup of $O_p(G)$ is permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

(3) $G$ is a $PST_c$-group if and only if $O_p(G)$ is $S$-permutable cyclic sensitive in $G$ for all primes $p$ dividing the order of $G$.

**Proof.** (1) Suppose that $G$ is a $T_c$ group. Let $K$ be any subgroup of $O_p(G)$ and $X$ normal cyclic subgroup of $K$. Then $X$ is a cyclic subnormal subgroup of $G$. Since, $G$ is a $T_c$-group then $X$ is normal in $G$. Hence, each subgroup of the Sylow $p$-subgroup $O_p(G)$ of $Fit(G)$ is normal cyclic sensitive in $G$, for all primes $p$ dividing the order of $G$. Conversely, let $X$ be a subnormal cyclic subgroup of $G$ and $X_p$ be a Sylow $p$ subgroup of $X$, for some prime $p$ dividing the order of $X$. Then $X_p$ is contained in $O_p(G)$. By a similar argument as in Theorem 3.3.1 we get that $O_p(G)$ is a Dedekind group and normal cyclic sensitive in $G$. Now, $X_p$ is a normal cyclic subgroup of $O_p(G)$. Thus, $X_p$ is a normal subgroup of $G$. Since $X$ is a direct product of its Sylow subgroups, then $X$ is normal in $G$, that is $G$ is a $T_c$-group.

(2) Suppose that $G$ is a $PT_c$ group. Let $K$ be any subgroup of $O_p(G)$ and $X$ permutable cyclic subgroup of $K$. Then $X$ is a cyclic subnormal subgroup of $G$. Since, $G$ is a $PT_c$-group then $X$ is permutable in $G$. Hence, each subgroup of the Sylow $p$-subgroup $O_p(G)$ of $Fit(G)$ is permutable cyclic sensitive in $G$, for all primes $p$ dividing the order of $G$. Conversely, let $X$ be a subnormal cyclic subgroup of $G$ and $X_p$ be a Sylow $p$ subgroup of $X$, for some prime $p$ dividing the order of $X$. Then $X_p$ is contained in $O_p(G)$. By a similar argument as in Theorem 3.3.1 we get that $O_p(G)$ is an Iwasawa group and permutable cyclic sensitive in $G$. Now, $X_p$ is permutable cyclic subgroup of $O_p(G)$. Thus, $X_p$ is permutable subgroup of $G$. Since $X$ is a direct
product of its Sylow subgroups, then $X$ is permutable in $G$, that is $G$ is a $\mathcal{PT}_c$-group.

(3) Suppose that $G$ is a $\mathcal{PST}_c$ group. Let $X$ be a cyclic $S$-permutable subgroup of $O_p(G)$. Then $X$ is subnormal cyclic subgroup of $G$ and $G$ being $\mathcal{PST}_c$-group implies that $X$ is $S$-permutable in $G$. Hence, $O_p(G)$ is $S$-permutable cyclic sensitive in $G$, for all primes $p$ dividing the order of $G$. Conversely, let $X$ be a subnormal cyclic subgroup of $G$ and $X_p$ be a Sylow $p$ subgroup of $X$, for some prime $p$ dividing the order of $X$. Then $X_p$ is $S$-permutable subgroup of $O_p(G)$. Thus, $X_p$ is $S$-permutable subgroup of $G$. Since $X$ is a direct product of its Sylow subgroups, then $X$ is $S$-permutable in $G$, that is $G$ is a $\mathcal{PST}_c$-group.

Note that in part (3) of Theorems 3.3.1 and 3.3.3 we do not need every subgroup of the $G_p$ or $O_p(G)$ to be $S$-permutable cyclic sensitive since every subgroup of $G_p$ or $O_p(G)$ is $S$-permutable in $G_p$ or $O_p(G)$. Also, Theorem 3.3.3 and Corollary 3.3.2 lead to the following result.

**Corollary 3.3.4.** Let $G$ be a finite group and $F := \text{Fit}(G)$.

1. $G$ is a $\mathcal{T}_c$-group if and only if $F$ is normal cyclic sensitive and a Dedekind group.

2. $G$ is a $\mathcal{PT}_c$-group if and only if $F$ is permutable cyclic sensitive and an Iwasawa group.

3. $G$ is a $\mathcal{PST}_c$-group if and only if $F$ is $S$-permutable cyclic sensitive.

