Some Applications of the Weierstrass Mean Value Theorem

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Some applications of the Weierstrass
mean value theorem

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1. Introduction. Let $f(z)$ be an analytic function holomorphic in some
domain $D$ of the complex plane. Then $f(z)$ is said to be univalent in $D$ if for any
two points in $D$ of affix $z_1$ and $z_2$ it is true that $f(z_1) = f(z_2)$ implies $z_1 = z_2$. Now
in order that $f(z)$ be univalent in $D$ it is necessary, but not sufficient, that $f'(z) \neq 0$ for $z \in D$. If, however, we specialize $D$ and require that $D$ be convex then
Re $\{f'(z)\} > 0$, $z \in D$, implies that $f(z)$ is univalent in $D$. This latter result was
first proved by J. Wolff [7] for the case in which $D$ is a half-plane and general-
alized by Noshiro and Warschawski [2], [6] to convex domains. On the other
hand Herzog and Piranian [1] demonstrated that if a domain $D$ has the prop-
erty that $f(z)$ is univalent in $D$ whenever Re $\{f'(z)\} > 0$ throughout $D$ then
the domain is not far from being convex. Thus if we propose to obtain condi-
tions for univalency in non-convex domains, we must expect some restrictions
on $f'(z)$ other than Re $\{f'(z)\} > 0$. In this note some conditions of this nature
are given, bearing mostly on Arg$f'(z)$ for $z \in D$. These conditions arise in a
natural manner from an application of a theorem due to Weierstrass [5]. The
results are similar to those obtained by Reade [3] whose results are based upon
a criterion for univalency due to Umezawa [4]. The theorem of Weierstrass
employed is the following

**Theorem** (Weierstrass). Let $C$ be a rectifiable curve joining the points of affix
$z = \alpha$ and $z = \beta$. Let $z = z(t)$ ($a \leqq t \leqq b$) be a parametric representation of $C$. Then
if $f(z)$ is continuous on $C$ and $g(z)$ is continuous and positive on $C$
$$\int_{a}^{b} f(z)g(z)dt = X\int_{a}^{b} g(z)dt$$
where the point of affix $X$ is contained in the convex hull of the set of values
which $f(z)$ assumes on $C$.

2. Applications of the Weierstrass theorem. For the sake of completeness
we begin with a proof of the Noshiro-Warschawski **Theorem**.

**Theorem** (Noshiro-Warschawski). A function regular in a convex domain $D$
is univalent in $D$ if Re $\{f'(z)\} > 0$ for $z \in D$.

**Proof.** Let $z_1$ and $z_2$ be two distinct points of $D$. Then upon integrating
the derivative of $f(z)$ along the line segment connecting $z_1$ and $z_2$ we have
Applications of the Weierstrass mean value theorem

\[ f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z) \, dz \]

\[ = (z_2 - z_1) \int_0^1 f'(z_1 + (z_2 - z_1)t) \, dt. \]

But by the Weierstrass theorem the latter integral has the value \( Z^* \), where \( Z^* \) is the affix of a point lying in the convex hull \( C^* \) of the image of the line segment \( z_1 z_2 \) under the mapping effected by \( f'(z) \). But since \( \text{Re}\{f'(z)\} > 0 \) for \( z \in D \) it follows that the origin is not in \( C^* \) and hence \( f(z_2) - f(z_1) = Z^*(z_2 - z_1) \neq 0 \) unless \( z_2 = z_1 \).

Remark. Clearly the theorem is still valid if for some \( 0 \leq \alpha \leq 2\pi \), \( \text{Re}\{e^{-i\alpha}f'(z)\} > 0 \) for \( z \in D \).

We next prove a theorem first given by Reade [3]. Our proof differs from Reade’s in that whereas Reade employed a condition for univalency due to Umezawa we use the mean value theorem of Weierstrass.