**Proof.**

1. Suppose that $G$ is a $\mathcal{T}_c$-group. Then by Lemma 1.7.2 $F$ is a Dedekind group. Let $X$ be a normal cyclic subgroup of $F$. Then $X$ is subnormal subgroup of $G$ and hence is normal in $G$, sine $G$ is a $\mathcal{T}_c$-group. Therefore, $F$ is normal cyclic sensitive in $G$. Conversely, let $X$ be a subnormal cyclic subgroup of $G$. Then $X$ is normal subgroup of $F$. Since $F$ is normal cyclic sensitive in $G$ then $X$ is a normal subgroup of $G$. Hence, $G$ is a $\mathcal{T}_c$-group.

2. Suppose that $G$ is a $\mathcal{PT}_c$-group. Then by Lemma 1.7.2 $F$ is an Iwasawa group. Let $X$ be a permutable cyclic subgroup of $F$. Then $X$ is subnormal subgroup of $G$ and hence is permutable in $G$, since $G$ is $\mathcal{PT}_c$-group. Therefore, $F$ is permutable cyclic sensitive in $G$. Conversely, let $X$ be a subnormal cyclic subgroup of $G$. Then $X$ is permutable subgroup of $F$, because $F$ is an Iwasawa group. Since $F$ is permutable cyclic sensitive in $G$ then $X$ is a permutable subgroup of $G$. Hence, $G$ is a $\mathcal{PT}_c$-group.

3. Suppose that $G$ is a $\mathcal{PST}_c$-group. Let $X$ be an $S$-permutable subgroup of $F$. Then $X$ is subnormal subgroup of $G$ and hence is $S$-permutable in $G$ because $G$ is...
a $\mathcal{PST}_c$-group. Hence $F$ is S-permutable cyclic sensitive. Conversely, let $X$ be a subnormal cyclic subgroup of $G$. Then $X$ is S-permutable cyclic subgroup of $F$ and $F$ is S-permutable cyclic sensitive. Hence $X$ is S-permutable in $G$. That is, $G$ is a $\mathcal{PST}_c$-group.

3.4 A note on direct products and the intersection map

Let $G_1$ and $G_2$ be any finite groups and $H_i$ an S-permutable sensitive in $G_i$. A possible question might be: under what conditions is $H_1 \times H_2$ S-permutable sensitive in $G_1 \times G_2$? While this question is interesting and challenging in its own right, direct products of solvable $\mathcal{PST}$-groups make it possible to answer at least part of this question. The same question applies to permutable (normal) sensitivity as well as to S-permutable (permutable or normal) cyclic sensitivity.

Lemma 3.4.1. Let $G_1$ and $G_2$ be any finite groups.

1. Every subgroup of $G_1 \times G_2$ is S-permutable sensitive in $G_1 \times G_2$ if and only if every subgroup of $G_i$ is S-permutable sensitive in $G_i$ and $|G_i|, |\gamma_s(G_j)| = 1$ for $i \neq j \in \{1, 2\}$.

2. Every subgroup of $G_1 \times G_2$ is S-permutable cyclic sensitive in $G_1 \times G_2$ if and only if every subgroup of $G_i$ is S-permutable cyclic sensitive in $G_i$.

Proof. (1). Suppose every subgroup of $G_1 \times G_2$ is S-permutable sensitive in $G_1 \times G_2$. Then by Theorem 3.1.2 $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group and hence each $G_i$ is a solvable $\mathcal{PST}$-group, which means by the same Theorem that every subgroup of $G_i$ is S-permutable sensitive. In addition, Theorem 2.1.2 implies that $|G_i|, |\gamma_s(G_j)| = 1$ for $i \neq j \in \{1, 2\}$. Conversely, if every subgroup of $G_i$ is S-permutable sensitive then $G_i$ is a solvable $\mathcal{PST}$-group. But, $|G_i|, |\gamma_s(G_j)| = 1$ for $i \neq j \in \{1, 2\}$, hence by Theorem 2.1.2 $G_1 \times G_2$ is a solvable $\mathcal{PST}$-group. Thus, by Theorem 3.1.2 every subgroup of $G_1 \times G_2$ is S-permutable sensitive in $G_1 \times G_2$.

(2). Follows by a direct application of Theorem 3.2.6 and the fact that a direct product of finitely many nilpotent groups is nilpotent.

We state two more results without a proof that are a direct consequence of Theorems 2.4.2, 3.2.7 and 2.6.1.
Corollary 3.4.2. Let $G_1$ and $G_2$ be finite solvable groups. Then, every subnormal subgroup of $G_1 \times G_2$ is $S$-permutable cyclic sensitive if and only if every subnormal subgroup of $G_i$ is $S$-permutable cyclic sensitive and $|\text{Fit}(G_i)|, |\gamma_s(G_j)| = 1$ for $i \neq j \in \{1, 2\}$.

Corollary 3.4.3. Let $G_1$ and $G_2$ be finite groups so that $|\text{Fit}(G_1)|, |\text{Fit}(G_2)| = 1$. Then, every subnormal subgroup of $G_1 \times G_2$ is $S$-permutable (normal) cyclic sensitive in $G_1 \times G_2$ if and only if every subnormal subgroup of $G_i$ is $S$-permutable (normal) cyclic sensitive in $G_i$.

The above results only indicate what can lead into a rich and full of research area. The potential of this interplay between direct products and the intersection map is clear and requires further investigation.
Appendix

This section is an introduction to the local structure of the classes \( \mathcal{PST}_c \), \( \mathcal{PT}_c \) and \( \mathcal{T}_c \). By analyzing local features of these groups we might be able to answer some of the question posed in Chapters 2 and 3. What follows is an analog (local) to [12].

**Definition 1.** Let \( G \) be a finite group and let \( p \) be a fixed prime number that divides the order of \( G \).

(i) The group \( G \) is called a \( \mathcal{PST}_c \) if every cyclic subnormal \( p \)-subgroup of \( G \) permutes with every Sylow subgroup of \( G \).

(ii) The group \( G \) is called a \( \mathcal{PT}_c \) if every cyclic subnormal \( p \)-subgroup of \( G \) permutes with every subgroup of \( G \).

(iii) The group \( G \) is called a \( \mathcal{T}_c \) if every cyclic subnormal \( p \)-subgroup of \( G \) is normal in \( G \).

We will say that a subgroup \( H \) of \( G \) is subpermutable of length 3 if \( H \) per \( H_1 \) per \( H_2 \) per \( G \) for some subgroups \( H_1 \) and \( H_2 \) of \( G \). Similarly, we can define subpermutability of a subgroup for any length \( n \). The reason this is emphasized is that in the next Lemma we need at most two subgroups in between to prove a claim.

**Lemma 1.** Let \( G \) be any finite group and let \( F_p := \text{Syl}_p(F) \).

(i) \( G \in \mathcal{PST}_c \) if and only if a cyclic \( p \)-subgroup of \( G \) is S-per \( G \) whenever it is S-per in some S-per subgroup of \( G \).

(ii) \( G \in \mathcal{PT}_c \) if and only if a cyclic \( p \)-subgroup of \( G \) is permutable in \( G \) whenever it is subpermutable in \( G \) of length at most 3 and \( F_p \) is an Iwasawa group.

(iii) \( G \in \mathcal{T}_c \) if and only if a cyclic \( p \)-subgroup of \( G \) is normal in \( G \) whenever it is subnormal in \( G \) with defect at most 3 and \( F_p \) is a Dedekind group.

**Proof.** (i). Suppose \( G \) is a \( \mathcal{PST}_c \) and let \( H \) be a cyclic \( p \)-subgroup of \( G \) such that \( H \) S-per \( K \) S-per \( G \). Then by Kegel’s Theorem \( H \) is subnormal in \( K \) and \( K \) is subnormal in \( G \). In particular \( H \) is subnormal subgroup of \( G \). Since \( G \in \mathcal{PST}_c \) then \( H \) S-per \( G \). Conversely, let \( H \) be any cyclic subnormal \( p \)-subgroup of \( G \). Then \( H \leq \text{Fit}(G) \). Since every Sylow subgroup of the \( \text{Fit}(G) \) is normal then \( H \) S-per \( \text{Fit}(G) \). But \( \text{Fit}(G) \) S-per \( G \), hence by assumption \( H \) S-per \( G \).
(ii). Suppose $G$ is a $\mathcal{PT}_{C_p}$ and let $H$ be a cyclic $p$-subgroup of $G$ such that $H$ per $K$ per $G$. Then by Ore’s Theorem $H$ is subnormal in $K$ and $K$ is subnormal in $G$. In particular $H$ is subnormal subgroup of $G$. Since $G \in \mathcal{PT}_{C_p}$ then $H$ per $G$. It remains to show that $F_p$ is an Iwasawa group. It is enough to show that any two cyclic subgroups of $F_p$ permute. Let $a, b \in F_p$ such that $a, b \neq 1$. Then $\langle a \rangle$ and $\langle b \rangle$ are subnormal $p$-subgroups of $G$, and therefore are permutable in $G$. In particular $\langle a \rangle \langle b \rangle = \langle b \rangle \langle a \rangle$. Hence, $F_p$ is an Iwasawa group.