Theorem 2.1. Let \( \varphi \) be fixed and \( 0 \leq \varphi \leq \pi \). Let \( D \) be a domain in which it is possible to join each pair of distinct points \( z_1, z_3 \) by a pair of straight line segments \( \overline{z_1 z_2}, \overline{z_2 z_3} \) lying in \( D \) such that

\[ |\text{Arg} \frac{z_3 - z_2}{z_2 - z_1}| \leq \varphi \]

\( \varphi \) being independent of \( z_1 \) and \( z_3 \) in \( D \). Then if \( f(z) \) is analytic in \( D \) and if for some \( 0 \leq \alpha \leq 2\pi \)

\[ \alpha \leq \text{Arg} f'(z) \leq \pi - \varphi + \alpha \]

for \( z \in D \), \( f(z) \) is univalent in \( D \).

Proof. Let \( z_1 \) and \( z_3 \) be two distinct points of \( D \) and let \( z_2 \neq z_1, \neq z_3 \) be a point of \( D \) such that the line segments \( \overline{z_1 z_2}, \overline{z_2 z_3} \) lie in \( D \) and satisfy (1). Now we may write

\[ f(z_3) - f(z_1) = \int_{z_1}^{z_3} f'(z) \, dz + \int_{z_3}^{z_2} f'(z) \, dz \]

\[ = (z_2 - z_1) \int_0^1 f'(z_1 + (z_2 - z_1)t) \, dt \]

\[ + (z_3 - z_2) \int_0^1 f'(z_2 + (z_3 - z_2)t) \, dt. \]

Hence by the Weierstrass mean value theorem we have

\[ f(z_3) - f(z_1) = Z_1^*(z_2 - z_1) + Z_2^*(z_3 - z_2) \]

\[ = Z_1^*(z_2 - z_1) \left[ 1 + \frac{Z_2^*}{Z_1^*} \frac{z_3 - z_2}{z_2 - z_1} \right], \]

since by hypothesis, \( |\text{Arg}(Z_2^*/Z_1^*)| < \pi - \varphi \) it follows from (1) that
Therefore the quantity in the brackets in the last member of (2) cannot vanish and, by (2), \( f(z_{3}) \neq f(z_{1}) \).

**Theorem 2.2.** Let \( D \) be a domain any two points of which may be joined by the arc (lying in \( D \)) of an ellipse \( z = z_{0} + e^{i\beta}(a \cos t + ib \sin t) \), for which \( t_{1} \leq t \leq t_{2} \), \( 0 \leq t_{1}, t_{2} \leq \frac{\pi}{2} \), and where \( a, b > 0, \beta \) real. \( (a, b, \beta \) may depend on the two points to be connected.) Then if \( f(z) \) is analytic in \( D \) and if for some \( 0 \leq \alpha \leq 2\pi \)

\[ \alpha < \text{Arg} f'(z) < \alpha + \frac{\pi}{2} \]

for all \( z \in D, f(z) \) is univalent in \( D \).

**Proof.** Let \( z_{1} \) and \( z_{3} \) be two points of \( D \) then

\[
f(z_{2}) - f(z_{1}) = \int_{z_{1}}^{z_{2}} f'(z) dz
\]

\[= e^{i\beta} \int_{t_{1}}^{t_{2}} f'(z)(-a \sin t + ib \cos t) dt,\]

where \( z_{k} = z_{0} + e^{i\beta}(a \cos t_{k} + ib \sin t_{k}) \), \( k = 1, 2 \).

Applying the Weierstrass theorem and simplifying yields

\[
f(z_{2}) - f(z_{1}) = e^{i\beta} \left[ -a \int_{t_{1}}^{t_{2}} f'(z) \sin t dt + ib \int_{t_{1}}^{t_{2}} f'(z) \cos t dt \right]
\]

\[= Z_{2}^{*} e^{i\beta} \left[ \frac{Z_{1}^{*}}{Z_{2}^{*}} \Re \{ e^{-i\beta}(z_{2} - z_{1}) \} + i \Im \{ e^{-i\beta}(z_{2} - z_{1}) \} \right],\]

where \( Z_{1}^{*} \) and \( Z_{2}^{*} \) are points in the convex hull of the sets which are images of the elliptic arcs from \( z_{1} \) to \( z_{2} \) under the mapping effected by \( f'(z) \). It results from the hypothesis that \( |\text{Arg}(Z_{1}^{*}/Z_{2}^{*})| < \frac{\pi}{2} \) and hence, since \( Z_{2}^{*} \neq 0 \), the right hand member of (3) is never zero unless \( z_{2} - z_{1} \) so that \( f(z) \) is univalent in \( D \).