Conversely, let $H$ be any cyclic subnormal subgroup of $G$, then $H \leq F_p$ and $F_p$ is an Iwasawa. Hence, $H$ per $F_p$ per $\text{Fit}(G)$ per $G$. Therefore by hypothesis $H$ is permutable in $G$ and so, $G$ is a $\mathcal{PT}_{C_p}$.

(iii). Suppose $G$ is a $\mathcal{T}_{C_p}$ and let $H$ be a cyclic $p$-subgroup of $G$ such that $H$ is subnormal in $G$ of defect at most 3. Trivially, then $H \unlhd G$. It remains to show that $F_p$ is a Dedekind group. It is enough to show that any cyclic subgroups of $F_p$ is normal. Let $a \in F_p$ such that $a \neq 1$. Then $\langle a \rangle$ is subnormal $p$-subgroups of $G$, and therefore normal in $G$. In particular $\langle a \rangle$ is normal subgroup of $F_p$. Hence, $F_p$ is a Dedekind group.

Conversely, let $H$ be any cyclic subnormal subgroup of $G$, then $H \leq F_p$ and $F_p$ is a Dedekind group. Hence, $H \leq F_p \leq \text{Fit}(G) \leq G$. Therefore by hypothesis $H$ is normal subgroup of $G$ and so, $G$ is a $\mathcal{T}_{C_p}$. □

For the next result we will need the following Theorem due to Huppert, [15] page 34.

**Theorem 1.** (Huppert [1961], [15]) If $G$ is a nonabelian finite $p$-group and $N$ is a group of power automorphisms acting on $G$ then $N$ is a $p$-group.

**Lemma 2.** Let $G$ be a finite $\mathcal{PST}_{C_p}$-group and $N$ be a normal $p$-subgroup of $G$. Then $p'$-elements of $G$ induce power automorphisms in $N$ and $G/C_G(N)$ is nilpotent.

**Proof.** Let $a \in N$ be such that $a \neq 1$ and let $Q \in Syl_Q(G)$ where $p \neq q$. Then $\langle a \rangle$ is subnormal cyclic $p$-subgroup of $G$. It follows that $\langle a \rangle Q = Q\langle a \rangle$ and $\langle a \rangle \in Syl_q(\langle a \rangle Q)$. This implies that $Q \leq N(\langle a \rangle)$. Therefore, we conclude that $p'$-elements of $G$ induce power automorphisms in $N$.

Next, if $N$ is not abelian then all $p'$-power automorphisms are trivial by Theorem [1]. This implies that $G/C_G(N)$ is a $p$-group, and $p$-groups are nilpotent. If $N$ is abelian then power automorphisms of $N$ must be in the $Z(\text{Aut}(N))$. Since $G/C_G(N)$ is a group of power automorphisms on $N$ then it is isomorphic to a subgroup of $Z(\text{Aut}(N))$ and hence is abelian, that is $G/C_G(N)$ is nilpotent. □
Lemma 3. Let $G$ be a finite solvable $\mathcal{PST}_{C_p}$-group and $L_p \in \text{Syl}_p(L)$ be normal subgroup of $G$. Then

(i) $L_p$ is abelian.

(ii) $G$ is $p$-supersolvable.

(iii) $L = L_p \times L_{p'}$, where $L_{p'}$ is a normal Hall-subgroup of $L$.

Proof. Let $F_p \in \text{Syl}_p(F)$. Since $F_p$ is subnormal subgroup of $G$ then by Lemma 1.2 $G/C_G(F_p)$ is nilpotent. But $L$ is the smallest normal subgroup of $G$ such that $G/L$ is nilpotent. This implies that $L \leq C_G(F_p)$. Also, $L_p \leq F_p$. Therefore, $[L_p, L] \leq [F_p, L] = 1$. Since $L_p$ is normal subgroup of $L$ and $[L_p, L] = 1$ then $L_p \leq Z(L)$. Hence, we conclude that $L_p$ is abelian.