**Definition.** A domain \( D \) is said to have property \( U \) with constant \( \theta = \theta(D) \) if when \( z_{1} \) and \( z_{2} \) in \( D \) are given there exists a constant \( \theta \) (independent of \( z_{1} \) and \( z_{2} \)), \( \theta < \pi \), and a sequence of points \( z_{1} = \zeta_{0}, \zeta_{1}, \zeta_{2}, \cdots, \zeta_{n} = z_{2} \) with the property that the segments \( \overline{\zeta_{0}\zeta_{1}}, \overline{\zeta_{1}\zeta_{2}}, \cdots, \overline{\zeta_{n-1}\zeta_{n}} \) all belong to \( D \) and

\[
|\text{Arg} \frac{\zeta_{k+1} - \zeta_{k}}{z_{2} - z_{1}}| \leq \theta, \quad k = 0, 1, \cdots, n - 1.
\]

**Remark.** A convex domain has the property \( U \) with constant \( \theta = 0 \).

**Theorem 2.3.** Let \( D \) be a domain having the property \( U \). Then if for some \( 0 \leq \alpha \leq 2\pi \), \( \text{Arg} f'(z) \) satisfies \( \alpha + \theta < \text{Arg} f'(z) < \pi + \alpha - \theta, \theta < \frac{\pi}{2} \), for \( z \) in \( D; f(z) \)
is univalent in $D$.

**Proof.** We may write

$$f(z_2) - f(z_1) = \sum_{k=0}^{n-1} \int_{z_k}^{z_{k+1}} f'(z)dz = (z_2 - z_1) \sum_{k=0}^{n-1} \lambda_k \frac{z_{k+1} - z_k}{z_2 - z_1},$$

where the vectors $\lambda_k$ satisfy $\alpha + \theta < \text{Arg} \lambda_k < \pi + \alpha - \theta$ for some $0 \leq \alpha \leq 2\pi$. The domain possesses the property $U$ and we have

$$\alpha < \text{Arg} \lambda_k \frac{z_{k+1} - z_k}{z_2 - z_1} < \alpha + \pi, \quad k = 0, 1, \ldots, n-1,$$

and consequently

$$\sum_{k=0}^{n-1} \lambda_k \frac{z_{k+1} - z_k}{z_2 - z_1} \neq 0$$

which implies $f(z)$ is univalent.

The following interesting theorem is obtained if $\theta(D) = \frac{\pi}{2}$.

**Theorem 2.4.** Let $D$ be a domain any two points of which may be joined by a finite number of line segments (lying in $D$) and parallel to the coordinate axis such that the $x$ and $y$ coordinates of the end points form either a non-decreasing or non-increasing sequence. Then if for some $0 \leq \alpha \leq 2\pi$

$$\alpha < \text{Arg} f'(z) < \alpha + \frac{\pi}{2}$$

for $z \in D$, $f(z)$ is univalent in $D$.

The proof follows along the same lines of Theorem 2.3 and 2.1.

**3. Examples.** We can easily construct examples for which Theorem 2.3 and 2.3 apply but for which Reade's theorem does not apply. For example consider the region in the first quadrant bounded by two confocal ellipses with center at the origin and with foci at $\pm 1$. If the difference between their corresponding semi axes is sufficiently small the region does not satisfy the criterion of Reade's theorem. Theorem 2.2 or 2.3 applied to the function $f(z) = z \sin^{-1}z + \sqrt{1-z^2}$ in this region shows that $f(z)$ is univalent in the region.

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