To show that $G$ is $p$-supersolvable we need to show that all $p$-chief factors of $G$ are cyclic. First we note that $G$ is solvable thus all $p$-chief factors are elementary abelian. If take any $p$-chief factor above $L$, i.e. if $H/K$ is a $p$-chief factor of $G$ such that $L \leq K \leq H \leq G$ then since $G/L$ is nilpotent then it is supersolvable and chief factors of $G/L$ are the chief factors of $G$ by the Third Isomorphism Theorem. Hence, all chief factors of $G$ above $L$ are cyclic, in particular $p$-chief factors are cyclic. Next we note that since $L_p$ is normal subgroup of $L$, then $L_p$ has a complement, in particular $L/L_p$ is a $p'$-group. Hence $p$-chief factors of $G$ between $L_p$ and $L$ are trivial. Let $H/K$ be a $p$-chief factor of $G$ below $L_p$ and let $P \in \text{Syl}_p(G)$. Then $PK/K \in \text{Syl}_p(G/K)$ and $H/K \cap Z(PK/K) \neq 1$. Let $\overline{P} \in H/K \cap Z(PK/K)$, then $\text{ord}(\overline{P}) = p$ because $H/K$ is elementary $p$-abelian. Since $O^p(G)$ acts by conjugation on $H/K$ and $PK/K$ normalizes $\langle \overline{P} \rangle$ then $\langle \overline{P} \rangle$ is a normal subgroup of $G/K$. But $H/K$ is minimal normal subgroup, hence $H/K = \langle \overline{P} \rangle$ and $|H/K| = p$. Therefore, $p$-chief factors of $G$ below $L_p$ are all cyclic. Hence, we conclude that $G$ is $p$-supersolvable.

Now, $G'$ of a $p$-supersolvable group is $p$-nilpotent, and $L \leq G'$. Hence $L$ is $p$-nilpotent which implies that $L_{p'}$ the complement of $L_p$ is a normal Hall-subgroup of $L$. Hence, $L = L_p \times L_{p'}$.

It is intuitive to suppose that if for each prime divisor $p$ a group $G$ is a $\mathcal{PST}_{C_p}$ then $G$ is a $\mathcal{PST}_c$. The intersection map of subgroups provides a straightforward proof to the latter.
Corollary 1. Let $G$ be a finite group and $F_p \in Syl_p(Fit(G))$ for some prime divisor $p$ of the order of $G$.

1. $G$ is a $\mathcal{T}_{C_p}$-group if and only if $F_p$ is normal cyclic sensitive and a Dedekind group.

2. $G$ is a $\mathcal{PT}_{C_p}$-group if and only if $F_p$ is permutable cyclic sensitive and an Iwasawa group.

3. $G$ is a $\mathcal{PST}_{C_p}$-group if and only if $F_p$ is $S$-permutable cyclic sensitive.

Proof. (1) Suppose that $G$ is a $\mathcal{T}_{C_p}$-group. Then by Lemma $F_p$ is a Dedekind group. Let $X$ be a normal cyclic subgroup of $F_p$. Then $X$ is subnormal subgroup of $G$ and hence is normal in $G$, since $G$ is a $\mathcal{T}_{C_p}$-group. Therefore, $F_p$ is normal cyclic sensitive in $G$. Conversely, let $X$ be a subnormal cyclic $p$-subgroup of $G$. Then $X$ is normal subgroup of $F_p$. Since $F_p$ is normal cyclic sensitive in $G$ then $X$ is a normal subgroup of $G$. Hence, $G$ is a $\mathcal{T}_{C_p}$-group.

(2) Suppose that $G$ is a $\mathcal{PT}_{C_p}$-group. Then by Lemma $F_p$ is an Iwasawa group. Let $X$ be a permutable cyclic subgroup of $F_p$. Then $X$ is subnormal subgroup of $G$ and hence is permutable in $G$, since $G$ is a $\mathcal{PT}_{C_p}$-group. Therefore, $F_p$ is permutable cyclic sensitive in $G$. Conversely, let $X$ be a subnormal cyclic $p$-subgroup of $G$. Then $X$ is permutable subgroup of $F_p$ since $F_p$ is an Iwasawa group by assumption. Since $F_p$ is permutable cyclic sensitive in $G$ then $X$ is a permutable subgroup of $G$. Hence, $G$ is a $\mathcal{PT}_{C_p}$-group.

(3) Suppose that $G$ is a $\mathcal{PST}_{C_p}$-group. Let $X$ be an $S$-permutable cyclic subgroup of $F_p$. Then $X$ is subnormal subgroup of $G$ and hence is $S$-permutable in $G$, sine $G$ is a $\mathcal{PST}_{C_p}$-group. Therefore, $F_p$ is $S$-permutable cyclic sensitive in $G$. Conversely, let $X$ be a subnormal cyclic $p$-subgroup of $G$. Then $X$ is $S$-permutable subgroup of $F_p$. Since $F_p$ is $S$-permutable cyclic sensitive in $G$ then $X$ is an $S$-permutable subgroup of $G$. Hence, $G$ is a $\mathcal{PST}_{C_p}$-group.

Theorem 2. Let $G$ be a finite group.

1. $G$ is a $\mathcal{PST}_c$ group if and only if $G$ is a $\mathcal{PST}_{C_p}$ for all prime divisors $p$ of the order of $G$.

2. $G$ is a $\mathcal{PT}_c$ group if and only if $G$ is a $\mathcal{PT}_{C_p}$ for all prime divisors $p$ of the order of $G$.

3. $G$ is a $\mathcal{T}_c$ group if and only if $G$ is a $\mathcal{T}_{C_p}$ for all prime divisors $p$ of the order of $G$. 

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Proof. (1) Suppose that $G$ is a $\mathcal{PST}_c$-group. Then by Theorem 3.3.3 every Sylow $p$ subgroup of $\text{Fit}(G)$ is S-permutable cyclic sensitive in $G$, for each prime divisor $p$ of the order of $G$. Hence, by Corollary 1.1 $G$ is a $\mathcal{PST}_{C_p}$ for each prime $p$. Conversely, suppose that $G$ is a $\mathcal{PST}_{C_p}$ for all prime divisors $p$ of the order of $G$. Then each Sylow subgroup of $\text{Fit}(G)$ is S-permutable cyclic sensitive by Corollary 1.1 and therefore by Theorem 3.3.3 $G$ is a $\mathcal{PST}_c$.

(2) Suppose that $G$ is a $\mathcal{PT}_c$-group. Then by Theorem 3.3.3 every subgroup of a Sylow $p$ subgroup of $\text{Fit}(G)$ is permutable cyclic sensitive in $G$, for each prime divisor $p$ of the order of $G$. By a similar argument as in Theorem 3.3.1 we get that each Sylow subgroup of the $\text{Fit}(G)$ is an Iwasawa group and permutable cyclic sensitive in $G$. Hence, by Corollary 1.1 $G$ is a $\mathcal{PT}_{C_p}$ for each prime $p$. Conversely, suppose that $G$ is a $\mathcal{PT}_{C_p}$ for all prime divisors $p$ of the order of $G$. Then each Sylow subgroup of $\text{Fit}(G)$ is permutable cyclic sensitive and an Iwasawa group by Corollary 1.1. Let $F_p$ be a Sylow $p$ subgroup of $\text{Fit}(G)$. Let $H$ be any subgroup of $F_p$ and $X$ any permutable cyclic subgroup of $H$. Since, $F_p$ is an Iwasawa group then $X$ is permutable subgroup of $F_p$. But $F_p$ is also permutable cyclic sensitive in $G$, hence $X$ is permutable subgroup of $G$. This means that every subgroup of each Sylow subgroup of the $\text{Fit}(G)$ is permutable cyclic sensitive in $G$. Therefore, by Theorem 3.3.3 $G$ is a $\mathcal{PT}_c$.

(3) Suppose that $G$ is a $\mathcal{T}_c$-group. Then by Theorem 3.3.3 every subgroup of a Sylow $p$ subgroup of $\text{Fit}(G)$ is normal cyclic sensitive in $G$, for each prime divisor $p$ of the order of $G$. By a similar argument as in Theorem 3.3.1 we get that each Sylow subgroup of the $\text{Fit}(G)$ is a Dedekind group and normal cyclic sensitive in $G$. Hence, by Corollary 1.1 $G$ is a $\mathcal{T}_{C_p}$ for each prime $p$. Conversely, suppose that $G$ is a $\mathcal{T}_{C_p}$ for all prime divisors $p$ of the order of $G$. Then each Sylow subgroup of $\text{Fit}(G)$ is normal cyclic sensitive and a Dedekind group by Corollary 1.1. Let $F_p$ be a Sylow $p$ subgroup of $\text{Fit}(G)$. Let $H$ be any subgroup of $F_p$ and $X$ any normal cyclic subgroup of $H$. Since, $F_p$ is a Dedekind group then $X$ is a normal subgroup of $F_p$. But $F_p$ is also normal cyclic sensitive in $G$, hence $X$ is a normal subgroup of $G$. This means that every subgroup of each Sylow subgroup of the $\text{Fit}(G)$ is normal cyclic sensitive in $G$. Therefore, by Theorem 3.3.3 $G$ is a $\mathcal{T}_c$. 

\[\Box\]
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